A NEW LOOK AT WHOLE-FOREST MODELLING

By

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Abstract

The harvest scheduling problem customarily known as Model II is reformulated as a control theory problem, and two ways of solving it using the numerical methods of linear programming are discussed. Viewing the problem in this way offers many insights and makes extensions to the basic problem relatively easy to handle. Extensions discussed include the risk of catastrophic loss through fire including the possibility of partial salvage of volumes burnt; the problem of a changing land base; the imposition of area constraints on standing forest; the inclusion of multiple timber types; and the problem of accessibility. The last two extensions are discussed in the context of the presence of the risk of fire. The mathematical relationship between the standard Model II linear program, and the control theory formulation is discussed and the numerical efficiency of the various methods is examined for simple problems

Additional Key Words: Model II, harvest scheduling, linear programming, fire risk.

1. INTRODUCTION

The problem of the scheduling of harvests from a forest comprising many even-age stands, and the related problem of making long-run projections of timber supply from such a forest, are of central importance in forest management. Both simulation and optimization techniques have been developed to address these problems, and large-scale package routines using both methods are widely employed.

Optimization routines (e.g. Timber RAM (Navon, 1971), MUSYC (Johnson & Jones, 1979), FORPLAN (Johnson, Jones and Kent, 1980)) mainly offer the choice of two alternative linear programming formulations termed Model I and Model II by Johnson & Scheurman (1977).

Simulation routines (e.g. TREES (Tedder, Schmidt and Gourley, 1980) are usually based on a procedure which numerically updates the state of the forest at the end of each period, after management options have been exercised. The updating takes into account the aging of the forest as well as harvesting and other management decisions.

This procedure, of using a dynamic equation to update the state of the forest, has been used as the basis of optimization analysis by McDonough and Park (1975) and Lyon and Sedjo (1983). The earlier of the above papers describes how the techniques of Optimal Control Theory (specifically the Discrete Maximum Principle (e.g. Halkin, 1966)) can be used to obtain an optimal solution to the scheduling problem. The latter paper uses the Maximum Principle to obtain numerically, a long-run projection of the regional supply of timber under optimal economic management. Because the objectives considered in both papers are non-linear the results obtained are not directly comparable to the results of an analysis using Model I or Model II linear programming formulations.

In this paper we consider the problem solved by the Model II formulation and show how it can be more conveniently formulated in a control theory framework i.e. with dynamic equations describing the period-to-period transitions of the forest, and a linear objective function to be maximized. One of the major advantages of this formulation is that it is much easier to comprehend than the formulation typically used (that is, as given in Johnson and Scheurman, 1977; and referred to henceforth as the Standard Form). A second major advantage is that many extensions of the basic Model II problem can more easily be handled using the control theory formulation. These extensions include: - (i) the possibility of random catastrophic loss by fire or other loss agent, (ii) the problem of changing land base, (iii) the inclusion of multiple timber types and regeneration options, and (iv) the inclusion of constraints on areas of standing forest for wildlife or recreational purposes etc. While most of these extensions can be handled in the Standard Model II Form (see Johnson, Jones and Kent, 1980 and Johnson & Stuart, 1985) the formulation of the corresponding area constraints is complicated. In fact an easy way to determine the Model II area constraints is to derive them from the control theory formulation.

The control theory formulation of the Model II problem is linear in both objective and its constraints. It can therefore be solved by linear programming techniques (e.g. the Revised Simplex Algorithm (Dantzig, 1963)). The linear program which arises is not the same as the Standard Model II Form linear program, although it will produce identical results. This is proved formally in the paper. Computational aspects of various methods of solving the control theory problem and of solving the Standard Model II Form problem are discussed. It is shown how certain dual variables determined incidentally in solving the control theory problem, provide shadow values for forested land i.e. they provide the net present value of standing timber, at the margin, in a forest-level (or "whole-forest") context.

In view of the results obtained the major conclusion of this paper is that the control theory formulation provides an attractive alternative to the Standard Model II linear programming formulation of the whole-forest scheduling problem.

The organization of the paper is as follows. In Section 2 the basic scheduling problem of Model II is formulated in a control theory framework and its solution by linear programming is discussed. In Section 3 the problem of catastrophic loss is considered, and it is shown how the feedback application of the optimal solution to a problem very similar to that in Section 2, provides an approximately optimal solution to the stochastic control problem and also provides estimates of long-run timber supply when the risk of fire or other catastrophic loss is present. Section 4 discusses the computational performance of the various methods of solution. Sections 5 through 7 discuss extensions to the basic model with the risk of catastrophic loss present. They include the possibility of multiple timber types and regeneration options, the possibility of salvage after a fire, and the problem of accessibility. Other extensions currently in FORPLAN and MUSYC can also be handled but are not explicitly discussed.

2 MODEL FORMULATION AND SOLUTION

In the Model I scheduling problem (Johnson & Scheurman, 1977), each timber type/age-class combination which contains hectares at the start of the problem forms a management unit, whose integrity is retained over the whole of the planning horizon. Each activity in the linear program represents a sequence of regeneration harvests for a single set of hectares in a management unit.

In contrast, in the Model II formulation hectares do not necessarily remain in the same management unit over the planning horizon. Rather each activity in the linear program represents the amount (number of hectares) of the forest of a given type which are regenerated at a specified time and harvested at another specified time for a single rotation only. Thus hectares from a management unit may be broken up and combined with hectares from other management units which are regeneration harvested at the same time.

In this paper we shall consider the problem addressed in Model II. In this section we shall formulate the basic Model II problem in a standard control theory fashion, i.e. with a <u>dynamic equation</u>, or equation of motion, of the form

$$\underset{\sim}{x}_{t+1} = f(\underset{\sim}{x}_t, \underset{\sim}{h}_t) \qquad (1)$$

(where $\underset{\sim}{x}_t$ is the <u>state-vector</u> denoting the areas in various age-classes at the start of time period t, and $\underset{\sim}{h}_t$ is the <u>control vector</u> of harvests in the various age-classes, in period t), and an objective, which we seek to maximize, of the form

$$J = \sum_{t=1}^{\infty} \alpha^t g(\underset{\sim}{h}_t) \qquad (2)$$

where α is the per-period discount factor and g is a revenue function. In addition we shall assume that there exist some flow constraints on the sequence of harvests $\{\underset{\sim}{h}_t\}$ taken from the forest.

Specifically we shall let

$$\underset{\sim}{x}_t = \begin{bmatrix} x_1^t \\ x_2^t \\ \vdots \\ x_k^t \end{bmatrix} \qquad (3)$$

where x_i^t $(i = 1,2,\ldots,k-1)$ denotes the area of the forest with even-age stands in age-class i at the beginning of period t (i.e. with stands of age between $(i-1)$ periods and i periods) and x_k^t is the area of the forest with even-age stands in age-class k, which we shall assume includes all stands of age $(k-1)$ periods or older. It will be assumed that k is chosen sufficiently large so that all stands in age-class k have approximately the same volume per unit area.

Also we let

$$\underset{\sim}{h}_t = \begin{bmatrix} h_1^t \\ h_2^t \\ \vdots \\ h_k^t \end{bmatrix} \qquad (4)$$

where h_i^t is the area clear-cut harvested in age-class i $(i = 1,\ldots,k)$ in period t.

To begin with we shall assume that all areas harvested in a period are regenerated and return to age-class 1 at the start of the following period. Thus the explicit form of the dynamic equation is:-

$$\underset{\sim}{x}_{t+1} = \begin{bmatrix} h_1^t + h_2^t + \ldots h_k^t \\ x_1^t - h_1^t \\ x_2^t - h_2^t \\ \vdots \\ (x_{k-1}^t - h_{k-1}^t) + (x_k^t - h_k^t) \end{bmatrix}, \tag{5}$$

which we can express conveniently in matrix notation (see also Lyon & Sedjo, 1983) as

$$\underset{\sim}{x}_{t+1} = R\underset{\sim}{x}_t - S\underset{\sim}{h}_t \tag{6}$$

where

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} -1 & -1 & \cdots & & -1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \end{bmatrix}. \tag{7}$$

We shall assume that the revenue function g in the objective (2) is of the linear form

$$g(\underset{\sim}{h}_t) = \underset{\sim}{V}' \cdot \underset{\sim}{h}_t \tag{8}$$

where $\underset{\sim}{V}' = (V_1, V_2, \ldots, V_k)$ is a <u>net-value-at-age</u> vector. The <u>volume</u> harvested in period t is of the form

$$H_t = \underset{\sim}{v}' \underset{\sim}{h}_t \tag{9}$$

where $\underset{\sim}{v}' = (v_1, v_2, \ldots, v_k)$ is a <u>volume-at-age</u> vector.

In order to ensure a controlled flow of timber from the forest constraints are usually imposed on the sequence of volumes $\{H_t\}$. These harvest flow constraints could be of several forms. Three possible forms that might be considered are:-

(a) <u>Upper and lower bound constraints</u>, of the form

$$a \leq H_t \leq b, \quad t = 1, 2, \ldots \tag{10}$$

where a and b are specified lower and upper limits on volume harvested.

(b) <u>Step constraints</u>, of the form

$$-\delta_1 \leq H_t - H_{t-1} \leq \delta_2, \quad t = 2, 3, \ldots \tag{11}$$

where δ_1 is the specified *maximum decrease* in volume cut and δ_2 is the specified *maximum increase* in volume cut. The condition of a non-declining flow would be specified by only the left-hand inequality with δ_1 set equal to zero.

(c) *Sequential flow constraints*, in which the maximum *proportional* increase or decrease in volume harvest is specified. These constraints would be of the form

$$(1-\gamma_1)H_{t-1} \leq H_t \leq (1+\gamma_2)H_{t-1} \qquad t = 2, 3, \ldots \tag{12}$$

where γ_1 $(0 \leq \gamma_1 \leq 1)$ is the maximum proportional decrease permitted, and γ_2 (≥ 0) is the maximum proportional increase permitted.

For the sake of illustration in what follows we shall assume that sequential flow constraints of the form (12) are imposed. The modifications necessary for alternative or additional flow constraints are obvious and will not be mentioned explicitly.

The problem of maximizing present discounted revenue from the forest, subject to sequential flow constraints can thus be expressed as:-

Maximize

$$J = \sum_{t=1}^{\infty} \alpha^t \underset{\sim}{V}' \cdot \underset{\sim}{h}_t \tag{13}$$

subject to

$$\underset{\sim}{x}_{t+1} = R\underset{\sim}{x}_t - S\underset{\sim}{h}_t \qquad t = 1, 2, \ldots \tag{14}$$

$$(1-\gamma_1)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \le \underset{\sim}{v}'\underset{\sim}{h}_t \le (1+\gamma_2)\underset{\sim}{v}'h_{t-1} \qquad t = 2, 3, \ldots \tag{15}$$

$$\text{and} \qquad \underset{\sim}{0} \le \underset{\sim}{h}_t \le x_t \qquad t = 1, 2, \ldots \tag{16}$$

and given an initial state $\underset{\sim}{x}_1$.

If there is existing harvesting activity taking place, with the total volume cut in the most recent period being H_0, and it is desirable to ensure continuity of supply, then there would be an additional constraint corresponding to (12) with $t = 1$.

The objective in (13) is for an infinite time-horizon. For practical purposes of solution one can consider a finite time horizon of N periods and include in the objective a terminal payoff corresponding to the net present value of standing timber at the end of the planning horizon. At first sight this may appear logically impossible to accomplish since one needs the solution to the infinite period optimization problem to determine the present value of the standing inventory. However after a suitable number of periods, under optimal management, the age-distribution of the forest will be close to a steady-state and an approximately even-flow of timber volume will be produced by the forest. The harvest flow constraints will thus be redundant and each stand of timber will be cut at its (stand-level) optimal cutting age. Thus in assessing the net present-value of one hectare of timber in age-class i $(i = 1,\ldots,k)$, at the end of the planning horizon, one can use its net present value as determined by stand-level theory (Faustmann 1849, Pearse 1967). Thus we can consider the problem of maximizing:-

$$J = \sum_{t=1}^{N} \alpha^{t} \underset{\sim}{V}'\underset{\sim}{h}_t + \alpha^{N+1} \underset{\sim}{r}'\underset{\sim}{x}_{N+1} \qquad (17)$$

subject to:-

$$\underset{\sim}{x}_{t+1} = R\underset{\sim}{x}_t - S\underset{\sim}{h}_t \qquad t = 1, \ldots, N \qquad (18)$$

$$0 \le \underset{\sim}{h}_t \le \underset{\sim}{x}_t \qquad t = 1, \ldots, N \qquad (19)$$

and $$(1-\gamma_1)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \le \underset{\sim}{v}'\underset{\sim}{h}_t \le (1+\gamma_2)\underset{\sim}{v}'\underset{\sim}{h}_{t-1}, \qquad t = 2, \ldots, N \qquad (20)$$

where $\underset{\sim}{r}' = (r_1, r_2, \ldots, r_k)$ is a vector of per-hectare net present values for areas in age-classes $1, 2, \ldots, k$. These values can be calculated either analytically or numerically using a Policy Improvement algorithm (e.g. Ross 1970, p. 125 ff). Details are given in Appendix 2. If there is existing harvest activity, in order to preserve continuity of supply there might be an additional constraint of the form (20), with $t = 1$.

