CHAOS IN THE GLYDÉN PROBLEM

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DMS-782-IR June 1997

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Abstract. We consider the Gyldén problem—a perturbation of the Kepler problem via an explicit function of time. For certain general classes of planar periodic perturbations, after proving a Poincaré-Melnikov-type criterion, we find a manifold of orbits in which the dynamics is given by the shift automorphism on the set of bi-infinite sequences with infinitely many symbols.

1. Introduction

In a note published in 1884 [G], the Swedish astronomer Johan August Hugo Gyldén (the same who in 1889 would compete for King Oscar's Prize—awarded to Poincaré—and then fuel the scandal that followed due to the essential mistake found in the prized paper, see [DH]) proposed a model aimed to describe the secular acceleration of the moon's longitude. He considered a perturbed Kepler problem in which the perturbation μ is an explicit function of time. In its most general form, this problem is given by the Hamiltonian function

$$H(p_1, p_2, q_1, q_2, t) = (p_1^2 + p_2^2)/2 - \big(1 - \epsilon\mu(t)\big)(q_1^2 + q_2^2)^{-1/2}. \tag{1}$$

Later on, Gyldén's model was used to tackle different physical phenomena: a central-force problem with variable mass, the motion of a test particle around a star whose radiation pressure is periodic, a two-body problem in a Dirac-type universe (i.e. one in which the gravitational constant is replaced by a parameter that varies in time), etc. More details on the physical significance of this model can be found in [Sa] and [D].

From the mathematical point of view, approximate solutions for slow variations of the perturbation function μ have been obtained by Hadjidemetriou [H], Verhulst [V1], and Cucu-Dumitrescu and Şelaru [CŞ]. The case of a periodic perturbation function has been tackled by Saslaw [Sa] and Şelaru, Cucu-Dumitrescu, and Mioc [ŞC1], [ŞC2]. Also, Şelaru and Mioc have applied KAM theory to this problem [ŞM].

In this paper we consider two classes of planar perturbations: the one for which μ is a periodic even function and the one for which $\mu(t) = A\cos(2\pi t/T) + B\sin(2\pi t/T)$, where A or B is nonzero, and $T > 0$ is the period. We show that in both cases the system exhibits chaos in the sense of symbolic dynamics. More precisely, we prove that the dynamics of a manifold of solutions is similar to that of a shift automorphism on the set of bi-infinite sequences with infinitely many symbols. To achieve this goal we first consider the unperturbed problem via McGehee transformations, determine its global flow, and find the Poincaré map near the homoclinic orbit to the degenerate periodic orbit at infinity. Then, using a result of McGehee [Mc1], we show that this Poincaré map possesses analytic stable and unstable manifolds. Pursuing the Poincaré-Melnikov method, we further prove a Melnikov-type criterion [Me], which tells under what conditions planar periodic perturbations make the stable and unstable manifolds of the periodic orbit at infinity for the Poincaré map split and intersect transversally. Then, using this criterion, we show that the Melnikov integral has zeroes for each of the above-described class of periodic perturbation functions, consequently the stable and unstable manifolds of the periodic orbit at infinity intersect transversely, giving rise to chaos in the neighborhood of the homoclinic orbit.

The above scenario has been previously used by Moser [Mo] to prove the existence of chaos in the Sitnikov problem and by Xia [X1] to put into the evidence the same phenomenon in the elliptic restricted three-body problem. In general, this method is hard to apply due to the difficulty of evaluating the Melnikov integral. However, for the above described classes of Gyldén systems, we manage to overcome the technical challenge.

