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ORIGINAL ARTICLE

Dynamics analysis of a spatiotemporal SI model

Mengxin Chen [a,*], Zhenyong Hu [b], Qianqian Zheng [c,*], Hari Mohan Srivastava [d,e,f,g]

[a] College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
[b] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China
[c] School of Science, Xuchang University, Xuchang 461000, China
[d] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
[e] Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
[f] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
[g] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

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Abstract In this article, we investigate a susceptible-infected (SI) model with the saturated treatment, the non-monotonic incidence rate, the logistic growth, and the homogeneous Neumann boundary conditions. The global existence and the uniform boundedness of the parabolic system are performed. After that, we investigate the global stability of the disease free equilibrium (DFE) and the endemic equilibrium (EE), respectively. In the end, we give a priori estimates, some propositions of the nonconstant steady states to the elliptic system. Meanwhile, we find that the diffusion rates of the susceptible and the infected population can affect the nonexistence of the nonconstant steady states. An interesting finding is DFE and basic reproduction number do not exist when the intrinsic growth rate of the susceptible class less than the rate of susceptible individuals being vaccinated.

1. Introduction

Over the past few decades, various diseases models have been modeled by scholars to characterize propagation dynamics among population. Such as, the SI model [1,2], the SIR model [3,4], the COVID-19 model [5,6], the SEIR model [7,8] etc. Of course, considering that the spread of disease may be affected by spatial location, some scholars proposed and investigated the spatial transmission epidemic models. Wang and Zhao [9] reported basic reproduction number and its computation formulae for diffusive epidemic models with compartmental structure. Magal et al. [10] employed a diffusive epidemic model to study the transmission dynamics of susceptible and infected populations with spatial heterogeneity environment. Renardy et al. [11] established a pipeline for structural identifiability analysis of a spatial epidemic model by employing a differential algebra framework and derive identifiability results. Qiang et al. [12] investigated the global threshold dynamics of a epidemic model in almost periodic environments and shown the diffusion rate could affect disease transmission.

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[*] Corresponding authors.

E-mail addresses: chmxdc@163.com (M. Chen), zhengqianqian35@163.com (Q. Zheng).

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More dynamical results about epidemic models with spatial effect, see Refs. [5,13–16] and reference cited therein.

In this paper, we investigate a diffusive SI model as follows.

$$\begin{cases} \frac{\partial S}{\partial t} - d_S\Delta S = rS\left(1-\frac{S}{K}\right) - \frac{\beta SI}{1+\alpha I^2} - \mu S, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} - d_I\Delta I = \frac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \frac{aeI}{1+beI}, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t \geqslant 0, \\ S(x,0) = S_0(x) \geqslant 0,\ I(x,0) = I_0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1}$$

where $S = S(x,t)$ and $I = I(x,t)$ are the susceptible humans and the infected humans at spatial position x and time t, respectively. Constants d_S and d_I are diffusion rates of the susceptible humans and the infected humans, respectively. The term $rS\left(1-\frac{S}{K}\right)$ represents the logistic growth function of the susceptible humans with the logistic growth rate r and the environmental carrying capacity K, see [17]. The nonlinear term $\frac{\beta SI}{1+\alpha I^2}$ describes the non-monotonic incidence rate [18,19], and constants α, β are the parameter that measures the inhibitory factors and the transmission rate of infection, respectively. Parameter μ denotes by the percentage of susceptible individuals being vaccinated. Moreover, the constants γ, d and δ describe the natural recovery rate, the mortality rate of infected individuals, and the death rate that induced by disease. The term $\frac{aeI}{1+beI}$ describes saturated treatment function [20,21], where a is cure rate, e is treatment control parameter and b represents delayed treatment of individuals. Also, denote Δ by the Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ with the smooth boundary $\partial\Omega, \nu$ is the outward unit normal vector along the boundary $\partial\Omega$. All parameters in diffusive model (1) are non-negative due to its practical meaning.

It is noticed that if the diffusion is absent, namely

$$\begin{cases} \frac{dS}{dt} = rS\left(1-\frac{S}{K}\right) - \frac{\beta SI}{1+\alpha I^2} - \mu S, \\ \frac{dI}{dt} = \frac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \frac{aeI}{1+beI}. \end{cases} \tag{2}$$

We can find that $(0,0)$ is a equilibrium of the epidemic model (2) and it admits a disease free equilibrium (DFE) denote by E_f and a endemic equilibrium (EE) denote by E_*. For such a temporal epidemic model, we need to point out that the stabilities of DFE/EE and primary bifurcation, including the transcritical bifurcation, the backward bifurcation, the saddle-node and Hopf bifurcation, Bogdanov-Takens are reported in [21]. For a epidemic model, the basic reproduction number is an important index to characterize some threshold dynamics which describes the average number of newly infected populations at the beginning of the infectious process [22]. By the technique of next generation matrix [23–25], the basic reproduction number, say $\mathscr{R}_0$, can be written as

$$\mathscr{R}_0 = \frac{\beta(r-\mu)K}{r(d+\delta+\gamma+ae)}. \tag{3}$$

This gives the logistic growth rate r should be greater than the percentage μ of susceptible individuals being vaccinated, i.e., $r > \mu$ always valid for the model (2).

