SOME FAMILIES OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

H.M. Srivastava & M.K. Aouf

DMS-701-IR

May 1995

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H.M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-Mail: HMSRI@UVVM.UVIC.CA

and

M.K. Aouf
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura
Egypt

Abstract

We introduce the subclass $\mathcal{P}(j, \lambda, \alpha, n)$ of starlike functions with negative coefficients by using the differential operator D^n which was considered by Sălăgean [11]. Coefficient inequalities, distortion theorems, closure theorems, and some properties involving the modified Hadamard products of several functions belonging to the class $\mathcal{P}(j, \lambda, \alpha, n)$ are obtained. We also determine the radii of close-to-convexity, starlikeness, and convexity for, and consider integral operators associated with, functions belonging to the class $\mathcal{P}(j, \lambda, \alpha, n)$. Finally, we extend some of the aforementioned distortion theorems to hold true for certain operators of fractional calculus (that is, fractional integral and fractional derivative).

1991 *Mathematics Subject Classification.* Primary 26A33, 30C45; Secondary 26A24.
Key words and phrases. Analytic functions, starlike functions, modified Hadamard products, integral operators, fractional integrals, fractional derivatives.

1. Introduction

Let $\mathcal{A}(j)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For a function $f(z)$ in $\mathcal{A}(j)$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N}). \tag{1.4}$$

The differential operator D^n was introduced by Sălăgean [11]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $\mathcal{A}(j)$ is in the class $\mathcal{Q}(j, \lambda, \alpha, n)$ if and only if

$$\Re\left\{\frac{(1-\lambda)\, z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1-\lambda)\, D^n f(z) + \lambda\, D^{n+1} f(z)}\right\} > \alpha \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{1.5}$$

for some α $(0 \leq \alpha < 1)$ and $\lambda (0 \leq \lambda \leq 1)$, and for all $z \in \mathcal{U}$.

Let $\mathcal{T}(j)$ denote the subclass of $\mathcal{A}(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \qquad (a_k \geq 0;\ \ j \in \mathbb{N}). \tag{1.6}$$

Further, we define the class $\mathcal{P}(j, \lambda, \alpha, n)$ by

$$\mathcal{P}(j, \lambda, \alpha, n) = \mathcal{Q}(j, \lambda, \alpha, n) \cap \mathcal{T}(j). \tag{1.7}$$

We note that, by specializing the parameters j, λ, α, and n, we obtain the following subclasses studied by various authors.

(i) $\mathcal{P}(j,\lambda,\alpha,0)=\mathcal{P}(j,\lambda,\alpha)$ (Altintaş [1]);

(ii) $\mathcal{P}(1,0,\alpha,0)=\mathcal{T}^*(\alpha)$ and $\mathcal{P}(1,1,\alpha,0)=\mathcal{P}(1,0,\alpha,1)=\mathcal{C}(\alpha)$ (Silverman [14]);

(iii) $\mathcal{P}(j,0,\alpha,0)=\mathcal{T}_\alpha(j)$ and $\mathcal{P}(j,1,\alpha,0)=\mathcal{P}(j,0,\alpha,1)=\mathcal{C}_\alpha(j)$ (Chatterjea [3] and Srivastava *et al.* [20]);

(iv) $\mathcal{P}(j,0,\alpha,n)=\mathcal{P}(j,\alpha,n)$, where $\mathcal{P}(j,\alpha,n)$ represents the class of functions $f(z)\in\mathcal{T}(j)$ satisfying the inequality:

$$\Re\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\}>\alpha \qquad (n\in\mathbb{N}_0;\ 0\leq\alpha<1;\ z\in\mathcal{U}); \tag{1.8}$$

(v) $\mathcal{P}(j,1,\alpha,n)=\mathcal{P}(j,\alpha,n+1)$, where $\mathcal{P}(j,\alpha,n+1)$ represents the class of functions $f(z)\in\mathcal{T}(j)$ satisfying the inequality:

$$\Re\left\{\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)}\right\}>\alpha \qquad (n\in\mathbb{N}_0;\ 0\leq\alpha<1;\ z\in\mathcal{U}). \tag{1.9}$$

The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of the general class $\mathcal{P}(j,\lambda,\alpha,n)$.

2. Coefficient Estimates and Other Properties of the Class $\mathcal{P}(j,\lambda,\alpha,n)$

Theorem 1. *Let the function* $f(z)$ *be defined by* (1.9). *Then* $f(z)\in\mathcal{P}(j,\lambda,\alpha,n)$ *if and only if*

