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Sequentially perfect and uniform one-factorizations of the complete graph

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Abstract

In this paper, we consider a weakening of the definitions of uniform and perfect one-factorizations of the complete graph. Basically, we want to order the $2n - 1$ one-factors of a one-factorization of the complete graph K_{2n} in such a way that the union of any two (cyclically) consecutive one-factors is always isomorphic to the same two-regular graph. This property is termed *sequentially uniform*; if this two-regular graph is a Hamiltonian cycle, then the property is termed *sequentially perfect*. We will discuss several methods for constructing sequentially uniform and sequentially perfect one-factorizations. In particular, we prove for any integer $n \geq 1$ that there is a sequentially perfect one-factorization of K_{2n} . As well, for any odd integer $m \geq 1$, we prove that there is a sequentially uniform one-factorization of $K_{2^t m}$ of type $(4, 4, \dots, 4)$ for all integers $t \geq 2 + \lceil \log_2 m \rceil$ (where type $(4, 4, \dots, 4)$ denotes a two-regular graph consisting of disjoint cycles of length four).

1 Introduction

A *one-factor* of a graph G is a subset of its edges which partitions the vertex set. A *one-factorization* of a graph G is a partition of its edges into one-factors. Any one-factorization

of the complete graph K_{2n} has $2n - 1$ one-factors, each of which has n edges. For a survey of one-factorizations of the complete graph, the reader is referred to [10], [14] or [15].

A one-factorization $\{F_0, \dots, F_{2n-2}\}$ of K_{2n} is *sequentially uniform* if the one-factors can be ordered (F_0, \dots, F_{2n-2}) so that the graphs with edge sets $F_i \cup F_{i+1}$ (subscripts taken modulo $2n - 1$) are isomorphic for all $0 \leq i \leq 2n - 2$. Since the union of two one-factors is a 2-regular graph which is 2-edge-colorable, it is isomorphic to a disjoint union of even cycles. We say the multiset $T = (k_1, \dots, k_r)$ is the *type* of a sequentially uniform one-factorization if $F_i \cup F_{i+1}$ is isomorphic to the disjoint union of cycles of lengths k_1, \dots, k_r , where $k_1 + \dots + k_r = 2n$. When the union of every two consecutive one-factors is a Hamiltonian cycle, the one-factorization is said to be *sequentially perfect*.

The idea to consider orderings of the one-factors in a one-factorization of K_{2n} is not entirely academic. In fact, an ordered one-factorization of K_{2n} is a schedule of play for a round-robin tournament (played in $2n - 1$ rounds). Round-robin tournaments possessing certain desired properties have been studied (see [15, Chapter 5], or [7]); however, to our knowledge, round robin tournaments with this “uniform” property have not been considered previously.

The definition above is a relaxation of the definition of uniform (perfect) one-factorization of K_{2n} , which requires that the union of *any* two one-factors be isomorphic (Hamiltonian, respectively). Much work has been done on perfect one-factorizations of K_{2n} ; for a survey, see Seah [13]. Perfect one-factorizations of K_{2n} are known to exist whenever n or $2n - 1$ is prime, and when $2n = 16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 16808, \text{ and } 29792$ (see [1]). Recently a few new perfect one-factorizations have been found, the smallest of which is in K_{530} (see [4, 9, 16]); however before this, no new perfect one-factorization of K_{2n} had been found since 1992 ([17]). The smallest value of $2n$ for which the existence of a perfect one-factorization of K_{2n} is unknown is $2n = 52$. We will show that sequentially perfect one-factorizations are much easier to produce and indeed we will produce a sequentially perfect one-factorization of K_{2n} for all $n \geq 1$.

Various uniform one-factorizations have been constructed from *Steiner triple systems*, [10]. For instance, when $n = 2^m$ for some positive m , the so-called binary projective Steiner triple systems provide a construction of uniform one-factorizations of K_{2n} of type $(4, 4, \dots, 4)$. There are also sporadic examples of *perfect* Steiner triple systems, [8], which give rise to uniform one-factorizations of type $(2n - 4, 4)$. When $v = 3^m$, uniform one-factorizations $(4, 6, \dots, 6)$ exist (these are Steiner one-factorizations from Hall triple systems) and when p is an odd prime there is a uniform one-factorization of K_{p^s+1} of type $(p + 1, 2p, \dots, 2p)$ which arises from the elementary abelian p -group (see [10]).

