# Classes of Sign Nonsingular Matrices With a Specified Number of Zero Entries

by

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We accept this thesis as conforming to the required standard

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## Abstract

An $n$-by-$n$ matrix $B_n$ is called *sign nonsingular* (SNS) if every matrix with the same sign pattern as $B_n$ is nonsingular. A given SNS matrix determines an equivalence class (with respect to transposition and multiplication by permutation and signature matrices) of SNS matrices, all of which have the same number of zero entries. Such a matrix is *maximal* if no zero entry can be set nonzero so that the resulting matrix is SNS. In addition, $B_n$ is fully indecomposable if it does not have an $(n-k)$-by-$k$ zero submatrix for some $k$, where $1 \leq k \leq n-1$.

Two new infinite classes of SNS matrices are identified: $H_n^{(p,q,r,s)}$ and $V_n^{(r)}$. These are defined in terms of the Hessenberg matrix, $H_n$, which, for fixed $n$, is known to determine the unique equivalence class with the minimum number of zero entries, namely $\binom{n-1}{2}$. Each member of these new classes is a fully indecomposable maximal SNS matrix having a specific number of zero entries.

It is proved, for $n \geq 5$, that the equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+1$ zero entries is unique and is represented by $H_n^{(2,1,0,0)}$. Similarly, for $n \geq 7$, it is proved that there are exactly two such equivalence classes with $\binom{n-1}{2}+2$ zero entries, which are represented by $H_n^{(3,1,0,0)}$ and $H_n^{(2,1,2,1)}$.

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To

God

who has been faithful

through it all

# Chapter 1

# Definitions

## 1.1 Matrix Definitions

These definitions come primarily from Horn and Johnson [HJ] The notation $A_{mn} \equiv [a_{ij}]$ specifies an $m$-by-$n$ matrix with entries from the set of real numbers If $m = n$ this is abbreviated to $A_n$ If either $m$ or $n$ equals zero, then the matrix is vacuous The notation $|A_{mn}|$ denotes the entry-wise absolute value of $A_{mn}$ and $A^T_{mn}$ denotes the transpose of $A_{mn}$ The function $sgn$ determines the sign of a matrix entry, specifically,

$$sgn(a_{ij}) = \begin{cases} +1 & \text{if } a_{ij} > 0 \\ -1 & \text{if } a_{ij} < 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\nu(A_{mn})$ denote the number of nonzero entries in $A_{mn}$ A matrix $A_{mn}$ is said to be *contained* in $B_{mn}$ if $a_{ij} \neq 0$ implies $b_{ij} = a_{ij}$. Note that $\nu(A_{mn}) \leq \nu(B_{mn})$. A matrix $A_{mn}$ is a *zero matrix* if every entry of $A_{mn}$ is zero and a *nonzero matrix* if some entry of $A_{mn}$ is nonzero.

For index sets $\alpha \subseteq M \equiv \{1, \ldots, m\}$ and $\beta \subseteq N \equiv \{1, \ldots, n\}$, the *submatrix* of $A_{mn}$ lying in the rows indicated by $\alpha$ and the columns indicated by $\beta$ is denoted by $A[\alpha, \beta]$. If $\alpha = \beta$, then $A[\alpha, \beta]$ is a *principal submatrix* of $A_{mn}$ and the notation is abbreviated to $A[\alpha]$. Principal submatrices are written as $A[n_1, \ldots, n_k]$ rather than $A[\{n_1, \ldots, n_k\}]$. The complements of $\alpha$ in $M$ and $\beta$ in $N$ are denoted by $\alpha^c$ and $\beta^c$, respectively and the submatrix $A[\alpha^c, \beta^c]$ is called the *complementary submatrix* of $A[\alpha, \beta]$. The notation $A_n \oplus B_m$ specifies a matrix $C_{n+m}$ where $C[1, \ldots, n] = A_n$, $C[n+1, \ldots, n+m] = B_m$ and $C[\{1, \ldots, n\}, \{n+1, \ldots, n+m\}]$ and $C[\{n+1, \ldots, n+m\}, \{1, \ldots, n\}]$ are zero submatrices.

In a square matrix, the entries that occur in positions $(i, i)$ for $i = 1, \ldots, n$ define the *main diagonal* of the matrix and each entry is called a *diagonal entry*. The entries that occur in positions $(i+1, i)$ for $i = 1, \ldots, n-1$ define the *subdiagonal* of the matrix and finally, the entries that occur in positions $(i, n-i+1)$ for $i = 1, \ldots, n$ define the *anti-diagonal* of the matrix.

A *diagonal matrix*, $A_n$, is a square matrix in which $a_{ij} = 0$ if $i \neq j$. A *signature matrix* is a diagonal matrix in which every diagonal entry is either $+1$ or $-1$. The $n$-by-$n$ diagonal matrix having all diagonal entries equal to 1 is called an *identity matrix* and is denoted by $I_n$.

A matrix is a *permutation matrix* if exactly one entry in each row and column is equal to 1, and all other entries are 0. It is often convenient to specify a permutation matrix, $P_n$, by a permutation of $N$. Thus, the matrix $P_n = (i_1, i_2, \ldots, i_n)$ is such that $p_{j,i_j} = 1$ for $j = 1, \ldots, n$ with all other entries being 0 [GVL]. For example, $(4, 1, 3, 2)$ denotes the permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

In some cases, when specifying a permutation matrix with this notation, a segment of the permutation may be vacuous. For example, if $p = 1$ and $q = 0$, the permutation matrix $Q_{p+q+1} = (p+1, p+2, \ldots, p+q, p, 1, 2, \ldots, p-1, p+q+1) = (1, 2)$, as the segments $p+1, p+2, \ldots, p+q$ and $1, 2, \ldots, p-1$ are both vacuous. This convention is used, for example, in Theorem 4.2.8.

A matrix $A_n$ with $n \geq 2$ is *partly decomposable* if there exist permutation matrices $P_n$ and $Q_n$ such that

$$P_n A_n Q_n = \begin{bmatrix} C_r & D_{r,n-r} \\ O_{n-r,r} & E_{n-r} \end{bmatrix}$$

where $1 \leq r \leq n-1$ and $O_{n-r,r}$ is a zero matrix. If $A_n$ is not partly decomposable, it is *fully indecomposable*. Thus, $A_n$ is fully indecomposable if and only if it does not have an $(n-k)$-by-$k$ zero submatrix for some $k$, where $1 \leq k \leq n-1$. A matrix $A_n$ with $n \geq 2$ is *reducible* if there exists a permutation matrix $P_n$ such that

$$P_n A_n P_n^T = \begin{bmatrix} C_r & D_{r,n-r} \\ O_{n-r,r} & E_{n-r} \end{bmatrix}$$

where $1 \leq r \leq n-1$ and $O_{n-r,r}$ is a zero matrix. If $A_n$ is not reducible, it is *irreducible*. It is sometimes possible to permute the rows (or columns) of a reducible matrix to obtain an irreducible one. For example, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is reducible but $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is irreducible. Notice that reducible implies partly decomposable and fully indecomposable implies irreducible. In fact, a matrix $A_n$ is irreducible if and only if $|A_n| + I_n$ is fully indecomposable [BR, Theorem 4.2.3].

A matrix with a nonzero main diagonal is said to be *k-connected* if there exists an $l$-by-$m$ zero submatrix with $l + m = n - k$ but no $l'$-by-$m'$ zero submatrix with $l' + m' > n - k$[LM1]. A matrix that is 0-connected is partly decomposable whereas a matrix that is $k$-connected for $k > 0$ is fully indecomposable.

## 1.2 Sign Nonsingularity (SNS) Definitions

The *determinant* of $A_n$ is defined as

$$\det A_n = \sum_{\sigma} \mathrm{s}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the summation is over all permutations $\sigma$ of $\{1, \ldots, n\}$ and $\mathrm{s}(\sigma)$ is $-1$ if the permutation is odd and $+1$ otherwise. Each term in this summation, namely

$s(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$, is called a *signed elementary product* from $A_n$. If det $A_n$ is nonzero, then $A_n$ is called *nonsingular*, otherwise it is *singular*.

**Definition 1.2.1** A matrix $A_n$ is *sign nonsingular* (SNS) if every matrix $B_n$ is nonsingular where $sgn(b_{ij}) = sgn(a_{ij})$ for all $i$ and $j$.

That is, every matrix with the same sign pattern as $A_n$ is nonsingular. This implies that when computing det $A_n$, every signed elementary product is weakly the same sign (i.e., the same sign or zero) and at least one is nonzero. Thus, as far as sign nonsingularity is concerned, it is clear that the magnitude of each entry of $A_n$ is not important, just the sign. Therefore, without loss of generality, a representative for each equivalence class of SNS matrices can be chosen so that each matrix entry is in $\{-1, 0, +1\}$.

The *permanent* of $A_n$ is defined as

$$\text{per } A_n = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the summation is over all permutations $\sigma$ of $\{1, \ldots, n\}$.

A matrix $B_n$ with entries from $\{0, 1\}$ is said to be *convertible* if there exists a matrix $A_n$ with entries from $\{-1, 0, +1\}$ such that $|A_n| = B_n$ and $|\det A_n| = \text{per}\,|A_n| > 0$. It is clear that when $A_n$ is a SNS matrix, $|A_n|$ is convertible, and if $B_n$ is convertible, then the nonzero entries can be signed so that the resulting matrix is SNS.

Even though determinant and permanent have similar definitions, the determinant has a computationally efficient algorithm ($O(n^3)$) whereas permanent does not (#P-Complete [Val]). Note that if $A_n$ is SNS, then $|\det A_n| = \text{per}\,|A_n| > 0$, and thus the permanent can be computed efficiently.

A SNS matrix $A_n$ is said to be *normalized* if $a_{ii} = -1$ for all $i = 1, \ldots, n$. A zero entry, $a_{ij}$, in a SNS matrix $A_n$ is called an *essential zero* if when $a_{ij}$ is replaced by $\pm 1$, the matrix $A_n$ is no longer SNS. If every zero entry in a SNS matrix is an essential zero, then the matrix is said to be *maximal*.

Two matrices, $A_n$ and $B_n$, are said to be *equivalent* (or in the same *equivalence class*) if $A_n$ can be transformed into $B_n$ by any combination of the following:

- transposition,
- multiplication by permutation matrices, or
- multiplication by signature matrices.

By the properties of the determinant, if a matrix $B_n$ is equivalent to a SNS matrix $A_n$, then $B_n$ is also SNS. Thus, a given SNS matrix determines an equivalence class of SNS matrices, all of which have the same number of zeros. If a matrix $A_n$ is not normalized, then there exists a matrix $B_n$ equivalent to $A_n$ that is normalized, thus, a representative for each equivalence class can be taken as a normalized SNS matrix. Equivalence classes of maximal SNS matrices are of most interest because non-maximal SNS matrices are obtained by setting to 0 one or more entries of a maximal SNS matrix. For small values of $n$, the number of equivalence classes is known (see, e g , [DGJOvD, Sh]) For example, when $n = 4$, there are three equivalence classes of fully indecomposable maximal SNS matrices and when $n = 5$ there are six (see Table 6 1)

## 1 3 Digraph Definitions

These definitions come mainly from Bondy and Murty [BM] A *signed directed graph*, $D = (V, E)$, hereafter called a *digraph*, is an ordered pair consisting of a nonempty set of vertices $V \equiv \{v_1, \ldots, v_n\}$ and a set of arcs (directed edges) $E \subseteq V \times V$ in which each arc $(v_i, v_j)$ is assigned either a + or a − sign An arc $(v_i, v_i)$ is allowed but multiple arcs are not The notations $V(D)$ and $E(D)$ denote the vertices and the arcs, respectively, of the specified digraph $D$ The function *sgn* is used to obtain the sign associated with an arc (Note that the context of *sgn* will distinguish the two types of values in its range The *sgn* of a matrix entry returns an integer from $\{-1, 0, +1\}$, whereas the *sgn* of an arc returns a sign from $\{-, +\}$) The vertices can be *ordered* with a bijective function $\alpha : \{1, \ldots, n\} \rightarrow V(D)$, where $n = |V(D)|$ An *ordered digraph*, denoted $D_\alpha$, is a digraph in which all the vertices are ordered by some bijection $\alpha$ Applying a bijection $\alpha$ to the vertices of an unordered digraph produces an ordered digraph A digraph $D'$ is a *subdigraph* of a digraph $D$ if $V(D') \subseteq V(D)$, $E(D') \subseteq E(D)$, and for every $(u, v) \in E(D')$ the vertices $u, v \in V(D')$

There is an obvious one-to-one correspondence between ordered digraphs on $n$ vertices and $n$-by-$n$ matrices with entries from $\{-1, 0, +1\}$ Given an ordered digraph $D_\alpha$, the $n$-by-$n$ *associated matrix*, $A(D_\alpha) \equiv [a_{ij}]$, is defined by

$$a_{ij} = \begin{cases} +1 & \text{if } (i,j) \in E(D_\alpha) \text{ and } sgn\,(i,j) = + \\ -1 & \text{if } (i,j) \in E(D_\alpha) \text{ and } sgn\,(i,j) = - \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \ldots, n$. On the other hand, given any matrix $A_n$, the *associated digraph*, $D(A_n)$, is defined as follows. Let $V(D(A_n))$ be the vertex set $\{1, \ldots, n\}$ and let $E(D(A_n))$ be the arc set with an arc $(i, j)$ from vertex $i$ to vertex $j$ if and only if $a_{ij} \neq 0$. The arc $(i, j)$ is labelled with $-$ if $sgn(a_{ij}) = -1$ and $+$ otherwise. The notation implies that $D(A_n)$ is an ordered digraph. Clearly, $A_n = A(D(A_n))$ and $D_\alpha = D(A(D_\alpha))$.

A (*simple directed*) *path* of length $k \geq 1$, say $((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1}))$, is a sequence of $k$ arcs in which all the vertices $v_i$ are distinct. For convenience, a path is represented by its vertex sequence $(v_1, v_2, \ldots, v_{k+1})$ rather than its arc sequence. A (*directed*) *cycle* of length $k \geq 1$, say $((v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1))$, is a sequence of $k$ arcs in which all the vertices $v_i$ are distinct. As with paths, a cycle is represented by its vertex sequence. When $k = 1$, the cycle is called a *self loop*. A *Hamilton cycle* is a cycle containing all the vertices of the digraph.

The (*directed*) *path sign* of a path is a sign ascribed to the entire path and is defined as $(-1)^m$, where $m$ is the number of arcs in the path that are negatively signed. The (*directed*) *cycle sign* is defined similarly.

A digraph is said to be *strongly connected* if for every pair of distinct vertices

$u$ and $v$, there exists a path from $u$ to $v$ It is well known that an ordered digraph, $D_\alpha$, is strongly connected if and only if its associated matrix, $A(D_\alpha)$, is irreducible If a digraph contains a Hamilton cycle, then the digraph is strongly connected (but the converse is not necessarily true)

The *indegree* of a vertex $v$ is the cardinality of the set $\{(z,v) \mid (z,v) \in E(D)$ and $z \neq v\}$ and the *outdegree* of a vertex $v$ is the cardinality of the set $\{(v,z) \mid (v,z) \in E(D)$ and $z \neq v\}$ Note that self loops are excluded The *total degree* of vertex $v$ is the sum of its indegree and outdegree [Th1] When examining $A(D_\alpha)$ for any ordering $\alpha$, it can be easily seen that the indegree of vertex $i$, denoted $d^-(i)$, is the number of nonzeros in column $i$ (excluding the diagonal entry) and the outdegree of vertex $i$, denoted $d^+(i)$, is the number of nonzeros in row $i$ (excluding the diagonal entry) Thus, for the $n$-by-$n$ matrix $A(D_\alpha) \equiv [a_{ij}]$, the total degree of vertex $i$ is $d(i) = \sum_{j=1}^n |a_{ij}| + \sum_{j=1}^n |a_{ji}| - 2|a_{ii}|$ The correspondence between matrices and digraphs clearly dictates that $\nu(A(D_\alpha)) = \frac{1}{2}\sum_{i=1}^n d(i) + \sum_{i=1}^n |a_{ii}|$

# Chapter 2

# Background and History

## 2.1 Motivation

Samuelson[Sa], in his 1947 book *Foundations of Economic Analysis*, formulated economic problems using qualitative reasonings and a qualitative calculus. This work includes the analysis of a system of linear equations where only the signs (but not the magnitudes) of the entries of the coefficient matrix $A_n$ are known. The solution of the quantitative problem when $\det A_n$ is nonzero, can be given, for example, by Cramer's rule. Using this same method on the qualitative problem requires $\det A_n$ to be nonzero for all magnitudes of the positive and negative entries of $A_n$. In the qualitative problem, $\det A_n$ is nonzero if and only if all nonzero signed elementary products have the same sign with at least one being nonzero. Thus, some method is needed to determine when this is true. In the 1960's, Lancaster and Gorman extended Samuelson's work [La1, La2, La3, Go].

More recently, SNS matrices have also arisen in ecology and biology (see for example [J, Se]), and have appeared in the following contexts: *sign solvability* (see for example, [KL, KLM, Man, May]), *sign stability* (see for example, [JKvD1, JKvD2, MQ]), *sign controllability* (see for example, [JMO]), *qualitative Schur complements* (see for example, [JM]), *Pfaffian orientations and Pfaffian graphs* (see for example, [Li, LP, VY, LR]) and *inverse determined matrices* (see for example, [Th2, Se, LM2]).