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}\,a_k\leq 1-\alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left|\frac{(1-\lambda)\,z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1-\lambda)\,D^n f(z) + \lambda\, D^{n+1} f(z)} - 1\right|$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)\{1+(k-1)\lambda\}\, a_k |z|^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k |z|^{k-1}}$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)\{1+(k-1)\lambda\}\, a_k}{1-\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k}$$

$$\leq 1-\alpha.$$

This shows that the values of the function

$$\Phi(z) = \frac{(1-\lambda)\,z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1-\lambda)\,D^n f(z) + \lambda\, D^{n+1} f(z)} \tag{2.2}$$

lie in a circle which is centered at $w = 1$ and whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ is in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then we have

$$\Re\left\{\frac{(1-\lambda)\,z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1-\lambda)\,D^n f(z) + \lambda\, D^{n+1} f(z)}\right\}$$

$$= \Re\left\{\frac{1-\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k\, z^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k\, z^{k-1}}\right\} > \alpha, \tag{2.3}$$

for some α $(0 \leq \alpha < 1)$, some λ $(\lambda \geq 0)$, $n \in \mathbb{N}_0$, and $z \in \mathcal{U}$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty} k^{n+1}\{1+(k-1)\lambda\}\, a_k \geq \alpha\left(1-\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k\right). \tag{2.4}$$

Thus we have the inequality (2.1).

Finally, the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}\, z^k \qquad (k \geq j+1;\ j \in \mathbb{N}) \tag{2.5}$$

is an extremal function for the assertion of Theorem 1.

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then*

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \qquad (k \geq j+1). \tag{2.6}$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

Theorem 2. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1$, $0 \leq \lambda \leq 1$, $j \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then*

$$\mathcal{P}(j, \lambda, \alpha_1, n) \supseteq \mathcal{P}(j, \lambda, \alpha_2, n). \tag{2.7}$$

Proof. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{P}(j, \lambda, \alpha_2, n)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha_2)\{1+(k-1)\lambda\}\, a_k \leq 1-\alpha_2 \tag{2.8}$$

and

$$\sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k \leq \frac{1-\alpha_2}{j+1-\alpha_2} < 1. \tag{2.9}$$

Consequently,

$$\begin{aligned} \sum_{k=j+1}^{\infty} k^n(k-\alpha_1)\{1+(k-1)\lambda\}\, a_k &= \sum_{k=j+1}^{\infty} k^n(k-\alpha_2)\{1+(k-1)\lambda\}\, a_k \\ &\quad + \delta \sum_{k=j+1}^{\infty} k^n\{1+(k-1)\lambda\}\, a_k \\ &\leq 1-\alpha_1. \end{aligned} \tag{2.10}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

Theorem 3. *Let $0 \leq \alpha < 1$, $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, $j \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then*

$$\mathcal{P}(j, \lambda_1, \alpha, n) \supseteq \mathcal{P}(j, \lambda_2, \alpha, n). \tag{2.11}$$

Proof. It follows from Theorem 1 that

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda_1\}\, a_k \leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda_2\}\, a_k$$

$$\leq 1-\alpha$$

for $f(z) \in \mathcal{P}(j, \lambda_2, \alpha, n)$.

Theorem 4. *For* $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $j \in \mathbb{N}$, *and* $n \in \mathbb{N}_0$,

$$\mathcal{P}(j, \lambda, \alpha, n+1) \subseteq \mathcal{P}(j, \lambda, \alpha, n). \tag{2.12}$$

The proof of Theorem 4 follows also from Theorem 1.

3. Distortion Theorems

Theorem 5. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{P}(j, \lambda, \alpha, n)$. *Then, for* $|z| = r < 1$,

$$|D^i f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)}\, r^{j+1} \tag{3.1}$$

and

$$|D^i f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)}\, r^{j+1} \tag{3.2}$$

$$(z \in \mathcal{U};\ \ 0 \leq i \leq n).$$

The equalities in (3.1) *and* (3.2) *are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)}\, z^{j+1} \qquad (z = \pm r). \tag{3.3}$$

Proof. Note that $f(z) \in \mathcal{P}(j, \lambda, \alpha, n)$ if and only if

$$D^i f(z) \in \mathcal{P}(j, \lambda, \alpha, n-i)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i\, a_k\, z^k. \tag{3.4}$$

By Theorem 1, we know that

$$(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\sum_{k=j+1}^{\infty} k^i\, a_k \leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}\, a_k \\ \leq 1-\alpha, \tag{3.5}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i\, a_k \leq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)}. \tag{3.6}$$

The assertions (3.1) and (3.2) of Theorem 5 would now follow readily from (3.4) and (3.6).

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)} z^{j+1}. \tag{3.7}$$

This completes the proof of Theorem 5.

Corollary 2. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{P}(j,\lambda,\alpha,n)$. *Then, for* $|z| = r < 1$,

$$|f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)(1+j\lambda)} r^{j+1} \tag{3.8}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)(1+j\lambda)} r^{j+1} \qquad (z \in \mathcal{U}). \tag{3.9}$$

The equalities in (3.8) *and* (3.9) *are attained for the function* $f(z)$ *given by* (3.3).

Proof. Taking $i = 0$ in Theorem 5, we immediately obtain (3.8) and (3.9).

