SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

The object of the present paper is to prove various distortion theorems for the fractional calculus of functions in the subclasses $\mathcal{R}[\alpha,\beta]$ and $\mathcal{C}[\alpha,\beta]$ consisting of prestarlike and normalized analytic functions with negative coefficients. Furthermore, distortion theorems involving a generalized fractional integral operator for functions in the subclasses $\mathcal{R}[\alpha,\beta]$ and $\mathcal{C}[\alpha,\beta]$ are given.

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1. Introduction, Definitions, and Preliminaries

Let $\mathcal{S}$ denote the class of (*normalized*) functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.1}$$

which are analytic and univalent in the *open* unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z)$ in $\mathcal{S}$ is said to be *starlike of order* α in $\mathcal{U}$ if and only if

$$\Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\quad 0 \leq \alpha < 1). \tag{1.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α in $\mathcal{U}$.

A function $f(z)$ in $\mathcal{S}$ is said to be *convex of order* α in $\mathcal{U}$ if and only if

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U};\quad 0 \leq \alpha < 1). \tag{1.3}$$

And we denote by $\mathcal{K}(\alpha)$ the class of all functions in $\mathcal{S}$ which are convex of order α in $\mathcal{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow z\, f'(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leq \alpha < 1). \tag{1.4}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were first introduced by Robertson [7], and were studied subsequently by Schild [8], MacGregor [1], and Pinchuk [6] (see also Srivastava and Owa [13]).

The function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (z \in \mathcal{U};\quad 0 \leq \alpha < 1) \tag{1.5}$$

is the well-known extremal function for $\mathcal{S}^*(\alpha)$. Setting

$$c_n(\alpha) = \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} \qquad (n \in \mathbb{N} \setminus \{1\};\quad \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.6}$$

$S_\alpha(z)$ can be written in the form:

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} c_n(\alpha)\, z^n. \tag{1.7}$$

Then we can see that $c_n(\alpha)$ is a decreasing function in α $(0 \leq \alpha < 1)$ and that

$$\lim_{n \longrightarrow \infty} c_n(\alpha) = \begin{cases} \infty & (\alpha < \frac{1}{2}) \\ 1 & (\alpha = \frac{1}{2}) \\ 0 & (\alpha > \frac{1}{2}) . \end{cases} \tag{1.8}$$

Let $f * g(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n \, z^n, \tag{1.9}$$

then

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n \, b_n \, z^n. \tag{1.10}$$

Let $\mathcal{R}(\alpha, \beta)$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ such that

$$f * \mathcal{S}_\alpha(z) \in \mathcal{S}^*(\beta) \qquad (0 \leq \alpha < 1; \quad 0 \leq \beta < 1).$$

Further, let $\mathcal{C}(\alpha, \beta)$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ such that

$$z \, f'(z) \in \mathcal{R}(\alpha, \beta) \qquad (0 \leq \alpha < 1; \quad 0 \leq \beta < 1).$$

The class $\mathcal{R}(\alpha, \beta)$ of *α-prestarlike functions of order β* was introduced by Sheil-Small *et al.* [9].

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n \, z^n \qquad (a_n \geq 0). \tag{1.11}$$

We denote by $\mathcal{R}[\alpha, \beta]$ and $\mathcal{C}[\alpha, \beta]$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{R}(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$ with the class $\mathcal{T}$, that is,

$$\mathcal{R}[\alpha, \beta] := \mathcal{R}(\alpha, \beta) \cap \mathcal{T} \tag{1.12}$$

and

$$\mathcal{C}[\alpha, \beta] := \mathcal{C}(\alpha, \beta) \cap \mathcal{T}. \tag{1.13}$$

The class $\mathcal{R}[\alpha, \beta]$ was studied recently by Silverman and Silvia [10] and Uralegaddi and Sarangi [15]. The class $\mathcal{C}[\alpha, \beta]$ was studied by Owa and Uralegaddi [5].

The following known results will be required in our investigation of the classes $\mathcal{R}[\alpha, \beta]$ and $\mathcal{C}[\alpha, \beta]$.

Lemma 1 (Silverman and Silvia [10]). *Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $\mathcal{R}[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} (n - \beta)\, c_n(\alpha)\, a_n \leq 1 - \beta. \tag{1.14}$$

The result is sharp.

Lemma 2 (Owa and Uralegaddi [5]). *Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $\mathcal{C}[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \beta)\, c_n(\alpha)\, a_n \leq 1 - \beta. \tag{1.15}$$

The result is sharp.

2. Operators of Fractional Calculus

We begin by recalling the following definitions of operators of fractional calculus (that is, fractional derivatives and fractional integrals) which were used by Owa ([2], [3]) and (more recently) by Srivastava and Owa [11] (see also Srivastava and Owa [12, p. 343]).

