

SUMS OF CERTAIN SERIES OF THE
RIEMANN ZETA FUNCTION[†]

By

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The object of the present paper is to investigate systematically several interesting families of summation formulas involving infinite series of the Riemann zeta function. Many of the various results, which are unified (and generalized) here in a remarkably simple manner, have received considerable attention in recent years. We also present a brief account of a number of analogous results associated with the (Hurwitz's) generalized zeta function.

1. INTRODUCTION AND DEFINITIONS

A classical (over two centuries old) theorem of Christian Goldbach (1690-1764), which was contained in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700-1782), has recently been revived as the following

PROBLEM (Shallit and Zikan [22]). Let \mathcal{S} be the set of nontrivial integer k th powers, i.e.,

$$(1.1) \quad \mathcal{S} = \{n^k \mid n \geq 2, k \geq 2\} = \{4, 8, 9, 16, 25, 27, 32, 36, \dots\}.$$

Show that

$$(1.2) \quad \sum_{\omega \in \mathcal{S}} (\omega-1)^{-1} = 1,$$

the sum being extended over all members ω of \mathcal{S} .

For the Riemann zeta function (see, e.g., Titchmarsh [26]):

$$(1.3) \quad \zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\operatorname{Re}(s) > 1), \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\operatorname{Re}(s) > 0; s \neq 1), \end{cases}$$

Goldbach's theorem (1.2) assumes the elegant form (cf. [22, p. 403]):

$$(1.4) \quad \sum_{k=2}^{\infty} \{\zeta(k)-1\} = 1.$$

Since $1 < \zeta(k) < 2$ for $k \geq 2$, we can easily rewrite the summation formula (1.4) in the more interesting form:

$$(1.5) \quad \sum_{k=2}^{\infty} f(\zeta(k)) = 1.$$

where $f(x) = x - [x]$ denotes the fractional part of the real number x . In fact, it is not difficult to show also that

$$(1.6) \quad \sum_{k=2}^{\infty} (-1)^k f(\zeta(k)) = \frac{1}{2},$$

$$(1.7) \quad \sum_{k=1}^{\infty} f(\zeta(2k)) = \frac{3}{4}, \quad \text{and} \quad \sum_{k=1}^{\infty} f(\zeta(2k+1)) = \frac{1}{4}.$$

Formula (1.5), and hence also (1.2) and (1.4), and its such interesting variations as (1.6) and (1.7) are, of course, equivalent to various (known or easily derivable) sums of double series (see, for details, Boole [4, p. 105, Exercise 10], Stieltjes [24, p. 300], Johnson [14, p. 479], Bromwich [5, p. 526, Example 6], Jordan [15, p. 340], Chrystal [7, p. 422, Exercise 18], and Hansen [13, p. 355]; see also Shallit and Zikan [22, p. 402]). In the present paper we aim at investigating systematically several related problems involving sums of series of $\zeta(s)$ and of the (Hurwitz's) generalized zeta function $\zeta(s, a)$ defined usually by (cf. [8, p. 24, Equation 1.10(1)])

$$(1.8) \quad \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\operatorname{Re}(s) > 1; a \neq 0, -1, -2, \dots),$$

so that, obviously,

$$(1.9) \quad \zeta(s, 1) = \zeta(s),$$

$$(1.10) \quad \zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s),$$

$$(1.11) \quad \frac{\partial}{\partial a} \{\zeta(s, a)\} = -s \zeta(s+1, a),$$

and

$$(1.12) \quad \zeta(s, a+N) = \zeta(s, a) - \sum_{n=0}^{N-1} \frac{1}{(n+a)^s} \quad (N = 1, 2, 3, \dots),$$

which, for $N = 1$, assumes a particularly simple (and useful) form.

