

MATRICES, DIGRAPHS AND DETERMINANTS

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**MATRICES, DIGRAPHS AND DETERMINANTS****JOHN S. MAYBEE<sup>1</sup>****D.D. OLESKY<sup>2</sup>****P. VAN DEN DRIESSCHE<sup>3</sup>****G. WIENER<sup>4</sup>**

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**Abstract.** We present a detailed account of various determinantal formulas in a graph–theoretic form involving paths and cycles in the digraph of the matrix. For cases in which the digraph has special local properties, for example a cutpoint or a bridge, we give particular formulas which are more efficient for computing the determinant than simply using the matrix representation. Applications are also given to characteristic determinants, general minors and cofactors.

Keywords: digraph, cycle, determinant, minor, cofactor.

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**1. Introduction.** The connection between the digraph of a matrix and the determinant of a matrix has been pointed out by many authors during the past three decades (see, e.g., [2], [3], [4], [5], [9], [10], [12], [15], [18], [19], [21]). These papers include applications to solving linear systems, spectra of graphs and solving qualitative problems; however, no systematic exposition of this subject has appeared. Our purpose here is to derive some fundamental formulas and give some new applications. The fundamental formulas are in section 2. The remaining sections are independent except that the cofactor formulas in section 7 depend on results on non-principal minors in section 6. Sections 3 and 4 contain formulas for digraphs with special local properties; applications are given in section 5. We conclude in section 8 with an example illustrating several of our formulas.

The motivation for this work comes from the fact that although the evaluation of a determinant may be very difficult using the matrix representation, the matrix digraph often indicates efficient means of evaluation. We give specific examples of this involving the graph-theoretic concepts of cutpoints, critical subdigraphs and bridges.

We now introduce our notation. With an  $n \times n$  matrix  $A = [a_{ij}]$  we associate the digraph  $D(A) = (V, \mathcal{A})$ , having vertex set  $V = \{1, 2, \dots, n\}$  and arc set  $\mathcal{A}$  containing the arc  $(i, j)$  iff  $a_{ij} \neq 0$  for  $i \neq j$ . In addition we suppose that there is a subset  $V_0 \subseteq V$  of distinguished vertices of  $D(A)$ . The vertex  $i \in V_0$  iff  $a_{ii} \neq 0$ . (It should be noted that, for reasons motivated by applications, some authors prefer to put  $(i, j)$  in  $\mathcal{A}$  iff  $a_{ji} \neq 0$ ; see [1] for example. The development of determinant formulas is the same in either case, however.) In order to fix our terminology, we shall call a sequence  $(i_1, i_2, \dots, i_r)$  of distinct vertices a *path* in  $D(A)$  if each of the arcs  $(i_1, i_2), (i_2, i_3), \dots, (i_{r-1}, i_r)$  belongs to  $\mathcal{A}$ . The length of such a path is  $r-1$ . We sometimes use a notation like  $p(i \rightarrow j)$  to denote a path in  $D(A)$  from  $i$  to  $j$ . The length of  $p$  will then be denoted by  $\ell(p)$  and the set of vertices belonging to the path will be denoted by  $V[p]$ . The set of vertices of  $D(A)$  *not* belonging to  $p$  will be denoted by  $V(p)$ . We call a sequence  $(i_1, i_2, \dots, i_r, i_1)$ , where  $i_1, i_2, \dots, i_r$  are distinct vertices of  $D(A)$  and each of the arcs  $(i_1, i_2), \dots, (i_r, i_1)$  belongs

to  $\mathcal{A}$ , a *cycle* of  $D(A)$ . Its length is  $r \geq 2$ . We also call the distinguished vertices of  $D(A)$  *cycles* of length one or 1-cycles. A cycle of  $D(A)$  will be denoted by  $c$ ; the length of  $c$  is  $\ell(c) \geq 1$ .  $V[c]$  is the set of vertices in  $c$  and  $V(c)$  the set of vertices of  $D(A)$  not in  $c$ .

Suppose  $I \subseteq V$ . We use the notation  $\langle I \rangle$  to denote the subdigraph of  $D(A)$  generated (induced) by the vertices in  $I$ ; that is, the arc set of this subdigraph is exactly the arcs of  $\mathcal{A}$  joining vertices of  $I$ . See [11] or [20] for a discussion of this concept. Usually we regard subsets of  $V$  as being ordered sets since they are subsets of the set of the first  $n$  integers. If  $I$  is a subset of  $V$ , we denote by  $A[I]$  the principal submatrix of  $A$  in the rows and columns defined by  $I$ . Similarly we denote by  $A(I)$  the complementary principal submatrix, i.e., the principal submatrix in the rows and columns defined by  $V-I$ . The determinants of these submatrices are denoted by  $\det A[I]$  and  $\det A(I)$ , respectively. These are *principal* minors of the matrix  $A$ . If  $I = \phi$ , we set  $\det A[I] = 1$ , and thus  $\det A(V) = 1$ . Note that  $\det A[V]$  is equal to the determinant of  $A$ , denoted by  $\det A$ . The relationship between principal submatrices of  $A$  and generated subdigraphs of  $D(A)$  is given by  $D(A[I]) = \langle I \rangle$ .

If  $p$  is a path in  $D(A)$ , we let  $A[p]$  denote the corresponding product of elements of  $A$ , which we call a *path of A*. Similarly, if  $c$  is a cycle of  $D(A)$ ,  $A[c]$  denotes the corresponding product of elements of  $A$ , which we call a *cycle of A*. (Note that if  $c = i$ , a distinguished vertex of  $D(A)$ , then  $A[c] = a_{ii}$ .) When  $p$  is a path of  $D(A)$  we denote by  $\det A[V(p)] = \det A(V[p])$  the principal minor of  $A$  in the rows and columns defined by  $V(p)$ , i.e., the indices not on the path. Similarly,  $\det A[V(c)]$  is defined for  $c$  a cycle of  $D(A)$ . We call  $\det A[V(p)]$  the *cominor of p* and  $\det A[V(c)]$  the *cominor of c*. Thus to each path and cycle of  $A$  there is associated a uniquely defined principal minor of  $A$  called the cominor of the path or cycle.

If  $I \subseteq V$  and  $J \subseteq V$  with  $|I| = |J|$ , we denote by  $A[I,J]$  the submatrix of  $A$  in rows  $I$  and columns  $J$  and by  $A(I,J)$  the complementary submatrix; note that

$A[I,I] = A[I]$  and  $A(I,I) = A(I)$ . Then  $\det A[I,J]$  and  $\det A(I,J)$  denote the corresponding determinants.

**2. Fundamental Formulas.** By definition, for  $A = [a_{ij}]$  an  $n \times n$  matrix,

$$\det A = \sum_{\phi} (\operatorname{sgn} \phi) \prod_{i=1}^n a_{i,\phi(i)}$$

where  $\phi$  is an arbitrary permutation of  $V$  and  $\operatorname{sgn} \phi$  is the sign of the permutation  $\phi$ . Any permutation  $\phi$  can be uniquely factored (up to the order of factors) into a product of disjoint permutation cycles. Let  $\phi = \phi_1, \phi_2, \dots, \phi_t$  be the unique factorization of  $\phi$ . Each of the  $\phi_j$ ,  $j = 1, 2, \dots, t$ , is actually a sequence  $\phi_j = (\ell_1, \ell_2, \dots, \ell_{r_j})$  of distinct integers. Thus, provided each of the arcs  $(\ell_1, \ell_2), (\ell_2, \ell_3), \dots, (\ell_{r_j}, \ell_1)$  belongs to  $\mathcal{A}$ ,  $\phi_j$  defines a unique cycle  $c_j$  of  $D(A)$ , namely,  $(\ell_1, \ell_2, \dots, \ell_{r_j}, \ell_1)$ . In this case  $\phi_j$  also determines a unique cycle  $A[c_j]$  of  $A$ . The sign of the permutation  $\phi$  can be computed from

$$\operatorname{sgn} \phi = (\operatorname{sgn} \phi_1)(\operatorname{sgn} \phi_2) \cdots (\operatorname{sgn} \phi_t).$$

Consequently we have the formula

$$(\operatorname{sgn} \phi) \prod_{i=1}^n a_{i,\phi(i)} = (\operatorname{sgn} \phi_1)A[c_1](\operatorname{sgn} \phi_2)A[c_2] \cdots (\operatorname{sgn} \phi_t)A[c_t].$$

This term is nonzero iff each of the permutation cycles  $\phi_1, \dots, \phi_t$  corresponds to a cycle of  $D(A)$ . Note that  $\operatorname{sgn} \phi_j$  is positive if  $\phi_j$  is a cycle of odd length and negative if  $\phi_j$  is a cycle of even length.

We now restrict  $\phi$  to permutations in which each permutation cycle corresponds to a cycle present in  $D(A)$ . Observe that the set of cycles  $f = \{c_1, c_2, \dots, c_t\}$  of  $D(A)$  defined by the factorization of  $\phi$  consists of pairwise disjoint cycles and every vertex of

$D(A)$  belongs to one of them. Such a set  $f$  is called a *factor of  $D(A)$* . (Graph theorists define a 1-factor for a digraph  $D$  to be a spanning subdigraph for which each vertex has indegree and outdegree equal to 1; see, e.g., [3]. Thus our use of the term factor coincides with the concept of a 1-factor as used in graph theory except that we use distinguished vertices in place of loops.) Thus there exists a one-to-one correspondence between the factors of  $D(A)$  and the nonzero terms in the expansion of  $\det A$ . With each factor  $f$  of  $D(A)$  we associate the unique integer  $\mu_f$  equal to the number of cycles of even length belonging to  $f$ . Also we set  $A[f]$  equal to the product of the  $A[c]$  over the cycles  $c$  in  $f$ .

From this discussion we can deduce the fundamental determinant formula in the following graph-theoretic form. (See, e.g., [2], [4], [9], [19]).

**THEOREM 1.** Let  $A$  be an  $n \times n$  matrix with digraph  $D(A)$ . Suppose  $D(A)$  has the factors  $f_k = \{c_{k1}, c_{k2}, \dots, c_{km_k}\}$ ,  $k = 1, 2, \dots, q$ , and let  $\mu_k$  be the number of cycles of even length in  $f_k$ . Then

$$(1) \quad \det A = \sum_{k=1}^q (-1)^{\mu_k} A[c_{k1}]A[c_{k2}] \cdots A[c_{km_k}] = \sum_{k=1}^q (-1)^{\mu_k} A[f_k]. \quad \blacksquare$$

We observe that (1) applied to  $\langle I \rangle$  gives a formula for computing  $\det A[I]$ . The formula (1) is simply a restatement of the formula for the determinant of a square matrix in graph-theoretic terms. It can have as many as  $n!$  terms in the event that  $D(A)$  is a complete digraph on  $n$  vertices and  $V_0 = V$ . Thus the utility of such a reformulation depends upon whatever special structural properties the matrix  $A$  may have, as defined by its digraph  $D(A)$ . Note that even if  $a_{ij} \neq 0$ , the term  $a_{ij}$  occurs in  $\det A$  if and only if  $i$  and  $j$  are in a cycle which is in a factor of  $D(A)$ . We present later some special cases for which (1) yields efficient formulas for  $\det A$ .