Sign nonsingular matrices have been studied from many different perspectives. The first digraph characterization of SNS matrices was given by Bassett et al. (see Section 2.2 and also [BMQ]). The even digraph characterization of sign nonsingularity (Section 2.3 and also [ST, Th1]) involves a specific forbidden subdigraph property of the associated digraph of the matrix. That is, a matrix is not SNS if and only if its associated digraph contains a forbidden subdigraph. The convertible matrix characterization (Section 2.4) uses a forbidden submatrix approach. That is, in order that a matrix be SNS, it cannot contain a specific submatrix under certain matrix operations (see also [Li, BR, BS, Sh] for details). Drew et al. obtained a characterization of SNS matrices using the associated signed bipartite graph[DGJOvD].

After discussing several of these characterizations, the final two sections (2.5 and 2.6) of this chapter provide important background necessary for the basis of this work. Section 2.5 shows that there is essentially only one way to affix signs to a convertible matrix so that it becomes a SNS matrix. Section 2.6 shows that the

Hessenberg matrix, $H_n$, is (up to equivalence) the unique $n$-by-$n$ SNS matrix with exactly $\binom{n-1}{2}$ zero entries.

## 2.2 Digraph Characterization of SNS Matrices

The digraph characterization of SNS matrices was given by Bassett, Maybee and Quirk[BMQ] in 1968. Their characterization requires the examination of every cycle in the associated digraph to determine if it has the correct cycle sign.

**Theorem 2.2.1** ([BMQ, Lemma 3]) A matrix, $A_n$, with every diagonal entry equal to $-1$ is SNS if and only if every cycle in $D(A_n)$ has negative cycle sign.

**Example 2.2.2** Let $A_4$ be the matrix

$$\begin{bmatrix} -1 & -1 & 0 & -1 \\ +1 & -1 & 0 & -1 \\ 0 & +1 & -1 & -1 \\ 0 & +1 & +1 & -1 \end{bmatrix}$$

Then, $D(A_4)$ is pictured in Figure 2.1 where negatively signed arcs are dashed and positively signed arcs are solid. All cycles in $D(A_4)$ have negative cycle sign and therefore, by Theorem 2.2.1, $A_4$ is SNS.

Using this characterization, Lady and Maybee [LaMa] proved that each essential zero in the matrix corresponds to a path condition in the associated digraph

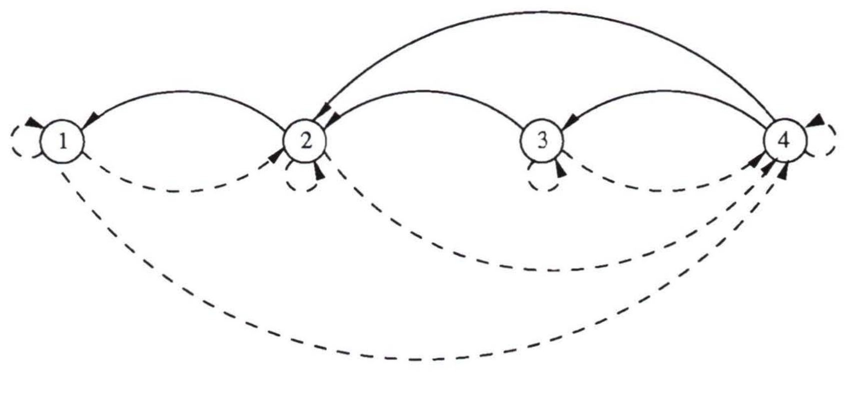


Figure 2.1: The digraph of the matrix in Example 2.2.2.

**Theorem 2.2.3** ([LaMa, Corollary of Lemma 3]) Let $A_n$ be a normalized SNS matrix. Then, $a_{ij} = 0$ is an essential zero if and only if there exist two paths in $D(A_n)$ from vertex $j$ to vertex $i$ having opposite path signs.

This result will be used extensively in later chapters.

## 2.3 Even Digraph Characterization of SNS Matrices

A digraph is called an *even digraph* if for every reassignment of signs to its arcs there exists some cycle with a positive cycle sign. Note that this definition is slightly different than the one in Thomassen[Th1], but is equivalent. Interpreting Theorem 2.2.1 in terms of even digraphs gives the following.

**Corollary 2.3.1** Let $D_\alpha$ be an ordered digraph with a negatively signed self loop at each vertex. Then, $D_\alpha$ is an even digraph if and only if $A(D_\alpha)$ is not SNS.

A *double cycle* on $n \geq 3$ vertices, denoted $C_n^*$, is a digraph with vertices $\{v_1, \ldots, v_n\}$ and arcs $(v_n, v_1)$, $(v_1, v_n)$, $(v_i, v_{i+1})$ and $(v_{i+1}, v_i)$ for $i = 1, \ldots, n-1$. If each cycle of length 2 is arbitrarily signed so that each has negative cycle sign, then one of the two cycles of length $n$ must have positive cycle sign whenever $n$ is odd. Therefore, $C_n^*$ is an even digraph whenever $n$ is odd.

An *edge subdivision* of a digraph $D$ is a digraph $D'$ where some arc $(u, v) \in E(D)$ is replaced by a path $P$ from $u$ to $v$ (with arcs signed arbitrarily) so that $V(D) \cap V(P) = \{u, v\}$. A *subdivision of D* is a digraph with some (possibly none) edge subdivisions of $D$. It is easy to see that when $n$ is odd, any subdivision of $C_n^*$ is an even digraph.

The *vertex splitting* of a vertex $x$ in a digraph $D$ is a digraph $D'$ where the vertex $x$ is deleted (and all arcs incident with $x$) and replaced by two vertices $x_1$ and $x_2$, the arcs $(z, x_1)$ are added for all $(z, x) \in E(D)$ with $sgn\,(z, x_1) = sgn\,(z, x)$, the arcs $(x_2, z)$ are added for all $(x, z) \in E(D)$ with $sgn\,(x_2, z) = sgn\,(x, z)$, and the arc $(x_1, x_2)$ is added with arbitrary sign. The digraph $D'$ is called a *splitting* of $D$. It is easy to see that when $n$ is odd, any splitting of $C_n^*$ is an even digraph.

Finally, a *weak odd double cycle* is a digraph obtained from a double cycle with an odd number of vertices by some (possibly none) splittings and some (possibly none) edge subdivisions. With this terminology, Seymour and Thomassen[ST] arrived at their characterization of even digraphs.

**Theorem 2.3.2** ([ST, Theorem 4.1]) A digraph is an even digraph if and only if it contains a weak odd double cycle.

A corollary that will be used extensively in Chapter 5 is the following.

**Corollary 2.3.3** Let $D_\alpha$ be an ordered digraph with a negatively signed self loop at each vertex. If $D_\alpha$ has some subdigraph that is a subdivision of $C_3^*$, then $A(D_\alpha)$ is not SNS.

*Proof.* Let $D_\alpha$ be as given in the statement of the corollary and suppose $D_\alpha$ has a subdigraph that is a subdivision of $C_3^*$. By Theorem 2.3.2, $D_\alpha$ is an even digraph and thus, by Corollary 2.3.1, $A(D_\alpha)$ is not SNS. □

## 2.4 Convertible Matrix Characterization of SNS Matrices

Let $A_n$ be a matrix with entries from $\{0,1\}$ with all $a_{ii} = 1$ such that for two distinct rows $i_1$ and $i_2$ and two distinct columns $j_1$ and $j_2$, $a_{i_1,j_1} = 1$, $a_{i_1,j_2} = 1$, $a_{i_2,j_1} = 1$, $a_{i_2,j_2} = 0$, $a_{i_1,k} = 0$ if $k \notin \{j_1, j_2\}$ and $a_{k,j_1} = 0$ if $k \notin \{i_1, i_2\}$. If $\tilde{A}_n$ is obtained from $A_n$ by replacing $a_{i_2,j_2}$ by 1, then the matrix $\tilde{A}[i_1^c, j_1^c]$ is called a *reduction*[Li] or a *contraction*[BS] of $A_n$. Using this definition, Little[Li] arrived at a characterization of convertible matrices (see also [BR, Theorem 7.5.7]).

**Theorem 2.4.1** (from [Li, Corollary 1]) A matrix $A_n$ is not convertible if and only if it is equivalent to a matrix $A'_n$ that is contained in a matrix of the form

$$\begin{bmatrix} I_m & X_{m,n-m} \\ Y_{n-m,m} & B_{n-m} \end{bmatrix}$$

where $0 \le m \le n-3$, $X_{m,n-m}$ and $Y_{n-m,m}$ are arbitrary matrices and $B_{n-m}$ can be transformed into $J_3$ by a series of reductions, where $J_3$ is the matrix of all 1's.

Brualdi and Shader [BS] note that there is a correspondence between Theorem 2.4.1 and Theorem 2.3.2 (since they are characterizing the same matrices) and they provide a sketch of the proof. Their paper also contains a discussion of the relationship between reduction, edge subdivision and vertex splitting.

## 2.5 Uniqueness of Sign Nonsingular Matrices

Brualdi and Shader[BS], in their work on convertible matrices, showed that there is essentially a unique way to sign a given convertible matrix to obtain a SNS matrix (see also Brualdi and Ryser[BR, Theorem 7.5.4]).

**Theorem 2.5.1** (from [BS, Theorem 2.1]) Let $B_n$ and $A_n$ be fully indecomposable SNS matrices such that $|B_n| = |A_n|$. Then, there exist signature matrices $T_n$ and $U_n$ such that $B_n = T_n A_n U_n$.

This concept can be generalized slightly as the following corollary shows.

**Corollary 2.5.2** Let $B_n$ and $A_n$ be fully indecomposable SNS matrices such that $|B_n|$ is contained in $|A_n|$. Then, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$. If, in addition, $B_n$ is maximal, then $B_n = T_n A_n U_n$.

*Proof.* Let $A'_n$ be contained in $A_n$ such that $|A'_n| = |B_n|$. Then, by Theorem 2.5.1, there exist signature matrices $T_n$ and $U_n$ such that $B_n = T_n A'_n U_n$ and so $B_n$ is contained in $T_n A_n U_n$. Clearly, if $B_n$ is maximal, then there is no proper containment and the result follows. □

## 2.6 The Hessenberg Class

**Definition 2.6.1** The Hessenberg matrix, $H_n$, is defined by

$$h_{ij} = \begin{cases} 0 & \text{for } i > j+1 \\ +1 & \text{for } i = j+1 \\ -1 & \text{for } i \leq j \end{cases}$$

**Example 2.6.2**

$$H_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ +1 & -1 & -1 & -1 \\ 0 & +1 & -1 & -1 \\ 0 & 0 & +1 & -1 \end{bmatrix}$$

**Observation 2.6.3** The matrix $H_n$ is fully indecomposable, maximal and has $\binom{n-1}{2}$ zero entries.

Gibson[Gi] showed that this is the minimum number of zero entries in any $n$-by-$n$ SNS matrix. Furthermore, he proved the following, showing that a SNS matrix with $\binom{n-1}{2}$ zero entries is essentially unique.

**Theorem 2.6.4** ([Gi, Corollary 2]) For $n \geq 1$, $B_n$ is a fully indecomposable maximal SNS matrix with $\binom{n-1}{2}$ zero entries if and only if $B_n$ is equivalent to $H_n$.

Related to the result of a minimum number of zero entries in a SNS matrix, Thomassen [Th1] defined a digraph called an *extended caterpillar*. First, let $n \geq k \geq 3$ and $1 < l < k$ all be fixed integers. Then, the extended caterpillar, $EC$, is given by

- $V(EC) = \{ v_1, \ldots, v_n \}$
- $E(EC)$ is set of arcs
  - $(v_i, v_i)$ for $i = 1, \ldots, n$,
  - $(v_i, v_{i+1})$ and $(v_{i+1}, v_i)$ for $i = 1, \ldots, k-1$,
  - $(v_l, v_i)$ and $(v_i, v_l)$ for $i = k+1, \ldots, n$,
  - $(v_i, v_j)$ for $i = 3, \ldots, k$ and $j = 1, \ldots, i-2$,
  - $(v_i, v_j)$ for $i = k+1, \ldots, n$ and $j = 1, \ldots, l-1$,
  - $(v_i, v_j)$ for $i = l+1, \ldots, k$ and $j = k+1, \ldots, n$, and
  - $(v_i, v_j)$ or $(v_j, v_i)$, but not both, for every distinct $i$, $j$ where $k+1 \leq i, j \leq n$

  with each self loop signed negatively, each cycle of length 2 arbitrarily signed so that the cycle sign is negative, and all other arcs uniquely signed so that their cycle signs are negative.

An extended caterpillar on $n$ vertices, as defined above, is a special case of the caterpillar discussed in [LuMa]. In that work, they show that the associated SNS matrix is maximal. Furthermore, it is clear that an extended caterpillar on $n$ vertices has $\frac{n^2+3n-2}{2} = n^2 - \binom{n-1}{2}$ arcs. Thus, the associated matrix of an extended caterpillar has $\binom{n-1}{2}$ zero entries and by Theorem 2.6.4, it is equivalent to $H_n$.

# Chapter 3

# Bordering a Matrix

One method for obtaining a SNS matrix from another SNS matrix is to border it in such a way that the property of being SNS is preserved. The first result presented considers the bordering of some SNS matrix containing essential zeros, under certain circumstances these essential zeros will cause certain entries in the bordering row or column also to be essential zeros. The second result shows that if a maximal partly decomposable SNS matrix is bordered to give a normalized SNS matrix, then the bordered matrix is partly decomposable. The third and most important bordering result involves the row/column stretch of a matrix. A certain row (respectively, column) of a given matrix is replicated and the bordering column (respectively, row) is given appropriate entries so as to preserve the property of being SNS. This operation is used in Chapter 4 to determine some properties of the two new classes of SNS matrices introduced there, and again in Chapter 5 for the uniqueness results.

## 3.1 Tools for Bordering

Essential zeros and zero submatrices in normalized SNS matrices are now considered.

**Lemma 3.1.1** Let $n \geq 4$ and $B_n$ be a normalized SNS matrix. If, for some fixed $k \geq 2$, $b_{1c} \neq 0$ and $b_{ic}$ is an essential zero in $B[k, \ldots, n]$, for some $i$ and $c$ satisfying $k \leq i, c \leq n$, then $b_{i1}$ is an essential zero in $B_n$. Similarly, if, for some fixed $k \geq 2$, $b_{r1} \neq 0$ and $b_{rj}$ is an essential zero in $B[k, \ldots, n]$, for some $r$ and $j$ satisfying $k \leq r, j \leq n$, then $b_{1j}$ is an essential zero in $B_n$.

*Proof.* Assume $b_{ic}$ is an essential zero in $B[k, \ldots, n]$ for some $i$ and $c$ where $k \leq i, c \leq n$. Then, by Theorem 2.2.3, there exists two paths, $(c, u_1, u_2, \ldots, i)$ and $(c, v_1, v_2, \ldots, i)$ in $D(B[k, \ldots, n])$, with opposite path signs. Since $B[k, \ldots, n]$ is a principal submatrix of $B_n$, $D(B[k, \ldots, n])$ is a subdigraph of $D(B_n)$ and these paths exist in $D(B_n)$. Appending $(1, c)$ onto both paths, yields two new paths, $(1, c, u_1, u_2, \ldots, i)$ and $(1, c, v_1, v_2, \ldots, i)$, with opposite path signs in $B_n$. Thus, $b_{i1}$ is an essential zero in $B_n$. A similar argument holds using $b_{r1}$ and $b_{rj}$ to show that $b_{1j}$ is an essential zero in $B_n$. □

**Lemma 3.1.2** Let $B_n$, $n \geq 2$, be a normalized SNS matrix. If $B[2, \ldots, n]$ is maximal and partly decomposable, then $B_n$ is partly decomposable.

*Proof.* If $n = 2, 3$ or $4$ then the result is vacuously true. For $n \geq 5$, assume $B[2, \ldots, n]$ is maximal and partly decomposable. Then, there exist permutation matrices $P_n$ and $Q_n$ such that $P_n B_n Q_n^T \equiv A_n$ is of the form

$$\left[\begin{array}{c|c|c} -1 & \multicolumn{2}{c}{V_{1,n-1}} \\ \hline \multirow{2}{*}{$U_{n-1,1}$} & W_k & X_{k,n-k-1} \\ \cline{2-3} & O_{n-k-1,k} & Y_{n-k-1} \end{array}\right]$$

for $0 < k < n-1$ and matrices $U_{n-1,1}$, $V_{1,n-1}$, $W_k$, $X_{k,n-k-1}$ and $Y_{n-k-1}$. Also, since $B[2, \ldots, n]$ is maximal, every zero entry is an essential zero.

If $a_{i1} \neq 0$ for some $i$, where $k+2 \leq i \leq n$, then by Lemma 3.1.1, $a_{1j} = 0$ for $j = 2, \ldots, k+1$ and $A_n$ is partly decomposable. If $a_{i1} = 0$ for all $i$, where $k+2 \leq i \leq n$, then $A_n$ is reducible and hence, partly decomposable. In both cases, $A_n$ partly decomposable implies $B_n$ is partly decomposable. □

An immediate corollary is the following.

**Corollary 3.1.3** Let $B_n$, $n \geq 2$, be a normalized SNS matrix. If $B_n$ is fully indecomposable, then $B[2, \ldots, n]$ is either fully indecomposable or not maximal.

## 3.2 The Row/Column Stretch of a Matrix

A row «column» stretch of an $n$-by-$n$ matrix is an $(n+1)$-by-$(n+1)$ matrix obtained by bordering it with one row and column as follows.