2. UNIFICATIONS (AND GENERALIZATIONS) OF THE SUMMATION FORMULAS (1.5) AND (1.6)

We begin by recalling the familiar binomial expansion:

$$(2.1) \quad \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} t^k = (1-t)^{-\lambda}, \quad |t| < 1,$$

where, as usual,

$$(2.2) \quad \binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!} \quad (n = 1, 2, 3, \dots)$$

for an arbitrary (real or complex) parameter λ . In view of (2.1), it is easily seen from the definitions (1.3) and (1.8) that

$$(2.3) \quad \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \{\zeta(\lambda+k)-1\} t^k = \zeta(\lambda, 2-t), \quad |t| < 2$$

or, equivalently, that (cf. Ramanujan [19, p. 78, Equation (15)] and Apostol [2, p. 240, Equation (7)])

$$(2.4) \quad \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \zeta(\lambda+k) t^k = \zeta(\lambda, 1-t), \quad |t| < 1.$$

For fixed $\lambda \neq 1$, the series in (2.3) and (2.4) converge absolutely for $|t| < 2$ and $|t| < 1$, respectively. Thus, by the principle of analytic continuation, formulas (2.3) and (2.4) are valid for all values of λ .

Formula (2.3) provides a unification (and generalization) of (1.5) and (1.6), and indeed also of a fairly large number of other summation formulas scattered in the literature. For example, in view of the relationships (1.9) and (1.12), (2.3) with $t = 1$ gives us a known result (cf. [13, p. 356, Equation (54.4.1)]) which generalizes (1.5), and a special case of (2.3) when $t = -1$ yields another known result (cf. [13, p. 356, Equation (54.4.2)]) which generalizes (1.6).

Several additional consequences of the general summation formulas (2.3) and (2.4) are worthy of note. First of all, replace the summation index k in (2.3) by $k + 1$, and set $\lambda = s - 1$, so that

$$(2.5) \quad \sum_{k=0}^{\infty} \binom{s+k-1}{k+1} \{\zeta(s+k)-1\} t^{k+1} = \zeta(s-1, 2-t) - \zeta(s-1) + 1, \quad |t| < 2,$$

which, for $t = 1$, reduces immediately to the following alternative form of the aforementioned generalization of (1.5):

$$(2.6) \quad \sum_{k=0}^{\infty} \binom{s+k-1}{k+1} \{\zeta(s+k)-1\} = 1.$$

Now it follows from the definition (2.2) that

$$(2.7) \quad \binom{s+k-1}{k+1} = \frac{(s-1)(s)_k}{(k+1)!}, \quad k \geq 0,$$

where, for convenience,

$$(2.8) \quad (s)_0 = 1 \quad \text{and} \quad (s)_k = s(s+1)(s+2) \cdots (s+k-1), \quad k = 1, 2, 3, \dots$$

Thus the formula (2.6) can be rewritten in the well-known form (cf. Landau [16, p. 274, Equation (3)] and Titchmarsh [26, p. 33, Equation (2.14.1)]):

$$(2.9) \quad \zeta(s) = 1 + \frac{1}{s-1} - \sum_{k=1}^{\infty} \frac{(s)_k}{(k+1)!} \{\zeta(s+k)-1\},$$

which is usually attributed to Edmund (Georg Hermann) Landau (1877-1938).

For $t = -1$, (2.5) readily yields

$$(2.10) \quad \zeta(s) = 1 + \frac{1}{2^{s-1}} \frac{1}{s-1} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(s)_k}{(k+1)!} \{\zeta(s+k)-1\},$$

which provides an interesting (presumably new) companion of Landau's formula (2.9).

Setting $t = \frac{1}{2}$ in (2.5), and making use of (1.12) with $a = \frac{1}{2}$ and $N = 1$, we obtain another series representation for $\zeta(s)$:

$$(2.11) \quad \zeta(s) = \frac{2^s - 1}{2^s - 2} + \frac{1}{2^s - 2} \sum_{k=1}^{\infty} \frac{(s)_k}{k! 2^k} \{\zeta(s+k)-1\},$$

which is believed to be new.

In their special cases when $s = 2$, (2.9) and (2.10) reduce simply to the summation formulas (1.5) and (1.6), respectively, while (2.10) similarly yields the elegant sum:

$$(2.12) \quad \sum_{k=2}^{\infty} \frac{k-1}{2^k} \{\zeta(k)-1\} = \frac{\pi^2}{8} - 1.$$

Next we turn to the summation formula (2.4) which (for $\lambda = s$ and $t = \frac{1}{2}$) readily yields the familiar result:

$$(2.13) \quad (1-2^{1-s})\zeta(s) = \sum_{k=1}^{\infty} \frac{(s)_k}{k!} \frac{\zeta(s+k)}{2^{s+k}},$$

which is attributed to Ramaswami (cf. [20, p. 166] and [26, p. 33, Equation (2.14.2)]). Furthermore, in its special case when $\lambda = s$ and $t = -\frac{1}{2}$, (2.4) gives us the following companion of (2.13):

$$(2.14) \quad (1-2^{1-s})\zeta(s) = 1 - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(s)_k}{k!} \frac{\zeta(s+k)}{2^{s+k}},$$

which was also given by Ramaswami [20, p. 166].