The sign  $(-1)^{\mu_k}$  appearing in (1) can also be written in another way. For  $1 \leq k \leq q$  and  $1 \leq j \leq m_k$ , the sign contributed by the cycle  $c_{kj}$  is  $(-1)^{\ell_{kj}+1}$ , where  $\ell_{kj}$  is the length of  $c_{kj}$ . But  $\ell_{k1} + \ell_{k2} + \cdots + \ell_{km_k} = n$  for each  $k$ . Consequently, as  $\mu_k$  and  $n + m_k$  are both even or odd, we can also write (1) in the form

$$(1') \quad \det A = (-1)^n \sum_{k=1}^q (-1)^{m_k} A[c_{k1}]A[c_{k2}] \cdots A[c_{km_k}],$$

where  $m_k$  is the number of cycles in the factor  $f_k$ . This is the form derived in [3], where the formula is attributed to Coates [2] and historical remarks are also given. At this point we observe that when  $A$  is a  $(0,1)$  matrix, our results can be stated in terms of the adjacency matrix of a digraph (or graph; see, e.g., [3]).

A classical tool in the theory of determinants is the expansion of  $\det A$  by rows or columns and, more generally, by the Laplace expansion formula. These tools have led to many useful theoretical results. By using the concept of a cycle in the digraph  $D(A)$ , theoretically useful expansions of  $\det A$  in terms of principal minors of  $A$  can be derived. We turn next to such expansions.

**THEOREM 2** ([18]). Let  $A$  be an  $n \times n$  matrix with directed graph  $D(A)$ . Let  $i$  be a fixed vertex in  $V$ , suppose the set of all cycles of  $D(A)$  containing the vertex  $i$  is  $\{c_1, c_2, \dots, c_q\}$  and the length of  $c_k$  is  $\ell_k$ . Then

$$(2) \quad \det A = \sum_{k=1}^q (-1)^{\ell_k+1} A[c_k] \det A[V(c_k)].$$

*Proof.* We can partition the set of factors of  $D(A)$  into subsets  $F_1, \dots, F_q$  according to which one of the cycles belongs to the factor. All factors containing  $c_k$  are placed in  $F_k$ . Each term in the expansion of  $\det A$  corresponding to a factor  $f \in F_k$  contains the product  $(\text{sgn } c_k)A[c_k]$  which equals  $(-1)^{\ell_k+1} A[c_k]$ . The remaining cycles in the factor  $f$  generate a factor of  $\langle V(c_k) \rangle$ . Thus, when we sum over all factors belonging to the set  $F_k$ , we generate the product  $(-1)^{\ell_k+1} A[c_k] \det A[V(c_k)]$ . Formula (2) now follows from (1) by summation on  $k$ . ■

Several applications of (2) are given in section 5. This formula can be looked upon as an expansion of the determinant of  $A$  relative to a fixed diagonal element, namely the  $i$ th diagonal element, i.e., relative to a fixed vertex of  $D(A)$ . Here is a generalization.

Let  $I$  be a fixed subset of  $V$ . A set  $f_I$  of disjoint cycles in  $D(A)$  *spans*  $I$  if every cycle in  $f_I$  contains at least one vertex of  $I$  and every vertex in  $I$  belongs to one of the cycles. Such a spanning set of cycles will be called *minimal* if the set of vertices in  $f_I$  is equal to  $I$ . Corresponding to each  $f_I$  we have a unique cominor  $\det A[V(f_I)]$ , where  $V(f_I)$  is the set of vertices not in  $f_I$ . If  $f_I$  is minimal, then  $\det A[V(f_I)] = \det A(I)$ . If  $f_I$  is not minimal, then  $\det A[V(f_I)]$  is a principal minor of  $A(I)$ . Denote by  $E_I$  the set of  $f_I$  which are spanning sets of cycles for  $I$ , and which are *not* minimal spanning sets of  $I$ .

**THEOREM 3** ([15]). Let  $A$  be an  $n \times n$  matrix with digraph  $D(A)$ . Then, in terms of the notation above, we have

$$(3) \quad \det A = \det A[I] \det A(I) + \sum_{f_I \in E_I} (-1)^{\mu(f_I)} A[f_I] \det A[V(f_I)],$$

where  $A[f_I]$  is the product of all  $A[c]$  for  $c \in f_I$  and  $\mu(f_I)$  is the number of cycles of even length in  $f_I$ .

*Proof.* From (1),  $\det A = \sum_{f_I} (-1)^{\mu(f_I)} A[f_I] \det A[V(f_I)]$ . Formula (3) follows by

separating minimal sets from those that are not minimal. ■

We may regard formula (3) as an expansion of  $\det A$  relative to a fixed set of vertices of  $D(A)$ . When  $I = \{i\}$ , this formula coincides with (2).

**3. The Cutpoint and Critical Subdigraph Formulas.** Our ability to relate the expansion of  $\det A$  to  $D(A)$  in (1)–(3) can be used to obtain useful special results in the event that  $D(A)$  has special local properties. Here and in section 4 we present some applications based upon this idea.

Recall that the vertex  $i$  of  $D(A)$  is called a *cutpoint* if the number of weak components of  $D(A) - \{i\}$  is larger than the number of weak components of  $D(A)$ . (We remind the reader that  $D(A) - \{i\}$  is obtained by removing the vertex  $i$  from  $D(A)$  together with any arcs of  $\mathcal{A}$  incident at  $i$ . Weak components are discussed in [11] and [20].) Suppose  $i$  is a cutpoint of  $D(A)$  and the components of  $D(A) - \{i\}$  are  $D_j$ ,  $1 \leq j \leq p(i)$ . Set  $I_j = V[D_j]$ , the vertex set of  $D_j$ , and let  $\bar{D}_j = \langle \bar{I}_j \rangle$  where  $\bar{I}_j = I_j \cup \{i\}$ ,  $1 \leq j \leq p(i)$ . Thus  $D_j = \bar{D}_j - \{i\}$ .

**THEOREM 4.** Let  $A$  be an  $n \times n$  matrix and suppose  $D(A)$  has a cutpoint  $i$ . Let  $p(i)$ ,  $I_j$  and  $\bar{I}_j$  be defined as above. Then

$$(4) \quad \det A = \sum_{j=1}^{p(i)} \left[ \det A[\bar{I}_j] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k] \right] - (p(i)-1) a_{ii} \prod_{k=1}^{p(i)} \det A[I_k].$$

*Proof.* Let  $\phi$  be a permutation of  $V$ . Suppose first that  $\phi(i) \neq i$ . Then  $\phi(i) \in I_j$  for some  $j$ . Factor  $\phi$  into the disjoint cycles  $\phi_1, \phi_2, \dots, \phi_t$ . There will be a unique cycle, say  $\phi_m$ , which moves  $i$ , i.e.,  $\phi_m(i) \neq i$ . Since  $i$  is a cutpoint of  $D(A)$ , the corresponding cycle  $c_m$  in  $D(A)$  must lie entirely in  $\bar{D}_j$  (because  $c_m - \{i\}$  is a path so must lie entirely in some weak component). Any other cycle  $c_k$  determined by  $\phi_k$  for  $k \neq m$  must lie entirely in some  $D_\ell$ ,  $1 \leq \ell \leq p(i)$ ,  $\ell \neq j$ . Therefore the product

$$(\operatorname{sgn} \phi_1)A[c_1](\operatorname{sgn} \phi_2)A[c_2] \cdots (\operatorname{sgn} \phi_t)A[c_t]$$

appears exactly once in the expansion of

$$\det A[\bar{I}_q] \prod_{\substack{k=1 \\ k \neq q}}^{p(i)} \det A[I_k]$$

when  $q = j$ , and does not appear in any of the terms when  $q \neq j$ ,  $1 \leq q \leq p(i)$ . Next suppose  $\phi(i) = i$ . In this case there is a unique cycle,  $\phi_m$  say, such that  $\phi_m(i) = i$ . But again because  $i$  is a cutpoint of  $D(A)$ , any cycle  $c_k \in D(A)$  determined by  $\phi_k$ ,  $k \neq m$ , must lie entirely in some  $D_\ell$ ,  $1 \leq \ell \leq p(i)$ . In this case we observe that the product must appear exactly once in each term

$$\det A[\bar{I}_j] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k], \quad 1 \leq j \leq p(i).$$

Since the permutation  $\phi$  either moves  $i$  or fixes  $i$ , every nonzero term in the expansion of  $\det A$  will appear at least once in

$$(5) \quad \sum_{j=1}^{p(i)} \det A[\bar{I}_j] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k].$$

Finally we note that the terms in  $\det A$  falling under the first case will be counted exactly once in (5). The terms in  $\det A$  falling under the second case will be counted  $p(i)$  times in (5). Since the expression  $a_{ii} \prod_{k=1}^{p(i)} \det A[I_k]$  comes from precisely those terms in the second case, formula (4) holds. ■

We illustrate formula (4) for a cluster of cycles in section 5 and also in our example in section 8.

We remark that, in the case where  $i$  is not a distinguished vertex of  $D(A)$ , the expression in (5) equals  $\det A$ . This can be viewed as a generalization of the formula for the determinant of the coalescence of two graphs without loops (see, e.g., [21]).

We now derive another form of the cutpoint formula (4). For each  $j = 1, 2, \dots, p(i)$  let  $\{c_{j1}, \dots, c_{jm_j}\}$  be the set of cycles of  $D(A)$  incident at the cutpoint  $i$  and such that  $V[c_{jk}] \cap I_j \neq \emptyset$ . Also let  $\ell_{jk}$  be the length of  $c_{jk}$ ,  $k = 1, 2, \dots, m_j$ . Then we can apply the vertex expansion formula (2) at the vertex  $i$  to evaluate each of the determinants  $\det A[\bar{I}_j]$ . To simplify the notation let us set  $I_{jk} = \bar{I}_j - V[c_{jk}]$  so that  $I_{jk} \subseteq \bar{I}_j$ . Then we have

$$\det A[\bar{I}_j] = \sum_{k=1}^{m_j} (-1)^{\ell_{jk}+1} A[c_{jk}] \det A[I_{jk}] + a_{ii} \det A[I_j].$$

Substituting this into (4) yields

$$\det A = \sum_{j=1}^{p(i)} \left[ \sum_{k=1}^{m_j} (-1)^{\ell_{jk}+1} A[c_{jk}] \det A[I_{jk}] + a_{ii} \det A[I_j] \right] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k] \\ - (p(i)-1)a_{ii} \prod_{k=1}^{p(i)} \det A[I_k],$$

which simplifies to

$$(4') \quad \det A = a_{ii} \prod_{k=1}^{p(i)} \det A[I_k] \\ + \sum_{j=1}^{p(i)} \left[ \sum_{k=1}^{m_j} (-1)^{\ell_{jk}+1} A[c_{jk}] \det A[I_{jk}] \right] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k].$$

Here is a special case of (4'). Suppose for all  $j = 1, 2, \dots, p(i)$ , there is a unique cycle  $c_j$  of length  $\ell_j$  incident at  $i$  such that  $V[c_j] \cap I_j \neq \emptyset$ . We then obtain the expansion

$$(6) \quad \det A = a_{ii} \prod_{k=1}^{p(i)} \det A[I_k] \\ + \sum_{j=1}^{p(i)} (-1)^{\ell_j+1} A[c_j] \det A[\bar{I}_j - V[c_j]] \prod_{\substack{k=1 \\ k \neq j}}^{p(i)} \det A[I_k].$$

The key property of a cutpoint which permits us to prove (4) and (4') is that each cycle in  $D(A)$  must be contained entirely within one of the sets  $\bar{D}_j$  and, hence, each factor of  $D(A)$  consists of factors of  $\bar{D}_j$  for some  $j$  and of  $D_k$  for  $k \neq j$ . If this property can be generalized to some larger subdigraph of  $D(A)$ , we say that  $D(A)$  has a critical subdigraph. More precisely we use the following concept.