In the above optimization problem both the objective and constraints are linear in the control variables $\underset{\sim}{h}_t$ and in the state variables $\underset{\sim}{x}_t$. The problem can thus be solved by standard Linear Programming (LP) techniques (e.g. using the Revised Simplex Algorighm (Dantzig (1963))).[1]

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[1] If one applies the Discrete Maximum Principle to the above control theory problem, one obtains the conditions implied by the Complementary Slackness Theorem of Linear Programming, which are necessary and sufficient conditions for an optimum. In other words a solution obtained by the Maximum Principle is identical to a solution obtained by Linear Programming.

Put in standard LP terms the problem is:-

(LP1) maximize $\underset{\sim}{c}'\underset{\sim}{u}$

subject to

$$\begin{aligned} A\underset{\sim}{u} \ &(\overset{=}{\leq})\ \underset{\sim}{b} \\ \text{and} \qquad \underset{\sim}{u} \ &\geq\ \underset{\sim}{0} \end{aligned} \tag{21}$$

where $\underset{\sim}{u}$ is the $(2N+1)k$-dimensional column vector of "activities" formed by vertically concatenating the $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{N+1}$ and the $\underset{\sim}{h}_1, \underset{\sim}{h}_2, \ldots, \underset{\sim}{h}_N$ vectors, i.e.

$$\underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_{N+1} \\ \underset{\sim}{h}_1 \\ \vdots \\ \underset{\sim}{h}_N \end{bmatrix}, \tag{22}$$

and $\underset{\sim}{c}'$ is the $(2N+1)k$-dimensional row vector

$$\underset{\sim}{c}' = [\underset{\sim}{0}', \underset{\sim}{0}', \ldots, \alpha^{N+1}\underset{\sim}{r}', \alpha\underset{\sim}{V}', \alpha^2\underset{\sim}{V}', \ldots, \alpha^N\underset{\sim}{V}']$$

(where $\underset{\sim}{0}'$ is a k-dimensional row vector of zeros). The constraint matrix A is of dimension $[2Nk+2(N-1)] \times [(2N+1)k]$ and is of the form

$$\left[\begin{array}{ccccccc|cccccc}
I & & & & & & & & & & & & \\
-R & I & & & & & & S & & & & & \\
 & -R & I & & & & & & S & & & & \\
 & & & \ddots & & & & & & & \ddots & & \\
 & & & & -R & I & & & & & & S & \\
 & & & & & -R & I & & & & & & S \\
\hline
-I & & & & & & & I & & & & & \\
 & -I & & & & & & & I & & & & \\
 & & & \ddots & & & & & & & \ddots & & \\
 & & & & & -I & & & & & & & I \\
\hline
 & & & & & & & \underset{\sim}{\ell}'_1 & -\underset{\sim}{v}' & & & & \\
 & & & & & & & -\underset{\sim}{\ell}'_2 & v' & & & & \\
 & & & & & & & & \underset{\sim}{\ell}'_1 & -\underset{\sim}{v}' & & & \\
 & & & & & & & & -\underset{\sim}{\ell}'_2 & \underset{\sim}{v}' & & & \\
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \underset{\sim}{\ell}'_1 & -\underset{\sim}{v}' \\
 & & & & & & & & & & & -\underset{\sim}{\ell}'_2 & \underset{\sim}{v}'
\end{array}\right] \tag{23}$$

where I is the $k \times k$ identity matrix, $\underset{\sim}{\ell}'_1 = (1-\gamma_1)\underset{\sim}{v}'$, $\underset{\sim}{\ell}'_2 = (1+\gamma_2)\underset{\sim}{v}'$, and blocks of zeros appear in the display in places other than those indicated by specific entries. The right-hand side vector $\underset{\sim}{b}$ in (21) is of dimension $2Nk + 2(N-1)$

and comprises zeros except in the first k positions where the initial state vector $\underset{\sim}{x}_1$ appears. The first Nk constraints are equalities, while the remainder are inequalities.

This linear program looks fundamentally different from the Standard Model II Form of Johnson & Scheurman (1977). It includes as activities not only the areas regeneration harvested, (the $\underset{\sim}{h}_t$), as in the Standard Form, but also the variables, $\underset{\sim}{x}_t$. However LP1 can be reduced to something very close to the linear program which arises in the Standard Model II Form by eliminating the $\underset{\sim}{x}_t$.

To eliminate the $\underset{\sim}{x}_t$ we can use the dynamic equation (6) iteratively. From (6)

$$\underset{\sim}{x}_2 = R\underset{\sim}{x}_1 - S\underset{\sim}{h}_1$$

$$\begin{aligned} \text{and} \quad \underset{\sim}{x}_3 &= R\underset{\sim}{x}_2 - S\underset{\sim}{h}_2 \\ &= R(R\underset{\sim}{x}_1 - S\underset{\sim}{h}_1) - S\underset{\sim}{h}_2 \\ &= R^2\underset{\sim}{x}_1 - (RS\underset{\sim}{h}_1 + S\underset{\sim}{h}_2), \end{aligned}$$

and by induction it can easily be established that

$$\underset{\sim}{x}_t = R^{t-1}\underset{\sim}{x}_1 - (R^{t-2}S\underset{\sim}{h}_1 + R^{t-3}S\underset{\sim}{h}_2 + \ldots + S\underset{\sim}{h}_{t-1}). \tag{24}$$

Using this, the constraints (19), $(\underset{\sim}{h}_t \le \underset{\sim}{x}_t)$ can be expressed as:-

$$R^{t-2}S\underset{\sim}{h}_1 + R^{t-3}S\underset{\sim}{h}_2 + \ldots + S\underset{\sim}{h}_{t-1} + \underset{\sim}{h}_t \le R^{t-1}\underset{\sim}{x}_1 \tag{25}$$

which does not involve the state vectors $\underset{\sim}{x}_t$, other than the initial state $\underset{\sim}{x}_1$.

Also $\underset{\sim}{x}_{N+1}$ can be eliminated from the objective (17) by using (24) to give

$$J = \alpha^{N+1} r' R\underset{\sim}{x}_1 + \sum_{t=1}^{N} \left(\alpha^t \underset{\sim}{V}' - \alpha^{N+1} \underset{\sim}{r}' R^{N-t} S\right) \underset{\sim}{h}_t . \tag{26}$$

The first term, $\alpha^{N+1} \underset{\sim}{r}' R\underset{\sim}{x}_1$, does not depend on the harvest vectors, and thus the maximization problem can be solved by solving the LP

(LP2) maximize $\underset{\sim}{c}^{*\prime}\underset{\sim}{u}^*$

subject to:-

$$\begin{aligned} A^* \underset{\sim}{u}^* &\leq \underset{\sim}{b}^* \\ \text{and} \qquad \underset{\sim}{u}^* &\geq \underset{\sim}{0} \end{aligned} \tag{27}$$

where now $\underset{\sim}{u}^{*\prime}$ is an Nk-dimensional column vectors of activities formed by vertically concatenating the harvest vectors $\underset{\sim}{h}_1, \ldots, \underset{\sim}{h}_N$. i.e.

$$\underset{\sim}{u}^* = \begin{bmatrix} \underset{\sim}{h}_1 \\ \underset{\sim}{h}_2 \\ \vdots \\ \underset{\sim}{h}_N \end{bmatrix},$$

and $\underset{\sim}{c}^{*\prime}$ is the Nk-dimensional row vector formed by horizontally concatenating the Nk-dimensional components $\left(\alpha^t \underset{\sim}{V}' - \alpha^{N+1} r' R^{N-t} S\right)$ for $t = 1, \ldots, N$. The constraint matrix A^* is of dimension $[Nk+2(N-1)] \times Nk$ and is of the form

$$\left[\begin{array}{ccccccc}
I & & & & & & \\
S & I & & & & & \\
RS & S & I & & & & \\
 & & & & & & \\
 & & & & & & \\
R^{N-2}S & R^{N-3}S & R^{N-4}S & \cdots\cdots & RS & S & I \\
\hdashline
\underset{\sim}{\ell}_1' & -\underset{\sim}{v}' & & & & & \\
\underset{\sim}{\ell}_2' & \underset{\sim}{v}' & & & & & \\
 & \underset{\sim}{\ell}_1' & -\underset{\sim}{v}' & & & & \\
 & -\underset{\sim}{\ell}_2' & \underset{\sim}{v}' & & & & \\
 & & & & & & \\
 & & & & & \underset{\sim}{\ell}_1' & -\underset{\sim}{v}' \\
 & & & & & -\underset{\sim}{\ell}_2' & \underset{\sim}{v}'
\end{array}\right]. \qquad (28)$$

The right-hand side vector $\underset{\sim}{b}^*$ is of dimension $NK + 2(N-1)$ and is of the form

$$\underset{\sim}{b}^* = \begin{bmatrix} \underset{\sim}{x}_1 \\ R\underset{\sim}{x}_1 \\ R^2\underset{\sim}{x}_1 \\ \vdots \\ R^N\underset{\sim}{x}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (29)$$

Reed and Errico (1985(b)) show that (28) contains redundant rows which can be removed, reducing problem size. Specifically one can remove the first $(k-2)$ rows from each of the first $(N-1)$ partition-rows. This reduces the row dimension of (28) to $4(N-1) + k$.

It is shown in Appendix 1, that the form, LP2, is essentially the equivalent to the linear program of the Standard Model II Form. The only difference lies in the fact that in LP2, all areas above a certain age k, are classified as belonging to one age-class. In the Standard Model II Form there is no such upper age-class. If we set k equal to the oldest possible age that trees could reach over the planning horizon, i.e. $k = M' + N$, where M' is the age of the oldest trees in the initial inventory, then this difference vanishes, and the two linear programs are identical.

The fact that the control problem, which we have formulated here in terms of a dynamic equation for a state variable, can be solved by a linear program (LP2), which is essentially the same as the Standard Model II Form, should come as no surprise; these are just two different ways of formulating the same optimization problem. The question naturally arises as to which is the better way of formulating the problem. We claim that the control theory formulation holds some significant advantages over that of the Standard Model II Form.

One major advantage of the control theory formulation is that it is intuitively much easier to understand than the Standard Model II Form. The dynamic equation (5) should be obvious to most users, whereas comprehension of the area constraints (a) and (b) of the Standard Form (Johnson & Scheurman, 1977, p. 5), may require considerable thought. The importance of the ease of understanding of the model to the user should not be underestimated.

A second advantage, which follows in part from the first, is that extensions to the basic model are much easier to incorporate in the control theory formulation. For example, it is shown in the next section how the problem of random catastrophic losses, such as destruction of areas through fire, can easily be handled by modifying the matrices R and S which occur in the dynamic equation (6). Such a modification to the Standard Model II Form is possible (Johnson and Stuart, 1985) but it requires considerably more detailed effort to correctly write down the area constraints. Indeed an easy way to determine the form of these constraints is to start with the dynamic equation and then work out the LP2 constraint matrix (28) and right-hand side vector (29).

Another extension, which is very easy to handle in the control theory formulation, is to include the possibility of a changing land base. For example, if it is anticipated that in period 2, areas $\underset{\sim}{A}' = (A_1, A_2, \ldots, A_k)$ in the various age-classes are to be removed from the land base, then the dynamic equation in the control theory formulation for period $t = 2$ would simply be modified to read

$$\underset{\sim}{x}_3 = R\underset{\sim}{x}_2 - S\underset{\sim}{h}_2 - \underset{\sim}{A}, \tag{30}$$

with all the other equations remaining unchanged. To handle such a simple change in land-base in the Standard Model II Form would introduce considerable complexity, as can be seen by determining the LP2 form for the above problem.