The remainder of this paper is organized as follows. In Section 2, we review the classical “starter” construction for one-factorizations and we show that sequentially perfect one-factorizations of K_{2n} exist for all n . In Section 3, we summarize existence results obtained by computer for small orders. In Section 4, we investigate the construction of sequentially uniform one-factorizations from so-called quotient starters in noncyclic abelian groups. Here we obtain interesting number-theoretic conditions that determine if the resulting one-factorizations can be ordered so that they are sequentially uniform. In Section 5,

we present a recursive product construction which yields infinite classes of sequentially uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \dots, 4)$, for any odd integer m .

2 Starters

We describe our main tool for finding sequentially uniform one-factorizations. Let Γ be an abelian group of order $2n - 1$, written additively. A *starter* in Γ is a set of $n - 1$ pairs $S = \{\{x_1, y_1\}, \dots, \{x_{n-1}, y_{n-1}\}\}$ such that every nonzero element of Γ appears as some x_i or y_i , and also as some difference $x_j - y_j$ or $y_j - x_j$. Let $S^* = S \cup \{\{0, \infty\}\}$ and define $x + \infty = \infty + x = \infty$ for all $x \in \Gamma$. Then $\{S^* + x : x \in \Gamma\}$ forms a one-factorization of K_{2n} (with vertex set $\Gamma \cup \{\infty\}$).

Many of the known constructions for (uniform and perfect) one-factorizations use starters in this way. In our first lemma we note the connection between starter-induced one-factorizations and sequentially uniform one-factorizations. Clearly, the order in which the 1-factors are listed is essential to the type of a sequentially uniform one-factorization. Thus we will sometimes refer to *ordered one-factorizations* in this context. Whenever we discuss sequentially uniform one-factorizations, we will always give the 1-factor ordering.

Lemma 2.1. *Let S be a starter in \mathbb{Z}_{2n-1} with $n \geq 1$. Then the ordered one-factorization of K_{2n} generated by S , namely $(S^*, S^* + 1, S^* + 2, \dots, S^* + (2n - 2))$ is sequentially uniform.*

Proof: For any $x \in \mathbb{Z}_{2n-1}$, we have $(S^* + x) \cup (S^* + (x + 1)) = x + (S^* \cup (S^* + 1))$, so all unions of two consecutive one-factors in the given order are isomorphic. \square

Remark: When $\gcd(k, 2n - 1) = 1$, the ordering $(S^*, S^* + k, S^* + 2k, \dots, S^* + (2n - 2)k)$ of the same one-factorization is also sequentially uniform. Note, however, that it is not necessarily of the same type as the ordered one-factorization $(S^*, S^* + 1, S^* + 2, \dots, S^* + (2n - 2))$.

The most well-known one-factorization of K_{2n} (called $GK(2n)$) is generated from the *patterned* starter $P = \{\{x, -x\} : x \in \mathbb{Z}_{2n-1}\}$ in the cyclic group \mathbb{Z}_{2n-1} . It is known when $2n - 1$ is prime that $GK(2n)$ is a perfect one-factorization and, in general, $GK(2n)$ is a uniform one-factorization for all $n \geq 1$. The cycle lengths in $P^* \cup (P^* + k)$ for $k \in \mathbb{Z}_{2n-1} \setminus \{0\}$ are now given.

Lemma 2.2. *Let P be the patterned starter in \mathbb{Z}_{2n-1} with $n \geq 1$. Let $k \in \mathbb{Z}_{2n-1} \setminus \{0\}$ with $\gcd(2n - 1, k) = d$. Then $P^* \cup (P^* + k)$ consists of a cycle of length $1 + (2n - 1)/d$ and $(d - 1)/2$ cycles of length $2(2n - 1)/d$.*

Proof: The cycle through infinity is $(\infty, 0, 2k, -2k, 4k, -4k, \dots, -k, k)$, which has length $1 + (2n - 1)/d$. All other cycles (if any) are of the form

$$(i, -i, 2k + i, -2k - i, 4k + i, -4k - i, \dots, -2k + i, 2k - i),$$

for $1 \leq i < d$. \square

Combining Lemmas 2.1 and 2.2 (with $d = 1$) we have the following result.

Theorem 2.3. *For every $n \geq 1$ there exists a sequentially perfect one-factorization of K_{2n} .*

Contrast this with the known results for perfect one-factorizations: the sporadic small values mentioned in the Introduction, and only two infinite classes (each of density zero).

3 Small orders

The one-factorizations of K_4 and K_6 are unique and in each case they are perfect. Hence both are sequentially perfect (the only possible type in these small cases). The one-factorization of K_8 obtained from the unique Steiner triple system of order 7 has type $(4, 4)$ while $GK(8)$ is a perfect one-factorization. Hence there exist sequentially uniform one-factorizations of K_8 of all possible types.