Let  $D$  be a digraph. The subdigraph  $D_0$  will be called a *critical subdigraph* of  $D$  if:

- (a)  $D_0 = \langle I_0 \rangle$  for some  $I_0 \subset V$ ;
- (b)  $\langle V - I_0 \rangle \equiv D - D_0$  has more weak components than  $D$ ; and
- (c) if  $D_j = \langle I_j \rangle$ ,  $j = 1, 2, \dots, p$  ( $p \geq 2$ ) are the weak components of  $D - D_0$ , then every factor of  $D$  consists of a factor of  $\langle I_0 \cup I_j \rangle$  for some fixed  $j$  together with factors of  $\langle I_k \rangle$  for  $k = 1, 2, \dots, p$  ( $k \neq j$ ).

We can now prove the following result.

**THEOREM 5.** Suppose the digraph  $D(A)$  of the matrix  $A$  has a critical subdigraph  $D_0$ . Then in the above notation

$$(7) \quad \det A = \sum_{j=1}^p \det A[I_0 \cup I_j] \prod_{\substack{k=1 \\ k \neq j}}^p \det A[I_k] - (p-1) \det A[I_0] \prod_{k=1}^p \det A[I_k].$$

*Proof.* By property (c), the sum on the right above contains every term in the expansion of  $\det A$ . However the term  $\det A[I_0] \prod_{k=1}^p \det A[I_k]$  occurs  $p$  times in the summation, hence it must be subtracted off  $(p-1)$  times, giving (7). ■

We shall see, by way of some examples, that Theorem 5 offers a substantial generalization of the cutpoint formulas (4) and (4'). On the other hand, it is certainly not clear even from the graph-theoretic point of view how to characterize a critical subdigraph. Here, however, is a sufficient condition that a subdigraph be critical.

**LEMMA 1.** Let  $D_0 = \langle I_0 \rangle$  be a subdigraph of  $D$  satisfying (a) and (b). If there exist vertices  $v_{\text{in}}$  and  $v_{\text{out}}$  of  $D_0$  such that every cycle  $c$  with  $V[c] \cap I_0 \neq \phi$  and  $V[c] \cap (V - I_0) \neq \phi$  enters  $D_0$  at  $v_{\text{in}}$  and leaves  $D_0$  at  $v_{\text{out}}$ , then  $D_0$  is a critical subdigraph of  $D$ .

*Proof.* If  $c$  is any cycle of  $D$ , then either  $V[c] \subset I_0$ ,  $V[c] \subset I_j$  for some fixed  $j \in \{1, 2, \dots, p\}$  where  $D_j = \langle I_j \rangle$  is the  $j$ th weak component of  $D - D_0$ , or  $V[c]$  satisfies the condition of the lemma. But in the last case it is clear that  $V[c] \cap (V - I_0) \subset I_j$  for

some fixed  $j$ . It follows that every factor of  $D$  consists of a factor of  $\langle I_0 \cup I_j \rangle$  for some fixed  $j$  together with factors of  $\langle I_k \rangle$  for  $k \neq j$ . ■

We now give two applications of the critical subdigraph formula.

Let  $D = (V, \mathcal{A})$  be a digraph. We call  $D$  a *ladder digraph* if

$V = V_1 \cup V_2 \cup \cdots \cup V_k$ , where  $V_i \cap V_j = \phi$ ,  $i \neq j$ ,  $k \geq 3$ , and every  $(x, y) \in \mathcal{A}$  is such that  $x \in V_i$ ,  $y \in V_j$  with  $|i-j| \leq 1$ . The subdigraphs  $\langle V_i \rangle$ ,  $i = 1, 2, \dots, k$ , are called the *rungs* of  $D$  and, for  $i = 2, 3, \dots, k-1$ , the *interior rungs* of  $D$ .

Let  $A$  be a block tridiagonal matrix, i.e.,

$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & 0 \\ C_1 & A_2 & B_2 & \cdots & 0 & 0 \\ 0 & C_2 & A_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{p-1} & B_{p-1} \\ 0 & 0 & 0 & \cdots & C_{p-1} & A_p \end{bmatrix}$$

where  $p \geq 3$ , and  $A_j$  is an  $r_j \times r_j$  block,  $j = 1, 2, \dots, p$ ,  $\sum_{j=1}^p r_j = n$ . Then we can write

$D(A) = (V, \mathcal{A})$  where  $V = V_1 \cup V_2 \cup \cdots \cup V_p$ , and setting  $r_i^* = \sum_{j=1}^{i-1} r_j$ ,  $r_1^* = 0$ ,

$V_i = (r_i^* + 1, \dots, r_i^* + r_i)$ ,  $i = 1, 2, \dots, p$ . Note that  $V_i \cap V_j = \phi$  if  $i \neq j$ . Also we have

$(i, j) \in \mathcal{A}$  iff  $i$  and  $j$  belong to  $V_k$  for some  $k = 1, 2, \dots, p$ , or  $i \in V_k$ ,  $j \in V_{k+1}$ ,

$k = 1, 2, \dots, p-1$ , or  $i \in V_k$ ,  $j \in V_{k-1}$ ,  $k = 2, 3, \dots, p$ . Thus, if  $A$  is block tridiagonal,

$D(A)$  is a ladder digraph. Conversely, if  $D(A)$  is a ladder digraph, there exists a permutation matrix  $P$  such that  $P^T A P$  is a block tridiagonal matrix.

We call the block tridiagonal matrix *critical* if  $D(A)$  is a ladder digraph and each interior rung of  $D$  is a critical subdigraph. The concepts are illustrated in Figure 1, where  $\otimes$  denotes a distinguished vertex. By Lemma 1, the subdigraphs  $\langle 3, 4 \rangle$ ,  $\langle 5, 6 \rangle$ ,  $\langle 7, 8 \rangle$  and

$\langle 9,10 \rangle$  are all critical.

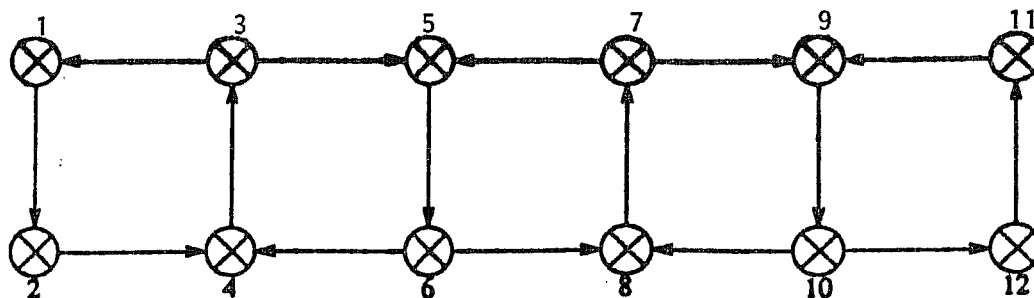


Figure 1.

A ladder digraph.

Now consider the interior rung  $\langle V_2 \rangle$ . Applying Theorem 5 we get

$$\det A = \det A[V_1 \cup V_2] \det A[V_3 \cup \dots \cup V_k] + \det A[V_1] \det A[V_2 \cup \dots \cup V_k] \\ - \det A[V_1] \det A[V_2] \det A[V_3 \cup \dots \cup V_k].$$

But we can apply Theorem 3 to  $\det A[V_1 \cup V_2]$ . In fact, let  $E_{1,2}$  be the set of all non-minimal sets of cycles which span  $V_1$  in  $\langle V_1 \cup V_2 \rangle$ . Then

$$\det A[V_1 \cup V_2] = \det A[V_1] \det A[V_2] + \sum_{f \in E_{1,2}} (-1)^{\mu(f)} A[f] \det A[V(f)].$$

Here each  $\det A[V(f)]$  is a principal minor of  $\det A[V_2]$ . Thus in this case formula (7) becomes

$$\det A = \det A[V_1] \det A[V_2 \cup \cdots \cup V_k] \\ + \sum_{f \in E_{1,2}} (-1)^{\mu(f)} A[f] \det A[V(f)] \det A[V_3 \cup \cdots \cup V_k].$$

Observe the analogy between this formula and the recurrence formula for the ordinary tridiagonal case. Also observe that for  $k > 3$  we can apply the same reasoning to  $\det A[V_2 \cup \cdots \cup V_k]$  using the interior rung  $\langle V_3 \rangle$ , etc. In this way we can associate a generalized recurrence relation with a critical ladder digraph.

As a second application, consider the following class of digraphs (see [17]). If  $D$  is a digraph and  $(x,y)$  is an arc of  $D$ , a *3-path operation* adds two new vertices  $z_1, z_2$  and three new arcs  $(y,z_1), (z_1,z_2), (z_2,x)$  to  $D$ . We call  $D$  a *directed 4-cockade* if it can be obtained from a 4-cycle by a finite sequence of 3-path operations. This is introduced for undirected graphs in [22]. It is easy to see that every directed 4-cockade is strongly connected and that every cycle has length four. We illustrate in Figure 2 a directed 4-cockade with all vertices distinguished. Observe that for  $D_0 = \langle x,y \rangle$ ,  $D - D_0$  has four weak components, but there do not exist vertices  $v_{\text{in}}$  and  $v_{\text{out}}$  in  $\langle x,y \rangle$ . This shows that the condition of Lemma 1 is not necessary for a subdigraph to be critical. However, we can use our formula (7) in this example as  $D_0$  is a critical subdigraph.