Owing to the diffusion effect being incorporated into the model, how to guarantee the positivity and boundedness of the solution (S, I) should be first addressed. Moreover, whether the diffusion rates d_S of the susceptible individual and d_I of the infected individual will affect the existence of the nonconstant steady states also is an important issue. Exploring the keys of these two problems will help us theoretically understand the dynamic profiles of the SI model (1) with diffusion effect. Consequently, the purpose of the present paper is exploring the dynamical behaviors of the diffusive SI model (1) as well as its elliptic system, respectively. To be precise, we first the positivity and uniform boundedness results of the solution $(S(x,t), I(x,t))$ of the SI model (1). The mixed quasi-monotone method, the strong maximum principle, etc. are used to help us to achieve these goals. We also give the conditions to ensure the global asymptotic stabilities of DFE and EE by construction some suitable Lyapunov time evolution functions, respectively. In the sequel, we turn our attention to the elliptic system of the SI model (1). We main want to establish some results of the steady state solution $(S(x), I(x))$. For instance, the priori estimates, the nonexistence of the steady state $(S(x), I(x))$, and so on. Particularly, we find that the diffusion coefficients the susceptible humans d_S and the infected humans d_I can affect the nonexistence of the positive steady state $(S(x), I(x))$ with certain conditions.

We design the remaining sections of the paper as follows. In Section 2, the positivity and uniform boundedness of the solution $(S(x,t), I(x,t))$ of the SI model (1) are established. In Section 3, we perform the global asymptotic stabilities of DFE and EE, respectively. In Section 4, one focuses on some dynamics phenomena of the steady state of the corresponding elliptic system. Finally, this paper ends with some conclusions performed in Section 5.

2. Positivity and uniform boundedness

In this section, the positivity and the uniform boundedness of the solution $(S(x,t), I(x,t))$ to the SI model (1) will be explored.

Theorem 2.1. Suppose that $0 < \mu < r, 0 < \beta < 2\sqrt{\alpha}(r-\mu)$ and $S_0(x), I_0(x) \geqslant 0$ in $\Omega \subset \mathbb{R}^N (N \geqslant 1)$. Then system (1) has a unique positive solution $u(x,t) > 0, v(x,t) > 0$ for $t \in (0, +\infty)$ and $x \in \overline{\Omega}$. Moreover,

$$(i) \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} S(\cdot,t) \leqslant K,\ \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} I(\cdot,t) \leqslant \frac{\beta K}{2\sqrt{\alpha}(\gamma+d+\delta)},$$

$$(ii) \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} S(\cdot,t) \geqslant \frac{K[2\sqrt{\alpha}(r-\mu)-\beta]}{2r\sqrt{\alpha}}.$$

Proof. Assume that $(\underline{S}(x,t), \underline{I}(x,t)) = (0,0)$ and $(\overline{S}(x,t), \overline{I}(x,t)) = \left(\widehat{S}(t), \widehat{I}(t)\right)$, where we define $\left(\widehat{S}(t), \widehat{I}(t)\right)$ by the unique solution of the following problem

$$\begin{cases} \frac{dS}{dt} = rS\left(1-\frac{S}{K}\right) - \mu S, \\ \frac{dv}{dt} = \frac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \frac{aeI}{1+beI}, \\ S(0) = \widehat{S}, I(0) = \widehat{I}, \end{cases}$$

where one claims that $\widehat{S} = \sup_{x\in\Omega} S_0(x)$ and $\widehat{I} = \sup_{x\in\Omega} I_0(x)$. We then have

$$\frac{\partial \overline{S}(x,t)}{\partial t} - d_S\Delta\overline{S}(x,t) - r\overline{S}(x,t)\left(1-\frac{\overline{S}(x,t)}{K}\right) + \frac{\beta\overline{S}(x,t)\underline{I}(x,t)}{1+\alpha\underline{I}^2(x,t)} + \mu\overline{S}(x,t) = 0$$

$$\geqslant 0 = \frac{\partial \underline{S}(x,t)}{\partial t} - d_S\Delta\underline{S}(x,t) - r\underline{S}(x,t)\left(1-\frac{\underline{S}(x,t)}{K}\right) - \frac{\beta\underline{S}(x,t)\overline{I}(x,t)}{1+\alpha\overline{I}^2(x,t)} - \mu\underline{S}(x,t),$$

and

$$\tfrac{\partial \overline{I}(x,t)}{\partial t} - d_I \Delta \overline{I}(x,t) - \tfrac{\beta \overline{S}(x,t)\overline{I}(x,t)}{1+\alpha \overline{I}^2(x,t)} + (\gamma + d + \delta)\overline{I}(x,t) + \tfrac{ae\overline{I}(x,t)}{1+be\overline{I}(x,t)} = 0$$
$$\geqslant 0 = \tfrac{\partial \underline{I}(x,t)}{\partial t} - d_I \Delta \underline{I}(x,t) - \tfrac{\beta \underline{S}(x,t)\underline{I}(x,t)}{1+\alpha \underline{I}^2(x,t)} + (\gamma + d + \delta)\underline{I}(x,t) + \tfrac{ae\underline{I}(x,t)}{1+be\underline{I}(x,t)}.$$