2. The Equations of Motion

Consider a unit point mass moving around a center in a Newtonian gravitational field perturbed by a periodic function of time. The equations of motion are described by the Hamiltonian (1), or in explicit form by the system:

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = p_2 \\ \dot{p}_1 = -\big(1-\epsilon\mu(t)\big)q_1(q_1^2+q_2^2)^{-3/2} \\ \dot{p}_2 = -\big(1-\epsilon\mu(t)\big)q_2(q_1^2+q_2^2)^{-3/2}, \end{cases} \tag{2}$$

for which (q_1, q_2) and (p_1, p_2) denote the position and the momentum of the moving particle, μ is the periodic perturbation function (of prime period $T > 0$), and $\epsilon \geq 0$ is a parameter. If $\epsilon = 0$, the equations (2) define the *unperturbd system*, whereas if ϵ is positive and small, the equations (2) are called the *perturbed system*. The latter expresses the planar periodic Gyldén problem and the former describes the classical Kepler problem; in both cases the center body and the moving particle have unit mass.

Under the inverse of the transformation

$$(0,\infty) \times S^1 \times \mathbb{R}^2 \longrightarrow (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2$$

$$(r, \theta, p_r, p_\theta) \to (q_1, q_2, p_1, p_2)$$

given by

$$\begin{cases} q_1 = r\cos\theta \\ q_2 = r\sin\theta \\ p_1 = p_r\cos\theta - (p_\theta/r)\sin\theta \\ p_2 = p_r\sin\theta + (p_\theta/r)\cos\theta, \end{cases} \tag{3}$$

which is a real analytic diffeomorphism, the equations (2) become

$$\begin{cases} \dot{r} = p_r \\ \dot{\theta} = r^{-2}p_\theta \\ \dot{p}_r = r^{-3}p_\theta^2 - r^{-2}\big(1-\epsilon\mu(t)\big) \\ \dot{p}_\theta = 0. \end{cases} \tag{4}$$

Notice that the equations (4) have the Hamiltonian

$$\bar{H}(p_r, p_\theta, r, \theta) = p_r^2/2 + p_\theta^2/(2r^2) - \big(1-\epsilon\mu(t)\big)/r. \tag{5}$$

Since $\dot{p}_\theta = 0$, it follows that $p_\theta = k$ (constant), so the integral of energy is

$$p_r^2/2 + k^2/(2r^2) - \big(1-\epsilon\mu(t)\big)/r = h, \tag{6}$$

where h is the energy constant. Also, in the equations (4) the last equation can be dropped.

We will further use *McGehee transformations of the first kind* *, given by the inverse of the function

$$(0,\infty)\times S^1\times \mathbb{R}\longrightarrow (0,\infty)\times S^1\times \mathbb{R}$$

$$(x,\theta,y)\to(r,\theta,p_r),$$

where

$$\begin{cases} r=1/x^2 \\ y=p_r. \end{cases} \tag{7}$$

The real analytic diffeomorphism (7) transforms the equations (4) into

$$\begin{cases} \dot x=-x^3y/2 \\ \dot\theta=kx^4 \\ \dot y=-x^4\big(1-\epsilon\mu(t)\big)+k^2x^6, \end{cases} \tag{8}$$

whose integral of energy (6) becomes

$$y^2/2+k^2x^4/2-\big(1-\epsilon\mu(t)\big)x^2=h. \tag{9}$$

Note that the McGehee transformations destroy the Hamiltonian structure of the equations of motion. In exchange they will allow us to understand the dynamics at infinity by bringing infinity to the origin of the system of coordinates. In fact, since the equations (8) make sense for $x=0$, the phase space $(0,\infty)\times S^1\times\mathbb{R}$ can be extended to $[0,\infty)\times S^1\times\mathbb{R}$.

The second equation in (8) can be solved independently on the other two, so from now on we will focus on the nonautonomous system

$$\begin{cases} \dot x=-x^3y/2 \\ \dot y=-x^4\big(1-\epsilon\mu(t)\big)+k^2x^6 \end{cases} \tag{10}$$

in the phase plane $[0,\infty)\times\mathbb{R}$.

3. The Kepler Problem in McGehee Coordinates

Let us first tackle the particular case of the unperturbed (autonomous) system, i.e. the Kepler problem in McGehee coordinates given by the equations (10) with $\epsilon=0$:

$$\begin{cases} \dot x=-x^3y/2 \\ \dot y=-x^4+k^2x^6, \end{cases} \tag{11}$$

with the energy relation

$$y^2/2+k^2x^4/2-x^2=h. \tag{12}$$

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* There are two kinds of McGehee transformations: the first ones, which we use here, have been defined in [Mc1] to address the concept of a "periodic orbit at infinity;" the second ones were considered in [Mc2] to blow up the triple-collision singularity of the rectilinear three-body problem.