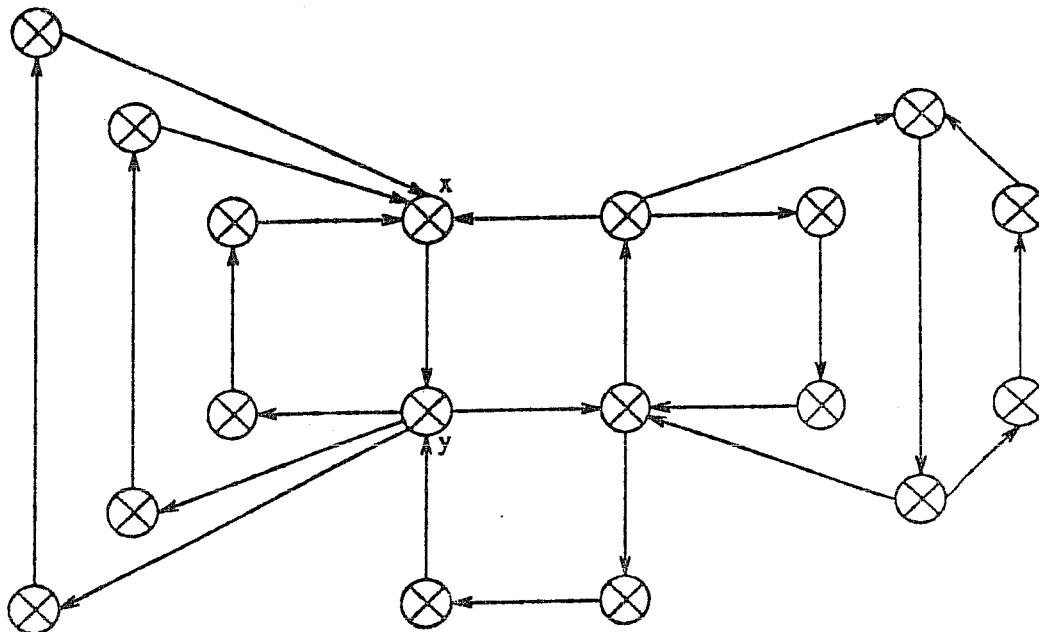


Figure 2.

A directed 4-cockade.

More generally, if  $\langle x,y \rangle$  is an arc of a directed 4-cockade  $D$  with all vertices distinguished, then  $D - \langle x,y \rangle$  has more weak components than  $D$  iff the arc  $\langle x,y \rangle$  belongs to at least two 4-cycles. Note that, as pointed out by a referee, such a digraph  $\langle x,y \rangle$  is not necessarily a critical subdigraph. However, we can prove that such an  $\langle x,y \rangle$  is not critical iff there is a 4-cycle in  $D - \{x\}$  which is not in  $D - \{y\}$ , and a 4-cycle in  $D - \{y\}$  which is not in  $D - \{x\}$ .

**4. The Bridge Formulas.** Another type of local behavior lending itself to a simple determinantal formula is the following. The arcs  $(i,j)$  and  $(j,i)$  constitute a *bridge* of the digraph  $D$  if their removal increases the number of weak components of  $D$ . Suppose  $D$  is a weakly connected digraph and that  $(i,j)$  and  $(j,i)$  constitute a bridge of  $D$ . Then  $i$  and  $j$  are in different weak components and there are exactly two weak components. Let  $\bar{D}_i, \bar{D}_j$  be the weak component of  $D(A) - \{(i,j), (j,i)\}$  containing  $i, j$ , respectively. Set  $D_i = \bar{D}_i - \{i\}$ ,  $D_j = \bar{D}_j - \{j\}$  and define  $\bar{I} = V[\bar{D}_i]$ ,  $\bar{J} = V[\bar{D}_j]$ ,  $I = V[D_i]$ ,  $J = V[D_j]$ .

**THEOREM 6.** Let  $A$  be an  $n \times n$  matrix with a weakly connected digraph  $D(A)$ . If the arcs  $(i,j)$  and  $(j,i)$  constitute a bridge of  $D(A)$  and the subsets  $I, J, \bar{I}, \bar{J}$  of  $V$  are defined as above, then

$$(8) \quad \det A = \det A[\bar{I}]\det A[\bar{J}] - a_{ij}a_{ji} \det A[I]\det A[J].$$

*Proof.* Any nonzero term  $(\text{sgn } \phi) \prod_{i=1}^n a_{i, \phi(i)}$  in the expansion of  $\det A$  is uniquely determined by its representation as a factor of  $D(A)$ . If  $f$  is a factor, then  $f$  falls into one of the following mutually exclusive classes:

- (a) Every cycle of  $f$  lies in  $\bar{D}_i$  or in  $\bar{D}_j$ ;
- (b)  $f$  contains the cycle  $(i,j,i)$  and every other cycle of  $f$  lies in either  $D_i$  or  $D_j$ .

The factors in (a) are uniquely determined by all the terms in  $\det A[\bar{I}]\det A[\bar{J}]$ , while those in (b) are uniquely determined by all the terms in  $a_{ij}a_{ji} \det A[I]\det A[J]$ . Formula (8) follows upon taking account of the sign of the 2-cycle  $a_{ij}a_{ji}$ . ■

When  $A$  is a  $(0,1)$  symmetric matrix, (8) can be viewed as a formula for computing the determinant of a graph from the determinants of subgraphs (see [3]).

The *bridge formula* of Theorem 6 can be looked upon as a special case of the following situation. Let  $c$  be a cycle of length  $\ell \geq 2$  of a weakly connected digraph  $D(A)$ , and assume that  $D(A) - \{\text{arcs of } c\}$  consists of a set of isolated points and a set  $\overline{D}_1, \dots, \overline{D}_p$  of disjoint subdigraphs each containing two or more points and exactly one point of  $c$ . Here  $0 \leq p \leq \ell$ . Let  $V[c] \cap V(\overline{D}_j) = \{x_j\}$ ,  $j = 1, 2, \dots, p$ , and  $I_0 = V[c] - \{x_1, \dots, x_p\}$ . We can assume that  $p \geq 1$  since  $p = 0$  implies that  $D(A) = c$ . Setting  $\overline{I}_j = V[\overline{D}_j]$  and  $I_j = V[\overline{D}_j] - \{x_j\}$ ,  $j = 1, 2, \dots, p$ , we have

$$(9) \quad \det A = \prod_{j=1}^p \det A[\overline{I}_j] \prod_{\sigma \in I_0} a_{\sigma\sigma} + (-1)^{\ell+1} A[c] \prod_{j=1}^p \det A[I_j].$$

This *generalized bridge formula* is an obvious extension of (8) and we omit the proof. When  $\ell = p = 2$  it reduces precisely to (8). Also, in the special case where  $p = \ell$  we have

$$\det A = \prod_{j=1}^{\ell} \det A[\overline{I}_j] + (-1)^{\ell+1} A[c] \prod_{j=1}^{\ell} \det A[I_j].$$

Next we present an application of this generalized bridge formula. Let  $c = (1, 2, \dots, \ell, 1)$  be a cycle of length  $\ell \geq 2$  and suppose there is at most one cycle  $c_j$ ,  $j = 1, 2, \dots, \ell$ , of length  $\ell_j \geq 2$  such that  $V[c_j] \cap V[c] = \{j\}$ . Setting  $\overline{I}_j = V[c_j]$  and  $I_j = V[c_j] - \{j\}$ , if in addition  $c_j$  is the only cycle of length  $\geq 2$  in  $\overline{I}_j$ , then we have

$$\det A[\overline{I}_j] = \prod_{\sigma \in \overline{I}_j} a_{\sigma\sigma} + (-1)^{\ell_j+1} A[c_j],$$

and  $\det A[I_j] = \prod_{\sigma \in I_j} a_{\sigma\sigma}$ . We then obtain from (9) the result

$$(10) \quad \det A = \prod_{j=1}^{\ell} \left[ \prod_{\sigma \in \bar{1}_j} a_{\sigma\sigma} + (-1)^{\ell_j+1} A[c_j] \right] + (-1)^{\ell+1} A[c] \prod_{\sigma=\ell+1}^n a_{\sigma\sigma}.$$

Observe also in connection with (10) that  $A[c] = a_{12}a_{23} \cdots a_{\ell-1,\ell}a_{\ell 1}$ . In the particular case where  $\ell_j = 2, j = 1, 2, \dots, \ell$ , we can write  $A[c_j] = a_{j,\ell+j}a_{\ell+j,j}, j = 1, 2, \dots, \ell$ , and thus

$$\det A = \prod_{j=1}^{\ell} \{a_{jj}a_{\ell+j,\ell+j}^{-1}a_{j,\ell+j}a_{\ell+j,j}\} + (-1)^{\ell+1} a_{12} \cdots a_{\ell 1} \prod_{j=\ell+1}^{2\ell} a_{jj}.$$

**5. Applications.** Let us begin with two applications of the expansion formula (2) relative to a vertex. First consider an  $n \times n$  matrix  $A = [a_{ij}]$  with  $a_{ij} \neq 0$  iff  $-2 \leq i-j \leq 1$ ; this is a special case of an upper Hessenberg matrix. The digraph of  $A$  is shown in Figure 3.

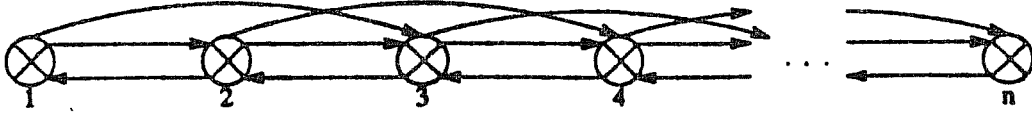


Figure 3.

The digraph  $D(A)$  for  $A = [a_{ij}]$  with  $a_{ij} \neq 0$  iff  $-2 \leq i-j \leq 1$ .

Let us denote the leading principal minor of  $A$  of order  $r$  by  $\det A_r$ ,  $r = 0, 1, 2, \dots, n$ , where  $\det A_0 = 1$ , and  $\det A_n = \det A$ . There are three cycles incident at vertex  $n$ , namely the 1 cycle at  $n$ , the 2-cycle  $(n, n-1, n)$  and the 3-cycle  $(n, n-1, n-2, n)$ . The corresponding cycles of  $A$  are  $a_{nn}$ ,  $a_{n,n-1}a_{n-1,n}$  and  $a_{n,n-1}a_{n-1,n-2}a_{n-2,n}$  with cominors  $\det A_{n-1}$ ,  $\det A_{n-2}$  and  $\det A_{n-3}$ , respectively. Consequently, we derive from formula (2) that

$$\det A = a_{nn} \det A_{n-1} - a_{n,n-1}a_{n-1,n} \det A_{n-2} + a_{n,n-1}a_{n-1,n-2}a_{n-2,n} \det A_{n-3}.$$

This formula expresses the determinant of  $A$  in terms of the determinants of three successive principal minors of  $A$ . Obviously the same reasoning can be applied to any of the generated subdigraphs  $\langle 1, 2, \dots, r \rangle$  for  $r \geq 3$ . In this way we obtain the recurrence formulas

$$(11) \quad \det A_r = a_{rr} \det A_{r-1} - a_{r,r-1} a_{r-1,r} \det A_{r-2} + a_{r,r-1} a_{r-1,r-2} a_{r-2,r} \det A_{r-3}$$

for  $r \geq 3$  with  $\det A_0 = 1$ ,  $\det A_1 = a_{11}$ ,  $\det A_2 = a_{22} \det A_1 - a_{12} a_{21} \det A_0$ .

