GENERATING NEW FIELD SOLUTIONS, BY ANALYTIC CONTINUATION, AND NEW TOPOLOGIES FROM OLD

GARY G. MILLER

DMS–524–IR

October 1989

GENERATING NEW FIELD SOLUTIONS, BY ANALYTIC CONTINUATION, AND NEW TOPOLOGIES FROM OLD

GARY G. MILLER

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C. V8W 2Y2

Abstract. By analytic continuation, even from real via complex to real, new fields can be obtained from old with ease. These *associated* fields are physically new for continuation is not a coordinate transformation. Here we put forward a general method of *transfer*, some of which is known, and give some simple examples of its application. Unlike other methods of generation, all properties, that are analytically expressible and continue to admit interpretation, *transfer* to hold in the new situation. General conditions, *e.g.*, vacuum, characteristic features, *e.g.*, Killing vectors, and additional fields, *e.g.*, Maxwell, transfer. Yet, the possible reversed signs of curvature and signature, the possible complexification of some features, and the possible appearance of spacetime warps can force attention to interpretation.

1. **Analytic functions.** Lagrange, inspired by the *deterministic* behavior of class $\mathscr{C}_\omega$ functions, promoted their exclusive use as *the* functions of physics, as well as, the term "analytic" to emphasize their significance. The common exact solutions of physics are such real or complex $f(\underline{x})$ of n real or complex variables $\underline{x} = (x^1,\cdots,x^n)$. A function is *analytic* on a domain U provided, say, it is expressible as a power series at each $\underline{x}$ in U that is convergent in some region about $\underline{x}$. We take *domain* U to be given by $0 \leq \|x^k - c^k\| < r^k$ for fixed $\underline{r}$ and center $\underline{c}$, $k = 1,\cdots,n$. Such a domain is a *polydisc* in $\mathbb{C}^n$, in the complex case, and an n–*interval* in $\mathbb{R}^n$, in the real case. A given function can be recognized as analytic in several variables by noting an analytic function of one variable is produced whenever all variables but one are fixed. For example, $f(u,v) = 1/(v-u^2)$ is an analytic function of two complex variables for $u^2 \neq v$. For various fixed values $v = a$, this restricts, on taking $u = x + i0$, to real functions on different real regions, *e.g.*, (i) $1/(1-x^2)$ for $|x| < 1$, (ii) $-1/(x^2-1)$ for $1 < |x|$, (iii) $-1/x^2$ for $x \neq 0$, and (iv) $-1/(x^2+1)$ for all real x. On the other hand, one might start with (i), extend this to $\mathbb{C}$ by replacing x by $x + iy$, and continue back to (ii) by restricting that complex function to the real function $1/(1-(^*x)^2)$ for $^*x = x + i0$ in the domain $1 < |^*x|^2$.

2. Analytic continuation. The *continuation theorem* for n variables asserts: analytic functions f and g agreeing on even a small real n–interval, in an overlap of their domains (polydiscs or intervals), express the same *single–valued* analytic function. By the familiar technique of expanding a power series about a new center, a given analytic $f(\underline{x})$ on U might be *directly continued* to an analytic function, generically denoted $*f(*\underline{x})$, on an overlapping domain $*U$. If $f(\underline{x})$ is a complex valued analytic function for $\underline{x} = \underline{u} + i\underline{v}$ in a polydisc U, where $\underline{u}$, $\underline{v}$ are in $\mathbb{R}^n$, that contains a point $\underline{a} + i\underline{0}$ of $\mathbb{R}^n$, we now let $*f(*x)$ be $f(*x)$ for $*x = \underline{u} + i\underline{0}$ and let $*U$ be the real n–interval of all such $*x$ of U provided this *restriction* $f(*\underline{x})$ is real valued. As well, such an $f(\underline{x})$ can be *restricted* to a real environment $*U$ of a point $a + ib$ in U, a copy of a real n–interval through that point, to provide a $*f(*x)$ when this restriction is real valued. If $f(x)$ is a given real valued analytic function on an n–interval U, replace each x^k by $x^k + i\,y^k$, say, in a power series expression of $f(\underline{x})$, to obtain a complex *extension* of $f(\underline{x})$ to an analytic function, again denoted $*f(*\underline{x})$, for $*\underline{x}$ in some polydisc $*U$. By an *admissible continuation* $*$ of a given analytic $f(\underline{x})$ on U to an analytic $*f(*x)$ on a domain $*U$, we mean any of the preceding direct continuations, restrictions or extensions — or any chain of these if only simply connected regions are considered so a *single–valued* analytic function is formed. For any admissible continuation $*$, by the continuation theorem, $f(\underline{x})$ and $*f(*\underline{x})$ express the same *single–valued* analytic function but on different domains U and $*U$.