Hence, we can see that $(\underline{S}(x,t), \underline{I}(x,t)) = (0,0)$ and $\left(\overline{S}(x,t), \overline{I}(x,t)\right) = \left(\widehat{S}(t), \widehat{I}(t)\right)$ are the lower and the upper solutions of system (1), respectively, and $0 \leqslant S_0(x) \leqslant \widehat{S} = \sup_{x\in\Omega} S_0(x)$ and $0 \leqslant I_0(x) \leqslant \widehat{I} = \sup_{x\in\Omega} I_0(x)$ hold. Denote by

$$f(S,I) := rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{1+\alpha I^2} - \mu S,\ g(S,I) :$$
$$= \frac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \frac{aeI}{1+beI},$$

and a set $\Pi = \{(S,I) : S \geqslant 0, 0 \leqslant I \leqslant 1/\sqrt{\alpha}\}$. Then some simple computations show that

$$f_I(S,I) = \frac{\beta S(\alpha I^2 - 1)}{(1+\alpha I^2)^2} \leq 0,\ g_S(S,I) = \frac{\beta I}{1+\alpha I^2} \geqslant 0.$$

Therefore, we claim that system (1) is a mixed quasi-monotone system and has a unique globally defined solution $(S(x,t), I(x,t))$, satisfying $0 \leqslant S(x,t) \leqslant \widehat{S}(t)$ and $0 \leqslant I(x,t) \leqslant \widehat{I}(t)$,due to Pao [26]. Then the strong maximum principle gives $S(x,t) > 0$ and $I(x,t) > 0$ for $x \in \overline{\Omega}$ and $t \in (0, +\infty)$.

Now we turn our attention to the boundedness of solutions $S(x,t)$ and $I(x,t)$ of the system (1). Here the main technique to establish the bouns of the solutions $S(x,t)$ and $I(x,t)$ is the comparison principle of parabolic equations. Using the S-equation of (1), we have

$$\begin{cases} \frac{\partial S}{\partial t} - d_S \Delta S \leqslant rS\left(1 - \frac{S}{K}\right), & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial v} = 0, & x \in \partial\Omega, t \geqslant 0, \\ S(x,0) = S_0(x) \geqslant 0, & x \in \Omega. \end{cases}$$

Hence, there are $\varepsilon_1 > 0$ and $T_1 > 0$, such that $S(x,t) \leqslant K + \varepsilon_1$ for $\forall x \in \overline{\Omega}$ and $t > T_1$. It then follows I-equation of system (1) that

$$\begin{cases} \frac{\partial I}{\partial t} - d_I \Delta I \leqslant \frac{\beta(K+\varepsilon_1)}{2\sqrt{\alpha}} - (\gamma + d + \delta)I, & x \in \Omega, t > T_1, \\ \frac{\partial I}{\partial v} = 0, & x \in \partial\Omega, t \geqslant T_1, \\ I(x,0) = I_0(x) \geqslant 0, & x \in \Omega. \end{cases}$$

We know that $I \leqslant \frac{\beta(K+\varepsilon_1)}{2\sqrt{\alpha}(\gamma+d+\delta)} + \varepsilon_2$ for $\varepsilon_2 > 0, T_2 > 0, \forall x \in \overline{\Omega}$ and $t > T_2$.

By using the S-equation of (1) again, we get

$$\begin{cases} \frac{\partial S}{\partial t} - d_S \Delta S \geqslant rS\left(1 - \frac{S}{K}\right) - \frac{\beta S}{2\sqrt{\alpha}} - \mu S, & x \in \Omega, t > T_3, \\ \frac{\partial S}{\partial v} = 0, & x \in \partial\Omega, t \geqslant T_3, \\ S(x,0) = S_0(x) \geqslant 0, & x \in \Omega. \end{cases}$$

As such, the comparison principle shows that $S \geqslant \frac{K[2\sqrt{\alpha}(r-\mu)-\beta]}{2r\sqrt{\alpha}} + \varepsilon_3$ for $\varepsilon_3 > 0, T_3 > 0, \forall x \in \overline{\Omega}$ and $t > T_3$. This ends the proof.

Theorem 2.2. Suppose that $d_S, d_I, > 0$, then we have the uniform boundedness results for any solution (S,I) of system (1).(i) There is a positive constant C_1 depending on the initial condition satisfying

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_1, \forall t \geqslant 0. \tag{4}$$

(ii) There is a positive constant C_2 independent of the initial condition fulfilling

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_2, \forall t \geqslant T, \tag{5}$$

for some large $T > 0$.