The recurrence formulas (11) may also be readily applied to the characteristic matrix  $A - \lambda I = A(\lambda)$  to yield

$$(12) \quad \det A_r(\lambda) = (a_{rr} - \lambda) \det A_{r-1}(\lambda) - a_{r,r-1} a_{r-1,r} \det A_{r-2}(\lambda) \\ + a_{r,r-1} a_{r-1,r-2} a_{r-2,r} \det A_{r-3}(\lambda),$$

for  $r \geq 3$  with initial conditions  $\det A_0(\lambda) = 1$ ,  $\det A_1(\lambda) = a_{11} - \lambda$ ,  $\det A_2(\lambda) = (a_{22} - \lambda) \det A_1(\lambda) - a_{12} a_{21} \det A_0(\lambda)$ . Note that when  $a_{r-2,r} = 0$  this reduces to the recurrence formulas for a tridiagonal matrix (see, e.g, [7]). These relations could be used to investigate the spectral properties of such an upper Hessenberg matrix  $A$ ; see *e.g.* [17].

A second example of the application of formula (2) arises from a modification of an econometric model currently used by the U.S. Department of Energy [16]. Suppose an  $n \times n$  matrix  $A = [a_{ij}]$  is such that  $a_{ij} \neq 0$  iff  $i = 1, j = 1, i = j$  or  $i = n$ . The digraph of  $A$  is shown in Figure 4.

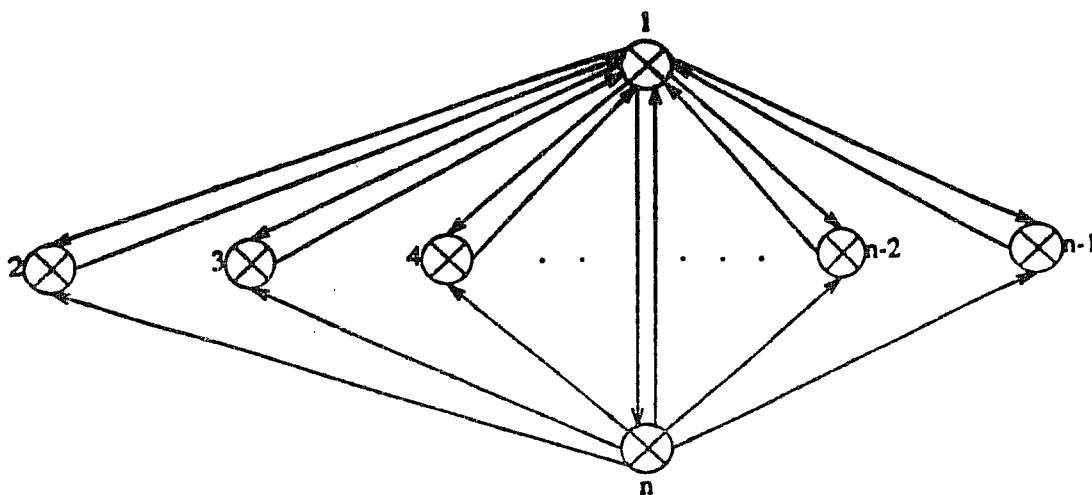


Figure 4.

The digraph  $D(A)$  for  $A = [a_{ij}]$  with  $a_{ij} \neq 0$  iff  $i = 1, j = 1, i = j$ , or  $i = n$

Here again each vertex of  $D(A)$  is distinguished. In this example every cycle of length  $\geq 2$  is incident at vertex 1. This means that the cominor of each such cycle, as well as the cominor of the 1-cycle at vertex 1 itself, is simply a product of elements of  $A$  on the principal diagonal. Choosing  $i = 1$  in formula (2), we obtain

$$(13) \quad \det A = a_{11} \prod_{i=2}^n a_{ii} - \sum_{i=2}^n a_{1i} a_{i1} \prod_{\substack{k=2 \\ k \neq i}}^n a_{kk} + \sum_{i=2}^{n-1} a_{1n} a_{ni} a_{i1} \prod_{\substack{k=2 \\ k \neq i}}^{n-1} a_{kk},$$

an explicit formula which can be readily evaluated.

Again we may apply our result to the matrix  $A(\lambda) = A - \lambda I$  to obtain the following formula for the characteristic determinant of  $A$ :

$$(14) \quad \det A(\lambda) = (a_{11} - \lambda) \prod_{i=2}^n (a_{ii} - \lambda) - \sum_{i=2}^n a_{1i} a_{i1} \prod_{\substack{k=2 \\ k \neq i}}^n (a_{kk} - \lambda) \\ + \sum_{i=2}^{n-1} a_{1n} a_{ni} a_{i1} \prod_{\substack{k=2 \\ k \neq i}}^{n-1} (a_{kk} - \lambda).$$

We note that this formula can be used to give information about the spectrum of  $A$  under various special hypotheses about the signs of the nonzero elements of  $A$ . In the formulas (13) and (14) we have separated off the factor involving  $a_{11}$  in the first term on the righthand side because this is the only place at which it occurs. All other elements along the principal diagonal appear in at least  $n-1$  terms.

We now derive an eigenvalue property from the critical subdigraph expansion (7). Suppose the matrix  $A$  has a critical subdigraph and, as above, set  $A(\lambda) = A - \lambda I$ . Then the expansion (7) becomes

$$(15) \quad \det A(\lambda) = \sum_{j=1}^p \det A[I_0 \cup I_j; \lambda] \prod_{\substack{k=1 \\ k \neq j}}^p \det A[I_k; \lambda] \\ - (p-1) \det A[I_0; \lambda] \prod_{k=1}^p \det A[I_k; \lambda],$$

where we have used the more compact notation  $A[I; \lambda]$  instead of  $A(\lambda)[I]$ .

**THEOREM 7.** Suppose the digraph of the matrix  $A$  has a critical subdigraph  $D_0$  and that  $\lambda_0$  is an eigenvalue of  $r$  of the submatrices  $A[I_k]$ ,  $2 \leq k \leq p$ . Then  $\lambda_0$  is an eigenvalue of  $A$  of multiplicity at least  $r-1$ .

*Proof.* By hypothesis  $\det A[I_k; \lambda_0] = 0$  for  $r \geq 2$  values of  $k$ . It follows that  $\lambda_0$  is a zero of each of the terms in the sum in formula (15)  $(r-1)$ -times. It is also a zero of the last term  $r$ -times. ■

We turn next to an application of formula (3). Let  $A$  be such that  $D(A)$  has a pair of pendant vertices  $k$  and  $\ell$  each joined to another vertex of  $D(A)$  by symmetric arcs. To be more specific, let  $i, j, k$  and  $\ell$  be distinct vertices of  $D(A)$  such that  $(i, k)$  and  $(k, i)$  are the only arcs of  $D(A)$  incident at  $k$ , and  $(j, \ell)$  and  $(\ell, j)$  the only arcs of  $D(A)$  incident at  $\ell$ . Applying (3) with  $I = \{k, \ell\}$ , and denoting  $A(\{k, \ell\})$  by  $A(k, \ell)$  we obtain

$$(16) \quad \det A = a_{kk} a_{\ell\ell} \det A(k, \ell) - a_{kk} a_j a_{\ell j} \det A(j, k, \ell) \\ - a_{\ell\ell} a_{ik} a_{ki} \det A(i, k, \ell) + a_{ik} a_{ki} a_j a_{\ell j} \det A(i, j, k, \ell).$$

Observe that in this formula the minors  $\det A(j, k, \ell)$ ,  $\det A(i, k, \ell)$  and  $\det A(i, j, k, \ell)$  are all principal minors of  $A(k, \ell)$ .

As an application of (4) consider the following. Let the matrix  $A$  be such that  $D(A)$  has all vertices distinguished and consists of  $m \geq 2$  cycles  $c_1, \dots, c_m$  of lengths  $\ell_1, \dots, \ell_m$ , respectively, all of which intersect at a single vertex and are otherwise disjoint. We call such a matrix a *cluster of cycles*. Without loss of generality, we label this unique cutpoint of  $D(A)$  vertex 1. Letting  $I_k, k = 1, 2, \dots, m$ , be the set of non-cutpoints of  $c_k$  and  $\bar{I}_k = I_k \cup \{1\}$ , we have

$$\det A[I_k] = \prod_{\sigma \in I_k} a_{\sigma\sigma} \quad \text{and} \quad \det A[\bar{I}_k] = \prod_{\sigma \in \bar{I}_k} a_{\sigma\sigma} + (-1)^{\ell_k+1} A[c_k],$$

$k = 1, 2, \dots, m$ . From (4) we then obtain

$$\det A = \sum_{j=1}^m \left[ \prod_{\sigma \in \bar{I}_j} a_{\sigma\sigma} + (-1)^{\ell_j+1} A[c_j] \right] \prod_{\substack{k=1 \\ k \neq j}}^m \prod_{\sigma \in I_k} a_{\sigma\sigma} - (m-1)a_{11} \prod_{k=2}^m a_{kk},$$

whence

$$(17) \quad \det A = a_{11} \prod_{k=2}^m a_{kk} + \sum_{j=1}^m (-1)^{\ell_j+1} A[c_j] \prod_{\sigma \notin \bar{I}_j} a_{\sigma\sigma}.$$

Setting  $A - \lambda I = A(\lambda)$  as before, we deduce from (17) the formula for the characteristic determinant of a cluster of cycles (with cutpoint at vertex 1), namely,

$$(18) \quad \det A(\lambda) = (a_{11} - \lambda) \prod_{k=2}^m (a_{kk} - \lambda) + \sum_{j=1}^m (-1)^{\ell_j+1} A[c_j] \prod_{\sigma \notin \bar{I}_j} (a_{\sigma\sigma} - \lambda).$$

**6. Non–principal Minors.** The expansion formulas derived in section 2 can be applied as in [14] to yield graph–theoretic insights into the expansion of an arbitrary minor of the matrix  $A$ . In particular this leads to a theoretically valuable formula for computing the matrix of cofactors of  $A$ ,  $\text{cof } A$ , and for computing  $A^{-1}$  whenever it exists.

As before, let  $D(A) = (V, \mathcal{A})$  with  $V_0 \subseteq V$  the set of distinguished vertices of  $D(A)$ . Suppose  $I \subseteq V, J \subseteq V$  with  $|I| = |J|, I \neq J$ . Let  $L = I \cup J, s = |L|$  and  $d(I, J) = |L| - |I|$ . Following [13], we call  $d(I, J)$  the *dispersion* of the pair of sets  $I$  and  $J$ . Note that  $d(I, J) + |I| \leq n$  and  $d(I, J) \geq 1$ . Let  $K = I \cap J$  (possibly empty). Then there exist nonempty sets  $I_0, J_0$  such that  $I = K \cup I_0$  and  $J = K \cup J_0$ , where  $|I_0| = |J_0| = d(I, J)$  and  $I_0 \cap J = J_0 \cap I = \phi$ . Note that  $L = K \cup I_0 \cup J_0$ .