Formulas (2.9) and (2.13) were rederived, using Eulerian integrals for Γ -functions, by Menon [18].

In case we add (2.4) to itself (with t replaced by $-t$), we obtain a known summation formula (cf. [13, p. 357, Equation (54.6.3)]):

$$(2.15) \quad \sum_{k=0}^{\infty} \binom{\lambda+2k-1}{2k} \zeta(\lambda+2k) t^{2k} = \frac{1}{2} \{ \zeta(\lambda, 1-t) + \zeta(\lambda, 1+t) \}, \quad |t| < 1,$$

while a similar subtraction yields

$$(2.16) \quad \sum_{k=0}^{\infty} \binom{\lambda+2k}{2k+1} \zeta(\lambda+2k+1) t^{2k+1} = \frac{1}{2} \{ \zeta(\lambda, 1-t) - \zeta(\lambda, 1+t) \}, \quad |t| < 1.$$

Various interesting special cases of (2.15) and (2.16) are also given in the literature. In particular, the special cases of (2.15) when

$$t = \frac{1}{2}, \quad t = \frac{1}{3}, \quad \text{and} \quad t = \frac{1}{6}$$

were considered by Ramaswami [20, p. 167, Equations (1), (3), and (4)] who also gave a special case of (2.16) when $t = \frac{1}{2}$ [20, p. 167, Equation (2)], and by Apostol [2] who proved various generalizations of Ramaswami's results.

By assigning suitable numerical values to the variable s in some of the aforementioned special cases of (2.15) and (2.16), Ramaswami [20] also evaluated a number of special sums including, for example,

$$(2.17) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2^k + 1} 2^{-2k} = \log 2 - \gamma,$$

where γ denotes the Euler-Mascheroni constant defined by

$$(2.18) \quad \gamma = \lim_{n \rightarrow \infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right\} \cong 0.5772156649\dots$$

Formula (2.17) is contained in a memoir of 1781 by Leonhard Euler (1707-1783) (cf. Glaisher [9, p. 28, Equation (8)]); it was rederived by Wilton [30, p. 92] who also gave a number of other sums. Furthermore, since

$$(2.19) \quad \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2^k - 1} = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right), \quad |x| < 1,$$

so that, for $x = \frac{1}{2}$,

$$(2.20) \quad \sum_{k=1}^{\infty} \frac{2^{-2k}}{2^k + 1} = \log 3 - 1,$$

the summation formula (2.17) is an immediate consequence of the following result (also contained in Euler's memoir of 1781 already referred to):

$$(2.21) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{(2k+1)2^{2k}} = 1 - \gamma - \log \frac{3}{2},$$

which was rederived in 1826 by Legendre ([17, p. 434]; see also Stieltjes [24, p. 302], and Glaisher [9, p. 28, Equation (9)] who recalls both (2.17) and (2.21) erroneously). Legendre [17, p. 434] also showed that

$$(2.22) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{2^k + 1} = 1 - \gamma - \frac{1}{2} \log 2.$$

Johnson [14] presented alternative (direct) proofs of the summation formulas (2.21) and (2.22), and obtained a number of additional results including, for example, the sum (Johnson [14, p. 480, Equation (8)]); see also Verma and Kaur [28, p. 181, Equation (D)]:

$$(2.23) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k) - 1}{k} = \log 2.$$

Formulas (2.22) and (2.23), together, imply the well-known result (contained in the aforementioned 1781 memoir by Euler):

$$(2.24) \quad \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} = 1 - \gamma,$$

which has appeared in several subsequent works by, for example, Glaisher [9, p. 28, Equation (4)], Johnson [14, p. 478, Equation (4)], Bromwich [5, p. 526, Example 6], Wilton [30, p. 93], Barnes and Kaufman [3] where it is posed as a problem, and Verma and Kaur [28, p. 181, Equation (A)] where it is rederived in a standard manner.