The time-rescaling transformation $ds = x^3 dt$, which is also a real analytic diffeomorphism, changes system (11) into

$$\begin{cases} x' = -y/2 \\ y' = -x + k^2x^3, \end{cases} \tag{13}$$

where the prime denotes differentiation with respect to the new time variable s. The equations (13) have two equilibria: $(0,0)$ and $(1/k^2, 0)$. For the origin, the eigenvalues of the attached linear system are $\pm\sqrt{2}/2$, so $(0,0)$ is a saddle; for the other equilibrium, the attached linear system has the eigenvalues $\pm i\sqrt{2}/2$, so no direct conclusion can be drawn about the nonlinear system. However, from the energy relation (12), we can tell that $(1/k^2, 0)$ is a center.

For the original system (11), the origin is a degenerate equilibrium, since every solution $(0, y)$ is an equilibrium. Thus in the phase-space picture, the periodic orbits correspond to $h < 0$, the heteroclinic ones to $h > 0$, and the homoclinic orbit has the energy constant $h = 0$. Let us tackle each case.

For $h < 0$ and $y \geq 0$, the periodic orbits are described by the equation

$$\dot{x} = -(1/2)x^3\sqrt{-k^2x^4 + 2x^2 + 2h}, \tag{14}$$

which if integrated yields

$$t = \frac{\sqrt{-k^2x^4 + 2x^2 + 2h}}{2hx^2} + \frac{1}{2\sqrt{2h}\sqrt{-h}} \arcsin \frac{x^2 + 2h}{x^2\sqrt{1 + 2hk^2}}, \tag{15}$$

for the admissible values of x. It looks hopeless to try to express x as an explicit function of t.

For $h = 0$ and $y \geq 0$, the homoclinic orbit is described by the equation

$$\dot{x} = -(1/2)x^4\sqrt{2 - k^2x^2},$$

which if integrated leads to

$$t = (1 + k^2x^2)\sqrt{2 - k^2x^2}/(3x^3),$$

for $x \in (0, \sqrt{2}/|k|]$. With the substitution $u = x^2$, we obtain the cubic equation

$$u^3 + (3/2)k^2u^2 - (9t^2 + k^6)/2 = 0,$$

which by Cardano's formula yields

$$u(t) = (1/2)\left[\left(3t + \sqrt{9t^2 + k^6}\right)^{2/3} + \left(3t - \sqrt{9t^2 + k^6}\right)^{2/3} - k^2\right].$$

Thus the homoclinic orbit is given by

$$\begin{cases} x(t) = \xi(t,k) = \sqrt{2}\left[\left(3t + \sqrt{9t^2 + k^6}\right)^{2/3} + \left(3t - \sqrt{9t^2 + k^6}\right)^{2/3} - k^2\right]^{-1/2} \\ y(t) = \eta(t,k) = \begin{cases} \left(k^2x^4(t) + 2x^2(t)\right)^{1/2}, & t \geq 0 \\ -\left(k^2x^4(t) + 2x^2(t)\right)^{1/2}, & t < 0, \end{cases} \end{cases} \tag{16}$$

for $k \neq 0$. The homoclinic orbit intersects the Ox axis at $x = \sqrt{2}/|k|$. Notice that for $k = 0$ the motion takes place on a line and that the solution leads to a collision and/or an ejection.

For $h > 0$ and $y \geq 0$, the heteroclinic orbits are also described by the equation (14). In this case, however, integrating (14) leads to

$$t = \frac{\sqrt{-k^2x^4 + 2x^2 + 2h}}{2hx^2} + \frac{1}{2\sqrt{2}h^{3/2}} \log \frac{2\sqrt{2h}\sqrt{-k^2x^4 + 2x^2 + 2h} + 2x + 4h}{x}, \tag{17}$$

for which it also looks hoples to write x as an explicit function of t.