Corollary 3. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{P}(j,\lambda,\alpha,n)$. *Then, for* $|z| = r < 1$,

$$|f'(z)| \geq 1 - \frac{1-\alpha}{(j+1)^{n-1}(j+1-\alpha)(1+j\lambda)} r^{j} \tag{3.10}$$

and

$$|f'(z)| \leq 1 + \frac{1-\alpha}{(j+1)^{n-1}(j+1-\alpha)(1+j\lambda)} r^{j} \qquad (z \in \mathcal{U}). \tag{3.11}$$

The equalities in (3.10) *and* (3.11) *are attained for the function* $f(z)$ *given by* (3.3).

Proof. Setting $i = 1$ in Theorem 5, and making use of the definition (1.3), we arrive at Corollary 3.

4. Convex Linear Combinations

In this section, we shall prove that the class $\mathcal{P}(j,\lambda,\alpha,n)$ is closed under convex linear combinations.

Theorem 6. *$\mathcal{P}(j,\lambda,\alpha,n)$ is a convex set.*

Proof. Let the functions

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{\nu,k}\, z^k \qquad (a_{\nu,k} \geq 0;\ \ \nu = 1,2) \tag{4.1}$$

be in the class $\mathcal{P}(j,\lambda,\alpha,n)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu\, f_1(z) + (1-\mu)\, f_2(z) \qquad (0 \leq \mu \leq 1) \tag{4.2}$$

is also in the class $\mathcal{P}(j,\lambda,\alpha,n)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} \{\mu\, a_{1,k} + (1-\mu)\, a_{2,k}\}\, z^k, \tag{4.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}\, \{\mu\, a_{1,k} + (1-\mu)\, a_{2,k}\} \leq 1-\alpha, \tag{4.4}$$

which implies that $h(z) \in \mathcal{P}(j,\lambda,\alpha,n)$. Hence $\mathcal{P}(j,\lambda,\alpha,n)$ is a convex set.

Theorem 7. *Let $f_j(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}\, z^k \qquad (k \geq j+1;\ \ n \in \mathbb{N}_0) \tag{4.5}$$

for $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. Then $f(z)$ is in the class $\mathcal{P}(j,\lambda,\alpha,n)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=j}^{\infty} \mu_k\, f_k(z), \tag{4.6}$$

where

$$\mu_k \geq 0 \quad (k \geq j) \qquad \text{and} \qquad \sum_{k=j}^{\infty} \mu_k = 1. \tag{4.7}$$

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z)$$
$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \mu_k z^k. \tag{4.8}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} \cdot \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}$$
$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1. \tag{4.9}$$

So, by Theorem 1, $f(z) \in \mathcal{P}(j, \lambda, \alpha, n)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \qquad (k \geq j+1;\ n \in \mathbb{N}_0). \tag{4.10}$$

Setting

$$\mu_k = \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} a_k \qquad (k \geq j+1;\ n \in \mathbb{N}_0) \tag{4.11}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k, \tag{4.12}$$

we can seen that $f(z)$ can be expressed in the form (4.6). This completes the proof of Theorem 7.

5. Radii of Close-to-Convexity, Starlikeness, and Convexity

Theorem 8. *Let the function $f(z)$ defined by* (1.6) *be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(\lambda, \alpha, n, \rho) := \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{5.1}$$

The result is sharp, the extremal function $f(z)$ being given by (2.5).

Proof. We must show that

$$|f'(z)-1| \leq 1-\rho \quad \text{for} \quad |z| < r_1(\lambda,\alpha,n,\rho),$$

where $r_1(\lambda,\alpha,n,\rho)$ is given by (5.1). Indeed we find from the definition (1.6) that

$$|f'(z)-1| \leq \sum_{k=j+1}^{\infty} k\, a_k\, |z|^{k-1}.$$

Thus

$$|f'(z)-1| \leq 1-\rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k\, |z|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right) |z|^{k-1} \leq \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^{n-1}(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{5.3}$$

Theorem 8 follows easily from (5.3).

Theorem 9. *Let the function $f(z)$ defined by* (1.6) *be in the class $\mathcal{P}(j,\lambda,\alpha,n)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(\lambda,\alpha,n,\rho) := \inf_k \left[\frac{(1-\rho)k^n(k-\alpha)\{1+(k-1)\lambda\}}{(k-\rho)(1-\alpha)}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left|\frac{z\,f'(z)}{f(z)} - 1\right| \leq 1-\rho \quad \text{for} \quad |z| < r_2(\lambda,\alpha,n,\rho),$$

where $r_2(\lambda, \alpha, n, \rho)$ is given by (5.4). Indeed we find, again from the definition (1.6), that

$$\left|\frac{z\,f'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{k=j+1}^{\infty} (k-1)\, a_k\, |z|^{k-1}}{1 - \sum\limits_{k=j+1}^{\infty} a_k\, |z|^{k-1}}.$$

Thus

$$\left|\frac{z\,f'(z)}{f(z)} - 1\right| \leq 1 - \rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\rho}{1-\rho}\right) a_k\, |z|^{k-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right) |z|^{k-1} \leq \frac{k^n (k-\alpha)\{1 + (k-1)\lambda\}}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^n (k-\alpha)\{1+(k-1)\lambda\}}{(k-\rho)(1-\alpha)}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{5.6}$$

Theorem 9 follows easily from (5.6).

