SOME PROPERTIES OF HOLOMORPHIC SELF-MAPS OF THE UNIT BALL IN $\mathbb{C}^n$

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Abstract

Let f be a holomorphic self-map of the unit ball in $\mathbb{C}^n$. In this note, we give information about relative positions of zeros and interior fixed-point sets of f, and the point $f(0)$.

1. Introduction

Let $f(z)$ be a holomorphic mapping from the unit ball in $\mathbb{C}^n$ into itself. Our interest is in finding information about locations of the fixed-point sets and the zero-sets of $f(z)$. When $f(z)$ is a holomorphic function from the unit disk in $\mathbb{C}$ into itself, C. Craig, Jr. and A.J. Macintyre [2] obtained several relations of positions of inner fixed points, zeros and the value $f(0)$. In this note, we will extend the results of Craig and Macintyre to the space $\mathbb{C}^n$. Combining these results, we will get more precise information concerning zero-sets and fixed-point sets.

2. Definitions and Preliminaries

The space $\mathbb{C}^n$ is the n-dimensional vector space of ordered n-tuples $z := (z_1, \cdots, z_n)$ of complex numbers with the inner product

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \bar{w}_j \qquad (z, w \in \mathbb{C}^n),$$

and the associated norm denoted by

$$\|z\| := \sqrt{\langle z, z \rangle} \qquad (z \in \mathbb{C}^n).$$

Let $\mathbb{B}_\gamma$ be the open ball in $\mathbb{C}^n$ with radius γ and center 0, *i.e.*,

$$\mathbb{B}_\gamma := \{z \in \mathbb{C}^n : \|z\| < \gamma\},$$

and let $\mathbb{B} := \mathbb{B}_1$. As usual, $\partial\mathbb{B}_\gamma$ and $\bar{\mathbb{B}}_\gamma$ denote the boundary and the closure of $\mathbb{B}_\gamma$, respectively.

Let $\mathcal{E}(\mathbb{B})$ be the family of holomorphic mappings from $\mathbb{B}$ to $\mathbb{B}$.

For an $f \in \mathcal{E}(\mathbb{B})$, we will denote the fixed-point set and the set of zeros of f by

$$F(f) := \{z \in \mathbb{B} : f(z) = z\}$$

and

$$z(f) := \{z \in \mathbb{B} : f(z) = 0\},$$

respectively. Then the fixed-point set $F(f)$ is affine ([3], p. 166, Theorem 8.2.3).

For $a \in \mathbb{B}$, let us put

$$P_a\, z := \begin{cases} \frac{\langle z, a\rangle}{\|a\|^2}\, a & (a \neq 0) \\ 0 & (a = 0), \end{cases}$$

$$Q_a := I - P_a \quad (I: \text{ the identity}), \quad s_a := \sqrt{1 - \|a\|^2},$$

and

$$\phi_a(z) := \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle} \qquad (z \in \mathbb{B}).$$

Then the following theorem holds:

Theorem A ([3], p. 26, Theorem 2.2.2). *Let $a \in \mathbb{B}$. Then the following relations hold true*:

(i) $\phi_a(0) = a$; $\quad \phi_a(a) = 0$;

(ii) $1 - \langle \phi_a(z), \phi_a(w)\rangle = \frac{(1-\|a\|^2)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)} \qquad (z, w \in \bar{\mathbb{B}})$;

(iii) $1 - \|\phi_a(z)\|^2 = \frac{(1-\|a\|^2)(1-\|z\|^2)}{|1-\langle z,a\rangle|^2} \qquad (z \in \bar{\mathbb{B}})$;

(iv) $\phi_a(\phi_a(z)) = z$;

(v) ϕ_a *is a homeomorphism of* $\bar{\mathbb{B}}$ *onto* $\bar{\mathbb{B}}$ *and an automorphism of* $\mathbb{B}$.

The following Schwarz lemma plays an essential rôle in our investigation.