**Definition 3.2.1** Let $A_n$ be a matrix with entries from $\{-1, 0, +1\}$ and let $r$ be any fixed integer where $1 \leq r \leq n$. Then, the *row stretch of $A_n$ on row $r$* is the matrix $B_{n+1}$ defined as follows

$$\begin{aligned} B[1, \ldots, n] &= A_n \\ b_{n+1,j} &= b_{rj} \quad \text{for } j = 1, \ldots, n \\ b_{n+1,n+1} &= -1 \\ b_{r,n+1} &= +1 \\ b_{i,n+1} &= 0 \quad \text{for } i = 1, \ldots, n \text{ and } i \neq r \end{aligned}$$

The *column stretch of $A_n$ on column $r$* is similarly defined

$$\begin{aligned} B[1, \ldots, n] &= A_n \\ b_{i,n+1} &= b_{ir} \quad \text{for } i = 1, \ldots, n \\ b_{n+1,n+1} &= -1 \\ b_{n+1,r} &= +1 \\ b_{n+1,j} &= 0 \quad \text{for } j = 1, \ldots, n \text{ and } j \neq r \end{aligned}$$

In Definition 3.2.1 the bordering row and column is inserted as row and column $n+1$ in $B_{n+1}$. This can be generalized to allow the bordering row and column to be inserted as a new row and column $k$ in $B_{n+1}$.

**Definition 3.2.2** Let $B_{n+1}$ be the row «column» stretch of $A_n$ on row «column» $r$. With $P_{n+1} = (1, 2, \ldots, k-1, n+1, k, k+1, \ldots, n)$ for some $k$, $1 \leq k \leq n+1$, $P_{n+1} B_{n+1} P_{n+1}^T$ is called the *row «column» stretch of $A_n$ on row «column» $r$ with the new row and column labelled as $k$*.

**Example 3.2.3**

$$\begin{bmatrix} -1 & -1 & -1 \\ +1 & -1 & -1 \\ 0 & +1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & -1 & 0 \\ +1 & -1 & -1 & 0 \\ 0 & +1 & -1 & +1 \\ 0 & +1 & -1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & 0 & -1 \\ +1 & -1 & 0 & -1 \\ 0 & +1 & -1 & -1 \\ 0 & +1 & +1 & -1 \end{bmatrix}$$

$A_n$ | The row stretch of $A_n$ on row 3 as in Definition 3.2.1 | The row stretch of $A_n$ on row 3 with the new row and column labelled as 3 as in Definition 3.2.2

The row stretch of $A_n$ on row $r$ with the new row and column labelled as $k$ is identical to the transpose of the column stretch of $A_n^T$ on column $r$ with the new row and column labelled as $k$. The row «column» stretch is a special case of the 1-linking of two matrices as presented by Lundy, Maybee, and Van Buskirk [LM1, LM2, LMVB]. In terms of convertible matrices, $|B_{n+1}|$ is called an *extension* of $|A_n|$ (see [BS, Sh]).

Let $A_n$ and $A'_n$ be two matrices that are equivalent under multiplication by permutation and signature matrices. The following observation shows that for any row stretch, $B_{n+1}$, of $A_n$, there is a row stretch, $B'_{n+1}$, of $A'_n$ that is equivalent to $B_{n+1}$.

**Observation 3.2.4** Let $1 \leq r, t \leq n$ and $P_n = (i_1, i_2, \ldots, i_n)$ where $i_t = r$ (so that row $t$ of $P_n A_n$ is row $r$ of $A_n$). Then, the row stretch of $A_n$ on row $r$ is equivalent to the row stretch on row $t$ of $P_n A_n$ (or indeed of $S_n P_n A_n Q_n^T T_n$, where $Q_n$ is an arbitrary permutation matrix and $S_n$ and $T_n$ are arbitrary signature matrices). Furthermore, when dealing with SNS matrices, $Q_n$, $S_n$ and $T_n$ can be chosen so that $S_n P_n A_n Q_n^T T_n$ is normalized.

A similar observation can be made for the column stretch.

**Theorem 3.2.5** Let $B_{n+1}$ be the row «column» stretch of $A_n$ on an arbitrary row «column». Then, $A_n$ is a fully indecomposable maximal SNS matrix if and only if $B_{n+1}$ is a fully indecomposable maximal SNS matrix.

*Proof.* From Observation 3.2.4, without loss of generality assume that $A_n$ is SNS and normalized and $B_{n+1}$ is the row stretch of $A_n$ on row 1 with the new row and column labelled as 1. Thus,

$$B_{n+1} \equiv \left[\begin{array}{c|c} -1 & w^T \\ \hline +1 & \\ 0 & A_n \\ \vdots & \\ 0 & \end{array}\right] \equiv \left[\begin{array}{c|c} -1 & w^T \\ \hline +1 & w^T \\ \hline 0 & \\ \vdots & X_{n-1,n} \\ 0 & \end{array}\right] \tag{3.1}$$

The Laplace determinant expansion of $B_{n+1}$ on column 1 results in

$$\det(B_{n+1}) = -2\det(A_n) \tag{3.2}$$

Then, as $A_n$ is SNS, (3.2) implies that $B_{n+1}$ is SNS.

Assume further that $A_n$ is maximal and fully indecomposable. The zeros in the last $n$ columns of $B_{n+1}$ are all essential in $B_{n+1}$ because $A_n$ is a maximal SNS matrix. As for the zeros in column 1 of $B_{n+1}$, suppose $b_{u1}$ is such a zero, where $3 \leq u \leq n+1$. Since $B[2, \ldots, n+1] = A_n$ is fully indecomposable, $D(B[2, \ldots, n+1])$ is strongly connected and there exists some path in $D(B_{n+1})$ from vertex 2 to vertex $u$, say $(2, u_1, u_2, \ldots, u)$. Since $B_{n+1}$ is the row stretch of $A_n$ on row 1, $sgn\,(1,2) = -$ and the arc $(1, u_1)$ exists in $D(B_{n+1})$ with $sgn\,(1, u_1) = sgn\,(2, u_1)$. Thus, $D(B_{n+1})$ contains two paths, $(1, u_1, u_2, \ldots, u)$ and $(1, 2, u_1, u_2, \ldots, u)$, which have opposite path signs and hence, $b_{u1}$ is an essential zero in $B_{n+1}$.

Since $D(B[2, \ldots, n+1])$ is strongly connected and the arcs $(1,2)$ and $(2,1)$ exist, $D(B_{n+1})$ is strongly connected and hence, $B_{n+1}$ is irreducible. Since $B_{n+1}$ is also normalized, it is fully indecomposable.

For the converse, without loss of generality assume that $B_{n+1}$ is the row stretch of $A_n$ on row 1 with the new row and column labelled as 1. Since $B_{n+1}$ is SNS, without loss of generality it may be assumed also to be normalized and thus, has form (3.1). Then, (3.2) implies that $A_n$ is SNS and normalized.

Assume further that $B_{n+1}$ is maximal and fully indecomposable. Then, every zero entry in $B_{n+1}$ is an essential zero. Suppose $b_{uv}$ is one such zero where $u, v \neq 1$. Therefore, there exist two paths of lengths at least 1 in $D(B_{n+1})$, say $(v, v_1, v_2, \ldots, u)$ and $(v, u_1, u_2, \ldots, u)$, having opposite path signs. If vertex 1 is not in either of these paths, clearly $b_{uv}$ is also an essential zero in $B[2, \ldots, n+1]$. If either (or both) of these paths does contain vertex 1, then the path must contain the path $(2,1,i)$, where $3 \leq i \leq n$. By using the arc $(2,i)$ instead of the path $(2,1,i)$ in the path(s), two new paths having opposite path signs are obtained that do not contain vertex 1 (as the sign of the paths $(2,1,i)$ and $(2,i)$ are the same). Thus, the zero is essential in $B[2,\ldots,n+1]$. Since $B[2,\ldots,n+1] = A_n$, it follows that $A_n$ is maximal.

Since $B_{n+1}$ is fully indecomposable, $D(B_{n+1})$ is strongly connected. Thus, there exists a path between any two vertices, $u, v \in V(D(B_{n+1}))$, where $u, v \neq 1$. If this path does not contain vertex 1, then there is nothing further to show. If this path does contain vertex 1, then the path must contain the path $(2,1,i)$. By using the arc $(2,i)$ instead of the path $(2,1,i)$ in the path, a new path is obtained that does not contain vertex 1. In both cases, $D(B[2,\ldots,n+1])$ is strongly connected and hence, $B[2,\ldots,n+1]$ is irreducible. Since $B[2,\ldots,n+1] = A_n$ is also normalized, it is fully indecomposable.

The column stretch result follows similarly. □

**Remark.** Note, however, that if $A_n$ is a maximal SNS matrix but is not fully indecomposable, then the row stretch of $A_n$ may not be a maximal SNS matrix. For example, let $A_4$ be the partly decomposable maximal SNS matrix

$$\left[\begin{array}{cc|cc} 0 & 0 & -1 & -1 \\ 0 & 0 & +1 & -1 \\ \hline -1 & -1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{array}\right]$$

and $B_5$ be the row stretch of $A_4$ on row 1. The matrix $B_5$ has 9 zero entries but all 5-by-5 partly decomposable maximal SNS matrices have 7 or 8 zero entries [DGJOvD]. Therefore, $B_5$ is not maximal.

Extensive research on the $k$-connectedness of SNS matrices has been done by Lundy and Maybee [LM1, LM2]. They showed that the row «column» stretch $B_{n+1}$ (of $A_n$ on row «column» $r$) is 1-connected, which is clear since if row and column $r$ of $B_{n+1}$ are deleted, then the matrix is partly decomposable. Recently, Thomassen [Th3] showed that all fully indecomposable maximal normalized SNS matrices are either 1-connected or 2-connected.

# Chapter 4

# New Classes of SNS Matrices

In this chapter, two new classes of $n$-by-$n$ SNS matrices are identified. Both classes are defined in terms of $H_n$ and depending on the parameters given, specific entries of $H_n$ are changed. These classes are used in the next chapter to prove the uniqueness results concerning SNS matrices having a specified number of zeros, but are also of independent interest.

## 4.1 The $V_n^{(r)}$ Class

**Definition 4.1.1** Let $n$ and $r$ be integers where $1 \leq r \leq n-3$ and $H_n$ be the SNS Hessenberg matrix as defined in Definition 2.6.1. Then, $V_n^{(r)} \equiv [v_{ij}^{(r)}]$ is defined by

$$
\begin{aligned}
v_{r+2,r}^{(r)} &= +1 \\
v_{r+3,r+1}^{(r)} &= +1 \\
v_{i,r+1}^{(r)} &= 0 \quad \text{for } i = 1, \ldots, r \\
v_{r+1,r+2}^{(r)} &= 0 \\
v_{r+2,j}^{(r)} &= 0 \quad \text{for } j = r+3, \ldots, n \\
v_{ij}^{(r)} &= h_{ij} \quad \text{otherwise.}
\end{aligned}
$$

**Remark.** The matrix $V_n^{(r)}$ is obtained from $H_n$ by replacing the zeros in positions $(r+2, r)$ and $(r+3, r+1)$ by $+1$ and, in addition, replacing certain negative entries of $H_n$ by 0 so that the resulting matrix is SNS. Here, the parameter $r$ represents the column in which to place the leftmost of the two new $+1$ entries.

**Example 4.1.2** The following two matrices are $V_5^{(1)}$ and $V_5^{(2)}$, respectively.

$$
\begin{bmatrix}
-1 & 0 & -1 & -1 & -1 \\
+1 & -1 & 0 & -1 & -1 \\
+1 & +1 & -1 & 0 & 0 \\
0 & +1 & +1 & -1 & -1 \\
0 & 0 & 0 & +1 & -1
\end{bmatrix}
\qquad
\begin{bmatrix}
-1 & -1 & 0 & -1 & -1 \\
+1 & -1 & 0 & -1 & -1 \\
0 & +1 & -1 & 0 & -1 \\
0 & +1 & +1 & -1 & 0 \\
0 & 0 & +1 & +1 & -1
\end{bmatrix}
$$

**Observation 4 1 3** Each matrix $V_n^{(r)}$, as defined above, is fully indecomposable Since all of the diagonal entries are nonzero, the existence of a Hamilton cycle in $D(V_n^{(r)})$ is sufficient to show this Such a cycle exists since every entry on the subdiagonal is nonzero and the entry in the $(1, n)$ position is nonzero

**Observation 4 1 4** Each matrix $V_n^{(r)}$ has exactly $\binom{n-1}{2} + (n-3)$ zeros

**Theorem 4 1 5** Let $V_n^{(r)}$ be a matrix as defined in Definition 4 1 1 Then, $V_n^{(r)}$ is a fully indecomposable maximal SNS matrix

*Proof* By inspection, the matrix

$$V_4^{(1)} = \begin{bmatrix} -1 & 0 & -1 & -1 \\ +1 & -1 & 0 & -1 \\ +1 & +1 & -1 & 0 \\ 0 & +1 & +1 & -1 \end{bmatrix}$$

is fully indecomposable, maximal and SNS Performing a row stretch on row 1 of $V_4^{(1)}$ with the new row and column labelled as 1 gives $V_5^{(2)}$ Repeating this operation $r-2$ times gives $V_{r+3}^{(r)}$ Performing a column stretch on the last column of $V_{r+3}^{(r)}$ with the new row and column labelled as $r+4$ gives $V_{r+4}^{(r)}$ Repeating this operation $n-r-4$ times gives $V_n^{(r)}$ Thus, $V_n^{(r)}$ is fully indecomposable, maximal and SNS by Theorem 3 2 5 □

It is interesting to note that the matrix $V_4^{(1)}$ is 2-connected whereas every other matrix in the $V_n^{(r)}$ class is 1-connected This is because every matrix in the $V_n^{(r)}$

class (except $V_4^{(1)}$) can be obtained by a series of row stretches and/or column stretches

The equivalence of different matrices within the $V_n^{(r)}$ class is now investigated The following theorem shows that $V_n^{(r)}$ is equivalent to $V_n^{(n-r-2)}$

**Theorem 4 1 6** Let $P_n$ be the permutation matrix with each anti-diagonal entry equal to 1 Then, $P_n V_n^{(r)^T} P_n^T = V_n^{(n-r-2)}$

*Proof* If $W_n = P_n V_n^{(r)^T} P_n^T$, then

$$w_{n-l+1,n-k+1} = v_{kl}^{(r)} \tag{4 1}$$

for $k, l = 1, \ldots, n$ Applying (4 1) to the entries of the matrix $V_n^{(r)}$ as specified in Definition 4 1 1 results in

$$\begin{aligned}
w_{n-r+1,n-r-1} &= +1 \\
w_{n-r,n-r-2} &= +1 \\
w_{n-r,n-i+1} &= 0 \quad \text{for } i = 1, \ldots, r \\
w_{n-r-1,n-r} &= 0 \\
w_{n-j+1,n-r-1} &= 0 \quad \text{for } j = r+3, \ldots, n \\
w_{n-j+1,n-i+1} &= h_{ij} \quad \text{otherwise,}
\end{aligned}$$

or equivalently,

$$
\begin{aligned}
w_{n-r,n-r-2} &= +1 \\
w_{n-r+1,n-r-1} &= +1 \\
w_{j',n-r-1} &= 0 \quad \text{for } j' = 1, \ldots, n-r-2 \\
w_{n-r-1,n-r} &= 0 \\
w_{n-r,i'} &= 0 \quad \text{for } i' = n-r+1, \ldots, n \\
w_{ij} &= h_{ij} \quad \text{otherwise (using (4.1))}
\end{aligned}
$$

From Definition 4.1.1, it can be seen that $W_n = V_n^{(n-r-2)}$. □

With respect to this equivalence, if $r \neq s$ and $1 \leq r, s \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, then $V_n^{(r)}$ and $V_n^{(s)}$ are not equivalent. Thus, the following can be seen.

**Observation 4.1.7** For a given $n$, there are $\left\lfloor \frac{n-2}{2} \right\rfloor$ nonequivalent matrices $V_n^{(r)}$ and each has the same number of zeros.

## 4.2 The $H_n^{(p,q,r,s)}$ Class

**Definition 4.2.1** Let $n > 0$ and $p, q, r, s \geq 0$ all be integers such that $p+q+r+s \leq n-1$. Let $H_n$ denote the SNS Hessenberg matrix as defined in Definition 2.6.1. Then, $H_n^{(p,q,r,s)} \equiv [h_{ij}^{(p,q,r,s)}]$ is defined as follows.

4.2.1.1 if $p > 0$ and $q > 0$, then for $i = 1, \ldots, q$

$$h_{p+i+1,p}^{(p,q,r,s)} = +1$$

$$h_{j,p+i}^{(p,q,r,s)} = 0 \quad \text{for } j = 1, \ldots, p,$$

4.2.1.2 if $r > 0$ and $s > 0$, then for $i = 1, \ldots, s$

$$h_{n-r+1,n-r-i}^{(p,q,r,s)} = +1$$

$$h_{n-r-i+1,n-r+j}^{(p,q,r,s)} = 0 \quad \text{for } j = 1, \ldots, r,$$

4.2.1.3 otherwise

$$h_{ij}^{(p,q,r,s)} = h_{ij}.$$

**Remark.** For $p > 0$ and $r > 0$, the matrix $H_n^{(p,q,r,s)}$ is obtained from $H_n$ by replacing $q$ zeros in column $p$ by $+1$ and $s$ zeros in row $n-r+1$ by $+1$. In addition, certain negative entries of $H_n$ are replaced by 0 so that $H_n^{(p,q,r,s)}$ is a SNS matrix (see Theorem 4.2.6). If $p$ or $q$ is 0, then no replacements as in (4.2.1.1) are made. Similarly, if $r$ or $s$ is 0, then no replacements as in (4.2.1.2) are made. Thus, for example, $H_n^{(p,0,r,0)} = H_n^{(0,q,0,s)} = H_n^{(0,0,0,0)} = H_n$ for any $p$ and $r$ satisfying $p+r \leq n-1$ and any $q$ and $s$ satisfying $q+s \leq n-1$.