Corollary 4. *Let the function $f(z)$ defined by* (1.6) *be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = r_3(\lambda, \alpha, n, \rho) := \inf_k \left[\frac{(1-\rho)k^{n-1} (k-\alpha)\{1+(k-1)\lambda\}}{(k-\rho)(1-\alpha)}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. Modified Hadamard Products

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by (4.1). The *modified* Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) := z - \sum_{k=j+1}^{\infty} a_{1,k}\, a_{2,k}\, z^k. \tag{6.1}$$

Theorem 10. *Let each of the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (4.1) be in the class $\mathcal{P}(j,\lambda,\alpha,n)$. Then*

$$(f_1 * f_2)(z) \in \mathcal{P}\left(j,\lambda,\beta(j,\lambda,\alpha,n),n\right),$$

where

$$\beta(j,\lambda,\alpha,n) := \frac{(j+1)^n(1+j\lambda) - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}{(j+1)^n(1+j\lambda) - \left(\frac{1-\alpha}{j+1-\alpha}\right)^2}. \tag{6.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [13],we need to find the largest $\beta = \beta(j,\lambda,\alpha,n)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\beta)\{1+(k-1)\lambda\}}{1-\beta}\, a_{1,k}\, a_{2,k} \leq 1. \tag{6.3}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\, a_{1,k} \leq 1 \tag{6.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\, a_{2,k} \leq 1, \tag{6.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} \sqrt{a_{1,k}\, a_{2,k}} \leq 1. \tag{6.6}$$

Thus it is sufficient to show that

$$\frac{k^n(k-\beta)}{1-\beta}\, a_{1,k}\, a_{2,k} \leq \frac{k^n(k-\alpha)}{1-\alpha} \sqrt{a_{1,k}\, a_{2,k}} \qquad (k \geq j+1), \tag{6.7}$$

that is, that

$$\sqrt{a_{1,k}\, a_{2,k}} \leq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)} \qquad (k \geq j+1). \tag{6.8}$$

Note that

$$\sqrt{a_{1,k}\, a_{2,k}} \leq \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \qquad (k \geq j+1). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \le \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)} \qquad (k \ge j+1), \tag{6.10}$$

or, equivalently, that

$$\beta \le \frac{k^n\{1+(k-1)\lambda\} - k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{k^n\{1+(k-1)\lambda\} - \left(\frac{1-\alpha}{k-\alpha}\right)^2} \qquad (k \ge j+1). \tag{6.11}$$

Since

$$\Lambda(k) := \frac{k^n\{1+(k-1)\lambda\} - k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{k^n\{1+(k-1)\lambda\} - \left(\frac{1-\alpha}{k-\alpha}\right)^2} \tag{6.12}$$

is an increasing function of k $(k \ge j+1)$, letting $k = j+1$ in (6.12) we obtain

$$\beta \le \Lambda(j+1) = \frac{(j+1)^n(1+j\lambda) - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}{(j+1)^n(1+j\lambda) - \left(\frac{1-\alpha}{j+1-\alpha}\right)^2}, \tag{6.13}$$

which proves the main assertion of Theorem 10.

Finally, by taking the functions

$$f_\nu(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1} \qquad (\nu = 1, 2), \tag{6.14}$$

we can see that the result is sharp.

Theorem 11. *Let*

$$f_1(z) \in \mathcal{P}(j, \lambda, \alpha, n) \quad \text{and} \quad f_2(z) \in \mathcal{P}(j, \lambda, \gamma, n).$$

Then

$$(f_1 * f_2)(z) \in \mathcal{P}\left(j, \lambda, \xi(j, \lambda, \alpha, \gamma, n), n\right),$$

where

$$\xi(j, \lambda, \alpha, \gamma, n) = \frac{(j+1)^n(1+j\lambda) - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)\left(\frac{1-\gamma}{j+1-\gamma}\right)}{(j+1)^n(1+j\lambda) - \left(\frac{1-\alpha}{j+1-\alpha}\right)\left(\frac{1-\gamma}{j+1-\gamma}\right)}. \tag{6.15}$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1} \tag{6.16}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n(j+1-\gamma)(1+j\lambda)} z^{j+1}. \tag{6.17}$$

Proof. Proceeding as in the proof of Theorem 10, we get

$$\xi \leq \frac{k^n\{1+(k-1)\lambda\} - k\left(\frac{1-\alpha}{k-\alpha}\right)\left(\frac{1-\gamma}{k-\gamma}\right)}{k^n\{1+(k-1)\lambda\} - \left(\frac{1-\alpha}{k-\alpha}\right)\left(\frac{1-\gamma}{k-\gamma}\right)} \qquad (k \geq j+1). \tag{6.18}$$

Since the right-hand side of (6.18) is an increasing function of k, setting $k = j+1$ in (6.18), we obtain (6.15).