Up to now we have understood the qualitative behavior of the unperturbed system (the Kepler problem in McGehee coordinates), but have only a very limited understanding of its analytic aspect. Since it looks hopeless to solve the unperturbed system (10) or even to understand its global flow, the natural questions that arise concern the existence of periodic, oscillatory, and dense orbits for the equations (10). A classical expansion theorem in perturbation theory (see e.g. [V2], Chapter 9) indicates that if periodic orbits of system (10) exist, they can be found only in the region within the homoclinic loop to $(0, 0)$, with $O(\epsilon)$ error of the unperturbed orbit. However, this result offers little insight into the problem. Therefore we now proceed with developing a stronger method, along the lines of Poincaré and Melnikov.

4. The Melnikov Function

Let us assume from now on that μ has also the property

$$\int_0^T \mu(t)dt = 0.$$

(In fact this condition does not restrain the generality since we can introduce in the Newtonian term of the interaction the eventual temporal average of the perturbation.)

Also notice that since μ is analytic, bounded, and T-periodic, it has the Fourier representation:

$$\mu(t) = \sum_{n=1}^{\infty} [C_n \cos(2\pi nt/T) + S_n \sin(2\pi nt/T)],$$

where the coefficients C_n and S_n are real for all $n \geq 1$.

Let us further rewrite the equations (10) as an autonomous system:

$$\begin{cases} \frac{dx}{d\tau} = -\frac{x^3y}{2} \\ \frac{dy}{d\tau} = -x^4 + k^2x^6 + \epsilon x^4\mu(t) \\ \frac{dt}{d\tau} = 1. \end{cases} \tag{18}$$

Consider further a global section $\Sigma^{\tau=\tau_0}$ and denote by $P_{\tau=\tau_0}(x, y)$ the first return of the Poincaré function defined on $\Sigma^{\tau=\tau_0}$. Notice that the origin O is a degenerate fixed point for $P_{\tau=\tau_0}$.

For degenerate situations of this type, McGehee has obtained a result that shows the existence of stable and unstable manifolds for the Poincaré map (see [Mc1], p. 82, system (7.3) and Proposition 11). With the help of this result we can now prove the following:

Lemma 1. *The stable and unstable manifold $W^s(\Lambda)$ and $W^u(\Lambda)$ of the periodic orbit $\Lambda = \{x = 0, y = 0, \tau \in [0, T)\}$ are analytic manifolds for $x > 0$ and for a fixed value of the angular momentum k.*

Proof. McGehee's result cannot be directly applied to system (18), so we will first consider the change of variables

$$\begin{cases} \bar{x} = x \\ \bar{t} = t \\ \bar{y} = y + \epsilon x^4 \sum_{n=1}^{\infty} \frac{T}{2\pi n}\left(C_n \sin \frac{2\pi n t}{T} - S_n \cos \frac{2\pi n t}{T}\right), \end{cases}$$

which fulfills the condition $\frac{D(x,y,t)}{D(\bar{x},\bar{y},\bar{t})} = 1$, and which transforms system (18) into

$$\begin{cases} \frac{d\bar{x}}{d\tau} = -\bar{x}^3\left(\frac{\bar{y}}{2} + g_1(\bar{x}, \bar{y}, \bar{t})\right) \\ \frac{d\bar{y}}{d\tau} = -\bar{x}^3\left(\bar{x} + g_2(\bar{x}, \bar{y}, \bar{t})\right) \\ \frac{d\bar{t}}{d\tau} = 1, \end{cases}$$

where the functions g_1, g_2 are periodic in $\bar{t}$, C^∞, and such that $|g_i(\bar{x}, \bar{y}, \bar{t})| \leq O\left(|(x, y)|^2\right)$, $i = 1, 2$.

McGehee's result can now be applied to the above equations and Lemma 1 follows. This completes the proof.