We have checked all starters in \mathbb{Z}_9 by computer and report that no ordering of the translates of any of these starters yields a sequentially uniform one-factorization of K_{10} of type $(4, 6)$. However, there does exist a uniform one-factorization of type $(4, 6)$ (it is one-factorization #1 in the list of all 396 non-isomorphic one-factorizations of K_{10} given in [1, p. 655]). Clearly this is also sequentially uniform of type $(4, 6)$ under any ordering of the one-factors. From Theorem 2.3 there exists a sequentially perfect one-factorization of K_{10} . Thus sequentially uniform one-factorizations of K_{10} exist for both possible types.

Obviously, the ordering of the one-factors can affect the type of the 2-factors formed from consecutive 1-factors in an ordered one-factorization. Given a starter S in \mathbb{Z}_{2n-1} , let $F_S(k)$ denote the ordered one-factorization $(S^*, S^* + k, S^* + 2k, \dots, S^* + (2n - 2)k)$ of K_{2n} . In the following examples we discuss sequentially uniform one-factorizations in K_{12} and K_{14} . In \mathbb{Z}_{13} we will give one starter which induces all possible types of ordered one-factorizations when different orderings are imposed on translates of that starter.

Example 3.1. *Given the following starter in \mathbb{Z}_{11} ,*

$$S = \{\{1, 2\}, \{3, 8\}, \{4, 6\}, \{5, 9\}, \{7, 10\}\},$$

$F_S(1)$ is sequentially uniform of type $(6, 6)$, $F_S(2)$ is sequentially uniform of type $(4, 8)$ and $F_S(3)$ is sequentially uniform of type (12) .

By checking all starters in \mathbb{Z}_{11} , we found that no ordering of any of the one-factorizations formed by these starters gave a sequentially uniform one-factorization of type $(4, 4, 4)$. However, Figure 1 provides a non-starter-induced ordered one-factorization which is sequentially uniform of this type.

In [12] it is found that there exist exactly five nonisomorphic perfect one-factorizations of K_{12} and in [2] a uniform one-factorization of type $(6, 6)$ is given. From the enumeration in [5], it is known that there exist no other uniform one-factorizations of K_{12} . Hence it is noteworthy that $F_S(2)$ (defined in Example 3.1) gives a sequentially uniform one-factorization of K_{12} of type $(8, 4)$ and Figure 1 gives a sequentially uniform one-factorization of type $(4, 4, 4)$.

Figure 1: A sequentially uniform one-factorization of K_{12} with type $(4, 4, 4)$

$$\begin{aligned}
 F_0 &: \{\{0, 1\}, \{2, 6\}, \{3, 4\}, \{7, 9\}, \{8, 10\}, \{5, 11\}\} \\
 F_1 &: \{\{0, 2\}, \{1, 6\}, \{3, 9\}, \{4, 7\}, \{5, 10\}, \{8, 11\}\} \\
 F_2 &: \{\{0, 3\}, \{1, 4\}, \{5, 8\}, \{6, 7\}, \{2, 9\}, \{10, 11\}\} \\
 F_3 &: \{\{0, 4\}, \{1, 3\}, \{2, 8\}, \{7, 10\}, \{6, 11\}, \{5, 9\}\} \\
 F_4 &: \{\{0, 5\}, \{1, 2\}, \{3, 8\}, \{4, 9\}, \{6, 10\}, \{7, 11\}\} \\
 F_5 &: \{\{0, 8\}, \{1, 7\}, \{2, 11\}, \{3, 5\}, \{4, 6\}, \{9, 10\}\} \\
 F_6 &: \{\{0, 6\}, \{1, 5\}, \{2, 10\}, \{4, 8\}, \{3, 7\}, \{9, 11\}\} \\
 F_7 &: \{\{0, 7\}, \{1, 10\}, \{2, 5\}, \{3, 6\}, \{4, 11\}, \{8, 9\}\} \\
 F_8 &: \{\{0, 9\}, \{1, 11\}, \{2, 3\}, \{4, 10\}, \{5, 6\}, \{7, 8\}\} \\
 F_9 &: \{\{0, 11\}, \{1, 9\}, \{2, 4\}, \{3, 10\}, \{6, 8\}, \{5, 7\}\} \\
 F_{10} &: \{\{0, 10\}, \{1, 8\}, \{2, 7\}, \{3, 11\}, \{4, 5\}, \{6, 9\}\}
 \end{aligned}$$

Example 3.2. *The following starter in \mathbb{Z}_{13} ,*

$$S = \{\{1, 10\}, \{2, 3\}, \{4, 9\}, \{5, 7\}, \{6, 12\}, \{8, 11\}\},$$

yields sequentially uniform one-factorizations of K_{14} of all possible types: namely (14) , $(10, 4)$, $(8, 6)$, and $(6, 4, 4)$. Specifically, $F_S(3)$ is sequentially uniform of type $(6, 4, 4)$, $F_S(1)$ is sequentially uniform of type $(8, 6)$, $F_S(2)$ is sequentially uniform of type $(10, 4)$ and $F_S(5)$ is sequentially uniform of type (14) .