**Example 4 2 2** The following two matrices are $H_6^{(0,0,3,2)}$ and $H_7^{(2,1,2,1)}$, respectively

$$\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ +1 & -1 & -1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ +1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix}$$

See also Example 2 2 2 for $H_4^{(2,1,0,0)}$

**Observation 4 2 3** Each matrix $H_n^{(p,q,r,s)}$, as defined above, is fully indecomposable Since all of the diagonal entries are nonzero, the existence of a Hamilton cycle in $D(H_n^{(p,q,r,s)})$ is sufficient to show this Such a cycle exists since every entry on the subdiagonal is nonzero and the entry in the $(1, n)$ position is nonzero

**Observation 4 2 4** The number of zeros in $H_n^{(p,q,r,s)}$ is

$$\begin{cases} \binom{n-1}{2} + q(p-1) + s(r-1) & \text{if } p > 0 \text{ and } r > 0 \\ \binom{n-1}{2} + q(p-1) & \text{if } p > 0 \text{ and } r = 0 \\ \binom{n-1}{2} + s(r-1) & \text{if } p = 0 \text{ and } r > 0 \end{cases}$$

**Observation 4.2.5** When considering the row or column stretch of the matrix $H_n^{(p,q,r,s)}$, the following observations can be made.

| | Original Matrix | Transformation | Transformed Matrix | See Note |
|---|---|---|---|---|
| 4.2.5.1 | $H_n^{(0,0,r,s)}$ | row stretch on row 1 | $H_{n+1}^{(0,0,r,s)}$ | 1 |
| 4.2.5.2 | $H_n^{(p,q,r,s)}$ | row stretch on row 1 | $H_{n+1}^{(p+1,q,r,s)}$ | 1 |
| 4.2.5.3 | $H_n^{(p,q,r,s)}$ | row stretch on row $p+1$ | $H_{n+1}^{(p,q+1,r,s)}$ | 1 |
| 4.2.5.4 | $H_n^{(p,q,0,0)}$ | column stretch on column $n$ | $H_{n+1}^{(p,q,0,0)}$ | 2 |
| 4.2.5.5 | $H_n^{(p,q,r,s)}$ | column stretch on column $n$ | $H_{n+1}^{(p,q,r+1,s)}$ | 2 |
| 4.2.5.6 | $H_n^{(p,q,r,s)}$ | column stretch on column $n-r$ | $H_{n+1}^{(p,q,r,s+1)}$ | 2 |

Note 1. The row stretch of a matrix on row $x$ requires the new row and column to be labelled as $x$.

Note 2. The column stretch of a matrix on column $x$ requires the new column and row to be labelled as $x+1$.

**Theorem 4.2.6** Let $H_n^{(p,q,r,s)}$ be as defined in Definition 4.2.1. Then, $H_n^{(p,q,r,s)}$ is a fully indecomposable maximal SNS matrix.

*Proof.* The proof is based extensively upon repeated applications of Observation 4.2.5 until the matrix $H_n^{(p,q,r,s)}$ has been constructed from the initial matrix $H_1^{(0,0,0,0)} = [-1]$. The construction is:

- begin with $H_1^{(0,0,0,0)}$

- apply Observation 4 2 5 1 $n-q-s-1$ times to obtain the matrix $H_{n-q-s}^{(0,0,0,0)}$ Note that this matrix can also be identified by $H_{n-q-s}^{(p,0,r,0)}$
- apply Observation 4 2 5 3 $q$ times to obtain the matrix $H_{n-s}^{(p,q,r,0)}$
- apply Observation 4 2 5 6 $s$ times to obtain the matrix $H_n^{(p,q,r,s)}$

Clearly, $H_1^{(0,0,0,0)}$ is a maximal SNS matrix and each subsequent matrix in the contruction is SNS and maximal by Theorem 3 2 5 Thus, $H_n^{(p,q,r,s)}$ is maximal and SNS The matrix $H_n^{(p,q,r,s)}$ is fully indecomposable as established by Observation 4 2 3 □

From the above proof, every matrix in the $H_n^{(p,q,r,s)}$ class is obtained by a series of row and/or column stretches Consequently, by [LM2], every matrix in the $H_n^{(p,q,r,s)}$ class is 1-connected

The equivalence of different matrices within the $H_n^{(p,q,r,s)}$ class is now investigated It is conjectured that two matrices $H_n^{(p,q,r,s)}$ and $H_n^{(p',q',r',s')}$ are equivalent if and only if the equivalence follows from either of the following two theorems In Theorem 4 2 7, the upper left block of the matrix and the lower right block are interchanged, whereas in Theorem 4 2 8 the equivalence in the upper left block of the matrix is obtained

The following result shows that $H_n^{(p,q,r,s)}$ is equivalent to $H_n^{(r,s,p,q)}$

**Theorem 4 2 7** Let $P_n$ be the permutation matrix with each anti-diagonal entry equal to 1 Then, $P_n H_n^{(p,q,r,s)^T} P_n^T = H_n^{(r,s,p,q)}$

*Proof.* If $W_n = P_n H_n^{(p,q,r,s)^T} P_n^T$, then

$$w_{n-l+1,n-k+1} = h_{kl}^{(p,q,r,s)} \tag{4.2}$$

for $k, l = 1, \ldots, n$. Applying (4.2) to the entries of the matrix $H_n^{(p,q,r,s)}$ as specified in Definition 4.2.1 results in

4.2.1.1 if $r > 0$ and $s > 0$, then for $i = 1, \ldots, s$

$$\begin{array}{lcll} w_{r+i+1,r} & = & +1 & \\ w_{r-j+1,r+i} & = & 0 & \text{for } j = 1, \ldots, r, \end{array}$$

4.2.1.2 if $p > 0$ and $q > 0$, then for $i = 1, \ldots, q$

$$\begin{array}{lcll} w_{n-p+1,n-p-i} & = & +1 & \\ w_{n-p-i+1,n-j+1} & = & 0 & \text{for } j = 1, \ldots, p, \end{array}$$

4.2.1.3 otherwise

$$w_{n-j+1,n-i+1} = h_{ij},$$

or equivalently,

4.2.1.1 if $r > 0$ and $s > 0$, then for $i = 1, \ldots, s$

$$\begin{array}{lcll} w_{r+i+1,r} & = & +1 & \\ w_{j',r+i} & = & 0 & \text{for } j' = 1, \ldots, r, \end{array}$$

4.2.1.2 if $p > 0$ and $q > 0$, then for $i = 1, \ldots, q$

$$w_{n-p+1,n-p-i} = +1$$

$$w_{n-p-i+1,n-p+j'} \quad = \quad 0 \qquad \text{for } j' = 1, \ldots, p,$$

4.2.1.3 otherwise

$$w_{ij} \quad = \quad h_{ij} \qquad \text{(using (4.2))}.$$

From Definition 4.2.1, it can be seen that $W_n = H_n^{(r,s,p,q)}$. □

The following result shows that $H_n^{(p,q,r,s)}$ is equivalent to $H_n^{(q+1,p-1,r,s)}$.

**Theorem 4.2.8** If $p > 0$, then there exist permutation matrices $P_n$ and $Q_n$ and a signature matrix $S_n$ such that $P_n H_n^{(p,q,r,s)} Q_n^T S_n = H_n^{(q+1,p-1,r,s)}$.

*Proof.* Let $S_n$ be the signature matrix with $s_{q+1,q+1} = -1$ and $s_{ii} = +1$, otherwise. Let the permutation matrices $P_n$ and $Q_n$ be defined as $P_n = (p+1, p+2, \ldots, p+q+1, 1, 2, \ldots, p) \oplus I_{n-(p+q+1)}$ and $Q_n = (p+1, p+2, \ldots, p+q, p, 1, 2, \ldots, p-1, p+q+1) \oplus I_{n-(p+q+1)}$. The matrix $H_n^{(p,q,r,s)}$ can be partitioned as

$$\left[\begin{array}{cc|c|c|c}
\multicolumn{2}{c|}{H_p^{(0,0,0,0)}} & O_{p,q} & M_{p,1} & M_{p+q+1,n-p-q-1} \\
\hline
O_{q+1,p-1} & R_{q+1,1} & \multicolumn{2}{c|}{H_{q+1}^{(0,0,0,0)}} & \\
\hline
\multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1}
\end{array}\right]$$

or alternatively,

$$\left[\begin{array}{c|c|c|c|c}
W_{p,p-1} & M_{p,1} & O_{p,q} & M_{p,1} & M_{p+q+1,n-p-q-1} \\
\hline
O_{q+1,p-1} & R_{q+1,1} & X_{q+1,q} & M_{q+1,1} & \\
\hline
\multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1}
\end{array}\right]$$

where the matrices $W, X, Y$ and $Z$ have entries from $\{-1, 0, +1\}$, which are known but unspecified. The matrices $M$, $O$ and $R$ are matrices with every entry $-1, 0$ and $+1$, respectively.

Thus, $P_n H_n^{(p,q,r,s)}$ has the form

$$\left[\begin{array}{c|c|c|c|c} O_{q+1,p-1} & R_{q+1,1} & X_{q+1,q} & M_{q+1,1} & M_{p+q+1,n-p-q-1} \\ \cline{1-4} W_{p,p-1} & M_{p,1} & O_{p,q} & M_{p,1} & \\ \hline \multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1} \end{array}\right],$$

$P_n H_n^{(p,q,r,s)} Q_n^T$ has the form

$$\left[\begin{array}{c|c|c|c|c} X_{q+1,q} & R_{q+1,1} & O_{q+1,p-1} & M_{q+1,1} & M_{p+q+1,n-p-q-1} \\ \cline{1-4} O_{p,q} & M_{p,1} & W_{p,p-1} & M_{p,1} & \\ \hline \multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1} \end{array}\right]$$

and finally $P_n H_n^{(p,q,r,s)} Q_n^T S_n$ has the form

$$\left[\begin{array}{c|c|c|c|c} X_{q+1,q} & M_{q+1,1} & O_{q+1,p-1} & M_{q+1,1} & M_{p+q+1,n-p-q-1} \\ \cline{1-4} O_{p,q} & R_{p,1} & W_{p,p-1} & M_{p,1} & \\ \hline \multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1} \end{array}\right]$$

or equivalently,

$$\left[\begin{array}{cc|c|c|c}
\multicolumn{2}{c|}{H_{q+1}^{(0,0,0,0)}} & O_{q+1,p-1} & M_{q+1,1} & M_{p+q+1,n-p-q-1} \\
\cline{1-4}
O_{p,q} & \multicolumn{1}{|c|}{R_{p,1}} & \multicolumn{2}{c|}{H_{p}^{(0,0,0,0)}} & \\
\hline
\multicolumn{3}{c|}{O_{n-p-q-1,p+q}} & Y_{n-p-q-1,1} & Z_{n-p-q-1,n-p-q-1}
\end{array}\right]$$

This final matrix is clearly $H_n^{(q+1,p-1,r,s)}$ □

Taking $p = 1$ in the above gives the equivalence of $H_n^{(1,q,r,s)}$ and $H_n^{(0,0,r,s)}$

**Corollary 4.2.9** There exist permutation matrices $P_n$ and $Q_n$ and a signature matrix $S_n$ such that $P_n H_n^{(1,q,r,s)} Q_n^T S_n = H_n^{(0,0,r,s)}$ when $n \geq 2$ and $q+r+s \leq n-2$.

*Proof* By Theorem 4.2.8, there exist matrices $P_n, Q_n$ and $S_n$ so that $H_n^{(1,q,r,s)}$ is equivalent to $H_n^{(q+1,0,r,s)}$ By the remark following Definition 4.2.1, $H_n^{(q+1,0,r,s)}$ defines the same matrix as $H_n^{(0,0,r,s)}$ □

Note that Observation 4.2.4, when applied to the matrix $H_n^{(1,q,0,0)}$, gives the value $\binom{n-1}{2}$ for the number of zeros. Since Theorem 2.6.4 implies that any matrix with $\binom{n-1}{2}$ zeros must be equivalent to $H_n$, this confirms Corollary 4.2.9 for $r = s = 0$.

# Chapter 5

# Uniqueness

The results of this chapter are motivated by the data given in Table 6.1, which was determined from the results of a computer search [LMVB]. The evidence suggests that for each $n$ beyond some threshold and for small positive integer values of $k$, there are just a few nonequivalent SNS matrices with exactly $\binom{n-1}{2}+k$ zero entries. Specifically, the evidence from the table suggests that for $n \geq 5$ there is only one matrix with exactly $\binom{n-1}{2}+1$ zero entries. In Section 5.1 it is proved that this is indeed correct. Also, the evidence from the table suggests that for $n \geq 5$ there are only two matrices with exactly $\binom{n-1}{2}+2$ zero entries. For $n \geq 6$, this is proved in Section 5.2.

Regarding the proofs of Theorem 5.1.2 and Theorem 5.2.3, one direction is trivial and the other follows the following general framework. This framework uses a combination of properties of submatrices of SNS matrices, uniqueness of other SNS matrices and simple counting, and can be summarized as follows.

- Begin with a normalized fully indecomposable, maximal, SNS matrix.
- Symmetrically permute the matrix so that the total degree of vertex 1 is minimal.
- By properties of submatrices of SNS matrices, the total degree is bounded and can assume only a few values.
- For each of these values, determine the number of nonzero entries in the rest of the matrix (excluding row and column 1). This number either is used to show that the matrix is equivalent to a known matrix (which is unique by some previous argument) or leads to a contradiction and this case can be discarded.

This framework is used in proving both Theorem 5.1.2 and Theorem 5.2.3.

This chapter makes extensive use of the row/column stretch operation. In particular, given a matrix $B_n$ with specified properties (e.g., $B[2, \ldots, n] = H_{n-1}$), it is required to show that $B_n$ is contained in a certain row «column» stretch of $B[2, \ldots, n]$. The following convention with regard to indexing is used throughout this chapter: entries of the submatrix $B[2, \ldots, n]$ inherit the indexing of $B_n$. That is, row $i$ «column $j$» of $B[2, \ldots, n]$ refers to the entries of row $i$ «column $j$» of $B_n$.

## 5.1 Uniqueness of $H_n^{(2,1,0,0)}$

The first theorem in this section (Theorem 5.1.1) identifies all the $n$-by-$n$ normalized fully indecomposable SNS matrices that have exactly $H_{n-1}$ in rows and columns

2 through $n$ The second theorem (Theorem 5 1 2) gives the desired uniqueness result (i e , for $n \geq 5$, any $n$-by-$n$ fully indecomposable maximal SNS matrix with exactly $\binom{n-1}{2} + 1$ zero entries must be equivalent to $H_n^{(2,1,0,0)}$)

**Theorem 5 1 1** Let $B_n$, $n \geq 2$, be a normalized fully indecomposable SNS matrix with $B[2, \ldots, n] = H_{n-1}$ Let $j \geq 2$ be the smallest integer such that $b_{1j} \neq 0$ and $i \geq 2$ be the largest integer such that $b_{i1} \neq 0$ Then, $i \leq j+1$ and there exist signature matrices $T_n$ and $U_n$ such that with $B'_n \equiv T_n B_n U_n$,

1 if $i = j+1$, then $B'_n$ is contained in some matrix $A_n$ that is either

   (a) the row stretch of $H_{n-1}$ on row $i$, or

   (b) the column stretch of $H_{n-1}$ on column $j$

   Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$

2 if $i \leq j$, then $B'_n$ is contained in some matrix $A_n$ that is equivalent to $H_n$ Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$ (in which case $i = j$)

**Note** In every part of this theorem, the new row and column that results from either a row stretch or column stretch of a matrix is labelled as 1

*Proof* If $i > j+1$, then in $D(B_n)$ the paths

$$(i, 1, j), (j, i), (j, j+1), (j+1, j), (j+1, i), (i, i-1, \ldots, j+1)$$

form a subdivision of $C_3^*$ and by Corollary 2 3 3, $B_n$ is not SNS Therefore, $i \leq j+1$ and the remainder of the proof consists of the two parts $i = j+1$ and $i \leq j$

Part 1. Assume $i = j + 1$.

As $b_{1j} \neq 0$ and $b_{i1} \neq 0$ imply $d^-(1) \geq 1$ and $d^+(1) \geq 1$, assume further that $d^-(1) \geq 2$ and $d^+(1) \geq 2$ by considering the existence of the two arcs $(1, v)$ and $(u, 1)$ for $n \geq v > j$ and $2 \leq u < i$. This assumption, however, induces the following subdivision of $C_3^*$ in $D(B_n)$

$$(1, v, v-1, \ldots, i), (i, 1), (1, j), (j, j-1, \ldots, u, 1), (i, j), (j, i)$$

and $B_n$ is not SNS by Corollary 2.3.3. Therefore, vertex 1 cannot have both indegree and outdegree at least two, and thus either $d^-(1) = 1$ or $d^+(1) = 1$.

If $d^-(1) = 1$, then let $A_n$ be the row stretch of $H_{n-1}$ on row $i$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B_n' \equiv T_n B_n U_n$ is contained in $A_n$. If $d^+(1) = 1$, the above argument can be repeated with $A_n$ equal to the column stretch of $H_{n-1}$ on column $j$, thereby completing the proof of part 1. In addition, $B_n'$ is maximal if and only if $B_n' = A_n$, which is maximal by Theorem 3.2.5.

Part 2. Assume $i \leq j$.