This completes the proof of Theorem 11.

Corollary 5. *Let the functions $f_\nu(z)$ defined by*

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{\nu,k}\, z^k \qquad (a_{\nu,k} \geq 0;\ \nu = 1,2,3) \tag{6.19}$$

be in the class $\mathcal{P}(j,\lambda,\alpha,n)$. Then

$$(f_1 * f_2 * f_3)(z) \in \mathcal{P}(j,\lambda,\delta(j,\lambda,\alpha,n),n),$$

where

$$\delta(j,\lambda,\alpha,n) = \frac{(j+1)^{2n}(1+j\lambda)^2 - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^3}{(j+1)^{2n}(1+j\lambda)^2 - \left(\frac{1-\alpha}{j+1-\alpha}\right)^3}. \tag{6.20}$$

The result is the best possible for the functions

$$f_\nu(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1} \qquad (\nu = 1,2,3). \tag{6.21}$$

Proof. From Theorem 10, we have

$$(f_1 * f_2)(z) \in \mathcal{P}(j,\lambda,\beta(j,\lambda,\alpha,n),n),$$

where β is given by (6.2). Now, using Theorem 11, we get

$$(f_1 * f_2 * f_3)(z) \in \mathcal{P}(j, \lambda, \delta(j, \lambda, \alpha, n), n),$$

where

$$\delta(j, \lambda, \alpha, n) = \frac{(j+1)^n(1+j\lambda) - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)\left(\frac{1-\beta}{j+1-\beta}\right)}{(j+1)^n(1+j\lambda) - \left(\frac{1-\alpha}{j+1-\alpha}\right)\left(\frac{1-\beta}{j+1-\beta}\right)}$$

$$= \frac{(j+1)^{2n}(1+j\lambda)^2 - (j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^3}{(j+1)^{2n}(1+j\lambda)^2 - \left(\frac{1-\alpha}{j+1-\alpha}\right)^3}.$$

This completes the proof of Corollary 5.

Theorem 12. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (4.1) be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} \left(a_{1,k}^2 + a_{2,k}^2\right) z^k \tag{6.22}$$

belongs to the class $\mathcal{P}(j, \lambda, \eta(j, \lambda, \alpha, n), n)$, where

$$\eta(j, \lambda, \alpha, n) = \frac{(j+1)^n(1+j\lambda) - 2(j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}{(j+1)^n(1+j\lambda) - 2\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}. \tag{6.23}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (6.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2 a_{1,k}^2$$
$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1))\lambda\}}{1-\alpha} a_{1,k}\right]^2 \leq 1 \tag{6.24}$$

and

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2 a_{2,k}^2$$
$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} a_{2,k}\right]^2 \leq 1. \tag{6.25}$$

It follows from (6.24) and (6.25) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} \right]^2 \left(a_{1,k}^2 + a_{2,k}^2 \right) \leq 1. \tag{6.26}$$

Therefore, we need to find the largest $\eta = \eta(j, \lambda, \alpha, n)$ such that

$$\frac{k^n(k-\eta)\{1+(k-1)\lambda\}}{1-\eta} \leq \frac{1}{2} \left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} \right]^2 \qquad (k \geq j+1), \tag{6.27}$$

that is,

$$\eta \leq \frac{k^n\{1+(k-1)\lambda\} - 2k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{k^n\{1+(k-1)\lambda\} - 2\left(\frac{1-\alpha}{k-\alpha}\right)^2} \qquad (k \geq j+1). \tag{6.28}$$

Since

$$\Omega(k) := \frac{k^n\{1+(k-1)\lambda\} - 2k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{k^n\{1+(k-1)\lambda\} - 2\left(\frac{1-\alpha}{k-\alpha}\right)^2} \tag{6.29}$$

is an increasing function of k $(k \geq j+1)$, we readily have

$$\eta \leq \Omega(j+1) = \frac{(j+1)^n(1+j\lambda) - 2(j+1)\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}{(j+1)^n(1+j\lambda) - 2\left(\frac{1-\alpha}{j+1-\alpha}\right)^2}, \tag{6.30}$$

and Theorem 12 follows at once.

7. A Family of Integral Operators

Theorem 13. *Let the function $f(z)$ defined by* (1.6) *be in the class $\mathcal{P}(j, \lambda, \alpha, n)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \qquad (c > -1) \tag{7.1}$$

also beongs to the class $\mathcal{P}(j, \lambda, \alpha, n)$.

Proof. From the representation (7.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k.$$

Therefore, we have

$$\begin{aligned}\sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}b_k &= \sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}\left(\frac{c+1}{c+k}\right) a_k \\ &\leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\lambda\}\, a_k \\ &\leq 1-\alpha,\end{aligned}$$

since $f(z) \in \mathcal{P}(j, \lambda, \alpha, n)$. Hence, by Theorem 1, $F(z) \in \mathcal{P}(j, \lambda, \alpha, n)$.