Recall that in the unperturbed case the stable and unstable manifolds of Λ are identical and form the homoclinic loop (16). We will see that in the perturbed case the two manifolds split and intersect transversally. The first step in formulating the necessary conditions of transverse intersections is to find suitable estimates of the stable and unstable manifolds of the perturbed system. This is done in the following:

Lemma 2. *The solutions $\left(x^{s,u}(\tau, \tau_0), y^{s,u}(\tau, \tau_0)\right)$ that remain in the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ and for which $\left(x^{s,u}(0, \tau_0), y^{s,u}(0, \tau_0)\right)$ belong to the Poincaré section at time $\tau = \tau_0$, admit, for ϵ sufficiently small, the estimates*

$$x^s(\tau, \tau_0) = \xi(\tau - \tau_0, k) + \epsilon x_1^s(\tau, \tau_0) + O(\epsilon^2), \ \tau \in [\tau_0, \infty),$$

$$y^s(\tau, \tau_0) = \eta(\tau - \tau_0, k) + \epsilon y_1^s(\tau, \tau_0) + O(\epsilon^2), \ \tau \in [\tau_0, \infty),$$

$$x^u(\tau, \tau_0) = \xi(\tau - \tau_0, k) + \epsilon x_1^u(\tau, \tau_0) + O(\epsilon^2), \ \tau \in (\infty, \tau_0],$$

$$y^u(\tau, \tau_0) = \eta(\tau - \tau_0, k) + \epsilon y_1^u(\tau, \tau_0) + O(\epsilon^2), \ \tau \in (\infty, \tau_0],$$

uniformly in the indicated time intervals. The functions $x_1^{s,u}(\tau, \tau_0)$, $y_1^{s,u}(\tau, \tau_0)$ are determined by the first variational equation of system (10) along the unperturbed homoclinic orbit (16).

The proof of Lemma 2, based on Gronwall's lemma and on the fact that the perturbed and unperturbed manifolds are C^∞-close, can be found in [GH], [Ho], or in [S]. The necessary condition for the proposed estimates to be valid is that the period T of the function μ is neither too big nor too small.

The vector

$$\mathbf{d}(\tau_0) = \big(x^s(\tau_0, \tau_0) - x^u(\tau_0, \tau_0), y^s(\tau_0, \tau_0) - y^u(\tau_0, \tau_0)\big)$$

measures the splitting of the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ in the Poincaré section $\Sigma^{\tau=\tau_0}$. Its projection onto the vector

$$\mathbf{H}_{0N} = \left(\frac{\partial H_0}{\partial x}(\xi(0,k), \eta(0,k)), \frac{\partial H_0}{\partial y}(\xi(0,k), \eta(0,k))\right),$$

normal to $H_0(x, y)$ at the point $\big(\xi(0,k), \eta(0,k)\big)$, approximates the separation of the two manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$, see [X1]. Denoting by $<,>$ the scalar product, in the first aproximation we have

$$< \mathbf{d}(\tau_0), \frac{\mathbf{H}_{0N}}{|\mathbf{H}_{0N}|} >=$$

$$\epsilon \frac{< \mathbf{H}_{0N}, \big(x_1^s(\tau_0, \tau_0) - x_1^u(\tau_0, \tau_0), y_1^s(\tau_0, \tau_0) - y_1^u(\tau_0, \tau_0)\big) >}{|\mathbf{H}_{0N}|} + O(\epsilon^2),$$

which, using Lemma 2 (see also [X1]), leads to

$$< \mathbf{d}(\tau_0), \frac{\mathbf{H}_{0N}}{|\mathbf{H}_{0N}|} >= \epsilon \frac{H_0\big(x^s(\tau_0, \tau_0), y^s(\tau_0, \tau_0)\big) - H_0\big(x^u(\tau_0, \tau_0), y^u(\tau_0, \tau_0)\big)}{|\mathbf{H}_{0N}|} + O(\epsilon^2)$$

$$= \frac{\epsilon}{|\mathbf{H}_{0N}|} \int_{-\infty}^{+\infty} \frac{dH_0}{d\tau} d\tau + O(\epsilon^2) = \frac{\epsilon}{|\mathbf{H}_{0N}|} M(\tau_0) + O(\epsilon^2),$$