Area constraints which control the age-class distribution of the forest for non-timber values (such as water, wildlife, or recreation) can easily be added to the control theory formulation by imposing constraints on the $\underset{\sim}{x}_t$

activities. To handle such constraints solely through constraining harvest activities of the Standard Model II Form would be complicated. Indeed, some models based upon the standard Model II Form set up additional "accounting" activities (see, for example, Johnson, Jones and Kent, 1980). These activities are nothing more than a form of the state variable $(\underset{\sim}{x}_t)$ of the control theory formulation.

In later sections of the paper extensions dealing with the problems of random catastrophic loss, accessibility, salvage and multiple timber types and regeneration options will be examined in more detail.

Having established that the <u>formulation</u> of the problem is easier using the control theory approach, the question naturally arises as to which is the better method of <u>solution</u> to the control theory problem -- LP1 or LP2? Leaving aside question of computation efficiency, which we shall discuss in Section 4, we claim that there are compelling reasons to use LP1 rather than LP2. The main reasons are three-fold. Firstly, modifications to the constraint and right-hand side vector to handle the extensions discussed above are more easily implemented in the LP1 form than the LP2 form.

Secondly in the LP1 form the <u>dual variables</u> at the optimum, corresponding to the dynamic equation constraints (first Nk rows of (23)) have a direct interpretation. They correspond to <u>shadow values</u> for land containing trees of each age, and at each time period throughout the planning horizon. For example the shadow value (dual variable) in age-class 5 in period 2 represents the amount by which the present value of the resource would be increased (decreased) if one hectare of land containing trees in age-class 5 were to be added to (deleted from) the land base in period 2. Thus the net present value of hectares of standing timber, at the margin, can be computed in the forest-level context, at any time period. This marginal forest-level net present value may be very different from the stand-level net present value. It will depend not only on the age (volume) of timber in a given hectare, but also on the age-distribution of stands throughout the forest, and on the harvest history, through the harvest flow constraints.

A third, although perhaps less important, reason for preferring LP1 over LP2, is that the solution to LP1 provides directly, the age-class distribution of areas, $\underset{\sim}{x}_t$ at every period up to the planning horizon, under optimal management. Of course this information can be recovered from the optimal harvest sequence in the LP2 form, but it requires extra computational steps in the form of a report writer.

Before discussing the relative computational efficiency of LP1 and LP2 we shall show how the dynamic model (7) can easily be modified to include the possibility of catastrophic losses due to fire or other loss agent.

3. HARVEST SCHEDULING IN THE PRESENCE OF THE RISK OF FIRE

The problem of incorporating the risk of destruction through fire or other catastrophic loss agent has in the past been largely ignored in forest scheduling models. This has recently been remedied by Johnson and Stuart (1985) and Reed and Errico (1985(b)). In the latter paper the model used was essentially the one described in Section 2 of the present paper, but with different forms to the matrices R and S.

Suppose that during period t, random proportions $p_1^t, p_2^t, \ldots, p_k^t$ of the areas in age-classes 1, ..., k are destroyed by fire. Subsequent to reforestation (or natural regeneration) these areas will end up in age-class 1 at the start of period t + 1. This can be modelled by changing the dynamic equation (6) to a stochastic difference equation

$$\underset{\sim}{x}_{t+1} = R_t \underset{\sim}{x}_t - S_t \underset{\sim}{h}_t \qquad (31)$$

where $\{R_t\}$ and $\{S_t\}$ are sequences of independent random matrices of the form

$$R_t = \begin{bmatrix} p_1^t, & p_2^t, & \cdots\cdots & & p_k^t \\ 1-p_1^t, & & & & \\ & 1-p_2^t, & & & \\ & & & & \\ & & & p_{k-1}^t, & 1-p_k^t \end{bmatrix} \qquad (32)$$

and

$$S_t = \begin{bmatrix} (-1+p_1^t), & (-1+p_2^t), & \cdots\cdots & & (-1+p_k^t) \\ (1-p_1^t), & & & & \\ & (1-p_2^t), & & & \\ & & & & \\ & & & (1-p_{k-1}^t), & (1-p_k^t) \end{bmatrix}. \qquad (33)$$

Details of the derivation of equation (30) are given in Reed and Errico (1985(b)) and will not be repeated here. However before proceeding it may help the reader to understand the model by expanding the right-hand side of equation (31). This gives:-

$$\underset{\sim}{x}_{t+1} = \begin{bmatrix} h_1^t+h_2^t+\ldots+h_k^t+p_1^t(x_1^t-h_1^t)+\ldots+p_k^t(x_k^t-h_k^t) \\ (1-p_1^t)(x_1^t-h_1^t) \\ (1-p_2^t)(x_2^t-h_2^t) \\ \vdots \\ (1-p_{k-1}^t)(x_{k-1}^t-h_{k-1}^t)+(1-p_k^t)(x_k^t-h_k^t) \end{bmatrix}. \qquad (34)$$

The area in the first age-class at the beginning of period $t + 1$ comprises those areas harvested in period t (the h_i^t's) plus random proportions of the areas <u>not</u> harvested (the $x_i^t-h_i^t$) which are presumed to be destroyed by fire. In age-class 2 at the beginning of period $t + 1$, there is the proportion of the area previously in age-class 1 which was <u>not</u> harvested and <u>not</u> destroyed by fire. A similar situation prevails for age-classes 3 through $k - 1$. In age-class k (which comprises all stands above a certain age), at the beginning of period $t + 1$, there are those areas previously in both age-class $k-1$ and age-class k which were not harvested nor destroyed by fire.

In this model, where the dynamics exhibit randomness, the objective will be to maximize the <u>expected</u> present value of the future streat of revenues from the forest,

$$J = E\left\{\sum_{t=1}^{\infty} \alpha^t \underset{\sim}{V}' \underset{\sim}{h}_t\right\}, \tag{35}$$

subject to the dyanmic equation (31) and the various flow constraints, etc. This is a problem in stochastic control theory, and an exact solution through the use of such techniques as Dynamic Programming etc., appears to be impractical. The optimal policy will be of a feedback nature i.e. the optimal harvest $\underset{\sim}{h}_t^*$ in period t, cannot be determined in advance but will depend on the current state $\underset{\sim}{x}_t$, and on previous harvests.

Although determination of the optimal feedback policy appears effectively impossible, Reed and Errico (1985(b)) show how an approximately optimal feedback policy can be obtained by using the certainty equivalence principle (see e.g. Chow, 1975). To do this the stochastic dynamic equation (31) is replaced by the deterministic equation

$$\underset{\sim}{x}_{t+1} = \bar{R}\underset{\sim}{x}_t - \bar{S}\underset{\sim}{h}_t, \tag{36}$$

where $\bar{R}$ and $\bar{S}$ are the expected values of the random matrices, R_t and S_t, and then the deterministic control problem, of maximizing

$$J = \sum_{t=1}^{\infty} \alpha^t \underset{\sim}{V}' \underset{\sim}{h}_t, \tag{37}$$

subject to (36) (t=1,2,...) and constraints of the form (15) and (16), is used to make current harvest-level decisions for a forest unit. At the time of subsequent harvest-level decisions the problem is re-run using a new initial state vector determined through re-inventory. (Changes in inventory will reflect *actual* fires and harvest activity, and aging).

Thus, the procedure is similar to that used for existing harvest scheduling models in that it is to be reiterated periodically; only, in this case, a deterministic control problem which incorporates losses due to catastrophic fire is used. See Reed and Errico (1985(b)) for details and an example simulation of this process.

It is shown in that paper how this procedure should provide a feedback policy close to the optimal policy for the stochastic control problem, provided that the variances-covariances of the random matrices R_t, and S_t are small, which will be the case if the total area of the forest is large, since R_t and S_t involve random *proportions*.

The deterministic control problems that have to be solved for this procedure are essentially the same as the one discussed in Section 2, only with the matrices R and S replaced by

$$\bar{R} = \begin{bmatrix} \bar{p}_1, & \bar{p}_2, & \dots, & & \bar{p}_k \\ 1-\bar{p}_1 & & & & \\ & 1-\bar{p}_2 & & & \\ & & \ddots & & \\ & & & 1-\bar{p}_{k-1}, & 1-\bar{p}_k \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} -1+\bar{p}_1, & -1+\bar{p}_2, & \dots, & & -1+\bar{p}_k \\ 1-\bar{p}_1 & & & & \\ & 1-\bar{p}_2 & & & \\ & & \ddots & & \\ & & & 1-\bar{p}_{k-1}, & 1-\bar{p}_k \end{bmatrix}$$

where $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_k$ are the *expected* proportions destroyed by fire in the age-classes $1, 2, \dots, k$ in each period. It follows that both LP1 and LP2 can be used to obtain a solution. The only novelty introduced is in the computation of the terminal payoff coefficients, $\underset{\sim}{r}'$. These correspond to the net present values of single hectares of forest of various ages, under optimal management when there is a risk of fire present. They can be determined analytically (using the methods of Reed and Errico, (1985a)) or numerically using, for example, the method of Martell (1980). An alternative numerical method based on a Policy Improvement algorithm (see e.g. Ross, 1970, p. 125 ff) is given in Appendix 2.

Besides providing the means for determining an approximately optimal feedback policy, the solution to the deterministic control problem provides <u>predictions</u> or <u>estimates</u> of future timber supply, in the face of uncertainty due to random losses.

The remarks in the last section pertaining to the relative merits of the LP1 and LP2 forms for solution apply equally in the case when there is fire risk present. In particular, LP1 provides shadow values for land in the whole-forest context, when fire-risk is present. The question of the relative computational efficiency of the two procedures is discussed in the next section.

4. SOLUTION CHARACTERISTICS

Solution of a linear programming problem using most solution software packages is mainly sensitive to the number of rows (constraints) and the number of non-zero coefficients (density) in the constraint matrix. An increase in the magnitude of either results in an increase in computing time necessary for successful solutions. Other factors which affect solution efficiency include the range and distribution of the values of coefficients. In general, models with coefficients which are normally distributed about a mean of 1 are more easily solved than others[2]. In this section we present some results for the model forms discussed previously. While the examples given here represent small problems relative to those of typical applications in Canada and the U.S. we feel they give an indication of the differences between each form. More thorough testing is required, however, for larger problems involving the constraints found typically in most large-scale linear programming forest scheduling exercises. We defer this to the future.

For the comparisons made in this section seven problems were tested. For all these problems a single-type forest was modelled, being that of white spruce (*Picea glauca* (Moench) Voss and *Picea engelmanii* Parry) on sites of site index 30+ m (reference age 100) of medium accessibility to mills in the Fort Nelson Timber Supply Area of north-eastern British Columbia. This is the same as in the example given by Reed and Errico (1985(b)) and details may be found there. The following items were identical for the seven problems:

(i) Objective (maximize harvested volume)
(ii) Initial age distribution of timber (see Figure 1)
(iii) Discount rate (0%)
(iv) Period length (20 years)
(v) Length of planning horizon (35 periods)
(vi) Harvest flow constraints (sequential flow - ±10% from period to period)
(vii) Accessibility restrictions (none)

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[2] Ketron Inc. MPS III events. Volume I, Number 1. Winter 1983

(viii) Regeneration delay (none)

(ix) Harvest age restrictions (none)

The only items which were varied were model form (Standard Model II Form, LP1, or LP2) and form of fire destruction proportions (constant zero, constant non-zero, and age dependent). To generate the Standard Model II form, the U.S. Forest Service model MUSYC (Johnson and Jones, 1979) was employed.

Table 1 details the model and fire destruction proportions along with problem dimensions, matrix density, and solution time for the execution step of linear programming solution package MPS III (Ketron, 1984), for the seven problems tested.