For large n , there are many more possible types than there are translates, so the starter in Example 3.2 is of particular interest. In the Appendix we give examples of sequentially uniform one-factorizations of K_{2n} of all possible types, for $14 \leq 2n \leq 24$.

4 Starters in non-cyclic groups

Many uniform and perfect one-factorizations are known to be starter-induced over a non-cyclic group; for example, see [6]. So it is natural to also expect sequentially uniform one-factorizations where the ordering is not cyclic. In this section we give a numerical condition that determines when certain starter-induced one-factorizations over non-cyclic groups are sequentially uniform.

Let q be an odd prime-power (not a prime) and write $q = 2rt + 1$, where t is odd. In order to eliminate trivial cases, we will assume that $t > 1$. Suppose ω is a generator of the multiplicative group of \mathbb{F}_q and let Q be the subgroup (of order t) generated by ω^{2r} . Suppose the cosets of Q are $C_i = \omega^i Q$, $i = 0, \dots, 2r - 1$. A starter S in \mathbb{F}_q is said to be an r -quotient starter if, whenever $\{x, y\}, \{x', y'\} \in S$ with $x, x' \in C_i$, it holds that $y/x = y'/x'$. An r -quotient starter S can be completely described by a list of *quotients* (a_0, \dots, a_{r-1}) , such that

$$S = \{\{x, a_i x\} : (a_i - 1)x \in C_i, i = 0, \dots, r - 1\}.$$

It is not hard to see that $S^* \cup (S^* + x)$ is isomorphic to $S^* \cup (S^* + y)$ whenever $x/y \in C_0 \cup C_r$. It follows that every 1-quotient starter yields a uniform one-factorization. We now show that, although r -quotient starters might not generate uniform one-factorizations when $r > 1$, [6], the resulting one-factorizations usually can be ordered in such a way that they are sequentially uniform.

Theorem 4.1. *Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2rt + 1$ and $t > 1$ is odd. Let S be any r -quotient starter in \mathbb{F}_q . Then the one-factorization generated by S can be ordered to be sequentially uniform if and only if the multiplicative order of p modulo t is equal to d .*

Proof: Let $q = p^d = 2rt + 1$ with t odd. C_0 is the multiplicative subgroup of \mathbb{F}_q^* generated by a primitive t th root of 1 in \mathbb{F}_q , say α . The splitting field of $x^t - 1$ over \mathbb{F}_p is \mathbb{F}_{p^e} , where e is the smallest positive integer such that $p^e \equiv 1 \pmod{t}$. Hence the extension field $\mathbb{F}_p(\alpha) = \mathbb{F}_q$ if and only if the multiplicative order of p modulo t , which we denote by $\text{ord}_t(p)$, is equal to d .

Suppose that $\text{ord}_t(p) = d$. Then $\mathbb{F}_p(\alpha) = \mathbb{F}_q$ and $1, \alpha, \dots, \alpha^{d-1}$ is a basis of \mathbb{F}_q over \mathbb{F}_p . Therefore, every element $x \in \mathbb{F}_q$ can be expressed uniquely as a d -tuple $(x_1, \dots, x_d) \in (\mathbb{Z}_p)^d$, where

$$x = \sum_{i=1}^d x_i \alpha^{i-1}.$$

Now, consider the graph on vertex set $(\mathbb{Z}_p)^d$ in which two vertices are adjacent if and only if they agree in $d - 1$ coordinates and their values in the remaining coordinate differ by 1 modulo p (this is a *Cayley graph* of the elementary abelian group of order p^d). It is not hard to check that this graph has a hamiltonian cycle, say $C = (y_1, y_2, \dots, y_{p^d}, y_1)$. The cycle C provides the desired ordering of \mathbb{F}_q because the difference between any two consecutive elements y_i and y_{i+1} is in $C_0 \cup C_r$ (note that one of $y_i - y_{i+1}$ and $y_{i+1} - y_i$ is a power of α and hence in C_0 , while the other is in C_r).

