AN ANALYSIS OF THE METRIC OF BACH AND WEYL

by

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ABSTRACT

In this thesis we refute the equivalence of the Weyl line mass solution of "length" $\frac{2GM}{c^2}$ to the exterior Schwarzschild solution. Equivalently, this work is a refutation of the interpretation, accepted for over fifty years, that the Weyl line mass solution of this critical "length" is spherically symmetric. This refutation is approached in three ways.

Firstly, the refutation is based on a examination of metric-coordinate interdependence. Specifically the role of a coordinate truncation im determining a metric geometry is examined.

Secondly, the refutation is based on the intrinsic, physical differences between the boundaries of the two solutions. In this approach intrinsic singularities of the Weyl solution are examined through analysis of the scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ and application of the elementary flatness criterion.

Thirdly, the symmetries of the two solutions are shown to be intrinsically different using Killing vector analysis. It is shown conclusively that the Weyl line mass solution of "length" $\frac{2GM}{c^2}$ is <u>not</u> a spherically symmetric solution.

Some of the works in the literature, which rely on the representation of a point mass by a line mass of characteristic "length" $\frac{2GM}{c^2}$ and which must be consequently modified in light of our reinterpretation, are discussed.

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CONVENTIONS AND NOTATIONS

The term "Schwarzschild" is used in referring to the form of the Weyl line mass solution for "length" $\frac{2GM}{c^2}$ commonly interpreted as being the bona fide Schwarzschild solution. Quotation marks indicate the authors contention that this form of the Weyl solution is not in fact the Schwarzschild solution.

Greek and Latin indices for vectors and tensors are used in accordance with the following convention: Greek for the four dimensions of space-time; lower case Latin for the three spatial dimensions of space-time; upper case Latin for the tetrad components of tensors in the differential forms section. Consequently the Greek indices run over $\mu = 1,2,3,4$ (with $\mu = 4$ the temporal component); the lower case Latin indices run over $i = 1,2,3$; and the upper case Latin indices run over $A = 1,2,3,4$.

Einstein summation convention is used throughout. For example,

$$A_\alpha A^\alpha = A_1 A^1 + A_2 A^2 + A_3 A^3 + A_4 A^4.$$

Ordinary derivatives are denoted by

$$\frac{\partial A^\mu}{\partial x^\rho} \equiv A^\mu{}_{,\rho}.$$

The covariant derivative of a vector or tensor is denoted in similar fashion with a semi-colon replacing the comma. For example, the covariant derivative of a vector, A^μ, is $A^\mu{}_{;\rho}$.

In the differential forms treatment of sections 5.2 and 5.3, $d\Omega$ refers to the exterior differential of a form Ω.

Elsewhere dA^{μ} refers to the ordinary differential. The covariant differential of a vector, A^{μ}, is denoted by DA^{μ}.

. The wedge product of differential forms is denoted by the symbol $\wedge$.

CHAPTER 1

INTRODUCTION

Bach and Weyl in their now famous paper (Bach and Weyl, 1922) demonstrated that the Weyl (or Levi-Civita) metric (Weyl, 1918, 1919 and Levi-Civita, 1919) for a line mass of "length" equal to the Schwarzschild radius could be transformed by means of a conformal mapping into "Schwarzschild" form. In the intervening fifty years the literature has accepted this transformation and its ramifications unquestioningly. The solution to the line mass of this particular length ($\frac{2GM}{c^2}$) has consequently come to be interpreted as the exterior Schwarzschild solution or, equivalently by Birkhoff's theorem (Birkhoff, 1923), the most general spherically symmetric solution in vacuum. This thesis shall be an attempt at a refutation of this argument. The hope is that in refutation of the traditional argument certain of its unappealling and bewildering consequences shall be removed from the relativity theory of Einstein and that consequently intuition and theory will be in more harmonious accord.

As Weyl (1952) so aptly put it: "Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection". This functional aspect of symmetry in comprehending the order of reality is epitomized in general relativity theory by spherical symmetry. Considering that one of the ultimate criteria for the validity of a theory is its ability to make correct predictions, one may go so far as to say general relativity's acceptability has developed in parallel with investigations into spherically symmetric space-times. In fact three of the four primary tests* of the theory are based on the Schwarzschild exterior solution (Schwarzschild, 1916), the only spherically symmetric solution in vacuum.

Indeed after over fifty years the ramifications of this spherically symmetric solution are far from exhausted. Speculation over the controversial phenomena of black holes and gravitational collapse, whose foundations lie in the Schwarzschild solution, keep this solution in the forefront of discussions of relativity.

Spherically symmetric space-times, in addition to being an integral part of general relativity theory, also are in the vanguard of relativistic cosmology. For example, the cosmological space-times of the Einstein, de Sitter, and Friedmann

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*The tests referred to are: the gravitational red shift, the anomalous perihelion rotation of Mercury, the deflection of light in a gravitational field, and the Shapiro radar test. The latter three are based on the Schwarzschild exterior solution, although one should point out that Schiff (1960) has claimed the deflection of light can be explained using only special relativity, the principle of equivalence, and geometrical optics.

universes are spherically symmetric. The Friedmann model is of special interest since it is believed to include the real universe as a special case. Given the importance of spherically symmetric space-times in relativity, it would be extremely unnerving to find a rampant anomaly confronting one's basic understanding of the concept of spherical symmetry.

Such an anomaly occurs in the traditional analysis of the Weyl solution for a line mass of "length" $\frac{2GM}{c^2}$. The traditional interpretation[1] of this solution as a spherically symmetric space-time conflicts with one's intuition on two accounts. Firstly, the general realitivistic spherically symmetric solution is obtained from the cylindrically, but not spherically, symmetric Newtonian potential of a line mass. (The details of the manner in which the Newtonian potential enters in the Weyl formalism will be given in Chapter 3.) Secondly and most irreconciliably, this anomalous behaviour occurs only at the particular "length" $\frac{2GM}{c^2}$.

In general relativity theory, coordinates are treated as labels identifying different events in space-time. They are merely a practical convenience and are not, in general, imbued with the intrinsic meanings which are given them in Euclidean geometry. For example, in general relativity the generalised cylindrical polar coordinates r, z, φ are not in general quantities which one can measure with a ruler as they are in Euclidean geometry. With this understanding one can attempt a reconciliation between intuition and the traditional interpretation of the line mass solution of "length" $\frac{2GM}{c^2}$ as a spher-

1. Bach and Weyl (1922), Bergmann (1942), Synge (1966), Robertson and Noonan (1968), and Takeno (1952,1966).

ically symmetric solution, Since the generalized cylindrical polar coordinates r, z,φ appearing in the relativistic solution do not retain their Euclidean meaning, the mass distribution in the general relativistic case may well not be the same as in the corresponding Newtonian case. Therefore, the symmetries in the two cases need not be the same.

Let us, however, subject this argument to more careful scrutiny. The argument goes that, although the Newtonian potential used in the Weyl formalism was obtained using constant mass density, the generalized cylindrical coordinates r and z lose their Euclidean meaning when placed in the general relativistic context and, consequently the general relativistic line mass, corresponding to the Newtonian constant mass density line mass, may have variable mass density. The argument is tractable but the connotation that this variable mass density can explain a spherical symmetry is not. As shall be demonstrated in Chapter 5, the line mass under consideration has a non-zero mass density throughout its length. That a line mass having non-zero mass density over the totality of its length, albeit variable, can produce an exactly spherically symmetric field is beyond the constructional capacity of the imagination. Intuition, experience, and Newtonian gravitational theory all support the conviction that an exactly spherically symmetric field can be produced only by a spherically symmetric body. *

* One could envisage a Newtonian potential having symmetries other than cylindrical losing the extra-cylindrical symmetries on introduction into the general relativistic Weyl formalism as a consequence of the differing mass density argument. Indeed, an example of this will be given in Chapter 6, the Weyl single mass center solution.

The rejoinder that intuition, experience, and Newtonian mechanics can only be invoked in the classical limit and that transcendent phenomena may explain the anomaly is precluded by the second and most disturbing point of the traditional interpretation of the Weyl line mass solution. The anomaly of obtaining a spherically symmetric relativistic solution from a Newtonian non-spherically symmetric solution is exhibited only for the characteristic "length" $\frac{2GM}{c^2}$. Contrast this to quantum mechanics. We know that in the limit in which characteristic distances and momenta involved in describing the motion of a particle are small enough so that the uncertainty principle becomes significant, the results of quantum mechanics diverge from those of Newtonian mechanics. Our intuitive concepts of the continuity of nature, based on the prejudice of macroscopic experience, must be supplanted.

In relativity theory, too, anomalies are encountered when one trespasses beyond the classical limit. One such anomaly is the phenomenon of the black hole (Oppenheimer and Snyder, 1939). As an object under goes gravitational collapse beyond the dimensions of a critical surface known as the event horizon, all its characteristics except mass, charge, and angular momentum are obliterated. Pertubations from sphericity, in the form of multipole moments, are radiated away decreasing exponentially with time (Doroshkevich, Zel'dovich, and Novikov, 1965, 1966). To an outside observer, details of the gravitational field, symmetry included, become obliterated after a transient period of the order of $\frac{2GM}{c^3}$.

One must note, however, an important common factor in the quantum mechanical and general relativistic phenomena cited. They are both "limit" phenomena, that is, after a certain limit has been passed they not only continue but become more exaggerated. Thus, since the anomaly with which we are concerned exhibits itself only at a characterisitic length, we cannot hope to explain it in the spirit of these transcendent phenomenon.* Another explanation must be sought.

A final attempt can be made at reconciling intuition with the construction of a general relativistic spherically symmetric solution from a non-spherically symmetric Newtonian potential. One can claim that, although Newtonian potentials of line masses are used in their development, the Weyl solutions corresponding to these potentials are not line mass solutions at all. For the characteristic length, for example, the Weyl solution is the solution of a sphere. However in this argument one replaces a conceptual difficulty with an even more grievous one. For if one accepts this explanation, one must abandon the most natural correspondence between the Newtonian and relativistic gravitation theories. Care must be exercised in not reading too much into this statement. It does not mean any

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* The black hole phenomenon was introduced as an illustrative example of relativistic "limit" phenomenon. It could never hope to find application in our problem since the Weyl line mass solution has no event horizon.

correspondence is precluded by the explanation given. Such an explanationwould violate one of the major criteria for the validity of a relativistic theory. The objection to the explanation is rather one based on the grounds of aesthetics and simplicity. Such an argument is not to be slighted since it has ample precedent in physics and one of the foundations of relativity theory, the Priniple of General Covariance, is based on just such an argument.

With the failure of the above explanations to satisfactorily reconcile intuition and the curiosity of a general relativistic spherically symmetric solution generated by a Newtonian non-spherically symmetric potential, we are logically led to question the hypothesis of spherical symmetry of the general relativistic solution. In this thesis we shall proceed with the investigation of this question in three ways:

i) By investigating the interrelationship of a metric and its coordinates. It shall be attempted to show heuristically that the coordinates in the "Schwarzschild" form of the Weyl line mass solution of "length" $\frac{2GM}{c^2}$ do <u>not</u> correspond to those in the bona fide sspherically symmetric Schwarzschild solution.

ii) By demonstrating that the <u>intrinsic</u> singularity of the line mass solution and that of the bona fide Schwarzschild solution do not correspond under mapping. This shall be attempted by the computation and analysis of characteristic scalars and by the application of the elementary flatness criterion.

iii) By employing the techniques of Killing vector analysis to test directly whether or not the symmetry group of the Weyl line mass solution of "length" $\frac{2GM}{c^2}$ is indeed the spherical symmetry group.

The first two approaches to the question are plausibility arguments. Their main value is in crystallizing one's doubts as to the validity of the traditional interpretation of the Weyl line mass solution of "length" $\frac{2GM}{c^2}$ and indicating wherein lies the source of misinterpretation. The third approach is conclusive and independent of conjecture.

In Chapter 2 the general relativistic formalism necessary for our work is developed. In Chapter 3 the specific solution of the Einstein gravitational theory on which our work focuses, the Weyl line mass solution, is constructed. The three subsequent chapters refute the traditional interpretation of the Weyl solution for a line mass of "length" $\frac{2GM}{c^2}$. In Chapter 4 the refutation is centered about the metric-coordinate interdependence of non-Euclidean geometry. In Chapter 5 the argument focuses on intrinsic singularities. In Chapter 6 Killing vector analysis is employed to demonstrate that the symmetries of the bona fide Schwarzschild solution and the characteristic line mass solution do <u>not</u> correspond. Chapter 7 is a summary of the work and a discussion of some of its ramifications.

CHAPTER 2

A HEURISTIC DEVELOPMENT OF GENERAL RELATIVITY IN VACUUM

2.1 Introduction

Einstein's general relativity theory has (unjustly) been labelled anything ranging from very difficult to incomprehensible. Indeed it is often claimed that the theory has been understood only by six people in the world since its inception. This reputation is extremely unfortunate since it acts as a psychological barrier to prospective students of the theory. The fact is that the basic ideas of the theory can be explained using only simple mathematical concepts and heuristic argumentation. Once the foundation of basic concepts has been laid, the mathemetical formalism, usually a major deterrent in the study of the theory, actually aids in the crystallization, refinement and extension of these basic ideas. In this chapter a heuristic development of the elements of general relativity theory necessary for our work will be attempted. Stress will be laid on ideas. The mathematics encountered shall be simple and minimal, serving the utilitarian purposes of illustrating ideas and solidifying heuristic arguments into mathematical expressions. In section 2.2 the reasons for a non-Euclidean, metric geometry will be probed. It shall be demonstrated that one is led in a <u>natural</u> way to postulate <u>a Riemannian</u> geometry for relativity space-times. In section 2.3 the Einstein, field equations, the formalism for relating the geometry to the material content of space, are developed.

2.2. Curvature of Space and the Riemann Metric.

One of the main features of general relativity is its contention that space is curved, or non-Euclidean. This idea is a source of mystification for many a novice but if approached correctly it is in fact quite simple. D.W.Sciama (1961) offers an approach to understanding which is beautiful in its simplicity. Consider a physicist interested in a disc claimed to be circular. He decides to check whether it is indeed circular by measuring its circumference and radius to see if they are in the right relation, that is, whether their ratio is 2π. He is careful to use a ruler small enough that error introduced in approximating the circle by the finite segments formed by the rulers is negligible. Despite this precaution, he finds the ratio much less than 2π. He is about to reject the contention that the disc is circular, when he realizes the sun had been shining unevenly on the disc, and its central portion which happened to be in shadow, was colder than the edge. Consequently, the rulers expanded less when measuring the radius than when measuring the circumference. The physicist now corrects his measurements for the temperature distribution of the disc and arrives at the conclusion that the disc is indeed circular.

But what if the physicist chose to regard the rulers as basic standards which never need to be corrected? He would find the ratio of circumference to radius does not have the Euclidean value 2π. However, the disc would still be circular in the sense that every point on its edge would be the same distance from its center (assuming the disc to be heated symmetrically). Hence the physicist would have discovered that a disc which is a circle, in the sense of all its radii being equal, can have a non-Euclidean circumference to radius ratio. He would be led to postulate a non-Euclidean geometry.

Of course the physicist would never choose this approach. He would rather choose to correct his rulers for thermal distortion partly to avoid the need to learn non-Euclidean geometry. However the preference is based on more solid grounds than sloth.

Suppose the physicist were to measure the disc with several rulers made of different materials. If he were to adopt the non-Euclidean geometry approach to the problem, considering his rulers as standards, he would find the amount of non-Euclidean geometry exhibited (deviation of the ratio of circumference to radius from 2π) would depend on the rulers he used. Thus the geometry of the disc would not be an intrinsic property of its thermal state. The physicist would be strongly advised to abandon this non-Euclidean approach here.

Consider measuring the geometry of space. The main distorting agent now is gravitation and we know from Galileo's experiment and the principle of equivalence that all bodies are equally affected by gravitation. If in analogy to our approach with the disc we choose to ignore gravitational distortion by making no corrections of our rulers, we get a non-Euclidean geometry. But this time the amount of non-Euclidean geometry will be independent of the rulers used, in other words, the geometry of space is an intrinsic property of its gravitational state. (In practice the distance between two points in space is determined by measuring the time taken by light or radio waves to travel from one point to the other. Thus these would be our "rulers" in measuring space). It was this line of reasoning that led Einstein to adopt a non-Euclidean view of space-time * and to seek a theory which relates the curvature of space (deviation from Euclidicity) to gravitation. Many authors claim that this non-Euclidean view is a matter of choice and that, in analogy with the disc, one could choose the alternate approach of Euclidean geometry and measurements corrected for the effects of gravitation. However this choice, as pointed out by Eddington (1959), is precluded by the fact that the process of correction assumes the pre-existence of a space whose properties cannot be ascertained by experiment - a metaphysical space assigned Euclidean properties a priori! This differs markedly from a space whose only distortions arise from thermal effects since, in that case, Euclidean properties are exhibited experimentally in regions of uniform temperature distribution.

Having been led to adopt the non-Euclidean view of space-time, we now seek the particular non-Euclidean geometry to

* The argument given here is for a non-Euclidean space. However, since Einstein realized his relativity theory would have to be constructed in a four-dimensional space-time, he made the obvious extension.

base a theory on. If we wish to incorporate the notions of distance and angle in our theory, a metric geometry is the proper choice. The question is now reduced to: What kind of metric geometry is appropriate for a relativity theory?

Euclidean metric geometry is characterized by the three-dimensional invariant line element (metric)

$$ds^2 = dx^2 + dy^2 + dz^2$$

where x,y,z are standard Cartesian coordinates. If one transforms to non-Cartesian coordinates,

$$x^1 = x^1(x,y,z)$$
$$x^2 = x^2(x,y,z)$$
$$x^3 = x^3(x,y,z)$$

one obtains the line element

$$ds^2 = g_{ik}\, dx^i dx^k$$

where the g_{ik} are functions of x^1, x^2, x^3. Thus a Euclidean space can be <u>disguised</u> as an <u>apparent curved</u> space by the use of curvilinear coordinates. An obvious extension is suggested. <u>Intrinsically</u> curved metric geometries can be given <u>most simply</u> by the expression

$$ds^2 = g_{ik}\, dx^i dx^k$$

However this three-dimensional extension of the Pythagorean expression for distance in Euclidean geometry to non-Euclidean geometry is doomed to failure by the results of the Michelson-Morley experiment. This experiment demonstrated that standards of length must change according to circumstances of motion. Thus the invariance of the three-dimensional line element which corresponds to invariance of length is a fallacy. One is forced to seek another invariant to base metric geometry upon.

The simplest and most obvious choice is a four-dimensional line element,

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

An important point must be made here. It is often suggested that the introduction of the fourth dimension (time) is only a mathematical artifice. Nothing could be further from the truth. Time, the fourth dimension, is united with the three spatial dimensions in an intrinsic way. As Eddington (1959) points out, in the external world the four dimensions are united. It is in the process of relation to an individual that the order falls apart into the distinct manifestations of space and time. An individual is representd by an elongated form in four-space of considerable extension in time and insignificant extension in space. It is when the world is related to such an indivual and his own asymmetry introduced in the relation that the order of events parallel to his own track appears in his experience to be differentiated from all other orders of events. This may be considered as a modified Aristotlian argument. A tree has no inherent length or duration (only four dimensional extent); it is the observer who imbues it with these qualities. (An excellent illustration of the "reality" of the fourth dimension is given in the discussions of "foreshortening" by Eddington (1959) and Rindler (1969)).

The choice of a Riemannian metric geometry

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

can be established uncontestably if one accepts special relativity and the principle of equivalence as true. The principle of equivalence states that we can locally abolish gravitational fields by attaching ourselves to a free falling body. Hence there exists a transformation of coordinates (actually an infinity of transformations) which makes a curved space-time metric equal the metric of gravitation-free space at a point. From special relativity we know this "flat" space metric is of the "pseudo-Euclidean" form,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

These observations enable us to invoke the theorem that any metric which corresponds locally to a "pseudo-Euclidean" metric must be Riemannian at the point of correspondence.

Having discovered the metric formalism of relativity theory, we now seek the mechanism which relates the geometry of space-time to the material content of space.

2.3. The Einstein Field Equations.

Newton's first law states that a body on which no forces act will move with a constant speed in a straight line (Law of Inertia). On subjection to scrutiny, this law runs into serious difficulties from the start. One of the fundamental criteria for any law is that it supply, within its framework, the possibility of experimental verification or rebuttal. It is here that the plausible sounding Law of Inertia encounters irreconciliable difficulty, for one can remove all forces acting on a body <u>except</u> gravitation. Newton suggested that one can isolate a body from gravitational forces by removing it sufficiently far from all other bodies. This counter fails for two reasons. Firstly, we know from astronomical evidence that we live in a populated universe and thus cannot remove a body indefinitely far from all other objects in the universe.