Observing, first, the numbers of columns (activities) it is seen that the Standard Model II Form has a larger number of columns than any of the LP2 examples (this is the form which it most closely resembles). This is due to the manner in which the Standard Model II Form handles its upper harvest age limit. A separate activity is required for each age class interval up to the maximum limit; in this case to the end of the problem. This is contrasted with the LP1 and LP2 forms where there is a single activity defined for timber which exceeds a particular upper age. As a consequence, in situations where there are no cutting-age restrictions (perhaps an unrealistic case) or where there is a very high upper age limit for harvesting, the number of variables required for the Standard Model II Form is much larger than for the similar LP2 problem. The LP1 problems of Table 1 have almost twice as many activities as the corresponding LP2 problems. This is due to the addition of the state variables $(\underset{\sim}{x}_t)$ to the problem.

Observing the numbers of rows (constraints), it is seen that the Standard

Model II Form has fewer rows than the other two forms tested. In actuality, the full LP2 form of (28) has a number of redundant rows which, when removed, reduce the number of rows to the size shown in Table 1. Even after the removal of these redundant rows from the LP2 form there are still more rows than in the Standard Model II Form. This is due to the different approaches to handling oldest-age timber. (See Appendix 1). As is the case with the numbers of columns, the LP1 examples also have almost twice as many rows as the LP2 form due to the inclusion of the state variables and their associated constraints.

Noting matrix densities, by far the most dense problem is that of the LP2 form where fire destruction proportions are age dependent. The reason for this large increase in density is that the matrix $\bar{R}^{\ell}\bar{S}$ of (24) and (28) (substituting $\bar{R}$ and $\bar{S}$ of (36) for R and S respectively) does not reduce to a simple sparse form when $\bar{p}_i$ is age dependent as it does when $\bar{p}_i$ is constant. Indeed, as the number of periods N increases the density of the problem increases. The LP1 form, in contrast, maintains a very low density (lower than the other two forms tested) regardless of the nature of the fire destruction proportions. This is because $\bar{R}$ and $\bar{S}$ remain in their original form in the constraint matrix (23) (replacing R and S with $\bar{R}$ and $\bar{S}$ respectively). The reason for the high density of the Standard Model II Form, relative to its associated LP2 form can be attributed to the necessary addition of slack variables to the LP2 problem. (This is optional in the Standard Model II Form. In the Standard Model II problem of Table 1 all constraint types for the area accounting rows are of the equality type -- implying conservation of the land base). The addition of slack variables has the effect of increasing the total number of possible matrix entries very rapidly while increasing the number of non-zero coefficients only very slightly (one per column added). The density for the problem therefore is reduced.

The ultimate effect of model size and density is perhaps best evaluated through comparison of a single statistic -- solution time. Computer storage requirements and ease of matrix generation may be additional concerns. We will ignore the former (all of the problems tested here used under 1000K) and briefly discuss the latter below. Observing the solution-time statistics of Table 1 it is seen that the fastest solving problems are those of the LP2 form where fire destruction proportions are constant with age. Despite the fact that the number of rows is slightly higher than that of the Standard Model II Form problem it would appear that the reduced density yields a slight reduction in solution time. Indeed, it would appear from the solution time for the LP2 problem with age dependent $\bar{p}_i$ that an increase in density can have a large impact upon solution time. (It has been the authors' experience that such a high density coupled with the larger number of rows of more typical problems results in a problem which is so difficult to solve that it may require over ten times the solution time of a similar LP2 problem with constant $\bar{p}_i$; if it can be solved at all.) Observing the solution statistics for the LP1 problems it is seen that where the fire destruction proportion is constant, the solution time is considerably higher than the associated Standard Model II Form and LP2 problems. However, for age dependent $\bar{p}_i$ (probably the more realistic case) the LP1 form appears to exhibit a stable behavior in that it requires no significant additional solution time, and therefore solves more quickly than the corresponding LP2 problem. This stable behaviour is desirable for larger problems.

To summarize these findings, where fire destruction proportions are constant for all ages (either zero or non-zero) there is little difference between the Standard Model II Form and the LP2 forms. Both are more efficient than the LP1 form in terms of solution time. Where fire destruction proportions are not constant with age, the LP2 form suffers from a much higher solution time; probably due in large part to increased density. In this case, the LP1 form appears to be more efficient due to its sparse form.

We add two final notes regarding the LP1 problem. Firstly, it is an easier problem to generate than the LP2 problem since fewer calculations are required to derive the coefficients and the constraint matrix has a more regular form. This results in reduced programming effort as well as lower computing costs for the generator step. Secondly, we believe that for more typical problems formulated in practice, where there are various constraints on the area states, such as for non-timber values, the LP1 form will show a less rapid increase in density than the LP2 or Standard Model II Forms when similar constraints are implemented through the control of harvest activities only. This maintenance of sparsity may become important for very large problems.

5. MULTIPLE TIMBER TYPES AND REGENERATION OPTIONS

In this section we discuss how the model of Sections 2 and 3 can be extended to cover the case of multiple timber types and regeneration options, with the risk of destruction through fire present.

We consider firstly the case of two timber types. We suppose that timber type 1 can adequately be described by k age-classes with a value-at-age vector

$$\underset{\sim}{V}' = (V_1,\ldots,V_k) \tag{38}$$

and a volume-at-age vector

$$\underset{\sim}{v}' = (v_1,v_2,\ldots,v_k). \tag{39}$$

Similarly we assume that timber type 2 can be described by ℓ age-classes with a value-at-age vector

$$\underset{\sim}{W}' = (W_1,W_2,\ldots,W_k) \tag{40}$$

and a volume-at-age vector

$$\underset{\sim}{w}' = (w_1,w_2,\ldots,w_k) . \tag{41}$$

The state of the forest at the beginning of period t will be described by a $(k+\ell)$-dimensional vector

$$\begin{bmatrix} {x_1}^t \\ \vdots \\ {x_k}^t \\ \cdots \\ {y_k}^t \\ \vdots \\ {y_\ell}^t \end{bmatrix}$$

where ${x_i}^t$ $(i = 1,\dots,k)$ denotes the area of timber type 1 at age i and ${y_i}^t$ $(i = 1,\dots,\ell)$ denotes the area of timber type 2 at age i.

We shall suppose that areas, ${a_1}^t, {a_2}^t, \dots, {a_k}^t$ are harvested from timber type 1 in period t, and regenerated as timber type 1 at the start of period $t + 1$, while areas ${b_1}^t, {b_2}^t, \dots, {b_k}^t$ are harvested from timber type 1 and regenerated as timber type 2. Similarly we shall suppose that areas ${c_1}^t, {c_2}^t, \dots, {c_\ell}^t$ are harvested from timber type 2 and regenerated as timber type 2, while areas ${d_1}^t, {d_2}^t, \dots, {d_\ell}^t$ are harvested from timber type 2 and regenerated as timber type 1.

We shall suppose that random proportions ${p_1}^t, {p_2}^t, \dots, {p_k}^t$ of the areas in timber type 1 are destroyed by fire in period t, while random proportions ${q_1}^t, {q_2}^t, \dots, {q_\ell}^t$ of the areas in timber type 2 are destroyed by fire in the same period.

The dynamics of the evolution of the forest can be described by the equation (similar to (5), and (34))

$$\left[\begin{array}{c} x_1^{t+1} \\ x_2^{t+1} \\ \vdots \\ x_k^{t-1} \\ \hdashline y_1^{t+1} \\ y_2^{t+1} \\ \vdots \\ y_\ell^{t+1} \end{array}\right] = \left[\begin{array}{l} (a_1^t+\ldots a_k^t) + (d_1^t+\ldots d_\ell^t) + (e_1^t+\ldots e_k^t) + (h_1^t+\ldots h_\ell^t) \\ (1-p_1^t)(x_1^t-a_1^t-b_1^t) \\ \vdots \\ (1-p_{k-1}^t)(x_{k-1}^t-a_{k-1}^t-b_{k-1}^t) + (1-p_k^t)(x_k^t-a_k^t-b_k^t) \\ \hdashline (c_1^t+\ldots c_\ell^t) + (b_1^t+\ldots b_k^t) + (g_1^t+\ldots g_\ell^t) + (f_1^t+\ldots f_k^t) \\ (1-q_1^t)(y_1^t-c_1^t-d_1^t) \\ \vdots \\ (1-q_{\ell-1}^t)(y_{\ell-1}^t-c_{\ell-1}^t-d_{\ell-1}^t) + (1-q_\ell^t)(y_\ell^t-c_\ell^t-d_\ell^t) \end{array}\right] \tag{42}$$

where: -

e_i^t represents area of timber type 1 of age i burnt in period t and regenerated as timber type 1 in period $t + 1$ $(i = 1, \ldots, k)$,

f_i^t represents area of timber type 1 of age i burnt in period t and regenerated as timber type 2 in period $t + 1$ $(i = 1, \ldots, k)$,

g_i^t represents area of timber type 2 of age i burnt in period t and regenerated as timber type 2 in period $t + 1$ $(i = 1, \ldots, \ell)$,

h_i^t represents area of timber type 2 of age i burnt in period t and regenerated as timber type 1 in period $t + 1$ $(i = 1, \ldots, \ell)$.

Clearly if all areas burnt are regenerated we have the extra constraints

$$e_i^{\,t} + f_i^{\,t} = p_i^{\,t}\left(x_i^{\,t} - a_i^{\,t} - b_i^{\,t}\right) \quad (i = 1,\ldots,k) \tag{43}$$

$$g_i^{\,t} + h_i^{\,t} = q_i^{\,t}\left(y_i^{\,t} - c_i^{\,t} - d_i^{\,t}\right) \quad (i = 1,\ldots,\ell) \tag{44}$$

The problem of maximizing expected discounted revenue from the resource can be expressed as: -

Maximize:

$$J = E\left\{\sum_{t=1}^{N} \alpha^t G^t + \alpha^{N+1}\left(\underset{\sim}{r}'\,\underset{\sim}{x}_{N+1} + \underset{\sim}{s}'\,y_{N+1}\right)\right\} \tag{45}$$

where

$$G_t = \underset{\sim}{V}'(\underset{\sim}{a}_t + \underset{\sim}{b}_t) + W'(\underset{\sim}{c}_t + \underset{\sim}{d}_t) \tag{46}$$

is the revenue earned in period t, and $\underset{\sim}{r}'$ and $\underset{\sim}{s}'$ are vectors of (stand-level) net present values of single hectares of timber at various ages in timber types 1 and 2. The maximization is subject to the constraints given by the dynamic equation (42) for $t = 1, \ldots, N$, the constraints (43) and (44) and harvest flow constraints

$$(1-\gamma_1)H_{t-1} \le H_t \le (1+\gamma_2)H_t \tag{47}$$

for $t = 2, \ldots, N$, where

$$H_t = \underset{\sim}{v}'(\underset{\sim}{a}_t+\underset{\sim}{b}_t) + w'(\underset{\sim}{c}_t+\underset{\sim}{d}_t) \tag{48}$$

represents the total volume harvested in period t. It should be noted that the control variables are

$$\underset{\sim}{a}_t = \left(a_1{}^t, a_2{}^t, \ldots, a_k{}^t\right)'$$

$$\underset{\sim}{b}_t = \left(b_1{}^t, b_2{}^t, \ldots, b_k{}^t\right)'$$

$$\underset{\sim}{c}_t = \left(c_1{}^t, c_2{}^t, \ldots, c_\ell{}^t\right)'$$

$$\underset{\sim}{d}_t = \left(d_1{}^t, d_2{}^t, \ldots, d_\ell{}^t\right)'$$

As in the single timber type case this is a problem in stochastic control since the $p_i{}^t$ and $q_i{}^t$ are random variables. However if we replace them by their expected values $\bar{p}_i$ $(i = 1,\ldots,k)$ and $\bar{q}_i$ $(i = 1,\ldots,\ell)$, the problem becomes one in deterministic control. As discussed in Section 3 we shall consider the deterministic problem where its optimal solution is used in a feedback manner to obtain an approximately optimal policy for the stochastic problem. The deterministic problem can be solved by linear programming using the LP1 form since the objective (45) and the constraints (42), (43), (44) and (47) are all linear in the variables $\underset{\sim}{x}_t$, $\underset{\sim}{y}_t$, $\underset{\sim}{a}_t$, $\underset{\sim}{b}_t$, $\underset{\sim}{c}_t$ and $\underset{\sim}{d}_t$. However the problem can be simplified considerably by adding together the equations with $x_1{}^{t+1}$ and $y_1{}^{t+1}$ on the l.h.s. of (42), and defining new variables

$$z_i^{\,t} = a_i^{\,t} + b_i^{\,t} \qquad i = 1, \ldots, k \tag{49}$$

$$u_i^{\,t} = c_i^{\,t} + d_i^{\,t} \qquad i = 1, \ldots, \ell \tag{50}$$

corresponding to total areas harvested in each age class in each timber type in each period.