Proof. To obtain the desire results in (4) and (5), we first verify that

$$\|S(\cdot,t)\|_{L^k(\Omega)} + \|I(\cdot,t)\|_{L^k(\Omega)} \leqslant C_1, \forall t \geqslant 0, \tag{6}$$

is valid. Then for $k = 1$, one has

$$\begin{aligned} &\tfrac{d}{dt}\textstyle\int_\Omega (S+I)dx \\ &= d_S \textstyle\int_\Omega \Delta S dx + d_I \int_\Omega \Delta I dx + r\int_\Omega S\left(1 - \frac{S}{K}\right)dx \\ &-\mu \textstyle\int_\Omega S dx - (\gamma + d + \delta)\int_\Omega I dx - ae\int_\Omega \frac{I}{1+beI}dx \\ &= r\textstyle\int_\Omega S\left(1 - \frac{S}{K}\right)dx - \mu\int_\Omega S dx - (\gamma + d + \delta)\int_\Omega I dx \\ &-ae\textstyle\int_\Omega \frac{I}{1+beI}dx \\ &\leqslant \tfrac{rK|\Omega|}{4} - m_1 \textstyle\int_\Omega (S+I)dx, \end{aligned}$$

where we denote by $m_1 = \min\{\mu, \gamma + d + \delta\}$. It follows that

$$\int_\Omega (S+I)dx \leqslant e^{-m_1 t}\int_\Omega (S_0(x) + I_0(x))dx + \frac{rK|\Omega|}{4}(1 - e^{-m_1 t}),$$

for $\forall x \in \overline{\Omega}, t \geqslant 0$. As such, we obtain

$$\limsup_{t\to\infty} \int_\Omega (S+I)dx \leqslant \frac{rK|\Omega|}{4},\ \forall x \in \overline{\Omega}. \tag{7}$$

Now we assume that (6) is valid for $k-1$. Multiply the S-equation by S^{k-1}, one yields

$$\begin{aligned} &\frac{1}{k}\frac{d}{dt}\int_\Omega S^k dx + d_S(k-1)\int_\Omega S^{k-2}|\nabla S|^2 dx \\ &\quad = \int_\Omega \left[rS^k\left(1 - \frac{S}{K}\right) - \frac{\beta S^k I}{1+\alpha I^2} - \mu S^k\right]dx. \end{aligned} \tag{8}$$

By the same fashion, multiply the I-equation by I^{k-1}, we get

$$\begin{aligned} &\frac{1}{k}\frac{d}{dt}\int_\Omega I^k dx + d_I(k-1)\int_\Omega I^{k-2}|\nabla I|^2 dx \\ &\quad = \int_\Omega \left[\frac{\beta S I^k}{1+\alpha I^2} - (\gamma + d + \delta)I^k - \frac{aeI^k}{1+beI}\right]dx. \end{aligned} \tag{9}$$

Accordingly, on the basis of (8) and (9), one concludes

$$\begin{aligned} &\frac{1}{k}\frac{d}{dt}\int_\Omega S^k dx + d_S(k-1)\int_\Omega S^{k-2}|\nabla S|^2 dx + \frac{1}{k}\frac{d}{dt}\int_\Omega I^k dx + d_I(k-1)\int_\Omega I^{k-2}|\nabla I|^2 dx \\ &= r\int_\Omega S^k\left(1 - \frac{S}{K}\right)dx - \mu\int_\Omega S^k dx - (\gamma + d + \delta)\int_\Omega I^k dx - ae\int_\Omega \frac{I^k}{1+beI}dx \\ &\leqslant \frac{rK}{4}\int_\Omega S^{k-1}dx - \mu\int_\Omega S^k dx - (\gamma + d + \mu)\int_\Omega I^k dx \\ &\leqslant \frac{rK}{4}\int_\Omega S^{k-1}dx - m_1\int_\Omega (S^k + I^k)dx. \end{aligned}$$

On the other hand, (6) is true for $k-1$. Thence there is a positive constant, say M_0, fulfilling $\frac{rK}{4}\int_\Omega S^{k-1}dx \leqslant M_0$. We have

$$\frac{d}{dt}\int_\Omega (S^k + I^k)dx \leqslant M_0 - m_1\int_\Omega (S^k + I^k)dx,$$

namely

$$\int_\Omega (S^k + I^k)dx \leqslant e^{-m_1 t}\int_\Omega (S_0^k(x) + I_0^k(x))dx + \frac{M_0}{m_1}(1 - e^{-m_1 t}), \tag{10}$$

for $\forall x \in \overline{\Omega}, t \geqslant 0$. Moreover, one can deduce from (10) that

$$\limsup_{t\to\infty} \int_\Omega (S^k + I^k)dx \leqslant \frac{M_0}{m_1}, \ \forall x \in \overline{\Omega},$$

for some large $T > 0$. By virtue of [27], one concludes that the proof is completed.