Theorem B ([3], p. 163, Theorem 8.1.4). *Let $f \in \mathcal{E}(\mathbb{B})$ and $a \in \mathbb{B}$. Then*

$$\|\phi_{f(a)}(f(z))\| \leq \|\phi_a(z)\| \qquad (z \in \mathbb{B}).$$

3. The Main Results

Theorem 1. *Let $f \in \mathcal{E}(\mathbb{B})$. If $f(0) \neq 0$, then $f(z)$ does not have zeros in $\mathbb{B}_{\|f(0)\|}$. If $f(z)$ is an automorphism, then $f(z)$ has a zero on the boundary $\partial\mathbb{B}_{\|f(0)\|}$.*

Proof. By Theorem B, we get

$$\|\phi_{f(0)}(f(z))\| \leq \|\phi_0(z)\| = \|z\|.$$

Thus, if $f(z)$ has a zero at $z = z_0$, then, by Theorem A(i), we have

$$\|f(0)\| = \|\phi_{f(0)}(0)\| \leq \|z_0\|.$$

Therefore, $f(z)$ does not have zeros in $\mathbb{B}_{\|f(0)\|}$.

If $f(z)$ is an automorphism of $\mathbb{B}$, then, by Theorem 2.2.5 in [3], $f(z)$ has a representation

$$f(z) = U\phi_a,$$

where $a = f(0)$ and U is a unitary transformation. So, by Theorem A, $f(z)$ has a zero on $\partial\mathbb{B}_{\|f(0)\|}$.

Example 1. Let $|a| < 1$. In $\mathbb{C}^2$, let us put

$$f(z_1, z_2) = \left(\frac{z_1 - a}{1 - \bar{a}z_1}, 0\right).$$

Then $f(z) \in \mathcal{E}(\mathbb{B})$, $f(0) = (-a, 0)$, $z(f) = (a, z_2)$ ($|z_2| < \sqrt{1 - |a|^2}$), and $f(z)$ is not an automorphism of $\mathbb{B}$. This example shows that the converse of Theorem 1 does not hold true.

Theorem 2. *Let $f \in \mathcal{E}(\mathbb{B})$. If $F(f) \neq \emptyset$, then $f(z)$ has no zeros in the set $\cup_{a \in F(f)} E_a$, where*

$$E_a := \left\{z \in \mathbb{B} : \|z\|^2 + |\langle z, a\rangle|^2 - \left(\langle z, a\rangle + \overline{\langle z, a\rangle}\right) < 0\right\}.$$

If f is an automorphism of $\mathbb{B}$, then $f(z)$ has a zero on the set E_a.

Proof. For any $a \in F(f)$, by Theorem B, we have

$$\|\phi_{f(a)}(f(z))\| \leq \|\phi_a(z)\|.$$

Thus

$$\|\phi_a(f(z))\| \leq \|\phi_a(z)\|.$$

If there exists a point $z_0 \in Z(f)$, then we get

$$\|\phi_a(f(z_0))\| = \|\phi_a(0)\| = \|a\| \leq \|\phi_a(z_0)\|.$$

Therefore, by Theorem A (iii),

$$\|a\|^2 \leq 1 - \frac{(1-\|a\|^2)(1-\|z_0\|^2)}{|1-\langle z_0, a\rangle|^2}.$$

From this we obtain

$$\|z_0\|^2 + |\langle z_0, a\rangle|^2 - \left(\langle z_0, a\rangle + \overline{\langle z_0, a\rangle}\right) \geq 0.$$

Thus $f(z)$ has no zeros in the set

$$E_a := \left\{z \in \mathbb{B} : \|z\|^2 + 1\langle z, a\rangle|^2 - \left(\langle z, a\rangle + \overline{\langle z, a\rangle}\right) < 0\right\}.$$

When $f(z)$ is an automorphism, we put $f(z) = U\,\phi_a(z)$ and $\phi_a(z)$ has a unique fixed point $a/(1+s)$ ([3], p. 34, Theorem 2.4.7). Thus $f(z)$ has a zero on the boundary $\partial\mathbb{B}_\gamma$.

