CHOICE NUMBERS FOR UNIONS OF GRAPHS

by

SHEILA FERNEYHOUGH
University of Victoria, November 1996

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We accept this thesis as conforming
to the required standard

[redacted]

Dr Gary MacGillivray, Supervisor (Department of Mathematics & Statistics)

[redacted]

Dr Ernest Cockayne, Member (Department of Mathematics & Statistics)

[redacted]

Dr Frank Ruskey, Outside Member (Department of Computer Science)

[redacted]

Dr Denis Hanson, External Examiner (Department of Mathematics & Statistics,
University of Regina, Regina, SK)

©Sheila Ferneyhough, November 1996

University of Victoria

All rights reserved This thesis may not be reproduced in whole or in part, by
photocopy or other means, without the permission of the author

Supervisor: Dr. Gary MacGillivray (Department of Mathematics and Statistics)

Abstract

The problem of determining choice numbers for unions of graphs is investigated, with particular attention being paid to the case where one of the graphs is 2-choosable. It is shown that all 2-choosable graphs are $(2k : k)$-choosable for all positive values of k. New results on $(a : b)$-choosability of planar graphs, bipartite graphs and Cartesian products of graphs are also presented.

Examiners:

[redacted]

Dr. Gary MacGillivray, Supervisor (Department of Mathematics & Statistics)

[redacted]

Dr. Ernest Cockayne, Member (Department of Mathematics & Statistics)

[redacted]

Dr. Frank Ruskey, Outside Member (Department of Computer Science)

[redacted]

Dr. Denis Hanson, External Examiner (Department of Mathematics & Statistics, University of Regina, Regina, SK)

Contents

List of Figures

Acknowledgement

I wish to express my gratitude to Dr. Gary MacGillivray for his humor, patience, support and assistance, which made this thesis that much easier to complete.

I would like to thank Dr. Ernest Cockayne, Dr. Frank Ruskey and Dr. Denis Hanson for the time they spent examining this thesis, and for their helpful suggestions.

Deep appreciation is expressed to my friends and colleagues for their support and advice. Heartfelt thanks is extended to Sadie and Sandra Kschischang for their never-ending warmth and emotional support. For his caring, encouragement and advice, I thank Jesse Stone, whose repeated requests for acknowledgement are finally being met. I owe a special thanks to my family for their continuing love, support and motivation.

Sheila Ferneyhough

Chapter 1

Introduction

Graph choosability involves the problem of properly colouring the vertices of a graph from prescribed sets of permissible colours. A finite graph G is *k-choosable* if for every assignment of k-sets $L(v)$ to the vertices of G, there is a proper vertex colouring m of $V(G)$ where $m(v) \in L(v)$ for all $v \in V(G)$. The *choice number* of G, $\chi_\ell(G)$, is the minimum integer k such that G is k-choosable. Choice numbers were first defined by Erdos, Rubin and Taylor [4], and independently by Vizing [16]. The former began this investigation in an attempt to prove the Dinitz conjecture, which was recently proved by Galvin (see [19]).

There is a definite beauty in the purely abstract study of choice numbers, but this does not preclude its having applications. Graph colouring problems often arise where the choice of colour for each vertex or edge is restricted to a particular set, leading naturally to the notions of choosability and $(a : b)$-choosability. As k-choosable implies k-colourable, any result on choice number implies a corresponding result on chromatic number, and working with choosability has sometimes led to more elegant proofs of classical colouring results. For example, Thomassen's result [13] that all planar graphs are 5-choosable implies Heawood's result that all planar graphs are 5-colorable. Thomassen's very elegant induction argument, which does not require Euler's Theorem and does not require re-colouring any previously coloured vertices, is most likely the simplest known proof of the five-color theorem.

This thesis investigates the problem of determining choice numbers for unions of graphs, with particular attention being paid to the case where one of the graphs is 2-choosable. Chapter 2 provides a summary of the graph theoretic definitions used in this thesis, including formal definitions of k-choosable and $(a:b)$-choosable graphs. Chapter 3 provides a survey of previous work on choice numbers and $(a:b)$-choosability. New results are presented in Chapter 4, including the main result that all 2-choosable graphs are $(2k:k)$-choosable for all positive values of k. Tuza and Voigt [15] have recently derived this result using different methods. New results on $(a:b)$-choosability of planar graphs, bipartite graphs and Cartesian products of graphs are also presented.

Chapter 2

Preliminaries

This chapter summarizes the graph theoretic definitions used in this thesis All graphs considered are assumed to be finite, simple and, without loss of generality, connected For further discussion of basic graph theory, the reader is referred to Bondy and Murty [3]

When unions of graphs are considered, it is assumed, without loss of generality, that each graph has the same set of vertices Given graph G and H, vertices $u, v \in V(G \cup H)$ are said to be *G-adjacent* if $(u, v) \in E(G)$ The average degree $\frac{|E(G)|}{|V(G)|}$ of a graph G is denoted by $\overline{d}(G)$ A graph is called *m-degenerate* if each of its subgraphs contains a vertex of degree at most m The *core* of a graph G, denoted by core(G), is obtained by successively removing vertices of degree 1 until none remain An example is given in Figure 2 1

The *theta* graph, $\theta_{a,b,c}$ is the graph consisting of three paths of lengths a, b and

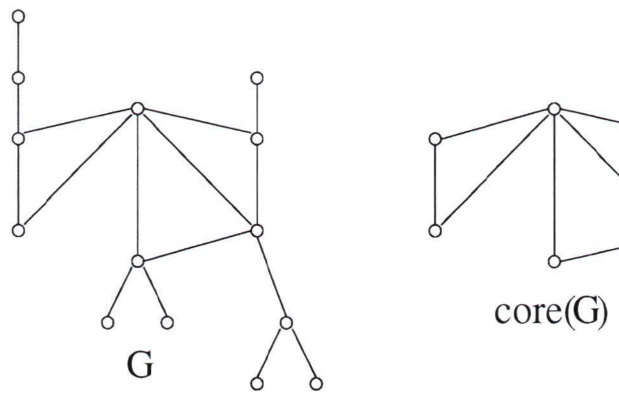

Figure 2.1. The Core of a Graph

c respectively, which share a pair of end vertices, but are otherwise vertex disjoint. The $\theta_{2,2,2m}$ graph is shown in Figure 2.2.

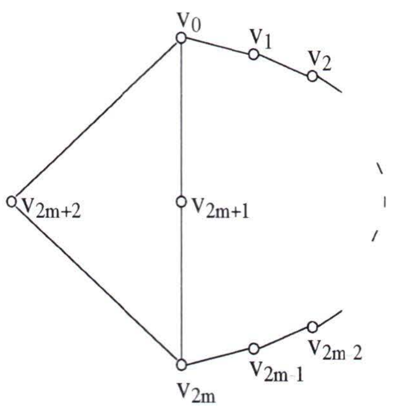

Figure 2.2. The Theta Graph

The complete s-partite graph consisting of s parts, each of size r is denoted by K_{r*s}. A *near-triangulation* is a planar embedding of a graph consisting of a cycle, and vertices and edges inside the cycle, such that each bounded face is bounded by a triangle.

The *Cartesian product* $G \square H$ of graphs G and H is defined by

$$V(G \square H) = V(G) \times V(H)$$

$$E(G \square H) = \{((g_1, h_1), (g_2, h_2)) : g_1 = g_2 \text{ and } (h_1, h_2) \in E(H), \text{ or } (g_1, g_2) \in E(G) \text{ and } h_1 = h_2\}$$

An example is given in Figure 2.3.

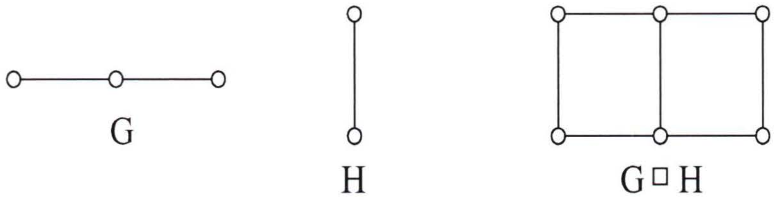

Figure 2.3: The Cartesian Product of Two Graphs

A *proper k-vertex-colouring* or *proper k-colouring* of a graph G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G, such that no two distinct adjacent vertices have the same colour. A graph is *k-colourable* if it has a proper k-colouring. The *chromatic number* $\chi(G)$ is the minimum k for which G is k-colourable.

The concept of vertex colouring can be generalized by imposing restrictions on the colours allowable for each vertex. A graph G is *k-choosable (k-list-colourable)* if for every assignment of k-sets $L(v)$ to the vertices of G, there is a proper vertex colouring m of $V(G)$ where $m(v) \in L(v)$ for each $v \in V(G)$. The *choice number (list-chromatic number)* $\chi_\ell(G)$ is the minimum integer k such that G is k-choosable.

For example, $\chi_\ell(C_4) = 2$. For any assignment of 2-sets to the vertices of C_4, there is a proper vertex colouring, where the colour on each vertex is chosen from its 2-set. Let $C_4 = v_0, v_1, v_2, v_3, v_0$. Suppose sets $L(v_0) = \{1,2\}$, $L(v_1) = \{1,3\}$, $L(v_2) = \{2,3\}$ and $L(v_3) = \{1,2\}$ are assigned to its vertices. Given this assignment, $m(v_0) = 2$, $m(v_1) = 3$, $m(v_2) = 2$ and $m(v_3) = 1$ is a proper vertex colouring where each $m(v_i) \in L(v_i)$.

The notion of choosability can be generalized further as follows. Let a and b be integers such that $a > b \geq 1$. A graph G is $(a:b)$-*choosable* [4] if for every assignment of a-sets $L(v)$ to the vertices of G, there exist b-subsets $M(v) \subset L(v)$ for every $v \in V(G)$ such that $M(u) \cap M(v) = \emptyset$ whenever u and v are adjacent in G. Note that a graph which is $(a:1)$-choosable is a-choosable. For a graph G with a fixed assignment of a-sets, such a choice of b-subsets is called an $(a:b)$-*choice*.

For example, the 4-cycle is $(4:2)$-choosable. For any assignment of 4-sets to the vertices of C_4, a 2-subset can be chosen from each 4-set so that subsets on adjacent vertices are disjoint. Let the vertices of C_4 be labelled as in the previous example. Suppose sets $L(v_0) = \{1,3,4,5\}$, $L(v_1) = \{1,3,4,5\}$, $L(v_2) = \{2,3,4,5\}$ and $L(v_3) = \{1,2,3,4\}$ are assigned to its vertices. Given this assignment, $M(v_0) = \{4,5\}$, $M(v_1) = \{1,3\}$, $M(v_2) = \{4,5\}$ and $M(v_3) = \{1,2\}$ is a $(4:2)$-choice.

Chapter 3

Previous Work

It is the purpose of this chapter to survey previous work on choice numbers. Proofs of known results are given if they were not found in the references. The unreferenced results given in this section are results which are well-known, despite not being explicitly stated in the references.

One possible assignment of sets to vertices of a graph G is $L(v) = \{1, 2, \ldots, k\}$ for each $v \in V(G)$. Thus $\chi_\ell(G) \geq \chi(G)$ [4]. There is no constant bound on the amount by which $\chi_\ell(G)$ can exceed $\chi(G)$. For any $k \geq 1$, $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ is 2-colourable, but not k-choosable [4]. A greedy colouring algorithm shows that any graph G is $(\Delta(G)+1)$-choosable, providing a Δ-dependent upper bound for $\chi_\ell(G)$.

3.1 Choice Numbers

This section presents known results on choice numbers for certain classes of graphs.

Theorem 3.1.1 *[4] For all $m \geq 1$, $\chi_\ell(\theta_{2,2,2m}) = 2$.*

As $C_{2m+2} \subset \theta_{2,2,2m}$, it follows that $\chi_\ell(C_{2m+2}) = 2$ for all $m \geq 1$ [4]. The next theorem characterizes the 2-choosable graphs.

Theorem 3.1.2 *[4] A graph G is 2-choosable if and only if $core(G) \in \{K_1, C_{2m+2}, \theta_{2,2,2m} : m \geq 1\}$.*

Gutner has observed that the problem of deciding if a given graph is 3-choosable is NP-hard (see [2], [9]) so that no such characterization is likely for graphs with choice number three or greater.

Theorem 3.1.3 *[4] For all $k \geq 2$, $\chi_\ell(K_{2*k}) = k$.*

Theorem 3.1.4 *[4] For all $n \geq 1$, $\chi_\ell(K_n) = n$.*

Proof. $n = \chi(K_n) \leq \chi_\ell(K_n) \leq \Delta(K_n) + 1 = n$. ∎

Theorem 3.1.5 *For all $m \geq 1$, C_{2m+1} is 3-choosable.*

Proof. $3 = \chi(C_{2m+1}) \leq \chi_\ell(C_{2m+1}) \leq \Delta(C_{2m+1}) + 1 = 3$. ∎

The following is the choosability version of Brooks' Theorem.