We conclude this section by recalling the formula (cf. Glaisher [9, p. 27, Equation (1)] and Johnson [14, p. 478, Equation (3)]):

$$(2.25) \quad \sum_{k=2}^{\infty} \frac{k-1}{k} \{\zeta(k)-1\} = \gamma,$$

which was given in Euler's memoir of 1769, and also the following results contained in Wilton's work [30]:

$$(2.26) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} = \log(2\pi) - 1;$$

$$(2.27) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2^{2k} k} = \log\left(\frac{1}{2} \pi\right).$$

Obviously, Euler's formula (2.25) follows immediately upon subtracting (2.24) from (1.4). Formula (2.27), on the other hand, complements such sums as (2.17) and (2.21).

3. ADDITIONAL CONSEQUENCES OF THE SUMMATION FORMULAS (2.3) AND (2.4)

Many of the summation formulas mentioned in the preceding sections would follow readily by suitably specializing the following straightforward consequences of (2.3):

$$(3.1) \quad \sum_{k=0}^{\infty} \binom{\lambda+2k-1}{2k} \{\zeta(\lambda+2k)-1\} t^{2k} = \frac{1}{2} \{\zeta(\lambda, 2-t) + \zeta(\lambda, 2+t)\}, \quad |t| < 2,$$

$$(3.2) \quad \sum_{k=0}^{\infty} \binom{\lambda+2k}{2k+1} \{\zeta(\lambda+2k+1)-1\} t^{2k+1} = \frac{1}{2} \{\zeta(\lambda, 2-t) - \zeta(\lambda, 2+t)\}, \quad |t| < 2,$$

which are derivable also from (2.15) and (2.16), respectively.

Now we replace the summation index k in (2.3) and (2.4) by $k+2$, set $\lambda = s-1$, and divide each side by t^2 . If we differentiate the resulting equations with respect to t , using the formulas (1.11) and (2.7), we finally obtain

$$(3.3) \quad \sum_{k=1}^{\infty} \frac{k \binom{s}{k+1}}{(k+2)!} \{\zeta(s+k+1)-1\} t^{k-1} \\ = t^{-2} \{\zeta(s, 2-t) + \zeta(s)-1\} - \frac{2t^{-3}}{s-1} \{\zeta(s-1, 2-t) - \zeta(s-1)+1\}, \quad 0 < |t| < 2,$$

and

$$(3.4) \quad \sum_{k=1}^{\infty} \frac{k \binom{s}{k+1}}{(k+2)!} \zeta(s+k+1) t^{k-1} = t^{-2} \{\zeta(s, 1-t) + \zeta(s)\} \\ - \frac{2t^{-3}}{s-1} \{\zeta(s-1, 1-t) - \zeta(s-1)\}, \quad 0 < |t| < 1,$$

respectively.

For $t = -1$, (3.3) readily yields

$$(3.5) \quad \zeta(s) = 1 + \frac{1}{2^{s+1}} \frac{s+3}{s-1} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \binom{s}{k+1}}{(k+2)!} \{\zeta(s+k+1)-1\},$$

while (3.4) formally reduces, when $t \rightarrow -1$, to the sum:

$$(3.6) \quad \zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k(s)_{k+1}}{(k+2)!} \zeta(s+k+1), \quad \text{Re}(s) < 1.$$

Formula (3.5) follows also from (2.10). As a matter of fact, Formulas (3.5) and (3.6) happen to be the main results in a recent paper by Singh and Verma [23] who prove each of these results in a markedly different manner.

By assigning suitable special values to the variable t in (3.3) and (3.4), we can deduce a large number of sums of series involving the zeta function. For example, for $t = 1$, (3.3) immediately yields the series representation:

$$(3.7) \quad \zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(k-1)(s)_k}{(k+1)!} \{\zeta(s+k)-1\},$$

which is deducible also from Landau's formula (2.9).