Remark. For any $a \in \mathbb{C}$ $(a \neq 0)$, the set

$$\left\{z \in \mathbb{C}^n : \|z\|^2 + |\langle z, a\rangle|^2 - \left(\langle z, a\rangle + \overline{\langle z, a\rangle}\right) < 0\right\}$$

is an elliptic surface through a and 0. Consequently, E_a is a non-empty set.

Theorem 3. *Let $f \in \mathcal{E}(\mathbb{B})$. If $f(0) \neq 0$, then $f(z)$ does not have fixed points in the ball $\mathbb{B}_\gamma$, where*

$$\gamma := \frac{1-\sqrt{1-\|f(0)\|^2}}{\|f(0)\|}.$$

If $F(f) \neq \emptyset$, then

$$F(f) \subset \left\{z \in \mathbb{B} : |1 - \langle z, f(0)\rangle| \leq \sqrt{1 - \|f(0)\|^2}\right\}.$$

Proof. By Theorem B,

$$\|\phi_{f(0)}(f(z))\| \leq \|\phi_0(z)\| = \|z\|.$$

Thus, if $F(f) \neq \emptyset$ and $z_0 \in F(f)$, then we obtain

$$\|\phi_{f(0)}(f(z_0))\| = \|\phi_{f(0)}(z_0)\| \leq \|z_0\|.$$

Therefore

$$\|\phi_{f(0)}(z_0)\|^2 = 1 - \frac{(1 - \|f(0)\|^2)(1 - \|z_0\|^2)}{|1 - \langle z_0, f(0)\rangle|^2} \leq \|z_0\|^2.$$

From this, we get the inequality

$$|1 - \langle z_0,\, f(0)\rangle|^2 \leq (1 - \|f(0)\|^2).$$

Thus $f(z)$ does not have fixed points in the ball

$$\mathbb{B}_\gamma := \left\{z \in \mathbb{B} : \|z\| < \frac{1 - \sqrt{1 - \|f(0)\|^2}}{\|f(0)\|}\right\}.$$

Example 2. In $\mathbb{C}^2$, let us put

$$f(z_1, z_2) := \left(\frac{1}{4}\,[1 + (1 + \delta)z_1]\,,\ \frac{1}{2}\,z_2 z_1\right),$$

$$(\delta > 0 : \text{sufficiently small}).$$

Then

$$F(f) = \left(\frac{1}{3 - \delta}, 0\right), \quad Z(f) = \left(-\frac{1}{1 + \delta}, 0\right),$$

and

$$f(0) = \left(\frac{1}{4}, 0\right).$$

Theorem 4. *Let $f \in \mathcal{E}(\mathbb{B})$. Suppose that x is the point of $\mathbb{B}$ such that $f(x) = \lambda x$ $(\lambda > 0)$ and $f(x) \neq x$. Then $f(z)$ does not have fixed points on the opposite side of $f(x)$ with respect to the cylindrical surface*

$$S_x := \left\{ z \in \mathbb{C}^n : |2\langle z, x\rangle - (1 + \|x\|^2)| = (1 - \|x\|^2) \right\}.$$

(The surface S_x is orthogonal to the radius $0x$ and through the point x.)