Definition 1. The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta \qquad (\lambda > 0), \tag{2.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta \qquad (0 \leq \lambda < 1), \tag{2.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the *fractional derivatve of order* $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n}\, D_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{2.3}$$

Our first set of distortion properties of the class $\mathcal{R}[\alpha,\beta]$, involving the fractional integral operator $D_z^{-\lambda}$, is contained in

Theorem 1. *Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then*

$$\left|D_z^{-\lambda}\, f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{1-\beta}{(2-\beta)(1-\alpha)(2+\lambda)}\, |z|\right\} \tag{2.4}$$

and

$$\left|D_z^{-\lambda}\, f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{1-\beta}{(2-\beta)(1-\alpha)(2+\lambda)}\, |z|\right\} \tag{2.5}$$

for $\lambda > 0$ and $z \in \mathcal{U}$. The results are sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\lambda)\, z^{-\lambda}\, D_z^{-\lambda} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}\, a_n\, z^n \\ &= z - \sum_{n=2}^{\infty} \Phi(n)\, a_n\, z^n, \end{aligned} \tag{2.6}$$

where

$$\Phi(n) = \frac{\Gamma(n+1)\,\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} \qquad (\lambda > 0; \quad n \in \mathbb{N} \setminus \{1\}). \tag{2.7}$$

We note that

$$0 < \Phi(n) \leq \Phi(2) = \frac{2}{2+\lambda} \qquad (\lambda > 0; \quad n \in \mathbb{N} \setminus \{1\}) \tag{2.8}$$

and that

$$c_{n+1}(\alpha) \geq c_n(\alpha) \qquad \left(0 \leq \alpha \leq \frac{1}{2}; \quad n \in \mathbb{N} \setminus \{1\}\right),$$

by means of (1.6). Consequently, by using Lemma 1, we have

$$(2-\beta)\, c_2(\alpha) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} (n-\beta)\, c_n(\alpha)\, a_n \leq 1-\beta,$$

which implies that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\beta}{2(2-\beta)(1-\alpha)}. \tag{2.9}$$

Therefore, by using (2.8) and (2.9), we find that

$$\begin{aligned} |F(z)| &\geq |z| - \Phi(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{1-\beta}{(2-\beta)(1-\alpha)(2+\lambda)}\,|z|^2 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} |F(z)| &\leq |z| + \Phi(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{1-\beta}{(2-\beta)(1-\alpha)(2+\lambda)}\,|z|^2, \end{aligned} \tag{2.11}$$

which prove the inequalities (2.4) and (2.5) of Theorem 1.

The equalities in (2.4) and (2.5) are attained by the function $f(z)$ given by

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{1-\beta}{(2-\beta)(1-\alpha)(2+\lambda)}\, z \right\} \tag{2.12}$$

or, equivalently, by

$$f(z) = z - \frac{1-\beta}{2(2-\beta)(1-\alpha)}\, z^2. \tag{2.13}$$

Thus we complete the proof of Theorem 1.

In the same way, we can prove the following distortion theorem for the class $\mathcal{C}[\alpha,\beta]$ by using Lemma 2 instead of Lemma 1.

Theorem 2. *Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{C}[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then*

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{1-\beta}{2(2-\beta)(1-\alpha)(2+\lambda)}\, |z| \right\} \tag{2.14}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{1-\beta}{2(2-\beta)(1-\alpha)(2+\lambda)} |z|\right\} \tag{2.15}$$

for $\lambda > 0$ and $z \in \mathcal{U}$. The results are sharp for the function $f(z)$ given by

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{1-\beta}{2(2-\beta)(1-\alpha)(2+\lambda)} z\right\} \tag{2.16}$$

or, equivalently, by

$$f(z) = z - \frac{1-\beta}{4(2-\beta)(1-\alpha)} z^2. \tag{2.17}$$

Next we state and prove

Theorem 3. *Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}[\alpha, \beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then*

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1-\beta}{(2-\beta)(1-\alpha)(2-\lambda)} |z|\right\} \tag{2.18}$$

and

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{1-\beta}{(2-\beta)(1-\alpha)(2-\lambda)} |z|\right\} \tag{2.19}$$

for $0 \leq \lambda < 1$ and $z \in U$. The results are sharp.