Theorem 3.1.6 *[4] The choice number of any connected graph which is neither complete nor an odd cycle does not exceed its maximum degree.*

Theorem 3.1.7 *[2] The choice number of any m-degenerate graph is at most $m+1$.*

By Theorem 3.1.7, the 0-degenerate graphs or empty graphs have choice number at most 1, and the 1-degenerate graphs or forests have choice number at most 2. In fact, equality holds in these two cases. As every planar graph is 5-degenerate, Theorem 3.1.7 implies that the choice number of any planar graph does not exceed 6. In fact, the choice number of a planar graph never exceeds 5 [13]. Every bipartite planar graph is 3-degenerate, so by Theorem 3.1.7, every such graph has choice number at most 4. In fact, the choice number of any bipartite planar graph is at most 3 [1].

Theorem 3.1.8 *[2] If G is a graph with $\overline{d}(G) \geq d$, and k is an integer such that $d > 4\binom{k^4}{k}log(2\binom{k^4}{k})$, then $\chi_\ell(G) > k$.*

In a private communication to B. Toft and D. Hanson, N. Alon stated the following corollary which provides a non-Δ-dependent upper bound for the choice number of the union of graphs.

Corollary 3.1.9 *If G and H are graphs, where $\chi_\ell(G) = k$ and $\chi_\ell(H) = j$, then $\chi_\ell(G \cup H) \leq 4\binom{k^4}{k}log(2\binom{k^4}{k}) + 4\binom{j^4}{j}log(2\binom{j^4}{j}) + 1$.*

3.2 $(a : b)$-Choosability

Theorem 3.2.1 *For all positive integers n and k, K_n is $(nk : k)$-choosable.*

Theorem 3.2.2 *For all $m \geq 1$, $k \geq 1$, C_{2m+1} is $(3k : k)$-choosable.*

Proof. Let $C_{2m+1} = v_0, v_1, \ldots, v_{2m}, v_0$. Suppose that $3k$-sets $L(v_i)$ have been assigned to the vertices of C_{2m+1}. As the path $v_0, v_1, \ldots, v_{2m-1}$ is $(2k : k)$-choosable, a $(3k : k)$-choice $M(v_0), M(v_1), \ldots, M(v_{2m-1})$, can be determined for the path. As $|L(v_{2m}) - M(v_{2m-1}) - M(v_0)| \geq 3k - k - k = k$, the $(3k : k)$-choice for the path can be extended to a $(3k : k)$-choice for the cycle. ∎

The following is a generalization of Theorem 3.1.6.

Theorem 3.2.3 *[6] Any connected graph G which is neither complete nor an odd cycle is $(k\Delta(G) : k)$-choosable for all $k \geq 1$.*

The next lemma is useful in determining choice numbers for unions of graphs.

Lemma 3.2.4 *[4] If G and H such that H is $(d : a)$-choosable and G is $(a : b)$-choosable, then $G \cup H$ is $(d : b)$-choosable.*

Proof. Suppose d-sets $L(v)$ have been assigned to the vertices of $G \cup H$. Choose an a-subset $M(v) \subset L(v)$ for each vertex, such that H-adjacent vertices have disjoint a-subsets. Such a choice can be determined because H is $(d : a)$-choosable. Now choose a b-subset $M'(v) \subset M(v)$ for each vertex, so that G-adjacent vertices have disjoint b-subsets. This choice can be determined because G is $(a : b)$-choosable.

The result is a $(d : b)$-choice for $G \cup H$. If u and v are adjacent in G, then clearly $M'(u)$ and $M'(v)$ are disjoint b-subsets. If u and v are adjacent in H, then $M'(u)$ and $M'(v)$ are disjoint as they were chosen, respectively, from the disjoint a-subsets $M(u)$ and $M(v)$. ∎

Conjecture 3.2.5 *[4] If a graph is $(a:b)$-choosable, then it is $(ak:bk)$-choosable for all $k \geq 1$.*

It is interesting to note that the converse of Conjecture 3.2.5 is not true. For example, $K_{2,4}$ is $(4:2)$-choosable, but not $(2:1)$-choosable [4].

If Conjecture 3.2.5 does hold, then by Lemma 3.2.4, $\chi_\ell(G \cup H) \leq \chi_\ell(G)\chi_\ell(H)$ for all graphs G and H. By Theorems 3.2.1 and 3.2.2, the conjecture holds for any complete graph or odd cycle. Gutner [5] partially proved the conjecture by showing that any $(2:1)$-choosable graph is $(4:2)$-choosable. Gutner and Tarsi [6] have also shown that any even cycle is $(2k:k)$-choosable. Recently, Tuza and Voigt [15] proved that any 2-choosable graph is $(2k:k)$-choosable.

3.3 Planar Graphs

Thomassen [13] has proved the conjecture of Erdős, Rubin and Taylor [4], that every planar graph is 5-choosable. This is a best possible result. Erdős, Rubin and Taylor [4] conjectured that there exist planar graphs which are not 4-choosable, which Voigt [17] proved by exhibiting a non-4-choosable planar graph on 238 vertices. Mirzakhani [11] has since found a graph on 63 vertices which is planar and not 4-choosable.

Alon and Tarsi [1] have shown that every bipartite planar graph is 3-choosable. This is a best possible result as there exist bipartite planar graphs which are not 2-choosable. If $\frac{a}{b} < 3$, then the bipartite planar graph, $K_{2,(\frac{a}{b})^2}$ is not $(a:b)$-

choosable [4] For example, $K_{2,4}$ is not 2-choosable.

Every planar graph of girth at least 5 is 3-choosable, in fact it is possible to precolour any 5-cycle in the graph and extend this to a proper colouring of the whole graph [14]. This is a best possible result as not every planar graph of girth 4 is 3-choosable [18].

If every 3-colourable planar graph was 4-choosable (T. Jensen, see [14]), then because every bipartite planar graph is 3-choosable, and because every planar graph is 5-choosable, this would imply that every k-colourable planar graph is $(k+1)$-choosable. Mirzakhani has disproved this by finding a 3-colourable planar graph which is not 4-choosable [11].

3.4 Bipartite Graphs

Known results for choice numbers of bipartite graphs are summarized in the following theorems.

Theorem 3.4.1 *[4] If $p \geq 1$ and $q \geq p$, then $K_{p,q}$ is $p+1$-choosable.*

Corollary 3.4.2 *[4] If $1 \leq p \leq 2$ and $q \geq p$, then $\chi_\ell(K_{p,q}) \leq 3$.*

Theorem 3.4.3 *[4] $\chi_\ell(K_{3,q}) \leq 3$ if and only if $q \leq 26$.*

Theorem 3.4.4 *[8], [10] $\chi_\ell(K_{4,q}) \leq 3$ if and only if $q \leq 18$.*

Theorem 3.4.5 *[12] $\chi_\ell(K_{5,q}) \leq 3$ if and only if $q \leq 12$.*

The result of Theorem 4 1 1 is best possible Consider $G = K_{2k} - M = K_{2*k}$, where M is a perfect matching on the vertices of G By Theorem 3 1 3, $\chi_\ell(G) = \chi_\ell(K_{2*k}) = k$ and by Theorem 3 1 4, $\chi_\ell(G \cup M) = \chi_\ell(K_{2k}) = k + k = 2k$

Theorem 4 1 2 *If G is a k-choosable graph, then $\chi_\ell(G \cup K_n) \leq k + n - 1$*

Proof Suppose $(k + n - 1)$-sets have been assigned to the vertices of $G \cup K_n$ If G and K_n have no common vertex, then $\chi_\ell(G \cup K_n) = \max\{k, n\} \leq k + n - 1$ Otherwise there exists a vertex $v \in V(G) \cap V(K_n)$ Colour each vertex of $K_n - \{v\}$ with a distinct arbitrary colour from its set, deleting the colour from all other sets Properly colour the uncoloured vertices of G using colours from their resulting sets of cardinality at least k ∎

The result of Theorem 4 1 2 is best possible Consider $G = K_{n+1} - K_{1,n} = K_n$ Now, $\chi_\ell(K_{1,n}) = 2$ and $\chi_\ell(G \cup K_{1,n}) = \chi_\ell(K_{n+1}) = n + 2 - 1 = n + 1$

Theorem 4 1 3 *If G is a k-choosable graph, then $\chi_\ell(G \cup C_n) \leq 2k + 1$*

Proof Let $C_n = P_n \cup e$, where e is an edge of C_n By Theorem 3 2 3 and Lemma 3 2 4, $\chi_\ell(G \cup P_n) \leq 2k$, and by Theorem 4 1 1, $\chi_\ell((G \cup P_n) \cup e) = \chi_\ell(G \cup C_n) \leq 2k+1$ ∎

The result of Theorem 4 1 3 is best possible Consider $G = \overline{K_{2m+1}}$, for some $m \geq 1$ $\chi_\ell(G) = 1$ and by Theorem 3 1 5, $\chi_\ell(G \cup C_{2m+1}) = \chi_\ell(C_{2m+1}) = 2(1) + 1 = 3$

4.2 $(a : b)$-Choosability

Lemma 4.2.1 *If a and b are positive integers such that $\frac{a}{b} \geq 2$, and G is a graph such that core(G) is $(a : b)$-choosable, then G is $(a : b)$-choosable.*

Proof. The proof is by induction on the number of vertices of G which are not in core(G). If G is a graph such that core(G) is $(a : b)$-choosable and there are no vertices which are not in core(G), then clearly G is $(a : b)$-choosable.

For some $k \geq 1$, let $S_{k-1} = \{G :$ core(G) is $(a : b)$-choosable and G has $k-1$ vertices not in core$(G)\}$. Assume that all graphs in S_{k-1} are $(a : b)$-choosable. Let G be a graph such that core(G) is $(a : b)$-choosable and G has k vertices not in core(G). Let v be a vertex of degree one in G. As core$(G - \{v\})$ is $(a : b)$-choosable and $G - \{v\}$ has $k-1$ vertices not in core$(G - \{v\})$, $G - \{v\} \in S_{k-1}$. Thus an $(a : b)$-choice can be determined for $G - \{v\}$. There are $|L(v)| - b \geq 2b - b = b$ elements in $L(v)$ which are not in the b-subset of the vertex adjacent to v. Thus the $(a : b)$-choice for $G - \{v\}$ can be extended to an $(a : b)$-choice for G. ∎

Theorem 4.2.2 *If G is a graph with $\chi_\ell(G) = j \geq 2$, then G is $(j + (k-1)(\alpha(G)+1) : k)$-choosable for any $k \geq 1$, where $\alpha(G) =$max$\{\Delta($core$(G)), 1\}$.*

Proof. The proof is by induction on k. For $k = 1$, G is $(j + (1-1)(\alpha(G)+1) : 1)$-choosable or $(j : 1)$-choosable. Suppose that for some $k \geq 2$, G is $(j + (k-2)(\alpha(G) + 1) : (k-1))$-choosable, and that sets $L(v)$ of cardinality $j + (k-1)(\alpha(G)+1)$ have been assigned to the vertices of G. Then a $(j + (k-1)(\alpha(G)+1) : (k-1))$-choice can be determined for core(G). Now for each vertex $v \in$ core(G), delete the elements of the $(k-1)$-subsets of adjacent vertices from $L(v)$. At least $j + (k-1)(\alpha(G)+1) - (k-1)(\alpha(G)+1) = j$ unchosen elements remain in each set. As G is j-choosable, a kth element can be chosen for each vertex in core(G). By Lemma 4.2.1, since $\frac{j+(k-1)(\alpha(G)+1)}{k} \geq \frac{2+(k-1)(1+1)}{k} = \frac{2k}{k} = 2$, the $(j + (k-1)(\alpha(G)+1) : k)$-choice for core($G$) can be extended to include vertices not in core(G). ∎

If G and H are graphs where $\alpha(G)$ =max$\{\Delta(\text{core}(G)), 1\}$, $\chi_\ell(G) = j \geq 2$ and $\chi_\ell(H) = k$, then by Lemma 3.2.4 and Theorem 4.2.2, $\chi_\ell(G \cup H) \leq j + (k-1)(\alpha(G)+1)$. The following bounds can be derived from this result.

1. If G is a forest, core(G) $= mK_1$ for some m, so $\chi_\ell(G \cup H) \leq 2 + (k-1)(1+1) = 2k$.

2. If core(G) is an odd cycle, then $\chi_\ell(G \cup H) \leq 3 + (k-1)(2+1) = 3k$.

3. If G is K_n where $n \geq 2$, then $\chi_\ell(G \cup H) \leq n + (k-1)((n-1)+1) = nk$.

Note that these results are not necessarily best possible. Consider the special case in which G is a forest consisting of m copies of $K_{1,n}$, where $m < k$. By repeated application of Theorem 4.1.1, $\chi_\ell(G \cup H) \leq k + m < 2k$. If G is an odd cycle and $k > 1$, then by Theorem 4.1.3, $\chi_\ell(G \cup H) \leq 2k + 1 < 3k$.