3. Global stability

In this section, we will establish the global stability results of the equilibria to the system (1). To this end, let

$$rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{1+\alpha I^2} - \mu S = 0, \text{ and } \frac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \frac{aeI}{1+beI} = 0.$$

Then we have

Lemma 1. ([21]) Assume that $r > \mu$ is valid, then(i) system (1) has always the equilibrium $E_0 = (0, 0)$;(ii) system (1) has a disease free equilibrium (DFE) $E_f = (S_f, 0) = \left(\frac{(r-\mu)K}{r}, 0\right)$;(iii) system (1) has endemic equilibria (EE) $E_* = (S_*, I_*)$ with $S_* = \frac{K\left[(r-\mu)\alpha I_*^2 - \beta I_* + r - \mu\right]}{r\left(1+\alpha I_*^2\right)}$ and I_* satisfies

$$a_0 I^5 + a_1 I^4 + a_2 I^3 + a_3 I^2 + a_4 I + a_5 = 0,$$

where

$$\begin{aligned}
a_0 &= r\alpha^2 be(d+\delta+\gamma),\\
a_1 &= r\alpha^2(d+\delta+\gamma) + aer\alpha^2,\\
a_2 &= be\alpha r(d+\delta+\gamma+ae)(1-\mathscr{R}_0) + be\alpha(r(d+\delta+\gamma) - ae),\\
a_3 &= r\alpha(d+\delta+\gamma+ae)(2-\mathscr{R}_0) + \tfrac{ber^2(d+\delta+\gamma+ae)^2\mathscr{R}_0^2}{K(r-\mu)^2},\\
a_4 &= ber(d+\delta+\gamma+ae)(1-\mathscr{R}_0) - rabe^2 + \tfrac{r^2(d+\delta+\gamma+ae)^2\mathscr{R}_0^2}{K(r-\mu)^2},\\
a_5 &= r(d+\delta+\gamma+ae)(1-\mathscr{R}_0),
\end{aligned}$$

and $\mathscr{R}_0$ is defined by the basic reproduction number, that is

$$\mathscr{R}_0 = \frac{\beta(r-\mu)K}{r(d+\delta+\gamma+ae)}.$$

Lemma 2. ([21]) Assume that $r > \mu$ is valid,(i) if $ber(d+\gamma+\delta) + \beta^2 K < \beta Kbe(r-\mu), r(d+\gamma+\delta) > ea, 4\alpha(r-\mu)^2 > \beta^2$ and $\mathscr{R}_0 < 1$ hold, then the number of endemic equilibria is 0 or 2;(ii) if $4\alpha(r-\mu)^2 > \beta^2$ and $\mathscr{R}_0 > 1$ hold, then the number of endemic equilibria is 1 or 3;(iii) if $ber(d+\gamma+\delta) + \beta^2 K < \beta Kbe(r-\mu)$, $r(d+\gamma+\delta) > ea$, $4\alpha(r-\mu)^2 > \beta^2$ and $\mathscr{R}_0 = 1$ hold, then system has a unique endemic equilibrium.

Theorem 3.1. Suppose that $\mu < r$ and $\frac{\beta S_f^2}{c_0} \leqslant (\gamma + d + \delta)$ are valid, then DFE $E_f = (S_f, 0)$ is globally asymptotically stable, where c_0 is a positive constant depending on r, α, β, μ and K.

Proof. Define

$$E(t) = \int_\Omega \left(\int_{S_f}^{S} \frac{S^2 - S_f^2}{S^2} dS + \int_0^I I dI \right) dx.$$

Then one yields

$$\begin{aligned}
\dot{E}(t) &= \int_\Omega \frac{\partial S}{\partial t}\left(1 - \frac{S_f^2}{S^2}\right)dx + \int_\Omega \frac{\partial I}{\partial t}dx\\
&= \int_\Omega d_S \Delta S\left(1 - \frac{S_f^2}{S^2}\right)dx + \int_\Omega d_I \Delta I dx\\
&+ \int_\Omega \left(1 - \frac{S_f^2}{S^2}\right)\left[rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{1+\alpha I^2} - \mu S\right]dx\\
&+ \int_\Omega \left(\frac{\beta SI}{1+\alpha I^2} - (\gamma+d+\delta)I - \frac{aeI}{1+beI}\right)dx\\
&= E_1 + E_2,
\end{aligned}$$

where

$$E_1 = \int_\Omega d_S \Delta S\left(1 - \frac{S_f^2}{S^2}\right)dx + \int_\Omega d_I \Delta I dx = -2d_S S_f^2 \int_\Omega \frac{|\nabla S|^2}{S^3}dx \leq 0,$$