Conversely, suppose that $\text{ord}_t(p) = e < d$. Then $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^e}$ which is a strict subfield of \mathbb{F}_q . Clearly $C_0 \cup C_r \subseteq \mathbb{F}_{p^e}$. Suppose that y_1, y_2, \dots is an ordering of the elements of \mathbb{F}_q such that adjacent elements always have a difference that is an element of $C_0 \cup C_r$. Without loss of generality we can take $y_1 = 0$. But then every element y_i is in the subfield \mathbb{F}_{p^e} , which is a contradiction. Hence, the desired ordering cannot exist. \square

It is interesting to note that the proof above does not depend on the structure of the starter S . Either all r -quotient starters in \mathbb{F}_q yield sequentially uniform one-factorizations or they all do not do so.

Example 4.2. *Let $q = 25$ so that $t = 3$ and $r = 4$. We have $\text{ord}_t(p) = 2 = d$, so Theorem 4.1 asserts that any 4-quotient starter will yield a sequentially uniform one-factorization. In particular, if we take $\mathbb{F}_{25} = \mathbb{Z}_5[x]/(x^2 + x + 2)$ then C_0 contains a basis $\{1, \alpha\}$ for the field, where $\alpha = x^8 = 3x + 1$. The field elements can be cyclically ordered*

$$0, 1, 2, 3, 4, 3x, \dots, 3x + 4, x, \dots, x + 4, 4x, \dots, 4x + 4, 2x, \dots, 2x + 4, 0$$

so that the difference of consecutive elements is either 1 or α .

Most applications of r -quotient starters use values of r that are powers of two (see, for example, [6]). It is interesting to determine the conditions under which the hypotheses of Theorem 4.1 are satisfied in this case. This is done in Lemma 4.3.

Lemma 4.3. *Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2^k t + 1$ and $t > 1$ is odd. Then one of the two following conditions hold:*

1. $\text{ord}_t(p) = d$, or
2. $p = 2^j - 1$ for some integer j (i.e., p is a Mersenne prime) and $d = 2$. (In this case, $\text{ord}_t(p) = 1$ is less than d .)

Proof: Suppose that $p^e \equiv 1 \pmod{t}$ for some positive integer $e < d$ (note that $e|d$). Let $p^e = bt + 1$ where b is a positive integer. Then

$$2^k t + 1 = q = (p^e)^{d/e} = (bt + 1)^{d/e} = cbt + 1$$

for some integer c . Hence, $b|2^k$, and therefore $b = 2^\ell$ for some positive integer $\ell \leq k$. So we have that $p^e = 2^\ell t + 1$.

Let $\rho = p^e$ and $f = d/e$. Then we have that

$$t = \frac{\rho^f - 1}{2^k} = \frac{\rho - 1}{2^\ell}.$$

Removing common factors, we obtain

$$\rho^{f-1} + \rho^{f-2} + \dots + \rho + 1 = 2^{k-\ell}. \tag{1}$$

Suppose that $k = \ell$. Then the right side of (1) is equal to 1, so $f = 1$ and $d = e$. This contradicts the assumption that $d > e$. Therefore $k > \ell$ and the right side of (1) is even.

Now, suppose that f is odd. Then the left side of (1) is odd, and we have a contradiction. Therefore f is even, and $\rho + 1$ is a factor of the left side of (1). This implies that $2^{k-\ell} \equiv 0 \pmod{\rho + 1}$, and hence $\rho = 2^j - 1$ for some integer $j \geq 2$. Then, after dividing (1) by the factor $\rho + 1$, we obtain the following equation:

$$\rho^{f-2} + \rho^{f-4} + \dots + \rho^2 + 1 = 2^{k-\ell-j}. \tag{2}$$

Suppose that $j < k - \ell$. Then the right side of (2) is even and $\rho^2 + 1$ is a factor of the left side of (2), so $\rho^2 = 2^i - 1$ for some integer $i \geq 2$. But $\rho = 2^j - 1$ where $j \geq 2$, so $\rho \equiv 3 \pmod{4}$. Then $\rho^2 \equiv 1 \pmod{4}$, which contradicts the fact that $\rho^2 = 2^i - 1$ where $i \geq 2$. Therefore we have that $j = k - \ell$. This implies that $f = 2$ and so $d = 2e$. So $\rho = 2^j - 1$ for some integer j and $q = \rho^2$.