Secondly and more fundamentally, even if it were possible to establish the standard motion of Newton's first law (straight line, constant speed trajectory in absence of forces) out beyond the regions of galactic matter, it would be of no use to the experimental physicist. The experimentalist wants the standard in his laboratory where he can <u>use</u> it.

Thus we are led to reformulate Newton's first law. Instead of the definition that standard motion is the motion exhibited by a body in the absence of forces, we adopt the definition that a body exhibits <u>standard motion</u> when all <u>abolishable</u> forces have been elimated. Therefore gravitation ceases to be regarded as a force and now combines with inertia in defining standards of motion. (The standards of motion are now, of course, much more complicated than the straight line, constant speed standards of Newton's first law).

Given that gravitation can be abolished <u>locally</u>, we see that the <u>intrinsic</u> part of the gravitational field (that which can <u>not</u> be abolished) is the part exhibited by the <u>relative</u> acceleration of freely falling bodies. (Free falling bodies are now taken to be those on which no forces <u>other</u> than gravity act).

In the neighborhood of any given event E_0, The Newtonian gravitational field can be given by:

$$\vec{f} = \vec{f}_0 + \Delta\vec{f}_0$$

where $\vec{f}_0$ is the gravitational field at E_0 and $\Delta\vec{f}_0$ is defined by the above equation. Since $\vec{f}_0$ can be abolished by going to a local inertial frame attached to a freely falling body, the tidal field, $\Delta \vec{f}_0$, must be the <u>intrinsic</u> part of the gravitational field. (The appelation, tidal field, originates in the fact that tides on earth are caused by just this sort of field). Considering Einstein's famous elevator, it is this tidal field which causes two free particles on a common horizontal to accelerate towards each other and eventually collide at the earth's center. Thus we can determine experimentally the existence or non-existence of an intrinsic gravitational field by the observation of two neighbouring free test particles. In this spirit we seek a reformulation of Newtonian gravitation in terms of the <u>intrinsically measurable</u> part of the gravitational field.

Every Newtonian gravitational field $\vec{f}$ is derivable from a potential φ :

$$f_i = -\frac{\partial \varphi}{\partial x^i}$$

The relative acceleration of two test particles, which is the criterion for the existence of an <u>intrinsic</u> gravitational field, is given by:

$$\frac{d^2 n^i}{dt^2} = df_i = -\frac{\partial^2 \varphi}{\partial x^i \partial x^j} n^j \qquad (2.1)$$

where n^i is the small connecting three-vector between the two particles. Thus it is the <u>second</u> derivatives of the potential that indicate an intrinsic field. We recall that the field is related to its sources by Poisson's equation:

$$\sum_{i=1}^{3} \varphi_{ii} = 4\pi G\rho$$

This is the field equation of Newtonian gravitational theory. For the problem treated in this work we shall be interested in only vacuum fields $(\rho = 0)$ so Poisson's equation reduces to Laplace's equation;

$$\sum_{i=1}^{3} \varphi_{,ii} = 0 \qquad (2.2)$$

However, we immediately notice defects in our "new" Newtonian gravitation theory. One defect has already been mentioned. The gravitational "field" has an infinite velocity of propagation contrary to the edicts of special relativity which place an upper bound of the velocity of light on such propagations. Another defect arises from the Newtonian energy concept for gravitation. Consider a sphere of mass M and radius ρ. Its self energy is a multiple of $\frac{M^2}{\rho}$, a quantity which is clearly unbounded as ρ approaches zero. Thus by shrinking the sphere we get out as much energy as we please. Hence Newtonian gravitational theory is inconsistent with a meaningful energy concept. Yet another objection to Newtonian gravitational theory can be made on aesthetic grounds. In a complete theory, in addition to the field equation, we need a conservation law dealing with the permanency of source. In our Newtonian theory this would require an additional statement whereas in an elegant theory like Maxwell's, conservation of sources follows directly from the structure of the equations. We are thus led to seek an alternate theory.

In developing an alterate gravitational theory, one of the primary postulates Einstein invoked was that free falling particles follow <u>geodesics</u>, lines of minimal length between any two points of curved four-space. The common sense of this proposal is borne out by considering an analogy with classical mechanics. In classical mechanics, free test particles confined to a smooth curved surface, and in the absence of external forces, trace out geodesics in three space. This analogy also illustrates why in a relativity theory we must consider geodesics in four-space. Geodesics in three space are uniquely determined by an initial direction, whereas we know gravitational orbits depend on both initial direction and initial velocity. Finally, special relativity theory provides a precedent for using geodesics in four-space to describe free particle trajectories.

According to the definition of the term "geodesic", the mathematical statement for a geodesic line is obtained by extremizing $\int ds$.

One finds (Landau and Lifshitz, 1962):

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds} = 0 \qquad (2.3a)$$

where $\Gamma^\mu_{\nu\sigma}$ (Christoffel symbol) is given by,

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\tau}\left[\frac{\partial g_{\tau\nu}}{\partial x^\sigma} + \frac{\partial g_{\tau\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\tau}\right] \qquad (2.3b)$$

and $x^\mu = x^\mu(s)$ is the parametric representation of the coordinates of the test particle in terms of its proper time s.

As in the development of the Newtonian gravitation theory we wish to consider the relative acceleration of two neighbouring test particles in a gravitational field. Since geodesics describe the trajectories of test particles, we consider two nearby and almost parallel geodesics. Letting n^μ be the vector joining them and orthogonal to both we obtain the equation of "geodesic deviation",

$$\frac{D}{ds}\left(\frac{D\eta^{\mu}}{ds}\right) = \left(R^{\mu}{}_{\nu\rho\sigma}\frac{dx^{\nu}}{ds}\frac{dx^{\rho}}{ds}\right)\eta^{\sigma} \quad (2.4)$$

where

$$\frac{DA^{\mu}}{ds} \equiv \frac{dA^{\mu}}{ds} + \Gamma^{\mu}_{\nu\sigma}A^{\nu}\frac{dx^{\sigma}}{ds}$$

and,

$$R^{\mu}{}_{\nu\rho\sigma} \equiv \frac{\partial}{\partial x^{\rho}}\Gamma^{\mu}_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\rho}.$$

(For details of derivation see Synge, 1966.)

The tensor $\frac{DA^{\mu}}{ds}$ is called the absolute derivative of A^{μ} and the tensor $R^{\mu}{}_{\nu\rho\sigma}$ is called the Riemann curvature tensor.

One now makes the observation that in a local inertial frame where the Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ vanish, $\frac{D}{ds}\left(\frac{D\eta^{\mu}}{ds}\right)$ reduces to $\frac{d^2\eta^{\mu}}{ds^2}$. Also ds is approximately equal to cdt for the "classical" limit of slow velocities. Thus comparing (2.1) with (2.4), we see that $R^{\mu}{}_{\nu\rho\sigma}U^{\nu}U^{\rho}\left(U^{\rho} = \frac{dx^{\rho}}{ds}\right)$ corresponds to $-\frac{\partial^2\varphi}{\partial x^i \partial x^j}$. Hence the analogue of Laplace's equation in space-time is:

$$R^{\mu}{}_{\nu\rho\mu}U^{\nu}U^{\rho} = 0.$$

If this is to hold for arbitrary U^{μ}, then

$$R^{\mu}{}_{\nu\rho\mu} \equiv R_{\nu\rho} = 0 \quad (2.5)$$

The tensor $R_{\nu\rho}$, defined by the above equation, is called the Ricci tensor. The field equations (2.5) are the famous <u>Einstein field equations in vacuum</u> (Einstein, 1915).

Using the definitions of the Ricci tensor and Riemann curvature tensor one finds $R_{\mu\nu} = R_{\nu\mu}$.

so that there are ten independent empty space field equations. At first this may seem as an overdetermination since there are also only ten independent metric tensor componets ($g_{\alpha\beta}$ is symmetric in α, β .), the implication being the $g_{\alpha\beta}$ are uniquely determined. This would conflict with the Principle of Covariance which states we should be able to change coordinates at will in space-time thus transforming the metric. For consistency with the Principle of Covariance we would like four degrees of freedom corresponding to the four arbitrary coordinates $x'^{\mu} = x'^{\mu}(x^{\nu})$ which specify a transformation. Fortunately, there are four differential identities (Bianchi Identities) which the field equations satisfy and thus four degrees of freedom in the determination of the $g_{\alpha\beta}$, as desired.

The metric tensor compontents, $g_{\alpha\beta}$, determined from the field equations are the analogues of the potential, φ, in Newtonian theory. Their role as potentials can be readily seen by comparing the expression for "acceleration" in Maxwellian theory (Φ denoting the four-vector potential)

$$\frac{d^2x^{\mu}}{ds^2} = \frac{1}{c} \sum_{\tau,\nu} g^{\mu\tau} \left(\frac{\partial \Phi}{\partial x^{\tau}} - \frac{\partial \Phi}{\partial x^{\nu}} \right) \frac{dx^{\nu}}{ds}$$

with the expression for "acceleration" in general relativity theory (2·3). The $g_{\alpha\beta}$'s clearly are in the roles of potentials and general relativity can be formally regarded as a gravitational field theory with tensor potentials.

Viewing the metric tensor components as potentials we see from the definition of $R_{\mu\nu}$ that the field equations are second order differential equations in analogy with Laplace's equation (2.2) in Newtonian theory.

A final,,extremely important observation is that the field equations (2.5) are non-linear, containing products of the g's and their derivatives. This is an essential characteristic if the field equations are to take into account mass-energy dependence. For there is a gravitational "binding energy", which in a manner similar to nuclear "binding energy", causes "mass defect". A body exerts a gravitational force equal to its constituent parts minus the mass equivalent of the energy necessary to separate the parts. In a linear theory the gravitational force of the whole would be equal to that of its constituent parts by virtue of the "superposition" of solutions in linear theory.

CHAPTER 3

THE WEYL LINE MASS METRIC

3.1. Introduction

In chapter 2, we developed the formalism, the Einstein field equations, for relating the geometry of curved space-time to the material content of space. In this chapter we shall develop the particular geometry of the universe of a line mass.

The Weyl axially symmetric formalism (Weyl, 1917, 1919) shall be developed in section 3.2 and particularized to the problem of the line mass in section 3.3. Also, the transformation of the line mass metric to "Schwarzschild" form will be given in section 3.3.

3.2. The Weyl Static Axially Symmetric Solutions.

In the development of any metric corresponding to a physical problem the first step is to make maximal use of the symmetries of the problem in order to reduce the metric to its simplest form. The Weyl solutions correspond to problems in which the metric is assumed to be:

1) stationary (the field produced by the body is a constant)
2) rotationally symmetric
3) invariant under reversal of time $(t \to -t)$

and 4) invariant under reversal of angle $(\varphi \to -\varphi;\ \varphi$ being the azimuthal angle.)

(Conditions 1) and 3) are often rephrased as the conditions for a static field.) Given this set of conditions we can now proceed to reduce the complexity of the metric.

Mathematically, assumption 1) implies that the metric is independent of the time coordinate, t. Assumption 3) implies the metric form, ds^2, contains dt only as a square. Thus,

$$g_{\alpha\beta} \neq g_{\alpha\beta}(t)$$
$$g_{4i} \equiv 0$$

Physically, assumptions 1) and 3), the static field conditions, imply that the body producing the field is fixed in the reference frame in which the metric potentials are independent of the time coordinate. Consequently both directions of time are equivalent.

Assumption 2), rotational symmetry, implies that with a proper choice of cordinate system the field is independent of azimuthal angle, φ. Thus,

$$g_{\alpha\beta} \neq g_{\alpha\beta}(\varphi)$$

Physically assumption 4) means that we are dealing with non-rotating matter. Mathematically it implies that the metric form, ds^2, contains $d\varphi$ only as a square. Thus,

$$g_{13} = g_{23} = g_{43} \equiv 0 .$$

Synthesizing the results obtained so far, the line element can be written as:

$$-ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^4)$$

where the g's are functions of (x^1, x^2) and where $x^3 \equiv \varphi$ and $x^4 \equiv t$.

Call $\quad \psi \equiv g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2$

We now note that except for the condition that they must be in the meridian plane, x^1 and x^2 are two completely arbitrary variables. Thus we have the freedom to impose two coordinate conditions corresponding to our freedom of preforming coordinate transformations in this plane. We choose

$$g_{11} = g_{22} \text{ and } g_{12} = 0.$$

Then ψ assumes the isothermal form:

$$\psi = \alpha^2[(dx^1)^2 + (dx^2)^2]$$

α being a function of (x^1, x^2). Our line element now assumes the form

$$-ds^2 = \alpha^2[(dx^1)^2 + (dx^2)^2] + \beta^2(dx^3)^2 - \gamma^2(dx^4)^2 \quad (3.1)$$

where (α, β, γ) are functions of (x^1, x^2).*

If one now calculates the components of the Ricci tensor, one finds that the only non-vanishing terms are:

$$R_{11} = \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{11}}{\beta} + \frac{\gamma_{11}}{\gamma} + \frac{\alpha_2}{\alpha}\left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right) - \frac{\alpha_1}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right),$$
$$R_{22} = \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{22}}{\beta} + \frac{\gamma_{22}}{\gamma} + \frac{\alpha_1}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) - \frac{\alpha_2}{\alpha}\left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right),$$
$$R_{12} = \frac{\beta_{12}}{\beta} + \frac{\gamma_{12}}{\gamma} - \frac{\alpha_2}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) - \frac{\alpha_1}{\alpha}\left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right),$$
$$R_{33} = \frac{\beta}{\alpha_2}\left\{\Delta\beta + \frac{1}{\gamma}(\beta_1\gamma_1 + \beta_2\gamma_2)\right\}, \quad (3.2)$$
$$R_{44} = -\frac{\gamma}{\alpha^2}\left\{\Delta\gamma + \frac{1}{\beta}(\beta_1\gamma_1 + \beta_2\gamma_2)\right\}$$

where the subscripts on the expressions on the right of the equalities denote partial derivatives with respect to x^1 and x^2, and

$$\Delta\beta = \beta_{11} + \beta_{22}, \quad \Delta\gamma = \gamma_{11} + \gamma_{22}$$

We note that

*The ingenious argument which follows for reducing the number of unknowns from three to two was first developed by Weyl (1917).

$$R^3_3 + R^4_4 = \beta^{-2} R_{33} - \gamma^{-2} R_{44} = \frac{1}{\alpha^2 \beta\gamma} \Delta(\beta\gamma).$$

We are interested in vacuum solutions where $R_{\mu\nu} = 0$. Thus,

$$\Delta(\beta\gamma) = 0.$$

This means that $\beta\gamma$ is a harmonic function of (x^1, x^2) which we denote by

$$\beta\gamma = r(x^1, x^2). \qquad (3.3)$$

From the theory of Complex Variables we know that there must exist a conjugate harmonic function $z(x^1, x^2)$, such that

$$r + iz = f(x^1 + ix^2)$$

where f is a an analytic function. We now make the transformation

$$(x^1, x^2) \rightarrow (r, z).$$

This transformation is conformal by virtue of f being analytic, hence it preserves the isothermal character of a quadratic form. Thus,

$$\alpha^2 [(dx^1)^2 + (dx^2)^2] = A(dr^2 + dz^2)$$

where A is a function of (r,z). But from (3.3) we have

$$\beta = \frac{r}{\gamma}$$

and so ds^2 of (3.1) becomes a form involving only <u>two</u> arbitrary functions:

$$-ds^2 = \alpha^2 (dr^2 + dz^2) + r^2 \gamma^{-2} d\varphi^2 - \gamma^2 dt^2$$

where (α, γ) are <u>redefined</u> functions of (r,z).

If we change notation from (α, γ) to (λ, ν) by the relations

$$\alpha = e^{\nu - \lambda}, \quad \beta = r e^{-\lambda}, \quad \gamma = e^{\lambda}$$

the metric becomes

$$-ds^2 = e^{2(\nu - \lambda)} (dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 - e^{2\lambda} dt^2. \qquad (3.4)$$

From eqn (3.2) we find

$$\frac{1}{2}(R_{11}+R_{22}) = \Delta\nu - \left(\Delta\lambda + \frac{\lambda_1}{r}\right) + \lambda_1^2 + \lambda_2^2;$$

$$\frac{1}{2}(R_{11}-R_{22}) = \lambda_1^2 - \lambda_2^2 - \frac{\nu_1}{r},$$

$$R_{12} = 2\lambda_1\lambda_2 - \frac{\nu_2}{r},$$

$$R^3_3 - R^4_4 = -\frac{2}{\alpha^2}\left(\Delta\lambda + \frac{\lambda_1}{r}\right),$$

$$R^3_3 + R^4_4 = 0 .$$

In vacuum, where $R_{\mu\nu}=0$, these equations are equivalent to

$$\Delta\lambda + \frac{\lambda_1}{r} = 0, \qquad (3.5a)$$

$$\nu_1 = r(\lambda_1^2 - \lambda_2^2), \quad \nu_2 = 2r\lambda_1\lambda_2 \qquad (3.5b)$$

$$\Delta\nu + \lambda_1^2 + \lambda_2^2 = 0 \qquad (3.5c)$$

Note that (3.5c) is implied by (3.5a) and (3.5b).

Thus to obtain a Weyl vacuum solution we choose a λ satisifying (3.5a), substitute its first derivatives into (3.5b), and integrate the resulting expressions in (3.5b), solving for ν. The functions ν and λ thus obtained are substituted into (3.4), giving the metric solutions.

If we write out equation (3.5a) explicitly

$$\frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0$$

we immediately notice that it is Laplace's equation in cylindrical polar coordinates for a function, λ, independent of φ. Thus we come to the rather startling conclusion that λ satisfies the same equation (Laplace's equation) as does the vacuum gravitational potential in Newtonian theory. Hence we can generate general relativistic solutions given Newtonian potentials, Φ. The procedure is clear. We first choose $\lambda = \frac{\Phi}{c^2}$ (the factor of $\frac{1}{c^2}$ is introduced in order that λ be dimensionless) and then follow the prescription given above to determine ν and subsequently the metric. We shall now apply this procedure to the problem of a line mass.

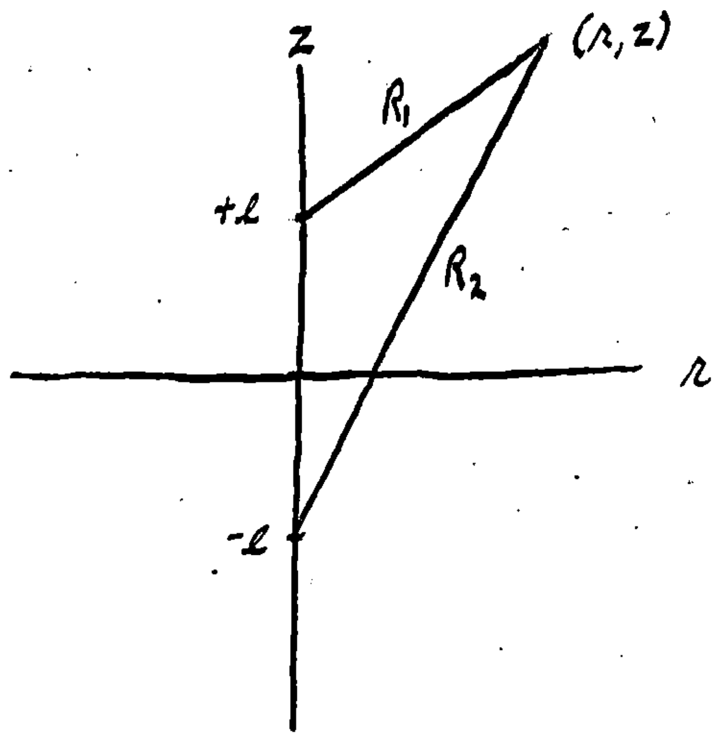
3.3. The Weyl Line Mass Solution.

The Newtonian potential for a line mass of constant mass density is given by

$$\Phi = -\frac{GM}{2\ell} \ln \frac{R_1 + R_2 + 2\ell}{R_1 + R_2 - 2\ell}$$

where

$$R_1^2 = (z-\ell)^2 + r^2$$
$$R_2^2 = (z+\ell)^2 + r^2$$



Applying the procedure given in section 3.2 we choose $\lambda = \frac{\sigma}{c^2}$ and integrate (3.5b) to find ν, obtaining

$$\lambda = \frac{\mu}{2l} \ln \frac{R_1 + R_2 - 2l}{R_1 + R_2 + 2l}$$

$$\nu = \frac{1}{2}\left(\frac{\mu}{l}\right)^2 \ln \frac{(R_1+R_2)^2 - 4l^2}{4R_1 R_2} \tag{3.6}$$

where $\mu \equiv \frac{GM}{c^2}$ (Robertson and Noonan, 1968; see also Bach and Weyl, 1922). Thus the Weyl line mass metric is given by (3.4) where the ν and λ appearing in the metric are given by (3.6).