Theorem 4.2.3 *If G is a graph where core(G) is neither complete nor an odd cycle then G is $(k\beta(G) : k)$-choosable for any $k \geq 1$, where $\beta(G)$ =max$\{\Delta(\text{core}(G)), 2\}$.*

Proof. By Theorem 3.2.3, core(G) is $(k\beta(G):k)$-choosable. As $\frac{k}{k}\beta(G) = \beta(G) \geq 2$, G is $(k\beta(G):k)$-choosable by Lemma 4.2.1. ∎

If G and H are graphs where core(G) is neither complete nor an odd cycle, $\beta(G) = \max\{\Delta(\text{core}(G)), 2\}$ and $\chi_\ell(H) = k$, then by Lemma 3.2.4 and Theorem 4.2.3, $\chi_\ell(G \cup H) \leq k\beta(G)$. The following bounds can be derived from this result.

1. If G is a forest, core(G) $= mK_1$ for some m. Thus $\chi_\ell(G \cup H) \leq 2k$.

2. If core(G) is an even cycle, then $\chi_\ell(G \cup H) \leq 2k$.

3. If $\chi_\ell(G) = 2$, then by Theorem 3.1.2, $\beta(G) \leq 3$. Thus $\chi_\ell(G \cup H) \leq 3k$.

Again, these bounds are not necessarily best possible.

4.3 2-Choosable Graphs

This section presents results for 2-choosable graphs, including algorithms which determine a $(2k:k)$-choice for the core of any 2-choosable graph, thereby proving that any 2-choosable graph is $(2k:k)$-choosable.

It is not difficult to determine a $(3k:k)$-choice for any 2-choosable graph.

Theorem 4.3.1 *If G is 2-choosable, then G is $(3k:k)$-choosable for any $k \geq 1$.*

Proof. If G is a graph where core(G) is $(3k:k)$-choosable, then by Lemma 4.2.1, as $\frac{3k}{k} = 3 > 2$, G is $(3k:k)$-choosable. Thus it suffices to show, using the characterization of 2-choosable graphs given by Theorem 3.1.2, that the core of any 2-choosable graph is $(3k : k)$-choosable. Clearly this is true for K_1.

For $m \geq 2$, let $C_{2m} = v_0, v_1, \ldots, v_{2m-1}, v_0$. Suppose that $3k$-sets $L(v_i)$ have been assigned to the vertices of C_{2m}. As the path $v_0, v_1, \ldots, v_{2m-2}$ is $(2k : k)$-choosable, a $(3k : k)$-choice $M(v_0), M(v_1), \ldots, M(v_{2m-2})$, can be determined for the path. As $|L(v_{2m-1}) - M(v_{2m-2}) - M(v_0)| \geq 3k - k - k = k$, the $(3k : k)$-choice for the path can be extended to a $(3k : k)$-choice for the cycle.

For some $m \geq 1$, let $\theta_{2,2,2m}$ consist of the paths v_0, v_{2m+2}, v_{2m} and v_0, v_{2m+1}, v_{2m} and $v_0, v_1, v_2, \ldots, v_{2m}$. Suppose that $3k$-sets have been assigned to $V(\theta_{2,2,2m})$. By the preceding argument, a $(3k : k)$-choice, $M(v_0), M(v_1), \ldots, M(v_{2m+1})$, can be determined for the $2m + 2$-cycle $v_0, v_1, \ldots, v_{2m+1}, v_0$. As $|L(v_{2m+2}) - M(v_{2m}) - M(v_0)| \geq 3k - k - k = k$, there exists a k-subset $M(v_{2m+2}) \subset L(v_{2m+2})$ which is disjoint from both $M(v_0)$ and $M(v_{2m})$. Therefore $\theta_{2,2,2m}$ is $(3k : k)$-choosable. ∎

In order to prove Conjecture 3.2.5 for 2-choosable graphs, it must be shown that all 2-choosable graphs are $(2k : k)$-choosable, for all $k \geq 1$. A simple proof of this fact has not yet been found. The following is a proof by induction on k for the case of the 4-cycle.

Theorem 4.3.2 *For any positive integer k, C_4 is $(2k : k)$-choosable.*

Proof. The proof is by induction on k. Let $C_4 = v_0, v_1, v_2, v_3, v_0$. Suppose that 2-sets $L(v_i)$ have been assigned to the vertices of C_4. Suppose there exist adjacent vertices

v_i and v_j such that $|L(v_i) \cap L(v_j)| \leq 1$. Choose an element $m(v_i) \in L(v_i) - L(v_j)$. For the path $C_4 - (v_i, v_j)$, $\{m(v_i)\}$ can be extended to a $(2:1)$-choice. As $m(v_i) \notin L(v_j)$, $m(v_i) \neq m(v_j)$, and therefore a $(2:1)$-choice has been determined for C_4.

Otherwise $|L(v_i) \cap L(v_j)| \geq 2$ for each pair v_i, v_j of adjacent vertices. Thus $L(v_i) = L(v_j)$ for each pair of adjacent vertices, and further, $L(v_i) = L(v_j)$ for each pair of vertices. Choose elements $m(v_0) = m(v_2) \in L(v_0)$, and $m(v_1) = m(v_3) = L(v_0) - \{m(v_0)\}$, determining a $(2:1)$-choice for C_4.

Suppose C_4 is $(2(k-1):(k-1))$-choosable for some $k \geq 2$, and $2k$-sets $L(v_i)$ have been assigned to the vertices of C_4. Suppose there exist adjacent vertices v_i and v_j such that $|L(v_i) \cap L(v_j)| \leq k$. Choose a k-subset $M(v_i) \subset L(v_i)$ such that $M(v_i) \cap L(v_j) = \emptyset$. For the path $C_4 - (v_i, v_j)$, $M(v_i)$ can be extended to a $(2k:k)$-choice. As $M(v_i) \cap L(v_j) = \emptyset$, $M(v_i) \cap M(v_j) = \emptyset$, and therefore a $(2k:k)$-choice has been determined for C_4.

Otherwise the sets of any two adjacent vertices intersect in at least $k+1$ elements, and in particular, $|L(v_0) \cap L(v_1)| \geq k+1$ and $|L(v_1) \cap L(v_2)| \geq k+1$. Now $|L(v_0) \cap L(v_1)| + |L(v_1) \cap L(v_2)| - |L(v_0) \cap L(v_1) \cap L(v_2)| \leq |L(v_1)| = 2k$. Therefore $|L(v_0) \cap L(v_1) \cap L(v_2)| \geq (k+1) + (k+1) - 2k = 2$, so certainly $|L(v_0) \cap L(v_2)| \geq 2$. Similarly, $|L(v_1) \cap L(v_3)| \geq 2$. Choose elements $m(v_0) = m(v_2) \in L(v_0) \cap L(v_2)$ and $m(v_1) = m(v_3) \in (L(v_1) \cap L(v_3)) - \{m(v_0)\}$, determining a $(2k:1)$-choice for C_4. Set $L'(v_i) = L(v_i) - \bigcup_{i=0}^{3}\{m(v_i)\}$ for each v_i. As $|\bigcup_{i=0}^{3}\{m(v_i)\}| = 2$, $|L'(v_i)| \geq 2k-2$ for each v_i. As C_4 is $(2(k-1):k-1)$-choosable, $(k-1)$-subsets $M'(v_i) \subset L'(v_i)$ can be chosen for each v_i so that adjacent vertices have disjoint $(k-1)$-subsets. Setting $M(v_i) = M'(v_i) \cup \{m(v_i)\}$ for each v_i yields a $(2k:k)$-choice for C_4. Thus C_4 is $(2k:k)$-choosable for all $k \geq 1$. ∎

Each of Algorithms 2 through 5 which follow acts on a particular class of 2-choosable graphs. If $2k$-sets have been assigned to the vertices of the input graph, then the appropriate algorithm can be used to output a $(2k:k)$-choice for that graph.

In Algorithms 2 through 5, an iteration involves removing edges of the input graph to obtain a spanning tree T with root r, choosing an element $m(r) \in L(r)$, and calling Algorithm 1, below. Algorithm 1 performs a breadth-first search on T, choosing and deleting an element from colour set $L(v)$ when reaching vertex v. If $d(r, v)$ is even, the chosen element is added to its choice set $M(v)$. The data structures for Algorithm 1 include R, which is a queue of vertices, and S, which is a set of vertices of T.

Algorithm 1

Input: tree T, root r, element $m(r)$, sets $M(v)$ and $L(v)$ for each $v \in V(T)$

Initialization: $S := \emptyset$, $i := 0$

BEGIN
    $R :=$ queue_insert(r, empty_queue);
    $M(r) := M(r) \cup m(r)$; (1.1)
    $L(r) := L(r) - m(r)$;
    WHILE not_empty(R) DO BEGIN
        $i := i + 1$;
        $v :=$ queue_remove(R);
        FOR each neighbour u of v DO

```
            IF u is not in R or S THEN BEGIN
                R := queue_insert(u,R);                      (1.2)
                IF m(v) ∈ L(u) THEN
                        m(u) := m(v)                         (1.3)
                ELSE
                        Choose m(u) ∈ L(u) − L(v);           (1.4)
                L(u) := L(u) − m(u);                         (1.5)
                IF i is even THEN
                        M(u) := M(u) ∪ m(u)                  (1.6)
            END;
        S := S ∪ {v}
    END
END.
```

Complexity $O(|V(T)|)$

That the time complexity of this algorithm is $O(|V(T)|)$ is clear from its breadth-first search nature, in which a fixed set of constant time operations is performed for each vertex of T.

For the proofs which follow, note that after each call to Algorithm 1, the cardinality of each colour set $L(v)$ has decreased by one, and for each vertex v such that $d(r,v)$ is even, the cardinality of $M(v)$ has increased by one.

Lemma 4.3.3 *Suppose there is a call to Algorithm 1 where $m(v) = x$ is added to $M(v)$. Then for any vertex u which is adjacent to v in T, x will not be in $L(u)$ on completion of the call, and x will not be added to $M(u)$ during the call.*

Proof. If $v = r$, then $d(r,v)$ is even. Otherwise, $m(v) = x$ and x is added to $M(v)$ at line (1.6), so $d(r,v)$ is even.

Suppose that $d(r, v) = d(r, u) - 1$. Then $m(v)$ is chosen before $m(u)$. If $m(v) = x \in L(u)$, then $m(u) = x$ at line (1.3). Now x is deleted from $L(u)$ at line (1.5), and as $d(r, u)$ is odd, u is not added to $M(u)$ at line (1.6).

Suppose that $d(r, v) - 1 = d(r, u)$. Then $m(u)$ is chosen before $m(v)$. Suppose that $m(u) = y$. Then y is deleted from $L(u)$ at line (1.5) and y is not added to $M(u)$ at line (1.6). Now $m(v) = x$ is chosen either at line (1.3) or line (1.4). If chosen at line (1.3), then $m(v) = m(u)$, so $y = x$. If chosen at line (1.4), then $y \neq x$ and $m(v) \in L(v) - L(u)$, so $x \notin L(u)$. ∎

Given an assignment of $2k$-sets to the vertices of a tree T, Algorithm 2 determines a $(2k : k)$-choice for T. The initialization includes choosing an arbitrary root $r \in V(T)$. Algorithm 1 is called k times by Algorithm 2, resulting in a choice of k elements for each vertex at an even distance from r. For each vertex at an odd distance from r, the remaining k elements in its colour set provide its choice set.

Algorithm 2

Input: k, T, sets $L(v)$ for each $v \in V(T)$

Initialization: Arbitrarily choose a root vertex, $r \in V(T)$, $i = 0$, $M(v) = \emptyset$ for each $v \in V(T)$

```
BEGIN
    FOR i = 1 TO k DO BEGIN
        Choose m(r) ∈ L(r);                    (2.1)
        Call Algorithm 1
    END;
```

FOR all $v \in V(T)$ such that $d(r, v)$ is odd DO
$M(v) = L(v)$ (2.2)
END

Output sets $M(v)$ for each $v \in V(T)$

Complexity $O(k|V(T)|)$

Theorem 4.3.4 *Any tree is $(2k : k)$-choosable for all $k \geq 1$*

Proof It suffices to show that the sets $M(v)$ for each $v \in V(T)$ are a $(2k : k)$-choice for T. As the root vertex r is fixed during Algorithm 2, there is a fixed set of vertices at an even distance from r, and a fixed set of vertices at an odd distance from r. So if v is a vertex such that $d(r, v)$ is odd, elements of $L(v)$ are added to $M(v)$ only at line (2.2)

After Algorithm 1 is called k times, $|L(v)| = k$ for each v, and $|M(v)| = k$ for each v such that $d(r, v)$ is even. By line (2.2), $|M(v)| = |L(v)| = k$ for all v such that $d(r, v)$ is odd. Therefore $M(v) \subset L(v)$ and $|M(v)| = k$ for each $v \in V(T)$.