However, it is easy to prove that the Diophantine equation $2^u - y^v = 1$ has no solution in positive integers with $u, v > 1$ [†]. See, for example, Cassels [3, Corollary 2]. Therefore,

[†]This result is a special case of *Catalan's Conjecture*, which states that the Diophantine equation $x^u - y^v = 1$ has no solution in positive integers with $u, v > 1$ except for $3^2 - 2^3 = 1$. Catalan's Conjecture was proven correct in 2002 by Mihăilescu (see Metsänkylä [11] for a recent exposition of the proof).

we can conclude that ρ is prime. Hence, $p = 2^j - 1$ is a Mersenne prime, $e = 1$ and $d = 2$. In this case, we have $q = p^2$. Then we have that

$$q - 1 = (p - 1)(p + 1) = (p - 1)2^j \equiv 0 \pmod{t}.$$

But t is odd, so $p \equiv 1 \pmod{t}$. Therefore $\text{ord}_t(p) = 1$. □

Example 4.4. Let $q = 961 = 31^2$ so $p = 31$ and $d = 2$. Here $p = 31 = 2^5 - 1$ is a Mersenne prime. We can write $q = 2^5 15 + 1$, so $t = 15$. We see that $\text{ord}_t(p) = 1 < 2$, as asserted by Lemma 4.3.

The following corollary is an immediate consequence of Theorem 4.1 and Lemma 4.3.

Corollary 4.5. Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2^k t + 1$ and $t > 1$ is odd. Let S be any 2^{k-1} -quotient starter in \mathbb{F}_q . Then the one-factorization generated by S can be ordered to be sequentially uniform if and only if it is not the case that p is a Mersenne prime and $d = 2$.

5 Product construction

We now recall the usual product construction for one-factorizations, and apply it to determine another infinite class of sequentially uniform one-factorizations.

Suppose that F is a one-factor on X and G is a one-factor on Y , where $|X| = 2n$ and $|Y| = 2m$. Define various one-factors of $X \times Y$ by

$$\begin{aligned} F^* &= \{ \{(x_i, y), (x'_i, y)\} : \{x_i, x'_i\} \in F, y \in Y \}, \\ G^* &= \{ \{(x, y_j), (x, y'_j)\} : x \in X, \{y_j, y'_j\} \in G \}, \\ FG &= \{ \{(x_i, y_j), (x'_i, y'_j)\} : \{x_i, x'_i\} \in F, \{y_j, y'_j\} \in G \}. \end{aligned}$$

Given one-factorizations $\mathcal{F} = \{F_0, \dots, F_{2n-2}\}$ and $\mathcal{G} = \{G_0, \dots, G_{2m-2}\}$ of K_{2n} and K_{2m} on the points X and Y , respectively, it is easy to see that

$$\begin{aligned} \mathcal{F}\mathcal{G} &= \{F_i G_j : i = 0, \dots, 2n - 2 \text{ and } j = 0, \dots, 2m - 2\} \\ &\quad \cup \{F_i^* : i = 0, \dots, 2n - 2\} \cup \{G_j^* : j = 0, \dots, 2m - 2\} \end{aligned}$$

is a one-factorization of $X \times Y$.

The following are easy lemmas about the cycle types of pairs of one-factors in $\mathcal{F}\mathcal{G}$.

Lemma 5.1. For any $i \in \{0, \dots, 2n - 2\}$ and $j \in \{0, \dots, 2m - 2\}$, the following all have cycle type $(4, 4, \dots, 4)$:

- (i) $F_i^* \cup G_j^*$,
- (ii) $F_i G_j \cup F_i^*$, and
- (iii) $F_i G_j \cup G_j^*$.

Lemma 5.2. *If $(F_0, F_1, \dots, F_{2n-2})$ is sequentially uniform of type $(4, 4, \dots, 4)$, then the following all have cycle type $(4, 4, \dots, 4)$:*

(i) $F_i^* \cup F_{i+1}^*$, and

(ii) $F_i G_j \cup F_{i+1} G_j$,

for any i, j , where the subscripts $i + 1$ are reduced modulo $2n - 1$.

We can use the above results to give a product construction for sequentially uniform one-factorizations of type $(4, 4, \dots, 4)$.

Theorem 5.3. *Suppose there exists a sequentially uniform one-factorization of K_{2n} of type $(4, 4, \dots, 4)$. Let $m \leq n$. Then there is a sequentially uniform one-factorization of K_{4mn} of type $(4, 4, \dots, 4)$.*