The special case $l = \mu \equiv \frac{GM}{c^2}$ constitutes the focal point of this work. Written out explicitly this line element (metric) is

$$ds^2 = \frac{R_1 + R_2 - 2\mu}{R_1 + R_2 + 2\mu} dt^2 - \frac{(R_1 + R_2 + 2\mu)^2}{4R_1 R_2}(dr^2 + dz^2) - \frac{R_1 + R_2 + 2\mu}{R_1 + R_2 - 2\mu} r^2 d\varphi^2 \tag{3.7}$$

where $c = 1$.

If we now transform to coordinates given by

$$\rho = \frac{R_1 + R_2 + 2\mu}{2}$$

$$\cos\theta = \frac{R_2 - R_1}{2\mu}$$

the line element (3.7) is transformed into

$$ds^2 = \left(1 - \frac{2\mu}{\rho}\right) dt^2 - \frac{1}{1 - \frac{2\mu}{\rho}} d\rho^2 - \rho^2 d\theta^2 - \rho^2 \sin^2\theta \, d\varphi^2 \tag{3.8}$$

where the ρ coordinate is only defined for $\rho > 2\mu$.

It is in the interpretation of this last expression (3.8) that this work differs irreconciliably from the literature (cf. footnote p.3). The standard interpretation is that (3.8) is the spherically symmetric Schwarzschild exterior solution. Thus, since the particular line mass solution (3.7) is transformable into (3.8), it is further claimed that (3.7) must be spherically symmetric. The remainder of this work shall be an attempt at the refutation of these latter statements.

CHAPTER 4

INTERPRETATION OF THE "SCHWARZSCHILD" METRIC.

4.1. Metric-Coordinate Interdependence.

In Chapter 2, it was shown that given the material content of space, one could determine the corresponding metric potentials, $g_{\alpha\beta}$, through the use of the Einstein field equations. It was further shown that once the metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ was thus determined the geometry of the four-space was completely defined. But the metric does more than define the geometry of space, it also describes all the geometrical properties of the coordinates which are used in its own expression. Consider the two-dimensional metric $ds^2 = dx_1^2 + x_1^2 dx_2^2$. The metric expression not only tells us that our two-dimensional space corresponds to a flat surface* but also that the coordinates x_1 and x_2 are the familar coordinates r and θ .

However, one must not be mislead by the simplicity of the two-dimensional example given above. Although it is still true that the metric describes all the geometrical properties of the coordinates in four-space, characterizing the coordinates can be a difficult task. Consider as an illustrative example the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2u}{\rho}\right)dt^2 - \frac{1}{\left(1 - \frac{2u}{\rho}\right)}d\rho^2 - \rho^2 d\theta^2 - \rho^2 \sin^2\theta\, d\varphi^2 \qquad (4.1)$$

---

* It should be noted that the metrical relations on a plane are not altered if the plane is rolled into a cylinder- the measures being confined to the two dimensional world. A truly two-dimensional being could not distinguish between a plane and a cylinder (except at the joining of the edges).

One can show that all the information necessary for characterizing the coordinates is contained in the metric. If there exists another set of coordinates $(\rho', \theta', \varphi', t')$ which can be used equally well for the coordinates $(\rho, \theta, \varphi, t)$ in (4.1), that is, which can preserve the form of the metric, then transformation from $(\rho, \theta, \varphi, t)$ to $(\rho', \theta', \varphi', t')$ gives

$$ds^2 = \left(1 - \frac{2u}{\rho'}\right)dt'^2 - \frac{1}{1-\frac{2u}{\rho'}}d\rho'^2 - \rho'^2 d\theta'^2 - \rho'^2 \sin^2\theta' d\varphi'^2 \qquad (4.2)$$

The only continuous family of transformations which transform (4.1) into (4.2) are the time translations $t' = t + t_0$ and θ, φ transformations corresponding to rigid rotations of the standard sphere. Thus it is demonstrated that the $(\rho, \theta, \varphi, t)$ in (4.1) are defined even more uniquely by the metric (4.1) than the rectangular coordinates of the Minkowski space (flat space).

Although the above argument demonstrates that the coordinates $(\rho, \theta, \varphi, t)$ are characterized implictly by the metric (4.1), it does not provide an algorithm for computing these coordinates at some physically defined space-time event. Except for a few simple cases, algorithms for operationally measuring the coordinate values of a given metric have <u>not</u> been devised. Such algorithms would have doubtful practicality anyway since an observer doesn't measure <u>coordinate</u> values but invariants. Thus the theorists' coordinate systems are merely intermediary practical aids in arriving at the physically significant invariants.

The important point to be stressed in the above discussion is that although the coordinates of a metric are defined implicity (to within a well-defined group) by the metric itself, the characterization of the coordinates must be approached with extreme caution.

In section 4.2 it shall be demonstrated that the coordinates of the Schwarzschild solution (4.1) do not correspond to those of the "Schwarzschild" form of the line mass of "length" 2μ (3.8) as one would be tempted to claim on a cursory investigation of the metric (3.8).

4.2. The "Schwarzschild" Metric.

In Chapter 3 it was found that the line mass metric for a "length" 2μ (3.7) could be transformed into "Schwarzschild" form

$$ds^2 = \left(1-\frac{2\mu}{\rho}\right)dt^2 - \frac{1}{1-\frac{2\mu}{\rho}}d\rho^2 - \rho^2 d\theta^2 - \rho^2 \sin^2\theta \, d\varphi^2 \quad (3.8)$$

where ρ is defined only for values $\rho \geq 2\mu$.

The traditional view claims that this metric corresponds exactly to the spherically symmetric Schwarzschild solution,

$$ds^2 = \left(1-\frac{2\mu}{\rho}\right)dt^2 - \frac{1}{1-\frac{2\mu}{\rho}}d\rho^2 - \rho^2 d\theta^2 - \rho^2 \sin^2\theta \, d\varphi^2 \quad (4.1)$$

in the region $\rho > 2\mu$.

However, since it is the metric itself which characterizes the coordinates (to within a well-defined group), one must exercise caution so as not to be misled into erroneous conclusions as a result of a cursory investigation. One must consider the metric statements in their totality.

Upon closer investigation of the metrics (3.8) and (4.1) we note an intrinsic disparity. In (3.8) the appending condition, where ρ is defined only for values $\rho > 2u$, is a statement about the intrinsic properties of the coordinate itself. On the other hand the appending condition of (4.1), in the region $\rho > 2u$, is merely a statement as to the region of space-time we are interested in. It does not state anything about the coordinate ρ. In fact for the metric (4.1) the coordinate ρ is defined for values $\rho > 0$, except for $\rho = 2u$. Thus the ρ's in (3.8) and (4.1) do not have the same spatial range. One may object that the definition of the ρ coordinate given implicitly by a metric expression is only unique to within a group of transformations. It may be suggested, that the truncation of the ρ values in (3.8) is a result of the transformation $\rho' = \rho + 2u$ where the primed ρ is the ρ appearing in (3.8) while the unprimed one is the ρ appearing in (4.1). However it is immediately evident that the metric form is changed by such a transformation and thus such a ρ' does not correspond to the ρ characterized by the Schwarzschild metric. Indeed there is no hope for any similar argument since, as we have shown in section 4.1, the $(\rho, \theta, \varphi, t)$ coordinates are defined by the Schwarzschild metric to within the transformations $t' = t + t_0$ and transformations corresponding to rigid rotations of the standard sphere. Neither of these transformations can explain a truncation of ρ value.

Further we notice from the development of the metric (3.8) in Chapter 3 that the degenerate surface $\rho = 2u$ (a constant) corresponds physically to the <u>line</u> mass. Thus we are led to abandon association of surfaces of constant ρ to surfaces of constant physical distance from the center of our source. In fact we are strongly inclined to regard surfaces of constant ρ physically as ellipsoids centered at the center of our source with their major axes lying along the line mass. This is to be contrasted with the bona fide Schwarzschild solution (4.1) whose constant ρ surfaces correspond to surfaces of constant physical distance from the center of the source. *

Thus since the ρ in the metric (3.8) does <u>not</u> belong to the group of ρ's <u>defined</u> by the Schwarzschild metric, the metric (3.8) can <u>not</u> be considered as the Schwarzschild form of the spherically symmetric vacuum solution. Further since the metric potentials, $g_{\alpha\beta}$ in (3.8) and (4.1) have exactly the same functional form, the metrics (3.8) and (4.1) can <u>not</u> be considered equivalent. For any attempt at establishing such an equivalence by transformation from the Schwarzschild ρ to the ρ in (3.8) will be invalidated by the consequent change in functional form of the metric potentials. Thus the two metrics must describe <u>intrinsically</u> different space-times.

An important by-product has been obtained in our discussion of the metrics (3.8) and (4.1). The discussion clearly illustrates that if one is to preserve the metric-coordinate interdependence of non-Euclidean geometry, one must include any appending conditions which deal with intrinsic properties of the coordinates in <u>one's criteria for judging whether two metrics are the same.</u>

* This does not assume that ρ is a true radial distance. In fact it is not. Radial distance is here given by $\int (1-\frac{2\mu}{\rho})^{-\frac{1}{2}} d\rho$. [Integration carried out between the two events of constant (t,θ,φ)]

CHAPTER 5

SINGULARITIES.

5.1. Introduction.

The subject of singularities is one of the least understood and most speculated on topics of the general theory of relativity. In large part, the difficulty of obtaining consensus of opinion regarding statements about singularities originates in their very definition. A singularity is defined as an irregular * region of the metric field. Thus the meaning of such statements as isolated singularities follow geodesics is unclear since quantities dealt with in a field theory, such as geodesics, are defined only for regular points of the field.

Another basic difficulty encountered in dealing with singularities is the determination of whether they are intrinsic or coordinate singularities. An intrinsic singularity is one which can not be removed by coordinate transformation. It is a physical property of the metric geometry and corresponds to a non-vacuum region of space-time $(T^{uv} \neq 0)$. A coordinate singularity, on the other hand is one due entirely to the choice of coordinate system.

* An irregular region of the metric field is one in which one or more of the metric components $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are not differentiable.

The practical difficulty in determining the nature of a singularity is clearly illustrated by the history of the Schwarzschild singularity at $\rho = 2\mu$. If one considers the Schwarzschild solution (4.1) one immediately notices that the metric field is singular at $\rho = 0$ and at $\rho = 2\mu$. From 1916, when Schwarzschild published his famous solution (4.1), until 1933 it was believed that the $\rho = 2\mu$ singularity was an intrinsic one. Finally, Le Maitre (1933) unmasked the fallacy of the then traditional interpretation by showing that the $\rho = 2\mu$ singularity is merely due to a particular choice of coordinate system. In Le Maitre's form of the spherically symmetric vacuum solution,

$$ds^2 = dT^2 - \frac{2\mu}{\rho}\, dR^2 - \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

where

$$\rho = \left[\frac{3\sqrt{2\mu}}{2}\,(T-R)\right]^{\frac{2}{3}}.$$

The only singularity occurs at $T = R$ which corresponds to the origin. *

The Schwarzschild singularities also effectively illustrate the physical difference between a coordinate and intrinsic singularity. An observer in a small, freely falling cabin would pass through the $\rho = 2\mu$ coordinate singularity without noticing anything special about it, while any observer would clearly recognize the non-vacuum characteristics at the intrinsic singularity $\rho = 0$.

* Actually Eddington (1929) first transformed the Schwarzschild metric into a form not singular at $\rho = 2\mu$ but the implication went unnoticed.

Since the transformations to remove coordinate singularities are not in general readily determinable, one must seek alternate tests of the nature of singularities. There are two such tests in common use. The first consists in calculating the invariants of the curvature tensor, $R^{\mu}{}_{\nu\rho\sigma}$. The curvature tensor corresponds to tide-producing field gradients (cf. Chapter 2). Thus a coordinate independent combination of the components of this tensor becoming infinite in a particular region of space-time implies that this region is <u>intrinsically</u> singular. Note however that the converse of this statement <u>isn't</u> necessarily true. Hence, unfortunately, this test will not always give conclusive results.

The second test commonly used in determining whether a given region of space-time is intrinsically singular or not is the <u>elementary flatness</u> criterion. In essence it is the application of the criterion that in a vacuum region $(T^{\mu\nu} = 0)$ the <u>local</u> characteristics of the space are Minkowskian. This test shall be discussed in more detail in section 5.4.

In this chapter we undertake an indirect proof that the <u>exterior</u> regions of the bona fide Schwarzschild solution and the "Schwarzschild" form of the Weyl line mass solution do not correspond under transformation. We first show that the boundary of the exterior Weyl line mass (the line itself) corresponds to an <u>intrinsic</u> singularity. Consequently the surface $\rho = 2\mu$ which corresponds to this boundary under the mapping preceding (3.8) differs physically from the merely coordinate singular surface $\rho = 2\mu$ of the exterior Schwarzschild solution. We then invoke the argument that if the boundaries of regions of space-time do not correspond physically under transformation then the regions themselves do not correspond physically under the given transformation. Assuming the validity of this argument, the proof is complete and we have reduced the demonstration of non-correspondence between the Schwarzschild solution and the "Schwarzschild" form of the Weyl line mass to the demonstration of the intrinsic singularness of the boundary of the Weyl line mass solution.

The intrinsic singularness of the boundary of the exterior Weyl line mass solution will be investigated using the two methods discussed above: analysis of the curvature invariants and application of the elementary flatness criterion.

In section 5.2 we develop briefly the powerful differential forms approach to Riemannian geometry. In section 5.3 this approach will be used to analyze the invariants formed from the curvature tensor in an attempt to locate the intrinsic singularities of the Weyl line mass solution. In section 5.4 the elementary flatness criterion shall be introduced and its use illustrated by a historic example. In section 5.5 the elementary flatness criterion will be applied to the Weyl line mass solution of "length" 2μ in order to investigate the intrinsic singularness of the line itself.

5.2. Differential Forms

In this section, the differential forms approach to Riemannian geometry will be developed. In the interest of brevity definitions of operations and the algebra of differential forms will be presupposed here. The reader unfamiliar with the operations and algebra of differential forms is referred to Appendix A for a cursory introduction.*

A differential form is a multilinear scalar function associated with a tensor which is effectively completely skew-symmetric. For example, if $T_{\alpha\beta}$ is a second-order tensor, then

$$\Phi(d\underset{\sim}{x}, d\underset{\sim}{y}) \equiv 2!\, T_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]}$$

is a differential form of degree two (a 2-form). The square brackets enclosing a set of indices indicate anti-symmetrization,

$$dx^{[\alpha} dy^{\beta]} \equiv \left(\frac{1}{2!}\right)\left(dx^{\alpha} dy^{\beta} - dx^{\beta} dy^{\alpha}\right).$$

Note that the anti-symmetrization of the square bracket operation effectively sifts out the skew part $T_{[\alpha\beta]}$ of the original tensor.

The traditional approach to Riemannian geometry relies heavily on the use of Christoffel symbols (2.3b) . The curvature of four-space is described by the Riemann tensor $R^{\alpha}_{\beta\gamma\delta}$ which is in turn described entirely in terms of the Christoffel symbols and their derivatives. This approach has two important drawbacks. Firstly, the Christoffel symbols have

*Those desiring more than an operative knowledge of the subject are referred to "Differential Forms in General Relativity" by W. Israel and "Differential Forms with Application to the Physical Sciences" by H. Flanders.

no invariant significance and secondly, they are extremely cumbersome to work with. The differential forms approach to Riemannian geometry allows the Riemann tensor components to be expressed in terms of invariants and their first partial derivatives. As shall be demonstrated later, this elegance not only resolves the first problem of the traditional approach but leads to computational facility thereby resolving the second.

Consider N linearly independent vector fields $\underset{\sim}{e}_{(1)}, \cdots \underset{\sim}{e}_{(N)}$ in a Riemannian N-space. Because of their linear independence, these vector fields form a complete vector basis (frame) at each point in the Riemannian N-space. The Ricci rotation coefficients γ^{A}_{BC} associated with the vector fields $\underset{\sim}{e}_{(A)}$ are a set of N^3 numbers, scalars under coordinate transformations, which give the components with respect to the frame of the covariant derivatives of the $\underset{\sim}{e}_{(A)}$. It is known from the classical theory of Ricci rotation coefficients that the Riemann tensor is expressible in terms of the γ^{A}_{BC} and their first partial derivatives. The differential forms approach to Riemannian geometry is a modified version of the Ricci rotation coefficient formalism in which attention is shifted from the basis vectors, Ricci rotation coefficients, and Riemann tensor components to sets of forms associated with them.

Consider N linearly independent vector fields $\underset{\sim}{e}_{(A)}(x^{\mu})$ in a Riemannian N-space. One defines the "frame components" (tetrad components in four dimensions) of any tensor $T_{\alpha\beta}$ by

$$T_{AB\ldots} \equiv T_{\alpha\beta\ldots}\, \ell_{(A)}{}^{\alpha} \ell_{(B)}{}^{\beta} \ldots$$

where the $\ell_{(A)}^{\alpha}$ are the contravariant components of the basis vector $\underset{\sim}{\ell}_{(A)}$.

We introduce the matrix of scalar products of the basis vectors

$$g_{AB} \equiv \underset{\sim}{\ell}_{(A)} \cdot \underset{\sim}{\ell}_{(B)}$$

or

$$g_{AB} = g_{\alpha\beta}\, \ell_{(A)}{}^{\alpha} \ell_{(B)}{}^{\beta} \qquad (5.2.1)$$

It is clear from (5.2.1) that g_{AB} are the frame components of the metric tensor.

Now since the $\underset{\sim}{\ell}_{(A)}$ are linearly independent, the matrix g_{AB} has a symmetric inverse g^{AB} such that $g^{AB} g_{AC} = \delta^{B}_{C}$. Defining the dual basis $\underset{\sim}{\ell}^{(A)}$ as

$$\underset{\sim}{\ell}^{(A)} \equiv g^{AB} \underset{\sim}{\ell}_{(B)}$$

it is readily apparent that

$$\underset{\sim}{\ell}^{(A)} \cdot \underset{\sim}{\ell}_{(B)} = \delta^{A}_{B}$$

so that $\ell_{\alpha}^{(A)}$ and $\ell_{(B)}{}^{\beta}$ are inverse matrices.

We now introduce the 1-forms

$$\theta^{A} = \ell_{\alpha}^{(A)} dx^{\alpha} \qquad (5.2.2a)$$

associated with the dual basis vectors $\underset{\sim}{\ell}^{(A)}$. Solving for dx^{α} one obtains

$$dx^{\alpha} = \ell_{(A)}{}^{\alpha} \theta^{A} \qquad (5.2.2b)$$

Since the $\underset{\sim}{\ell}^{(A)}$ form a complete vector basis, the θ^{A} form a basis for all 1-forms. Explicitly one can expand any 1-form as a linear combination of the θ^{A} thus:

$$A_{\alpha} dx^{\alpha} = A_{\alpha} \ell_{(A)}{}^{\alpha} \theta^{(A)} = A_{A} \theta^{A}.$$

Similarly $\theta^A \wedge \theta^B \equiv 2!\, e^{(A)}_{\alpha} e^{(B)}_{\beta}\, dx^{[\alpha} dy^{\beta]}$

form a basis for all 2-forms. Explicitly one can expand any 2-form as a linear combination of the $\theta^A \wedge \theta^B$ thus:

$$2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = F_{AB}\, \theta^A \wedge \theta^B.$$

From (5.2.1) and (5.2.2.) the metric tensor of Riemannian geometry

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

becomes

$$ds^2 = g_{AB}\, \theta^A \theta^B.$$

In a curvilinear coordinate system the ordinary differential of a vector A^u, dA^u, is not a vector since dA^u is the difference of vectors located at different points in space. If we define a differential δA^u as the change in the vector A^u due to an infinitesimal parallel displacement of A^u from x^u to $x^u + dx^u$, the infinitesimal

$$DA^u \equiv dA^u - \delta A^u$$

is a vector since it is the difference between two vectors located at the same point. The differential, DA^u, is called the covariant differential of A^u.