There is a fixed set of vertices at an odd distance from r, so if v is a vertex such that $d(r, v)$ is odd, elements of $L(v)$ are added to $M(v)$ only at line (2.2). Thus if $x \notin L(v)$ at any stage of Algorithm 2, then x is not in $M(v)$ on completion of the algorithm. By Lemma 4.3.3, if x is added to $M(v)$ during a call to Algorithm 1, then for any vertex u which is adjacent to v, x will not be in $L(u)$ on completion of the call, and x will not be added to $M(u)$ during the call. Therefore for each $x \in M(v)$, where $d(r, v)$ is even, $x \notin M(u)$ for any u adjacent to v.

Thus the sets $M(v)$ are a $(2k : k)$-choice for T. ∎

For some $m \geq 2$, let $C_{2m} = v_0, v_1, \ldots, v_{2m-1}, v_0$. Given an assignment of $2k$-sets to the vertices of C_{2m}, Algorithm 3 determines a $(2k : k)$-choice for the cycle. The algorithm searches for a pair u, v of adjacent vertices which have distinct colour sets. If such a pair exists, then $T := C_{2m} - (u, v)$ and Algorithm 1 is called. This search is iterated $i \leq k$ times, while such a pair of vertices exists, resulting in a choice set of i elements for each even-indexed vertex. If $i = k$, then for each vertex at an odd distance from r, the remaining k elements in its colour set provide its choice set. If $i < k$, then for each vertex at an even distance from r, an additional $k - i$ elements for its choice set are chosen from $L(v_0)$. For each vertex at an odd distance from r, the remaining $2k - i$ elements of its colour set, excluding the $k - i$ elements chosen above, provide a choice set of k elements.

Algorithm 3

Input: k, sets $L(v)$ for each $v \in V(C_{2m})$

Initialization: $i := 0$, $M(v) := \emptyset$ for each $v \in V(C_{2m})$

BEGIN
  WHILE [$i < k$] AND [there exists an even j, $0 \leq j < 2m - 1$ such that
    either $L(v_j) \neq L(v_{j+1})$ OR $L(v_j) \neq L(v_{j-1})$] DO BEGIN
      IF there exists an even j such that $L(v_j) \neq L(v_{j+1})$ DO BEGIN
        $T := C_{2m} - (v_j, v_{j+1})$,
        $r := v_j$,
        $m(r) := x$ where $x \in L(v_j) - L(v_{j+1})$ (3.1)
      END

ELSE IF there exists an even j such that $L(v_j) \neq L(v_{j-1})$ DO BEGIN
$T := C_{2m} - (v_j, v_{j-1})$,
$r := v_j$,
$m(r) := x$ where $x \in L(v_j) - L(v_{j-1})$ (3.2)
END,
Call Algorithm 1; (3.3)
$i := i + 1$
END,
IF $i = k$ THEN
$M(v_j) := L(v_j)$ for each odd j (3.4)
ELSE BEGIN
Choose $M'(v_0) \subset L(v_0)$ where $|M'(v_0)| = k - i$, (3.5)
For each even j, $M(v_j) := M(v_j) \cup M'(v_0)$, (3.6)
For each odd j, $M(v_j) := L(v_1) - M'(v_0)$ (3.7)
END
END.

Output: sets $M(v)$ for each $v \in V(C_{2m})$

Complexity: $O(mk)$

Algorithm 3 is iterated at most k times, where each iteration consists of a fixed set of operations, including at most one call to Algorithm 1. As the searches take time $O(|V(T)|) = O(m)$, the other operations take constant time, and Algorithm 1 always acts on a tree T where $|V(T)| = 2m$, the time complexity of Algorithm 3 is $O(mk)$.

Theorem 4.3.5 *For all $k \geq 1$, C_{2m} is $(2k : k)$-choosable.*

Proof. It suffices to show that the sets $M(v)$ for each $v \in V(C_{2m})$ are a $(2k : k)$-choice for C_{2m}. For any call to Algorithm 1, $r = v_j$ where j is even, so the vertices at an even distance from r are v_j such that j is even, and those at an odd distance from r are v_j such that j is odd. Suppose that $v_j \in V(C_{2m})$ where j is odd. Elements of $L(v_j)$ are added to $M(v_j)$ only at line (3.4) or (3.7).

Suppose that Algorithm 3 calls Algorithm 1 exactly i times. Then $|L(v_j)| = 2k - i$ for all v_j, and $|M(v_j)| = i$ for all even j. If $i = k$, then $|L(v_j)| = k$ for each v_j, and $|M(v_j)| = k$ for each even j. By line (3.4), $|M(v_j)| = |L(v_j)| = k$ for each odd j. Thus $M(v_j) \subset L(v_j)$ and $|M(v_j)| = k$ for each $v_j \in V(C_{2m})$.

If $i < k$, then for all even j, $L(v_j) = L(v_{j+1})$ and $L(v_j) = L(v_{j-1})$. Thus all vertices have the same set. At line (3.6), $|M(v_j)| = |M(v_j) \cup M'(v_0)| = $ i + (k - i) = k for each even j. As $M'(v_0) \subset L(v_0) = L(v_j)$, $M(v_j) \subset L(v_j)$ for each even j. At line (3.7), $|M(v_j)| = |L(v_1) - M'(v_0)| = 2k - i - (k - i) = k$ for each odd j. As $L(v_1) - M'(v_0) \subseteq L(v_1) = L(v_j)$, $M(v_j) \subset L(v_j)$ for all odd j.

The vertices at an odd distance from r are v_j where j is odd, so if $v_j \in V(C_{2m})$ and j is odd, then elements of $L(v_j)$ are added to $M(v_j)$ only at line (3.4) or (3.7). Thus if at any stage of Algorithm 3, $x \notin L(v_j)$, then x is not in $M(v_j)$ on completion of the algorithm. Suppose that $m(v) = x$ is added to $M(v)$ during a call to Algorithm 1 which acts on T. Then by Lemma 4.3.3, for any u which is adjacent to v in T, x will not be in $M(u)$ on completion of Algorithm 3.

Consider a call to Algorithm 1 where $T = C_{2m} - (v_j, v_{j+1})$, and j is even. Then by line (3.1), $x \in L(v_j) - L(v_{j+1})$. At line (1.1), x will be added to $M(v_j)$. As $x \notin L(v_{j+1})$ and $j + 1$ is odd, x will not be in $M(v_{j+1})$ on completion of Algorithm 3.

Similarly if a call is made when $T = C_{2m} - (v_j, v_{j-1})$

Suppose $x \in M(v_j)$ was chosen at line (3.5). Then at line (3.6), $M(v_j) = M(v_j) \cup M'(v_0)$, where $x \in M'(v_0)$. For each odd j, $M(v_j) = L(v_1) - M'(v_0)$ at line (3.7), so $x \notin M(v_j)$.

Therefore $M(u) \cap M(v) = \emptyset$ for each pair of adjacent vertices, and the sets $M(v)$ are a $(2k : k)$ choice for C_{2m}. ∎

Let $\theta_{2,2,2}$ consist of the paths v_0, v_4, v_2 and v_0, v_3, v_2 and v_0, v_1, v_2. Given an assignment of $2k$-sets to the vertices of $\theta_{2,2,2}$, Algorithm 4 determines a $(2k : k)$-choice for the graph. The algorithm searches for a pair of vertices, v_s, v_t, adjacent to v_0 (or v_2) such that $L(v_0)$ (or $L(v_2)$) contains an element which is not in $L(v_s) \cap L(v_t)$. If such vertices exist, $T = \theta_{2,2,2} - (v_0, v_s) - (v_0, v_t)$ (or $T = \theta_{2,2,2} - (v_2, v_s) - (v_2, v_t)$) and Algorithm 1 is called. This search is iterated $i \leq k$ times, while such a pair of vertices exists, resulting in choice sets of i elements for v_0 and v_2. If $i = k$, then for each of v_1, v_3 and v_4, the remaining k elements in its colour set provide its choice set. If $i < k$, then for v_0 and v_2, an additional $k - i$ elements for its choice set are chosen from $L(v_0) \cap L(v_2)$. For each of v_1, v_3 and v_4, the remaining $2k - i$ elements of its colour set, excluding the $k - i$ elements chosen above, provide a choice set of k elements.

Algorithm 4

Input: k, sets $L(v)$ for each $v \in V(\theta_{2,2,2})$

Initialization $i := 0$, $M(v) := \emptyset$ for each $v \in V(\theta_{2,2,2})$

BEGIN
  WHILE $[i < k]$ AND [there exists $x \in L(v_0) - L(v_s) - L(v_t)$ OR there exists $x \in L(v_2) - L(v_s) - L(v_t)$ where $s, t \in \{1,3,4\}$, $s \neq t$] DO BEGIN
    IF there exists $x \in L(v_0) - L(v_s) - L(v_t)$, where $s, t \in \{1,3,4\}$, $s \neq t$ THEN BEGIN
      $T := \theta_{2,2,2} - (v_0, v_s) - (v_0, v_t)$, (4.1)
      $r := v_0$,
      $m(r) := x$ (4.2)
    END
    ELSE IF there exists $x \in L(v_2) - L(v_s) - L(v_t)$, where $s, t \in \{1,3,4\}$, $s \neq t$ THEN BEGIN
      $T := \theta_{2,2,2} - (v_2, v_s) - (v_2, v_t)$,
      $r := v_2$,
      $m(r) := x$
    END;
    Call Algorithm 1; (4.3)
    Set $i := i + 1$
  END;
  IF $i = k$ THEN
    $M(v_j) := L(v_j)$ for $j = 1, 3, 4$ (4.4)
  ELSE BEGIN
    Choose $M'(v_0) = M'(v_2) \subset L(v_0) \cap L(v_2)$ where $|M'(v_0)| = k - i$; (4.5)
    Choose $M'(v_j) \subseteq L(v_j) - M'(v_0)$, where $|M'(v_j)| = k$, for $j = 1, 3, 4$; (4.6)
    $M(v_j) := M(v_j) \cup M'(v_j)$ for each vertex v_j (4.7)
  END
END.

Output sets $M(v)$ for each $v \in V(\theta_{2,2,2})$

Complexity $O(k)$

Algorithm 4 is iterated at most k times, where each iteration consists of a fixed set of operations, including at most one call to Algorithm 1. As the searches take time $O(|V(T)|) = O(1)$, the other operations take constant time, and Algorithm 1 always acts on a tree T where $|V(T)| = 5$, the time complexity of Algorithm 4 is $O(k)$.

Theorem 4.3.6 *For all $k \geq 1$, $\theta_{2,2,2}$ is $(2k:k)$-choosable.*

Proof. It suffices to show that the sets $M(v)$ for each $v \in V(\theta_{2,2,2})$ are a $(2k:k)$-choice for $\theta_{2,2,2}$. For any call to Algorithm 1, $r = v_0$ or $r = v_2$, so the vertices at an even distance from r are v_0 and v_2, and those at an odd distance from r are v_1, v_3 and v_4. For $j = 1$, 3 and 4, elements of $L(v_j)$ are added to $M(v_j)$ only at line (4.4) or (4.7).

Suppose that Algorithm 4 calls Algorithm 1 exactly i times. Then $|L(v_j)| = i$ for all v_j, and $|M(v_j)| = i$ for $j = 0$ and $j = 2$. If $i = k$, then $|L(v_j)| = k$ for each v_j, and $|M(v_j)| = k$ for $j = 0$ and $j = 2$. At line (4.4), $|M(v_j)| = |L(v_j)| = k$ for $j = 1$, 3 and 4. Thus $M(v_j) \subset L(v_j)$ and $|M(v_j)| = k$ for each $v_j \in V(\theta_{2,2,2})$.

If $i < k$, it must be shown that at line (4.5), $|L(v_0) \cap L(v_2)| \geq k - i$. Each element of $L(v_0)$ occurs in at least two of $L(v_1)$, $L(v_3)$, $L(v_4)$ and each element of $L(v_2)$ occurs in at least two of $L(v_1)$, $L(v_3)$, $L(v_4)$. Therefore $2|L(v_0)| + 2|L(v_2)| - 2|L(v_0) \cap L(v_2)| \leq |L(v_1)| + |L(v_3)| + |L(v_4)|$. As $|L(v_j)| = 2k - i$ for each v_j, $2(2k - i) + 2(2k - i) - 2|L(v_0) \cap L(v_2)| \leq (2k - i) + (2k - i) + (2k - i)$. Therefore $|L(v_0) \cap L(v_2)| \geq k - \frac{i}{2} > k - i$.