Theorem 14. *Let the function*

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k\, z^k \qquad (a_k \geq 0;\ \ j \in \mathbb{N})$$

be in the class $\mathcal{P}(j, \lambda, \alpha, n)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ given by (7.1) *is univalent in $|z| < R^*$, where*

$$R^* := \inf_k \left[\frac{(k-\alpha)k^{n-1}\{1+(k-1)\lambda\}(c+1)}{(1-\alpha)(c+k)}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{7.2}$$

The result is sharp.

Proof. From (7.1), we have

$$f(z) = \frac{z^{1-c}\{z^c\, F(z)\}'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k\, z^k.$$

In order to obtain the required result, it suffices to show that

$$|f'(z) - 1| < 1 \quad \text{whenever} \quad |z| < R^*,$$

where R^* is given by (7.2). Now

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1}\, a_k\, |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1}\, a_k\, |z|^{k-1} < 1. \tag{7.3}$$

But Theorem 1 confirms that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\, a_k \leq 1. \tag{7.4}$$

Hence (7.3) will be satisfied if

$$\frac{k(c+k)}{c+1}\, |z|^{k-1} < \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}$$

that is, if

$$|z| < \left[\frac{(k-\alpha)k^{n-1}\{1+(k-1)\lambda\}(c+1)}{(1-\alpha)(c+k)}\right]^{1/(k-1)} \qquad (k \geq j+1). \tag{7.5}$$

Therefore, the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n(k-\alpha)\{1+(k-1)\lambda\}(c+1)}\, z^k \qquad (k \geq j+1). \tag{7.6}$$

8. Applications of Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (*cf.*, *e.g.*, [2], [4, Chapter 13], [5], [6], [7], [9], [10], [12], [15], [16, p. 21 *et seq.*], [18], [20], [22], and [24]). We find it to be convenient to recall here the following devinitions which were used earlier by Owa [8] (and, subsequently, by Srivastava and Owa [17]).

Definition 1. The *fractional integral of order* μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, d\zeta \qquad (\mu > 0), \tag{8.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The *fractional derivative of order* μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \qquad (0 \leq \mu < 1), \tag{8.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $n+\mu$ is defined by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \qquad (0 \leq \mu < 1;\ n \in \mathbb{N}_0). \tag{8.3}$$

Theorem 15. *Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then*

$$|D_z^{-\mu}\left(D^i f(z)\right)| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{1 - \frac{(1-\alpha)\,\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\,\Gamma(j+2+\mu)} |z|^j\right\} \tag{8.4}$$

and

$$|D_z^{-\mu}\left(D^i f(z)\right)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{(1-\alpha)\,\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\,\Gamma(j+2+\mu)} |z|^j\right) \tag{8.5}$$

$$(\mu > 0;\ 0 \leq i \leq n;\ z \in \mathcal{U}).$$

The result is sharp.

Proof. Note that

$$f(z) \in \mathcal{P}(j, \lambda, \alpha, n) \Leftrightarrow D^i f(z) \in \mathcal{P}(j, \lambda, \alpha, n-i)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i\, a_k\, z^k. \tag{8.6}$$

By Theorem 1, we know that

$$\begin{aligned}(j+1)^{n-i}(j+1-\alpha)(1+j\lambda) &\sum_{k=j+1}^{\infty} k^i\, a_k \\ &\leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)\{1+(k-1)\,\lambda\}\, a_k \\ &\leq 1-\alpha,\end{aligned} \tag{8.7}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i\, a_k \leq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)}. \tag{8.8}$$

Let

$$\begin{aligned} F(z) &= \Gamma(2+\mu)\, z^{-\mu}\, D_z^{-\mu}\left(D^i f(z)\right) \\ &= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}\, k^i\, a_k\, z^k \\ &= z - \sum_{k=j+1}^{\infty} \Psi(k)\, k^i\, a_k\, z^k, \end{aligned} \tag{8.9}$$

where

$$\Psi(k) := \frac{\Gamma(k+1)\,\Gamma(2+\mu)}{\Gamma(k+1+\mu)} \qquad (k \geq j+1). \tag{8.10}$$

Since

$$0 < \Psi(k) \leq \Psi(j+1) = \frac{\Gamma(j+2)\,\Gamma(2+\mu)}{\Gamma(j+2+\mu)}. \tag{8.11}$$

Therefore, by using (8.8) and (8.11), we see that

$$\begin{aligned} |F(z)| &\geq |z| - \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i\, a_k \\ &\geq |z| - \frac{(1-\alpha)\,\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\,\Gamma(j+2+\mu)}\, |z|^{j+1} \end{aligned} \tag{8.12}$$

and

$$\begin{aligned} |F(z)| &\leq |z| + \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i\, a_k \\ &\leq |z| + \frac{(1-\alpha)\,\Gamma(j+2)\,\Gamma(2+\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\,\Gamma(j+2+\mu)}\, |z|^{j+1}, \end{aligned} \tag{8.13}$$

which prove the inequalities (8.4) and (8.5) of Theorem 15.