Consider the covariant differential of the vector $\underset{\sim}{e}_{(B)}$, $D\underset{\sim}{e}_{(B)}$. Since by construction it is a vector, it can be expressed as a linear combination of the base vectors;

$$D\underset{\sim}{e}_{(B)} = \omega^{C}_{B}\, \underset{\sim}{e}_{(C)}. \qquad (5.2.3)$$

Taking the scalar product of both sides of (5.2.3) with $\underset{\sim}{e}^{(A)}$ we obtain

$$\omega^{A}_{B} = \underset{\sim}{e}^{(A)} \cdot D\underset{\sim}{e}_{(B)}. \qquad (5.2.4)$$

Now take the covariant differential of $\delta^A_B = \underset{\sim}{e}^{(A)} \cdot \underset{\sim}{e}_{(B)}$,

$$D(\delta^A_B) = 0 .$$

Also,

$$D(\delta^A_B) = D(\underset{\sim}{e}^{(A)} \cdot \underset{\sim}{e}_{(B)}) = \underset{\sim}{e}^{(A)} \cdot D\underset{\sim}{e}_{(B)} + \underset{\sim}{e}_{(B)} \cdot D\underset{\sim}{e}^{(A)} .$$

So,

$$\underset{\sim}{e}^{(A)} \cdot D\underset{\sim}{e}_{(B)} + \underset{\sim}{e}_{(B)} \cdot D\underset{\sim}{e}^{(A)} = 0 \qquad (5.2.5)$$

From (5.2.4) and (5.2.5) we obtain

$$\omega^A_B = \underset{\sim}{e}^{(A)} \cdot D\underset{\sim}{e}_{(B)} = -\underset{\sim}{e}_{(B)} \cdot D\underset{\sim}{e}^{(A)}$$

$$= -e^{(A)}_{\beta;\gamma}\, e^{\beta}_{(B)}\, dx^\gamma, \quad (D\underset{\sim}{e}^{(A)} = \underset{\sim}{e}^{(A)}_{;\gamma} dx^\gamma) \qquad (5.2.6)$$

The definition of the Ricci rotation coefficients, γ^A_{BC}, is

$$\gamma^A_{BC} \equiv -e^{(A)}_{\beta;\gamma}\, e^{\beta}_{(B)}\, e^{\gamma}_{(C)} \qquad (5.2.7)$$

From (5.2.2), (5.2.6) and (5.2.7) we have

$$\omega^A_B = \gamma^A_{BC}\, \theta^C \qquad (5.2.8)$$

Thus we have found the "connection 1-forms", ω^A_B, associated with the Ricci rotation coefficients, γ^A_{BC}.

From (5.2.6) we infer

$$D\underset{\sim}{e}^{(A)} = -\omega^A_B\, \underset{\sim}{e}^{(B)} ,$$

or, using (5.2.7), the expanded version

$$e^{(A)}_{\beta;\gamma} = -\gamma^A_{BC}\, e^{(B)}_{\beta}\, e^{(C)}_{\gamma} . \qquad (5.2.9)$$

Now let us take the differential * of both sides of $g_{AB} = \underset{\sim}{e}_{(A)} \cdot \underset{\sim}{e}_{(B)}$.

$$dg_{AB} = Dg_{AB} = \underset{\sim}{e}_{(A)} \cdot D\underset{\sim}{e}_{(B)} + \underset{\sim}{e}_{(B)} \cdot D\underset{\sim}{e}_{(A)}$$

$$= \underset{\sim}{e}_{(A)} \cdot \underset{\sim}{e}_{(C)} \omega^{C}_{B} + \underset{\sim}{e}_{(B)} \cdot \underset{\sim}{e}_{(C)} \omega^{C}_{A}.$$

If we agree to raise and lower capital Latin induces with the matrices g^{AB} and g_{AB} respectively, then

$$dg_{AB} = \omega_{AB} + \omega_{BA}. \qquad (5.2.10)$$

Let us now evalute the exterior differential of $\theta^{A} = e^{(A)}_{\beta} dx^{\beta}$,

$$d\theta^{A} = 2!\, e^{(A)}_{\beta;\gamma} dx^{[\gamma} dy^{\beta]} \qquad \text{(See Appendix A)}$$

$$= -2!\, \gamma^{A}_{BC} e^{(C)}_{\gamma} e^{(B)}_{\beta} dx^{[\gamma} dy^{\beta]} \qquad \text{(from 5.2.9)}$$

$$= -\gamma^{A}_{BC}\, \theta^{C} \wedge \theta^{B} \qquad \text{(See Appendix A)}$$

But from (5.2.8), $\omega^{A}_{B} = \gamma^{A}_{BC} \theta^{C}$, so

$$d\theta^{A} = -\omega^{A}_{B} \wedge \theta^{B}. \qquad (5.2.11)$$

Equations (5.2.11) are referred to as the "<u>first equat=ions of structure</u>".

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*Here it is immaterial whether we interpret differential as the covariant, ordinary, or exterior differential since g_{AB} is a <u>scalar</u> and thus there is no distinction between the different differentials.

Equations (5.2.10) and (5.2.11) completely determine the connection 1-forms, $\omega^{A}{}_{B}$. In practice one usually chooses an orthonormal frame for which g_{AB} is a constant matrix. Equation (5.2.10) then becomes trivially $\omega_{AB} = -\omega_{BA}$ and we are essentially left with only (5.2.11) to solve.

In the same spirit in which we found the connection between the Ricci rotation coefficients and the exterior differentials of the 1-forms, θ^{A}, we proceed to find the connection between the Riemann tensor and the exterior differentials of the connection 1-forms, $\omega^{A}{}_{B}$.

By definition (See Appendix A), the exterior differential of

$$\omega^{A}{}_{B} = \gamma^{A}{}_{BC}\, e^{(C)}{}_{\gamma}\, dx^{\gamma} \qquad (5.2.12)$$

is

$$d\omega^{A}{}_{B} = 2!\,(\gamma^{A}{}_{BC}\, e^{(C)}{}_{\gamma})_{;\delta}\, dx^{[\delta} dy^{\gamma]}. \qquad (5.2.13)$$

But the Riemann tensor is defined by the relation:

$$e^{(A)}{}_{\alpha} R^{\alpha}{}_{\beta\gamma\delta} = 2 e^{(A)}{}_{\beta;[\gamma\delta]}. \qquad (5.2.14)$$

From (5.2.9),

$$e^{(A)}{}_{\beta;\gamma\delta} = -(\gamma^{A}{}_{BC}\, e^{(C)}{}_{\gamma})_{;\delta}\, e^{(B)}{}_{\beta} - \gamma^{A}{}_{BC}\, e^{(C)}{}_{\gamma}\, e^{(B)}{}_{\beta;\delta} \qquad (5.2.15)$$

Changing dummy indices in the last term of (5.2.15) and again using (5.2.9), expression (5.2.15) becomes

$$e^{(A)}{}_{\beta;\gamma\delta} = -(\gamma^{A}{}_{BC}\, e^{(C)}{}_{\gamma})_{;\delta}\, e^{(B)}{}_{\beta} + \gamma^{A}{}_{HC}\, e^{(C)}{}_{\gamma}\, \gamma^{H}{}_{BD}\, e^{(D)}{}_{\delta}\, e^{(B)}{}_{\beta}. \qquad (5.2.16)$$

Multiply both sides of (5.2.16) by $dx^{[\gamma} dy^{\delta]}$ and consider the left and right hand sides of the resulting expression separately.

For the left hand side

$$e^{(A)}{}_{\beta;\gamma\delta}\, dx^{[\gamma} dy^{\delta]}$$

$$= e^{(A)}{}_{\beta;[\gamma\delta]}\, dx^{\gamma} dy^{\delta}$$

$$= \frac{e^{(A)}_{\alpha}}{2} R^{\alpha}{}_{\beta\gamma\delta}\, dx^{\gamma} dy^{\delta} \qquad \text{[from (5.2.14)]}$$

$$= \frac{e^{(A)}_{\alpha}}{2} R^{\alpha}{}_{\beta\gamma\delta}\, dx^{[\gamma} dy^{\delta]} \qquad \text{[since } R^{\alpha}{}_{\beta\gamma\delta} = -R^{\alpha}{}_{\beta\delta\gamma}\text{]}$$

For the right hand side noting the definition of the wedge product (Appendix A), (5.2.12) and (5.2.13) we obtain

$$\tfrac{1}{2}(d\omega^{A}{}_{B} + \omega^{A}{}_{H} \wedge \omega^{H}{}_{B})\, e^{(B)}{}_{\beta}.$$

Equating the left and right hand expressions,

$$e^{(A)}{}_{\alpha} R^{\alpha}{}_{\beta\gamma\delta}\, dx^{[\gamma} dy^{\delta]} = (d\omega^{A}{}_{B} + \omega^{A}{}_{H} \wedge \omega^{H}{}_{B})\, e^{(B)}_{\beta} \qquad (5.2.17)$$

Now if we define the <u>curvature 2-forms</u> as

$$\Omega^{A}{}_{B} \equiv \tfrac{1}{2} R^{A}{}_{BCD}\, \theta^{C} \wedge \theta^{D} \qquad (5.2.18)$$

then by virtue of the definition of the wedge product (Appendix A) and (5.2.18), expression (5.2.17) becomes

$$\Omega^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B}. \qquad (5.2.19)$$

The equations (5.2.19) are referred to as the "second equations of structure". All <u>frame components</u> $R^{A}{}_{BCD}$ of the Riemann tensor can be recovered from the 2-forms $\Omega^{A}{}_{B}$ by expanding them in terms of the basis $\theta^C \wedge \theta^D$ and comparing with (5.2.18).

In summary, Riemannian geometry has been reexpressed in the invariant formalism of differential forms. We have gained the elegance of invariance and the flexibility of an arbitrary choice of vector frame $\underset{\sim}{e}^{(A)}$. (The traditional approach is recovered if we tie the frame to the coordinate net: $\underset{\sim}{e}^{(A)} = \text{grad}\, x^A$.) By choosing the reference frame such that $g_{AB} = \underset{\sim}{e}_{(A)} \cdot \underset{\sim}{e}_{(B)}$ is a constant, the connection 1-forms have the property $\omega_{AB} = -\omega_{BA}$ (cf. 5.2.10). Thus there are only six nontrivial ω^A_B in a four space as opposed to forty Christoffel symbols. The computation of the Riemann tensor is thus greatly simplified.

5.3. $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ and the Intrinsic Singularity of the Line Mass.

In this section we shall use the powerful differential forms formalism built up in section 5.2. to analyze the non-finite regions of the invariant, $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ formed from the Riemann curvature tensor. We treat the general case of a line mass of arbitrary length and then specialize to the particular "length" 2μ. Only sufficient highlights of the development to illustrate the use of the differential forms formalism are included here. Details of the computations involved can be found in Appendix B.

The Weyl line mass element given by (3.4) and (3.6) can be expressed in terms of the coordinates ξ, η defined by

$$R_1 + R_2 = 2\ell \cosh \xi$$
$$R_2 - R_1 = 2\ell \cos \eta \qquad (5.3.1)$$

as

$$ds^2 = \left(\tanh\frac{\xi}{2}\right)^{2\psi} dt^2 - \ell^2 \left(\coth\frac{\xi}{2}\right)^{2\psi} \Big[(\sinh\xi)^{2\psi^2} \times (\sinh^2\xi + \sin^2\eta)^{1-\psi^2} (d\xi^2 + d\eta^2) + \sinh^2\xi \sin^2\eta \, d\varphi^2 \Big] \qquad (5.3.2)$$

where $\psi = \frac{\mu}{\ell}$.

Choosing our basis 1-forms as

$$\theta_1 = \left(\tanh\frac{\xi}{2}\right)^{\psi} dt$$
$$\theta_2 = \ell \left(\coth\frac{\xi}{2}\right)^{\psi} (\sinh\xi)^{\psi^2} (\sinh^2\xi + \sin^2\eta)^{\frac{1-\psi^2}{2}} d\xi$$
$$\theta_3 = \ell \left(\coth\frac{\xi}{2}\right)^{\psi} (\sinh\xi)^{\psi^2} (\sinh^2\xi + \sin^2\eta)^{\frac{1-\psi^2}{2}} d\eta$$
$$\theta_4 = \ell \left(\coth\frac{\xi}{2}\right)^{\psi} \sinh\xi \sin\eta \, d\varphi \qquad (5.3.3)$$

the line element (5.3.2) becomes

$$ds^2 = (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2.$$

Thus the frame components of the metric tensor are

$$g_{11} = -g_{22} = -g_{33} = -g_{44} = 1, \qquad (5.3.4)$$

all other $g_{AB} = 0. \quad (A \neq B)$

Also,

$$g^{AB} = g_{AB} \qquad (5.3.5)$$

Noticing the g_{AB}'s are constants we have from (5.2.10)

$$\omega_{AB} = -\omega_{BA}.$$

Taking the exterior differentials of (5.3.3), expressing these in terms of the basis 2-forms $\theta^A \wedge \theta^B$ and comparing the expressions to (5.2.11) we obtain a unique set of connection 1-forms $\omega^A{}_B$ (See Appendix B).

We then form the exterior differentials of the connection 1-forms $\omega^A{}_B$, the wedge products $\omega^A{}_C \wedge \omega^C{}_B$, and combine these expressions to obtain

$$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

(The second equations of structure).

If we then express these curvature 2-forms $\Omega^A{}_B$ in terms of the basis $\theta^C \wedge \theta^D$ and compare the resulting expressions to (5.2.18), we obtain the frame components of the Riemann tensor, $R^A{}_{BCD}$ (See Appendix B).

Now, as we would expect,

$$\begin{aligned} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} &= g^{\alpha\nu} g^{\beta\rho} g^{\gamma\xi} g^{\delta\eta} R_{VREN}\, e^{(V)}_\nu e^{(R)}_\rho e^{(E)}_\xi e^{(N)}_\eta \\ &\quad R_{ABCD}\, e^{(A)}_\alpha e^{(B)}_\beta e^{(C)}_\gamma e^{(D)}_\delta \\ &= g^{AV} g^{BR} g^{CE} g^{DN} R_{VREN} R_{ABCD} \\ &= R^{ABCD} R_{ABCD}. \end{aligned}$$

But from (5.3.4) and (5.3.5), we make the additional observation that

$$\begin{aligned} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} &= R^{ABCD} R_{ABCD} \\ &= g^{GB} g^{HC} g^{LD} g_{AF} R^A{}_{GHL} R^F{}_{BCD} \end{aligned}$$

$$= 4\left[(R^1{}_{212})^2 + (R^1{}_{213})^2 + (R^1{}_{312})^2 + (R^1{}_{313})^2 + (R^1{}_{414})^2 + (R^2{}_{323})^2 + (R^2{}_{424})^2 + (R^2{}_{434})^2 + (R^3{}_{424})^2 + (R^3{}_{434})^2\right]. \qquad (5.3.6)$$

Thus, since $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is composed entirely of the sum of squares of certain frame components and we consider only <u>real</u> numbers, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ approaches infinity if and only if at least one of these frame components approaches infinity.

Consider the Riemann frame component $R^1{}_{231}$:

$$R^1{}_{231} = \frac{\psi(\psi^2-1)\sin\eta\cos\eta}{(2)^{2\psi^2+1}\,\ell^2\left(\cosh\frac{\xi}{2}\right)^{2\psi+1+2\psi^2}\left(\sinh\frac{\xi}{2}\right)^{2\psi^2-2\psi+1}\left(\sinh^2\xi+\sin^2\eta\right)^{2-\psi^2}}$$

For $\psi \neq 0, 1$, this frame component is clearly infinite when

$$\sinh\frac{\xi}{2} = 0,$$

or,

$$\xi = 0.$$

From (5.3.1) we see that $\xi = 0$ corresponds to

$$R_1 + R_2 = 2\ell.$$

This is the expression for the locus of the line mass. Thus, from the argument following (5.3.6), we have shown that for $\psi \neq 0, 1$ the invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is infinite in the region corresponding to the line mass. Consequently, the Weyl line mass element (5.3.2) has an <u>intrinsic</u> singularity along the line mass for $\gamma \neq 0,1$ (as expected). For $\gamma = 0$ the frame component $R^{1}_{231} = 0$ as is true for all Riemann frame components

$$R^{A}_{\ BCD} = 0 .$$ (See Appendix B.)

This again corresponds to our intuitive feelings since $\gamma = 0$ corresponds to a "line mass" of <u>zero</u> mass density and space in the absence of matter has an identically vanishing Riemann tensor.

For $\gamma = 1$ the determination of the non-finite regions of $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is not so simple. For as γ approaches 1 and ξ approaches 0 certain frame components of the Riemann tensor (such as R^{1}_{231}) become indeterminates of the form $\frac{0}{0}$. One can argue that these frame components are infinite at $\xi = 0$ since upon the addition to or subtraction from $\gamma = 1$ of an infinitesimal density ϵ these Riemann components become infinite at $\xi = 0$. Appealing to a continuity argument this would imply these frame components are also infinite at $\gamma = 1$ for $\xi = 0$. Thus we are strongly inclined to claim that there exists an <u>intrinsic</u> singularity at $\xi = 0$, that is, along the line mass, for any <u>non-zero</u> density γ including the "Schwarzschild" line mass density $\gamma = 1$. We shall <u>prove</u> the correctness of this inclination using the elementary flatness criterion.

5.4. The Elementary Flatness Criterion.

The elementary flatness criterion, succinctly stated, is the condition that for any infinitesimally small space-like circle the ratio of circumference to radius shall be 2π, in regions where $T^{\mu\nu}=0$ (vacuum). This is merely a mathematical statement that locally the characteristics of space are Minkowskian in vacuum.

Possibly the most important application to date of the elementary flatness criterion is that of Einstein and Rosen (Einstein and Rosen, 1936) to the Silberstein two mass center solution (Silberstein, 1936). Silberstein had used the Weyl formalism developed in Chapter 3 to arrive at a solution for two static mass centers. Specifically he found that, for this problem, the λ satisfying Laplace's equation and appearing in the Weyl metric (3.4) is

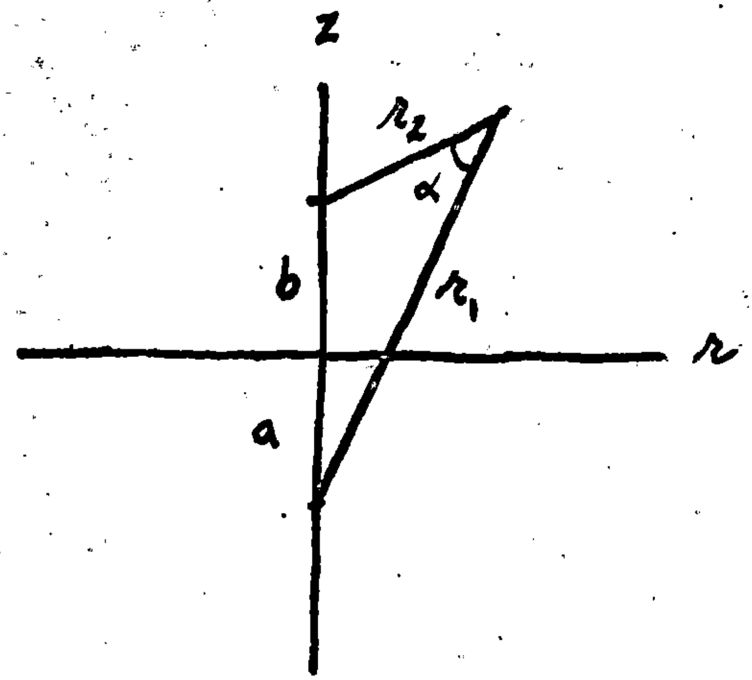
$$\lambda = -\frac{L_1}{r_1} - \frac{L_2}{r_2} \qquad (5.4.1)$$

where L_1, L_2 are constants and

$$r_1^2 = r^2 + (z+a)^2$$

$$r_2^2 = r^2 + (z-b)^2,$$

the a and b being constants (See figure below).



By an ingenious argument he was able to determine the ν appearing in (3.4),

$$\nu = \frac{-r^2}{2}\left(\frac{L_1^2}{r_1^4}+\frac{L_2^2}{r_2^4}\right)+2\,\frac{L_1L_2}{D^2}\left[\left(1-\frac{D^2r^2}{r_1^2r_2^2}\right)^{\frac{1}{2}}-1\right] \quad (5.4.2)$$

where $D = a+b =$ distance separating the two mass centers.

Silberstein noticed that the only <u>explicit</u> singularities in his solution (3.4), (5.4.1), (5.4.2) occur at $r_1 = 0$ and $r_2 = 0$, that is, at the mass centers themselves. He concluded (incorrectly) that these were the <u>only</u> singularities in his solution and that consequently there was a complete absence of matter and stress everywhere in his solution except at the mass centers. This condition that, in the absence of any intervening medium, two mass points are fixed relative to each other, instead of falling towards each other, "flagrantly contradicts man's most ancient, primitive experience." (Silberstein, 1936). Thus Silberstein concluded that either the Einstein field equations are incorrect or one cannot consider "material particles" (mass points) as singularities of the field.