At line (4.7), $|M(v_j)| = |M(v_j) \cup M'(v_j)| = i + (k - i) = k$ for $j = 0$ and $j = 2$. As $M'(v_j) \subset L(v_0) \cap L(v_2)$, $M(v_j) \subset L(v_j)$ for each even j. At line (4.7), $|M(v_j)| =$

$|M(v_j) \cup M'(v_j)| = |M'(v_j)| = k$ for $j = 1, 3, 4$. As $M'(v_j) \subseteq L(v_j) - M'(v_0)$, $M(v_j) \subset L(v_j)$ for $j = 1, 3, 4$.

The vertices at an odd distance from r are v_1, v_3 and v_4. For $j = 1, 3$ or 4, elements of $L(v_j)$ are added to $M(v_j)$ only at line (4.4) or (4.7). Thus if at any stage of Algorithm 4, $x \notin L(v_j)$, then x is not in $M(v_j)$ on completion of the algorithm. Suppose that $m(v) = x$ is added to $M(v)$ during a call to Algorithm 1. Then by Lemma 4.3.3, for any u which is adjacent to v in T, x will not be in $M(u)$ on completion of Algorithm 4.

Consider a call to Algorithm 1 where $T = \theta_{2,2,2} - (v_0, v_s) - (v_0, v_t)$. By line (4.2), $m(v_0) = x$, and x will be added to $M(v_0)$ at line (1.1). As $x \in L(v_0) - L(v_s) - L(v_t)$, $x \notin L(v_s)$ and $x \notin L(v_t)$, so x will not be in $M(v_s)$ or in $M(v_t)$ on completion of Algorithm 4. Similarly if a call is made when $T = \theta_{2,2,2} - (v_2, v_s) - (v_2, v_t)$.

Suppose $x \in M(v_0)$ was chosen at line (4.5). Then at line (4.7), $M(v_0) = M(v_0) \cup M'(v_0)$, where $x \in M'(v_0)$. For $j = 1$, 3 and 4, $M(v_j) = M(v_j) \cup M'(v_j) = M'(v_j)$, where $M'(v_j) \subseteq L(v_j) - M'(v_0)$. Thus $x \notin M'(v_j)$, so $x \notin M(v_j)$.

Thus $M(v) \cap M(u) = \emptyset$ for each pair of adjacent vertices, and the sets $M(v)$ are a $(2k : k)$ choice for $\theta_{2,2,2}$. ∎

For some $m \geq 1$, let $\theta_{2,2,2m}$ consist of the paths v_0, v_{2m+2}, v_{2m} and v_0, v_{2m+1}, v_{2m} and $v_0, v_1, v_2, \ldots, v_{2m}$. Given an assignment of $2k$-sets to the vertices of $\theta_{2,2,2m}$, Algorithm 5 determines a $(2k : k)$-choice for the graph. For each iteration, a rooted subtree T of $\theta_{2,2,2m}$ is chosen and Algorithm 1 is called. If the root r is such that $d(r, v_0)$ is even, then i is incremented after the call. Otherwise, i' is incremented. This iteration occurs $i + i'$ times, while $i \leq k$, $i' \leq k$, and certain other conditions are satisfied, resulting in choice sets of i elements for vertices v_j such that $j = 0, 2, \ldots, 2m$ and choice sets of i' elements for v_j such that $j = 1, 3, \ldots, 2m+1, 2m+2$. If $i = k$, then for each of $v_1, v_3, \ldots, v_{2m+1}, v_{2m+2}$, the remaining $k - i'$ elements in its colour set provide the remainder of its choice set. If $i' = k$, then for each of $v_0, v_2, \ldots, v_{2m}$, the remaining $k - i$ elements in its colour set provide the rest of its choice set. If $i < k$ and $i' < k$, then ordered pairs of elements $\{(x_{2m+1}, x_{2m+2}) : x_{2m+1} \in L(v_{2m+1}), x_{2m+2} \in L(v_{2m+2})\}$ are formed, reducing the remainder of the $(2k : k)$-choice for $\theta_{2,2,2m}$ to a $(2k : k)$-choice for C_{2m+2}.

Algorithm 5

Input: k, m, sets $L(v)$ for each $v \in V(\theta_{2,2,2m})$

Initialization: $i := 0$, $i' := 0$, $M(v) := \emptyset$ for each $v \in V(\theta_{2,2,2m})$, $L'(v_{2m+1}) := \emptyset$

BEGIN
WHILE $[i < k]$ AND $[i' < k]$ AND [there exists $x \in L(v_0) - L(v_s) - L(v_t)$ such that $s, t \in \{1, 2m+1, 2m+2\}$, $s \neq t$ OR there exists $x \in L(v_{2m}) - L(v_s) - L(v_t)$ such that $s, t \in \{2m-1, 2m+1, 2m+2\}$, $s \neq t$ OR there exists an element x which is contained in exactly three of $L(v_0)$, $L(v_{2m})$, $L(v_{2m+1})$, $L(v_{2m+2})$] DO BEGIN
IF there exists $x \in L(v_0) - L(v_s) - L(v_t)$, $s, t \in \{1, 2m+1, 2m+2\}$, $s \neq t$ THEN BEGIN
$T := \theta_{2,2,2m} - (v_0, v_s) - (v_0, v_t)$, (5.1)
$r := v_0$,
$m(r) := x$ (5.2)
END
ELSE IF there exists $x \in L(v_{2m}) - L(v_s) - L(v_t)$, where $s, t \in$

$\{2m-1, 2m+1, 2m+2\}$, $s \neq t$ THEN BEGIN

$T := \theta_{2,2,2m} - (v_{2m}, v_s) - (v_{2m}, v_t)$, (5.3)

$r := v_{2m}$,

$m(r) := x$

END

ELSE IF there exists $x \in (L(v_0) \cap L(v_{2m+1}) \cap L(v_{2m+2})) - L(v_{2m})$ THEN BEGIN

$T := \theta_{2,2,2m} - (v_{2m}, v_{2m+1}) - (v_{2m}, v_{2m+2})$, (5.4)

$r := v_{2m+1}$,

$m(r) := x$ (5.5)

END

ELSE IF there exists $x \in (L(v_{2m}) \cap L(v_{2m+1}) \cap L(v_{2m+2})) - L(v_0)$ THEN BEGIN

$T := \theta_{2,2,2m} - (v_0, v_{2m+1}) - (v_0, v_{2m+2})$ (5.6)

$r := v_{2m+1}$,

$m(r) := x$

END

ELSE IF there exists $x \in (L(v_0) \cap L(v_{2m}) \cap L(v_{2m+1})) - L(v_{2m+2})$ THEN BEGIN

Find the minimum $t \geq 0$ such that $x \in L(v_j)$ for $j = t, t+1, \ldots, 2m$ (5.7)

IF t is even THEN BEGIN

$T := \theta_{2,2,2m} - (v_t, v_{t-1}) - (v_{2m}, v_{2m+2})$ (5.8)

$r := v_{2m}$,

$m(r) := x$ (5.9)

END

ELSE BEGIN

$m(v_{2m+2}) := y \in L(v_{2m+2}) - L(v_0)$,

$C := \theta_{2,2,2m} - (v_{2m}, v_{2m+2}) - (v_0, v_{2m+2})$,

IF $y \notin L(v_{2m})$ THEN BEGIN

$T := C - (v_t, v_{t-1})$, (5.10)

$r := v_t$,

$m(r) := x$ (5.11)

$M(v_{2m+2}) := M(v_{2m+2}) \cup \{y\}$

END

END

ELSE BEGIN

Find the minimum $u > 0$ so that $y \in L(v_j)$ for $j = u$,

$u+1, \ldots, 2m$ (5.12)

$T := C - (v_u, v_{u-1})$ (5.13)

$r := v_u$,

$m(r) := y$ (5.14)

IF u is odd THEN

$M(v_{2m+2}) := M(v_{2m+2}) \cup \{y\}$ (5.15)

END;

$L(v_{2m+2}) := L(v_{2m+2}) - \{y\}$, (5.16)

END;

ELSE IF there exists $x \in (L(v_0) \cap L(v_{2m}) \cap L(v_{2m+2})) - L(v_{2m+1})$ THEN BEGIN

Find the minimum $t \geq 0$ such that $x \in L(v_j)$ for $j = t$, $t+1, \ldots, 2m$

IF t is even THEN BEGIN

$T := \theta_{2,2,2m} - (v_t, v_{t-1}) - (v_{2m}, v_{2m+1})$ (5.17)

$r := v_{2m}$,

$m(r) := x$

END

ELSE BEGIN

$m(v_{2m+1}) := y \in L(v_{2m+1}) - L(v_0)$,

$C := \theta_{2,2,2m} - (v_{2m}, v_{2m+1}) - (v_0, v_{2m+1})$,

IF $y \notin L(v_{2m})$ THEN BEGIN

$T := C - (v_t, v_{t-1})$, (5.18)

$r := v_t$,

$m(r) := x$

$M(v_{2m+1}) := M(v_{2m+1}) \cup \{y\}$

END

END

ELSE BEGIN

Find the minimum $u > 0$ so that $y \in L(v_j)$ for $j = u$, $u+1, \ldots, 2m$

$T := C - (v_u, v_{u-1})$ (5.19)

$r := v_u$;

$m(r) := y$

IF u is odd THEN

$M(v_{2m+1}) := M(v_{2m+1}) \cup \{y\}$

END,

$L(v_{2m+1}) := L(v_{2m+1}) - \{y\}$, (5.20)

```
        END;
        Call Algorithm 1;                                                  (5.21)
        IF r ∈ {v_0, v_2, ..., v_2m} THEN
            i := i + 1                                                     (5.22)
        ELSE
            i' := i' + 1;
    IF i = k THEN
        M(v_j) := M(v_j) ∪ L(v_j) for j = 2m + 2 and each odd j            (5.23)
    ELSE IF i' = k THEN
        M(v_j) := M(v_j) ∪ L(v_j) for each even j, 0 ≤ j ≤ 2m              (5.24)
    ELSE BEGIN {pairing}
        FOR all x ∈ L(v_2m+1) DO
            IF x ∈ L(v_2m+2) THEN BEGIN
                L'(v_2m+1) := L'(v_2m+1) ∪ {(x, x)}                        (5.25)
                L(v_2m+1) := L(v_2m+1) − {x}
                L(v_2m+2) := L(v_2m+2) − {x}
            END
            ELSE IF x ∈ L(v_0) THEN BEGIN
                L'(v_2m+1) := L'(v_2m+1) ∪ {(x, y)}, where y ∈ L(v_2m+2)−
                    L(v_0)                                                 (5.26)
                L(v_2m+1) := L(v_2m+1) − {x}
                L(v_2m+2) := L(v_2m+2) − {y}
            END
            ELSE BEGIN
                L'(v_2m+1) := L'(v_2m+1) ∪ {(x, y)}, where y ∈ L(v_2m+2)−
                    L(v_2m)                                                (5.27)
                L(v_2m+1) := L(v_2m+1) − {x}
                L(v_2m+2) := L(v_2m+2) − {y}
            END;
        Call Algorithm 6;
        FOR each (x, y) ∈ M(v_2m+1) DO BEGIN
            M(v_2m+1) := M(v_2m+1) ∪ {x};
            M(v_2m+2) := M(v_2m+2) ∪ {y}
        END
    END
END.
```

Output: sets $M(v)$ for each $v \in V(\theta_{2,2,2m})$

Complexity: $O(mk)$

Algorithm 5 is iterated at most $2k$ times, where each iteration consists of a fixed set of operations, including at most one call to Algorithm 1, either directly from Algorithm 5, or indirectly via Algorithm 6. As the searches take time $O(|V(T)|) = O(m)$, the other operations take constant time, and Algorithm 1 always acts on a tree T where $|V(T)| = 2m + 2$, the time complexity of Algorithm 5 is $O(mk)$.