Let $P_n = (j, 1, 2, \ldots, j-1, j+1, j+2, \ldots, n)$, $Q_n = (j-1, 1, 2, \ldots, j-2, j, j+1, \ldots, n)$ and $S_n = -I_1 \oplus I_{n-1}$. Then, with $A_n \equiv P_n H_n Q_n^T S_n$, it is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then,

$B_n' \equiv T_n B_n U_n$ is contained in $A_n$. Furthermore, $B_n'$ is maximal if and only if $B_n' = A_n$, which is maximal since it is equivalent to $H_n$. Equality is possible only when $i = j$ as $b_{k1}' = 0$ for $k = i+1, \ldots, n$ and $a_{k1} = +1$ for $k = 2, \ldots, j$.

□

For $n \leq 3$, there are no fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + 1$ zero entries. However, for $n = 4$, there are two such matrices, namely $H_4^{(2,1,0,0)}$ and $V_4^{(1)}$ (see Table 6.1). The case $n \geq 5$ is now considered.

**Theorem 5.1.2** For $n \geq 5$, $B_n$ is a fully indecomposable maximal SNS matrix with $\binom{n-1}{2} + 1$ zero entries if and only if $B_n$ is equivalent to $H_n^{(2,1,0,0)}$.

*Proof.* Theorem 4.2.6 and Observation 4.2.4 show that $H_n^{(2,1,0,0)}$ is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2} + 1$ zero entries.

For the converse, assume $B_n$ is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2} + 1$ zero entries. Also, without loss of generality, assume $B_n$ is normalized and $d(1) = \min_{1 \leq i \leq n} d(i)$.

Then, $d(1) \geq n-1$, otherwise $B[2, \ldots, n]$ contains at least $(n-1)^2 - \binom{n-2}{2} + 1$ nonzero entries and is therefore not SNS. Also, $d(1) \leq n$ by Thomassen [Th1, Proposition 2.3], so $d(1) = n$ or $n-1$.

By assuming $d(1) = n$, Thomassen [Th1, Theorem 2.7] implies (since $n \geq 5$) that $D(B_n)$ is a subdigraph of an extended caterpillar on $n$ vertices, $EC$. But $EC$ has the same number of arcs as $D(H_n)$ and by Theorem 2.6.4, the associated matrix of $EC$ is equivalent to $H_n$. Without loss of generality, $EC = D(H_n)$ and $D(B_n)$ is a subdigraph of $D(H_n)$. Since $\nu(B_n) = \nu(H_n) - 1$, $B_n$ cannot be maximal.

Therefore, the assumption $d(1) = n$ is false and $d(1) = n - 1$.

With $d(1) = n - 1$ and $b_{11} = -1$, $B[2, \ldots, n]$ must be SNS and have exactly $(n-1)^2 - \binom{n-2}{2}$ nonzero entries. But this submatrix must be equivalent to $H_{n-1}$, as it is the unique matrix with this many zero entries, and thus, without loss of generality, $B[2, \ldots, n] = H_{n-1}$. Thus far, the form of $B_n$ has been deduced as

$$\left[\begin{array}{c|c} -1 & w^T \\ \hline u & H_{n-1} \end{array}\right]$$

and the remainder of the proof will determine the correct positions and signing for the $n - 1$ nonzero entries in $u$ and $w$.

Let $j \geq 2$ be the smallest integer such that $b_{1j} \neq 0$ and $i \geq 2$ be the largest integer such that $b_{i1} \neq 0$. Clearly such indices exist, otherwise $B_n$ is partly decomposable. Now, each part of Theorem 5.1.1 can be applied with the proviso that $B_n$ is maximal. There exist signature matrices $T_n$ and $U_n$ such that $B'_n \equiv T_n B_n U_n$ satisfies the specified conditions in the following two cases.

If $i \leq j$, then $B'_n$ is equivalent to $H_n$. However, this is impossible as $\nu(B'_n) = \nu(H_n) - 1$, therefore, $i = j + 1$ and either of two cases follows.

Case 1. The matrix $B'_n$ is the row stretch of $H_{n-1}$ on row $i$ with the new row and column labelled as 1. In this case, the only choice of $i$ that results in $d(1) = n - 1$ is $i = 4$. With this, $P_n B'_n P_n^T = H_n^{(2,1,0,0)}$ by Definition 4.2.1, where $P_n = (2, 3, 1, 4, 5, \ldots, n)$.

Case 2 The matrix $B'_n$ is the column stretch of $H_{n-1}$ on column $j$ with the new row and column labelled as 1 In this case, the only choice of $j$ that results in $d(1) = n-1$ is $j = n-2$ With this, $P_n B'_n P_n^T = H_n^{(0,0,2,1)}$ by Definition 4 2 1, where $P_n = (2, 3, \quad , n-2, 1, n-1, n)$ By Theorem 4 2 7, $H_n^{(0,0,2,1)}$ is equivalent to $H_n^{(2,1,0,0)}$

Therefore, any fully indecomposable maximal SNS matrix with $\binom{n-1}{2} + 1$ zero entries is equivalent to $H_n^{(2,1,0,0)}$ □

## 5.2 Uniqueness of $H_n^{(3,1,0,0)}$ and $H_n^{(2,1,2,1)}$

The first theorem in this section (Theorem 5 2 1) identifies all the $n$-by-$n$ normalized fully indecomposable SNS matrices that contain a matrix that is permutation similar to $H_{n-1}^{(2,1,0,0)}$ in rows and columns 2 through $n$ The second theorem (Theorem 5 2 2) identifies all those $n$-by-$n$ matrices that have rows and columns 2 through $n$ identical to that of $H_{n-1}$ except for one nonzero that has been changed to zero The third theorem (Theorem 5 2 3) gives the desired uniqueness result (i e , for $n \geq 7$, any $n$-by-$n$ fully indecomposable maximal SNS matrix with exactly $\binom{n-1}{2} + 2$ zero entries must be equivalent to either $H_n^{(3,1,0,0)}$ or $H_n^{(2,1,2,1)}$)

**Theorem 5 2 1** Let $n \geq 5$, $R_{n-1} = (3, 1, 2, 4, 5, \quad , n-1)$, and $B_n$ be a normalized fully indecomposable SNS matrix with $B[2, \quad , n] = W_{n-1}$, where $W_{n-1} = R_{n-1} H_{n-1}^{(2,1,0,0)} R_{n-1}^T$ Let $j_s, j_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{1,j_s} \neq 0$ and $b_{1,j_l} \neq 0$ and let $i_s, i_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{i_s,1} \neq 0$ and $b_{i_l,1} \neq 0$. Then, there exist signature matrices $T_n$ and $U_n$ such that with $B'_n \equiv T_n B_n U_n$,

1. if $i_l = 2$, then $B'_n$ is contained in some matrix $A_n$ that is the row stretch of $W_{n-1}$ on row $i_l$. Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$.

2. if $i_l \geq 3$ and $j_s = 2$, then $i_l = 5$ and either

   (a) $i_l = i_s = 5$, in which case $B'_n$ is contained in some matrix $A_n$ that is the row stretch of $W_{n-1}$ on row $i_l$, or

   (b) $j_s = j_l = 2$, in which case $B'_n$ is contained in some matrix $A_n$ that is the column stretch of $W_{n-1}$ on column $j_s$.

   Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$.

3. if $i_l \geq 3$ and $j_s = 3$, then $i_l \leq 4$ and either

   (a) $i_l = i_s$, in which case $B'_n$ is contained in some matrix $A_n$ that is the row stretch of $W_{n-1}$ on row $i_l$, or

   (b) $j_s = j_l = 3$, in which case $B'_n$ is contained in some matrix $A_n$ that is the column stretch of $W_{n-1}$ on column $j_s$.

   Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$.

4. if $i_l \geq 3$ and $j_s = 4$, then $i_l \leq 5$ and either

   (a) $i_l = i_s$, in which case $B'_n$ is contained in some matrix $A_n$ that is the row stretch of $W_{n-1}$ on row $i_l$ and $B'_n$ is never maximal.

(b) $j_s = j_l = 4$, in which case $B'_n$ is contained in some matrix $A_n$ that is the column stretch of $W_{n-1}$ on column $j_s$. Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$.

(c) $i_l = 4$ and $i_s = 3$, in which case $B'_n$ is contained in some matrix $P_n A_n P_n^T$ where $P_n = (3, 2, 1, 4, 5, \ldots, n)$ and $A_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Furthermore, $B'_n$ is maximal if and only if $B'_n = P_n A_n P_n^T$.

(d) $i_l = 5$ and $i_s = 2$, in which case $B'_n$ is contained in some matrix $P_n A_n P_n^T$ where $P_n = (2, 1, 3, 4, \ldots, n)$ and $A_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Furthermore, $B'_n$ is maximal if and only if $B'_n = P_n A_n P_n^T$.

5. if $i_l \geq 3$ and $j_s \geq 5$, then either

(a) $i_l = j_s + 1$, in which case $B'_n$ is contained in some matrix $A_n$ that is either

i. the row stretch of $W_{n-1}$ on row $i_l$, or

ii. the column stretch of $W_{n-1}$ on column $j_s$.

Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$.

(b) $i_l \leq j_s$, in which case $B'_n$ is contained in some matrix $A_n$ that is equivalent to $H_n^{(2,1,0,0)}$. Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$ (in which case $i_l = j_s$).

**Note.** In every part of this theorem, the new row and column that results from either a row stretch or column stretch of a matrix are labelled as 1.

*Proof* The proof consists of five parts corresponding to the five statements above The following note imposes an important restriction on the indices $i_l$ and $j_s$ and applies to all but the first two parts of the proof

**Note** With $j_s \geq 3$, if $i_l > j_s + 1$ then the paths

$$(i_l, 1, j_s), (j_s, i_l), (j_s, j_s + 1), (j_s + 1, j_s), (j_s + 1, i_l), (i_l, i_l - 1, \quad , j_s + 1)$$

form a subdivision of $C_3^*$ and $B_n$ is not SNS by Corollary 2 3 3 Therefore,

$$i_l \leq j_s + 1 \tag{5 1}$$

Part 1 Assume $i_l = 2$

Then, $b_{2,1} \neq 0$ and since $b_{2,3}$ is an essential zero, Lemma 3 1 1 implies $b_{1,3}$ is an essential zero Let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l$ It is easily verified that $|B_n|$ is contained in $|A_n|$ By Corollary 2 5 2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B'_n \equiv T_n B_n U_n$ is contained in $A_n$ Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$, which is maximal by Theorem 3 2 5

Part 2 Assume $i_l \geq 3$ and $j_s = 2$

As $b_{1,2} \neq 0$ and $b_{k2}$ is an essential zero for $k = 3$, $k = 4$ or $k \geq 6$, Lemma 3 1 1 implies that $b_{k1}$ is an essential zero Therefore, $i_l = 5$ Since $b_{5,3}$ is an essential zero and $b_{5,1} \neq 0$, Lemma 3 1 1 implies $b_{1,3}$ is an essential zero Now, as $b_{1,j_s} \neq 0$ and $b_{i_l,1} \neq 0$ imply $d^-(1) \geq 1$ and $d^+(1) \geq 1$, assume further that $d^-(1) \geq 2$ and $d^+(1) \geq 2$ by considering the existence of the two other arcs $(1, j_l)$ and $(i_s, 1)$ for $n \geq j_l \geq 4$ and $i_s = 2$. This assumption, however, leads to the following contradictions. For $j_l = 4$, the paths

$$(1,4,5),(5,1),(1,2),(2,1),(2,5),(5,2)$$

and for $j_l \geq 5$, the paths

$$(1, j_l, j_l - 1, \ldots, 5),(5,1),(1,2),(2,1),(2,5),(5,2)$$

form subdivisions of $C_3^*$ implying that $B_n$ is not SNS by Corollary 2.3.3. Therefore, vertex 1 cannot have both indegree and outdegree at least two, and thus either $d^-(1) = 1$ ($i_l = i_s$) or $d^+(1) = 1$ ($j_s = j_l$).

If $d^-(1) = 1$, then let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B'_n \equiv T_n B_n U_n$ is contained in $A_n$. If $d^+(1) = 1$, then the above argument can be repeated with $A_n$ equal to the column stretch of $W_{n-1}$ on column $j_s$, thereby completing the proof of part 2. In addition, $B'_n$ is maximal if and only if $B'_n = A_n$, which is maximal by Theorem 3.2.5.

Part 3. Assume $i_l \geq 3$ and $j_s = 3$.

Since $b_{1,3} \neq 0$ and $b_{2,3}$ is an essential zero, Lemma 3.1.1 implies $b_{2,1} = 0$. Therefore, $3 \leq i_s \leq i_l$ and $3 \leq i_l \leq 4$ by (5.1). Since $b_{i_l,1} \neq 0$ and $b_{i_l,2}$ is an essential zero, Lemma 3.1.1 implies $b_{1,2}$ is an essential zero.

If $d^-(1) = 1$ ($i_l = i_s$), then let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B_n' \equiv T_n B_n U_n$ is contained in $A_n$. If $d^+(1) = 1$, then the above argument can be repeated with $A_n$ equal to the column stretch of $W_{n-1}$ on column $j_s$. In addition, $B_n'$ is maximal if and only if $B_n' = A_n$, which is maximal by Theorem 3.2.5.

Now, as $b_{1,3} \neq 0$ and $b_{i_l,1} \neq 0$ imply that $d^-(1) \geq 1$ and $d^+(1) \geq 1$, assume further that $d^-(1) \geq 2$ and $d^+(1) \geq 2$ by considering the existence of the two other arcs $(1, j_l)$ and $(i_s, 1)$ for $n \geq j_l > j_s$ and $3 \leq i_s < i_l$. Then, $i_l = 4$, $i_s = 3$ and the subdivision of $C_3^*$

$$(1,3),(3,1),(3,4),(4,3),(4,1),(1,j_l,j_l-1,\ldots,4)$$

is induced, which, by Corollary 2.3.3, implies $B_n$ is not SNS. Therefore, vertex 1 cannot have both the indegree and outdegree at least two and thus, either $d^-(1) = 1$ or $d^+(1) = 1$ as previous displayed.

Part 4. Assume $i_l \geq 3$ and $j_s = 4$.

By (5.1), $i_l = 3, 4$ or $5$. If $d^-(1) = 1$ ($i_l = i_s$), then let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_nA_nU_n$, then, $B'_n \equiv T_nB_nU_n$ is contained in $A_n$. In this case, $B'_n$ is never maximal as either $a_{1,2}$ or $a_{1,3}$ is nonzero but $b_{1,2}$ and $b_{1,3}$ are both zero.

If $d^+(1) = 1$ ($j_s = j_l$), then let $A_n$ be the column stretch of $W_{n-1}$ on column $j_s$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_nA_nU_n$, then, $B'_n \equiv T_nB_nU_n$ is contained in $A_n$. Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$, which is maximal by Theorem 3.2.5.

Now, as $b_{1,j_s} \neq 0$ and $b_{i_l,1} \neq 0$ imply $d^-(1) \geq 1$ and $d^+(1) \geq 1$, assume further that $d^-(1) \geq 2$ and $d^+(1) \geq 2$ by considering the existence of the two other arcs $(1, j_l)$ and $(i_s, 1)$ for $n \geq j_l > j_s$ and $2 \leq i_s < i_l$.

For $i_l = 3$ (and thus $i_s = 2$), this assumption induces a subdivision of $C_3^*$, namely,

$$(1,4),(4,3,1),(1,j_l,j_l-1,\ldots,5),(5,2,1),(4,5),(5,4)$$

and by Corollary 2.3.3, $B_n$ is not SNS.

For $i_l = 4$ and $i_s = 2$, the paths

$$(1,4),(4,1),(1,j_l,j_l-1,\ldots,5),(5,2,1),(4,5),(5,4)$$

form a subdivision of $C_3^*$ and $B_n$ is not SNS (Corollary 2.3.3). For $i_l = 4$ and $i_s = 3$, let $P_n = (3, 2, 1, 4, 5, \ldots, n)$ and let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l = 4$. It is easily verified that $|B_n|$ is contained in $|P_n A_n P_n^T|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n P_n A_n P_n^T U_n$; then, $B'_n \equiv T_n B_n U_n$ is contained in $P_n A_n P_n^T$. Furthermore, $B'_n$ is maximal if and only if $B'_n = P_n A_n P_n^T$ where $A_n$ is maximal by Theorem 3.2.5.

For $i_l = 5$, the following two subdivisions of $C_3^*$ imply that $b_{4,1}$ and $b_{3,1}$, respectively, are essential zeros:

$$(1,4), (4,1), (4,5), (5,4), (1, j_l, j_l - 1, \ldots, 5), (5,1)$$

and

$$(1,4), (4,3,1), (4,5), (5,4), (1, j_l, j_l - 1, \ldots, 5), (5,1).$$

Thus, $i_s = 2$. Let $P_n = (2, 1, 3, 4, \ldots, n)$ and $A_n$ be the row stretch of $W_{n-1}$ on row $i_l = 5$. It is easily verified that $|B_n|$ is contained in $|P_n A_n P_n^T|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n P_n A_n P_n^T U_n$; then, $B'_n \equiv T_n B_n U_n$ is contained in $P_n A_n P_n^T$. Furthermore, $B'_n$ is maximal if and only if $B'_n = P_n A_n P_n^T$ where $A_n$ is maximal by Theorem 3.2.5.