The equalities in (8.4) and (8.5) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\left(D^i f(z)\right) = \frac{z^{1+\mu}}{\Gamma(2+\mu)} \left\{1 - \frac{(1-\alpha)\,\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2+\mu)}\, z^j\right\} \tag{8.14}$$

or, equivalently, by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)}\, z^{j+1}. \tag{8.15}$$

Thus we complete the proof of Theorem 15.

Taking $i = 0$ in Theorem 15, we have

Corollary 6. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{P}(j, \lambda, \alpha, n)$. *Then*

$$|D_z^{-\mu} f(z)| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n(j+1-\alpha)(1+j\lambda)\Gamma(j+2+\mu)} |z|^j\right\} \tag{8.16}$$

and

$$|D_z^{-\mu} f(z)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n(j+1-\alpha)(1+j\lambda)\Gamma(j+2+\mu)} |z|^j\right\} \tag{8.17}$$

$$(\mu > 0; \ z \in \mathcal{U}).$$

The equalities in (8.16) *and* (8.17) *are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1}. \tag{8.18}$$

Theorem 16. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $\mathcal{P}(j, \lambda, \alpha, n)$. *Then*

$$|D_z^{\mu} (D^i f(z))| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)} |z|^j\right\} \tag{8.19}$$

and

$$|D_z^{\mu} (D^i f(z))| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)} |z|^j\right\} \tag{8.20}$$

$$(0 \leq \mu < 1; \ 0 \leq i \leq n-1; \ z \in \mathcal{U}).$$

The result is sharp.

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\mu) z^\mu D_z^\mu (D^i f(z)) \\ &= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)} k^i a_k z^k \\ &= z - \sum_{k=j+1}^{\infty} \Theta(k) k^{i+1} a_k z^k, \end{aligned} \tag{8.21}$$

where

$$\Theta(k) := \frac{\Gamma(k)\Gamma(2-\mu)}{\Gamma(k+1-\mu)} \qquad (k \geq j+1). \tag{8.22}$$

It is easily seen from (8.22) that

$$0 < \Theta(k) \leq \Theta(j+1) = \frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+2-\mu)}. \tag{8.23}$$

Consequently, with the aid of (8.8) and (8.23), we have

$$\begin{aligned} |G(z)| &\geq |z| - \Theta(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1}\, a_k \\ &\geq |z| - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)}\, |z|^{j+1} \end{aligned} \tag{8.24}$$

and

$$\begin{aligned} |G(z)| &\leq |z| + \Theta(j+1)\, |z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1}\, a_k \\ &\leq |z| + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)}\, |z|^{j+1}. \end{aligned} \tag{8.25}$$

Now (8.19) and (8.20) follow from (8.24) and (8.25), respectively.

Since the equalities in (8.19) and (8.20) are attained for the function $f(z)$ given by

$$D_z^{\mu}\left(D^i f(z)\right) = \frac{z^{1-\mu}}{\Gamma(2-\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)}\, z^j\right\} \tag{8.26}$$

or for the function $D^i f(z)$ given by (8.15), the proof of Theorem 16 is thus completed.

Taking $i = 0$ in Theorem 16, we have

Corollary 7. *Let the function $f(z)$ defined by* (1.6) *be in the class $\mathcal{P}(j, \lambda, \alpha, n)$. Then*

$$|D_z^{\mu} f(z)| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)}\, |z|^j\right\} \tag{8.27}$$

and

$$|D_z^{\mu} f(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 + \frac{(1-\alpha)\Gamma(j+1)\Gamma(2-\beta)}{(j+1)^{n}(j+1-\alpha)(1+j\lambda)\Gamma(j+2-\mu)}\, |z|^j\right\} \tag{8.28}$$

$$(0 \leq \mu < 1;\ \ z \in \mathcal{U}).$$

The equalities in (8.27) *and* (8.28) *are attained for the function* $f(z)$ *given by* (8.18).

9. A Generalized Fractional Integral Operator

We need the following definition of a generalized fractional integral operator given by Srivastava *et al.* [23].

Definition 4. For real numbers $\mu > 0$, γ, and δ, the *generalized fractional integral operator* $I_{0,z}^{\mu,\gamma,\delta}$ is defined by

$$I_{0,z}^{\mu,\gamma,\delta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-\zeta)^{\mu-1} F\left(\mu+\gamma, -\delta; \mu; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \tag{9.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}) \qquad (z \to 0),$$

with $\epsilon > \max\{0, \gamma-\delta\} - 1$. Here $F(a,b;c;z)$ is the Gauss hypergeometric function defined by

$$F(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{9.2}$$

where $(\lambda)_k$ is the Pochhammer symbol given by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}), \end{cases} \tag{9.3}$$

and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Remark 1. When $\gamma = -\mu$, we have

$$I_{0,z}^{\mu,-\mu,\delta} f(z) = D_z^{-\mu} f(z),$$

where the fractional integral operator $D_z^{-\mu}$ is given by Definition 1.