Einstein and Rosen's refutation of Silberstein's conclusion was based on the elementary flatness criterion. For convenience they introduced the angle α formed by the two radius-vectors r_1, and r_2,

$$r_1 r_2 \sin\alpha = Dr \, . \quad (5.4.3)$$

Expression (5.4.2) then becomes

$$v = \frac{-r^2}{2}\left(\frac{L_1^2}{r_1^4} + \frac{L_2^2}{r_2^4}\right) + \frac{2L_1L_2}{D^2}\left[\pm\cos\alpha - 1\right] \quad (5.4.4)$$

Einstein and Rosen noted that according to Silberstein's development the square root is always taken to be positive. However this leads to discontinuous derivatives for v on the surface $\alpha = \frac{\pi}{2}$ in violation of the conditions of regularity. After further investigation, Einstein and Rosen found the correct solution did not contain the ambiguity in sign and the corrected equation (5.4.4) could be written with the bracketed expression as

$$\left[\cos\alpha - 1\right] \quad (5.4.4')$$

Now applying the elementary flatness criterion to an infinitesimal circle in the plane t = constant, z = constant, with center at $r = 0$ one obtains (using 3.4)

$$\text{Circumference} = \int_{t,z,r=\text{constant}} ds = \int_{t,z,r=\text{constant}} r e^{-\lambda} d\varphi$$

$$= 2\pi r e^{-\lambda},$$

and,

$$\text{Radius} = \int_{t,z,\phi=\text{constant}} ds = \int_{t,z,\varphi=\text{constant}} e^{(\nu-\lambda)} dr$$

$$= e^{(\nu-\lambda)} r.$$

Thus the ratio of circumference to radius $\frac{C}{R}$ is

$$\frac{C}{R} = 2\pi e^{-\nu} \quad (5.4.5)$$

This satisfies the criterion $\frac{C}{R} = 2\pi$ only if $\nu = 0$ for $r = 0$.

But from (5.4.4'), $\nu = 0$ for $r = 0$ only if $\alpha = 0$. From the definition of α this corresponds physically to the portion of the axis defined by the mass centers (z-axis) outside the segment joining the two centers. For the segment joining the two mass centers, $\alpha = \pi$. But here $\nu \neq 0$ and hence the elementary flatness criterion is violated. One is thus forced to conclude that this region corresponds to an intrinsic singularity and hence there exists matter (or stress) in this region. Silberstein's conclusion is refuted.

5.5. Application of the Elementary Flatness Criterion to the Problem.

For the metric (3.7) we again, as in section 5.4, consider the z-axis. Considering an infinitesimally small circle in the plane t = constant, Z = constant and centered at $r = 0$ we obtain for the ratio of circumference to radius

$$\frac{C}{R} = 2\pi e^{-V} \quad (5.4.5)$$

where

$$V = \frac{1}{2} \ln \frac{(R_1+R_2)^2 - 4\mu^2}{4R_1R_2} \quad (5.5.1)$$

[Expression (5.4.5) was developed in section 5.4 for the Weyl element (3.4) in general when considering an infinitesimally small circle in the plane t = constant, Z = constant and centered at $r = 0$. The particular V (5.5.1) for our problem was obtained by setting $\ell = \mu$ in (3.6).]

Thus for the Weyl line mass of "length" 2μ the elementary flatness criterion is satisfied (for $r = 0$) only if

$$V = 0 = \frac{1}{2} \ln \frac{(R_1+R_2)^2 - 4\mu^2}{4R_1R_2}$$

This implies that

$$(R_1+R_2)^2 - 4\mu^2 = 4R_1R_2$$

$$R_1^2 + R_2^2 - 2R_1R_2 = 4\mu^2$$

$$\pm (R_1 - R_2) = 2\mu \quad (5.5.2)$$

The only regions where condition (5.5.2) is met are on the z-axis <u>outside</u> the line mass. Thus the <u>region</u> of the line mass itself violates the elementary flatness criterion and consequently must be an <u>intrinsic</u> singularity of the Weyl line mass solution for "length" 2μ.

CHAPTER 6

GEOMETRICAL STRUCTURES, SYMMETRY AND KILLING VECTORS.

6.1. Introduction.

For ages geometry's relationship to experimental science has been a bone of contention for philosophers, mathematicians, and scientists. Historically, Euclidean geometry had its inception as an experimental science with the Egyptians and Babylonians. Later, it was codified by Euclid and consequently was developed into a subject of pure thought. By the time of Immanuel Kant's "Critique of Pure Reason", the historic origins of geometry had been forgotten and Euclidean geometry was accepted as one of the a priori categories of thought.

However, not long after Kant, mathematicians developed non-Euclidean geometries. Noteworthy among these is that of Bolyai and Lobatsheysky (1829), a non-Euclidean geometry developed by modifying the parallel postulate. The geometries developed in the ensuing years were not experimental but did demonstrate that Euclidean geometry was not the only conceivable one.

It took the genius of Einstein to reinstate geometry as an experimental science in his general theory of relativity. Einstein in his famous principle of equivalence stated that it is impossible to distinguish locally, by an experimental observation, between a uniform gravitational field and a uniformly accelerated frame of reference in the absence of a field. Thus one can describe gravitational effects either in terms of a flat space-time with a gravitational field present, or in terms of a curved four-dimensional space-time without a gravitational field. The second alternative is the foundation for the geometric interpretation of gravity and directly reinstates geometry to its former role as an experimental science. It is this interpretation that led to the elegant theory of general relativity.

The goals of this chapter are to show the relationship between physical objects and geometric structures, to show how the imposition of geometrical structure on a space selects a characteristic symmetry subgroup for the space, and finally to show that the symmetry subgroups of the Schwarzschild solution and the Weyl line mass solution are not the same even for the critical "length" 2μ . The plan follows.

In section 6.2 we shall review some basic topological and geometrical concepts necessary for this work. Also we will attempt to show heuristically how geometric structures corresponding to physical objects are superimposed on basic topological spaces to give them geometries.

In section 6.3 intrinsic symmetries of geometrical objects will be formulated in mathematical terms and Killing's equation will be developed.

In section 6.4 simple applications of Killing vector analysis for the determination of intrinsic symmetries will be demonstrated.

In section 6.5 Killing vector analysis will be applied to the Weyl axi-symmetric static solution of a line mass with "length" $2u$ .The intrinsic symmetry will be demonstrated to be non-spherical in contrast to the Schwarzschild solution.

6.2. Topological and Geometrical Concepts.

The task of the physicist is to construct models or theories that correspond to the physical world. Of all models used, the most successful to date are the ones employing the space-time concept. Indeed most physicists accept a theory as fundamental only if it does make explicit use of the space-time concept.

The foundation of all space-time theories is the use of the space-time point (event). It corresponds to where and when a physical event takes place. The distinguishing feature of a space-time point is that it has no distinguishing features; all points of space time are assumed equivalent.

In order to be of use, however, space-time must be more than just a collection of points. In physical theories, it is endowed with a number of properties. Of prime importance are its topological properties. The topological properties of a space are those which are unaffected by arbitrary continuous deformations of the space. The common example given is that of the topologies of a sphere and a torus. They are topologically unequivalent since it is impossible to deform a torus into a sphere.

Common to all space-time theories to date is the <u>assumption</u> that <u>locally</u> the topology of space-time is that of a

Euclidean four-plane. Note that this is a local statement. The global topology of space-time is not entirely at our disposal but is restricted by the type of geometry we wish to impose. In contrast to the time of Kant, when Euclidean geometry was sacred, now even the contention of local Euclidean topology is being contested.

Wheeler and Misner have constructed models for electric charge by actually introducing non-Euclidean manifolds, containing worm holes. We shall however assume Euclidean topology. As a consequence it is possible to map points of any small but finite region of space-time onto the points of a corresponding region of the Euclidean four-plane in a one-to-one bicontinuous manner. This is equivalent to assuming that the points of space-time constitute a space-time manifold. This is a local property, however, and it may well be impossible to map the whole manifold onto a single Euclidean four-plane.

As we will make extensive use of coordinates and mappings it may be wise to stress a few ideas of import here. Coordinate systems are merely an aid in calculating. Intrinsic geometry is independent of the coordinate system used. In principle there are no grounds for the preference of one coordinate system over any other*. (Principle of General Covariance.)

Once we have coordinated the space-time manifold it

* In practice coordinate systems which exhibit the symmetries of one's space-time are preferable.

is possible to characterize a mapping of this manifold onto itself. By a mapping we shall mean each point of the manifold is associated with some other point of the manifold.

$$x^{\mu} \rightarrow x'^{\mu} = x'^{\mu}(x)$$

We shall limit ourselves to one-to-one bicontinuous mappings.

Up to now we have talked only about a bare space-time manifold. This is an empty concept as far as describing physical systems. In order to develop a world picture we must introduce the notion of a geometrical object. Euclid, in developing his geometry characterized geometrical objects in a purely axiomatic way, thereby emphasizing the geometrical nature of the objects defined. This method, however, doesn't lend itself to analytic techniques. Therefore we shall approach geometrical objects in terms of their transformation properties.

We shall demand of a geometrical object, y, which undergoes transformation to an object, y', that it satisfy the following conditions:

(1) If y is transformed to y' by a mapping $x'^{\mu}(x)$ and y' is transformed into y'' by a mapping $x''^{\mu}(x')$ then y will be transformed into y'' by the mapping $x''^{\mu}(x'^{\mu}(x))$, that is, the product mapping.

(2) y is transformed into itself by the identity mapping.

(3) If y is transformed into y' by a given mapping, then y' will be transformed into y by the inverse of this mapping.

The physical grounds for these stipulations on the geometrical objects is to ensure that all the intrinsic properties (physics) of these objects and all relations between them are preserved under mappings.

The most familiar geometrical objects to physicists are those belonging to the linear, homogeneous transformation class. These are tensors (quantities obeying the transformation law

$$T'^{\cdots\mu\cdots}_{\cdots\nu\cdots} = \cdots\frac{\partial x'^{\mu}}{\partial x^{\rho}}\cdots\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\cdots T^{\cdots\rho\cdots}_{\cdots\sigma\cdots}\ )$$

and tensor densities (quantities obeying the transformation law

$$\tau'^{\mu\cdots}_{\nu\cdots} = \left|\frac{\partial x}{\partial x'}\right|^{w}\frac{\partial x'^{\mu}}{\partial x^{\rho}}\cdots\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\cdots\tau^{\rho\cdots}_{\sigma\cdots}\ ).$$

These are geometrical objects which are associated with actual physical structures (e.g. the association between the antisymmetric electromagnetic tensor and the electromagnetic field). Their introduction imbues our space with some geometric structure but not enough. The difficulty arises in the fact that we have no geometrical means for deciding how a tensor field changes as it moves over the manifold. The mathematical basis for this problem is that the ordinary derivatives of tensors and tensor densities do not form geometrical objects in general. (One cannot add or subtract tensor at different points of the manifold, an operation necassary to construct a derivative.) This situation can be rectified by the introduction of the notion of parallel transport of a tensor which in turn leads to the construction of an affine geometry.

Given a vector field $A^\mu(x)$, the components of A^μ at $x^\mu + dx^\mu$ are related to the components at x^μ by

$$A^\mu(x+dx) = A^\mu(x) + dA^\mu(x)$$
$$= A^\mu(x) + \frac{\partial A^\mu}{\partial x^\nu} dx^\nu.$$

The dA^μ, being the difference of two vectors at <u>different</u> points of the manifold, do not constitute a geometrical object. In curvilinear coordinates, in order to obtain a differential of a vector which behaves like a geometrical object, it is necessary the two vectors to be subtracted from one another be located at the same point. One of the vectors must be "parallelly" translated to the point where the second is located, allowing us to determine the difference of the two vectors at the same point in space. We define the vector at $x^\mu + dx^\mu$, parallel to $A^\mu(x)$ to be $A^\mu(x) + \delta A^\mu(x)$. The δA^μ is not a vector but the difference $dA^\mu(x) - \delta A^\mu(x)$ clearly is, being the difference of two vectors at the <u>same</u> point. In order that our generalization of parallel transport correspond to our ordinary notions of parallel transport in Euclidean geometry we require that δA^μ vanish when either dx^μ or $A^\mu(x)$ vanishes. This is accomplished most easily by requiring $\delta A^\mu(x)$ to be bilinear in $A^\mu(x)$ and dx^μ that is,

$$\delta A^\mu(x) = -\Gamma^\mu_{\rho\sigma}(x) A^\rho(x) dx^\sigma.$$

The quantities $\Gamma^\mu_{\rho\sigma}(x)$ constitute the components of a new geometrical object called the affine connection. The components are almost completely arbitrary but once they are given our manifold is endowed with an affine geometry. Using the affine connections we can construct a new derivative, the coderivative $A^{\mu}{}_{;\nu}$ of a contravariant vector $A^{\mu}(x)$,

$$A^{\mu}{}_{;\nu} \equiv \frac{A^{\mu}(x+dx) - (A^{\mu}+\delta A^{\mu})}{dx^{\nu}}$$

$$= A^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\rho\nu} A^{\rho}.$$

The transformation properties,

$$\Gamma'^{\lambda}_{\alpha\beta} = \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\beta}} \Gamma^{\mu}_{\rho\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha} \partial x'^{\beta}}$$

are immediately derivable from the imposition that $dA^{\mu} - \delta A^{\mu}$ transform like a contravarient vector.

If we impose the conditions that the coderivative of a scalar equal the ordinary derivative and the product rule of differentiation hold for covariant differentiation, we can construct the coderivative of an arbitrary tensor $T^{\mu\cdots}_{\nu\cdots}$.

$$T^{\mu\cdots}_{\nu\cdots;\rho} = T^{\mu\cdots}_{\nu\cdots,\rho} + \Gamma^{\mu}_{\sigma\rho} T^{\sigma\cdots}_{\nu\cdots} - \Gamma^{\sigma}_{\nu\rho} T^{\mu\cdots}_{\sigma\cdots} + \cdots$$

Once we have imbued our manifold with an affine geometry through the introduction of the affine connections, we can construct an important family of curves known as the affine geodesics. We define these curves thus: Given a point P lying on the curve, take any vector proportional to the tangent to the curve at P and transport this vector parallel to itself along the curve to another point P' on the curve. If the parallel-transported vector is proportional to the tangent to the curve at P' for every point P' and every starting point P, the curve is an affine geodesic. The affine geodesic between the two points is thus the "straightest" line between the two points. Algebraically, if the coordinates of points lying on the curve are given parametrically as

$$x^\mu = \xi^\mu(\tau),$$

where τ is any continuous parameter defined along the curve, the only restriction being τ must increase monotonically as one proceeds along the curve in a fixed direction, then

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{d\tau}\frac{d\xi^\sigma}{d\tau} = \alpha(\tau)\frac{d\xi^\mu}{d\tau}.$$

There exists a parameter, $s(\tau)$, such that the equation of the geodesic becomes

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{ds}\frac{d\xi^\sigma}{ds} = 0.$$

This affine parameter, $s(\tau)$, can be used to define the length of a segment of the geodesic between points P and P' as $\int_{P'}^{P} ds$. However we cannot compare lengths of segments belonging to different geodesics because this affine parameter is not unique. (If s is an affine parameter then so is $as+b$, where a and b are arbitrary constants.)

One of the fundamental assumptions of general relativity is that a free particle will move along a geodesic in four-dimensional space-time. Using affine geometry we were able to construct affine geodesics but these in themselves are not enough to describe the geometry of space. As mentioned above affine geometry in itself is not enough to compare distances along different geodesics. A mechanism must be developed for assigning invariant distances between points in space. A metrical geometry must be constructed which will add the concepts of distance and angle to that of parallelism developed in the affine geometry.

A metrical structure is given to a manifold by assigning to each pair of neighbouring points x^{μ} and $x^{\mu}+dx^{\mu}$ a distance ds. This distance, ds, will in general depend on x^{μ} and dx^{μ}. In order to satisfy our physical intuition, ds should vanish when $dx^{\mu}=0$. Since the distance, ds, is a local distance, homogeneity of degree 1 in dx^{μ} is suggested. There are still an infinity of metric geometries available but if we use simplicity as a criterion we are led to the choice made by Riemann,

$$ds^2 = g_{\alpha\beta}(x)\, dx^{\alpha} dx^{\beta}$$

which is analogous to the Pythagorean expression for distance in Euclidean geometry.

If we impose the natural requirement that the distance, ds, between two neighbouring points remain unchanged under mapping (ds a scalar), then we categorize the $g_{\alpha\beta}$ as a covariant tensor of second rank. We further assume that the $g_{\alpha\beta}$ is symmetric since antisymmetric components can't contribute to ds. Thus the metric tensor has ten independent components.

It is a natural question to ask at this point if there is any relationship between the metric tensor $g_{\alpha\beta}$, describing distance-angle relationships, and the affine connections describing the concept of parallel transport. There is if we require that the affine geodesic, related to the concept of the "straightest line", be equal to the metric geodesic, related to the concept of shortest distance between two points. The equation for the curve of shortest distance between two points in a space is obtained by extremizing the line element $(\delta \int ds = 0)$. For the Riemannian metric,

$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ one gets

$$\frac{d^2x^\mu}{ds^2} + \left\{ {\mu \atop \rho\sigma} \right\} \frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0$$

where $\left\{ {\mu \atop \rho\sigma} \right\}$ is known as the Christoffel symbol and is defined as

$$\left\{ {\mu \atop \rho\sigma} \right\} = \frac{1}{2} g^{\mu\nu}(g_{\nu\rho,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu}).$$

Comparing this with the affine geodesic equation we get the desired correspondence between the affine geodesic and the line of shortest distance if

$$\Gamma^\mu_{\rho\sigma} \equiv \left\{ {\mu \atop \rho\sigma} \right\}.$$

Thus the entire geometric structure of general relativity is contained in the ten components of the metric tensor.

In synopsis, the imposition of various geometric objects on a base manifold superimposes a geometric structure on the manifold, a geometry. In general relativity the geometric structure is contained in the metric tensor $g_{\alpha\beta}$.

6.3. Intrinsic Symmetry and Killing's Equation.

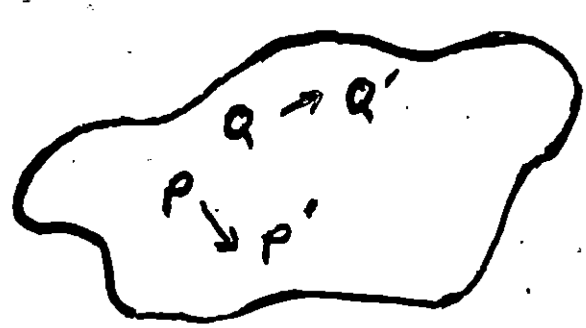
The concept of symmetry is so deeply inbred and interwoven with experience that it is difficult to explain its meaning in words which convey more than the word itself. Even a child has an intuitive understanding of what is meant by a statement like a ball is round. However intuition has its limitation and so the mathematician and physicist must translate the concept into mathematical terms. We shall construct the mathematical statement for the symmetry properties of space in a systematic manner starting with the bare manifold and showing how the introduction of geometrical structure affects the possible symmetry properties of space.

As stated previously the distinguishing feature of the bare manifold is that it has no distinguishing features. There are no geometric signposts to aid one in distinguishing one point of the manifold from another. Thus under a mapping of a point P to a point P', the topology at P' being replaced by the topology at P, there is no change on the manifold. The bare manifold allows the most general topological transformation possible. Thus its symmetry is that of all possible topological transformations. Analyticably if our space is coordinatized with the point P assigned the coordinates $\{x^{\mu}\}$ and P' assigned $\{x'^{\nu}\}$ we can express the entire mapping or transformation by

$$x'^{\mu} = x'^{\mu}(\{x^{\nu}\}).$$

For a bare manifold any set of four functions $x'^{\mu}(\{x^{\nu}\})$ generates a topological transformation.

We can represent an infinitisimal mapping by a vector field over the manifold. The elements of the vector field are formed by connecting each point P to the infinitisimally displaced point P' with a short arrow from P to P'.



We then use this vector field as a generator of the possible group of motions. Of course for a bare manifold any vector field generates a possible motion.

Now if we impose a geometric structure such as an affinity or metric tensor, on our space what happens to its symmetry properties? The imposition of a geometric structure assigns "sign-posts" throughout our manifold. We compare these for all points before and after a mapping. Only if they compare exactly can we say that the mapping represents a symmetry of the space. This parallels our intuitive approach to symmetries. Given a cylinder, for example, one observes that upon rotating it about a certain axis (its axis of symmetry) its shape appears unaltered while if one rotates it about some other axis its shape appears altered (surface elements at a given point change). Thus the first motion corresponds to a symmetry group while the second does not.