where $M(\tau_0)$ is the Melnikov function (see [GH], [Me], or [S])

$$\begin{aligned} M(\tau_0) &= \frac{1}{\epsilon} \int_{-\infty}^{+\infty} \frac{dH_0\big(\xi(\tau - \tau_0, k), \eta(\tau - \tau_0, k)\big)}{d\tau} d\tau \\ &= \int_{-\infty}^{+\infty} \xi^4(\tau - \tau_0, k)\eta(\tau - \tau_0, k)\mu(\tau) d\tau \\ &= \int_{-\infty}^{+\infty} \xi^4(\tau, k)\eta(\tau, k)\mu(\tau + \tau_0) d\tau. \end{aligned} \tag{19}$$

Substituting the Fourier series representation of μ into the Melnikov function (19), we get

$$M(\tau_0) = \sum_{n=1}^{\infty} \Bigg[\left(C_n \cos\frac{2\pi n \tau_0}{T} + S_n \sin\frac{2\pi n \tau_0}{T}\right) \int_{-\infty}^{+\infty} \xi^4(\tau, k)\eta(\tau, k) \cos\frac{2\pi n \tau}{T} d\tau +$$

$$\left(-C_n \sin\frac{2\pi n \tau_0}{T} + S_n \cos\frac{2\pi n \tau_0}{T}\right) \int_{-\infty}^{+\infty} \xi^4(\tau, k)\eta(\tau, k) \sin\frac{2\pi n \tau}{T} d\tau \Bigg] =$$

$$\sum_{n=1}^{\infty} \left(-C_n \sin\frac{2\pi n \tau_0}{T} + S_n \cos\frac{2\pi n \tau_0}{T}\right) \int_{-\infty}^{+\infty} \xi^4(\tau, k)\eta(\tau, k) \sin\frac{2\pi n \tau}{T} d\tau,$$

where the first of the integrals cancels because the function under the integral is odd. We further have

$$M'(\tau_0) = \frac{dM}{d\tau}(\tau_0) = -\sum_{n=1}^{\infty} \frac{2\pi n}{T} \left(C_n \cos \frac{2\pi n \tau_0}{T} + S_n \sin \frac{2\pi n \tau_0}{T} \right) I_n,$$

where $I_n > 0$ stands for

$$I_n = \int_{-\infty}^{+\infty} \xi^4(\tau, k)\eta(\tau, k) \sin \frac{2\pi n \tau}{T} d\tau.$$

The actual form of I_n will not be involved in our future endeavors. However, for the sake of completeness, we compute I_n in the Appendix at the end of our paper.

Taking now into account the form of the vector $\mathbf{H}_{0N}$ and the equations (16), a straightforward computation leads to

$$|\mathbf{H}_{0N}| = \sqrt{2k^2\xi^3(0,k) - 2\xi(0,k) + \eta^2(0,k)} = 2\sqrt{2}/|k|,$$

so we have

$$< \mathbf{d}(\tau_0), \frac{\mathbf{H}_{0N}}{|\mathbf{H}_{0N}|} >= \epsilon \frac{|k|\, M(\tau_0)}{2\sqrt{2}} + O(\epsilon^2).$$

The above formula will allow us to reach our conclusions.

6. The Main Results

As we have seen in the previous section, the necessary conditions for the transverse intersections of $W^s(\Lambda)$ and $W^u(\Lambda)$ are:

i) ϵ sufficiently small;

ii) μ T-periodic, analytic, and bounded;

iii) $|k|$ large enough such that $\max \frac{|k|M(\tau_0)}{2\sqrt{2}}$ is bounded (in other words, the behavior of the function $< \mathbf{d}(\tau_0), \frac{\mathbf{H}_{0N}}{|\mathbf{H}_{0N}|} >$ must be dominated by that of the term $O(\epsilon^2)$);

iv) There exists $\tau_0 \in [0, T)$ such that $M(\tau_0) = 0$, the derivative M' is defined and differentiable at τ_0, and $M'(\tau_0) \neq 0$.