4. SUMMATION FORMULAS INVOLVING SERIES OF $\zeta(k)/k$

In the theory of Γ -functions, it is fairly well known that (see, e.g., Erdélyi et al. [8, p. 45, Equation 1.17(2)] and Jordan [15, p. 62, Equation (2)])

$$(4.1) \quad \log \Gamma(1+t) = -\gamma t + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{t^k}{k}, \quad |t| < 1,$$

or, equivalently, that (cf. Abramowitz and Stegun [1, p. 256, Equation (6.1.35)])

$$(4.2) \quad \log \Gamma(2+t) = (1-\gamma)t + \sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} \frac{t^k}{k}, \quad |t| < 2.$$

For $t \rightarrow 1$, (4.1) reduces immediately to a classical result (see Jordan [15, p. 62] and Erdélyi et al. [8, p. 45, Equation 1.17(3)]), and (4.2) with $t = 1$ yields another sum (cf. Verma [27]; see also Verma and Kaur [28, p. 182, Equation (1)]) which is, of course, equivalent to the aforementioned classical result.

The special case of (4.2) when $t = -1$ gives us the well-known result (2.24) which, in conjunction with (4.2) with $t = 1$, would immediately yield the summation formulas (2.22) and (2.23). Furthermore, the obvious special cases of (4.1) when $t = \pm \frac{1}{2}$, together, yield the results (2.17) and (2.27), and indeed also the summation formula (2.21).

Finally, we set $t = \pm \frac{3}{2}$ in (4.2), and we obtain the sums:

$$(4.3) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k} \left(\frac{3}{2}\right)^k = \log \frac{15}{8} + \frac{1}{2} \log \pi - \frac{3}{2} (1-\gamma)$$

and

$$(4.4) \quad \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} \left(\frac{3}{2}\right)^k = \frac{1}{2} \log \pi + \frac{3}{2} (1-\gamma),$$

which do not seem to have been recorded earlier.

With a view to simplifying the derivation of such summation formulas as (2.17), (2.21), (2.22), (2.23), and (2.27), we can make use of some well-known consequences of (4.1) and (4.2) recorded, among others, by Hansen [13, p. 356, Equations (54.5.3) and (54.5.8)].

5. SUMMATION FORMULAS INVOLVING SERIES OF $\zeta(k)/(k+1)$

By differentiating (4.1) with respect to t , we obtain (see, e.g., Erdélyi et al. [8, p. 45, Equation 1.17(5)] and Jordan [15, p. 327, Equation (2)])

$$(5.1) \quad \psi(1+t) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) t^{k-1}, \quad |t| < 1,$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Now multiply (5.1) by t and integrate both sides between $t = 0$ and $t = z$; we thus find that

$$(5.2) \quad \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k+1}}{k+1} = z \log \Gamma(1+z) + \frac{1}{2} \gamma z^2 - \int_0^z \log \Gamma(1+t) dt, \quad |z| < 1.$$

In precisely the same manner, we find from (4.2) that

$$(5.3) \quad \sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} \frac{z^{k+1}}{k+1} \\ = z \log \Gamma(2+z) + \frac{1}{2} (\gamma-1) z^2 - \int_0^z \log \Gamma(2+t) dt, \quad |z| < 2.$$

Since [11, p. 661, Entry 6.441(1)]

$$(5.4) \quad \int_p^{p+1} \log \Gamma(q+t) dt = \frac{1}{2} \log(2\pi) + (p+q) \{\log(p+q)-1\},$$

it is readily seen from (5.2) with $z \rightarrow 1$ that

$$(5.5) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} = 1 + \frac{1}{2} \gamma - \frac{1}{2} \log(2\pi),$$

which was proved by Suryanarayana [25], and again by Singh and Verma [23, p. 5, Section 4]. The method of derivation of (5.5) by these earlier workers is fairly standard in the theory of the Riemann zeta function. By the same method, Suryanarayana [25, p. 143, Equation (14)] claimed to have summed the obviously divergent alternating series

$$\sum_{k=2}^{\infty} (-1)^k \zeta(k),$$

whose corrected (convergent) version is precisely the same as the well-known result (1.6).