The objective then becomes

$$J = \sum_{t=1}^{\infty} \alpha^t (\underset{\sim}{V}' \underset{\sim}{z}_t + \underset{\sim}{W}' \underset{\sim}{u}_t) + \alpha^{N+1} (\underset{\sim}{r}' \underset{\sim}{x}_{N+1} + \underset{\sim}{s}' \underset{\sim}{y}_{N+1}) \tag{51}$$

and the constraints corresponding to the dynamic equations (42), and to (43) and (44) become

$$\begin{aligned}
x_1^{\,t+1} + y_1^{\,t+1} &= \left(z_1^{\,t} + \ldots + z_k^{\,t}\right) + \left(u_1^{\,t} + \ldots + u_\ell^{\,t}\right) + \left(\bar{p}_1\left(x_1^{\,t} - z_1^{\,t}\right) + \ldots + \bar{p}_k\left(x_k^{\,t} - z_k^{\,t}\right)\right) \\
&\qquad + \left(\bar{q}_1\left(y_1^{\,t} - u_1^{\,t}\right) + \ldots + \bar{q}_\ell\left(y_\ell^{\,t} - u_\ell^{\,t}\right)\right) \\
x_2^{\,t+1} &= (1-\bar{p}_1)\left(x_1^{\,t} - z_1^{\,t}\right) \\
&\vdots \\
x_k^{\,t+1} &= (1-\bar{p}_{k-1})\left(x_{k-1}^{\,t} - z_{k-1}^{\,t}\right) + (1-\bar{p}_k)\left(x_k^{\,t} - z_k^{\,t}\right) \\
y_2^{\,t+1} &= (1-\bar{q}_1)\left(y_1^{\,t} - u_1^{\,t}\right) \\
&\vdots \\
y_\ell^{\,t+1} &= (1-\bar{q}_{\ell-1})\left(y_{\ell-1}^{\,t} - u_{\ell-1}^{\,t}\right) + (1-\bar{q}_\ell)\left(y_\ell^{\,t} - u_\ell^{\,t}\right).
\end{aligned} \tag{52}$$

This problem is linear in the variables $\underset{\sim}{u}_t$, $\underset{\sim}{z}_t$, $\underset{\sim}{x}_t$ and $\underset{\sim}{y}_t$ $(t = 1,...)$ both in the objective (51) and in the constraints (47) & (52). To solve it we can use the Simplex Algorithm of linear programming.

The solution will give explicitly only the total harvests from each age-class in each timber type (the z_i^t, and u_i^t) and the state of the forest (the x_i^t and y_i^t) at the beginning of each period, for the optimal policy. However information as to how the areas cut and burnt are regenerated under optimal management can be recovered from equations (42), (43) and (44).

An example of the method was run using the two volume-age relationships given in Table 2. For the sake of the example, value was assumed to be equal to volume. Timber type 1 corresponds to interior B.C. spruce as used in Section 4. Timber type 2 is a hypothetical type with a volume age curve derived from that of timber type 1, simply by reducing the volume at any given age by fifteen percent. Thus timber type 2 might represent a slower growing species. To offset this we have assumed that type 2 is less susceptible to destruction through fire than type 1. Specifically we have assumed that for all age-classes of type 1 the per annum fire probability os 0.01, while for all age-classes of type 2 it is 0.0065. The fire-adjusted volume rotation curves (VRCs) (see Reed and Errico 1985(a)) which can be used to determine maximum long-run average yields are shown in Figure 2. It can be seen that for lower rotation ages (below 80 years) the long-run average yield (LRAY) of type 1 is greater than that of type 2, but for higher rotation ages the situation is reversed. The differences, however, are only very slight.

In the example it was assumed that the initial inventory contained only stands of timber type 1. The inventory used corresponds to the current inventory of pure spruce, as described above, in the Fort Nelson Timber Supply Area and is the same as that used in Reed and Errico (1985(b)). The initial inventory is displayed in the top part (period 1) of Figure 3. The remainder of Figure 3 shows how the forest inventory evolves under optimal management using a 5 percent per annum discount rate and sequential flow constraints of ±10 percent of volume per period, and assuming fixed rather than random rates of fire as described above. It can be seen that all volumes cut are regenerated as type 2. Thus under optimal management the forest is eventually converted to a pure type 2 forest. The gain in yield through switching to type 2, although positive is very small, as the fire-adjusted volume-rotation curves of Fig. 3 would suggest. Roughly speaking, in this example, the effects of a reduction in the fire probability from 0.01 per annum to 0.0065 per annum, is equivalent to an increase in the volume growth curve slightly in excess of 15 percent. This would suggest that in fire-prone regions or for fire-prone species, silvicultural activities could be profitably directed towards reducing the risk of fire as much as toward increasing growth in volume.

The problem of multiple (as opposed to dual) timber types or regeneration options can be handled in essentially the same way. For a problem with m timber types, each described by k age-classes, and with a planning horizon of N periods using terminal payoffs, the resulting linear programming problem, if sequential flow constraints of the form (47) are included, would have $\{(k-1)m+2\} \times N$ rows (constraints) and km columns (activities). If there are restrictions on regeneration (e.g. that only a certain timber type can be regenerated after a fire) they can be incorporated as extra constraints in the problem, or the appropriate activities can be removed from the formulation.

6. SALVAGE

In the model described in Section 3, it is assumed that after a fire a stand is completely destroyed. In practice, very often, some usable timber can be salvaged after a fire. This can fairly easily be incorporated into the model.

Suppose that after a fire in a stand in age-class i in period t a proportion θ_i^t of the usable volume can be salvaged. Given proportions burnt, $p_1^t, \ldots, p_k^t$ in period t, and a harvest $\underset{\sim}{h}_t$, the value of total usable volume cut (harvest plus salvage) in that period would be

$$\underset{\sim}{V}'\underset{\sim}{h}_t + \sum_{i=1}^{k} V_i\, \theta_i^t\, p_i^t \left(x_i^t - h_i^t\right). \tag{53}$$

This quantity would be a random variable, since the fire and the salvage proportions would be random variables. However following the procedure discussed in Section 2, we can replace these random variables by their expected values and consider the corresponding deterministic control problem, and then apply its solution in a feedback way.

Under these assumptions the total discounted value of timber cut over an infinite time horizon would be

$$J = \sum_{t=1}^{\infty} \alpha^t \left\{ \underset{\sim}{V}'\underset{\sim}{h}_t + \sum_{i=1}^{k} V_i\, \bar{\theta}_i\, \bar{p}_i \left(x_i^t - h_i^t\right) \right\} \tag{54}$$

where $\bar{\theta}_i = E\left(\theta_i^t\right)$ and $\bar{p}_i = E\left(p_i^t\right)$ for $i = 1, \ldots, k$. We would want to maximize (54) subject to constraints (36) and (16) and the harvest flow constraints.

We can consider the salvage volume to be either:-

(a) extra to the regular harvest and not included in the flow constraints, or,

(b) included in the flow constraints.

In case (a) the harvest flow constraints would simply be the constraints (15). In case (b) they would be of the form

$$(1-\gamma_1)H_{t-1} \leq H_t \leq (1+\gamma_2)H_{t-1} \tag{55}$$

where H_t would be given by (53).

Using terminal payoffs in the manner discussed earlier (the net present value of a stand would now include salvage -- see Reed and Errico (1985(b)), we can reduce the problem to one with a finite time horizon and solve it by linear programming, using the LP1 form.

As an example we have considered a "forest" comprising pure spruce stands of the Fort Nelson area of B.C., with the initial inventory as discussed in Section 5. Again value was assumed equal to volume. The per annum fire rate considered was one percent (age independent) and the discount rate used was 5 percent per annum. Two distinct salvage scenarios were considered:-

(i) expected salvage of 25 percent of volume burnt for ages 70 and older, and zero percent of volume burnt for ages less than 70,

(ii) expected salvage of 75 percent of volume burnt for ages 70 and older and zero percent of volume burnt for ages less than 70.

The predicted harvest trajectories both with salvage included and not included in the flow constraints, along with that for no salvage are shown in Figs. 4 and 5. Also shown is the trajectory for optimal harvests with no fire. Figure 4 shows these predictions for the case (i) of 25% salvage, while Fig. 5 shows the case (ii) of 75% salvage. It can be seen that the predicted trajectory when the salvage is included in the flow constraints lies between that predicted when there is fire but no salvage and that predicted when there is no fire. From Fig. 5 it can be seen that being able to salvage on average 75% of the timber burnt of age 70 and older, recovers about half of the loss in optimal harvest yield in each period due to fire. When salvage volume is considered to be outside of the flow constraints, the resulting optimal cutting policy is very close to that with no salvage present. The extra volume harvested each period can be accounted for, more or less completely, through salvage. The overall effect is that there are considerably greater total harvests in early periods before the salvageable old-growth timber present in the initial inventory, has been liquidated.

One can think of allowing the salvage to be outside the flow constraints as effectively relaxing those constraints. In consequence time-discounting can have a stronger effect, resulting in greater harvests early on, at the expense of reduced harvests later.

It should be noted that an actual harvest trajectory, obtained by applying the first period harvest of the optimal deterministic policy in a feedback manner, will likely deviate more from the predicted harvest trajectory, in the case when salvage is present, than in the case when it is not. The reason for this is that the randomness due to fires enters directly into the total harvest in a given period (see (53)), rather than only indirectly through the state variable $\underset{\sim}{x}_t$. This additional variance in actual sample paths could, quite possibly, be considerable. When anticipated salvage is included in the flow constraints, it is quite possible that, for an actual sample path, the period to period fluctuation may be greater than the specified flow limits, because of the variation in actual amounts burnt and salvaged.

The question of whether salvage should or should not be included in the harvest flow constraints raises the whole question of the reasons for, and the influence of, such constraints. In forest-level harvest scheduling models, both with and without the presence of the risk of fire, the influence of harvest flow constraints on the optimal solution, is considerable, and to a large extent seems to override the influence of the discount rate. The opportunity cost, in terms of foregone revenue, of imposing flow constraints is fairly easy to assess. The benefits are harder to quantify in simple economic terms. Ultimately the chosen trade-off between evenness of supply and long-run yeild will reflect social, political and other preferences, as much as economic criteria. The question of how salvage should be treated, should be addressed in this light.

7. ACCESSIBILITY WHEN RISK OF FIRE PRESENT

In this section we consider the case of accessing roadless areas where the rate of access or road construction is determined prior to the harvest scheduling exercise. Most forest-level optimization models treat the accessibility problem through the simple application of a series of constraints which denote fixed amounts of hectares available for harvest in a given period. In these models, the only manner in which age-class one hectares can be created is after a harvest which logically can only take place in accessible areas. Thus, inaccessible timber only undergoes an aging process. When the risk of fire is present the problem is not so simple. Now there is the situation where age-class one hectares may occur through harvest or fire where fire can occur on inaccessible areas as well; therefore the inaccessible timber undergoes a dynamic process which includes aging and regeneration. Thus fewer hectares than anticipated may actually be harvestable after roading since some have been burnt.

One way of viewing the problem of accessibility, which makes it relatively easy to model, is as a problem with a changing land base -- extra hectares are added to the land base as new areas of the forest become accessible. As we have seen in Section 2 the problem of a changing land base is very easily handled by appropriate modifications to the dynamic equation for the system.

If we let $\underset{\sim}{x}_t$ denote the areas (by age) of the forest accessible in period t, then the dynamics of the system are described by the equation

$$\underset{\sim}{x}_{t+1} = R_t(\underset{\sim}{x}_t+\underset{\sim}{M}_t) - S_t\underset{\sim}{h}_t \tag{56}$$

where $M'_t = (M_1^{\,t}, M_2^{\,t}, \ldots, M_k^{\,t})$ is a vector of areas (by age) assumed to be newly roaded during period t, and R_t and S_t are random matrices as given in (32) and (33). To estimate the age distribution of those areas to be roaded <u>in the future</u>, the effects of aging and of fire must be considered.