It is clear that the imposition of geometric structure on a bare maniflod reduces the symmetry group from all topological transformations to some subgroup. Since in general relativity one deals with a metric geometry characterized by the metric tensor, $g_{\mu\nu}$, we shall develop mathematical conditions that a mapping must satisfy in order to leave the metric structure and thus the geometry of our space unchanged. Since every finite transformation can be built up from a sequence of infinitesimal transformations, there will be no loss of generality if we consider only the case of infinitesimal motions of points into new points.

The vector field which carries the point P with coordinates $\{x^\mu\}$ over into the point P' with coordinates $\{x^\mu + \xi^\mu(x)\}$ is given by ξ^μ. (Refer to description of vector field above). Since the metric characterizing our geometry, is a second rank covariant tensor, it transforms as

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}.$$

Here the x'^μ are the coordinates of the point P' so

$$x'^\mu = x^\mu + \xi^\mu(x).$$

Since the ξ's are infinitesimal quantities, the inverse transformation is

$$x^\mu = x'^\mu - \xi^\mu(x')$$

Thus to first order in the infinitesimals, ξ^μ, the metric assigned to P' is

$$g'_{\mu\nu} = g_{\mu\nu} - g_{\mu\rho}\xi^\rho_{,\nu} - g_{\rho\nu}\xi^\rho_{,\mu} \qquad (6.3.1)$$

where all quantities on the right hand side of the equation are to be evaluated at the point P.

The orginal metric at point P' can be expressed in terms of quantities at point P by making a Taylor series expansion of the metric about the point P:

$$g_{\mu\nu}(x^\mu + \xi^\mu) = g_{\mu\nu}(x) + g_{\mu\nu,\rho}\xi^\rho \qquad (6.3.2)$$

to first order in the infinitesimals ξ^μ.

The necessary and sufficient condition that the motion generated by $\xi^\rho(x)$ leave the geometry unchanged is that the two metrics given by (6.3.1) and (6.3.2) be the same, that is,

$$g_{\mu\nu} - g_{\mu\rho}\xi^\rho_{,\nu} - g_{\rho\nu}\xi^\rho_{,\mu} = g_{\mu\nu} + g_{\mu\nu,\rho}\xi^\rho$$

or

$$-g_{\mu\rho}\xi^\rho_{,\nu} - g_{\rho\nu}\xi^\rho_{,\mu} - g_{\mu\nu,\rho}\xi^\rho = 0. \qquad (6.3.3)$$

This equation is known as Killing's equation and the vector fields satisfying it as Killing vector fields.

Using the definition of covariant differentiation the compact form

$$-\xi_{\mu;\nu} - \xi_{\nu;\mu} = 0 \qquad (6.3.4)$$

can be shown to be equivalent to (6.3.3).

$$0 = -\xi_{\mu;\nu} - \xi_{\nu;\mu}$$

$$= -g_{\mu\rho}\xi^{\rho}{}_{;\nu} - g_{\rho\nu}\xi^{\rho}{}_{;\mu} \quad (g_{\mu\rho;\nu}=0)$$

$$= -g_{\mu\rho}\xi^{\rho}{}_{,\nu} - g_{\rho\nu}\xi^{\rho}{}_{,\mu} - g_{\mu\rho}\Gamma^{\rho}_{\lambda\nu}\xi^{\lambda} - g_{\rho\nu}\Gamma^{\rho}_{\lambda\mu}\xi^{\lambda}$$

But we have defined $\Gamma^{\rho}_{\lambda\nu} \equiv \frac{1}{2}g^{\rho\sigma}(g_{\sigma\lambda,\nu} + g_{\sigma\nu,\lambda} - g_{\lambda\nu,\sigma})$ so

$$0 = -g_{\mu\rho}\xi^{\rho}{}_{,\nu} - g_{\rho\nu}\xi^{\rho}{}_{,\mu} - \tfrac{1}{2}g_{\mu\rho}g^{\rho\sigma}(g_{\sigma\lambda,\nu} + g_{\sigma\nu,\lambda} - g_{\lambda\nu,\sigma})\xi^{\lambda} - \tfrac{1}{2}g_{\rho\nu}g^{\rho\sigma}(g_{\sigma\lambda,\mu} + g_{\sigma\mu,\lambda} - g_{\lambda\mu,\sigma})\xi^{\lambda}$$

$$= -g_{\mu\rho}\xi^{\rho}{}_{,\nu} - g_{\rho\nu}\xi^{\rho}{}_{,\mu} - \tfrac{1}{2}\delta^{\sigma}_{\mu}(g_{\sigma\lambda,\nu} + g_{\sigma\nu,\lambda} - g_{\lambda\nu,\sigma})\xi^{\lambda} - \tfrac{1}{2}\delta^{\sigma}_{\nu}(g_{\sigma\lambda,\mu} + g_{\sigma\mu,\lambda} - g_{\lambda\mu,\sigma})\xi^{\lambda}$$

$$= -g_{\mu\rho}\xi^{\rho}{}_{,\nu} - g_{\rho\nu}\xi^{\rho}{}_{,\mu} - g_{\mu\nu,\lambda}\xi^{\lambda}.$$

The demonstration that the Killing equation can be put into the form (6.3.4) is motivated by much more than compactness of formulation. Since equation (6.3.4) contains only geometrical objects (ξ^{μ}) and geometric operations (covariant differentiation), it is a geometric equation and is independent of coordinization. Thus the Killing vectors, being the generators of infinitesimal invariant motions of the space, determine the intrinsic symmetry properties of the space. Conversely, the intrinsic symmetry properties determine the infinitesimal generators.

6.4. Illustrative Examples of the Application of Killing Vector Analysis.

On the introduction of a new set of coordinates $x'^{\mu} = x'^{\mu}(x^{\nu})$ the Killing vector transforms as

$$\xi'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \xi^{\nu}(x).$$

Because of this transformation property it is clear that the Killing vector may take on an infinity of forms depending on the coordinate system used. Thus in order to make an unequivocal statement about intrinsic symmetry properties using Killing vectors we must know the form of our coordinates. In a metric geometry this information is contained in the metric itself. Consider, for sake of simplicity, a metric geometry given by

$$ds^2 = (dx^1)^2 + \sin^2(x^1)(dx^2)^2 \qquad (6.4.1)$$

An observer taking a number of measures of the length ds between adjacent points (x_1, x_2) and $(x_1 + dx_1, x_2 + dx_2)$ and finding the above formula fits these measures will know that his coordinate system is the usual polar coordinate system with x_1, x_2 being the numbers denoted by the polar coordinates θ, φ. With this information we are ready to examine the intrinsic symmetry of the metric. For this particular metric the Killing equations (6.3.3) yield the following system of equations,

for $\mu = \nu = 1$, $\xi'_{,1} = 0$

for $\mu=1, \nu=2, \; -\xi^1_{,2} - \sin^2(\chi^1)\,\xi^2_{,1} = 0$

for $\mu=2, \nu=2, \; -\sin^2(\chi^1)\,\xi^2_{,2} - \sin\chi^1\cos\chi^1\,\xi^1 = 0$

The solution to this system of equations is

$$\xi^1 = A\cos\chi^2 + B\sin\chi^2$$
$$\xi^2 = -\cot\chi^1\,(A\sin\chi^2 - B\cos\chi^2) + D \qquad (6.4.2)$$

where A,B,D are arbitrary constants.

We know the rotation group in three dimensions is given in Cartesian forms as

$$x^r \to x'^r = x^r + \varepsilon^{rs} x^s$$

where $\varepsilon^{rs} = -\varepsilon^{sr}$ otherwise arbitrary.

The Killing vector characterizing spherical symmetry is thus given in Cartesian forms as

$$\xi^r = \varepsilon^{rs} x^s.$$

Transforming this to polar coordinates we obtain

$$\xi'^1 = \varepsilon^{13}\cos\varphi + \varepsilon^{23}\sin\varphi$$
$$\xi'^2 = -\cot\theta \cdot (\varepsilon^{13}\sin\varphi - \varepsilon^{23}\cos\varphi) \qquad (6.4.3)$$
$$\xi'^3 = 0$$

We have already stated that we can make the identification $(x^1, x^2) = (\theta, \varphi)$, and since $\varepsilon^{13}, \varepsilon^{23}$ are arbitrary constants there is no prohibition on identifying ε^{13} with A and ε^{23} with B. Thus because of the equality of (6.4.2) and (6.4.3), the metric (6.4.1) belongs to the spherical symmetry group, that is, the three dimensional rotation group.

A note of caution should be interjected here. Although for a given intrinsic symmetry and a given coordinate system the Killing vector structure is defined uniquely (indeed this is the property which allows us to analyze the symmetry group of a metric), we must not assume that for a given intrinsic symmetry the Killing vector structure is uniquely attached to a certain coordinate system. In fact, for a given symmetry th there is an infinity of coordinate systems which have the same Killing vector structure. The basis for this is that in general relativity generalized coordinates are used. For instance when one speaks of spherical polars the coordinates $(\rho, \theta, \varphi, t)$ are not in general quantities one would measure with a ruler and proctractor. Only in the asymptotic limit do these quantities approach their usual meaning. It is heuristically evident that there can exist an infinity of asymptotic approaches and thus an infinity of coordinates which approach the standard forms. As an example consider the case of the Schwarzschild solution in regular and isotropic form:

regular
$$\begin{cases} g_{44} = 1 - \frac{2\mu}{\rho} \\ g_{4r} = 0 \\ g_{rs} = -\delta_{rs} - \frac{2\mu/\rho}{1 - \frac{2\mu}{\rho}} \frac{x^r x^s}{\rho^2} \end{cases}$$

and

isotropic
$$\begin{cases} g_{44} = \left(\frac{1 - \frac{\mu}{2\rho}}{1 + \frac{\mu}{2\rho}} \right)^2 \\ g_{4r} = 0 \\ g_{rs} = -\left(1 + \frac{\mu}{2\rho}\right)^4 \delta_{rs} \end{cases}$$

where the metric tensors have been given in generalized

Cartesian coordinates and

$$\rho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2.$$

The transformation from the regular to isotropic form is given by:

$$x^r = \left(1 + \frac{\mu}{2\rho'}\right)^2 x'^r, \quad x^4 = x'^4$$

where the primes refer to the isotropic coordinates.

We note that here we have two different solutions to the same physical problem whose coordinates differ markedly but approach one another in the asymptotic limit $(\rho' \to \infty)$. What about the Killing vector structures?

We know that in Cartesian forms the Killing vector structure associated with spherical symmetry is $\xi'^r = \varepsilon^{rs} x'^s$, $\xi'^4 = 0$ where $\varepsilon^{rs} = -\varepsilon^{sr}$. Looking at the ξ^1 component given by the transformation law,

$$\xi^1 = \frac{\partial x^1}{\partial x'^\nu} \xi'^\nu$$

we get

$$\xi^1 = \frac{\partial x^1}{\partial x'^1}(\varepsilon^{12} x'^2 + \varepsilon^{13} x'^3) + \frac{\partial x^1}{\partial x'^2}(-\varepsilon^{12} x'^1 + \varepsilon^{23} x'^3)$$
$$+ \frac{\partial x^1}{\partial x'^3}(-\varepsilon^{13} x'^1 - \varepsilon^{23} x'^2)$$

Calling $f = \left[\left(1 + \frac{\mu}{2\rho'}\right)^2\right]_{,\rho'}$ and $g = \left(1 + \frac{\mu}{2\rho'}\right)^2$ we obtain

$$\xi^1 = \left[\frac{(x'^1)^2}{\rho'} f + g\right](\varepsilon^{12} x'^2 + \varepsilon^{13} x'^3)$$
$$+ \left[\frac{f}{\rho'} x'^1 x'^2\right](-\varepsilon^{12} x'^1 + \varepsilon^{23} x'^3)$$
$$+ \left[\frac{f}{\rho'} x'^3 x'^1\right](-\varepsilon^{13} x'^1 - \varepsilon^{23} x'^2)$$
$$= g(\varepsilon^{12} x'^2 + \varepsilon^{13} x'^3)$$
$$= \varepsilon^{12} x^2 + \varepsilon^{13} x^3.$$

We see that ξ^{1} has the same structure as ξ'^{1}. Similarly the other components of the Killing vector can be shown to have equivalent structures in the primed and unprimed coordinates. Thus it is demonstrated that the Killing vector structure is not linked to one coordinate system but to a class of generalized coordinates.

Finally for historical perspective we shall apply Killing vector analysis to the Weyl solution for a single mass center. (Silberstein, 1935).

From Chapter 3 we know the line element will be of the form

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - r^2 e^{-2\lambda} d\varphi^2$$

where λ is a solution of Laplace's equation and

$$\nu = \int r\left([\lambda_1^{\,2} - \lambda_2^{\,2}]\,dr + 2\lambda_1\lambda_2\,dz\right).$$

For a single mass center the solution of Laplace's equation is:

$$\lambda = -\frac{L}{\rho},$$

where $\rho^2 = r^2 + z^2$. We readily integrate for ν obtaining

$$\nu = -\frac{L^2 r^2}{2\rho^4}.$$

The question of interest is this: does this line element possess spherical symmetry? The Killing vector structure for spherical symmetry in generalized cylindrical polar coordinates (r, z, φ, t) is

$$\xi^{\mu} = \left(\varepsilon^{13} z\cos\varphi + \varepsilon^{23} z\sin\varphi,\ -\varepsilon^{13} r\cos\varphi - \varepsilon^{23} r\sin\varphi,\ -\varepsilon^{12} - \varepsilon^{13}\frac{z}{r}\sin\varphi + \varepsilon^{23}\frac{z}{r}\cos\varphi,\ 0\right). \quad (6.4.4)$$

Investigating whether this structure is consistent with the Weyl solution for a single mass center, we apply the Killing equation test for $\mu = \nu = 1$. The condition for consistency thus obtained is:

$$-2g_{1\rho}\xi^{\rho}_{,1} - g_{11,\rho}\xi^{\rho} = 0$$

where the ξ^{ρ} is given by (6.4.4).

Noting that $\xi^{1}_{,1} = 0$ and that for the metric considered $g_{\mu\nu}$ is diagonal, the condition reduces to

$$-g_{11,\rho}\xi^{\rho} = 0.$$

Explicity, since g_{11} is independent of φ and t,

$$\left(e^{\frac{2L}{\rho}} e^{-\frac{L^2 r^2}{\rho^4}}\right)_{,r} \left(\varepsilon^{13} z \cos\varphi + \varepsilon^{23} z \sin\varphi\right)$$
$$+ \left(e^{\frac{2L}{\rho}} e^{-\frac{L^2 r^2}{\rho^4}}\right)_{,z} \left(-\varepsilon^{13} r \cos\varphi - \varepsilon^{23} r \sin\varphi\right) = 0.$$

After some algebraic manifestation we obtain the necessary condition

$$-\frac{2L^2}{\rho^4} r z e^{\frac{2L}{\rho}} e^{-\frac{L^2 r^2}{\rho^4}} \left(\varepsilon^{13}\cos\varphi + \varepsilon^{23}\sin\varphi\right) = 0.$$

This condition is <u>not</u> valid in general.

Thus we see the Killing vector structure for spherical symmetry is <u>not</u> consistent with the Weyl single mass solution. Therefore we are forced to conclude this solution is <u>not</u> spherically symmetric.* This conclusion at first seems troublesome since we started with the Newtonian potential for a single mass

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*Silberstein (1935) noticed that the single mass center solution could not be transformed into the Schwarzschild solution and thus could not be spherically symmetric. Anderson (1967) suggests the use of Killing vector analysis as a formal proof.

point. But it must be borne in mind that even a point may have multipole structure which happens to be the case here. Physically this has its source in the fact that the r and z coordinates do not retain their Euclidean maening in general relativity and thus mass distributions in the general relativistic case need not be the same as in the Newtonian case.

The imposition of an axially symmetric structure via the Weyl form of the Einstein field equations appears to be the fundamental influence on the metric structure. Although the $-1/\rho$ potential is both spherically and axially symmetric in the Newtonian sense, it is only the axially symmetric [illegible] attribute which is retained in the framework of the Weyl equations.

The important lesson here is that one cannot assume that the symmetries of the Newtonian potential which ohe uses in developing the Weyl solutions also apply to the general relativistic line element.

6.5. Application of Killing Vector Analysis to the Weyl Solution of a Line Mass of "Length" 2μ.

Having developed the techiques of Killing vector analysis and seen their application to simple problems, we now proceed to use these techniques to determine if the Weyl solution of a line mass of "length" 2μ (the Schwarzschild radius) is in fact spherically symmetric as claimed throughout the literature. We shall proceed in two ways:

i) Determining whether or not the Killing vector structure for spherical symmetry and this particular Weyl solution are compatible with each other;

icular Weyl solution are compatible with each other;

ii) Determining whether or not the Killing vector structure for the "Schwarzschild" form of this particular Weyl solution is that of spherical symmetry.

Both of the above methods are equivalent since by definition intrinsic symmetry properties are unaltered by transformation. However the second method is included since it particularly illuminates the source of a possible misinterpretation of the symmetry of the problem.

Proceeding in a manner analogous to that adopted in analyzing the Weyl single mass center solution in section 6.4, we inquire if the Weyl line mass solution for "length" 2μ is consistent with the Killing vector structure for spherical symmetry. The Killing vector structure for spherical symmetry in generalized cylindrical polars is given by (6.4.4). The metric for the particular Weyl solution under consideration is given by (3.7). We now apply the Killing equation test for $\mu = \nu = 4$.

The condition for consistency thus obtained is:

$$-2g_{44}\,\xi^4{}_{,4} - g_{44,\rho}\,\xi^\rho = 0 . \tag{6.5.1}$$

Since $\xi^4 = 0$, the condition reduces to

$$g_{44,s}\,\xi^s = 0 \tag{6.5.2}$$

Note that the derivatives are with respect to the generalized cylindrical polars (i.e.

$$g_{44,4} = \frac{\partial g_{44}}{\partial t}\,;\; g_{44,1} = \frac{\partial g_{44}}{\partial r}\,;\; g_{44,2} = \frac{\partial g_{44}}{\partial z}\,;\; g_{44,3} = \frac{\partial g_{44}}{\partial \varphi}\,).$$

Thus, upon noting g_{44} is independent of φ [cf. (3.7)], the condition (6.5.2) can be written out explicitly as

$$g_{44,1}\,\xi^1 + g_{44,2}\,\xi^2 = 0 . \tag{6.5.3}$$

Labelling $\dfrac{\partial g_{44}}{\partial R_1}$ as $f(R_1 R_2)$ and noting from g_{44}'s form

(3.7) that $\frac{\partial g_{44}}{\partial R_1} = \frac{\partial g_{44}}{\partial R_2} = f$, we use the chain rule obtaining for condition (6.5.3),

$$\left(f\frac{\partial R_1}{\partial r} + f\frac{\partial R_2}{\partial r}\right)\xi^1 + \left(f\frac{\partial R_1}{\partial z} + f\frac{\partial R_2}{\partial z}\right)\xi^2 = 0 \quad (6.5.4)$$

Cancelling the $f(R_1,R_2)$ from equation (6.5.4),* and also observing from the definition of R_1, R_2 (p.28) that

$$\frac{\partial R_1}{\partial r} = \frac{r}{R_1} \; ; \; \frac{\partial R_2}{\partial r} = \frac{r}{R_2}$$

$$\frac{\partial R_1}{\partial z} = \frac{(z-\ell)}{R_1} \; ; \; \frac{\partial R_2}{\partial z} = \frac{(z+\ell)}{R_2}$$

we obtain the necessary condition

$$\left(\frac{r}{R_1} + \frac{r}{R_2}\right)\xi^1 + \left[\frac{(z-\ell)}{R_1} + \frac{(z+\ell)}{R_2}\right]\xi^2 = 0.$$

Inserting the ξ^μ(6.4.4) to be tested, the condition becomes

$$\left(\frac{r}{R_1} + \frac{r}{R_2}\right)(\varepsilon^{13} z\cos\varphi + \varepsilon^{23} z\sin\varphi) + \left[\frac{(z-\ell)}{R_1} + \frac{(z+\ell)}{R_2}\right](-\varepsilon^{13} r\cos\varphi - \varepsilon^{23} r\sin\varphi) = 0. \quad (6.5.5)$$

Cancelling the common factor of r* and noting since $\varepsilon^{13}, \varepsilon^{23}$ are arbitrary their coefficients must separately equal zero for condition (6.5.5) to be valid, we obtain the necessary condition for the ε^{13} coefficient

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) z\cos\varphi - \left[\frac{(z-\ell)}{R_1} + \frac{(z+\ell)}{R_2}\right]\cos\varphi = 0.$$

Cancelling the common factor of $\cos\varphi$* and simplifying we obtain the necessary condition

$$\ell\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = 0.$$

This in turn is only valid if

$$R_1 = R_2$$

a condition true in general only for a line of zero length $(\ell = 0)$!