Applying the elementary integral (5.4), the special cases of (5.3) when $z = \pm 1$ can easily be rewritten in the forms:

$$(5.6) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k + 1} = \frac{3}{2} + \frac{1}{2} \gamma - \frac{1}{2} \log(8\pi)$$

and

$$(5.7) \quad \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k + 1} = \frac{3}{2} - \frac{1}{2} \gamma - \frac{1}{2} \log(2\pi),$$

which, together, yield the following sums:

$$(5.8) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k) - 1}{2k + 1} = \frac{3}{2} - \frac{1}{2} \log(4\pi),$$

$$(5.9) \quad \sum_{k=2}^{\infty} \frac{\zeta(2k-1) - 1}{k} = \log 2 - \gamma.$$

Formula (5.6) follows trivially from the known result (5.5) which also yields the equivalent sum:

$$(5.10) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) + 1}{k + 1} = \frac{1}{2} + \frac{1}{2} \gamma - \frac{1}{2} \log\left(\frac{1}{2} \pi\right).$$

Formulas (5.6) and (5.7), and indeed also the well-known result (2.24), happen to be the main results in a recent paper by Verma and Kaur [28, p. 181, Equations (A), (B), and (C)] who also state an erroneous version of the sum (5.9) above [28, p. 181, Equation (F)]. Furthermore, the summation formulas (5.6) and (5.7) appeared more recently as a problem (see [6]).

Such summation formulas as (5.8) would follow more rapidly if we multiply both sides of (5.1) by t and integrate the resulting equation from $t = -z$ to $t = z$. We thus find from (5.1) that (cf., e.g., [13, p. 356, Equation (54.5.4)] for an alternate form)

$$(5.11) \quad \sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{2k + 1} = \frac{1}{2} \log\left(\frac{\pi z}{\sin \pi z}\right) - \frac{1}{2z} \int_{-z}^z \log \Gamma(1+t) dt, \quad 0 < |z| < 1.$$

As an example of the use of the summation formula (5.11), we set $z = \frac{1}{2}$ and evaluate the resulting integral by means of (5.4). We thus obtain the sum:

$$(5.12) \quad \sum_{k=1}^{\infty} \zeta(2k) \frac{2^{-2k}}{2k+1} = \frac{1}{2} - \frac{1}{2} \log 2.$$

Finally, we subtract the series (5.12) from the series (5.8), and we find that

$$(5.13) \quad \sum_{k=1}^{\infty} \frac{(1-2^{-2k})\zeta(2k) - 1}{2k+1} = 1 - \frac{1}{2} \log(2\pi)$$

or, equivalently, that (see Robbins [21])

$$(5.14) \quad \frac{1}{2} \log(2\pi) = 1 - \sum_{m=1}^{\infty} \left\{ \frac{1}{3(2m+1)^2} + \frac{1}{5(2m+1)^4} + \dots \right\}.$$

6. MISCELLANEOUS SUMMATION FORMULAS AND GENERALIZATIONS

Making use of the elementary identity:

$$(6.1) \quad \frac{\lambda k + \mu}{k(k+1)} = \frac{\mu}{k} + \frac{\lambda - \mu}{k+1},$$

the various summation formulas established in the preceding sections can be applied to deduce sums of series involving, for example, $\zeta(k)/\{k(k+1)\}$. In particular, the summation formulas (2.24) and (5.7) lead us in this way to the sum:

$$(6.2) \quad \sum_{k=2}^{\infty} \frac{\lambda k + \mu}{k(k+1)} \{\zeta(k) - 1\} = \mu - \frac{1}{2} (\lambda + \mu)\gamma + \frac{1}{2} (\lambda - \mu) \{3 - \log(2\pi)\},$$

while (4.4) and (5.6) would similarly yield the sum:

$$(6.3) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\lambda k + \mu}{k(k+1)} \{\zeta(k) - 1\} = \mu(\log 2 - 1) + \frac{1}{2} (\lambda + \mu)\gamma + \frac{1}{2} (\lambda - \mu) \{3 - \log(8\pi)\}.$$

By assigning suitable special values to the arbitrary constants λ and μ , we can obtain a number of interesting summation formulas as immediate consequences of (6.2) and (6.3). For instance, the special case of (6.2) when $\lambda = -\mu = 1$ yields a known sum (see Chrystal [7, p. 372, Equation (18)]).