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\lambda)\, z^{\lambda}\, D_z^{\lambda} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\, a_n\, z^n \\ &= z - \sum_{n=2}^{\infty} \Psi(n)\, n\, a_n\, z^n, \end{aligned} \tag{2.20}$$

where

$$\Psi(n) = \frac{\Gamma(n)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \qquad (0 \leq \lambda < 1; \quad n \in \mathbb{N} \setminus \{1\}). \tag{2.21}$$

From (2.9) and Lemma 1, it follows that

$$2(1-\alpha) \sum_{n=2}^{\infty} n\, a_n \leq (1-\beta) + 2\beta(1-\alpha) \sum_{n=2}^{\infty} a_n \leq \frac{2(1-\beta)}{2-\beta}.$$

Consequently, we have

$$\sum_{n=2}^{\infty} n\, a_n \leq \frac{1-\beta}{(2-\beta)\,(1-\alpha)}. \tag{2.22}$$

Note also that

$$0 < \Psi(n) \leq \Psi(2) = \frac{1}{2-\lambda} \qquad (0 \leq \lambda < 1; \quad n \in \mathbb{N} \setminus \{1\}). \tag{2.23}$$

Therefore, by using (2.22) and (2.23), we observe that

$$\begin{aligned} |G(z)| &\geq |z| - \Psi(2)\,|z|^2 \sum_{n=2}^{\infty} n\, a_n \\ &\geq |z| - \frac{1-\beta}{(2-\beta)\,(1-\alpha)\,(2-\lambda)}\; |z|^2 \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} |G(z)| &\leq |z| + \Psi(2)\; |z|^2 \sum_{n=2}^{\infty} n\, a_n \\ &\leq |z| + \frac{1-\beta}{(2-\beta)\,(1-\alpha)\,(2-\lambda)}\; |z|^2. \end{aligned} \tag{2.25}$$

Thus the proof of Theorem 3 is completed upon noting that the equalities in (2.18) and (2.19) are attained by the function $f(z)$ defined by

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1-\beta}{(2-\beta)\,(1-\alpha)\,(2-\lambda)}\; z\right\} \tag{2.26}$$

or, equivalently, by the function $f(z)$ given by (2.13).

With the aid of Lemma 2, we can similarly prove

Theorem 4. *Let the function $f(z)$ defined by (1.11) be in the class $C[\alpha, \beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then*

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1-\beta}{2(2-\beta)\,(1-\alpha)\,(2-\lambda)}\; |z|\right\} \tag{2.27}$$

and

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{1-\beta}{2(2-\beta)\,(1-\alpha)\,(2-\lambda)}\; |z|\right\} \tag{2.28}$$

for $0 \leq \lambda < 1$ *and* $z \in \mathcal{U}$. *The results are sharp for the function* $f(z)$ *defined by*

$$D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1-\beta}{2(2-\beta)(1-\alpha)(2-\lambda)} z\right\} \tag{2.29}$$

or, equivalently, for the function $f(z)$ *given by* (2.17).

3. A Generalized Fractional Integral Operator

We now recall the following definition of a generalized fractional integral operator introduced by Srivastava *et al.* [14] (see also Srivastava and Owa [12, p. 344]).

Definition 4. For real numbers $\eta > 0$, γ, and δ, the *generalized fractional integral operator* $I_{0,z}^{\eta,\gamma,\delta}$ of order η is defined, for a function $f(z)$, by

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{z^{-\eta-\gamma}}{\Gamma(\eta)} \int_0^z (z-\zeta)^{\eta-1} F\left(\eta+\gamma, -\delta; \eta; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \tag{3.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^\epsilon) \qquad (z \longrightarrow 0),$$

$$\epsilon > \max(0, \gamma-\delta) - 1,$$

and

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (z \in \mathcal{U}), \tag{3.2}$$

$(\nu)_n$ being the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)\cdots(\nu+n-1) & (n \in \mathbb{N}), \end{cases} \tag{3.3}$$

provided further that the multiplicity of $(z-\zeta)^{\eta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Remark 1. For $\gamma = -\eta$, we readily have

$$I_{0,z}^{\eta,-\eta,\delta} f(z) = D_z^{-\eta} f(z) \qquad (\eta > 0),$$

where the fractional integral operator $D_z^{-\lambda}$ $(\lambda > 0)$ is given by Definition 1.

In order to prove our results involving the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$, we shall require the following lemma due to Srivastava *et al.* [14].