Proof. By Theorem B, we have

$$\|\phi_{f(x)}(f(z))\| \leq \|\phi_x(z)\|.$$

And, by the assumption,

$$\begin{aligned}\|\phi_{f(x)}(f(z))\|^2 &= \|\phi_{\lambda x}(f(z))\|^2 \\ &= 1 - \frac{(1 - \lambda^2 \|x\|^2)(1 - \|f(z)\|^2)}{|1 - \lambda\langle f(z), x\rangle|^2}.\end{aligned}$$

So, if $f(z)$ has a fixed point z_0, then we obtain

$$\frac{(1 - \|x\|^2)(1 - \|z_0\|^2)}{|1 - \langle z_0, x\rangle|^2} \leq \frac{(1 - \lambda^2\|x\|^2)(1 - \|z_0\|^2)}{|1 - \lambda\langle z_0, x\rangle|^2}.$$

Thus we get

$$\frac{1 - \|x\|^2}{|1 - \langle z_0, x\rangle|^2} \leq \frac{1 - \lambda^2\|x\|^2}{|1 - \lambda\langle z_0, x\rangle|^2},$$

and so

$$\begin{aligned}0 &\leq (1 - \lambda^2\|x\|^2)\,|1 - \langle z_0, x\rangle|^2 - (1 - \|x\|^2)\,|1 - \lambda\langle z_0, x\rangle|^2 \\ &= \left\{(1 + \lambda)|\langle z_0, x\rangle|^2 - (1 + \lambda\|x\|^2)\left(\langle z_0, x\rangle + \langle \overline{z_0, x}\rangle\right) + (1 + \lambda)\|x\|^2\right\}(1 - \lambda).\end{aligned}$$

Here we put

$$\begin{aligned}\Phi_\lambda(X) &:= (1 + \lambda)|X|^2 - (1 + \lambda\|x\|^2)(X + \bar{X}) + (1 + \lambda)\|x\|^2 \\ &= (1 + \lambda)\left|X - \frac{1 + \lambda\|x\|^2}{1 + \lambda}\right|^2 - \frac{(1 - \|x\|^2)(1 - \lambda^2\|x\|^2)}{1 + \lambda}.\end{aligned}$$

Then $\Phi_\lambda(X) \to \Phi_1(X)$ $(\lambda \to 1)$, where

$$\Phi_1(X) := 2\left|X - \frac{1 + \|x\|^2}{2}\right|^2 - \frac{(1 - \|x\|^2)^2}{2}.$$

Thus S_x can be represented as

$$S_x = \{z \in \mathbb{C}^n : \Phi_1(\langle z, x\rangle) = 0\},$$

and, clearly, S_x is orthogonal to the radius $0x$, and the point z_0 satisfies the inequality

$$\Phi_\lambda(\langle z_0, x\rangle)(1-\lambda) \geq 0.$$

Here, there are two cases to consider:

Case I. $\lambda > 1$. Thus $\Phi_\lambda(\langle z_0, x\rangle) \leq 0$, and we can easily find that

$$\{X \in \mathbb{C} : \Phi_\lambda(X) \leq 0\} \subset \{X \in \mathbb{C} : \Phi_1(X) \leq 0\}.$$

Since $f(x) \in \{z \in \mathbb{C}^n : \Phi_1(\langle z, x\rangle) \leq 0\}$, the point z_0 lies on the same side as $f(x)$ with respect to S_x.

Case II. $0 < \lambda < 1$. $\Phi_\lambda(\langle z_0, x\rangle) \geq 0$ and

$$\{X \in \mathbb{C} : \Phi_\lambda(X) \geq 0\} \subset \{X \in \mathbb{C} : \Phi_1(X) \geq 0\}.$$

And $f(x) \in \{z \in \mathbb{C}^n \mid \Phi_1(\langle z, x\rangle) \geq 0\}$. Thus z_0 lies on the same side as $f(x)$ with respect to S_x.

Consequently, in both cases, $f(z)$ does not have fixed points on the opposite side of $f(x)$ with respect to S_x, and the proof of Theorem 4 is completed.