Alternatively, with a view to obtaining sums of series involving $\zeta(k)/\{k(k+1)\}$ as consequences of the well-known formulas (4.1) and (4.2), we merely integrate both sides of (4.1) and (4.2) from $t = 0$ to $t = z$, and we get

$$(6.4) \quad \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^k}{k(k+1)} = \frac{1}{2} \gamma z + \frac{1}{z} \int_0^z \log \Gamma(1+t) dt, \quad 0 < |z| < 1;$$

$$(6.5) \quad \sum_{k=2}^{\infty} (-1)^k \{\zeta(k) - 1\} \frac{z^k}{k(k+1)} = \frac{1}{2} (\gamma - 1) z + \frac{1}{z} \int_0^z \log \Gamma(2+t) dt, \quad 0 < |z| < 2.$$

On the other hand, by integrating (4.1) from $t = -z$ to $t = z$, we have (cf. e.g., [13, p. 356, Equation (54.5.5)] for an alternate form)

$$(6.6) \quad \sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{k(2k+1)} = \frac{1}{z} \int_{-z}^z \log \Gamma(1+t) dt, \quad 0 < |z| < 1,$$

which obviously contains Wilton's formula (2.26) as a special case.

Multiplying (5.3) by λ/z , and (6.5) by μ , and adding the resulting equations, we obtain the following unification (and generalization) of the summation formulas (6.2) and (6.3), and indeed also of (6.5):

$$(6.7) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\lambda k + \mu}{k(k+1)} \{\zeta(k) - 1\} z^k \\ = \lambda \log \Gamma(2+z) + \frac{1}{2} (\lambda + \mu) (\gamma - 1) z - \frac{\lambda - \mu}{z} \int_0^z \log \Gamma(2+t) dt, \quad 0 < |z| < 2,$$

which, in view of (5.4), would yield (6.2) and (6.3) in its special cases when $z = -1$ and $z = 1$, respectively.

In the special case of (6.6) when $z = \frac{1}{2}$, if we evaluate the resulting integral by means of (5.4), we shall readily obtain a summation formula given by Wilton [30, p. 91]. Wilton's result was posed as a problem over four decades later (see [12]); it follows immediately upon setting $a = 1$ in Burnside's formula (cf., e.g., Wilton [30, p. 91, Equation (3)]); see also Erdélyi et al. [8, p. 48, Equation 1.18(11)]:

$$(6.8) \quad \sum_{k=1}^{\infty} \zeta(2k, a) \frac{2^{-2k}}{k(2k+1)} \\ = \log(2\pi) + (2a-1) \left\{ \log\left(a - \frac{1}{2}\right) - 1 \right\} - 2 \log \Gamma(a), \quad \operatorname{Re}(a) > -\frac{1}{2},$$

involving the generalized zeta function defined by (1.8). As a matter of fact, Wilton [30] rederived Burnside's formula (6.8) as a consequence of the following straightforward generalization of the expansions (2.3) and (2.4):

$$(6.9) \quad \sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} \zeta(\lambda+k, a) t^k = \zeta(\lambda, a-t), \quad |t| < |a|.$$

In terms of the generalized zeta function $\zeta(s, a)$, it is also known that (cf. Whittaker and Watson [29, p. 276]; see also Gradshteyn and Ryzhik [11, p. 1074, Entry 9.532])

$$(6.10) \quad \sum_{k=2}^{\infty} (-1)^k \zeta(k, a) \frac{t^k}{k} = \log \Gamma(a+t) - \log \Gamma(a) - t\psi(a), \quad |t| < |a|.$$

Since $\psi(1) = -\gamma$, (6.10) would reduce immediately to (4.1) and (4.2) upon setting $a = 1$ and $a = 2$, respectively.