and

$$\begin{aligned}
E_2 &= \int_\Omega \left(1 - \tfrac{S_f^2}{S^2}\right)\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx\\
&+ \int_\Omega \left(\tfrac{\beta SI}{1+\alpha I^2} - (\gamma+d+\delta)I - \tfrac{aeI}{1+beI}\right)dx\\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - \mu S\right]dx - \int_\Omega \tfrac{S_f^2}{S^2}\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx\\
&- \int_\Omega \left((\gamma+d+\delta)I + \tfrac{aeI}{1+beI}\right)dx\\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - \mu S\right]dx - \int_\Omega \left[r\tfrac{S_f^2}{S}\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta S_f^2 I}{S\left(1+\alpha I^2\right)} - \tfrac{\mu S_f^2}{S}\right]dx\\
&- \int_\Omega \left((\gamma+d+\delta)I + \tfrac{aeI}{1+beI}\right)dx\\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - r\tfrac{S_f^2}{S}\left(1 - \tfrac{S}{K}\right) + \tfrac{\mu S_f^2}{S} - \mu S\right]dx\\
&+ \int_\Omega \left(\tfrac{\beta S_f^2 I}{S\left(1+\alpha I^2\right)} - (\gamma+d+\delta)I - \tfrac{aeI}{1+beI}\right)dx\\
&= r\int_\Omega \left[S\left(1 - \tfrac{S}{K}\right) - \tfrac{S_f^2}{S}\left(1 - \tfrac{S}{K}\right) + \tfrac{S_f^2}{S}\left(1 - \tfrac{S_f}{K}\right) - S\left(1 - \tfrac{S_f}{K}\right)\right]dx\\
&+ \int_\Omega \left(\tfrac{\beta S_f^2 I}{S\left(1+\alpha I^2\right)} - (\gamma+d+\delta)I - \tfrac{aeI}{1+beI}\right)dx\\
&= -\tfrac{r}{K}\int_\Omega \tfrac{S+S_f}{S}\left(S - S_f\right)^2 dx + \int_\Omega \left(\tfrac{\beta S_f^2 I}{S\left(1+\alpha I^2\right)} - (\gamma+d+\delta)I - \tfrac{aeI}{1+beI}\right)dx\\
&\leqslant -\tfrac{r}{K}\int_\Omega \tfrac{S+S_f}{S}\left(S - S_f\right)^2 dx + \int_\Omega I\left(\tfrac{\beta S_f^2}{S} - (\gamma+d+\delta)\right)dx.
\end{aligned}$$

Now due to Theorem 2.1 we know that there is a positive constants, say c_0, satisfying $S(x,t) \geqslant c_0$ for some $t > 0$ and $x \in \overline{\Omega}$. Consequently, if condition $\frac{\beta S_f^2}{c_0} \leqslant (\gamma + d + \delta)$ holds, one infers $\frac{\beta S_f^2}{S} - (\gamma + d + \delta) \leqslant 0$, namely $E_2 \leqslant 0$ is valid. Therefore, DFE $E_f = (S_f, 0)$ is globally asymptotically stable. The proof is completed.

Theorem 3.2. Suppose that $\mu < r$ and $\beta S_f \leqslant (\gamma + d + \delta)$ are valid, then DFE $E_f = (S_f, 0)$ is globally asymptotically stable.

Proof. Define

$$V(t) = \int_\Omega \left(S - S_f - S_f \ln\frac{S}{S_f}\right)dx + \int_\Omega I dx.$$

Then one yields

$$\begin{aligned}
\dot{V}(t) &= \int_\Omega \tfrac{\partial S}{\partial t}\left(1 - \tfrac{S_f}{S}\right)dx + \int_\Omega \tfrac{\partial I}{\partial t}dx\\
&= \int_\Omega d_S \Delta S\left(1 - \tfrac{S_f}{S}\right)dx + \int_\Omega d_I \Delta I dx\\
&+ \int_\Omega \left(1 - \tfrac{S_f}{S}\right)\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx\\
&+ \int_\Omega \left(\tfrac{\beta SI}{1+\alpha I^2} - (\gamma+d+\delta)I - \tfrac{aeI}{1+beI}\right)dx\\
&= V_1 + V_2,
\end{aligned}$$

where

$$V_1 = \int_\Omega d_S \Delta S\left(1 - \frac{S_f}{S}\right)dx + \int_\Omega d_I \Delta I dx = -d_S S_f \int_\Omega \frac{|\nabla S|^2}{S^2} dx \le 0,$$

and

$$\begin{aligned}
V_2 &= \int_\Omega \left(1 - \tfrac{S_f}{S}\right)\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx \\
&+ \int_\Omega \left(\tfrac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \tfrac{aeI}{1+beI}\right)dx \\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - \mu S\right]dx - \int_\Omega \tfrac{S_f}{S}\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx \\
&- \int_\Omega \left((\gamma + d + \delta)I + \tfrac{aeI}{1+beI}\right)dx \\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - \mu S\right]dx - \int_\Omega \left[rS_f\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta S_f I}{1+\alpha I^2} - \mu S_f\right]dx \\
&- \int_\Omega \left((\gamma + d + \delta)I + \tfrac{aeI}{1+beI}\right)dx \\
&= \int_\Omega \left[rS\left(1 - \tfrac{S}{K}\right) - rS_f\left(1 - \tfrac{S}{K}\right) + \mu S_f - \mu S\right]dx \\
&+ \int_\Omega \left(\tfrac{\beta S_f I}{1+\alpha I^2} - (\gamma + d + \delta)I - \tfrac{aeI}{1+beI}\right)dx \\
&= r\int_\Omega \left[S\left(1 - \tfrac{S}{K}\right) - S_f\left(1 - \tfrac{S}{K}\right) + S_f\left(1 - \tfrac{S_f}{K}\right) - S\left(1 - \tfrac{S_f}{K}\right)\right]dx \\
&+ \int_\Omega \left(\tfrac{\beta S_f I}{1+\alpha I^2} - (\gamma + d + \delta)I - \tfrac{aeI}{1+beI}\right)dx \\
&= -\tfrac{r}{K}\int_\Omega (S - S_f)^2 dx + \int_\Omega \left(\tfrac{\beta S_f I}{1+\alpha I^2} - (\gamma + d + \delta)I - \tfrac{aeI}{1+beI}\right)dx \\
&\leqslant -\tfrac{r}{K}\int_\Omega (S - S_f)^2 dx + \int_\Omega I\big(\beta S_f - (\gamma + d + \delta)\big)dx.
\end{aligned}$$

Thence, we can claim that $V_2 \leqslant 0$ if $\beta S_f \leqslant (\gamma + d + \delta)$ is valid, and DFE $E_f = (S_f, 0)$ is globally asymptotically stable. This ends the proof.