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* It is the <u>general</u> validity of (6.5.1) that is being tested in this development.

Since we are considering the case $\ell = \mu \equiv \frac{GM}{c^2}$, zero length implies zero mass. Thus the Weyl line mass solution for "length" 2μ is not compatible with the Killing vector structure for spherical symmetry except for the trival case of mass-less space (flat space). The solution, therefore, does not possess spherical symmetry and cannot be considered as the Schwarzschild solution which is the general relativistic solution for spherical symmetry.

The only assumption used in the development above was that the metric (3.7) was in generalized cylindrical coordinates. This, however, is seen to be the case both from the development of the metric (cf. Chapter 3) and from the fact that the Killing vector structure for cylindrical symmetry in general cylindrical polars $\xi^{\mu} = (0, 0, \varepsilon'^{2}, 0)$ is compatible with the Weyl metric which is indisputedly cylindrically symmetric. If the metric were given in any coordinates but generalized cylindrical polars the Killing vector structure would have to correspond to these coordinates and would be altered in form according to

$$\xi'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \xi^{\nu}$$

Thus the Killing vector structure $\xi^{\mu} = (0, 0, \varepsilon'^{2}, 0)$ would no longer be compatible with the metric. We see that if some symmetry properties of a metric are known a priori, the criterion for compatibility of metric and Killing vector structure can be used to gain information as to the character of the coordinates. We shall make use of this in the second approach to the analysis of the Weyl solution.

The Weyl line mass solution for "length" 2μ can be transformed into "Schwarzschild" form

$$ds^2 = \left(1 - \frac{2\mu}{\rho}\right)dt^2 - \left(\frac{1}{1-\frac{2\mu}{\rho}}\right)d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad \rho > 2\mu$$

by the transformation preceding (3.8). The Killing vector structure compatible with this metric is

$$\xi^{\mu} = (0, B\sin\varphi + A\cos\varphi, B\cot\theta\cos\varphi - A\cot\theta\sin\varphi + D, 0)$$

where A,B and D are arbitrary constants. Herein lies the source of possible misinterpretations. The structure is immediately recognizable as that corresponding to spherical symmetry in generalized spherical polar coordinates $(\rho, \theta, \varphi, t)$. This is a seeming contradiction to the analysis above wherein we concluded that the particular Weyl solution under consideration was not spherically symmetric. Since transformations can not alter intrinsic symmetry properties it appears that we are in a dilemma. However, one must note that the Killing vector structure compatible with the "Schwarzschild" form of the solution is that of spherical symmetry in generalized spherical polar coordinates. Thus the proof of spherical symmetry hinges on whether or not the coordinates $(\rho, \theta, \varphi, t)$ are generalized spherical polars. As discussed in Chapter 4, serious intuitive doubt is raised by the consideration that ρ has an abridged range of possible values ($2\mu < \rho < \infty$) whereas the spherical radial coordinate ranges from zero to infinity. We can now, in addition, show analytically that our intuitive doubt is well founded. We can show that the coordinates $(\rho, \theta, \varphi, t)$ are not generalized spherical polars through the consideration of Killing vector structures.

We have seen previously that the Killing vector structure for spherical symmetry in generalized cylindrical polars (r,z,φ,t) is

$$\xi^{\mu} = (\varepsilon^{13} z\cos\varphi + \varepsilon^{23} z\sin\varphi, -\varepsilon^{13} r\cos\varphi - \varepsilon^{23} r\sin\varphi, -\varepsilon^{12} - \varepsilon^{13}\frac{z}{r}\sin\varphi + \varepsilon^{23}\frac{z}{r}\cos\varphi, 0)$$

But if in fact (ρ,θ,φ,t) were generalized spherical polars obtained from generalized cylindrical polars (r,z,φ,t) by the transformation preceding (3.8), then this Killing vector structure would have to transform to

$$\xi^{\mu} = (0, B\sin\varphi + A\cos\varphi, B\cot\theta\cos\varphi - A\cot\theta\sin\varphi + D, 0)$$

where A, B are constants.

Investigating the third component of the Killing vector after transformation $\left(\xi'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\xi^{\nu}\right)$, we obtain

$$\xi'^{3} = \frac{\partial x'^{3}}{\partial x^{\nu}}\xi^{\nu}$$

$$\xi'^{3} = \xi^{3} \quad \text{(since } x'^{3} = x^{3} = \varphi)$$

$$\xi'^{3} = -\varepsilon^{12} - \varepsilon^{13}\cot\theta \cdot \frac{(\rho-\mu)}{\sqrt{\rho^2-2\mu\rho}}\sin\varphi + \varepsilon^{23}\frac{(\rho-\mu)}{\sqrt{\rho^2-2\mu\rho}}\cot\theta\cos\varphi .$$

This does not have the desired form

$$\xi'^{3} = B\cot\theta\cos\varphi - A\cot\theta\sin\varphi + D$$

unless $\mu = 0$ (the trival case of a mass-less space).

Thus the assumption that $(\rho, \theta, \varphi, t)$ are generalized spherical polar coordinates is invalid and consequently the basis for the claim that the "Schwarzschild" form of the Weyl solution is spherically symmetric is refuted.

CHAPTER 7

SUMMARY AND CONCLUSIONS.

In this thesis we have refuted the alleged equivalence of the Weyl line mass solution for "length" $2\mu\left(\equiv\frac{2GM}{c^2}\right)$ and the Schwarzschild solution.

In Chapter 4 this refutation was based on the interdependence of metric and coordinates in Riemannian geometry. It was demonstrated that the coordinates of the "Schwarzschild" form of the Weyl line mass solution for "length" 2μ (3.8) do not lie in the group characterized by the bona fide Schwarzschild metric (4.1). The argument was made that since the two metrics (3.8) and (4.1) have the same functional form but intrinsically different variables (coordinates) they must be non-equivalent.

In Chapter 5 the refutation was based on the assumption that regions of space-time correspond under transformation only if their boundaries correspond physically under the given transformation. The scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ was analyzed using differential forms and found to be indeterminate along the line-mass for the Weyl line mass solution of characteristic "length" 2μ. A continuity argument was invoked to suggest that the scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ is infinite along the line mass, this implying directly that the boundary formed by the line mass itself is a physical, intrinsic singularity. This claim, that the boundary formed by the line mass is a physical singularity, was subsequently demonstrated *conclusively* using the elementary flatness criterion. Noting that this boundary transforms to the boundary surface $\rho = 2\mu$ under the transformation preceding (3.8) and also that the $\rho = 2\mu$ boundary of the Schwarzschild solution is *not* a physical, intrinsic singularity, we were forced to conclude (re the assumption above) that the exterior regions ($\rho > 2\mu$) of (3.8) and (4.1) do *not* correspond.

The most direct and conclusive refutation of the claimed equivalence of (3.8) and (4.1) was executed in Chapter 6. It was shown through the use of Killing vector analysis that the Weyl solution does not possess the same intrinsic symmetry (spherical symmetry) as the Schwarzschild solution (4.1). Thus the solutions (3.8) and (4.1) are intrinsically different.

The refutation of the alleged equivalence of the Weyl line mass solution for "length" 2μ and the Schwarzschild solution is important in itself. It removes a conceptual difficulty due entirely to a misinterpretation imbedded in the literature for over fifty years. With the refutation, theory and intuition are reconciled on one of the basic grounds of experience, symmetry. As one would instinctively expect, a line mass *is* differentiable from a spherical mass to an exterior observer.

The refutation in this work has further consequence, however, than aligning theory with intuition. Much work has been built on the concept of a line mass of "length" 2μ representing a single point particle. This must be appropriately modified and the consequent interpretations altered.

The most obvious extension of the representation of a single mass point by a line mass of "length" 2μ is the representation of two point particles by two such line masses (Bach and Weyl, 1922; Robertson and Noonan, 1968). In view of the refutation in this work, the resultant solution must now be relegated to the realm of line mass solutions and the claim that it is a solution of two point masses must be dismissed. The actual two point mass solution is that given by Silberstein (1936) and discussed in Chapter 5.

Similarly, the field of a collinear set of spherically symmetric masses (Isreal and Khan, 1964), derived assuming that point masses can be represented by line masses of "length" 2μ, must be reinterpreted as the field of a collinear set of line masses.

In discussions of gravitational collapse it is invariably stated that bodies lose their asymmetries as they collapse through the Schwarzschild event horizon. However this has only been proven for small perturbations from spherical symmetry and a few highly simplified cases of large departure from spherical symmetry. One would be tempted to offer the line mass solutions as examples exhibiting this phenomenon. Under the traditional interpretation that the line mass of "length" 2μ exhibits spherical symmetry, one could argue that as a line mass collapses, in quasi-static stages, to a critical "length" 2μ it becomes more and more spherical. Thus the line mass solutions would offer a substantiation of the current view of gravitational collapse. However in this work we have demonstrated that the traditional interpretation of the static line mass solution for "length" 2μ as a spherically symmetric solution is erroneous. Therefore the argument above would fail.

Israel (1967) in his discussion of possible instability in the self-closure phenomenon in gravitational collapse assumes the Weyl line mass solution for "length" 2μ to be a disguised form of the Schwarzschild metric. The analysis which follows from the assumption must be altered or discarded.

Takeno (1952, 1966), using his characteristic system for the determination of spherical symmetry, is led to claim that the Weyl line mass solution for "length" 2μ is spherically symmetric. This thesis takes issue with the characteristic system in its present form.

Tantamount to the validity of Takeno's characteristic system is the condition that any solution transformable to the form

$$ds^2 = -A(\rho,t)\,d\rho^2 - B(\rho,t)\,d\Sigma^2 + C(\rho,t)\,dt^2, \quad (7.1)$$

where $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, is spherically symmetric.*
However, this is true only if the coordinates in (7.1) are generalized spherical polar coordinates. If in transforming a metric to the form (7.1), qualifications as to the intrinsic properties of the coordinates must be introduced, it is entirely conceivable that the coordinates in (7.1) may not be generalized spherical polars. In Chapter 4 and again in Chapter 6 it was argued that the coordinates in the "Schwarzschild" form of the line mass solution for "length" 2μ are not generalized spherical polars. Thus this particular solution

* The ρ coordinate appearing here is denoted by the symbol, r, in Takeno's work.

is an example of a metric transformable to the form (7.1) and yet <u>not</u> spherically symmetric because of qualifications placed on the coordinates (in this case an appending condition on the range of the ρ coordinate).

The analyses in this thesis clearly illustrate the extreme care which must be exercised in interpreting the metric and its coordinates. Further work is suggested in pinpointing the role which qualifications on coordinates play in a metric geometry. Gedanken experiments exhibiting local differences introduced in metrics by coordinate qualifications seem a logical route to follow. Specifically it is suggested that such experiments be devised to illustrate the differences, other than symmetry discussed in this work, between the "Schwarzschild" form of the line mass solution and the bona fide Schwarzschild solution.

APPENDIX A

BASIC DEFINITIONS, OPERATIONS AND ALGEBRA OF DIFFERENTIAL FORMS.

A differential form is a multilinear scalar function associated with a tensor which is effectively skew-symmetric. For example a differential form of degree one, or a 1-form is

$$\theta(\underset{\sim}{dx}) \equiv A_\alpha dx^\alpha \qquad (A.1)$$

where A_α is the first order tensor with which the 1-form, θ, is associated. A differential form of degree two, or a 2-form is

$$\phi(\underset{\sim}{dx}, \underset{\sim}{dy}) \equiv 2!\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]}$$

where $F_{\alpha\beta}$ is the second order tensor with which the 2-form, ϕ, is associated. Higher order forms are defined similarly. The square bracket notation used above indicates anti-symmetrization,

$$dx^{[\alpha} dy^{\beta]} \equiv \left(\frac{1}{2!}\right)\left(dx^\alpha dy^\beta - dx^\beta dy^\alpha\right) \qquad (A.2)$$

The exterior product, or wedge product, of a p-form and a q-form is the (p+q)-form obtained by taking the tensor product of the associated tensors and anti-symmetrizing. For example

$$\psi \equiv \phi \wedge \theta = 3!\, F_{\alpha\beta} A_\gamma\, dx^{[\alpha} dy^\beta dz^{\gamma]}$$

The exterior differential of a p-form is the (p+1)-form obtained by taking the partial or covariant derivative of the associated p^{th} order tensor. For example,

$$d\phi(\underset{\sim}{dx}, \underset{\sim}{dy}, \underset{\sim}{dz}) = 3!\, F_{\beta\gamma;\alpha}\, dx^{[\alpha} dy^\beta dz^{\gamma]}$$

$$= 3!\, F_{\beta\gamma,\alpha}\, dx^{[\alpha} dy^\beta dz^{\gamma]}$$

If α, β are forms of degree a,b respectively,

$$\alpha \wedge \beta = (-1)^{ab}\, \beta \wedge \alpha \qquad (A.3)$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta . \qquad \text{(A.4)}$$

From (A.3) if θ is an odd-form,

$$\theta \wedge \theta = 0.$$

From (A.4), if f is a scalar and Ω any form,

$$d(f\Omega) = f d\Omega + df \wedge \Omega .$$

Also since applying exterior differentiation twice in succession involves the skew part of $\partial^2 / \partial x^\alpha \partial x^\beta$ which vanishes,

$$d^2 \Omega \equiv 0$$

for any form Ω.

The frame components (tetrad components) of any tensor $T_{\alpha\beta \dots}$ are defined by

$$T_{AB \dots} \equiv T_{\alpha\beta \dots} \, e_{(A)}{}^{\alpha} e_{(B)}{}^{\beta} \dots .$$

APPENDIX B

DETAILS OF THE ANALYSIS OF $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$

In section 5.3 a synopsis of the analysis of the scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ was presented. Herein the detailed calculations referred to in that section are given.

The Weyl line mass solution (5.3.2) can be written as

$$ds^2 = (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2$$

in terms of the 1-forms

$$\begin{aligned}
\theta^1 &= \left(\tanh\tfrac{\xi}{2}\right)^{\psi} dt \\
\theta^2 &= \ell\left(\coth\tfrac{\xi}{2}\right)^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}d\xi \\
\theta^3 &= \ell\left(\coth\tfrac{\xi}{2}\right)^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}d\eta \\
\theta^4 &= \ell\left(\coth\tfrac{\xi}{2}\right)^{\psi}\sinh\xi\sin\eta\, d\varphi
\end{aligned} \qquad (5.3.3)$$

Exterior differentiating the one forms (5.3.3),

$$d\theta^1 = \frac{\psi}{2}\left(\tanh\tfrac{\xi}{2}\right)^{\psi-1}\operatorname{sech}^2\tfrac{\xi}{2}\; d\xi\wedge dt$$

$$= \frac{\frac{\psi}{2}\left(\tanh\frac{\xi}{2}\right)^{\psi-1}\operatorname{sech}^2\frac{\xi}{2}\quad \theta^2\wedge\theta^1}{\ell\left(\coth\frac{\xi}{2}\right)^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}\left(\tanh\frac{\xi}{2}\right)^{\psi}}$$

$$= \frac{\psi}{2\ell}\,\frac{\operatorname{sech}^2\frac{\xi}{2}\quad \theta^2\wedge\theta^1}{\left(\coth\frac{\xi}{2}\right)^{\psi-1}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}$$

$$d\theta^2 = \ell\,(\coth\tfrac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}\left(\tfrac{1-\psi^2}{2}\right)(\sinh^2\xi+\sin^2\eta)^{-\left(\frac{\psi^2+1}{2}\right)}$$
$$\times\, 2\sin\eta\cos\eta\; d\eta\wedge d\xi$$

$$= \frac{(1-\psi^2)\sin\eta\cos\eta}{\ell(\coth\frac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\psi^2}{2}}}\;\theta^3\wedge\theta^2$$

$$d\theta^3 = \left\{\begin{array}{l} -\frac{\ell}{2}\,\psi\,(\coth\frac{\xi}{2})^{\psi-1}\operatorname{csch}^2\frac{\xi}{2}\,(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}} \\ +\,\ell\,(\coth\frac{\xi}{2})^{\psi}\psi^2(\sinh\xi)^{\psi^2-1}\cosh\xi\,(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}} \\ +\,\ell\,(\coth\frac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}(1-\psi^2)(\sinh^2\xi+\sin^2\eta)^{-\left(\frac{\psi^2+1}{2}\right)}\sinh\xi\cosh\xi \end{array}\right\} d\xi\wedge d\eta$$

$$= \frac{(\coth\frac{\xi}{2})^{-\psi-1}(\sinh\xi)^{-\psi^2-1}(\sinh^2\xi+\sin^2\eta)^{\frac{\psi^2-3}{2}}}{\ell}$$

$$\times\left\{\begin{array}{l} -\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi\,(\sinh^2\xi+\sin^2\eta) \\ +\,\coth\frac{\xi}{2}\,\psi^2\cosh\xi\,(\sinh^2\xi+\sin^2\eta) \\ +\,\coth\frac{\xi}{2}\sinh\xi\,(1-\psi^2)\sinh\xi\cosh\xi \end{array}\right\}\theta^2\wedge\theta^3$$

$$d\theta^4 = -\frac{\ell}{2}\psi\left(\coth\frac{\xi}{2}\right)^{\psi-1}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi\sin\eta\, d\xi\wedge d\varphi$$
$$+\ell\left(\coth\frac{\xi}{2}\right)^{\psi}\cosh\xi\sin\eta\; d\xi\wedge d\varphi$$
$$+\ell\left(\coth\frac{\xi}{2}\right)^{\psi}\sinh\xi\cos\eta\; d\eta\wedge d\varphi$$

$$= \frac{-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi + \coth\frac{\xi}{2}\cosh\xi}{\ell\left(\coth\frac{\xi}{2}\right)^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^2\wedge\theta^4$$

$$+\frac{\cot\eta}{\ell\left(\coth\frac{\xi}{2}\right)^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^3\wedge\theta^4 .$$

However, since

$$d\theta^A = -\omega^A{}_B\wedge\theta^B \qquad (5.2.11)$$

we obtain, upon examination of the exterior differentials above,

$$\omega^1{}_2 = \frac{\psi}{2\ell}\,\frac{\operatorname{sech}^2\frac{\xi}{2}}{\left(\coth\frac{\xi}{2}\right)^{\psi-1}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^1$$
$$+ A\theta^2$$

$$\omega^1{}_1 = 0$$

$$\omega^1{}_3 = B\theta^3$$

$$\omega^1{}_4 = C\theta^4$$

$$\omega^2_1 = D\theta^1$$

$$\omega^2_2 = 0$$

$$\omega^2_3 = \frac{(1-\gamma^2)\sin\eta\cos\eta}{\ell(\coth\frac{\xi}{2})^{\gamma}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\gamma^2}{2}}}\theta^2$$

$$+E\theta^3$$

$$\omega^2_4 = F\theta^4$$

$$\omega^3_1 = G\theta^1$$

$$\omega^3_2 = \frac{\left\{\begin{array}{l} -\frac{\gamma}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\gamma^2\cosh\xi(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\sinh\xi\,(1-\gamma^2)\sinh\xi\cosh\xi \end{array}\right\}}{\ell(\coth\frac{\xi}{2})^{\gamma+1}(\sinh\xi)^{\gamma^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\gamma^2}{2}}}\theta^3$$

$$+H\theta^2$$

$$\omega^3_3 = 0$$

$$\omega^3_4 = I\theta^4$$

$$\omega^4{}_1 = J\theta^1$$

$$\omega^4{}_2 = \frac{-\frac{\gamma^2}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi + \coth\frac{\xi}{2}\cosh\xi}{l\left(\coth\frac{\xi}{2}\right)^{\gamma+1}(\sinh\xi)^{\gamma^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,\theta^4$$

$$+ K\theta^2$$

$$\omega^4{}_3 = \frac{\cot\eta}{l\left(\coth\frac{\xi}{2}\right)^{\gamma}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,\theta^4$$

$$+ L\theta^3$$

$$\omega^4{}_4 = 0$$

where A, B, $\cdots$, L are, as yet, undetermined functions of ξ, η.