Part 5 Assume $i_l \geq 3$ and $j_s \geq 5$

By (5.1), $i_l \leq j_s + 1$, so consider the two cases independently $i_l = j_s + 1$ and $i_l \leq j_s$

Case (a) Assume $i_l = j_s + 1$

As $b_{1,j_s} \neq 0$ and $b_{i_l,1} \neq 0$ imply $d^-(1) \geq 1$ and $d^+(1) \geq 1$, assume further that $d^-(1) \geq 2$ and $d^+(1) \geq 2$ by considering the existence of the two other arcs $(1, j_l)$ and $(i_s, 1)$ for $n \geq j_l > j_s$ and $2 \leq i_s < i_l$. This assumption, however, induces the subdivision of $C_3^*$

$$(1, j_l, j_l - 1, \ldots, i_l), (i_l, 1), (1, j_s), (j_s, j_s - 1, \ldots, 5, 2, 1), (i_l, j_s), (j_s, i_l)$$

for $i_s = 2$ and

$$(1, j_l, j_l - 1, \ldots, i_l), (i_l, 1), (1, j_s), (j_s, j_s - 1, \ldots, i_s, 1), (i_l, j_s), (j_s, i_l)$$

for $i_s \geq 3$ and by Corollary 2.3.3, $B_n$ is not SNS. Therefore, vertex 1 cannot have both the indegree and outdegree at least two and thus, either $d^-(1) = 1$ $(i_l = i_s)$ or $d^+(1) = 1$ $(j_s = j_l)$.

If $d^-(1) = 1$, then let $A_n$ be the row stretch of $W_{n-1}$ on row $i_l$. It is easily verified that $|B_n|$ is contained in $|A_n|$. By Corollary 2.5.2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B_n' \equiv T_n B_n U_n$ is contained in $A_n$. If $d^+(1) = 1$, then the above argument can be repeated with $A_n$ equal to the column stretch of $W_{n-1}$

on column $j_s$, thereby completing the proof of case 1 In addition, $B'_n$ is maximal if and only if $B'_n = A_n$, which is maximal by Theorem 3 2 5

Case (b) Assume $i_l \leq j_s$

Let $P_n = (j_s, 1, 2, \ldots, j_s{-}1, j_s{+}1, j_s{+}2, \ldots, n)$, $Q_n = (j_s{-}1, 1, 2, \ldots, j_s{-}2, j_s, j_s + 1, \ldots, n)$, $S_n = -I_1 \oplus I_{n-1}$ and $\hat{R}_n = (1, 4, 2, 3, 5, 6, \ldots, n)$ Note that $\hat{R}_n \equiv I_1 \oplus R_{n-1}$ Then, with $A_n \equiv \hat{R}_n P_n H_n^{(2,1,0,0)} Q_n^T S_n \hat{R}_n^T$, it is easily verified that $|B_n|$ is contained in $|A_n|$ By Corollary 2 5 2, there exist signature matrices $T_n$ and $U_n$ such that $B_n$ is contained in $T_n A_n U_n$, then, $B'_n \equiv T_n B_n U_n$ is contained in $A_n$ Furthermore, $B'_n$ is maximal if and only if $B'_n = A_n$, which is maximal by Theorem 4 2 6 Equality is possible only when $i_l = j_s$ as $b'_{k1} = 0$ for $k = i_l + 1, \ldots, n$ and $a_{k1} = +1$ for $k = 2, \ldots, j_s$

□

In the following theorem, a path from $j$ to $i$ passing through vertex 1 is represented by a vertex sequence $(j, v', v, 1, u, u', i)$ where $j \leq v' \leq v \leq n$ and $2 \leq u \leq u' \leq i$ With this notation, if equality of two adjacent parameters occurs, these are coalesced and specified only once For example, $(j, j, j, 1, i, i, i)$ denotes the path $(j, 1, i)$, and $(j, v', v, 1, u, u, i)$ denotes the path $(j, v', v, 1, u, i)$

**Theorem 5 2 2** Let $B_n$, $n \geq 5$, be a fully indecomposable maximal SNS matrix, $b_{1,1} = -1$, and $B[2, \ldots, n]$ contained in $H_{n-1}$ where $\nu(B[2, \ldots, n]) = \nu(H_{n-1}) - 1$ Then, $B_n$ is equivalent to either $H_n$ or $V_n^{(r)}$ for some integer $r$ where $1 \leq r \leq n-3$

*Proof.* As $B[2,\ldots,n]$ is contained in $H_{n-1}$, let $B[2,\ldots,n]$ be $H_{n-1}$ with the $b_{ij}$ entry replaced by zero. Clearly, $b_{ij}$ is not an essential zero in $B[2,\ldots,n]$ and $i \leq j+1$. The proof consists of four parts based on the location of this zero: $i = j+1$, $i = j$, $i = j-1$ and $i \leq j-2$.

Part 1. Assume $i = j+1$.

Each zero entry, $b_{kl}$, where $k = 4,\ldots,n$ and $l = 2,\ldots,k-2$, is an essential zero in $B[2,\ldots,n]$ due to the existence of two paths having opposing path signs in $D(B[2,\ldots,n])$, namely $(l,k)$ with a negative path sign and $(l,l+1,k)$ with a positive path sign. Thus, every zero entry in $B[2,\ldots,n]$ is essential except for $b_{ij}$.

Since $B_n$ is fully indecomposable but $B[2,\ldots,n]$ is not, there exists some path in $B_n$ of the form $(v,1,u)$ for $i \leq v \leq n$ and $2 \leq u \leq j = i-1$.

If $v \geq i+1$, then as each $b_{vk}$ is an essential zero in $B[2,\ldots,n]$ for $k = 2,\ldots,v-2$, Lemma 3.1.1 implies $b_{1k} = 0$. This contradicts the necessity that $b_{1u} \neq 0$ and therefore, $v = i$. Since $b_{i1} \neq 0$ and $b_{ik}$ is an essential zero in $B[2,\ldots,n]$ for $k = 2,\ldots,j-1$, Lemma 3.1.1 implies $b_{1k}$ is an essential zero. Thus, $u = j$. For $2 \leq j \leq n-2$, since $b_{1j} \neq 0$ and $b_{kj}$ is an essential zero in $B[2,\ldots,n]$ for $k = j+2,\ldots,n$, Lemma 3.1.1 implies $b_{k1}$ is an essential zero. Let $P_n = (j,1,2,\ldots,j-1,j+1,j+2,\ldots,n)$. Then, it is easily verified that $|B_n|$ is contained in $|P_n H_n P_n^T|$. By Corollary 2.5.2 and since $B_n$ is maximal, there exist signature matrices $T_n$ and $U_n$ such that $B_n = T_n P_n H_n P_n^T U_n$. Therefore, $B_n$ is equivalent to $H_n$.

Part 2. Assume $i = j$.

When $i = n$, the matrix $B_n$ is equivalent to a matrix $\hat{B}_n$ with $i = 2$ under the transformation $\hat{B}_n = R_n B_n^T R_n^T$, where $R_n = (1, n, n-1, \ldots, 2)$. Thus, the case $i = n$ is covered by Case 2a below. By permuting $B_n$ to remove the zero from the diagonal, the classical digraph characterization theorem (Theorem 2.2.1) can be used. Without loss of generality, assume that $B_n$ is $B_n Q_n^T S_n$ where $Q_n = (1, n, 2, 3, \ldots, n-1)$ and $S_n = I_2 \oplus -I_{n-2}$. Thus, the zero that is not essential is in position $(i', i'+1)$ for some $i' = 2, 3, \ldots, n-1$.

Since $b_{i',i'+1}$ is not an essential zero in $B[2, \ldots, n]$, all paths from $i'+1$ to $i'$ in $D(B[2, \ldots, n])$ have the same path sign. From the path $(i'+1, 2, i')$ if $i' \neq 2$ or $(i'+1, 2)$ if $i' = 2$, the path sign is negative. However, since $b_{i',i'+1}$ is an essential zero in $B_n$, without loss of generality, there exists a path having one of the following forms and a positive path sign:

$$(i'+1, v, 1, u, i') \quad \text{for} \quad i'+1 \leq v \leq n \quad \text{and} \quad 3 \leq u \leq i' \tag{5.2}$$

$$(i'+1, 2, v, 1, u, i') \quad \text{for} \quad 2 \leq v \leq i'-1 \quad \text{and} \quad 3 \leq u \leq i' \tag{5.3}$$

$$(i'+1, v, 1, u, 2, i') \quad \text{for} \quad i'+1 \leq v \leq n \quad \text{and} \quad \begin{array}{l} i'+2 \leq u \leq n \\ \text{or } u = 2 \end{array} \tag{5.4}$$

For $i' = 2$, $b_{i',i'+1} = 0$ implies that $B[2, \ldots, n]$ is partly decomposable and will be considered as Case 2a below. The other values of $i'$ are considered as Case 2b below.

Case (a) Assume $i' = 2$.

Each zero entry, $b_{kl}$, where $k = 4, \ldots, n$ and $l = 3, \ldots, k-1$, is an essential zero in $B[2, \ldots, n]$ due to the existence of two paths having opposing path signs in $D(B[2, \ldots, n])$, namely $(l, k)$ with a positive path sign and $(l, 2, k)$ with a negative path sign. Thus, every zero entry in $B[2, \ldots, n]$ is essential except for $b_{2,3}$.

With $i' = 2$, paths of the form (5.2) or (5.3) do not exist and (5.4) becomes $(3, v, 1, u, 2)$ for $3 \leq v \leq n$ and $4 \leq u \leq n$ or $u = 2$. Since $b_{1,3} \neq 0$ (otherwise $B_n$ is partly decomposable) and $b_{k3}$ is an essential zero for $k = 4, 5, \ldots, n$, Lemma 3.1.1 implies that $b_{k1}$ is an essential zero. Thus, $v = 3$ and (5.4) reduces further to just $(3, 1, u, 2)$ for $4 \leq u \leq n$ or $u = 2$ with $sgn\,(3,1) = -\,sgn\,(1,3)$. Since $B_n$ is fully indecomposable, $b_{2,1} \neq 0$ with $sgn\,(2,1) = -\,sgn\,(1,3,2)$. Let $P_n = (2, 3, 1, 4, 5, \ldots, n)$, $Q_n = (2, n, 1, 3, 4, \ldots, n-1)$ and $S_n = I_3 \oplus -I_{n-3}$. Then, it is easily verified that $|B_n|$ is contained in $|P_n H_n Q_n^T S_n|$. By Corollary 2.5.2 and since $B_n$ is maximal, there exist signature matrices $T_n$ and $U_n$ such that $B_n = T_n P_n H_n Q_n^T S_n U_n$. Therefore, $B_n$ is equivalent to $H_n$.

Case (b) Assume $i' = 3, 4, \ldots, n-1$.

For each $i'$, each zero entry, $b_{kl}$, where $k = 4, \ldots, n$ and $l = 3, \ldots, k-1$, is an essential zero in $B[2, \ldots, n]$ (except when $k = i' + 1$ and $l = i'$ simultaneously) due to the existence of two paths having opposing path signs in $D(B[2, \ldots, n])$, namely $(l, k)$ with a positive path sign and

$(l,2,k)$ with a negative path sign. Thus, every zero entry in $B[2,\ldots,n]$ is essential except for $b_{i',i'+1}$ and $b_{i'+1,i'}$.

Paths of the form (5.3) and (5.4) induce the positively signed cycles $(v,1,u,i',2,v)$ and $(v,1,u,2,i'+1,v)$, respectively, resulting in $B_n$ not being SNS. Therefore, only paths of the form (5.2) need be considered. With (5.2), if $u \neq i'$ or $v \neq i'+1$, then the subdivision of $C_3^*$ is formed

$$(v,2),(2,v),(u,2),(2,u),(v,1,u),(u,v)$$

implying, by Corollary 2.3.3, that $B_n$ is not SNS. This leaves path (5.2) as $(i'+1,1,i')$ with a positive path sign.

On the other hand, since $b_{i'+1,i'}$ is an essential zero in $B_n$ but not in $B[2,\ldots,n]$, some path exists in $D(B_n)$ from vertex $i'$ to vertex $i'+1$ passing through vertex 1 and having path sign equal to $-\,sgn\,(i',2,i'+1)$, which is always positive. Without loss of generality, this path has one of the following forms:

$$(i',w,1,x,i'+1) \quad \text{for} \quad \begin{array}{l} i'+1 < w \leq n \\ \text{or } w = 1 \end{array} \quad \text{and} \quad \begin{array}{l} 3 \leq x < i' \\ \text{or } x = 1 \end{array} \tag{5.5}$$

$$(i',2,w,1,x,i'+1) \quad \text{for} \quad \begin{array}{l} 3 \leq w < i' \\ \text{or } w = 1 \end{array} \quad \text{and} \quad \begin{array}{l} 3 \leq x < i' \\ \text{or } x = 1 \end{array} \tag{5.6}$$

$$(i',w,1,x,2,i'+1) \quad \text{for} \quad \begin{array}{l} i'+1 < w \leq n \\ \text{or } w = 1 \end{array} \quad \text{and} \quad \begin{array}{l} i'+1 < x \leq n \\ \text{or } x = 1 \end{array} \tag{5.7}$$

Paths of the form (5.6) and (5.7) induce the positively signed cycles $(i'+1, 2, w, 1, x, i'+1)$ and $(i', w, 1, x, 2, i')$, respectively, and $B_n$ is not SNS. Therefore, only paths of the form (5.5) need be considered. With (5.5), the following two cycles exist and have positive cycle sign: $(i'+1, w, 1, x, i'+1)$ for $w \neq 1$ and $(i', 1, x, i')$ for $w = 1$ and $x \neq 1$. Therefore, both $w$ and $x$ are restricted to being 1 without violating the SNS property of $B_n$ and (5.5) becomes $(i', 1, i'+1)$ with a positive path sign. Since $(i', 1)$ and $(1, i')$ must be oppositely signed, $sgn\,(i', 1) = sgn\,(1, i'+1) = -\,sgn\,(i'+1, 1) = -\,sgn\,(1, i')$.

Let $P_n = (i', 2, 1, 3, 4, \ldots, i'-1, i'+1, i'+2, \ldots, n)$. When $i' = 3$, let $Q_n = (3, n, 1, 2, 4, 5, \ldots, n-1)$ and $S_n = I_3 \oplus -I_{n-3}$; otherwise, let $Q_n = (i'-1, n, 1, 2, 3, 4, \ldots, i'-2, i', i'+1, \ldots, n-1)$ and $S_n = -I_1 \oplus I_2 \oplus -I_{n-3}$. Then, it is easily verified that $|P_n V_n^{(i'-2)} Q_n^T S_n|$ is contained in $|B_n|$. By Corollary 2.5.2 and since $P_n V_n^{(i'-2)} Q_n^T S_n$ is maximal, there exist signature matrices $T_n$ and $U_n$ such that $P_n V_n^{(i'-2)} Q_n^T S_n = T_n B_n U_n$. Therefore, $B_n$ is equivalent to $V_n^{(i'-2)}$.

In the one case, $B_n$ is equivalent to $H_n$ and in the other case, $B_n$ is equivalent to $V_n^{(r)}$ for some integer $r$, where $1 \leq r \leq n-3$.

Part 3. Assume $i = j - 1$.

When $i = n-1$, the matrix $B_n$ is equivalent to a matrix $\hat{B}_n$ with $i = 2$ under the transformation $\hat{B}_n = R_n B_n^T R_n^T$, where $R_n = (1, n, n-1, \ldots, 2)$. Thus, the case $i = n-1$ is covered by Case 3a below.

Since $b_{ij}$ is not an essential zero in $B[2,\ldots,n]$, all paths from $j$ to $i$ in $D(B[2,\ldots,n])$ have the same path sign. From the path $(j,i)$ this sign is positive. However, since $b_{ij}$ is an essential zero in $B_n$, there exists a negatively signed path from vertex $j$ to $i$ passing through vertex 1. Without loss of generality, the path has the following form:

$$(j,v',v,1,u,u',i) \text{ for } 2 \le u \le u' \le i \text{ and } j \le v' \le v \le n \qquad (5.8)$$

With (5.8), nine forms of paths can be generated as summarized by the following table where each row represents the three distinct variations on $2 \le u \le u' \le i$ and each column represents the three distinct variations on $j \le v' \le v \le n$. Thus, for example, (5.8.8) denotes a path of the form $(j,v,1,u,u',i)$ for $2 \le u < u' < i$ and $j < v \le n$. As a consequence, $sgn\,(v,1,u) = +$ as is stated in the table.

| | $j = v' = v \le n$ | $j < v' = v \le n$ | $j < v' < v \le n$ |
|---|---|---|---|
| $2 \le u = u' = i$ | (5.8.1) $sgn\,(j,1,i) = -$ | (5.8.2) $sgn\,(v,1,i) = +$ | (5.8.3) $sgn\,(v,1,i) = -$ |
| $2 \le u = u' < i$ | (5.8.4) $sgn\,(j,1,u) = +$ | (5.8.5) $sgn\,(v,1,u) = -$ | (5.8.6) $sgn\,(v,1,u) = +$ |
| $2 \le u < u' < i$ | (5.8.7) $sgn\,(j,1,u) = -$ | (5.8.8) $sgn\,(v,1,u) = +$ | (5.8.9) $sgn\,(v,1,u) = -$ |

(Note: the case $2 \le u < u' = i$ is identical to the case $2 \le u = u' < i$ in the table. Similarly, $j = v' < v \le n$ is identical to $j < v' = v \le n$.)

For paths of the forms (5.8.3), (5.8.5), (5.8.7) or (5.8.9), $sgn\,(v,1,u) = -$

and the cycle $(v, 1, u, v)$ exists having positive cycle sign. For a path of the form (5.8.1), either the cycle $(j, 1, i, j+1, j)$ for $j \neq n$ or the cycle $(j, 1, i, i-1, j)$ for $i \neq 2$ exists having positive cycle sign. The situation $i = 2$ and $j = n$ simultaneously is not possible as $i = j - 1$ and $n \geq 5$. For paths of the forms (5.8.6) or (5.8.8), $sgn\,(v, 1, u) = +$ and the cycles $(v, 1, u, u', v)$ and $(v, 1, u, i, v)$, respectively, exist having positive cycle sign. The status of (5.8.2) or (5.8.4) is unknown at this point. The remainder of this part considers these two paths and consists of two cases based upon the value of $i$.