Remark 3.1. Both Theorem 3.1 and Theorem 3.2 show that the DFE $E_f = (S_f, 0)$ is globally asymptotically stable, but we need to emphasize is that the hypothesis conditions in Theorem 3.2 are simpler than conditions in Theorem 3.1 due to the different Lyapunov functions.

Next we suppose that (iii) in Lemma 2 holds such that the model (1) has a unique endemic equilibrium EE.

Theorem 3.3. Suppose that (iii) in Lemma 2 and hypothesis $\frac{rS_*}{K} \geqslant \frac{\alpha\beta I_* C_0^2}{4}$ are satisfied, then endemic equilibrium EE is globally asymptotically stable, where C_0 is a positive constants depends on $\alpha, \beta, d, \gamma, \delta$ and K.

Proof. Define

$$L(t) = \int_\Omega \left(S - S_* - S_* \ln\frac{S}{S_*}\right)dx + \int_\Omega \left(I - I_* - I_* \ln\frac{I}{I_*}\right)dx.$$

Then direct computation gives

$$\begin{aligned}
\dot{L}(t) &= \int_\Omega \tfrac{\partial S}{\partial t}\left(1 - \tfrac{S_*}{S}\right)dx + \int_\Omega \tfrac{\partial I}{\partial t}\left(1 - \tfrac{I_*}{I}\right)dx \\
&= \int_\Omega d_S \Delta S\left(1 - \tfrac{S_*}{S}\right)dx + \int_\Omega d_I \Delta I\left(1 - \tfrac{I_*}{I}\right)dx \\
&+ \int_\Omega \left(1 - \tfrac{S_*}{S}\right)\left[rS\left(1 - \tfrac{S}{K}\right) - \tfrac{\beta SI}{1+\alpha I^2} - \mu S\right]dx \\
&+ \int_\Omega \left(1 - \tfrac{I_*}{I}\right)\left(\tfrac{\beta SI}{1+\alpha I^2} - (\gamma + d + \delta)I - \tfrac{aeI}{1+beI}\right)dx \\
&= L_1 + L_2 + L_3,
\end{aligned}$$

where

$$\begin{aligned}
L_1 &= \int_\Omega d_S \Delta S\left(1 - \frac{S_*}{S}\right)dx + \int_\Omega d_I \Delta I\left(1 - \frac{I_*}{I}\right)dx \\
&= -d_S \int_\Omega \nabla S d\left(1 - \frac{S_*}{S}\right) - d_I \int_\Omega \nabla I d\left(1 - \frac{I_*}{I}\right) \\
&= -d_S S_* \int_\Omega \frac{|\nabla S|^2}{S^2} dx - d_I I_* \int_\Omega \frac{|\nabla I|^2}{I^2} dx \leqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
L_2 &= \int_\Omega \left(1 - \frac{S_*}{S}\right)\left[rS\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{1 + \alpha I^2} - \mu S\right]dx \\
&= \int_\Omega (S - S_*)\left[r\left(1 - \frac{S}{K}\right) - \frac{\beta I}{1 + \alpha I^2} - r\left(1 - \frac{S_*}{K}\right) + \frac{\beta I_*}{1 + \alpha I_*^2}\right]dx \\
&= -\int_\Omega \frac{r}{K}(S - S_*)^2 dx - \int_\Omega \frac{\beta(\alpha I I_* - 1)(S - S_*)(I - I_*)}{\left(1 + \alpha I_*^2\right)\left(1 + \alpha I^2\right)}dx,
\end{aligned}$$

and

$$\begin{aligned}
L_3 &= \int_\Omega (I - I_*)\left(\tfrac{\beta S}{1+\alpha I^2} - (\gamma + d + \delta) - \tfrac{ae}{1+beI}\right) \\
&= \int_\Omega (I - I_*)\left(\tfrac{\beta S}{1+\alpha I^2} - \tfrac{\beta S_*}{1+\alpha I_*^2} + \tfrac{ae}{1+beI_*} - \tfrac{ae}{1+beI}\right)dx \\
&= -\int_\Omega \tfrac{ae(I-I_*)^2}{(1+beI_*)(1+beI)}dx - \int_\Omega \tfrac{\alpha\beta S_*(I+I_*)(I-I_*)^2}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)}dx \\
&\qquad + \int_\Omega \tfrac{\beta\left(1+\alpha I_*^2\right)(S-S_*)(I-I_*)}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)}dx.
\end{aligned}$$