We define the  $\langle I, J \rangle$ –generated subdigraph of  $D(A)$  as follows. Start with  $\langle L \rangle$  and delete all arcs of  $\langle L \rangle$  incident *to* a vertex of  $I_0$  and all arcs of  $\langle L \rangle$  incident *from* a vertex of  $J_0$ . In order to relate this subdigraph of  $D(A)$  to a submatrix of  $A$ , consider the principal submatrix  $A[L]$ . In this submatrix set to zero all elements in the rows corresponding to  $J_0$  and all elements in the columns corresponding to  $I_0$ . Call the resulting submatrix  $A[L; I, J]$ . Observe that the nonzero elements of  $A[L; I, J]$  all appear in the rows  $K \cup I_0$  and in the columns  $K \cup J_0$ , and they are precisely the same as the elements in the submatrix  $A[I, J]$  of  $A$ . Note that  $\det A[L; I, J] = 0$  because it has  $d(I, J)$  rows of zeros and  $d(I, J)$  columns of zeros.

The next step in our construction follows that in [14]. Set  $d(I, J) = r$ ,  $I_0 = \{i_1, i_2, \dots, i_r\}$  and  $J_0 = \{j_1, j_2, \dots, j_r\}$ , where  $i_1 < i_2 < \dots < i_r$  and  $j_1 < j_2 < \dots < j_r$ . In the matrix  $A[L; I, J]$  replace each of the zeros in the positions  $(j_\sigma, i_\sigma)$ ,  $\sigma = 1, 2, \dots, r$ , with a one. Call the resulting matrix  $\hat{A}[L; I, J]$ .

**LEMMA 2.** For the matrix  $\hat{A}[L; I, J]$  defined as above,

$$\det \hat{A}[L; I, J] = (-1)^{\mu(I, J)} \det A[I, J]$$

where  $\mu(I,J) = \sum_{\sigma=1}^r (\tau(i_\sigma) + \tau(j_\sigma))$  and  $\tau(i_\sigma), \tau(j_\sigma)$  are the relative positions of  $i_\sigma, j_\sigma$ , respectively, in the ordered set  $L$ .

*Proof.* Denoting  $L$  by  $\{\ell_1, \ell_2, \dots, \ell_s\}$  with  $\ell_1 < \ell_2 < \dots < \ell_s$ , let  $\tau$  denote the function such that  $\tau(\ell_k) = k$ ,  $1 \leq k \leq s$ . Since  $I_0, J_0 \subseteq L$ ,  $\tau(i_\sigma)$  and  $\tau(j_\sigma)$  denote, respectively, the positions of  $i_\sigma$  and  $j_\sigma$  ( $1 \leq \sigma \leq r$ ) in the ordered set  $L$ . The result now follows from the structure of  $\hat{A}[L;I,J]$ . ■

In order to obtain a graph-theoretic understanding of this lemma, observe that inserting the ones in the matrix  $A[L;I,J]$  as was done above can be interpreted in terms of the digraph  $\langle I,J \rangle$  as adding the arcs  $(j_\sigma, i_\sigma)$ ,  $\sigma = 1, 2, \dots, r$ . Denoting the resulting digraph by  $\langle \overline{I}, \overline{J} \rangle$ , observe that  $\langle \overline{I}, \overline{J} \rangle$  is *not* in general a subdigraph of  $D(A)$  but that  $D(\hat{A}[L;I,J]) = \langle \overline{I}, \overline{J} \rangle$ . We can interpret the calculation of  $\det \hat{A}[L;I,J]$  graph theoretically with the help of  $\langle \overline{I}, \overline{J} \rangle$ . Corresponding to a factor (a set of cycles) of  $\langle \overline{I}, \overline{J} \rangle$  is a "factor" which is a set of cycles and paths of  $\langle I,J \rangle$ . The extension of this notion of factor is used below in the context of non-principal minors.

**THEOREM 8.** Let  $A$  be an  $n \times n$  matrix with  $I, J, L, s, \mu(I,J)$  and  $\langle \overline{I}, \overline{J} \rangle$  defined as above. A non-principal minor of  $A$  is given by

$$(19) \quad \det A[I,J] = (-1)^{\mu(I,J)} (-1)^s \sum_k (-1)^{\nu_k} A[p_{k1}] \cdots A[p_{kr}] A[c_{k1}] \cdots A[c_{km_k}],$$

where the sum is taken over all factors  $f_k$  of  $\langle \overline{I}, \overline{J} \rangle$ , and  $\nu_k$  is the number of cycles of  $f_k$ .

*Proof.* Clearly the terms in the expansion of  $\det \hat{A}[L;I,J]$  correspond to the factors of  $\langle \overline{I}, \overline{J} \rangle$ . The distinguished vertices of  $\langle \overline{I}, \overline{J} \rangle$  are found in  $V_0 \cap K$ . Therefore every cycle of  $\langle \overline{I}, \overline{J} \rangle$  containing either of the vertices  $j_\sigma$  or  $i_\sigma$ ,  $\sigma = 1, 2, \dots, r$ , must contain the arc  $(j_\sigma, i_\sigma)$  because  $j_\sigma$  is a sink vertex of  $\langle I, J \rangle$  and  $i_\sigma$  is a source vertex of  $\langle I, J \rangle$ . It follows that every factor of  $\langle \overline{I}, \overline{J} \rangle$  contains all of the arcs  $(j_\sigma, i_\sigma)$ ,  $\sigma = 1, 2, \dots, r$ . But this implies that each factor of  $\langle \overline{I}, \overline{J} \rangle$  contains a set of paths  $p_1, \dots, p_r$  in  $\langle I, J \rangle$  having the following properties (see [14]):

- (a)  $p_1, \dots, p_r$  are disjoint;
- (b) each  $p_\sigma$  starts at a vertex of  $I_0$  and ends at a vertex of  $J_0$ ; and
- (c) each  $p_\sigma$  contains no other vertices in either  $I_0$  or  $J_0$ .

Set  $\nu_k$  equal to the number of cycles in the factor  $f_k$  of  $\langle \overline{I}, \overline{J} \rangle$ . Then the sum of the lengths of the cycles in  $f_k$  is equal to  $s$  and the sign they contribute to the factor  $f_k$  is  $(-1)^{s+\nu_k}$ . Corresponding to the factor  $f_k$  of  $\langle \overline{I}, \overline{J} \rangle$  is a "factor" of  $\langle I, J \rangle$  which we may write in the form  $\hat{f}_k = \{p_{k1}, \dots, p_{kr}, c_{k1}, \dots, c_{km_k}\}$  as the element of  $A$  corresponding to each  $(j_\sigma, i_\sigma)$  has the value 1,  $\sigma = 1, 2, \dots, r$ . The  $r$  paths in  $\hat{f}_k$  come from the  $r$  cycles of  $\langle \overline{I}, \overline{J} \rangle$  containing the arcs  $(j_\sigma, i_\sigma)$ ,  $\sigma = 1, 2, \dots, r$ . We associate with the factor  $\hat{f}_k$  the sign  $(-1)^{s+\nu_k}$ , and thus using (1') and Lemma 2 we have formula (19) for the non-principal minor  $\det A[I, J]$ . ■

Since  $\hat{A}[L;I,J]$  only plays an auxiliary role for the purpose of computing  $\det A[I, J]$ , we can interpret the computation of the determinant graph theoretically in terms of  $D(A[I, J]) = \langle I, J \rangle$ . Observe that each of the cycles  $c_{k1}, \dots, c_{km_k}$  in (19) belongs to the subdigraph  $\langle K \rangle$ . Therefore let us partition the factors  $\hat{f}_k$  according to the set of  $r$  paths in the factor; thus two factors having the same set of paths are put into the same class. Denote by  $V(k;L)$  the complementary set of indices in  $L$  to the set contained in

the union of the paths  $p_{k1}, p_{k2}, \dots, p_{kr}$ . Note that  $V(k;L)$  is uniquely defined. We can now modify (19) to the form

$$(19') \quad \det A[I,J] = (-1)^{\mu(I,J)} \sum_k (-1)^{\mu_k} A[p_{k1}] \cdots A[p_{kr}] \det A[V(k;L)],$$

where the sum is over all distinct sets of paths in  $\langle I,J \rangle$  and  $\mu_k$  equals the number of cycles of even length generated by the paths  $p_{k1}, \dots, p_{kr}$  in  $\langle \overline{I}, \overline{J} \rangle$ . This result is used in [14] to show that if  $A$  is an  $M$ -matrix with its graph having no simple cycle of length greater than three, then the sign of any minor depends only on this graph (and not on the magnitudes of the matrix entries). The formulas of this section illustrate the fact that the expansions of non-principal minors involve paths in  $D(A)$ , whereas the expansions of principal minors involve only cycles (see [18], [19]).

We give now two special cases of Theorem 8. First consider a minor with maximum possible dispersion, i.e., the case  $I \cap J = \phi$ .

**COROLLARY 8.1.** Let  $A$  be an  $n \times n$  matrix and  $d(I,J) = |I| = r$ . Then a minor of maximum dispersion of  $A$  is given by

$$(20) \quad \det A[I,J] = (-1)^r \sum_k (-1)^{\nu_k} \prod_{\sigma=1}^r a_{i_{\sigma}, f_k(i_{\sigma})},$$

where the sum is taken over all factors  $f_k$  of  $\langle \overline{I}, \overline{J} \rangle$ ,  $\nu_k$  is the number of cycles in  $f_k$  and  $f_k(i_{\sigma})$  is the element of  $J$  which is the endpoint of the arc with initial point  $i_{\sigma}$  for given  $f_k$ .

*Proof.* In this case the graph  $\langle I, J \rangle$  is a directed bipartite graph, i.e., every arc of  $\langle I, J \rangle$  has the form  $(i_\sigma, j_\sigma)$  where  $i_\sigma \in I$  and  $j_\sigma \in J$ . For each factor  $f_k$  of  $\langle \overline{I}, \overline{J} \rangle$  every cycle has length  $2\ell$  for some  $\ell$ , since  $\langle \overline{I}, \overline{J} \rangle$  is a bipartite digraph. Thus each cycle of  $f_k$  contributes a negative sign. Letting  $I = \{i_1, i_2, \dots, i_r\}$  and  $J = \{j_1, j_2, \dots, j_r\}$ , consider the mapping  $\tau$  on  $I \cup J$  for which  $\tau(I \cup J) = \{1, 2, \dots, 2r\}$ . Now  $\tau(i_\sigma) + \tau(j_\sigma)$  is even if both  $\tau(i_\sigma)$  and  $\tau(j_\sigma)$  are even or if they are both odd, and  $\tau(i_\sigma) + \tau(j_\sigma)$  is odd if one of  $\tau(i_\sigma), \tau(j_\sigma)$  is odd and the other even. But, if  $r$  is odd, there must be an odd number of differences with one even and one odd and, if  $r$  is even, an even number of such

differences. It follows that  $\mu(I, J) = \sum_{\sigma=1}^r (\tau(i_\sigma) + \tau(j_\sigma))$  has the same parity as  $r$ . As  $s$  is even, we obtain the expansion formula (20) in the maximum dispersion case. ■

Now consider the case of a minor of dispersion one. Such minors are sometimes called almost principal minors (see [13]). There is, however, some confusion in the literature concerning this term. Apparently Gantmacher and Krein [7] had a narrower concept in mind when they introduced almost principal minors. We use the term in the broad sense here.