Algorithm 6

Input: k, sets $L(v)$ for each $v \in V(C_{2m+2})$

BEGIN
  WHILE [$i < k$] AND [there exists an even j, $0 \leq j < 2m + 1$ such that either $L(v_j) \neq L(v_{j+1})$ OR $L(v_j) \neq L(v_{j-1})$] DO BEGIN
    IF there exists an even j such that $L(v_j) \neq L(v_{j+1})$ DO BEGIN
      $T := C_{2m+2} - (v_j, v_{j+1})$;
      $r := v_j$;
      $m(r) := x$ where $x \in L(v_j) - L(v_{j+1})$ (6.1)
    END
    ELSE IF there exists an even j such that $L(v_j) \neq L(v_{j-1})$ DO BEGIN
      $T := C_{2m+2} - (v_j, v_{j-1})$;
      $r := v_j$;
      $m(r) := x$ where $x \in L(v_j) - L(v_{j-1})$ (6.2)
    END;
    Call Algorithm 1; (6.3)
    $i := i + 1$
  END;
  IF $i = k$ THEN BEGIN
    $M(v_j) := M(v_j) \cup L(v_j)$ for each odd $j < 2m + 1$; (6.4)
    $M(v_{2m+1}) := L(v_{2m+1})$
  ELSE BEGIN
    Choose $M'(v_0) \subset L(v_0)$ where $|M'(v_0)| = k - i$; (6.5)
    $M'(v_1) := L(v_1) - M'(v_0)$;

For each even j, $M(v_j) := M(v_j) \cup M'(v_0)$, (6.6)
For each odd $j < 2m+1$, $M(v_j) := M(v_1) - M'(v_1)$, (6.7)
$M(v_{2m+1}) \subseteq L(v_{2m+1}) - M'(v_0)$, where $|M(v_{2m+1}| = k - i'$ (6.8)
END
END

Note that Algorithm 6 is almost identical to Algorithm 3. Algorithm 6 acts on a $2m+2$-cycle, rather than a $2m$-cycle, and no initialization is required for Algorithm 6.

As the elements of all other sets are singletons, while the elements of $L(v_{2m+1})$ are ordered pairs, some definitions are required. Define $L(v_j) = L(v_{2m+1})$ if and only if for each $x \in L(v_j)$, there exists $(y, z) \in L(v_{2m+1})$ such that either $y = x$ or $z = x$ (or both). Define $L(v_j) - L(v_{2m+1}) = \{x : x \in L(v_j)$ and for all $(y, z) \in L(v_{2m+1}),\ y \neq x$ and $z \neq x\}$. Define $L(v_{2m+1}) - M'(v_0) = \{(y, z) : (y, z) \in L(v_{2m+1}),\ y \notin M'(v_0)$ and $z \notin M'(v_0)\}$. In Algorithm 1, define $m(v_{2m+1}) \in L(u)$ if and only if $m(v_{2m+1}) = (y, z)$, where $y \in L(u)$ or $z \in L(u)$ (or both).

Lemma 4.3.7 *If the elements of $L(v_{2m+1})$ and $L(v_{2m+2})$ are paired in Algorithm 5, then every $x \in L(v_0) \cup L(v_{2m}) \cup L(v_{2m+1}) \cup L(v_{2m+2})$ is contained in exactly two or in exactly four of $L(v_0)$, $L(v_{2m})$, $L(v_{2m+1})$, $L(v_{2m+2})$.*

Proof If the elements of $L(v_{2m+1})$ and $L(v_{2m+2})$ are paired in Algorithm 5, then the following conditions are satisfied.

1. If $x \in L(v_0)$ then x is in at least two of $L(v_1)$, $L(v_{2m+1})$, $L(v_{2m+2})$;

2. If $x \in L(v_{2m})$ then x is in at least two of $L(v_{2m-1})$, $L(v_{2m+1})$, $L(v_{2m+2})$;

3. If $x \in L(v_0) \cap L(v_{2m+1}) \cap L(v_{2m+2})$, then $x \in L(v_{2m})$,

4. If $x \in L(v_{2m}) \cap L(v_{2m+1}) \cap L(v_{2m+2})$, then $x \in L(v_0)$,

5. If $x \in L(v_0) \cap L(v_{2m}) \cap L(v_{2m+1})$, then $x \in L(v_{2m+2})$,

6. If $x \in L(v_0) \cap L(v_{2m}) \cap L(v_{2m+2})$, then $x \in L(v_{2m+1})$

By conditions 3 through 6, if x is in any three of $L(v_0)$, $L(v_{2m})$, $L(v_{2m+1})$, $L(v_{2m+2})$, then x is in all four of them. By conditions 1 and 2, if $x \in L(v_0)$ or $x \in L(v_{2m})$, then $x \in L(v_{2m+1})$ or $x \in L(v_{2m+2})$. Thus it suffices to show that if $x \in L(v_{2m+1})$ or $x \in L(v_{2m+2})$, then $x \in L(v_0)$, or $x \in L(v_{2m})$

$$\begin{aligned}
|L(v_0)| &= |L(v_0) \cap L(v_{2m+1})| + |L(v_0) \cap L(v_{2m+2})| - \\
&\quad |L(v_0) \cap L(v_{2m+1}) \cap L(v_{2m+2}) \cap L(v_{2m})| \\
|L(v_{2m})| &= |L(v_{2m}) \cap L(v_{2m+1})| + |L(v_{2m}) \cap L(v_{2m+2})| - \\
&\quad |L(v_{2m}) \cap L(v_{2m+1}) \cap L(v_{2m+2}) \cap L(v_0)| \\
|L(v_{2m+1})| &= |L(v_{2m+1}) - L(v_0) - L(v_{2m})| + |L(v_{2m+1}) \cap L(v_0)| + \\
&\quad |L(v_{2m+1}) \cap L(v_{2m})| - |L(v_{2m+1}) \cap L(v_0) \cap L(v_{2m}) \cap L(v_{2m+2})| \\
|L(v_{2m+2})| &= |L(v_{2m+2}) - L(v_0) - L(v_{2m})| + |L(v_{2m+2}) \cap L(v_0)| + \\
&\quad |L(v_{2m+2}) \cap L(v_{2m})| - |L(v_{2m+2}) \cap L(v_0) \cap L(v_{2m}) \cap L(v_{2m+1})|
\end{aligned}$$

As $|L(v_0)| = |L(v_{2m})| = |L(v_{2m+1})| = |L(v_{2m+2})|$, $0 = |L(v_{2m+1})| + |L(v_{2m+2})| - |L(v_0)| - |L(v_{2m})| = |L(v_{2m+1}) - L(v_0) - L(v_{2m})| + |L(v_{2m+2}) - L(v_0) - L(v_{2m})|$

Therefore $|L(v_{2m+1}) - L(v_0) - L(v_{2m})| = 0$ and $|L(v_{2m+2}) - L(v_0) - L(v_{2m})| = 0$, so any $x \in L(v_0) \cup L(v_{2m}) \cup L(v_{2m+1}) \cup L(v_{2m+2})$ is contained in exactly two or in exactly four of $L(v_0)$, $L(v_{2m})$, $L(v_{2m+1})$, $L(v_{2m+2})$. ∎

Lemma 4.3.8 *If the elements of $L(v_{2m+1})$ and $L(v_{2m+2})$ are paired in Algorithm 5, then $|L(v_0) \cap L(v_{2m+1})| = |L(v_{2m}) \cap L(v_{2m+2})|$ and $|L(v_0) \cap L(v_{2m+2})| = |L(v_{2m}) \cap L(v_{2m+1})|$.*

Proof. $0 = |L(v_{2m+1})| + |L(v_0)| - |L(v_{2m+2})| - |L(v_{2m})| = 2|L(v_0) \cap L(v_{2m+1})| - 2|L(v_{2m}) \cap L(v_{2m+2})|$. Thus $|L(v_0) \cap L(v_{2m+1})| = |L(v_{2m}) \cap L(v_{2m+2})|$. Similarly for $|L(v_0) \cap L(v_{2m+2})| = |L(v_{2m}) \cap L(v_{2m+1})|$. ∎

Lemma 4.3.9 *The elements of $L(v_{2m+1})$ can be paired with elements of $L(v_{2m+2})$ in Algorithm 5.*

Proof. Suppose the elements of $L(v_{2m+1})$ and $L(v_{2m+2})$ are paired in Algorithm 5. If $x \in L(v_{2m+1}) \cap L(v_{2m+2})$ then (x, x) is paired at line (5.25). Otherwise $x \in L(v_{2m+1})$ and $x \notin L(v_{2m+2})$. If $x \in L(v_{2m+1}) \cap L(v_0)$ then by Lemma 4.3.8, there exists $y \in L(v_{2m+2}) - L(v_0)$ and (x, y) is paired at line (5.26). Otherwise $x \in L(v_{2m+1})$, $x \notin L(v_{2m+2})$ and $x \notin L(v_0)$, so $x \in L(v_{2m+1}) \cap L(v_{2m})$. By Lemma 4.3.8, there exists $y \in L(v_{2m+2}) - L(v_{2m})$ and (x, y) is paired at line (5.27). Thus the elements of $L(v_{2m+1})$ can be paired with the elements of $L(v_{2m+2})$. ∎

Lemma 4.3.10 *Suppose there is a call to Algorithm 1 where $m(r) = x$. If there exists a path $r = u_1, u_2, \ldots, u_\ell$ in T where $x \in u_i$ for $i = 1$ to ℓ, then $m(u_\ell) = x$ during the call to Algorithm 1.*

Proof The proof is by induction on ℓ. Suppose that a call is made to Algorithm 1 where $m(r) = x$. If $\ell = 1$, then $m(r) = m(u_1) = x$ during the call to Algorithm 1.

Suppose that if there exists a path $r = u_1, u_2, \ldots, u_\ell$ in T where $x \in u_i$ for $i = 1$ to ℓ, then $m(u_{\ell-1}) = x$ during the call to Algorithm 1. Now $m(u_{\ell-1}) = x \in L(u_\ell)$, so $m(u_\ell) = x$ at line (1.3). ∎

Theorem 4.3.11 *For all $m \geq 1$, $k \geq 1$, $\theta_{2,2,2m}$ is $(2k, k)$-choosable.*

Proof It will be shown that the sets $M(v)$ for each $v \in V(\theta_{2,2,2m})$ are a $(2k, k)$-choice for $\theta_{2,2,2m}$. After Algorithm 1 is called $i+i'$ times, $|L(v_j)| = 2k-i-i'$ for all v_j. For $j = 0, 2, \ldots, 2m$, $|M(v_j)| = i$, and for $j = 1, 3, \ldots, 2m+1, 2m+2$, $|M(v_j)| = i'$. If $i = k$, then $|L(v_j)| = k - i'$ for each v_j, and for $j = 0, 2, \ldots, 2m$, $|M(v_j)| = k$. Therefore $|M(v_j)| = |M(v_j) \cup L(v_j)| = (i'+k) - i' = k$ for $j = 1, 3, \ldots, 2m+1, 2m+2$. If $i' = k$, then $|L(v_j)| = k - i$ for each v_j, and for $j = 1, 3, \ldots, 2m+1, 2m+2$, $|M(v_j)| = k$. Therefore, $|M(v_j)| = |M(v_j) \cup L(v_j)| = (i + k) - i = k$ for $j = 0, 2, \ldots, 2m$. If $i = k$ or $i' = k$, then $M(v_j) \subset L(v_j)$ and $|M(v_j)| = k$ for each v_j. If $i < k$ and $i' < k$, then the elements of $L(v_{2m+1})$ and $L(v_{2m+2})$ are paired.

By Lemma 4.3.3, if $m(v) = x$ is added to $M(v)$ during a call to Algorithm 1, then for any u which is adjacent to v in T, x will not be in $L(u)$ after the call, and x will not be added to $M(u)$ during the call. Thus it suffices to consider choices for vertices which are adjacent in $\theta_{2,2,2m}$, but which are not adjacent in T.

Suppose there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_0, v_s) - (v_0, v_t)$ (line (5.1)). By line (5.2), $m(v_0) = x$ and x is added to $M(v_0)$ at line (1.1). As $x \in L(v_0) - L(v_s) - L(v_t)$, $x \notin L(v_s)$ and $x \notin L(v_t)$. Similarly if there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_s) - (v_{2m}, v_t)$ (line (5.3)).

Suppose there is a call made to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_{2m+1}) - (v_{2m}, v_{2m+2})$ (line (5.4)). By line (5.5), $m(v_{2m+1}) = x$ and x is added to $M(v_{2m+1})$ at line (1.1). As $x \in (L(v_0) \cap L(v_{2m+1}) \cap L(v_{2m+2})) - L(v_{2m})$, $x \notin L(v_{2m})$, but $x \in L(v_0)$ and $x \in L(v_{2m+2})$. The next choice will be $m(v_0)$, and the choice after that will be $m(v_{2m+2})$. As $m(v_{2m+1}) = x \in L(v_0)$, $m(v_0) = x$ at line (1.3). As $m(v_0) = x \in L(v_{2m+2})$, $m(v_{2m+2}) = x$ at line (1.3). As $d(v_{2m+1}, v_{2m+2})$ is even, x will be added to $M(v_{2m+2})$ at line (1.6). Similarly if there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_0, v_{2m+1}) - (v_0, v_{2m+2})$ (line (5.6)).

Suppose there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_t, v_{t-1}) - (v_{2m}, v_{2m+2})$ (line (5.8)). By line (5.9), $m(v_{2m}) = x$ and x is added to $M(v_{2m})$ at line (1.1). As $x \in (L(v_0) \cap L(v_{2m}) \cap L(v_{2m+1})) - L(v_{2m+2})$, $x \notin L(v_{2m+2})$. There exists a path $r = v_{2m}, v_{2m-1}, \ldots, v_{t-1}, v_t$ in T where x is in $L(v_i)$ for each v_i in the path. So by Lemma 4.3.10, $m(v_t) = x$ during the call to Algorithm 1. As $d(v_{2m}, v_t)$ is even, x is added to $M(v_t)$ at line (1.6). As the minimum t as chosen at line (5.7), $x \notin L(v_{t-1})$. Similarly if there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_t, v_{t-1}) - (v_{2m}, v_{2m+1})$

(line (5.17)).