In order to prove our result involving the generalized fractional integral operator $I_{0,z}^{\mu,\gamma,\delta}$, we recall here the following lemma due to Srivastava *et al.* [23].

Lemma 1. *If* $\mu > 0$ *and* $\kappa > \gamma - \delta - 1$, *then*

$$I_{0,z}^{\mu,\gamma,\delta}\, z^{\kappa} = \frac{\Gamma(\kappa+1)\Gamma(\kappa-\gamma+\delta+1)}{\Gamma(\kappa-\gamma+1)\Gamma(\kappa+\mu+\delta+1)}\, z^{\kappa-\gamma}. \tag{9.4}$$

With the aid of Lemma 1, we now prove

Theorem 17. *Let* $\mu > 0$, $\gamma < 2$, $\mu+\delta > -2$, $\gamma - \delta < 2$, $\gamma(\mu+\delta) \leq \mu(j+2)$, *and* $j \in \mathbb{N}$. *If the function* $f(z)$ *defined by* (1.6) *is in the class* $\mathcal{P}(j,\gamma,\alpha,n)$, *then*

$$\begin{aligned} |I_{0,z}^{\mu,\gamma,\delta}\, f(z)| \geq \frac{\Gamma(2-\gamma+\delta)\, |z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\mu+\delta)} \\ \cdot \left\{1 - \frac{(1-\alpha)(2-\gamma+\delta)_j(2)_j}{(j+1)^n(j+1-\alpha)(1+j\lambda)(2-\gamma)_j(2+\mu+\delta)_j}\, |z|^j\right\} \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} |I_{0,z}^{\mu,\gamma,\delta}\, f(z)| \leq \frac{\Gamma(2-\gamma+\delta)\, |z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\mu+\delta)} \\ \cdot \left\{1 + \frac{(1-\alpha)(2-\gamma+\delta)_j(2)_j}{(j+1)^n(j+1-\alpha)(1+j\lambda)(2-\gamma)_j(2+\mu+\delta)_j}\, |z|^j\right\} \end{aligned} \tag{9.6}$$

for $z \in \mathcal{U}_0$, *where*

$$\mathcal{U}_0 := \begin{cases} \mathcal{U} & (\gamma \leq 1) \\ \mathcal{U} - \{0\} & (\gamma > 1). \end{cases} \tag{9.7}$$

The equalities in (9.5) *and* (9.6) *are attained for the function* $f(z)$ *given by* (8.18).

Proof. By using Lemma 1, we have

$$\begin{aligned} I_{0,z}^{\mu,\gamma,\delta}\, f(z) = \frac{\Gamma(2-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(2+\mu+\delta)}\, z^{1-\gamma} \\ - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\mu+\delta+1)}\, a_k\, z^{k-\gamma} \quad (z \in \mathcal{U}_0). \end{aligned} \tag{9.8}$$

Letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\gamma)\,\Gamma(2+\mu+\delta)}{\Gamma(2-\gamma+\delta)}\, z^{\gamma}\, I_{0,z}^{\mu,\gamma,\delta}\, f(z) \\ &= z - \sum_{k=j+1}^{\infty} \Xi(k)\, a_k\, z^k, \end{aligned} \tag{9.9}$$

where

$$\Xi(k) := \frac{(2-\gamma+\delta)_{k-1}\,(2)_{k-1}}{(2-\gamma)_{k-1}(2+\mu+\delta)_{k-1}} \qquad (k \geq j+1), \tag{9.10}$$

we can see that $\Xi(k)$ is non-increasing for integers $k \geq j+1$, and we have

$$0 < \Xi(k) \leq \Xi(j+1) = \frac{(2-\gamma+\delta)_j\,(2)_j}{(2-\gamma)_j(2+\mu+\delta)_j}. \tag{9.11}$$

Therefore, by using (8.8) (with $i = 0$) and (9.11), we have

$$\begin{aligned} |H(z)| &\geq |z| - \Xi(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} a_k \\ &\geq |z| - \frac{(1-\alpha)(2-\gamma+\delta)_j\,(2)_j}{(j+1)^n(j+1-\alpha)(1+j\lambda)(2-\gamma)_j(2+\mu+\delta)_j}\,|z|^{j+1} \end{aligned} \tag{9.12}$$

and

$$\begin{aligned} |H(z)| &\leq |z| + \Xi(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} a_k \\ &\leq |z| + \frac{(1-\alpha)(2-\gamma+\delta)_j\,(2)_j}{(j+1)^n(j+1-\alpha)(1+j\lambda)(2-\gamma)_j(2+\mu+\delta)_j}\,|z|^{j+1} \end{aligned} \tag{9.13}$$

for $z \in \mathcal{U}_0$, where $\mathcal{U}_0$ is defined by (9.7). This completes the proof of Theorem 17.

Remark 2. Taking $\gamma = -\mu$ in Theorem 17, we get the assertion of Corollary 6.

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

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H.M. Srivastava:

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

E-Mail: HMSRI@UVVM.UVIC.CA

M.K. Aouf:

Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura
Egypt