**COROLLARY 8.2.** Let  $A$  be an  $n \times n$  matrix. Then, with  $I = K \cup \{i_0\}$  and  $J = K \cup \{j_0\}$ , a minor of dispersion one of  $A$  is given by

$$(21) \quad \det A[I, J] = (-1)^{\tau(i_0) + \tau(j_0)} \sum_{k=1}^m (-1)^{\ell_k} A[p_k(i_0 \rightarrow j_0)] \det A[V(p_k, K)],$$

where the sum is over all distinct paths from  $i_0$  to  $j_0$  in  $\langle I, J \rangle$ ;  $\tau(i_0), \tau(j_0)$  is the position of  $i_0, j_0$ , respectively, in the ordered set  $I \cup J$ ;  $\ell_k$  is the length of path  $p_k$  from  $i_0$  to  $j_0$ ; and  $V(p_k, K)$  is the set of vertices of  $\langle I, J \rangle$  not belonging to  $p_k$ .

*Proof.* When  $I = K \cup \{i_0\}$ ,  $J = K \cup \{j_0\}$  each "factor" of  $\langle I, J \rangle$  consists of a product of cycles of  $\langle K \rangle$  and a path  $p(i_0 \rightarrow j_0)$ . The path contributes sign  $(-1)^\ell$  where  $\ell$  is the length of the path. We can partition the factors of  $\langle I, J \rangle$  into equivalence classes by holding the path  $p(i_0 \rightarrow j_0)$  fixed and permitting the cycles of  $\langle K \rangle$  to vary. Let  $p_1(i_0 \rightarrow j_0), \dots, p_m(i_0 \rightarrow j_0)$  be the distinct paths from  $i_0$  to  $j_0$  in  $\langle I, J \rangle$ . Let  $V(p_k, K)$  be the set of vertices of  $\langle I, J \rangle$  not belonging to  $p_k$ ; these vertices are all in  $\langle K \rangle$ , hence the notation. Then we have (21) as a general formula for an almost principal minor. ■

Note that when  $|I| = 1$  the almost principal minor is the nondiagonal element  $a_{i_0 j_0}$  of  $A$ . In this case  $L = \{i_0, j_0\}$  and we have  $\tau(\min\{i_0, j_0\}) = 1$ ,  $\tau(\max\{i_0, j_0\}) = 2$  and  $\ell_k = 1$ . Thus  $(-1)^{\tau(i_0) + \tau(j_0)} = (-1)^{\ell_k} = -1$  so that (21) yields  $\det A[i_0, j_0] = a_{i_0 j_0}$ , as it must. Note that if  $A$  is an  $M$ -matrix, then each term in the summation of (21) is nonnegative, so that  $\det A[I, J] \det A[J, I] \geq 0$  for any  $I, J$  with  $d(I, J) = 1$ . When this inequality holds,  $A$  is called weakly sign symmetric.

**7. Cofactor Formulas.** We now use Corollary 8.2 to prove the following results, where the cofactor of  $a_{ij}$  is denoted by  $A_{ij}$ , and the matrix  $\text{cof } A = [A_{ij}]$ .

**THEOREM 9** ([15]). Let  $A$  be an  $n \times n$  matrix with digraph  $D(A)$ . Let  $a_{ij}$  with  $i \neq j$  be an arbitrary non-diagonal element of  $A$ . Then the cofactor of  $a_{ij}$  is given by

$$(22) \quad A_{ij} = \sum_k (-1)^{\ell_k} A_{[p_k(j-i)]} \det A[V(p_k)],$$

where the sum is taken over all paths in  $D(A)$  from  $j$  to  $i$ , and  $\ell_k$  is the length of path  $p_k$ .

*Proof.* Let us apply the formula (21) to the almost principal minor  $\det A(i,j)$ , i.e., to the almost principal minor  $\det A[I,J]$  where  $I = V - \{i\}$ ,  $J = V - \{j\}$ . We then have  $L = V$ ,  $i_0 = j$  and  $j_0 = i$ ,  $\tau(j) = j$ ,  $\tau(i) = i$ . Note that the set of all paths from  $j$  to  $i$  in  $\langle I,J \rangle$  is the same as the set of all paths from  $j$  to  $i$  in  $D(A)$ . Therefore we obtain the result

$$\det A(i,j) = (-1)^{i+j} \sum_k (-1)^{\ell_k} A_{[p_k(j-i)]} \det A[V(p_k)].$$

But the cofactor of  $a_{ij}$  is  $A_{ij} = (-1)^{i+j} \det A(i,j)$ , so the formula (22) follows at once. ■

**COROLLARY 9.1.** Let  $A$  be an  $n \times n$  nonsingular matrix with digraph  $D(A)$  and  $A^{-1} = [\alpha_{ij}]$ . Then we have

$$(23a) \quad \alpha_{ii} = \det A(i) / \det A,$$

and

$$(23b) \quad \alpha_{ij} = \frac{1}{\det A} \sum_k (-1)^{\ell_k} A[p_k(i \rightarrow j)] \det A[V(p_k)], \quad i \neq j,$$

where the sum is taken over all paths in  $D(A)$  from  $i$  to  $j$ , and  $\ell_k$  is the length of path  $p_k$ .

*Proof.* The formulas (23) follow at once from (22) and the fact that  $(\det A)A^{-1} = (\text{cof } A)^T$ . ■

We use these formulas to prove, in the following corollary, that if a matrix is nonsingular, irreducible and every vertex is distinguished, then, if cancellations are ignored, its inverse matrix is full. Other proofs of this have recently been given ([6], [8]) in the context of sparse matrices.

**COROLLARY 9.2.** Let  $A = [a_{ij}]$  be an  $n \times n$  irreducible matrix with  $a_{ii} \neq 0$  for all  $i \in V$ , and suppose  $A^{-1} = [\alpha_{ij}]$  exists. Then, ignoring cancellations,  $\alpha_{ij} \neq 0$  for all  $i, j \in V$ .

*Proof.* Suppose  $\alpha_{ii} = 0$ . Then from (23a),  $\det A(i) = 0$ . As cancellations are ignored, this implies that at least one of  $a_{pp} = 0$ ,  $p \in V - \{i\}$ , which is a contradiction.

Suppose  $\alpha_{ij} = 0$ ,  $i \neq j$ . Then from (23b),  $A[p_k(i \rightarrow j)] = 0$  for each path  $p_k$  in  $D(A)$  from  $i$  to  $j$ . (Note that as each  $a_{ii}$  is assumed nonzero and cancellations are ignored,  $\det A[V(p_k)]$  is nonzero.) Thus there is no path from  $i$  to  $j$  in  $D(A)$ . So  $A$  is reducible, which is a contradiction. ■

Note that it is possible for every vertex to be not distinguished, but the inverse matrix to be full.

The basic cofactor formula is presented above as equation (22) of Theorem 9. We now elaborate on this result and indicate some applications.

Since  $A(\text{cof } A)^T = (\det A)I$ ,

$$\sum_{k=1}^n a_{ik}A_{jk} = \begin{cases} 0 & \text{if } i \neq j, \\ \det A & \text{if } i = j. \end{cases}$$

For  $i = j$  we have

$$\begin{aligned} \det A &= \sum_{k=1}^n a_{ik}A_{ik} = a_{ii}A_{ii} + \sum_{k \neq i} a_{ik}A_{ik} \\ &= a_{ii} \det A(i) + \sum_{k \neq i} a_{ik} \sum_m (-1)^{\ell_{mk}} A_{[p_{mk}(k \rightarrow i)]} \det A[V(p_{mk})], \end{aligned}$$

where  $m$  is taken over all paths in  $D(A)$  from  $k$  to  $i$ . Clearly for each  $k$  such that both  $a_{ik} \neq 0$  and there exists at least one path  $p(k \rightarrow i)$  in  $D(A)$ , the product  $a_{ik}A_{[p_{mk}(k \rightarrow i)]}$  is a cycle containing the index  $i$ . The sign attached is  $(-1)^{\ell+1}$  where  $\ell$  is the length of the cycle. So we have rederived Theorem 2 using the cofactor formula.

Now for  $i \neq j$  we have

$$0 = \sum_{k=1}^n a_{ik}A_{jk} = a_{ii}A_{ji} + a_{ij}A_{jj} + \sum_{k \neq i, j} a_{ik}A_{jk}.$$

Thus

$$0 = a_{ii} \sum_k (-1)^{\ell_k} A[p_k(i \rightarrow j)] \det A[V(p_k)] + a_{ij} \det A(j) \\ + \sum_{k \neq i, j} a_{ik} \sum_m (-1)^{\ell_{mk}} A[p_{mk}(k \rightarrow j)] \det A[V(p_{mk})].$$

Now the sum in the first term is over all paths in  $D(A)$  from  $i$  to  $j$ . On the other hand, a given path from  $i$  to  $j$  in  $D(A)$  appears exactly once in the third term. Observe that, since  $p_{mk}(k \rightarrow j)$  does not contain the vertex  $i$ , the set  $V(p_{mk})$  does. On the other hand,  $V(p_k)$  appearing in the first term does not include the vertex  $i$ . Thus we have the following identity:

$$0 = a_{ij} \det A(j) + \sum_k (-1)^{\ell_k} A[p_k(i \rightarrow j)] \{a_{ii} \det A[V(p_k)] - \det A[V(p_k) \cup \{i\}]\}.$$

In the remainder of this section, we consider particular cases of the cofactor formulas when  $D(A)$  has special local properties.

Consider first the case where  $D(A)$  has the cutpoint  $i$ ; see section 3 for notation.

There are four cases to consider in evaluating  $\text{cof } A$ .

**Case (i):**  $\sigma \in I_j, \tau \in I_k$  for  $j \neq k$ . We have

$$A_{\sigma\tau} = A[\bar{I}_j]_{\sigma i} A[\bar{I}_k]_{i\tau} \prod_{m \neq j, k} \det A[I_m].$$

**Case (ii):**  $\sigma = i$ ,  $\tau \in I_j$  ( $\tau = i$ ,  $\sigma \in I_j$  is analogous). We have

$$A_{\sigma\tau} = A[\bar{I}_j]_{\sigma\tau} \prod_{k \neq j} \det A[I_k].$$

**Case (iii):**  $\sigma, \tau \in I_j$ ,  $\sigma \neq \tau$ . We now obtain

$$\begin{aligned} A_{\sigma\tau} &= A[\bar{I}_j]_{\sigma\tau} \prod_{k \neq j} \det A[I_k] + \sum_{k \neq j} A[I_j]_{\sigma\tau} \det A[\bar{I}_k] \prod_{m \neq j, k} \det A[I_m] \\ &\quad - (p(i)-1)a_{ii} A[I_j]_{\sigma\tau} \prod_{k \neq j} \det A[I_k]. \end{aligned}$$

**Case (iv):**  $\sigma = \tau$ . If  $\sigma = i$ , then  $A_{ii} = \prod_{k=1}^{p(i)} \det A[I_k]$ . If  $\sigma \in I_j$ , we obtain

$$\begin{aligned} A_{\sigma\sigma} &= \det A[\bar{I}_j - \{\sigma\}] \prod_{k \neq j} \det A[I_k] \\ &\quad + \sum_{k \neq j} \det A[\bar{I}_k] \det A[I_j - \{\sigma\}] \prod_{m \neq j, k} \det A[I_m] \\ &\quad - (p(i)-1)a_{ii} \det A[I_j - \{\sigma\}] \prod_{k \neq j} \det A[I_k]. \end{aligned}$$

Next consider the cofactors of  $A$  when  $D(A)$  contains the bridge consisting of the arcs  $(i,j)$  and  $(j,i)$ ; see section 4 for notation. There are now three cases to consider in evaluating the  $A_{\sigma\tau}$ .

