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EXACT TRAVELING-WAVE SOLUTIONS FOR LINEAR AND NON-LINEAR HEAT TRANSFER EQUATIONS

by

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The exact traveling-wave solutions for the linear and non-linear heat transfer equations at several different excess temperatures are addressed and investigated in this paper.

Key words: *heat transfer equations, travelling-wave transformation, excess temperatures, exact solution*

Introduction

Ordinary differential equations (ODE) and partial differential equations (PDE) were used to describe the thermal problems in engineering sciences (see [1] and several related earlier references which are cited therein). Especially in heat transfer problems, the PDE [2] were adopted to govern the excess temperature fields in materials. In recent years, many different techniques were developed to derive the exact solutions for the heat transfer equations, such as the tanh method [3], exp-function method [4], (G'/G)-expansion method [5], heat-balance integral method [6], traveling-wave transformation method (TTM) [7, 8], and other methods [9-12].

However, the traveling-wave solutions of the heat transfer problems at several different excess temperatures have not yet been investigated. Motivated by the previous investigations, the aim of the present paper is to propose the traveling-wave solutions for the linear and nonlinear heat transfer equations.

The method applied

In order to introduce the concept of the traveling-wave solution, we consider the following PDE with respect to ξ and τ :

$$\aleph \left[\frac{\partial \Theta(\xi, \tau)}{\partial \tau}, \frac{\partial^2 \Theta(\xi, \tau)}{\partial \xi^2}, \Theta^n(\xi, \tau) \right] = 0 \quad (1)$$

where n is a positive integer.

Following the argument in [7, 8], we set up the TTM, which is given by:

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$$\omega = \xi - \gamma\tau \quad (2)$$

where γ is a constant.

With the aid of the following chain rules:

$$\frac{\partial\Theta(\xi, \tau)}{\partial\tau} = -\gamma \frac{\partial\Theta(\omega)}{\partial\omega} \quad (3)$$

and

$$\frac{\partial^2\Theta(\xi, \tau)}{\partial\xi^2} = \frac{\partial^2\Theta(\omega)}{\partial\omega^2} \quad (4)$$

Equation (1) can be transformed into the ODE with respect to ω , which is given by:

$$\aleph \left[-\gamma \frac{d\Theta(\omega)}{d\omega}, \frac{d^2\Theta(\omega)}{d\omega^2}, \Theta^n(\omega) \right] = 0 \quad (5)$$

After obtaining the solution of eq. (5) by using the mathematical software, if we substitute eq. (2) into the obtained solution, we get the traveling-wave solution.

Traveling-wave solutions for linear and non-linear heat transfer problems

At first, we consider the linear heat transfer equation at the low excess temperature as follows, [2]:

$$\frac{\partial\Xi(\xi, \tau)}{\partial\tau} = \alpha \frac{\partial^2\Xi(\xi, \tau)}{\partial\xi^2} - \beta\Xi(\xi, \tau) \quad (6)$$

where α is the heat-diffusion coefficient, β – a constant, and $\Xi(\xi, \tau)$ – the excess temperature.

Following eq. (2), eq. (6) can be written:

$$-\gamma \frac{d\Xi(\omega)}{d\omega} = \alpha \frac{d^2\Xi(\omega)}{d\omega^2} - \beta\Xi(\omega) \quad (7)$$

With the help of the integrating-factor method [13] or the MATLAB software, the exact solution of eq. (7) is given by, [13]:

$$\Xi(\omega) = \begin{cases} \varpi_1 e^{-k_1\omega} + \varpi_2 e^{-k_2\omega}, & \gamma^2 + 4\alpha\beta > 0, \\ (\varpi_3 + \varpi_4\omega) e^{-\lambda\omega/2}, & \gamma^2 + 4\alpha\beta = 0, \\ e^{-\lambda\omega/2} (\varpi_5 \cos \varphi\omega + \varpi_6 \sin \varphi\omega), & \gamma^2 + 4\alpha\beta < 0 \end{cases} \quad (8)$$

where $\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5$ and ϖ_6 are constants, $\lambda = -\gamma / \alpha$, $k_1 = [-\gamma + (\gamma^2 + 4\alpha\beta)^{1/2}] / 2\alpha$, $k_2 = [-\gamma - (\gamma^2 + 4\alpha\beta)^{1/2}] / 2\alpha$ and $\varphi = [-(\gamma^2 + 4\alpha\beta)^{1/2}] / 2\alpha$.