Suppose there is a call made to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_{2m+2}) - (v_0, v_{2m+2}) - (v_t, v_{t-1})$ (line (5.10)). By line (5.11), $m(v_t) = x$ and x is added to $M(v_t)$ at line (1.1). As the minimum t was chosen at line (5.7), $x \notin L(v_{t-1})$. There exists a path $r = v_t, v_{t+1}, \ldots, v_{2m}, v_{2m+1}, v_0$ in T where x is in $L(v_i)$ for each v_i in the path. So by Lemma 4.3.10, $m(v_{2m}) = x$ and $m(v_0) = x$ during the call to Algorithm 1. As $x \in (L(v_0) \cap L(v_{2m}) \cap L(v_{2m+1})) - L(v_{2m+2})$, $x \notin L(v_{2m+2})$. Similarly if there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_{2m+1}) - (v_0, v_{2m+1}) - (v_t, v_{t-1})$ (line (5.18)).

Suppose there is a call made to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_{2m+2}) - (v_0, v_{2m+2}) - (v_u, v_{u-1})$ (line (5.13)). By line (5.14), $m(v_u) = y$ and y is added to $M(v_u)$ at line (1.1). As the minimum u was chosen at line (5.12), $y \notin L(v_{u-1})$. There exists a path $r = v_u, v_{u+1}, \ldots, v_{2m}$ in T where y is in $L(v_i)$ for each v_i in the path. So by Lemma 4.3.10, $m(v_{2m}) = y$ during the call to Algorithm 1. As $y \in L(v_{2m+2}) \cap L(v_0)$, $y \notin L(v_0)$. Suppose that u is even. Then since $d(v_u, v_{2m})$ is even, y is added to $M(v_{2m})$ at line (1.6). As $y \notin L(v_0)$, some $m(v_0) = z \in L(v_0)$ is added to $M(v_0)$ at line (1.6). As u is even, y is not added to $M(v_{2m+2})$ at line (5.15), and y is deleted from $L(v_{2m+2})$ at line (5.16). Suppose that u is odd. Then y is added to $M(v_{2m+2})$ at line (5.15). At line (1.5), y is deleted from $L(v_{2m})$, and as $d(v_u, v_{2m})$ is odd, y is not added to $M(v_{2m})$ at line (1.6). Similarly if there is a call to Algorithm 1 where $T = \theta_{2,2,2m} - (v_{2m}, v_{2m+1}) - (v_0, v_{2m+1}) - (v_u, v_{u-1})$ (line (5.19)). ∎

An example demonstrating the application of Algorithm 5 to a $\theta_{2,2,4}$ graph follows Suppose sets $L(v_0) = \{1,2,3,4\}$, $L(v_1) = \{1,2,5,6\}$, $L(v_2) = \{1,3,4,5\}$, $L(v_3) = \{4,5,6,7\}$, $L(v_4) = \{5,6,7,8\}$, $L(v_5) = \{1,2,4,8\}$ and $L(v_6) = \{3,5,6,7\}$ are assigned to the vertices of $\theta_{2,2,4}$

The algorithm begins with a search for an element of $L(v_0)$ which is not in the colour sets of at least two of the vertices adjacent to v_0 Elements $1,2 \in L(v_0)$ do not satisfy this criterion, but element $3 \in L(v_0)$ does It is contained in neither $L(v_1)$ nor $L(v_5)$ At line (5 1), tree T is obtained by removing edges (v_0, v_1) and (v_0, v_5) Vertex v_0 is chosen as the root of T, and at line (5 2), $m(v_0) = 3$ is chosen from its colour set At line (5 21), Algorithm 1 is called, beginning a breadth-first search on T, where an element will be chosen from each colour set

Algorithm 1 begins by adding $m(v_0) = 3$ to choice set $M(v_0)$, and deleting it from set $L(v_0)$ Then the neighbour v_6 of v_0 is considered Because $m(v_0) = 3 \in L(v_6)$, $m(v_6) = 3$ is chosen at line (1 3) Element 3 is deleted from $L(v_6)$ at line (1 5) The neighbour v_4 of v_6 is considered next As $m(v_6) = 3 \notin L(v_4)$, an element of $L(v_4)$ which is not in $L(v_6)$ must be chosen at line (1 4) Element $m(v_4) = 8$ satisfies this criterion It is deleted from $L(v_4)$ at line (1 5), and because v_4 is at an even distance from the root v_0, element 8 is added to choice set $M(v_4)$ at line (1 6) Continuing in this fashion, $m(v_5) = 8$ and $m(v_3) = 4$ are chosen, and are deleted from sets $L(v_5)$ and $L(v_3)$, respectively Next, $m(v_2) = 4$ is chosen, deleted from $L(v_2)$ and added to $M(v_2)$ Finally, $m(v_1) = 2$ is chosen and deleted from $L(v_1)$, completing this call to Algorithm 1

As root v_0 was chosen, i is incremented at line (5.22). Because $i < 2$, this iteration of Algorithm 5 is completed, and there is a return to the beginning of the loop.

Again, there is a search for an element in $L(v_0)$ which is not in the sets of at least two of its adjacent vertices. As element 2 was deleted from $L(v_1)$, this element is in neither $L(v_1)$ nor $L(v_6)$, thus satisfying the criterion. At line (5.1), tree T is obtained by removing edges (v_0, v_1) and (v_0, v_6) from the $\theta_{2,2,4}$ graph. The root of T becomes vertex v_0, $m(v_0) = 2$ is chosen, and Algorithm 1 is called.

First, $m(v_0) = 2$ is added to $M(v_0)$ and is deleted from $L(v_0)$. Now $m(v_5) = 2$ is chosen and deleted from $L(v_5)$. Then, $m(v_4) = 5$ is chosen, deleted from $L(v_4)$ and added to $M(v_4)$. Next, $m(v_6) = 5$ and $m(v_3) = 5$ are chosen, and are deleted from sets $L(v_6)$ and $L(v_3)$, respectively. Then, $m(v_2) = 5$ is chosen, deleted from $L(v_2)$ and added to $M(v_2)$. Finally, $m(v_1) = 5$ is chosen and deleted from $L(v_1)$, completing this call to Algorithm 1.

Once again, i is incremented at line (5.22). Now $i = 2$, which means that the choice sets for the vertices at an even distance from v_0 have been completed. They are $M(v_0) = \{2, 3\}$, $M(v_2) = \{4, 5\}$ and $M(v_4) = \{5, 8\}$. At line (5.23), the choice sets for the remaining vertices are completed by adding their remaining elements to their respective choice sets. Set $L(v_1)$ had elements 2 and 5 deleted, and has elements 1 and 6 remaining, so choice set $M(v_1)$ becomes $\{1, 6\}$. Similarly, $M(v_3) = \{6, 7\}$, $M(v_5) = \{1, 4\}$ and $M(v_6) = \{6, 7\}$. This completes the choices, and Algorithm 5.

Theorem 4.3.12 *Any 2-choosable graph is $(2k : k)$-choosable for any $k \geq 1$.*

Proof. By Theorems 4.3.4 through 4.3.11, and the characterization of 2-choosable graphs given by Theorem 3.1.2, the core of any graph which is 2-choosable is $(2k : k)$-choosable for all $k \geq 1$. By Lemma 4.2.1, any graph whose core is $(2k : k)$-choosable is $(2k : k)$-choosable. ∎

If G is a k-choosable graph and H is a 2-choosable graph, by Theorem 4.3.12 and Lemma 3.2.4, $\chi_\ell(G \cup H) \leq 2k$.

4.4 Planar Graphs

The proof of Theorem 4.4.1 is based on Thomassen's proof [13] that any planar graph is 5-choosable.

Theorem 4.4.1 *For any positive integer k and any planar graph G, G is $(5k : k)$-choosable.*

Proof. Let G be a near-triangulation with outer cycle $C = v_0, v_1, \ldots, v_{p-1}, v_0$. Assume that v_0 and v_1 have k-subsets $M(v_0) = \{1, 2, \ldots, k\}$ and $M(v_1) = \{k+1, k+2, \ldots, 2k\}$, and that $L(v)$ is a set of at least $3k$ colours if $v \in V(C) - \{v_0, v_1\}$ and at least $5k$ colours if $v \in V(G - C)$. It suffices to show that the k-subsets of v_0 and v_1 can be extended to a collection of subsets $\{M(v) : v \in V(G)\}$ where $M(v) \subset L(v)$ and $M(v_i) \cap M(v_j) = \emptyset$ whenever v_i and v_j are adjacent. The proof is by induction on $|V(G)|$.

If $p-1=3$ and $G=C$ then $|L(v_2)| \geq 3k$ and $|L(v_2)-M(v_0)-M(v_1)| \geq 3k-k-k=k$. Thus $M(v_2) \subset L(v_2)$ can be chosen such that $M(v_2)$ is disjoint from both $M(v_0)$ and $M(v_1)$.

If C has a chord (v_i, v_j), where $1 \leq i \leq j-2 \leq p-2$ $(v_p = v_0)$, then the induction hypothesis can be applied to the cycle $v_0, v_1, \ldots, v_i, v_j, v_{j+1}, \ldots, v_{p-1}, v_0$ and its interior. Then by making the assumption that $M(v_i)$ and $M(v_j)$ are the k-subsets chosen for the former cycle, the induction hypothesis can be applied to $v_j, v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ and its interior.

So assume that C has no chord. Let $v_0, u_0, u_1, \ldots, u_m, v_{p-2}$ be the neighbours of v_{p-1} in that clockwise order around v_{p-1}. As the interior of C is triangulated, G contains the path $P = v_0, u_0, u_1, \ldots, u_m, v_{p-2}$. As C is chordless, $P \cup (C - v_{p-1})$ is a cycle C'. Let S be a $2k$-subset in $L(v_{p-1}) - \{1, 2, \ldots, k\}$. Define $L'(u_i) = L(u_i) - S$ for $0 \leq i \leq m$ and $L'(v) = L(v)$ if $v \in V(G) - \{u_0, u_1, \ldots, u_m\}$. Apply the induction hypothesis to C' and its interior and the new sets L'. As $|S - M(v_{p-2})| \geq 2k - k = k$ and S and any of $M(v_0), M(u_0), M(u_1), \ldots, M(u_m)$ are disjoint, the $(5k : k)$-choice can be completed by letting $M(v_{p-1})$ be a k-set in $S - M(v_{p-2})$. ∎

Thomassen's result that all planar graphs are 5-choosable [13] follows directly from Theorem 4.4.1. It follows from Theorem 4.4.1 and Lemma 3.2.4 that if G is k-choosable and H is planar, then $\chi_\ell(G \cup H) \leq 5k$.

4.5 Bipartite Graphs

Theorem 4.5.1 *For all $k \geq 1$, $p \geq 1$ and $q \geq p$, $K_{p,q}$ is $((p+1)k : k)$-choosable.*

Proof. Assign $(p+1)k$-sets to $V(K_{p,q})$. Arbitrarily choose k-subsets $M(v) \subset L(v)$ for all vertices of degree q. For each vertex v of degree p, set $L'(v) = L(v) - M(x_1) - M(x_2) - \cdots - M(x_p)$ where $x_1, x_2, \ldots, x_p$ are the vertices adjacent to v. As $|L'(v)| \geq (p+1)k - pk = k$, a k-subset $M(v) \subset L'(v)$ can be chosen for each vertex of degree p. ∎

Theorem 3.4.1 follows directly from Theorem 4.5.1.

In the proofs of Theorems 4.5.2 and 4.5.3, $\cup$ denotes the union of disjoint sets.

Theorem 4.5.2 *If $\frac{a}{b} < p+1$ then $K_{p,\binom{a}{b}^p}$ is not $(a : b)$-choosable.*

Proof. It suffices to give an assignment of a-sets to $V(K_{p,\binom{a}{b}^p})$ for which no $(a : b)$-choice is possible. Let $v_1, v_2, \ldots, v_p$ be the vertices of degree $\binom{a}{b}^p$ and $u_1, u_2, \ldots, u_{\binom{a}{b}^p}$ the vertices of degree p. Assign a-sets $L(v_i)$ to $v_1, v_2, \ldots, v_p$ such that the sets are pairwise disjoint. If $\frac{a}{b} < p$, let $L' = \emptyset$. Otherwise let L' be an $(a - bp)$-set where $L' \cap L(v_i) = \emptyset$ for each i, $1 \leq i \leq p$. Assign to each vertex u_i, $1 \leq i \leq \binom{a}{b}^p$, a distinct set $L(u_i) = L_i(v_1) \cup L_i(v_2) \cup \cdots \cup L_i(v_p) \cup L'$, where $|L_i(v_j)| = b$ and $L_i(v_j) \subset L(v_j)$ for $1 \leq j \leq p$, and each of the $\binom{a}{b}^p$ possible sets that can be constructed in this way is assigned to some vertex u_i. If $\frac{a}{b} < p$, then $|L(u_i)| = pb > a$. Otherwise $|L(u_i)| = pb + (a - bp) = a$.