Case (i):  $\sigma \in \bar{I}$ ,  $\tau \in \bar{J}$  ( $\sigma \in \bar{J}$ ,  $\tau \in \bar{I}$  is done analogously). Let  $p$  be an arbitrary path in  $D(A)$  from  $\tau$  to  $\sigma$ . We can write  $p(\tau \rightarrow \sigma) = p'(\tau \rightarrow j)(j,i)p''(i \rightarrow \sigma)$ . Then  $p'(\tau \rightarrow j)$

is contained in  $\bar{D}_j$  and  $p''(i \rightarrow \sigma)$  in  $\bar{D}_i$ . Therefore from (22) we may write

$$\begin{aligned} A_{\sigma\tau} &= \sum_k (-1)^{\ell_k} A_{[p_k(\tau \rightarrow \sigma)]} \det A[V(p_k)] \\ &= \sum_k (-1)^{\ell(p'_k)+1+\ell(p''_k)} A_{[p'_k(\tau \rightarrow j)]} a_{ij} A_{[p''_k(i \rightarrow \sigma)]} \det A[\bar{J}-V[p']] \det A[\bar{I}-V[p'']], \end{aligned}$$

which is equivalent to  $A_{\sigma\tau} = -a_{ji} A[\bar{I}]_{\sigma i} A[\bar{J}]_{j\tau}$ . This expansion means that for  $\sigma \in \bar{I}$  and  $\tau \in \bar{J}$ , the cofactor  $A_{\sigma\tau}$  can be written as the product of  $-a_{ji}$  and certain cofactors of the smaller matrices  $A[\bar{I}]$  and  $A[\bar{J}]$ .

Case (ii):  $\sigma, \tau \in \bar{I}$ ,  $\sigma \neq \tau$  ( $\sigma, \tau \in \bar{J}$ ,  $\sigma \neq \tau$  can be done analogously). If  $\sigma$  and  $\tau$  both differ from  $i$ , we obtain

$$A_{\sigma\tau} = A[\bar{I}]_{\sigma\tau} \det A[\bar{J}] - a_{ij} a_{ji} A[I]_{\sigma\tau} \det A[J].$$

For  $\sigma = i$  ( $\tau = i$  is done analogously), this reduces to

$$A_{\sigma\tau} = A[\bar{I}]_{\sigma\tau} \det A[\bar{J}].$$

Case (iii):  $\sigma = \tau$ . If  $\sigma \neq i$ ,  $\sigma \neq j$ ,  $\sigma \in I$  ( $\sigma \in J$  is analogous), then

$$A_{\sigma\sigma} = A[\bar{I}]_{\sigma\sigma} \det A[\bar{J}] - a_{ij} a_{ji} A[I]_{\sigma\sigma} \det A[J].$$

For  $\sigma = i$  ( $\sigma = j$  is done analogously), then  $A_{\sigma\sigma} = \det A[I] \det A[\bar{J}]$ .

From these three cases, we see that, when  $D(A)$  has the bridge  $\{(i,j), (j,i)\}$ , the matrix  $\text{cof } A$  is completely determined by the matrices  $\text{cof } A[\bar{I}]$ ,  $\text{cof } A[I]$ ,  $\text{cof } A[\bar{J}]$ ,  $\text{cof } A[J]$ , the principal minors  $\det A[\bar{I}]$ ,  $\det A[I]$ ,  $\det A[\bar{J}]$ ,  $\det A[J]$  and the elements  $a_{ij}$ ,  $a_{ji}$ .

As a final application of our methods we mention the following double bridge formula. Given a matrix  $A$ , let  $i, j, k, \ell$  be distinct vertices in  $D(A)$ . A *double bridge* is a subset  $B$  of arcs of  $D(A)$  such that  $B \subseteq \{(i,k), (k,i), (j,\ell), (\ell,j)\}$ ,  $B$  contains at least one arc from  $\{(i,k), (k,i)\}$  and at least one arc from  $\{(j,\ell), (\ell,j)\}$ , and  $D(A)-B$  has more weak components than  $D(A)$  with  $\{i,j\}$  in one weak component and  $\{k,\ell\}$  in another.

Let  $D(A)$  be weakly connected and have a double bridge. Suppose  $\langle I_1 \cup \{i,j\} \rangle$  and  $\langle I_2 \cup \{k,\ell\} \rangle$  are the subdigraphs of  $D(A)-B$  containing  $\{i,j\}$  and  $\{k,\ell\}$ , respectively. Letting  $\bar{I}_1 = I_1 \cup \{i,j\}$  and  $\bar{I}_2 = I_2 \cup \{k,\ell\}$ , then the cycles of a factor  $f$  of  $D(A)$  may be categorized as follows:

- (i)  $f$  contains cycles lying entirely in  $\langle \bar{I}_1 \rangle$  or in  $\langle \bar{I}_2 \rangle$ ;
- (ii)  $f$  contains the two 2-cycles  $(i,k,i)$ ,  $(j,\ell,j)$  and cycles lying entirely in  $\langle \bar{I}_1 \rangle$  or in  $\langle \bar{I}_2 \rangle$ ;
- (iii)  $f$  contains the 2-cycle  $(j,\ell,j)$  and cycles lying either in  $\langle I_1 \cup \{i\} \rangle$  or in  $\langle I_2 \cup \{k\} \rangle$ ;
- (iv)  $f$  contains the 2-cycle  $(i,k,i)$  and cycles lying either in  $\langle I_1 \cup \{j\} \rangle$  or in  $\langle I_2 \cup \{\ell\} \rangle$ ;
- (v)  $f$  contains cycles which lie partly in  $\langle \bar{I}_1 \rangle$  and partly in  $\langle \bar{I}_2 \rangle$ .

Thus

$$\begin{aligned}
\det A &= \det A[\bar{I}_1] \det A[\bar{I}_2] + a_{ik} a_{ki} a_j a_{\ell j} \det A[I_1] \det A[I_2] \\
&\quad - a_j a_{\ell j} \det A[I_1 \cup \{i\}] \det A[I_2 \cup \{k\}] - a_{ik} a_{ki} \det A[I_1 \cup \{j\}] \det A[I_2 \cup \{\ell\}] \\
&\quad - a_j a_{ki} A[\bar{I}_1]_{ji} A[\bar{I}_2]_{k\ell} - a_{ik} a_{\ell j} A[\bar{I}_1]_{ij} A[\bar{I}_2]_{\ell k},
\end{aligned}$$

where the first four terms correspond, respectively, to categories (i)–(iv) and the last two terms correspond to the two types of factors in (v).

8. **An Example.** We conclude with a  $7 \times 7$  example which illustrates several of the formulas given previously. Figure 5 displays the strongly connected digraph  $D(A)$  for our example; all the vertices are distinguished. We use  $A[i,j]$  to denote  $A[\{i,j\}]$  and similar notation for other principal minors.

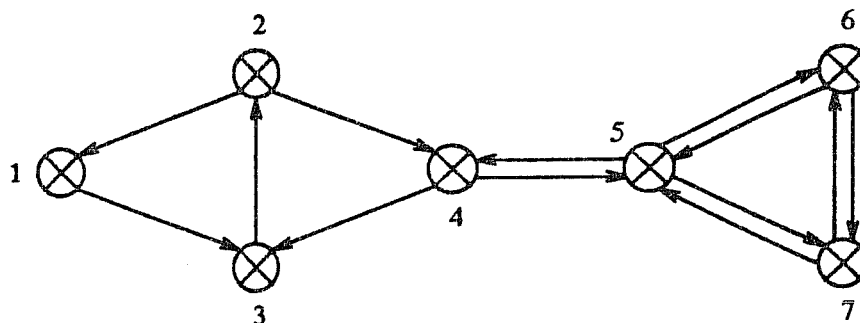


Figure 5

The digraph for our  $7 \times 7$  example.

Vertex 5 of  $D(A)$  is a cutpoint and the number of weak components in  $D(A) - \{5\}$  is 2. Taking  $I_1 = \{1,2,3,4\}$  and  $I_2 = \{6,7\}$ , our cutpoint formula (4) gives

$$\det A = \det A[1,2,3,4,5] \det A[6,7] + \det A[5,6,7] \det A[1,2,3,4] \\ - a_{55} \det A[1,2,3,4] \det A[6,7].$$

Working with cycles, our formula (4') gives

$$\det A = \{a_{55} \det A[1,2,3,4] - a_{45}a_{54} \det A[1,2,3]\} \det A[6,7] \\ + \{a_{57}a_{76}a_{65} + a_{56}a_{67}a_{75} - a_{65}a_{56}a_{77} - a_{75}a_{57}a_{66}\} \det A[1,2,3,4].$$

We can also regard  $D_0 = \langle 2,3 \rangle$  as a critical subdigraph of  $D(A)$ . Then, taking  $I_1 = \{1\}$  and  $I_2 = \{4,5,6,7\}$ , our critical subdigraph formula (7) gives

$$\begin{aligned} \det A &= \det A [1,2,3] \det A [4,5,6,7] + a_{11} \det A [2,3,4,5,6,7] \\ &\quad - a_{11} \det A [2,3] \det A [4,5,6,7]. \end{aligned}$$

The arcs  $(4,5)$  and  $(5,4)$  constitute a bridge of  $D(A)$ , with subsets  $I = \{1,2,3\}$  and  $J = \{6,7\}$ . Formula (8) then yields

$$\det A = \det A [1,2,3,4] \det A [5,6,7] - a_{45} a_{54} \det A [1,2,3] \det A [6,7].$$

Expanding about row 4 of  $A$ , we have, by the usual cofactor expansion,  $\det A = a_{43} A_{43} + a_{44} A_{44} + a_{45} A_{45}$ . Clearly  $A_{44} = \det A [1,2,3] \det A [5,6,7]$ , and from our cofactor formula (22)  $A_{43} = a_{32} a_{24} a_{11} \det A [5,6,7]$ , and  $A_{45} = -a_{54} \det A [1,2,3] \det A [6,7]$ .

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