Case (a) Assume $i = 2$.

Each zero entry, $b_{kl}$, where $k = 5, \ldots, n$ and $l = 2, \ldots, k-2$, is an essential zero in $B[2, \ldots, n]$ due to the existence of two paths having opposing path signs in $D(B[2, \ldots, n])$, namely $(l, k)$ with a negative path sign and $(l, k-1, k)$ with a positive path sign. Thus, every zero entry in $B[2, \ldots, n]$ is essential except for the entries $b_{4,2}$ and $b_{2,3}$.

Since $b_{4,2}$ is an essential zero in $B_n$ but not in $B[2, \ldots, n]$, some path exists in $D(B_n)$ from vertex 2 to vertex 4 passing through vertex 1 and having path sign equal to $-\,sgn\,(2, 4)$, which is always positive. Without loss of generality, the path has one of the following three forms.

$$(2, 1, x, x-1, \ldots, 4) \text{ for } 4 \leq x \leq n, \tag{5.9}$$

$$(2, w', w, 1, x, x-1, \ldots, 4) \text{ for } 5 \leq w' \leq w \leq n \text{ and } 4 \leq x < w' \tag{5.10}$$

or

$$(2, 1, 3, 4) \tag{5.11}$$

With paths of the forms (5.9) and (5.10), the positively signed cycles $(2, 1, x, x-1, \ldots, 2)$ and $(w, 1, x, w', w)$ exist, respectively, and thus, (5.11) must be the path.

On the other hand, with $i = 2$, path (5.8.4) does not exist and path (5.8.2) becomes $(3, v, 1, 2)$ for $3 < v \leq n$. As $b_{k2}$ is an essential zero for $5 \leq k \leq n$, Lemma 3.1.1 implies $b_{k1} = 0$. Thus, $v = 4$. Therefore, the two paths that cause $b_{2,3}$ and $b_{4,2}$ to be essential zeros in $B_n$ are $(3, 4, 1, 2)$ and $(2, 1, 3, 4)$, respectively.

Let $P_n = (3, 1, 2, 4, 5, \ldots, n)$. Then, it is easily verified that $|P_n V_n^{(1)} P_n^T|$ is contained in $|B_n|$. By Corollary 2.5.2 and since $P_n V_n^{(1)} P_n^T$ is maximal (Theorem 4.1.5), there exist signature matrices $T_n$ and $U_n$ such that $P_n V_n^{(1)} P_n^T = T_n B_n U_n$. Therefore, $B_n$ is equivalent to $V_n^{(1)}$.

Case (b). Assume $i = 3, 4, \ldots, n-2$.

For each $i$, each zero entry, $b_{kl}$, where $k = 4, \ldots, n$ and $l = 2, \ldots, k-2$, is an essential zero in $B[2, \ldots, n]$ due to the existence of two paths having opposing path signs in $D(B[2, \ldots, n])$, namely $(l, k)$ with a negative path sign and one of $(l, k-1, k)$, $(l, l+1, k+1, k)$, or $(l, l-1, k-1, k)$ with a positive path sign. The path $(l, k-1, k)$ determines if $b_{kl}$ is an essential zero for all $k$ and $l$ specified except for the entries $b_{lj}$ for appropriate $l$ and $b_{j+1,i}$. The path $(l, l+1, k+1, k)$ resolves the entries $b_{lj}$ and both paths $(l, l+1, k+1, k)$ and $(l, l-1, k-1, k)$ are needed to resolve the entry $b_{j+1,i}$. Thus, every zero entry in $B[2, \ldots, n]$ is essential except for $b_{ij}$.

For paths of the form (5.8.2), since $b_{1i} \neq 0$ and $b_{k2}$ is an essential zero for $k = i+2, \ldots, n$, Lemma 3.1.1 implies $b_{k1}$ is an essential zero. But as $i = j-1$, the arc $(v, 1)$ cannot exist for the interval specified. For paths of the form (5.8.4), since $b_{j1} \neq 0$ and $b_{jk}$ is an essential zero for $k = 2, \ldots, j-2$, Lemma 3.1.1 implies $b_{1k}$ is an essential zero. But as $i = j-1$, the arc $(1, u)$ cannot exist for the interval specified.

In the one case, $B_n$ is equivalent to $V_n^{(1)}$ and in the other, a contradiction results.

Part 4. Assume $i \leq j-2$.

Since $b_{ij}$ is not an essential zero in $B[2, \ldots, n]$, all paths from $j$ to $i$ in $D(B[2, \ldots, n])$ have the same path sign. From the path $(j, j-1, \ldots, i)$ the sign is positive. However, since $b_{ij}$ is an essential zero in $B_n$, there exists a negatively signed path from vertex $j$ to $i$ passing through vertex 1. Without loss of generality, the path has one of the four following forms:

$$(j, v', v, 1, u, u', i) \text{ for } 2 \leq u \leq u' \leq i \text{ and } j \leq v' \leq v \leq n \qquad (5.12)$$

$$(j, v', v, 1, u, u-1, \ldots, i) \text{ for } i+1 \leq u < j \leq v' \leq v \leq n \tag{5.13}$$

$$(j, j-1, \ldots, v, 1, u, u', i) \text{ for } 2 \leq u \leq u' \leq i < v \leq j-1 \tag{5.14}$$

$$(j, j-1, \ldots, v, 1, u, u-1, \ldots, i) \text{ for } i+1 \leq u < v \leq j-1 \tag{5.15}$$

A path of the form $(j, \ldots, v, 1, u, \ldots, i)$ for $2 \leq v \leq i$ or $j \leq u \leq n$ cannot exist as the path would not be simple, that is, either $i$ or $j$ would appear twice in the path.

Assume the path is of the form (5.12). Suppose $sgn\,(v, 1, u) = +$. Then, the positively signed cycle $(v, 1, u, i+1, v)$ is formed. On the other hand, suppose $sgn\,(v, 1, u) = -$. Then, for $u \neq i$ or $v \neq j$, the cycle $(v, 1, u, v)$ has positive cycle sign. For $u = i \neq 2$, the cycle $(v, 1, u, u-1, v)$ has positive cycle sign. For $v = j \neq n$, the cycle $(v, 1, u, v+1, v)$ has positive cycle sign. Finally, for $u = i = 2$ and $v = j = n$ and since $n \geq 5$, the cycle $(v, 1, u, u+1, v-1, v)$ exists and has positive cycle sign. On the other hand, by assuming the path is of the form (5.13), the cycle $(v, 1, u, j, v', v)$ exists and has positive cycle sign. Similarly, a path of the form (5.14) induces the positively signed cycle $(v, 1, u, u', i, v)$. Finally, with a path of the form (5.15), the cycle $(v, 1, u, v)$ exists and has positive cycle sign. Therefore, none of these paths exist, $b_{ij}$ is not an essential zero implying that $B_n$ is not maximal.

The conclusions for each part of the theorem clearly dictate the result of the theorem. □

It is interesting to examine the pattern of those positions in B[2, ..., n] that result in fully indecomposable maximal SNS matrices as determined by Theorem 5.2.2. Let $B_n$ be as specified in the theorem. Then, the positions indicated by $\star$ below mark all those locations in $H_{n-1}$ where a nonzero entry can be set to 0 giving such a matrix $B_n$.

$$B[2, \ldots, n] = \begin{bmatrix} \star & \star & & & & & \\ \star & \star & & & & & \\ & & \star & & & & \\ & & & \ddots & & & \\ & & & \star & \star & & \\ & & & & & \ddots & & \\ & & & & & & \star & \star \\ & & & & & & \star & \star \end{bmatrix}$$

The specific SNS matrix that is equivalent to $B_n$ when any one of the above $\star$ entries is set to 0 is given schematically by the following:

$$\begin{bmatrix} H_n & V_n^{(1)} & & & & & \\ H_n & V_n^{(1)} & & & & & \\ & & V_n^{(2)} & & & & \\ & & & \ddots & & & \\ & & & H_n & V_n^{(r)} & & \\ & & & & & \ddots & & \\ & & & & & & V_n^{(n-3)} & V_n^{(1)} \\ & & & & & & H_n & H_n \end{bmatrix}$$

For example, if position (3,2) of $B_n$ (and hence $B[2, \ldots, n]$) is set to 0 and the rest of the conditions of the theorem are satisfied, then $B_n$ must be equivalent to $H_n$. Note that the entries in the lower left of $B[2, \ldots, n]$ are already zero and cannot be used to satisfy the conditions of the theorem and the entries in the upper right corner of the matrix result in contradictions when set to 0.

Theorem 4.1.6 implies that $V_n^{(n-3)}$ is equivalent to $V_n^{(1)}$ (and similar equivalences for the other members of the class $V_n^{(r)}$ specified here). This results in a schematic pattern that is symmetric about its anti-diagonal.

In order to state the next theorem, an anomalous SNS matrix, $\mathcal{M}_6^{(3)}$, must be defined. It appears to be a degenerate form of the $H_n^{(2,1,2,1)}$ class of matrices as it has all the properties of the class (such as the number of zero entries, maximality, fully indecomposability, etc.) but it cannot be transformed into any member of the class. Let

$$\mathcal{M}_6^{(3)} = \begin{bmatrix} -1 & -1 & -1 & 0 & -1 & -1 \\ +1 & -1 & -1 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix}$$

For $n \leq 4$, there are no fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + 2$ zero entries. However, for $n = 5$, there are two such matrices, namely $H_5^{(3,1,0,0)}$ and $V_5^{(1)}$ (see Table 6.1). The case $n \geq 6$ is now considered.

**Theorem 5.2.3** For $n \geq 6$, $B_n$ is a fully indecomposable maximal SNS matrix with $\binom{n-1}{2}+2$ zero entries if and only if

1. for $n = 6$, $B_n$ is equivalent to $H_6^{(3,1,0,0)}$ or $\mathcal{M}_6^{(3)}$,

2. for $n \geq 7$, $B_n$ is equivalent to $H_n^{(3,1,0,0)}$ or $H_n^{(2,1,2,1)}$.

*Proof.* Theorem 4.2.6 and Observation 4.2.4 show that both $H_n^{(3,1,0,0)}$ for $n \geq 6$ and $H_n^{(2,1,2,1)}$ for $n \geq 7$ are fully indecomposable maximal SNS matrices having $\binom{n-1}{2}+2$ zero entries. By inspection, $\mathcal{M}_6^{(3)}$ is also a fully indecomposable maximal SNS matrix having $\binom{n-1}{2}+2$ zero entries.

For the converse, assume $B_n$ is a fully indecomposable maximal SNS matrix having $\binom{n-1}{2}+2$ zero entries. Also, without loss of generality, assume $B_n$ is normalized and $d(1) = \min_{1\leq i\leq n} d(i)$.

Then, $d(1) \geq n-2$, otherwise $B[2,\ldots,n]$ contains at least $(n-1)^2 - \binom{n-2}{2}+1$ nonzeros and is therefore not SNS. Also, $d(1) \leq n$ by Thomassen [Th1, Proposition 2.3], so $d(1) = n-2$, $n-1$, or $n$.

As in Theorem 5.1.2, if $d(1) = n$, then by Thomassen [Th1, Theorem 2.7], without loss of generality $D(B_n)$ is a subdigraph of $D(H_n)$. Since $\nu(B_n) = \nu(H_n) - 2$, this contradicts the fact that $B_n$ is maximal. Therefore, the assumption that $d(1) = n$ is false.

With $d(1) = n-2$ and $b_{11} = -1$, $B[2,\ldots,n]$ must be SNS and have exactly $(n-1)^2 - \binom{n-2}{2}$ nonzero entries. But this submatrix must be equivalent to $H_{n-1}$, as it is the unique matrix with this many zero entries, and thus, without loss of generality, $B[2, \ldots, n] = H_{n-1}$. Thus far, the form of $B_n$ has been deduced as

$$\left[\begin{array}{c|c} -1 & w^T \\ \hline u & H_{n-1} \end{array}\right]$$

and the remainder of the proof will determine the correct positions and signing for the $n-2$ nonzero entries in $u$ and $w$.

Let $j \geq 2$ be the smallest integer such that $b_{1j} \neq 0$ and $i \geq 2$ be the largest integer such that $b_{i1} \neq 0$. Clearly such indices exist, otherwise $B_n$ is partly decomposable. Now, each part of Theorem 5.1.1 can be applied with the proviso that $B_n$ is maximal. There exist signature matrices $T_n$ and $U_n$ such that $B'_n \equiv T_n B_n U_n$ satisfies the specified conditions in the following two cases.

If $i \leq j$, then $B'_n$ is equivalent to $H_n$. However, this is impossible as $\nu(B'_n) = \nu(H_n) - 2$, therefore, $i = j + 1$ and either of two cases follows.

Case 1. The matrix $B'_n$ is the row stretch of $H_{n-1}$ on row $i$ with the new row and column labelled as 1. In this case, the only choice of $i$ that results in $d(1) = n-2$ is $i = 5$. With this, $P_n B'_n P_n^T = H_n^{(3,1,0,0)}$ by Definition 4.2.1, where $P_n = (2, 3, 4, 1, 5, 6, \ldots, n)$.

Case 2. The matrix $B'_n$ is the column stretch of $H_{n-1}$ on column $j$ with the new row and column labelled as 1. In this case, the only choice of $j$ that results in $d(1) = n-2$ is $j = n-3$. With this, $P_n B'_n P_n^T = H_n^{(0,0,3,1)}$ by Definition 4.2.1, where $P_n = (2, 3, \ldots, n-3, 1, n-2, n-1, n)$. By Theorem 4.2.7, $H_n^{(0,0,3,1)}$ is equivalent to $H_n^{(3,1,0,0)}$.

This completes the possibility $d(1) = n-2$.

Finally, with $d(1) = n-1$, $B[2,\ldots,n]$ has exactly $(n-1)^2 - \binom{n-2}{2} - 1$ nonzero entries. Since $B_n$ is fully indecomposable, Corollary 3.1.3 states that $B[2,\ldots,n]$ is either not maximal or fully indecomposable.

First, assume $B[2,\ldots,n]$ is not maximal. Then, setting some zero entry nonzero gives a SNS matrix with $(n-1)^2 - \binom{n-2}{2}$ nonzero entries and by Theorem 2.6.4, this must be equivalent to $H_{n-1}$. Therefore, the only way for $B[2,\ldots,n]$ to be not maximal is to assume $B[2,\ldots,n]$ is contained in $H_{n-1}$, where $\nu(B[2,\ldots,n]) = \nu(H_{n-1}) - 1$. Since $B_n$ is maximal, Theorem 5.2.2 states that $B_n$ is equivalent to either $H_n$ or $V_n^{(r)}$ for some integer $r$ where $1 \leq r \leq n-3$. However, $\nu(B_n) = \nu(H_n) - 2$ and for $n \geq 6$, $\nu(B_n) > \nu(V_n^{(r)})$. Thus, $B_n$ cannot be equivalent to either $H_n$ or $V_n^{(r)}$, and $B[2,\ldots,n]$ must be maximal.

Second, assume $B[2,\ldots,n] \equiv W_{n-1}$ is fully indecomposable and maximal. As this submatrix has $\binom{n-2}{2} + 1$ zeros, by Theorem 5.1.2, $W_{n-1}$ is equivalent to $H_{n-1}^{(2,1,0,0)}$. Without loss of generality, $W_{n-1} = R_{n-1} H_{n-1}^{(2,1,0,0)} R_{n-1}^T$, where $R_{n-1} = (3,1,2,4,5,\ldots,n-1)$. Let $j_s, j_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{1,j_s} \neq 0$ and $b_{1,j_l} \neq 0$ and $i_s, i_l \geq 2$ be the smallest and largest integers, respectively, such that $b_{i_s,1} \neq 0$ and $b_{i_l,1} \neq 0$. Clearly such indices exist, otherwise $B_n$ is partly decomposable. Now, each part of Theorem 5.2.1 can be applied with the proviso that $B_n$ is maximal. There exist signature matrices $T_n$ and $U_n$ such that $B_n' \equiv T_n B_n U_n$ satisfies the specified conditions in each of the following parts. Note also that the new row and column is labelled as 1 in each of the following

Part 1. If $i_l = 2$, then $B'_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Thus, with $P_n = (3,4,1,2,5,6,\ldots,n)$, $P_n B'_n P_n^T = H_n^{(2,2,0,0)}$ by Definition 4.2.1 and by Theorem 4.2.8, $H_n^{(2,2,0,0)}$ is equivalent to $H_n^{(3,1,0,0)}$.

Part 2. If $i_l \geq 3$ and $j_s = 2$, then $i_l = 5$ and either of two cases follows.

Case (a). The matrix $B'_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Then, $P_n B'_n Q_n^T S_n = H_n^{(3,1,0,0)}$ by Definition 4.2.1, where $P_n = (5,1,2,3,4,6,7,\ldots,n)$, $Q_n = (1,2,4,3,5,6,\ldots,n)$ and $S_n = -I_3 \oplus I_{n-3}$.

Case (b). The matrix $B'_n$ is the column stretch of $W_{n-1}$ on column $j_s$. But then, $d(1) \neq n-1$ as column $j_s$ has only two nonzero entries.

Part 3. If $i_l \geq 3$ and $j_s = 3$, then $i_l \leq 4$ and either of two cases follows.