Suppose that at the beginning of the problem (in period 1) the total extent of the forest can be partitioned into regions corresponding to those parts currently accessible, those parts to be roaded during period 1, those parts to be roaded during period 2, ..., etc. Let $\underset{\sim}{a}_t$ describe the areas (by current age) which are to be roaded in period t, $t = 1, 2, 3, \ldots$. Long range plans for road development may be available for this. It follows that the actual future areas by age (in period t) which will be newly roaded in period t, will be

$$M_t = R_{t-1} \, R_{t-2} \cdots R_1 \, \underset{\sim}{a}_t \tag{57}$$

where $R_1, R_2, \ldots, R_{t-1}$ are random matrices of the form (32). Using the expected values of these matrices, one obtains predictions of the areas, by period t age, to be roaded in period t:

$$\hat{M}_t = \bar{R}^{t-1} \, \underset{\sim}{a}_t \tag{58}$$

Substituting this in (56), and again using expected values of R_t and S_t, we get a deterministic dynamic equation of the form

$$\underset{\sim}{x}_{t+1} = \bar{R}\underset{\sim}{x}_t - \bar{S}\underset{\sim}{h}_t + \bar{R}^t \, \underset{\sim}{a}_t, \qquad t = 1, 2, \ldots \tag{59}$$

This equation would be used in place of (36) in the linear program.

8. DISCUSSION

In this paper we have presented an alternative control theory formulation to the harvest scheduling problem addressed by the Model II linear programming procedure. We have discussed two methods of solution of this problem, LP1 and LP2, both using the numerical methods of linear programming. We claim that the control theory formulation, coupled with the LP1 method of solution presents an attractive alternative to the Standard Model II method.

One of the main advantages which the control theory formulation possesses is that it is easy to comprehend. Once the dynamics of the process are understood, many extensions are easily handled. In the paper the following extensions are discussed: - (i) the problem of a changing land base, (ii) the inclusion of constraints on areas of standing forest for wildlife and recreational purposes, etc., (iii) the possibility of random catastrophic losses through fire or other loss agent, including the possibility of partial salvage of volumes burnt, (iv) the inclusion of multiple timber types and regeneration options, and (v) the problem of accessibility. The last two of these extensions are discussed in the context of there being a risk of catastrophic destruction present. While all of the above extensions can be handled in one way or another in the Standard Model II formulation, the determination of the appropriate area constraints, in that form, can be extremely complicated. Indeed perhaps the easiest way to derive them is to start with the dynamic equations, and corresponding constraints, and then eliminate the state variables recursively, in the manner discussed in Section 2.

We have presented two methods, LP1 and LP2, of solving the control theory problem. While, in simple problems, when there is age independent risk of catastrophic loss present (including the case of no risk) LP2 performs better than LP1, in terms of computation time (and indeed better than the Standard Model II linear program, when there is no risk present), this is not the case for more realistic problems when fire destruction probabilities are age-dependent. In such situations, LP1 is likely to outperform LP2. This is because of the fact that the constraint matrix for LP1 remains sparse, whereas for LP2 it becomes very dense. Typically problems with many constraints and a high density are slow to solve.

There are other advantages to the LP1 method. Firstly programming a matrix generator is easier, and secondly generation of the linear program is quicker than with the LP2 form. Thirdly, the values of the dual variables at the optimum, corresponding to the dynamic equation equality constraints, provide net present values for forested land at the margin in a forest-wide context. Also the dual variables corresponding to constraints on standing timber, provide the marginal opportunity costs (in terms of foregone harvests) of reserving land for non-timber uses. Finally, the output of LP1 provides directly the age-class distribution of the forest at every period, under optimal management. To obtain this using LP2 or the Standard Form, extra computational steps in the form of a report writer are required.

While recent developments in FORPLAN (Johnson and Stuart, 1985) will allow for the inclusion of catastrophic mortality, using the Standard Model II procedure, the relative computational performance of that method, and of the LP1 method have yet to be investigated. Aside from computational aspects we feel that the control theory formulation presented here, with the LP1 method of solution, merits attention as an alternative approach to the forest-level harvest scheduling problem. At the very least it provides insights into the Model II problem especially with respect to extensions and how they may be incorporated into the standard linear programming procedure.

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Appendix I

Equivalence of LP2 with the Standard Model II Form

Model II, Form II of Johnson and Schuerman (1977) is

$$\text{maximize} \quad \sum_{j=1}^{N} \sum_{i=-M}^{j-1} D_{ij}\, y_{ij} + \sum_{i=-M}^{N} E_{iN}\, z_{iN}$$

subject to:-

1. area constraints

$$\text{(a)} \quad \sum_{j=1}^{N} y_{ij} + z_{iN} = A_i \qquad i = -M, \ldots, 0$$

$$\text{(b)} \quad \sum_{k=j+2}^{N} y_{jk} + z_{jN} = \sum_{i=-M}^{j-1} y_{ij} \qquad j = 1, \ldots, N$$

2. harvest flow constraints

$$(1-\gamma_1)H_j \le H_{j+1} \le (1+\gamma_2)H_j \qquad j = 1, \ldots, N-1$$

where

D_{ij} = discounted net revenue per hectare of areas regenerated in period i and harvested in period j (equivalent to $\alpha^j V_{j-i}$ in our notation)

E_{iN} = discounted value per hectare of areas regenerated in period i and left as ending inventory at the end of period N (equivalent to $\alpha^{N+1} r_{N+1-i}$ in our notation)

y_{ij} = area regenerated in period i and harvested in period j (equivalent to h_{j-i}^{j} in our notation)

A_i = area present in period one that was regenerated in period i $(i = -M,\ldots,0)$, with $M+1$ being the age of the oldest timber in the initial (period 1) inventory, (A_{-j} is equivalent to x_1^{j+1} in our notation)

z_{jN} = area regenerated in period j and left as ending inventory after the harvest in period N (equivalent to x_{N-j+1}^{N+1} in our notation)

In general the Standard Model II Form allows for the specification of a minimum cutting age. For the sake of simplicity we have set this equal to one, in establishing the equivalence with our model. To establish a minimum cutting age in our model we would simply have to constrain some of the components of the harvest vectors $\underset{\sim}{h}_t$ to be equal to zero or delete them from the formulation.

The Standard Model II Form, unlike the model of this paper, does not classify all areas with stands above a certain age, into a single age-class. To accommodate for this difference in models we shall have to modify our model slightly by allowing for stands of age up to $M + N + 1$, and modifying the matrices R and S to be the $M + N + 1$ dimensional square matrices

$$R = \begin{bmatrix} 0, & 0, & \cdots & 0 \\ 1, & & & \\ & 1, & & \\ & & \ddots & \\ & & 1, & 0 \end{bmatrix}, \quad S = \begin{bmatrix} -1, & -1, & \ldots & -1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

The initial and terminal vectors of our model expressed in terms of Standard Model II Form parameters are the $M + N + 1$ dimensional vectors

$$\underset{\sim}{x}_1 = \begin{bmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \underset{\sim}{x}_{N+1} = \begin{bmatrix} z_{NN} \\ z_{N-1,N} \\ \vdots \\ z_{-M,N} \end{bmatrix} \tag{A1.1}$$

We now derive the Standard Model II Form from LP2. When there is no upper age class, all of the constraints (25) in LP2 can be derived from (24) with $t = N + 1$, i.e. from

$$\underset{\sim}{x}_{N+1} = R^N \underset{\sim}{x}_1 - \left(R^{N-1} S\underset{\sim}{h}_1 + R^{N-2} S\underset{\sim}{h}_2 + \ldots + RS \underset{\sim}{h}_{N-1} + S\underset{\sim}{h}_N \right) \tag{A1.2}$$

This equation represent $M + N + 1$ equality constraints. The first N of these are equivalent to the area constraints (b), while the last $M + 1$ are equivalent to the area constraints (a) of the Standard Model II Form. This can be established by multiplying out the right hand side of (A1.2), and using the equivalences in (A1.1) and the equivalence

$$h_i^{\,j} = x_{j-i,j} \quad i = 1, \ldots, M, \quad j = 1, \ldots, M + N + 1 \qquad \text{(A1.3)}$$

To do this for general M and N is quite tedious. For the sake of illustration we do it here for the case N = 2, M = 3. (A1.2) gives in this case

$$\underset{\sim}{x}_3 = R^2\, \underset{\sim}{x}_1 - RS\, \underset{\sim}{h}_1 - S\underset{\sim}{h}_2$$

i.e. using (A1.1),

$$\begin{bmatrix} z_{22} \\ z_{12} \\ z_{02} \\ z_{-12} \\ z_{-22} \\ z_{-32} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ A_{-1} \\ A_{-2} \\ A_{-3} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1^{\,1} \\ h_2^{\,1} \\ h_3^{\,1} \\ h_4^{\,1} \\ h_5^{\,1} \\ h_6^{\,1} \end{bmatrix}$$

$$- \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} h_1^{\,2} \\ h_2^{\,2} \\ h_3^{\,2} \\ h_4^{\,2} \\ h_5^{\,2} \\ h_6^{\,2} \end{bmatrix}$$

Using (A1.3) this gives six equations

$$z_{22} = y_{12} + y_{02} + y_{-12} + y_{-22} + y_{-32} + y_{-42}$$

$$z_{12} = y_{01} + y_{-11} + y_{-21} + y_{-31} + y_{-41} + y_{-51} - y_{12}$$

$$z_{02} = A_0 - y_{01} - y_{02}$$

$$z_{-12} = A_{-1} - y_{-11} - y_{-12}$$

$$z_{-22} = A_{-2} - y_{-21} - y_{-22}$$

$$z_{-32} = A_{-3} - y_{-31} - y_{-32}$$

The last four equations can be written

$$\sum_{j=1}^{2} y_{ij} + z_{iN} = A_i \qquad i = -3, -2, -1, 0$$

which are the area constraints (a) of Model II, while the first two equations are

$$\sum_{k=2}^{2} y_{1k} + z_{12} = \sum_{i=-5}^{0} y_{i1}$$

$$z_{22} = \sum_{i=-4}^{1} y_{i2}$$

which are seen to be the two area constraints (b) once we recognize that

$$y_{-51} = y_{-41} = y_{-42} = 0,$$

since there were no hectares regenerated in periods -5 and -4 available for harvest.

The sequential flow constraints, and the non-negativity constraints are the same in both models. The objective of LP2 is

$$J = \sum_{t=1}^{N} \alpha^{t} \underset{\sim}{v}' \underset{\sim}{h}_{t} + \alpha^{N+1} r' \underset{\sim}{x}_{N+1}$$

The first term can be re-expressed as

$$\sum_{t=1}^{N} \sum_{s=1}^{M+N+1} \alpha^{t} V_{s} h_{s}^{t} = \sum_{t=1}^{N} \sum_{s=1}^{M+N+1} \alpha^{t} V_{s} y_{t-s,t}$$

By a change of summation variables

$$i = t - s, \quad j = t$$

the sum can be written as

$$\sum_{j=1}^{N} \sum_{i=j-(M+N+1)} \alpha^{j} V_{j-i} y_{ij} = \sum_{j=1}^{N} \sum_{i=-M}^{j-1} D_{ij} y_{ij},$$

since $i \geq -M$ ($-M$ is the time of generation of the oldest trees in the initial inventory). This is the first term of the objective function of the Standard Model II Form. Similarly the second term of our model, corresponding to terminal payoffs for areas left standing at the end of the planning horizon, is equivalent to the second term in the objective of the Standard Model II Form.

Appendix 2

Computation of Terminal Payoff Coefficients

As discussed in Sections 2 and 3, the terminal payoff coefficients correspond to the net present values of single hectares of forest of various ages. These net present values are calculated in a stand-level context, rather than in a forest-level one. For the sake of generality we shall consider the coefficients for the case when there is a risk of fire present. Their values for the case when there is no fire risk (Section 2) can be obtained by setting the fire probability parameters equal to zero.

By using a similar approach to that in Reed (1984) and Reed and Errico (1985a), only in discrete-time rather than continuous time, it can be shown that when there is an age-independent per period probability of fire, p, the net present value of a stand of age i periods is

$$\begin{cases} \dfrac{V_i(\alpha q)^{i_0-1}}{q^2(1-\alpha)\left(1-(\alpha q)^{i_0-1}\right)}\{q(1-\alpha)+p(\alpha q)^i\} & i < i_0 \\ \\ V_i + \alpha r_i & i \geq i_0 \end{cases} \tag{A2.1}$$

where $q = 1 - p$, and i_0 is the optimal single stand cutting age (Reed & Errico (1985(a)).