Theorem 5. *Let $f(z) \in \mathcal{E}(\mathbb{B})$. Suppose that there exists a point $x \in \mathbb{B}$ ($x \neq 0$) such that $f(x) = -\lambda x$ ($\lambda > 0$). If $\lambda \geq 1$, then $f(z)$ does not have fixed points on the opposite side of $f(x)$ (same side as x) with respect to the hyperplane*

$$H_x := \{z \in \mathbb{C}^n : \Re\langle z, x\rangle = 0\}.$$

If $1 > \lambda > 0$, $f(z)$ does not have fixed points in the domain

$$D_x := \left\{z \in \mathbb{B} : \left|\langle z, x\rangle - \frac{1-\lambda\|x\|^2}{1-\lambda}\right| < K\right\},$$

where

$$K = \frac{\sqrt{(1-\|x\|^2)(1-\lambda^2\|x\|^2)}}{1-\lambda}.$$

Proof. By the same way as in the proof of Theorem 4, if $f(z)$ has a fixed point z_0, then

$$\frac{1-\|x\|^2}{|1-\langle z_0, x\rangle|^2} \leq \frac{1-\lambda^2\|x\|^2}{|1+\lambda\langle z_0, x\rangle|^2},$$

so, putting $X := \langle z_0, x\rangle$,we get

$$\Phi_\lambda(X) := (1-\lambda)|X|^2 - (1-\lambda\|x\|^2)(X+\bar{X}) + (1-\lambda)\|x\|^2 \geq 0.$$

If $\gamma \geq 1$, then there are two cases to consider:

Case I: $\lambda = 1$. $\Phi_1(X) = -(1-\|x\|^2)(X+\bar{X})$. So

$$\Phi_1(X) \geq 0 \Longleftrightarrow \Re(X) \leq 0 \Longleftrightarrow \Re\langle z_0, x\rangle \leq 0.$$

Case II: $\lambda > 1$.

$$\Phi_\lambda(X) = (1-\lambda)\left|X + \frac{1-\lambda\|x\|^2}{\lambda-1}\right| - \frac{(1-\|x\|^2)(1-\lambda^2\|x\|^2)}{1-\lambda}.$$

$$\Phi_\lambda(X) \geq 0 \Longleftrightarrow \left|X + \frac{1-\lambda\|x\|^2}{\lambda-1}\right| \leq \frac{\sqrt{(1-\|x\|^2)(1-\lambda^2\|x\|^2)}}{\lambda-1},$$

this implies $\Re X \leq 0$ (*i.e.*, $\Re\langle z_0, x\rangle \leq 0$).

Thus, in Cases I and II, $f(z)$ does not have fixed points on the opposite side of $f(x)$ with respect to the hyperplane H_x.

If $0 < \lambda < 1$,

$$\Phi_\lambda(X) = (1-\lambda)\left|X - \frac{1-\lambda\|x\|^2}{1-\lambda}\right|^2 - \frac{(1-\|x\|^2)(1-\lambda^2\|x\|^2)}{1-\lambda}.$$

$$\Phi_\lambda(X) \geq 0 \Longleftrightarrow \left|X - \frac{1-\lambda\|x\|^2}{1-\lambda}\right|^2 \geq \frac{\sqrt{(1-\|x\|^2)(1-\lambda^2\|x\|^2)}}{1-\lambda}.$$

Thus $f(z)$ does not have fixed points in D_x.

Remark. In Theorem 5, in the case when $0 < \gamma < 1$, since

$$\frac{(1-\lambda)\|x\|^2}{2(1-\lambda\|x\|^2)} < \|x\|,$$

the set

$$\left\{ z \in \mathbb{B} : \Re\langle z, x\rangle > \frac{(1-\lambda)\|x\|^2}{2(1-\lambda\|x\|^2)} \right\}$$

is non-empty and contained in D_x. Thus $f(z)$ has no fixed points on the same side as x with respect to the hyperplane

$$\left\{ z \in \mathbb{C}^n : \Re\langle z, x\rangle = \frac{(1-\lambda)\|x\|^2}{2(1-\lambda\|x\|^2)} \right\}.$$

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