Substituting eq. (2) into eq. (8), we obtain

$$\Xi(\xi, \tau) = \begin{cases} \varpi_1 e^{-k_1(\xi-\gamma\tau)} + \varpi_2 e^{-k_2(\xi-\gamma\tau)}, & \gamma^2 + 4\alpha\beta > 0, \\ [\varpi_3 + \varpi_4(\xi - \gamma\tau)] e^{-\lambda(\xi-\gamma\tau)/2}, & \gamma^2 + 4\alpha\beta = 0, \\ e^{-\lambda(\xi-\gamma\tau)/2} [\varpi_5 \cos \varphi(\xi - \gamma\tau) + \varpi_6 \sin \varphi(\xi - \gamma\tau)], & \gamma^2 + 4\alpha\beta < 0 \end{cases} \quad (9)$$

The graphs of the traveling-wave solutions in eq. (6) are illustrated in figs. 1(a)-1(c).

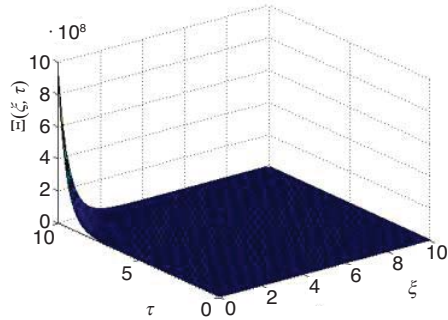


Figure 1(a). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta > 0$

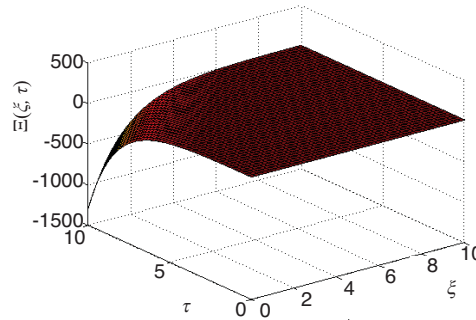


Figure 1(b). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta = 0$

As the second example, let us consider the following non-linear heat transfer equation at the high excess temperature (see [14, p. 159]):

$$\frac{\partial \Xi(\xi, \tau)}{\partial \tau} = \alpha \frac{\partial^2 \Xi(\xi, \tau)}{\partial \xi^2} - \kappa \Xi^4(\xi, \tau) \quad (10)$$

where α is the thermal diffusivity, κ – a constant, and $\Xi(\xi, \tau)$ – the excess temperature.

In view of eqs. (2)-(4), we can structure the non-linear ODE in the form:

$$-\gamma \frac{d\Xi(\omega)}{d\omega} = \alpha \frac{d^2 \Xi(\omega)}{d\omega^2} - \kappa \Xi^4(\omega) \quad (11)$$

which leads to

$$\frac{d^2 \Xi(\omega)}{d\omega^2} + \frac{\gamma}{\alpha} \frac{d\Xi(\omega)}{d\omega} - \frac{\kappa}{\alpha} \Xi^4(\omega) = 0 \quad (12)$$

With the aid of MATLAB software, the solution of eq. (12) can be written as:

$$\Xi(\omega) = A_1 e^{-a\omega} - \frac{b(a^5 \omega^5 - 5a^4 \omega^4 + 20a^3 \omega^3 - 60a^2 \omega^2 + 120a\omega - 120) + A_2}{5a^6} \quad (13)$$

where A_1 and A_2 are two constants, $a = \gamma/\alpha$ and $b = -\kappa/\alpha$.