Proof: We use all the notation above, with $(F_0, F_1, \dots, F_{2n-2})$ sequentially uniform of type $(4, 4, \dots, 4)$ and \mathcal{G} any one-factorization of K_{2m} . The ordered one-factorization

$$\begin{aligned} & (G_0^*, F_0 G_0, F_1 G_0, \dots, F_{2n-2} G_0, F_{2n-2}^*, \\ & \quad G_1^*, F_1 G_1, F_2 G_1, \dots, F_0 G_1, F_0^*, \\ & \quad G_2^*, F_2 G_2, F_3 G_2, \dots, F_1 G_2, F_1^*, \\ & \quad \vdots \\ & \quad G_{2m-2}^*, F_{2m-2} G_{2m-2}, \dots, F_{2m-3} G_{2m-2}, F_{2m-3}^*, \\ & \quad \quad F_{2m-2}^*, F_{2m-1}^*, \dots, F_{2n-3}^*) \end{aligned}$$

of K_{4mn} is sequentially uniform of type $(4, 4, \dots, 4)$ by Lemmas 5.1 and 5.2. □

By applying the above product construction with $2n$ a power of 2 — for which the existence of *uniform* one-factorizations of type $(4, 4, \dots, 4)$ are known — one immediately has the following corollary.

Corollary 5.4. *For any odd integer $m \geq 1$, there is a sequentially uniform one-factorization of $K_{2^t m}$ of type $(4, 4, \dots, 4)$ for all integers $t \geq 2 + \lceil \log_2 m \rceil$.*

Let $t_0 = t_0(m)$ denote the smallest integer such that there is a sequentially uniform one-factorization of $K_{2^t m}$ of type $(4, 4, \dots, 4)$ for all integers $t \geq t_0$. Corollary 5.4 provides an explicit upper bound on $t_0(m)$; however, for a particular value of m , we might be able to give a better bound on $t_0(m)$. For example, the sequentially perfect one-factorization of K_4 shows that $t_0(1) = 2$, the sequentially uniform one-factorization of K_{12} of type $(4, 4, 4)$ given in Figure 1 yields $t_0(3) = 2$, and the sequentially uniform one-factorization of K_{20} of type $(4, 4, 4, 4, 4)$ exhibited in the Appendix gives $t_0(5) = 2$. In fact, we conjecture that $t_0(m) = 2$ for all odd integers $m \geq 1$.

As a final note, we observe that the existence results for sequentially uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \dots, 4)$ provide an interesting contrast to those for uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \dots, 4)$, which exist only when $m = 1$ (see Cameron [2, Proposition 4.3]).

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Appendix

Below is a table giving all possible types for sequentially uniform one-factorizations of K_{2n} , $14 \leq 2n \leq 24$. Each type is realized by the ordered 1-factorization $F_S(1)$ corresponding to the starter $S = \{\{x_i, x_i + i\} : i = 1, \dots, n - 1\}$ in \mathbb{Z}_{2n-1} .