To show that this assignment does not admit an $(a:b)$-choice, arbitrarily choose b-subsets $M(v_i) \subset L(v_i)$ for $i = 1, 2, \ldots, p$. As the sets $L(v_i)$ are pairwise disjoint, $\bigcup_{i=1}^{p} M(v_i) = pb$. There exists j, $1 \leq j \leq \binom{a}{b}^p$ such that $L(u_j) = \bigcup_{i=1}^{p} M(v_i) \cup L'$. If $\frac{a}{b} < p$, $L(u_j) = \bigcup_{i=1}^{p} M(v_i)$ and no b-subset $M(u_j) \subset L(u_j)$ can be chosen which is disjoint from each $M(v_i)$. Otherwise, $L(u_j) = \bigcup_{i=1}^{p} M(v_i) \cup L'$ contains only $a - bp < b$ elements which are not in any $M(v_i)$. Again, no b-subset $M(u_j) \subset L(u_j)$ can be chosen which is disjoint from each $M(v_i)$. ∎

For $a = 2$, $b = 1$, set $p = 2$. By Theorem 4.5.2, $K_{2,4}$ is not $(2:1)$-choosable. This is a best possible result in the sense that $K_{2,3} = \theta_{2,2,2}$ is $(2:1)$-choosable. For $a = 3$, $b = 1$, set $p = 3$. By Theorem 4.5.2, $K_{3,27}$ is not $(3:1)$-choosable. This is a best possible result in the sense that, by Theorem 3.4.3, $K_{3,26}$ is $(3:1)$-choosable.

Theorem 4.5.3 *If* $\frac{a}{b} < p + 1$ *then* $K_{p+1,\binom{a}{b}^p - \binom{a-b}{b}^p}$ *is not* $(a:b)$*-choosable.*

Proof. It suffices to give an assignment of a-sets to $V(K_{p+1,\binom{a}{b}^p - \binom{a-b}{b}^p})$ for which no $(a:b)$-choice is possible. Let $v_1, v_2, \ldots, v_{p+1}$ be the vertices of degree $\binom{a}{b}^p - \binom{a-b}{b}^p$ and let $u_1, u_2, \ldots, u_{\binom{a}{b}^p - \binom{a-b}{b}^p}$ be the vertices of degree $p + 1$. Assign a-sets $L(v_i)$ to $v_1, v_2, \ldots, v_p$ such that the sets are pairwise disjoint. If $\frac{a}{b} < p$, let $L'' = \emptyset$. Otherwise let L'' be a set of cardinality $a - bp$ where $L'' \cap L(v_i) = \emptyset$ for each i, $1 \leq i \leq p$. Let $L(v_{p+1}) = \bigcup_{i=1}^{p} L'(v_i) \cup L''$ where $|L'(v_i)| = b$ and $L'(v_i) \subset L(v_i)$ for each v_i. Assign to each vertex u_i, $1 \leq i \leq \binom{a}{b}^p - \binom{a-b}{b}^p$, a distinct set $L(u_i) = L_i(v_1) \cup L_i(v_2) \cup \ldots \cup L_i(v_p) \cup L''$, where $|L_i(v_j)| = b$, $L_i(v_j) \subset L(v_j)$ for $1 \leq j \leq p$ and

$L_i(v_j) \not\subseteq L(v_j) - L'(v_j)$ for at least one j, and each of the $\binom{a}{b}^p - \binom{a-b}{b}^p$ possible sets that can be constructed in this way is assigned to some vertex u_i. If $\frac{a}{b} < p$, then $|L(u_i)| = pb > a$. Otherwise, $|L(u_i)| = pb + (a - bp) = a$.

To show that this assignment does not admit an $(a : b)$-choice, arbitrarily choose b-subsets $M(v_i) \subset L(v_i)$ for $i = 1, 2, \ldots, p+1$. Suppose there exists j such that $M(v_j) \not\subseteq L(v_j) - L'(v_j)$. Then $\bigcup_{i=1}^{p} M(v_i) \cup L'' = L(u_t)$ for some t, $1 \leq t \leq \binom{a}{b}^p - \binom{a-b}{b}^p$. Suppose that $\frac{a}{b} < p$. Then $L(u_t) = \bigcup_{i=1}^{p} M(v_i)$ and no b-subset $M(u_t) \subset L(u_t)$ can be chosen which is disjoint from each $M(v_i)$. Otherwise, $L(u_t) = \bigcup_{i=1}^{p} M(v_i) \cup L''$ contains at most $a - bp < b$ elements which are not in any $M(v_i)$. Again, no b-subset $M(u_t) \subset L(u_t)$ can be chosen which is disjoint from each $M(v_i)$.

Otherwise $M(v_j) \subseteq L(v_j) - L'(v_j)$ for each v_j, $1 \leq j \leq p$. Now $M(v_{p+1})$ consists of b elements from $L(v_{p+1}) = \bigcup_{i=1}^{p} L'(v_i) \cup L''$. Let $x \in M(v_{p+1}) - L''$. For some ℓ, $x \in L'(v_\ell)$. Let $y \in M(v_\ell)$, and consider $(\bigcup_{i=1}^{p} M(v_i)) \cup \{x\} - \{y\}$. As $x \in M(v_{p+1})$, x cannot be in the choice set of u_j, so no element of $(\bigcup_{i=1}^{p} M(v_i)) \cup \{x\} - \{y\}$ can be in the choice set of any u_j. Now for some j, $(\bigcup_{i=1}^{p} M(v_i)) \cup \{x\} - \{y\} \cup L'' = L(u_j)$. If $\frac{a}{b} < p$, then $L(u_j) = (\bigcup_{i=1}^{p} M(v_i)) \cup \{x\} - \{y\}$ and no b-subset $M(u_j) \subset L(u_j)$ can be chosen which is disjoint from each $M(v_i)$. Otherwise $L(u_j) = (\bigcup_{i=1}^{p} M(v_i)) \cup \{x\} - \{y\} \cup L''$ contains at most $a - bp < b$ elements which are not in any $M(v_i)$. Again, no b-subset $M(u_j) \subset L(u_j)$ can be chosen which is disjoint from each $M(v_i)$. ∎

For $a = 2$, $b = 1$, set $p = 2$. By Theorem 4.5.3, $K_{3,3}$ is not $(2 : 1)$-choosable. This is a best possible result in the sense that $K_{3,2} = \theta_{2,2,2}$ is $(2 : 1)$-choosable. For $a = 3$,

$b = 1$, set $p = 3$ By Theorem 4 5 3, $K_{4,19}$ is not $(3 : 1)$-choosable This is a best possible result in the sense that, by Theorem 3 4 4, $K_{4,18}$ is $(3 : 1)$-choosable

4.6 Cartesian Graph Products

Lemma 4 6 1 *If H is a $(d : a)$-choosable graph and G is an $(a : b)$-choosable graph, then $G \Box H$ is $(d : b)$-choosable*

Proof Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$ Suppose d-sets have been assigned to $V(G \Box H)$ For $i = 1$ to n, let H_i be the copy of H induced by $\{(v_i, u_j) : u_j \in H\}$ For each H_i, choose an a-subset from each vertex, such that H_i-adjacent vertices have disjoint a-subsets Such a choice can be determined because each H_i is $(d : a)$-choosable For $j = 1$ to m, let G_j be the copy of G induced by $\{(v_i, u_j) : v_i \in G\}$ For each G_j, choose a b-subset from the a-subset of each vertex, so that G_j-adjacent vertices have disjoint b-subsets This choice can be determined because each G_j is $(a : b)$-choosable The resulting choice makes the b-subsets disjoint on adjacent vertices of $G \Box H$

Suppose (v_i, u_j) and (v_k, u_ℓ) are adjacent in $G \Box H$ Then if $v_i = v_k$ and $(u_j, u_\ell) \in E(H)$, then in particular, $(u_j, u_\ell) \in E(H_i) = E(H_k)$ So (v_i, u_j) and (v_k, u_ℓ) have disjoint b-subsets, as their b-subsets were chosen from disjoint a-subsets

Otherwise $u_j = u_\ell$ and $(v_i, v_k) \in E(G)$, and in particular, $(v_i, v_k) \in E(G_j) = E(G_\ell)$ So (v_i, u_j) and (v_k, u_ℓ) have disjoint b-subsets, as they were chosen so that

G_j-adjacent vertices would have disjoint b-subsets. Thus $G \Box H$ is $(d \cdot b)$-choosable. ∎

If G is a k-choosable graph and H is a 2-choosable graph, then by Theorem 4.3.12 and Lemma 4.6.1, $\chi_\ell(G \Box H) \leq 2k$. By Theorem 4.4.1 and Lemma 4.6.1, if G is k-choosable and H is planar, then $\chi_\ell(G \Box H) \leq 5k$.

If Conjecture 3.2.5 holds, then by Lemma 4.6.1, $\chi_\ell(G \Box H) \leq \chi_\ell(G)\chi_\ell(H)$ for all graphs G and H.

Bibliography

[1] N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992) 125-134.

[2] N. Alon, *Restricted colorings of graphs*, Proc. 14th British Combinatorics Conference. Cambridge University Press (1993) 1-33.

[3] J.A. Bondy and U.S.R. Murty, **Graph Theory with Applications**. North-Holland, New York (1976).

[4] P. Erdős, A.L. Rubin and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium **26** (1979) 125-157.

[5] S. Gutner, M.Sc. Thesis, Tel Aviv University (1992).

[6] S. Gutner and M. Tarsi, *Some results on* $(a:b)$*-choosability* (1995), manuscript.

[7] D. Hanson, G. MacGillivray and B. Toft, *Choosability of bipartite graphs*, Ars Combinatoria **44** (1996) 183-192.

[8] D.G. Hoffman and P.D. Johnson Jr., *On the choice number of* $K_{m,n}$, Congressus Numerantium **98** (1993) 105-111.

[9] J. Kratochvil and Z. Tuza, *Algorithmic complexity of list colorings*, Discrete Appl. Math. **50** (1994) 297-302.

[10] N.V.R. Mahadev, F.S. Roberts and P. Santhanakrishnan, 3-*Choosable complete bipartite graphs*. DIMACS Technical Report 91-62 (1991).

[11] M. Mirzakhani, *A small non-4-choosable planar graph*, Bulletin of the ICA, **17** (1996) 15-18.

[12] A.M. Shende and B. Tesman, 3-*Choosability of* $K_{5,q}$, Technical Report #94-9, Dept. of Computer Science, Bucknell University (1994).

[13] C. Thomassen, *Every planar graph is 5-choosable*, J. Comb. Theory (B) **62** (1994) 180-181.

[14] C. Thomassen, *3-List-coloring planar graphs of girth 5*, J. Comb. Theory (B) **64** (1995) 101-107.

[15] Zs. Tuza and M. Voigt, *Every 2-choosable graph is $(2m, m)$-choosable*, Journal of Graph Theory, **22** (1996) 245-252.

[16] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors* (Russian), Diskret. Analiz **29** (1976) 3-10.

[17] M. Voigt, *List colorings of planar graphs*, Discrete Math. **120** (1993) 215-219.

[18] M. Voigt, *A not 3-choosable planar graph without 3-cycles*, Discrete Math. **146** (1995) 325-328.

[19] D. B. West, *Open problems*, SIAM Discrete Math. Newsletter, (Spring 1994) 5-7.

VITA

Surname: Ferneyhough — Given Names: Sheila
Place of Birth: Nanaimo, BC, Canada — Date of Birth: June 9, 1968

Educational Institutions Attended

University of Victoria	1994 to 1996
Simon Fraser University	1990 to 1994
Capilano College	1989 to 1990
Malaspina College	1985 to 1987

Degree Awarded

B Sc	Simon Fraser University 1994

Honours and Awards

University of Victoria Fellowship	1994-1996

PARTIAL COPYRIGHT LICENSE

I hereby grant the right to lend my thesis to users of the University of Victoria Library, and to make single copies only for such users or in response to a request from the Library of any other university, or similar institution, on its behalf or for one of its users. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by me or a member of the University designated by me. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis: **CHOICE NUMBERS OF UNIONS OF GRAPHS**

Author [redacted]
(Signature)

FERNEYHOUGH SHEILA

(Name in Block Letters)

November 23, 1997

(Date)