Lemma 3. *If* $\eta > 0$ *and* $\kappa > \gamma - \delta - 1$, *then*

$$I_{0,z}^{\eta,\gamma,\delta}\ z^{\kappa} = \frac{\Gamma(\kappa+1)\,\Gamma(\kappa-\gamma+\delta+1)}{\Gamma(\kappa-\gamma+1)\,\Gamma(\kappa+\eta+\delta+1)}\ z^{\kappa-\gamma}. \tag{3.4}$$

With the aid of Lemma 3, we prove

Theorem 5. *Let* $\eta > 0$, $\gamma < 2$, $\eta + \delta > -2$, $\gamma - \delta < 2$, *and* $\gamma(\eta+\delta) \leq 3\eta$. *If the function* $f(z)$ *defined by* (1.11) *is in the class* $\mathcal{R}[\alpha,\beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta < 1$, *then*

$$\begin{aligned}\left|I_{0,z}^{\eta,\gamma,\delta}\ f(z)\right| &\geq \frac{\Gamma(2-\gamma+\delta)\,|z|^{1-\gamma}}{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)} \\ &\quad \cdot \left\{1 - \frac{(1-\beta)\,(2-\gamma+\delta)}{(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z|\right\}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\left|I_{0,z}^{\eta,\gamma,\delta}\ f(z)\right| &\leq \frac{\Gamma(2-\gamma+\delta)\ |z|^{1-\gamma}}{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)} \\ &\quad \cdot \left\{1 + \frac{(1-\beta)\,(2-\gamma+\delta)}{(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z|\right\}\end{aligned} \tag{3.6}$$

for $z \in \mathcal{U}_0$, *where*

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & (\gamma \leq 1) \\ \mathcal{U} \setminus \{0\} & (\gamma > 1). \end{cases} \tag{3.7}$$

The equalities in (3.5) *and* (3.6) *are attained by the function* $f(z)$ *given by* (2.13).

Proof. By using Lemma 3, we have

$$\begin{aligned} I_{0,z}^{\eta,\gamma,\delta}\ f(z) &= \frac{\Gamma(2-\gamma+\delta)}{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)}\ z^{1-\gamma} \\ &\quad - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\,\Gamma(n+\eta+\delta+1)}\ a_n\ z^{n-\gamma}. \end{aligned} \tag{3.8}$$

Letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)}{\Gamma(2-\gamma+\delta)}\ z^{\gamma}\ I_{0,z}^{\eta,\gamma,\delta}\ f(z) \\ &= z - \sum_{n=2}^{\infty} h(n)\ a_n\ z^n, \end{aligned} \tag{3.9}$$

where

$$h(n) = \frac{(2-\gamma+\delta)_{n-1}\, n!}{(2-\gamma)_{n-1}\,(2+\eta+\delta)_{n-1}} \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{3.10}$$

we can see that $h(n)$ is non-increasing for integers n $(n \geq 2)$, and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\gamma+\delta)}{(2-\gamma)\,(2+\eta+\delta)}. \tag{3.11}$$

Therefore, by using (2.9) and (3.11), we have

$$\begin{aligned} |H(z)| &\geq |z| - h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{(1-\beta)\,(2-\gamma+\delta)}{(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z|^2 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} |H(z)| &\leq |z| + h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{(1-\beta)\,(2-\gamma+\delta)}{(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z|^2. \end{aligned} \tag{3.13}$$

This completes the proof of Theorem 5.

Similarly, by applying Lemma 2 (instead of Lemma 1) to the function $f(z)$ belonging to the class $\mathcal{C}[\alpha, \beta]$, we can derive

Theorem 6. *Let $\eta > 0$, $\gamma < 2$, $\eta + \delta > -2$, $\gamma - \delta < 2$, and $\gamma(\eta+\delta) \leq 3\eta$. If the function $f(z)$ defined by* (1.11) *is in the class $\mathcal{C}[\alpha, \beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$, then*

$$\begin{aligned} \left| I_{0,z}^{\eta,\gamma,\delta}\ f(z) \right| &\geq \frac{\Gamma(2-\gamma+\delta)\ |z|^{1-\gamma}}{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)} \\ &\quad \cdot \left\{ 1 - \frac{(1-\beta)\,(2-\gamma+\delta)}{2(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z| \right\} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \left| I_{0,z}^{\eta,\gamma,\delta}\ f(z) \right| &\leq \frac{\Gamma(2-\gamma+\delta)\ |z|^{1-\gamma}}{\Gamma(2-\gamma)\,\Gamma(2+\eta+\delta)} \\ &\quad \cdot \left\{ 1 + \frac{(1-\beta)\,(2-\gamma+\delta)}{2(2-\beta)\,(1-\alpha)\,(2-\gamma)\,(2+\eta+\delta)}\ |z| \right\} \end{aligned} \tag{3.15}$$

for $z \in \mathcal{U}_0$*, where* $\mathcal{U}_0$ *is defined by* (3.7). *The equalities in* (3.14) *and* (3.15) *are attained by the function* $f(z)$ *given by* (2.17).

Remark 2. For $\beta = \alpha$, Theorem 1 and Theorem 3 would reduce immediately to the corresponding results obtained by Owa and Al-Bassam [4].

Remark 3. In view of Remark 1, upon setting $\gamma = -\eta = -\lambda$ in Theorems 5 and 6, we get the assertions of Theorems 1 and 2, respectively.

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