3. Algebraic transfer. *Let* $f(\underline{x})$ *and* $g(\underline{x})$ *be analytic on* U, *and let* $*$ *be an admissible continuation of both to* $*f(*\underline{x})$ *and* $*g(*\underline{x})$ *on* $*U$. *Then,* $*(f+g) = *f + *g$, $*(fg) = (*f)(*g)$, $*(1/f) = 1/*f$ *away from zeros,* $*k = k$, k *constant, and any polynomial combination of quantities continues to the same polynomial combination of the continuations of those quantities.* To see the first equation, as an indication of proof, consider a direct continuation $*$. Then, on the overlap of U and $*U$, $f = *f$ and $g = *g$. Thus, $f + g = *f + *g$ on the overlap, and, by the continuation theorem, $f + g$, $*(f+g)$ and $(*f) + (*g)$ express the same *single–valued* function. Thus $*(f+g) = *f + *g$. ■

In the sequel, let $f(\underline{x}), g(\underline{x}), \cdots, h(\underline{x})$ be any analytic functions on a domain U and let $*$ be an admissible continuation of these to $*f(*\underline{x}), *g(*\underline{x}), \cdots, *h(*\underline{x})$ on a domain $*U$.

4. Transfer of identities (analytic permanence). *Let* $G(u,v,\cdots,w)$ *be any analytic function. Then*

$$G(f(\underline{x}),g(\underline{x}),\cdots,h(\underline{x})) = 0, \text{ for all } \underline{x} \text{ in } U, \tag{1}$$

just in case

$$G(^*f(^*\underline{x}),^*g(^*\underline{x}),\cdots,^*h(^*\underline{x})) = 0, \text{ for all } ^*\underline{x} \text{ in } ^*U \tag{2}$$

provided, of course, the compositions are defined.

To see this, note compositions of analytic functions are analytic, and an analytic function that vanishes throughout a domain vanishes everywhere, by the continuation theorem. ■

5. Transfer of derivatives. *The* k*th derivative* $f_{,k}(\underline{x})$ *continues to the* k*th derivative of* $^*f(^*\underline{x})$, $^*(f_{,k}) = (^*f)_{,k}$ *for* $^*\underline{x}$ *in* $^*U$. To see this for the case of extension or restriction, note the Cauchy–Riemann relations yield

$$\frac{\partial f}{\partial z^k} = \frac{\partial(\mathrm{Re}(f))}{\partial x^k} + i\,\frac{\partial(\mathrm{Im}(f))}{\partial x^k} \text{ for } z^k = x^k + i\,y^k. \tag{3}$$

[Had restriction to the pure imaginary been "admissible", $^*(f_{,k})$ would have been $(-i)(^*f)_{,k}$.] For the case of direct continuation, the transfer of derivatives is a basic result. ■

6. Transfer of differential equations. *All partial differential equations*

$$G[f,g,\cdots,h;f_{11};\cdots;f_{,kj\cdots p};\cdots;g_{,rs\cdots t};\cdots;h_{,mm\cdots m}] = 0 \tag{4}$$

that hold for $f(\underline{x}),g(\underline{x}),\cdots,h(\underline{x})$ *transfer unchanged to hold for* $^*f(^*\underline{x}),^*g(^*\underline{x}),\cdots,^*h(^*\underline{x})$ *under an admissible continuation* *, by Th. 4, Th. 5. [Had restriction to the pure imaginary been "admissible", a Schrödinger equation could be transferred (Wick rotation) to a Diffusion equation.]

7. Transfer of metric quantities. Let $g_{ij}(\underline{u})$ be given metric components that are real analytic functions of $\underline{u} = (x^1,\cdots,x^n;a^1,\cdots,a^m)$ in a domain U where $\underline{x} = (x^1,\cdots,x^n)$ are real spacetime coordinates and $\underline{a} = (a^1,\cdots,a^m)$ are optional parameters. Assume $g(\underline{u}) = \det g_{ij}(\underline{u})$ is nonsingular. Let * be an admissible continuation of the metric components to analytic real metric components $h_{ij}(*\underline{u}) = *g_{ij}(*\underline{u})$ where $*\underline{u} = (\underline{y},\underline{b})$ ranges through *U, $\underline{y} = (y^1,\cdots y^n)$ are coordinates and $\underline{b} = (b^1,\cdots,b^m)$ are optional parameters. Since the determinant is a polynomial combination, we have $*g = *(\det g_{ij}) = \det(*g_{ij}) = \det(h_{ij}) = h$. Since derivatives transfer, $*(g_{ij,k}) = h_{ij,k}$ and $*(g_{ij,km}) = h_{ij,km}$. *Thus, all formal polynomial combinations of* g_{ij}, g^{ij} *and their partial derivatives transfer to the corresponding combinations of the metric coefficients* h_{ij}, h^{ij} *and their derivatives.* For example, the connection coefficients $\Gamma^i{}_{jk}$ associated with g_{ij} continue to the corresponding connection coefficients of h_{ij} as do all standard metric quantities $R^i{}_{jkm}$, etc., for these as well are such polynomial combinations. However, signature(g_{ij}) need not be signature(h_{ij}), and quantities can experience sign reversal on transfer. Because of this, care must be given to interpreting, say, curvature components as that same combination.