By employing the rather elementary techniques illustrated fairly fully in this section, and in the preceding sections, we can easily derive appropriate generalizations of the various summation formulas considered in this paper as useful consequences of (6.9) and (6.10). In addition to the generalizations recorded by Hansen [13, p. 358, Equations (54.11.2), (54.11.3), and (54.11.4)], we have

$$(6.11) \quad \sum_{k=2}^{\infty} (-1)^k \zeta(k, a) \frac{z^k}{k(k+1)} = \frac{1}{z} \int_0^z \log \Gamma(a+t) dt - \log \Gamma(a) - \frac{1}{2} z\psi(a), \quad 0 < |z| < |a|.$$

It is easily seen from (6.10) and (6.11) that

$$(6.12) \quad \sum_{k=2}^{\infty} (-1)^k \frac{\lambda k + \mu}{k(k+1)} \zeta(k, a) z^k = \lambda \log \Gamma(a+z) - \mu \log \Gamma(a) \\ - \frac{1}{2} (\lambda + \mu) z\psi(a) - \frac{\lambda - \mu}{z} \int_0^z \log \Gamma(a+t) dt, \quad 0 < |z| < |a|.$$

Setting $z = -1$ in (6.12), and evaluating the resulting integral by means of (5.4), we obtain the summation formula:

$$(6.13) \quad \sum_{k=2}^{\infty} \frac{\lambda k + \mu}{k(k+1)} \zeta(k, a) = \lambda \log \Gamma(a-1) - \mu \log \Gamma(a) + \frac{1}{2} (\lambda + \mu)\psi(a) \\ - \frac{1}{2} (\lambda - \mu) \log(2\pi) - (\lambda - \mu)(a-1) \{ \log(a-1) - 1 \}, \quad |\arg(a-1)| < \pi.$$

For $\lambda = -\mu = 1$, this last result (6.13) reduces immediately to Binet's formula (cf. Whittaker and Watson [29, p. 261, Example 18]; see also Erdélyi et al. [8, p. 48, Equation 1.18(10)]):

$$(6.14) \quad \sum_{k=2}^{\infty} \frac{k-1}{k(k+1)} \zeta(k, a) = 2 \log \Gamma(a) - (2a-1) \log(a-1) \\ - \log(2\pi) + 2(a-1), \quad |\arg(a-1)| < \pi,$$

which, for $a = 2$, yields the aforementioned known sum [7, p. 372, Equation (18)].

The summation formula

$$(6.15) \quad \sum_{k=1}^{\infty} \frac{\lambda k + \mu}{k(2k+1)} \zeta(2k, a) z^{2k} = \frac{1}{2} \lambda \{ \log \Gamma(a+z) + \log \Gamma(a-z) \} \\ - 2\mu \log \Gamma(a) - \frac{\lambda - 2\mu}{2z} \int_{-z}^z \log \Gamma(a+t) dt, \quad 0 < |z| < |a|$$

would follow from known results like (6.10) and (6.11) [13, p. 358, Equations (54.11.2) and (54.11.4)] in essentially the same manner as the sum (6.12). Setting $z = \frac{1}{2}$ in (6.15), and evaluating the resulting integral by means of (5.4), we deduce the following interesting generalization of several results including, for example, Burnside's formula (6.8):

$$(6.16) \quad \sum_{k=1}^{\infty} \frac{\lambda k + \mu}{k(2k+1)} \zeta(2k, a) 2^{-2k} = \lambda \log \Gamma(a + \frac{1}{2}) - 2\mu \log \Gamma(a) - \frac{1}{2} (\lambda - 2\mu) \log(2\pi) \\ - \{ \lambda a - \mu(2a-1) \} \log(a - \frac{1}{2}) + (\lambda - 2\mu) (a - \frac{1}{2}), \quad \text{Re}(a) > -\frac{1}{2},$$

which indeed yields (6.8) for $\lambda = \mu - 1 = 0$.

Finally, by integrating a known result [13, p. 358, Equation (54.11.3)] from $t = 0$ to $t = z$, we obtain the summation formula:

$$(6.17) \quad \sum_{k=1}^{\infty} \zeta(2k+1, a) \frac{z^{2k+2}}{(k+1)(2k+1)} = \int_0^z \{ \log \Gamma(a-t) - \log \Gamma(a+t) \} dt + z^2 \psi(a), \quad |z| < |a|,$$

which, for $z = 1$, yields

$$(6.18) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k+1, a)}{(k+1)(2k+1)} = (a-1) \log(a-1) - a \log a + \psi(a) + 1, \quad |\arg(a-1)| < \pi,$$

where we have made use of the elementary integral (5.4).

This last result (6.18) reduces, when $a \rightarrow 1$, to an elegant summation formula considered, for instance, by Glaisher [10, p. 9, §18] and Ramanujan [19, p. 73].

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