We have thus proved the following criterion:

Theorem 3. *For the equations (10), which define the Gyldén problem, for μ T-periodic, analytic, and bounded, for a sufficiently large value of the angular momentum k, and for a sufficiently small $\epsilon > 0$, if there exists a $\tau_0 \in [0, T)$ such that $M(\tau_0) = 0$ and $M'(\tau_0) \neq 0$, then the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ of the Poincaré map (18) intersect transversally.*

From Theorem 3 we can now draw the following conclusion:

Corollary 4. *For the equations (10), which define the Gyldén problem, for μ even, T-periodic, analytic, and bounded (i.e. $\mu(t) = \sum_{n=1}^{\infty} C_n \cos(2\pi n t/T)$), for a sufficiently large* *value of the angular momentum k, and for a sufficiently small $\epsilon > 0$, the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ of the Poincaré map (18) intersect transversally. This happens generically, except possibly for functions μ that belong to a certain set, of Lebesgue measure zero, included in the set of all functions μ with the above mentioned properties.*

Proof. Observe that the Melnikov function becomes in this case

$$M(\tau_0) = -\sum_{n=1}^{\infty} C_n I_n \sin\frac{2\pi n\tau_0}{T},$$

and its derivative has the form

$$M'(\tau_0) = -\sum_{n=1}^{\infty} \frac{2\pi n C_n I_n}{T} \cos\frac{2\pi n\tau_0}{T} I_n.$$

Notice that $M(\tau_{0i}) = 0$, $i = 1, 2$, at least for the values $\tau_{01} = 0$, $\tau_{02} = T/2$, whereas for the derivative of the Melnikov function at these points we get:

$$M'(\tau_{01}) = -\sum_{n=1}^{\infty} \frac{2\pi n C_n}{T} I_n,\ M'(\tau_{02}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\pi n C_n}{T} I_n.$$

Clearly, $M'(\tau_{0i}) \neq 0$ for most of the coefficients $C_1, C_2, \ldots$. More precisely, $M'(\tau_{0i}) \neq 0$ in the space of coefficients, $\{(C_1, C_2, \ldots) | C_i \in \mathbb{R}\}$, except possibly for a subset of Lebesgue measure zero. This completes the proof.

Another consequence we can draw from Theorem 3 is the following:

Corollary 5. *For the equations (10), which define the Gyldén problem, for functions μ that are T-periodic, analytic, bounded and have a single harmonic (i.e. $\mu(t) = A\cos(2\pi t/T) + B\sin(2\pi t/T)$, with $\sqrt{A^2 + B^2} \neq 0$), for a sufficiently large value of the angular momentum k, and for a sufficiently small $\epsilon > 0$, the manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ of the Poincaré map (18) intersect transversally.*

Proof. The Melnikov function becomes in this case

$$M(\tau_0) = \left(-A\sin\frac{2\pi\tau_0}{T} + B\cos\frac{2\pi n\tau_0}{T}\right) I_1,$$

and its derivative takes the form

$$M'(\tau_0) = -\frac{2\pi}{T}\left(A\cos\frac{2\pi\tau_0}{T} + B\sin\frac{2\pi n\tau_0}{T}\right) I_1.$$

We can consider $B(A^2 + B^2)^{-1/2} = \sin\alpha$ and $A(A^2 + B^2)^{-1/2} = \cos\alpha$, and obtain the expressions

$$M(\tau_0) = I_1\sqrt{A^2 + B^2}\sin(\alpha - 2\pi\tau_0/T),$$

$$M'(\tau_0) = -I_1\sqrt{A^2+B^2}(2\pi/T)\cos(\alpha - 2\pi\tau_0/T)$$

for the Melnikov function and its derivative. The equation $M(\tau_0) = 0$ admits the only solutions $\tau_{01} = \alpha T/2\pi, \tau_{02} = (\alpha - \pi)T/2\pi$, for which

$$M'(\tau_{01}) = -I_1\sqrt{A^2+B^2}(2\pi/T) \neq 0,\ M'(\tau_{02}) = I_1\sqrt{A^2+B^2}(2\pi/T) \neq 0.$$

By Theorem 3 the conclusion of Corollary 5 follows. This completes the proof.