8. **Transfer of general conditions.** *All algebraic identities held by quantities built from the metric* g_{ij} *transfer to hold for the corresponding metric quantities of* h_{ij} , by Th. 4. For example, *if* g_{ij} *is a vacuum spacetime metric and* h_{ij} *has a spacetime signature, then* h_{ij} *is a vacuum spacetime*, since the Ricci flat condition, $R_{ij} = 0$, must transfer.

9. **Transfer of features.** Locally, a solution of a system of partial differential equations transfers to a corresponding solution of the same equations by Th. 6. In some cases, that formally continued solution may be complex valued or otherwise lack interpretation for the purposes at hand. With this in mind, *Killing vectors transfer to Killing vectors*, *geodesics transfer to geodesics*, etc., if these, local solutions of differential equations, *admit interpretation* upon continuation.

10. **Transfer of additional fields.** *Maxwell and other fields transfer from a spacetime given by* g_{ij} *to an associated spacetime* h_{ij} since, if those field quantities are continued to the same coordinate domain of h_{ij}, then the system of field equations, as well as all identities involving the field components and metric quantities, transfer subject to interpretation.

11. A simple example: Spherical to pseudospherical transfer. Let

$$ds^2 = A(u,v)\left[\frac{dx^2}{1-x^2} + (1-x^2)\,d\varphi^2\right] + B(u,v)du^2 - C(u,v)dv^2 \qquad (5)$$

for the case (i), $|x| < 1$, where A, B and C are analytic real positive functions, express *the general case of spherical spatial symmetry.* Admissibly continue to each of the cases (ii), (iii) and (iv), presented in section 1. By transforming coordinates to $x = \cos\theta$ and the continued variables to $\cosh\theta$ for (ii), e^{θ} for (iii) and $\sinh\theta$ for (iv):

$$ds^2 = A(u,v)(d\theta^2 + \sin^2\theta\, d\varphi^2) + B(u,v)du^2 - C(u,v)dv^2 \qquad (6)$$
$$ds^2 = -A(u,v)(d\theta^2 + \sinh^2\theta\, d\varphi^2) - C(u,v)dv^2 + B(u,v)du^2 \qquad (7)$$
$$ds^2 = -A(u,v)(d\theta^2 + e^{2\theta}\, d\varphi^2) - C(u,v)dv^2 + B(u,v)du^2 \qquad (8)$$
$$ds^2 = -A(u,v)(d\theta^2 + \cosh^2\theta\, d\varphi^2) - C(u,v)dv^2 + B(u,v)du^2. \qquad (9)$$

If one of these is a vacuum spacetime, all are, from Th. 8. For example, there are analogues of the Schwarzschild solution for cases (7), (8) and (9), as is known. The point here, though, is that such results are immediate and also that the admissible continuation transfers features between such associated spacetimes.

12. Generation of new topologies. *In the transfer from one spacetime to an associated spacetime a new topology may appear.* This is the case in transferring from say (6) to (9). Taking u and v fixed, (6) yields an ordinary sphere while (9) yields two otherwise topologically Euclidean plane universes joined by a wormhole. By forming quotients, the locally isometric (7), (8) and (9), can be seen to be metrics of infinitely many examples of spacetime warps.

References

Hille, E., *Analytic Function Theory*, Ginn (1962).

Kramer, D. *et al.*, *Exact Solutions of Einstein's Field Equations*, Cambridge (1980).

Thurston, W.P., "Three dimensional manifolds, Kleinian groups and hyperbolic geometry", *Bulletin Amer. Math. Soc.* **6**(1982), 357–381.

Vladimirov, V.S., *Methods of the Theory of Functions of Many Complex Variables*, M.I.T. (1966).