Case (a). The matrix $B'_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Then, for $i_l = 3$, $P_n B'_n P_n^T = H_n^{(3,1,0,0)}$ by Definition 4.2.1, where $P_n = (1,3,4,2,5,6,\ldots,n)$ and for $i_l = 4$, $P_n B'_n Q_n^T S_n = H_n^{(3,1,0,0)}$ by Definition 4.2.1, where $P_n = (1,4,3,2,5,6,\ldots,n)$, $Q_n = (1,3,4,2,5,6,\ldots,n)$ and $S_n = I_1 \oplus -I_1 \oplus I_{n-2}$.

Case (b). The matrix $B'_n$ is the column stretch of $W_{n-1}$ on column $j_s$. But then, $d(1) \neq n-1$ as column $j_s$ has only two nonzero entries.

Part 4. If $i_l \geq 3$ and $j_s = 4$, then $i_l \leq 5$ and either of four cases follows.

Case (a). When $i_l = i_s$, $B'_n$ cannot be maximal, so this case cannot apply.

Case (b) The matrix $B'_n$ is the column stretch of $W_{n-1}$ on column $j_s$ when $j_s = j_l = 4$. For $n \geq 7$, $d(1) \neq n-1$ but, for $n = 6$,

$$B'_n = \begin{bmatrix} -1 & 0 & 0 & +1 & 0 & 0 \\ +1 & -1 & 0 & +1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & +1 & -1 & -1 & -1 \\ +1 & +1 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix}$$

and with $P_6 = (3,4,1,2,5,6)$, $P_6 B'_n P_6^T = \mathcal{M}_6^{(3)}$.

Case (c) The matrix $B'_n = R_n A_n R_n^T$ where $R_n = (3,2,1,4,5,\ldots,n)$ and $A_n$ is the row stretch of $W_{n-1}$ on row $i_l$ when $i_l = 4$ and $i_s = 3$. With $P_n$, $Q_n$ and $S_n$ as in Part 3a above, $P_n A_n Q_n^T S_n = H_n^{(3,1,0,0)}$, and thus $B'_n$ is equivalent to $H_n^{(3,1,0,0)}$.

Case (d) The matrix $B'_n = R_n A_n R_n^T$ where $R_n = (2,1,3,4,\ldots,n)$ and $A_n$ is the row stretch of $W_{n-1}$ on row $i_l$ when $i_l = 5$ and $i_s = 2$. With $P_n$, $Q_n$ and $S_n$ as in Part 2a above, $P_n A_n Q_n^T S_n = H_n^{(3,1,0,0)}$, and thus $B'_n$ is equivalent to $H_n^{(3,1,0,0)}$.

Part 5 If $i_l \geq 3$ and $j_s \geq 5$, then if $i_l \leq j_s$ also, then $B'_n$ is equivalent to $H_n^{(2,1,0,0)}$. However, this contradicts the number of nonzeros in $B'_n$ as $\nu(B'_n) = \nu(H_n^{(2,1,0,0)}) - 1$. Therefore, $i_l = j_s + 1$ and either of two cases follows.

Case (a) The matrix $B'_n$ is the row stretch of $W_{n-1}$ on row $i_l$. Since $i_l \geq 6$, every such row contains at least three zeros and thus, $d(1) \neq n-1$.

Case (b) The matrix $B'_n$ is the column stretch of $W_{n-1}$ on column $j_s$. In this case, the only choice of $j_s$ that results in $d(1) = n-1$ is $j_s = n-2$ and hence, $n \geq 7$. With this, $P_n B'_n P_n^T = H_n^{(2,1,2,1)}$ by Definition 4.2.1, where $P_n = (3, 4, 2, 5, 6, \ldots, n-2, 1, n-1, n)$.

Thus, any fully indecomposable maximal SNS matrix with $\binom{n-1}{2}+2$ zero entries must be equivalent to

1. $H_6^{(3,1,0,0)}$ or $\mathcal{M}_6^{(3)}$ for $n = 6$,

2. $H_n^{(3,1,0,0)}$ or $H_n^{(2,1,2,1)}$ for $n \geq 7$.

□

# Chapter 6

# Summary and Future Work

## 6.1 Summary

This work was initiated in response to the generation of all the fully indecomposable maximal SNS matrices of dimensions 1 through 9 by Lundy, Maybee and Van Buskirk[LMVB]. The results of an analysis of this data with regard to the number of equivalence classes containing a specified number of zero entries are given in Table 6.1. For fixed $n$, the known lower bound of $\binom{n-1}{2}$ for the number of zero entries is tight by Gibson[Gi] but a tight upper bound is not known. As proven by Gibson (Theorem 2.6.4), $H_n$ determines the unique equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}$ zero entries. The data in Table 6.1 shows that there is exactly one equivalence class of fully indecomposable maximal SNS matrices with $\binom{n-1}{2}+1$ zero entries for $5 \leq n \leq 9$. In Theorem 5.1.2, this is proven for all $n \geq 5$, and moreover this class is represented by $H_n^{(2,1,0,0)}$. Similarly, the data in Table 6.1 shows that there are exactly two equivalence classes of fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + 2$ zeros for $5 \leq n \leq 9$. In Theorem 5.2.3, this is proven for all $n \geq 6$, and moreover $H_n^{(3,1,0,0)}$ and $H_n^{(2,1,2,1)}$ are the representatives for these classes for all $n \geq 7$.

In Chapter 3, the row (respectively, column) stretch $B_{n+1}$ of a matrix $A_n$ is defined, and it is proven in Theorem 3.2.5 that $B_{n+1}$ is a fully indecomposable maximal SNS matrix if and only if $A_n$ is a fully indecomposable maximal SNS matrix. In Chapter 4, two classes of SNS matrices, $V_n^{(r)}$ and $H_n^{(p,q,r,s)}$, are defined, providing a convenient categorization of some of the SNS matrices found by Lundy et al. [LMVB].

Chapter 5 contains the characterizations of all fully indecomposable SNS matrices obtained by bordering $H_{n-1}$ or a matrix that is permutation similar to $H_{n-1}^{(2,1,0,0)}$ (Theorem 5.1.1 and Theorem 5.2.1, respectively). These theorems are used to prove the main results of this thesis (Theorems 5.1.2 and 5.2.3), but they are also of independent interest.

Table 6.1 The categorization of all fully indecomposable maximal SNS matrices of dimensions 1 through 9 by dimension and number of zero entries.

The numbers of equivalence classes were obtained from the raw data produced by Lundy et al [LMVB]. The matrices denoted $\mathcal{M}_n^{(j)}$ are listed in the Appendix, they are related to the conjectures of Table 6.2, but are not members of either of the classes $V_n^{(r)}$ or $H_n^{(p,q,r,s)}$. The matrices $\mathcal{M}_n^{(j)}$ are not identified as a class, the parameter $j$ is used only as a label.

| Matrix Dimension | Number of Zero Entries | Number of Equivalence Classes | Equivalence Class Membership |
|---|---|---|---|
| 1 | 0 | 1 | $H_1$ |
| 2 | 0 | 1 | $H_2$ |
| 3 | 1 | 1 | $H_3$ |
| 4 | 3 | 1 | $H_4$ |
| | 4 | 2 | $H_4^{(2,1,0,0)}$, $V_4^{(1)}$ |
| 5 | 6 | 1 | $H_5$ |
| | 7 | 1 | $H_5^{(2,1,0,0)}$ |
| | 8 | 2 | $H_5^{(3,1,0,0)}$, $V_5^{(1)}$ |
| | 9 | 2 | $\mathcal{M}_5^{(1)}$, $\mathcal{M}_5^{(2)}$ |
| 6 | 10 | 1 | $H_6$ |
| | 11 | 1 | $H_6^{(2,1,0,0)}$ |
| | 12 | 2 | $H_6^{(3,1,0,0)}$, $\mathcal{M}_6^{(3)}$ |
| | 13 | 3 | $H_6^{(4,1,0,0)}$, $V_6^{(1)}$, $V_6^{(2)}$ |
| | 14 | 6 | $H_6^{(3,2,0,0)}$, $\mathcal{M}_6^{(4)}$, $\mathcal{M}_6^{(5)}$, $\mathcal{M}_6^{(6)}$, $\mathcal{M}_6^{(7)}$, $\mathcal{M}_6^{(8)}$ |
| | 15 | 4 | |
| | 16 | 4 | |

| Matrix Dimension | Number of Zero Entries | Number of Equivalence Classes | Equivalence Class Membership |
|---|---|---|---|
| 7 | 15 | 1 | $H_7$ |
| | 16 | 1 | $H_7^{(2,1,0,0)}$ |
| | 17 | 2 | $H_7^{(3,1,0,0)}$, $H_7^{(2,1,2,1)}$ |
| | 18 | 2 | $H_7^{(4,1,0,0)}$, $\mathcal{M}_7^{(9)}$ |
| | 19 | 5 | $H_7^{(5,1,0,0)}$, $H_7^{(3,2,0,0)}$, $V_7^{(1)}$, $V_7^{(2)}$, $\mathcal{M}_7^{(10)}$ |
| | 20 | 6 | |
| | 21 | 11 | $H_7^{(4,2,0,0)}$ |
| | 22 | 12 | |
| | 23 | 15 | |
| | 24 | 8 | |
| | 25 | 8 | |
| | 26 | 0 | |
| | 27 | 0 | |
| | 28 | 1 | |

| Matrix Dimension | Number of Zero Entries | Number of Equivalence Classes | Equivalence Class Membership |
|---|---|---|---|
| 8 | 21 | 1 | $H_8$ |
| | 22 | 1 | $H_8^{(2,1,0,0)}$ |
| | 23 | 2 | $H_8^{(3,1,0,0)}$, $H_8^{(2,1,2,1)}$ |
| | 24 | 2 | $H_8^{(4,1,0,0)}$, $H_8^{(3,1,2,1)}$ |
| | 25 | 5 | $H_8^{(5,1,0,0)}$, $H_8^{(3,2,0,0)}$, $\mathcal{M}_8^{(11)}$, $\mathcal{M}_8^{(12)}$, $\mathcal{M}_8^{(13)}$ |
| | 26 | 7 | $H_8^{(6,1,0,0)}$, $V_8^{(1)}$, $V_8^{(2)}$, $V_8^{(3)}$ |
| | 27 | 9 | $H_8^{(4,2,0,0)}$ |
| | 28 | 13 | |
| | 29 | 20 | $H_8^{(5,2,0,0)}$ |
| | 30 | 32 | $H_8^{(4,3,0,0)}$ |
| | 31 | 29 | |
| | 32 | 50 | |
| | 33 | 45 | |
| | 34 | 53 | |
| | 35 | 34 | |
| | 36 | 37 | |
| | 37 | 0 | |
| | 38 | 1 | |

| Matrix Dimension | Number of Zero Entries | Number of Equivalence Classes | Equivalence Class Membership |
|---|---|---|---|
| 9 | 28 | 1 | $H_9$ |
| | 29 | 1 | $H_9^{(2,1,0,0)}$ |
| | 30 | 2 | $H_9^{(3,1,0,0)}$, $H_9^{(2,1,2,1)}$ |
| | 31 | 2 | $H_9^{(4,1,0,0)}$, $H_9^{(3,1,2,1)}$ |
| | 32 | 5 | $H_9^{(5,1,0,0)}$, $H_9^{(3,2,0,0)}$, $H_9^{(4,1,2,1)}$, $H_9^{(3,1,3,1)}$, $\mathcal{M}_9^{(14)}$ |
| | 33 | 6 | $H_9^{(6,1,0,0)}$, $H_9^{(3,2,2,1)}$ |
| | 34 | 10 | $H_9^{(7,1,0,0)}$, $H_9^{(4,2,0,0)}$, $V_9^{(1)}$, $V_9^{(2)}$, $V_9^{(3)}$ |
| | 35 | 13 | |
| | 36 | 17 | $H_9^{(5,2,0,0)}$ |
| | 37 | 26 | $H_9^{(4,3,0,0)}$ |
| | 38 | 41 | $H_9^{(6,2,0,0)}$ |
| | 39 | 49 | |
| | 40 | 67 | $H_9^{(4,4,0,0)}$ |
| | 41 | 85 | |
| | 42 | 118 | |
| | 43 | 149 | |
| | 44 | 181 | |
| | 45 | 225 | |
| | 46 | 244 | |
| | 47 | 252 | |
| | 48 | 218 | |
| | 49 | 192 | |
| | 50 | 2 | |
| | 51 | 1 | |

## 6.2 Future Work

The two classes of matrices, $H_n^{(p,q,r,s)}$ and $V_n^{(r)}$, defined in this work classify only a small subset of the SNS matrices known from the data of Lundy et al [LMVB] Other infinite classes of SNS matrices could be identified and their properties investigated

The current status regarding the number of nonequivalent fully indecomposable maximal $n$-by-$n$ SNS matrices with $\binom{n-1}{2} + k$ zero entries for $0 \leq k \leq 4$ is as given in Table 6 2 Thus, for example, we conjecture that for each $n \geq 7$, there are 5 such matrices having $\binom{n-1}{2}+4$ zero entries The method of proof used in Theorems 5 1 2 and 5 2 3 does not easily extend to values of $k \geq 3$ and thus, other methods need to be devised in order to show the correctness of these two conjectures For $n \geq 5$, it can be seen by using Observation 4 2 5 that $H_n^{(3,1,0,0)}$ can be obtained by two successive row and/or column stretches of $H_{n-2}$ The first row (or column) stretch of $H_{n-2}$ produces a matrix equivalent to $H_{n-1}^{(2,1,0,0)}$ and the second produces a matrix equivalent to $H_n^{(3,1,0,0)}$ For $n \geq 7$, a similar observation can be made for $H_n^{(2,1,2,1)}$ This suggests that the conjectures of Table 6 2 might be proved by doing several row and/or column stretches of $H_{n-3}$ or $H_{n-4}$ However, this appears to lead to a large number of cases to be considered Also, all matrices generated in this manner are 1-connected, and thus other bordering techniques leading to 2-connected SNS matrices should be investigated

Table 6.2 The known or conjectured number of nonequivalent fully indecomposable maximal SNS matrices with $\binom{n-1}{2} + k$ zero entries, where $0 \leq k \leq 4$

| Number of zeros $\binom{n-1}{2} + k$ where $k$ is | Number of Distinct Matrices (Equivalence Classes) | Status |
|---|---|---|
| 0 | 1 | Known for $n \geq 1$ (Theorem 2.6.4) |
| 1 | 1 | Known for $n \geq 5$ (Theorem 5.1.2) |
| 2 | 2 | Known for $n \geq 6$ (Theorem 5.2.3) |
| 3 | 2 | Conjectured for $n \geq 7$ |
| 4 | 5 | Conjectured for $n \geq 7$ |

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# Appendix A

# The SNS Matrices $\mathcal{M}_n^{(j)}$

The following fully indecomposable maximal SNS matrices represent the equivalence classes in Table 6 1 that are not members of $H_n^{(p,q,r,s)}$ or $V_n^{(r)}$ but that have $\binom{n-1}{2} + k$ zero entries, where $2 \leq k \leq 4$ They are related to the conjectures in Table 6 2 The matrices $\mathcal{M}_n^{(j)}$ are not identified as a class, the parameter $j$ is used only as a label

The matrix $\mathcal{M}_6^{(3)}$ has already been defined in Section 5 2 but is restated here for completeness

$$\mathcal{M}_5^{(1)} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ +1 & -1 & -1 & 0 & 0 \\ 0 & +1 & -1 & -1 & -1 \\ 0 & -1 & +1 & -1 & -1 \\ 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_5^{(2)} = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ +1 & -1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & -1 \\ 0 & 0 & +1 & -1 & -1 \\ -1 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(3)} = \begin{bmatrix} -1 & -1 & -1 & 0 & -1 & -1 \\ +1 & -1 & -1 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(4)} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ +1 & -1 & 0 & -1 & 0 & 0 \\ 0 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & -1 & -1 & -1 \\ 0 & -1 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(5)} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ +1 & -1 & 0 & 0 & 0 & -1 \\ 0 & +1 & -1 & -1 & 0 & -1 \\ 0 & +1 & +1 & -1 & 0 & -1 \\ 0 & +1 & 0 & +1 & -1 & -1 \\ 0 & +1 & 0 & +1 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(6)} = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 & 0 \\ +1 & -1 & -1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 & -1 & -1 \\ 0 & 0 & +1 & -1 & -1 & -1 \\ 0 & -1 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(7)} = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & -1 \\ +1 & -1 & 0 & 0 & -1 & -1 \\ 0 & +1 & -1 & -1 & 0 & -1 \\ 0 & +1 & +1 & -1 & 0 & -1 \\ 0 & +1 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_6^{(8)} = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 & 0 \\ +1 & -1 & 0 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & -1 & -1 \\ 0 & 0 & +1 & -1 & -1 & -1 \\ -1 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_7^{(9)} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & -1 & -1 \\ +1 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & +1 & +1 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_7^{(10)} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ +1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & +1 & -1 & -1 & 0 & -1 & -1 \\ 0 & +1 & +1 & -1 & 0 & -1 & -1 \\ 0 & +1 & 0 & +1 & -1 & -1 & -1 \\ 0 & +1 & 0 & +1 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_8^{(11)} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ +1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & +1 & -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & +1 & +1 & -1 & 0 & -1 & -1 & -1 \\ 0 & +1 & 0 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & 0 & +1 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_8^{(12)} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ +1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & 0 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_8^{(13)} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ +1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & 0 & +1 & -1 & -1 & -1 \\ 0 & +1 & +1 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix},$$

$$\mathcal{M}_9^{(14)} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ +1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & +1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & +1 & +1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & +1 & 0 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & +1 & 0 & +1 & +1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \end{bmatrix}$$

# Index of Definitions and Notation

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