| type | (x_1, \dots, x_{n-1}) | type | (x_1, \dots, x_{n-1}) |
|-----------------|----------------------------------|--------------------|--|
| (14) | (1, 5, 8, 6, 12, 3) | (22) | (1, 3, 4, 10, 11, 13, 8, 12, 9, 17) |
| (10, 4) | (1, 9, 4, 6, 3, 12) | (18, 4) | (1, 3, 4, 13, 11, 12, 8, 6, 10, 20) |
| (8, 6) | (1, 5, 9, 6, 3, 11) | (16, 6) | (1, 3, 4, 10, 11, 12, 13, 9, 6, 19) |
| (6, 4, 4) | (7, 2, 9, 1, 6, 10) | (14, 8) | (1, 3, 4, 12, 9, 11, 13, 10, 6, 19) |
| (16) | (1, 4, 8, 9, 7, 14, 3) | (14, 4, 4) | (1, 4, 7, 11, 12, 14, 19, 8, 9, 3) |
| (12, 4) | (1, 4, 8, 9, 5, 12, 7) | (12, 10) | (1, 3, 6, 11, 13, 10, 12, 20, 8, 4) |
| (10, 6) | (1, 7, 11, 4, 5, 12, 6) | (12, 6, 4) | (1, 3, 7, 15, 11, 8, 13, 4, 9, 17) |
| (8, 8) | (1, 3, 7, 9, 6, 8, 12) | (10, 8, 4) | (1, 3, 7, 11, 14, 6, 13, 9, 16, 8) |
| (8, 4, 4) | (2, 11, 9, 4, 5, 1, 14) | (10, 6, 6) | (1, 3, 7, 16, 12, 8, 6, 11, 9, 15) |
| (6, 6, 4) | (3, 11, 2, 6, 7, 8, 9) | (10, 4, 4, 4) | (1, 5, 11, 13, 19, 6, 8, 10, 16, 20) |
| (4, 4, 4, 4) | (3, 6, 11, 12, 5, 7, 2) | (8, 8, 6) | (1, 3, 12, 6, 13, 14, 4, 9, 7, 19) |
| (18) | (1, 3, 7, 12, 8, 9, 4, 6) | (8, 6, 4, 4) | (1, 3, 10, 16, 12, 9, 4, 6, 19, 8) |
| (14, 4) | (1, 3, 11, 8, 4, 10, 6, 7) | (6, 6, 6, 4) | (1, 8, 6, 11, 12, 19, 13, 18, 7, 14) |
| (12, 6) | (1, 5, 8, 12, 9, 4, 13, 15) | (6, 4, 4, 4, 4) | (1, 11, 5, 16, 9, 6, 17, 10, 19, 15) |
| (10, 8) | (1, 3, 6, 11, 8, 10, 7, 4) | (24) | (1, 3, 4, 9, 11, 15, 12, 14, 8, 10, 18) |
| (10, 4, 4) | (1, 13, 5, 7, 9, 4, 16, 12) | (20, 4) | (1, 3, 4, 13, 10, 16, 12, 6, 11, 8, 21) |
| (8, 6, 4) | (1, 4, 12, 5, 8, 10, 7, 3) | (18, 6) | (1, 3, 4, 10, 16, 13, 8, 9, 11, 12, 18) |
| (6, 6, 6) | (1, 6, 10, 5, 11, 15, 7, 12) | (16, 8) | (1, 3, 4, 10, 12, 15, 11, 8, 13, 19, 9) |
| (6, 4, 4, 4) | (3, 7, 12, 2, 8, 10, 11, 14) | (16, 4, 4) | (1, 3, 6, 10, 16, 11, 15, 12, 4, 8, 19) |
| (20) | (1, 3, 6, 11, 13, 8, 10, 4, 7) | (14, 10) | (1, 3, 4, 13, 16, 8, 11, 12, 6, 9, 22) |
| (16, 4) | (1, 3, 9, 13, 10, 8, 4, 18, 16) | (14, 6, 4) | (1, 3, 6, 10, 12, 16, 13, 11, 21, 8, 4) |
| (14, 6) | (1, 3, 9, 7, 13, 8, 10, 15, 16) | (12, 12) | (1, 3, 4, 10, 12, 16, 13, 11, 6, 8, 21) |
| (12, 8) | (1, 3, 9, 13, 6, 10, 8, 18, 14) | (12, 8, 4) | (1, 3, 4, 13, 10, 12, 9, 14, 20, 11, 8) |
| (12, 4, 4) | (1, 5, 12, 6, 9, 17, 11, 8, 13) | (12, 6, 6) | (1, 3, 4, 15, 9, 10, 11, 12, 13, 21, 6) |
| (10, 10) | (1, 3, 12, 9, 11, 4, 7, 17, 18) | (12, 4, 4, 4) | (1, 3, 13, 15, 4, 6, 14, 10, 8, 20, 11) |
| (10, 6, 4) | (1, 4, 10, 11, 7, 18, 9, 14, 8) | (10, 10, 4) | (1, 3, 6, 15, 16, 8, 11, 12, 4, 7, 22) |
| (8, 8, 4) | (1, 3, 12, 10, 6, 7, 16, 9, 18) | (10, 8, 6) | (1, 3, 4, 11, 9, 16, 12, 13, 8, 10, 18) |
| (8, 6, 6) | (1, 4, 5, 13, 9, 10, 11, 7, 3) | (10, 6, 4, 4) | (1, 3, 4, 12, 15, 13, 10, 6, 9, 21, 11) |
| (8, 4, 4, 4) | (2, 5, 11, 4, 10, 12, 13, 9, 16) | (8, 8, 8) | (1, 3, 4, 11, 16, 8, 10, 12, 13, 9, 18) |
| (6, 6, 4, 4) | (2, 14, 8, 13, 7, 4, 18, 1, 15) | (8, 8, 4, 4) | (1, 3, 19, 14, 10, 7, 4, 12, 8, 6, 21) |
| (4, 4, 4, 4, 4) | (6, 16, 17, 11, 4, 8, 3, 5, 12) | (8, 6, 6, 4) | (1, 3, 9, 13, 22, 15, 7, 10, 11, 6, 8) |
| | | (8, 4, 4, 4, 4) | (1, 4, 16, 10, 13, 9, 5, 22, 17, 11, 20) |
| | | (6, 6, 6, 6) | (1, 5, 6, 12, 15, 21, 10, 11, 13, 8, 3) |
| | | (6, 6, 4, 4, 4) | (1, 6, 14, 7, 13, 15, 3, 12, 19, 22, 16) |
| | | (4, 4, 4, 4, 4, 4) | (5, 17, 9, 11, 13, 21, 1, 22, 16, 10, 3) |