In the case of age-dependent fire probabilities the above formulas need some modification.

An alternative procedure is to calculate the vector of per hectare net present values $\underset{\sim}{r}' = (r_1, r_2, \ldots, r_k)$, numerically using an iterative procedure. One possible such procedure is a Policy Improvement Algorithm (see e.g. Ross, 1970, p. 12 ff). In this procedure one starts with an initial guess $\underset{\sim}{r}'_0$ for $\underset{\sim}{r}$. One then calculates an "improved" value $\underset{\sim}{r}'_1$ as

$$\underset{\sim}{r}'_1 = \max\{\alpha \underset{\sim}{r}'_0 \bar{R},\ \alpha \underset{\sim}{r}'_0 E + \underset{\sim}{V}'\} \qquad (A2.2)$$

where the maximization is element-wise and where $\bar{R}$ is as given in Section 2, $\underset{\sim}{V}'$ is the value-at-age vector, and

$$E = \bar{R} - \bar{S} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & . & . & \cdots & 0 \\ \vdots & & & & \\ 0 & . & . & \cdots & 0 \end{bmatrix} \qquad (A2.3)$$

The procedure is then repeated iteratively until convergence, i.e. until the difference between successive values $\underset{\sim}{r}'_{n+1}$ and $\underset{\sim}{r}_n$ is less than some tolerance value.

The procedure can be interpreted in the following way: suppose one is managing a single stand and there is only one period remaining to the end of the planning horizon, at which time a terminal payoff equal to the values in $\underset{\sim}{r}'_0$ corresponding to the age of the stand, will be realized. The two values in braces on the r.h.s. of (A2.2) represent the expected present value of stands of ages 1, 2, ..., k given that: - (a) no harvest is undertaken, and (b) a harvest is undertaken. The maximum of these values gives the expected present value of the stand under optimal management. In repeated iterations one moves further and further forward in time, until the effect of the initial terminal payoff values $\underset{\sim}{r}'_0$ becomes neglible, and the corresponding limiting value of $\underset{\sim}{r}'$ represents the expected present value of the stand under optimal management over an infinite time horizon.

Interestingly enough, one can derive this algorithm, by considering the dual of the LP1 problem with no harvest flow constraints. The values $\underset{\sim}{r}'$ are the shadow prices of timber in the various age classes. It can be shown that the dual problem can be solved by an algorithm like that described above where the initial values $\underset{\sim}{r}'_0$ correspond to the terminal payoff coefficients used in the LP problem.

Table 1: Size and solution statistics for the Standard Model II, LP1 and LP2 linear programming formulations.

Model Form	Average per annum probability of distribution through fire $\bar{p}_i$	Number of Columns	Number of Rows	Density %	Solution Time (sec)
Standard Model II	0 all ages	674	116	4.87	3.97
LP1	0 all ages	1311	655	.41	6.61
LP1	.01 all ages	1311	655	.44	5.80
LP1	.015 age < 70 .005 age ≥ 70	1311	655	.44	6.65
LP2	0 all ages	509	149	2.99	3.34
LP2	.01 all ages	509	149	2.99	3.50
LP2	.015 age < 70 .005 age ≥ 70	509	149	2.99	7.77

Notes:

1. Column dimensions include slack and surplus variables.
2. Density is defined as the percentage of nonzero constraint coefficients to the total number of possible entries for the problem. This includes slack and surplus variables.
3. Solution times are for the MPS III (Ketron, 1984) execution step only, operating on an IBM 3081 Model K processor.

Figure Captions

Figure 1 Initial age distribution of spruce stands on sites of site index 30+ m. (reference age 100) of medium accessibility to mills in the Fort Nelson Timber Supply Area of British Columbia. Units of area are 10000 ha and units of age are years.

Figure 2 Fire-adjusted volume rotation curves (Reed and Errico 1985a) for the two timber types discussed in Section 5. Units of volume are m^3/ha.

Figure 3 Projected age-distributions of timber types I and II under optimal management, with a discount rate of 5 percent per annum. Note how areas of timber type I are harvested and regenerated with timber type II.

Figure 4 Projected optimal harvest trajectories when there is an average rate of fires of one percent per annum, and a discount rate of 5 percent per annum. The trajectories correspond to:- (a) no salvage, (b) 25% salvage for stands of age 70 or over which are burnt, with salvage considered as part of the harvest flow, and (c) 25% salvage for stands of age 70 or over which are burnt, with salvage considered extra to the harvest flow. Also shown (d) is the optimal trajectory when there is no fire risk. Units of harvest volume are 100,000 m^3, and periods are of length 20 years.

<u>Figure 5</u> Projected optimal harvest trajectories when there is an average rate of fires of one percent per annum, and a discount rate of 5 percent per annum. The trajectories correspond to:- (a) no salvage, (b) 75% salvage for stands of age 70 or over which are burnt, with salvage considered as part of the harvest flow, (c) 75% salvage for stands of age 70 or over which are burnt with salvage considered extra to the harvest flow. Also shown (d) is the optimal trajectory when there is no fire risk. Units of harvest volume are 100,000 m^3 and periods are of length 20 years.

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Fig 1

STARTING INENTORY

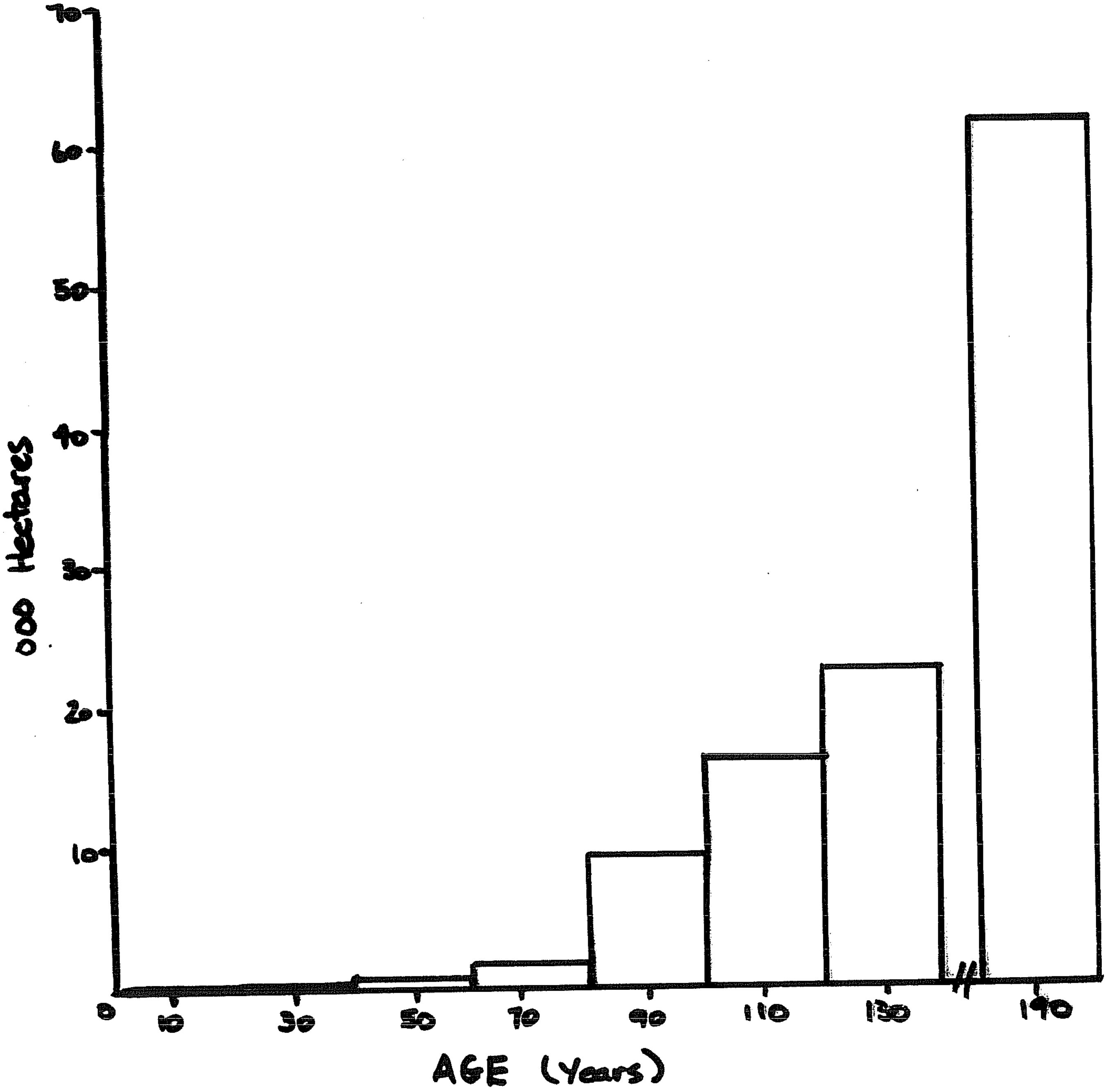


Fig2.