Thus, clearly, we easily obtain the traveling-wave solution for eq. (10) as:

$$\begin{aligned} \Xi(\xi, \tau) = & A_1 e^{-a(\xi-\gamma\tau)} - \frac{b}{5a^6} \left[a^5 (\xi-\gamma\tau)^5 - 5a^4 (\xi-\gamma\tau)^4 + 20a^3 (\xi-\gamma\tau)^3 \right] + \\ & + \frac{b}{5a^6} \left[60a^2 (\xi-\gamma\tau)^2 - 120a(\xi-\gamma\tau) + 120 \right] + \frac{A_2}{5a^6}. \end{aligned} \quad (14)$$

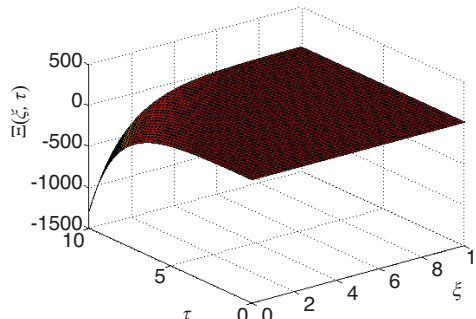


Figure 1(c). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta < 0$

The graphs of the traveling-wave solutions in eq. (10) are depicted in figs. 2(a)-2(c).

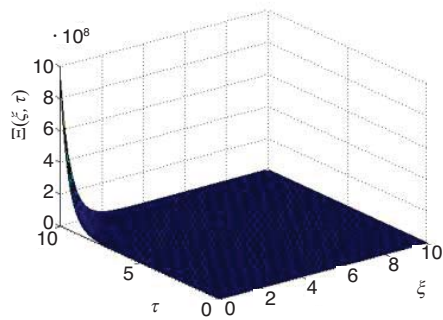


Figure 2(a). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 1$, $a = 1$, $b = -1$, $\gamma = 1$, and $A_2 = 0$

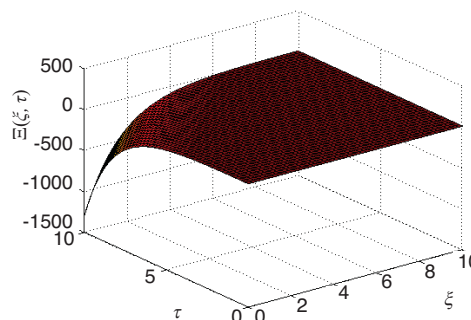


Figure 2(b). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 2$, $a = 1$, $b = -1$, $\gamma = 1$, and $A_2 = 0$

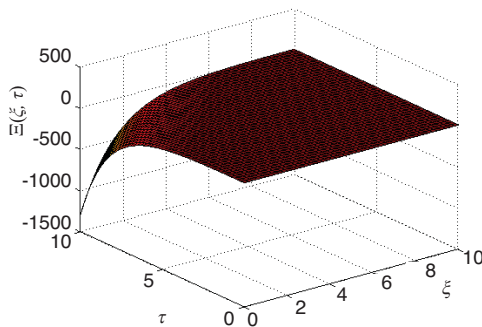


Figure 2(c). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 1$, $a = 1$, $b = -1$, $\gamma = 2$, and $A_2 = 0$

Conclusion

In our present work, we firstly investigated the linear and non-linear heat transfer equations at several different excess temperatures. With the help of the TTM, we transformed the linear and non-linear PDE arising in the heat transfer problems into the linear and non-linear ODE, respectively. We then obtained the solutions of the linear and nonlinear ODE by using the MATLAB software. Finally, the traveling-wave solutions of these heat transfer equations with the graphs are presented. The obtained results are given to reveal the efficiency of the techniques used in this paper.

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Nomenclature

α – heat-diffusion coefficient, [$\text{Wm}^{-1}\text{K}^{-1}$]
 β – constant, [$1/\text{s}$]
 κ – constant, [K^3s^{-1}]

ξ – space co-ordinate, [m]
 $\Xi(\xi, \tau)$ – excess temperature, [K]
 τ – time co-ordinate, [s]

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