We will close our endeavors with a symbolic-dynamics description of the above results, which will alow us a better understanding of the dynamics of the Gyldén problem. For this, consider a general Poincaré map P for a transversal homoclinic orbit Λ and let $\mathcal{R}$ be a small rectangle having two if its sides on the stable and unstable manifolds of Λ, respectively, at a point at which the two manifolds intersect transversally. For a point q in $\mathcal{R}$, let n be the smallest positive integer for which $P^n(q)$ belongs to $\mathcal{R}$, and denote by $\mathcal{Q}$ the set of all those q for which such an n (which depends on q) exists. In [Mo], Moser shows that $\mathcal{Q}$ is nonempty and proves that a shift automorphism on infinitely many symbols can be embedded in the set $\mathcal{Q}$. The precise formulation of this result is as follows:

Theorem 6. *Let $\mathcal{T}(q) = P^n(q)$, for all $q \in \mathcal{Q}$, and let $\mathcal{B}$ denote the set of bi-infinite sequences on infinitely many symbols. Then there is an invariant set $I \subset \mathcal{Q}$ for the map $\mathcal{T}$, homeomorphic to $\mathcal{B}$, such that the map $\mathcal{T}$ is topologically conjugate to the shift automorphism on $\mathcal{B}$.*

This result has two main consequences. First it proves the existence of periodic orbits of any period, of dense orbits, as well as that of oscillatory ones, or of any other orbits that can be put into correspondence with a symbolic sequence. Second, it shows that real analytic first integrals do not exist. All of these characterize chaotic motion in the Gyldén problem and put into the evidence some of the complicated dynamics in the phase-space of the equations (10).

Notice that due to the low dimension of system (10), transition-tori phenomena (like the Arnold diffusion—see [A], [M], [X2], [X3]) cannot occur. However, such subtle dynamical aspects may very well characterize higher dimensional systems of the same type, i.e. n-body Gyldén problems with $n \geq 3$.

Appendix

In this appendix we compute I_n. For this define

$$J_\omega = \int_{-\infty}^{\infty} x^4 y \sin\omega\tau d\tau$$

along the unperturbed orbit (16). From the equations (11), integrating J_ω by parts, we have

$$J_\omega = -\int_{-\infty}^{\infty} \frac{d}{d\tau}(x^2)\sin\omega\tau d\tau = \omega\int_{-\infty}^{\infty}\frac{\cos\omega\tau}{r}d\tau.$$

Using the angular momentum integral and the expression of r as function of the true anomaly v for unperturbed parabolic motions, given by $r = \frac{1}{1+cosv}$, we obtain

$$J_\omega = \frac{\omega k}{2} \int_{-\pi}^{\pi} \left(1 + \tan^2(v/2)\right) \cos \omega\tau dv.$$

Denoting $\tan(v/2) = \alpha$ and taking into account the fact that parabolic motion implies that $\tau = (k^3/4)(\alpha + \alpha^3/3)$, we get

$$J_\omega = \omega k^3 \int_{-\infty}^{\infty} \cos[(\omega k^3/4)(s + s^3/3)],$$

or

$$J_\omega = \frac{2\pi\omega k^3}{3} \left[\tilde{I}\left(-\frac{1}{3}, \frac{\omega k^3}{6}\right) - \tilde{I}\left(\frac{1}{3}, \frac{\omega k^3}{6}\right) \right], \tag{20}$$

where $\tilde{I}(n, z)$ is the modified Bessel function of the first kind (see [AS]).

Thus, $I_n = J_{\frac{2\pi n}{T}}$, with $J_{\frac{2\pi n}{T}}$ given by (20).

Acknowledgement. Florin Diacu was supported in part by Grant OGP0122045 of the National Science and Engineering Research Council of Canada.

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