Fig2

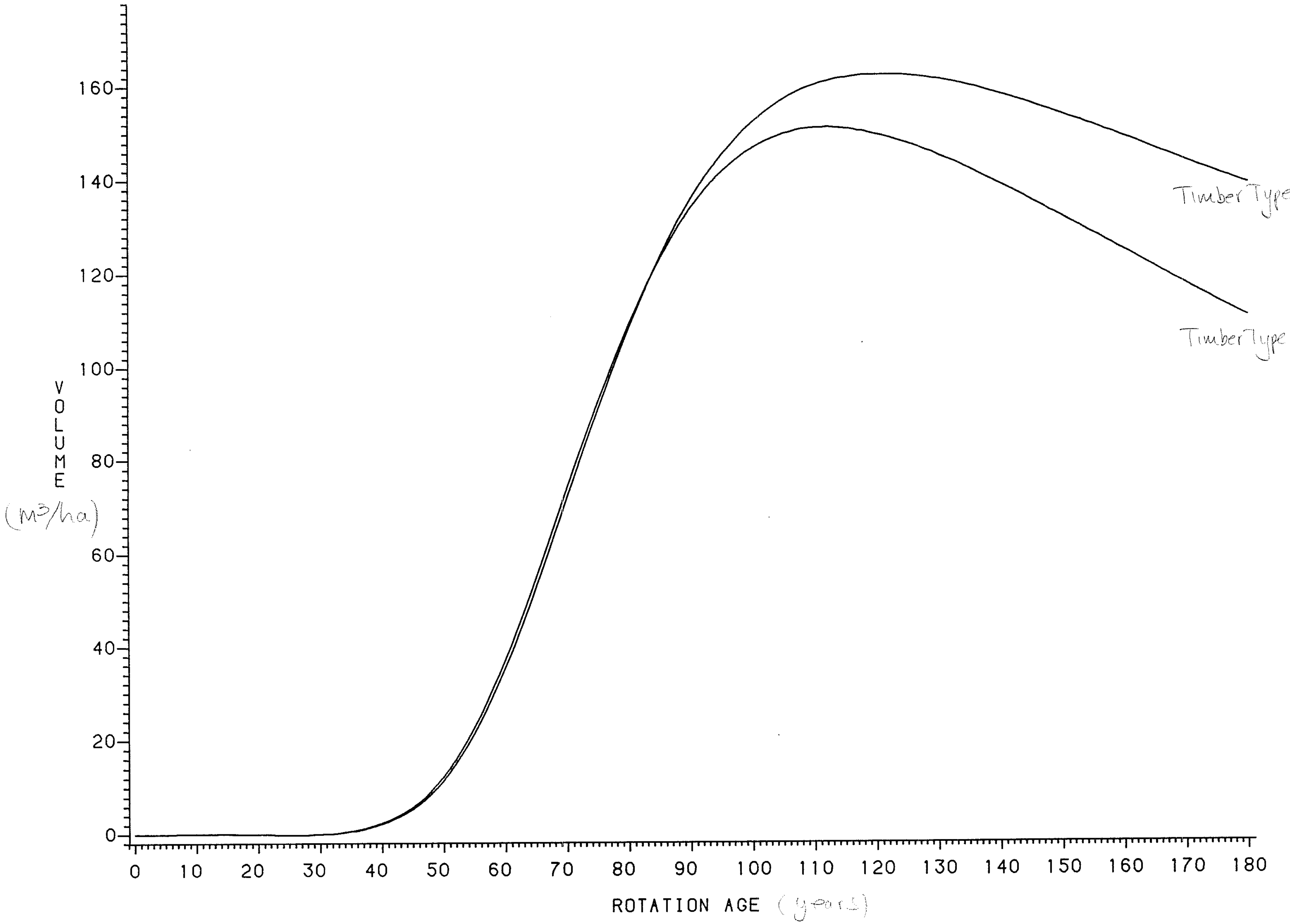
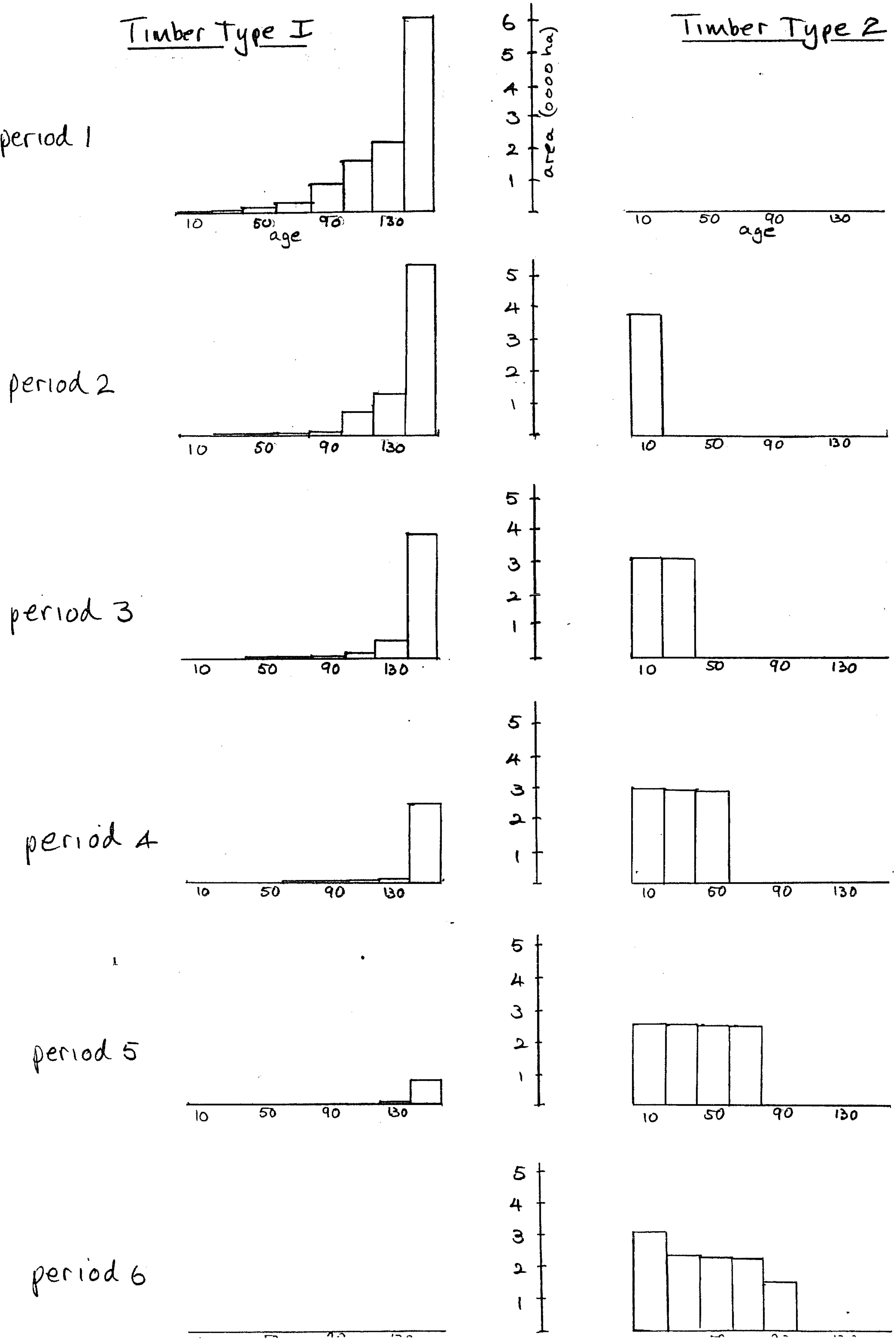

160
140
120
100
80
60
40
20
0
VOLUME
(m³/ha)
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
ROTATION AGE (years)
Timber Type
Timber Type

Fig. 3


Timber Type I
Timber Type 2
period 1
period 2
period 3
period 4
period 5
period 6
area (000 ha)
6
5
4
3
2
1
age
10
50
90
130

25% salvage

5% discou

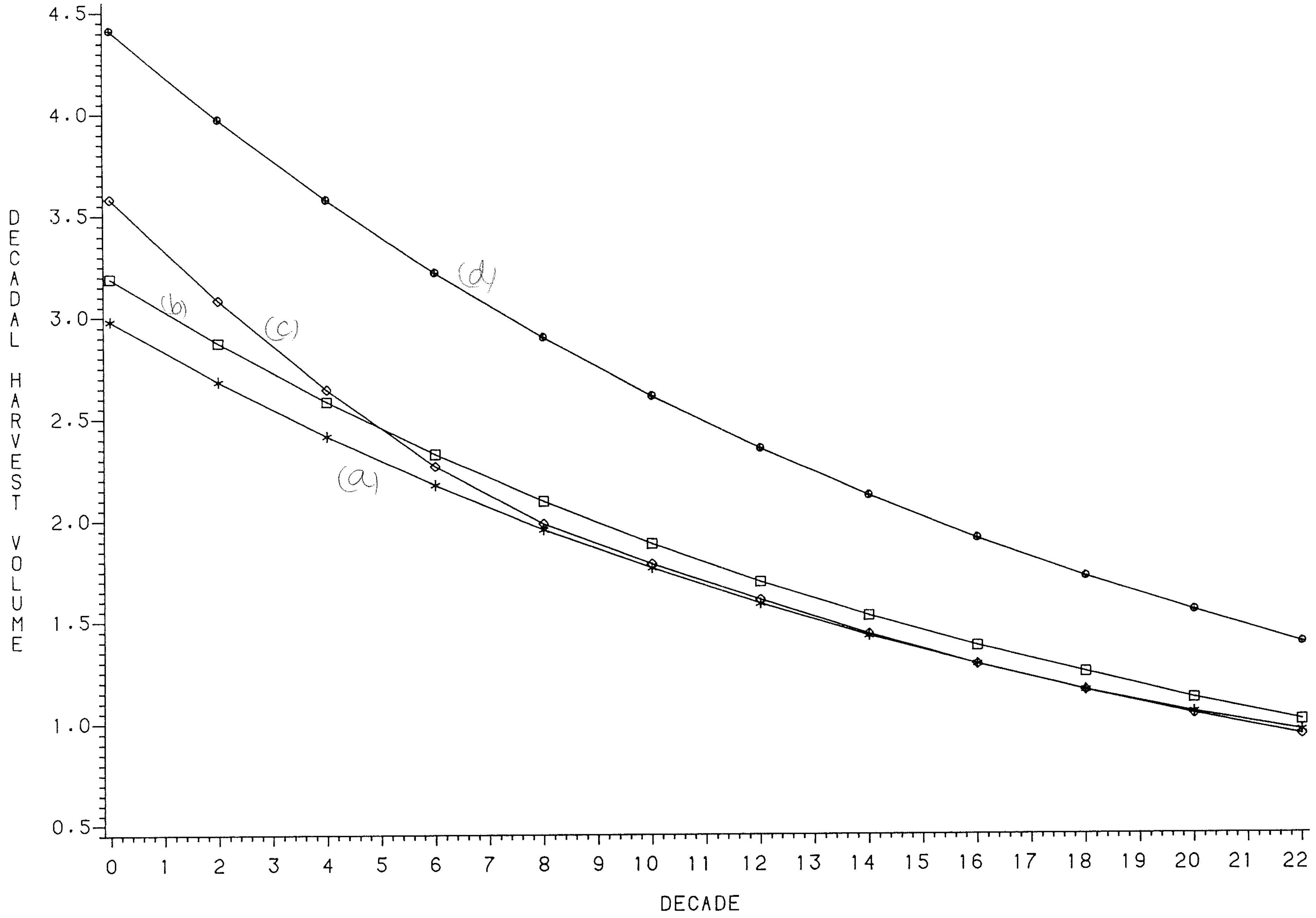


Fig 4

75% salvage 5%

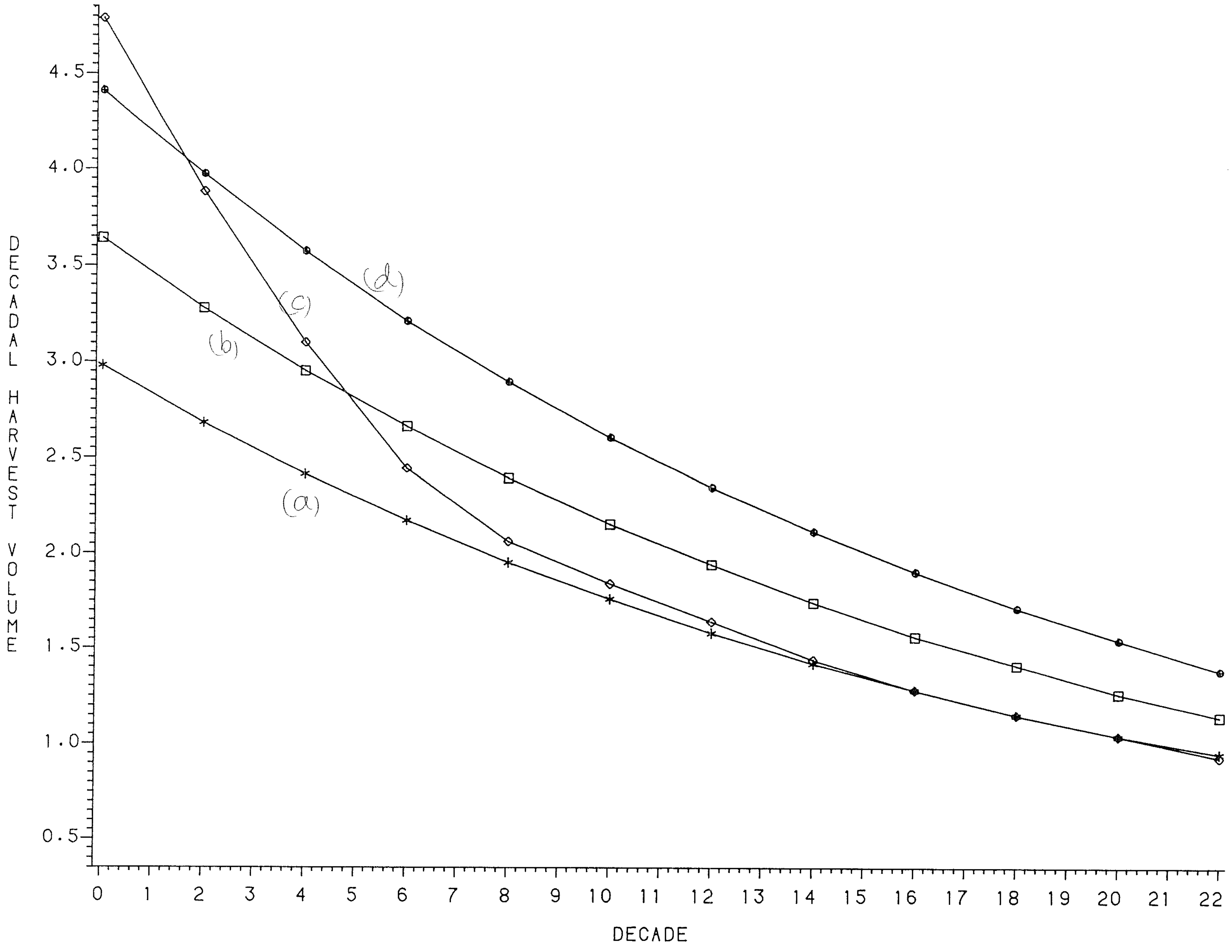


Fig 5