Now,

$$\omega^1{}_2 = g^{1B}\omega_{B2} = g^{11}\omega_{12} \qquad \text{(since } g^{AB} \text{ is diagonal)}$$

$$= -g^{11}\omega_{21} \qquad [\text{since } \omega_{AB} = -\omega_{BA} \ (5.2.10)]$$

$$= g^{22}\omega_{21} \qquad [\text{from } (5.3.4)]$$

$$= g^{2B}\omega_{B1} \qquad \text{(since } g^{AB} \text{ is diagonal)}$$

$$= \omega^2{}_1 .$$

Similarly,

$$\omega^1_3 = \omega^3_1$$

$$\omega^1_4 = \omega^4_1$$

$$\omega^2_1 = \omega^1_2$$

$$\omega^2_3 = -\omega^3_2$$

$$\omega^2_4 = -\omega^4_2$$

$$\omega^3_4 = -\omega^4_3 .$$

So,

$$\omega^1_2 = \omega^2_1 = \frac{\gamma}{2\ell} \frac{\operatorname{sech}^2\frac{\xi}{2}}{(\coth\frac{\xi}{2})^{\gamma-1}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,\theta^1$$

$$\omega^1_1 = \omega^2_2 = \omega^3_3 = \omega^4_4 = 0$$

$$\omega^1_3 = \omega^3_1 = 0$$

$$\omega^1_4 = \omega^4_1 = 0$$

$$\omega^2_3 = -\omega^3_2 = \frac{(1-\gamma^2)\sin\eta\cos\eta}{\ell(\coth\frac{\xi}{2})^{\gamma}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\gamma^2}{2}}}\,\theta^2$$

$$-\frac{\left\{\begin{array}{l} -\frac{\gamma}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\gamma^2\cosh\xi(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\sinh\xi(1-\gamma^2)\sinh\xi\cosh\xi \end{array}\right\}\theta^3}{\ell(\coth\frac{\xi}{2})^{\gamma+1}(\sinh\xi)^{\gamma^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\gamma^2}{2}}}$$

$$\omega^2{}_4 = -\omega^4{}_2 = \frac{\frac{\gamma}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi - \coth\frac{\xi}{2}\cosh\xi}{\ell(\coth\frac{\xi}{2})^{\gamma+1}(\sinh\xi)^{\gamma^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,\theta^4$$

$$\omega^3{}_4 = -\omega^4{}_3 = \frac{-\cot\eta}{\ell(\coth\frac{\xi}{2})^{\gamma}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,\theta^4$$

Using the connection 1-forms, $\omega^A{}_B$, above, we form the curvature 2-forms

$$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B .$$

We obtain

$$\Omega^1{}_1 = d\omega^1{}_1 + \omega^1{}_1\wedge\omega^1{}_1 + \omega^1{}_2\wedge\omega^2{}_1 + \omega^1{}_3\wedge\omega^3{}_1 + \omega^1{}_4\wedge\omega^4{}_1$$

$$= 0 \qquad \text{(cf. } \omega^A{}_B \text{ above and note } \theta\wedge\theta = 0, \text{ if } \theta \text{ is a 1-form).}$$

Similarly,

$$\Omega^2{}_2 = \Omega^3{}_3 = \Omega^4{}_4 = 0 .$$

Also,

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^1{}_1\wedge\omega^1{}_2 + \omega^1{}_2\wedge\omega^2{}_2 + \omega^1{}_3\wedge\omega^3{}_2 + \omega^1{}_4\wedge\omega^4{}_2$$

$$= d\omega^1{}_2$$

$$= \frac{\gamma}{2\ell}\,\frac{\operatorname{sech}^2\frac{\xi}{2}}{(\coth\frac{\xi}{2})^{\gamma-1}(\sinh\xi)^{\gamma^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\gamma^2}{2}}}\,d\theta^1$$

$$+\frac{\nu}{2\ell}\frac{\left(-\operatorname{sech}^2\frac{\xi}{2}\tanh\frac{\xi}{2}\right)d\xi}{\left(\coth\frac{\xi}{2}\right)^{\nu-1}(\sinh\xi)^{\nu^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\nu^2}{2}}}\wedge\theta^1$$

$$+\frac{\nu}{2\ell}\frac{\operatorname{sech}^2\frac{\xi}{2}\ \left(\frac{\nu-1}{2}\right)\operatorname{csch}^2\frac{\xi}{2}\ \ d\xi}{\left(\coth\frac{\xi}{2}\right)^{\nu}(\sinh\xi)^{\nu^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\nu^2}{2}}}\wedge\theta^1$$

$$+\frac{\nu}{2\ell}\frac{\operatorname{sech}^2\frac{\xi}{2}\ (-\nu^2)\cosh\xi\ \ d\xi}{\left(\coth\frac{\xi}{2}\right)^{\nu-1}(\sinh\xi)^{\nu^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\nu^2}{2}}}\wedge\theta^1$$

$$+\frac{\nu}{2\ell}\frac{\operatorname{sech}^2\frac{\xi}{2}\ (\nu^2-1)\sinh\xi\cosh\xi\ \ d\xi}{\left(\coth\frac{\xi}{2}\right)^{\nu-1}(\sinh\xi)^{\nu^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\nu^2}{2}}}\wedge\theta^1$$

$$+\frac{\nu}{2\ell}\frac{\operatorname{sech}^2\frac{\xi}{2}\ (\nu^2-1)\sin\eta\cos\eta\ d\eta}{\left(\coth\frac{\xi}{2}\right)^{\nu-1}(\sinh\xi)^{\nu^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\nu^2}{2}}}\wedge\theta^1$$

$$=\frac{\nu^2}{4\ell^2}\frac{\operatorname{sech}^4\frac{\xi}{2}}{\left(\coth\frac{\xi}{2}\right)^{2(\nu-1)}(\sinh\xi)^{2\nu^2}(\sinh^2\xi+\sin^2\eta)^{1-\nu^2}}\theta^2\wedge\theta^1$$

$$+\frac{\nu}{2\ell^2}\frac{\left(-\operatorname{sech}^2\frac{\xi}{2}\tanh\frac{\xi}{2}\right)}{\left(\coth\frac{\xi}{2}\right)^{2\nu-1}(\sinh\xi)^{2\nu^2}(\sinh^2\xi+\sin^2\eta)^{1-\nu^2}}\theta^2\wedge\theta^1$$

$$+\frac{\nu}{2\ell^2}\frac{\operatorname{sech}^2\frac{\xi}{2}\ \left(\frac{\nu-1}{2}\right)\operatorname{csch}^2\frac{\xi}{2}}{\left(\coth\frac{\xi}{2}\right)^{2\nu}(\sinh\xi)^{2\nu^2}(\sinh^2\xi+\sin^2\eta)^{1-\nu^2}}\theta^2\wedge\theta^1$$

$$+\frac{\nu}{2\ell^2}\frac{\operatorname{sech}^2\frac{\xi}{2}\ (-\nu^2)\cosh\xi}{\left(\coth\frac{\xi}{2}\right)^{2\nu-1}(\sinh\xi)^{2\nu^2+1}(\sinh^2\xi+\sin^2\eta)^{1-\nu^2}}\theta^2\wedge\theta^1$$

$$+\frac{\nu}{2\ell^2}\frac{\operatorname{sech}^2\frac{\xi}{2}\ (1-\nu^2)(-1)\cosh\xi}{\left(\coth\frac{\xi}{2}\right)^{2\nu-1}(\sinh\xi)^{2\nu^2-1}(\sinh^2\xi+\sin^2\eta)^{2-\nu^2}}\theta^2\wedge\theta^1$$

$$+ \frac{\psi}{2\ell^2} \frac{\operatorname{sech}^2\frac{\xi}{2}\,(\psi^2-1)\cos\eta\sin\eta}{(\coth\frac{\xi}{2})^{2\psi-1}(\sinh\xi)^{2\psi^2}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}\,\theta^3\wedge\theta^1$$

$$\Omega^1{}_2 = \frac{\psi}{2\ell^2}\,\frac{\operatorname{sech}^2\frac{\xi}{2}}{(\coth\frac{\xi}{2})^{2\psi}(\sinh\xi)^{2\psi^2}(\sinh^2\xi+\sin^2\eta)^{1-\psi^2}}$$

$$\times\left\{\begin{aligned}&\left(\psi-\tfrac{1}{2}\right)\operatorname{csch}^2\tfrac{\xi}{2} - 1 + \coth\tfrac{\xi}{2}\coth\xi\,(-\psi^2)\\ &+ \frac{(\psi^2-1)\coth\frac{\xi}{2}\cosh\xi\sinh\xi}{\sinh^2\xi+\sin^2\eta}\end{aligned}\right\}\theta^2\wedge\theta^1$$

$$+ \frac{\psi(\psi^2-1)\operatorname{sech}^2\frac{\xi}{2}\sin\eta\cos\eta}{2\ell^2(\coth\frac{\xi}{2})^{2\psi-1}(\sinh\xi)^{2\psi^2}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}\,\theta^3\wedge\theta^1$$

$$\Omega^1{}_3 = \omega^1{}_2\wedge\omega^2{}_3$$

$$= \frac{\psi(1-\psi^2)\operatorname{sech}^2\frac{\xi}{2}\sin\eta\cos\eta}{2\ell^2(\coth\frac{\xi}{2})^{2\psi-1}(\sinh\xi)^{2\psi^2}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}\,\theta^1\wedge\theta^2$$

$$- \frac{\psi\operatorname{sech}^2\frac{\xi}{2}}{2\ell^2(\coth\frac{\xi}{2})^{2\psi}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}$$

$$\times\left\{\begin{aligned}&-\tfrac{\psi}{2}\operatorname{csch}^2\tfrac{\xi}{2}\sinh\xi\,(\sinh^2\xi+\sin^2\eta)\\ &+\coth\tfrac{\xi}{2}\,\psi^2\cosh\xi\,(\sinh^2\xi+\sin^2\eta)\\ &+\coth\tfrac{\xi}{2}\sinh\xi\,(1-\psi^2)\sinh\xi\cosh\xi\end{aligned}\right\}\theta^1\wedge\theta^3$$

$$\Omega^1{}_4 = \omega^1{}_2 \wedge \omega^2{}_4$$

$$= \frac{\Psi \operatorname{sech}^2\frac{\xi}{2}\left(\frac{\Psi}{2}\operatorname{csch}^2\frac{\xi}{2}\operatorname{senh}\xi - \coth\frac{\xi}{2}\cosh\xi\right)}{2\ell^2(\coth\frac{\xi}{2})^{2\Psi}(\operatorname{senh}\xi)^{2\Psi^2+1}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{1-\Psi^2}}\,\theta^1\wedge\theta^4$$

$$\Omega^2{}_3 = d\omega^2{}_3$$

$$= \frac{(1-\Psi^2)^2\operatorname{sen}^2\eta\cos^2\eta}{\ell^2(\coth\frac{\xi}{2})^{2\Psi}(\operatorname{senh}\xi)^{2\Psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{3-\Psi^2}}\,\theta^3\wedge\theta^2$$

$$- \left\{\begin{array}{l} -\frac{\Psi}{2}\operatorname{csch}^2\frac{\xi}{2}\operatorname{senh}\xi\,(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta) \\ +\coth\frac{\xi}{2}\,\Psi^2\cosh\xi\,(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta) \\ +\coth\frac{\xi}{2}\operatorname{senh}\xi\,(1-\Psi^2)\operatorname{senh}\xi\cosh\xi \end{array}\right\}^2$$

$$\times \frac{1}{\ell^2(\coth\frac{\xi}{2})^{2\Psi+2}(\operatorname{senh}\xi)^{2\Psi^2+2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{3-\Psi^2}}\,\theta^2\wedge\theta^3$$

$$+ \frac{(1-\Psi^2)\left\{\cos^2\eta-\operatorname{sen}^2\eta+(\Psi^2-3)(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{-1}\operatorname{sen}^2\eta\cos^2\eta\right\}}{\ell^2(\coth\frac{\xi}{2})^{2\Psi}(\operatorname{senh}\xi)^{2\Psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{2-\Psi^2}}\,\theta^3\wedge\theta^2$$

$$
-\left\{\begin{aligned}
&\tfrac{\psi}{2}\operatorname{csch}^2\tfrac{\xi}{2}\coth\tfrac{\xi}{2}\sinh\xi(\sinh^2\xi+\sin^2\eta)\\
&-\tfrac{\psi}{2}\operatorname{csch}^2\tfrac{\xi}{2}\cosh\xi(\sinh^2\xi+\sin^2\eta)\\
&-\psi\operatorname{csch}^2\tfrac{\xi}{2}\sinh^2\xi\cosh\xi\\
&-\tfrac{\psi^2}{2}\operatorname{csch}^2\tfrac{\xi}{2}\cosh\xi(\sinh^2\xi+\sin^2\eta)\\
&+\psi^2\coth\tfrac{\xi}{2}\sinh\xi(\sinh^2\xi+\sin^2\eta)\\
&+2\psi^2\coth\tfrac{\xi}{2}\cosh^2\xi\sinh\xi\\
&+\tfrac{(\psi^2-1)}{2}\operatorname{csch}^2\tfrac{\xi}{2}\sinh^2\xi\cosh\xi\\
&+(1-\psi^2)\coth\tfrac{\xi}{2}\,(2\sinh\xi\cosh^2\xi+\sinh^3\xi)
\end{aligned}\right\}\theta^2\wedge\theta^3
$$

$$
\times\ \frac{1}{\ell^2(\coth\frac{\xi}{2})^{2\psi+1}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}
$$

$$
-\left\{\begin{aligned}
&-\tfrac{\psi}{2}\operatorname{csch}^2\tfrac{\xi}{2}\sinh\xi(\sinh^2\xi+\sin^2\eta)\\
&+\psi^2\coth\tfrac{\xi}{2}\cosh\xi(\sinh^2\xi+\sin^2\eta)\\
&+(1-\psi^2)\coth\tfrac{\xi}{2}\sinh^2\xi\cosh\xi
\end{aligned}\right\}
$$

$$
\times\frac{\left[\frac{(\psi+1)}{2}\operatorname{csch}^2\frac{\xi}{2}\tanh\frac{\xi}{2}-(\psi^2+1)\coth\xi+\frac{(\psi^2-3)\sinh\xi\cosh\xi}{\sinh^2\xi+\sin^2\eta}\right]}{\ell^2(\coth\frac{\xi}{2})^{2\psi+1}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}\,\theta^2\wedge\theta^3
$$

$$\Omega^2_4 = d\omega^2_4 + \omega^2_3 \wedge \omega^3_4$$

$$= \frac{\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi - \coth\frac{\xi}{2}\cosh\xi}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}$$

$$\times \left\{ \frac{-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi + \coth\frac{\xi}{2}\cosh\xi}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^2\wedge\theta^4 + \frac{\cot\eta}{\ell(\coth\frac{\xi}{2})^{\psi}\sinh\xi^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^3\wedge\theta^4 \right\}$$

$$+ \frac{\left(-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\coth\frac{\xi}{2}\sinh\xi + \frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\cosh\xi + \frac{1}{2}\operatorname{csch}^2\frac{\xi}{2}\cosh\xi - \coth\frac{\xi}{2}\sinh\xi\right)}{\ell^2(\coth\frac{\xi}{2})^{2\psi+1}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{1-\psi^2}}\,\theta^2\wedge\theta^4$$

$$+ \left(\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi - \coth\frac{\xi}{2}\cosh\xi\right)$$

$$\times \frac{\left\{\frac{(\psi+1)\operatorname{csch}^2\frac{\xi}{2}}{2\coth\frac{\xi}{2}} - (\psi^2+1)\coth\xi + \frac{(\psi^2-1)\sinh\xi\cosh\xi}{(\sinh^2\xi+\sin^2\eta)}\right\}}{\ell^2(\coth\frac{\xi}{2})^{2\psi+1}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{1-\psi^2}}\,\theta^2\wedge\theta^4$$

$$+ \frac{\left(\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi - \coth\frac{\xi}{2}\cosh\xi\right)(\psi^2-1)\sin\eta\cos\eta}{\ell^2(\coth\frac{\xi}{2})^{2\psi+1}(\sinh\xi)^{2\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}\,\theta^3\wedge\theta^4$$

$$\left[\frac{(1-\psi^2)\sin\eta\cos\eta}{\ell(\coth\frac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\psi^2}{2}}}\,\theta^2\right.$$

$$\left. + \frac{-\left\{\begin{array}{l} -\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi\,(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\,(\psi^2)\cosh\xi\,(\sinh^2\xi+\sin^2\eta) \\ +\coth\frac{\xi}{2}\sinh^2\xi\cosh\xi\,(1-\psi^2)\end{array}\right\}}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\psi^2}{2}}}\,\theta^3\right]$$

$$\wedge\ \frac{(-\cot\eta)}{\ell(\coth\frac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\,\theta^4$$

$$\Omega^3{}_4 = d\omega^3{}_4 + \omega^3{}_2 \wedge \omega^2{}_4$$

$$= -\left\{ \frac{-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\operatorname{senh}\xi + \coth\frac{\xi}{2}\cosh\xi}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\operatorname{senh}\xi)^{\psi^2+1}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{\frac{1-\psi^2}{2}}}\theta^3\wedge\theta^4 \right.$$

$$\left. + \frac{\cot\eta}{\ell(\coth\frac{\xi}{2})^{\psi}(\operatorname{senh}\xi)^{\psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{\frac{1-\psi^2}{2}}}\theta^3\wedge\theta^4 \right\}$$

$$\times \frac{\cot\eta}{\ell(\coth\frac{\xi}{2})^{\psi}(\operatorname{senh}\xi)^{\psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{\frac{1-\psi^2}{2}}}$$

$$+ \left[\csc^2\eta - \frac{(\psi^2-1)\operatorname{sen}\eta\cos\eta\cot\eta}{\operatorname{senh}^2\xi+\operatorname{sen}^2\eta}\right]$$

$$\times \frac{1}{\ell^2(\coth\frac{\xi}{2})^{2\psi}(\operatorname{senh}\xi)^{2\psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{1-\psi^2}}\theta^3\wedge\theta^4$$

$$- \frac{\cot\eta\left[\tanh\frac{\xi}{2}\left(\frac{\psi}{2}\right)\operatorname{csch}^2\frac{\xi}{2} - \psi^2\coth\xi + (\psi^2-1)\frac{\operatorname{senh}\xi\cosh\xi}{(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)}\right]}{\ell^2(\coth\frac{\xi}{2})^{2\psi}(\operatorname{senh}\xi)^{2\psi^2}(\operatorname{senh}^2\xi+\operatorname{sen}^2\eta)^{1-\psi^2}}\theta^2\wedge\theta^4$$

$$\left[\frac{\left\{\begin{array}{l}-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi\,(\sinh^2\xi+\sin^2\eta)\\ +\psi^2\coth\frac{\xi}{2}\cosh\xi(\sinh^2\xi+\sin^2\eta)\\ +\coth\frac{\xi}{2}\sinh^2\xi\,(1-\psi^2)\cosh\xi\end{array}\right\}}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\psi^2}{2}}}\theta^3 - \frac{(1-\psi^2)\sin\eta\cos\eta}{\ell(\coth\frac{\xi}{2})^{\psi}(\sinh\xi)^{\psi^2}(\sinh^2\xi+\sin^2\eta)^{\frac{3-\psi^2}{2}}}\theta^2\right]$$

$$\wedge\ \frac{\left(\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi-\coth\frac{\xi}{2}\cosh\xi\right)}{\ell(\coth\frac{\xi}{2})^{\psi+1}(\sinh\xi)^{\psi^2+1}(\sinh^2\xi+\sin^2\eta)^{\frac{1-\psi^2}{2}}}\theta^4$$

Noting that $\Omega^A{}_B = \frac{1}{2}R^A{}_{BCD}\,\theta^C\wedge\theta^D$ (5.2.18) and that the Riemann tensor is skew-symmetric in both index pairs, we can obtain the $R^A{}_{BCD}$ by identification with the coefficients of the appropriate 2-forms $\theta^C\wedge\theta^D$ in the $\Omega^A{}_B$ above. For example,

$$R^1{}_{231} = \frac{\psi(\psi^2-1)\operatorname{sech}^2\frac{\xi}{2}\sin\eta\cos\eta}{2\ell^2(\coth\frac{\xi}{2})^{2\psi-1}(\sinh\xi)^{2\psi^2}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}$$

$$\frac{\psi(\psi^2-1)\sin\eta\cos\eta}{(2)^{2\psi^2+1}\ell^2(\cosh\frac{\xi}{2})^{2\psi^2+2\psi+1}(\sinh\frac{\xi}{2})^{2\psi^2-2\psi+1}(\sinh^2\xi+\sin^2\eta)^{2-\psi^2}}$$

The R^{A}_{BCD} are not listed here explicity due to their extremely cumbersome form and the fact that any R^{A}_{BCD} component derired can be immediately read from the appropiate Ω^{A}_{B} form.

Finally, the determination of the non-finite regions of $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ can be reduced to a consideration of the non-finite regions of these R^{A}_{BCD} (cf. 5.3.6 and the argument following thereafter).

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