As such, one yields

$$\begin{aligned}
L_2 + L_3 &= -\int_\Omega \tfrac{r}{K}(S - S_*)^2 dx \\
&- \int_\Omega \tfrac{ae(I-I_*)^2}{(1+beI_*)(1+beI)}dx - \int_\Omega \tfrac{\alpha\beta S_*(I+I_*)(I-I_*)^2}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)}dx \\
&- \int_\Omega \tfrac{\beta[\alpha I_*(I-I_*)-2](S-S_*)(I-I_*)}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)}dx \\
&= -\int_\Omega X^T Q X dx,
\end{aligned}$$

where we define $X = \begin{pmatrix} S - S_* \\ I - I_* \end{pmatrix}$ and the matrix

$$Q = \begin{pmatrix} \frac{r}{K} & \frac{\beta[\alpha I_*(I-I_*)-2]}{2\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} \\ \frac{\beta[\alpha I_*(I-I_*)-2]}{2\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} & \frac{ae}{(1+beI_*)(1+beI)} + \frac{\alpha\beta S_*(I+I_*)}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} \end{pmatrix}.$$

We can find that $TraceQ = \frac{r}{K} + \frac{ae}{(1+beI_*)(1+beI)} + \frac{\alpha\beta S_*(I+I_*)}{\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} > 0$ and

$$\begin{aligned}
DetQ &= \frac{aer}{K(1 + beI_*)(1 + beI)} + \frac{\alpha\beta r S_*(I + I_*)}{K\left(1 + \alpha I_*^2\right)\left(1 + \alpha I^2\right)} \\
&- \frac{\beta^2[\alpha I_*(I - I_*) - 2]^2}{4\left(1 + \alpha I_*^2\right)^2\left(1 + \alpha I^2\right)^2}.
\end{aligned}$$

Now we show that $DetQ > 0$ with some assumptions.Indeed, one only need to guarantee $\frac{\alpha\beta r S_*(I+I_*)}{K} - \frac{\beta^2[\alpha I_*(I-I_*)-2]^2}{4\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} \geqslant 0$ is valid.Obviously, $\frac{\alpha\beta r S_*(I+I_*)}{K} > \frac{\alpha\beta r S_* I_*}{K}$ is true.On the other hand, $\frac{\beta^2[\alpha I_*(I-I_*)-2]^2}{4\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} < \frac{\alpha^2\beta^2 I_*^2 I^2}{4} \leqslant \frac{\alpha^2\beta^2 I_*^2 C_0^2}{4}$, where C_0 is a positive constant depends on $\alpha, \beta, d, \gamma, \delta$ and K due to Theorem 2.1. Consequently, if we restrict $\frac{\alpha\beta r S_* I_*}{K} \geqslant \frac{\alpha^2\beta^2 I_*^2 C_0^2}{4}$, then we can obtain $\frac{\alpha\beta r S_*(I+I_*)}{K} > \frac{\alpha\beta r S_* I_*}{K} \geqslant \frac{\alpha^2\beta^2 I_*^2 C_0^2}{4} > \frac{\beta^2[\alpha I_*(I-I_*)-2]^2}{4\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)}$, namely$\frac{\alpha\beta r S_*(I+I_*)}{K\left(1+\alpha I_*^2\right)\left(1+\alpha I^2\right)} - \frac{\beta^2[\alpha I_*(I-I_*)-2]^2}{4\left(1+\alpha I_*^2\right)^2\left(1+\alpha I^2\right)^2} > 0$ is valid. One claims that $DetQ > 0$, and the matrix Q is positively definite. It follows that $V_2 + V_3 = -\int_\Omega X^T Q X dx < 0$, and therefore $\dot{V} = V_1 + V_2 + V_3 \leqslant 0$ holds. The endemic equilibrium EE is globally asymptotically stable. This finishes the proof.

4. Positive steady states

In this section, we consider the problem of the steady states to the following system

$$\begin{cases} -d_S \Delta S(x) = rS(x)\left(1 - \frac{S(x)}{K}\right) - \frac{\beta S(x) I(x)}{1+\alpha I^2(x)} - \mu S(x), & x \in \Omega, \\ -d_I \Delta I(x) = \frac{\beta S(x) I(x)}{1+\alpha I^2(x)} - (\gamma + d + \delta) I(x) - \frac{aeI(x)}{1+beI(x)}, & x \in \Omega, \\ \frac{\partial S(x)}{\partial v} = \frac{\partial I(x)}{\partial v} = 0, & x \in \partial\Omega. \end{cases} \tag{11}$$

4.1. A priori estimates

Theorem 4.1. Suppose that $0 < \mu < r, 0 < \beta < 2\sqrt{\alpha}(r - \mu)$ are valid, then for any positive solution $(S(x), I(x))$ of system (11), one has

$$\frac{K[2\sqrt{\alpha}(r-\mu) - \beta]}{2r\sqrt{\alpha}} \leqslant S(x) \leqslant K, \ 0 < I(x) \leqslant \frac{\beta K}{2\sqrt{\alpha}(\gamma + d + \delta)}.$$

Proof. Suppose that $(S(x), I(x))$ is a positive solution of system (11), and let

$$S(x_1) = \max_{x \in \overline{\Omega}} S(x), \ I(y_1) = \max_{x \in \overline{\Omega}} I(x).$$

Due to the maximum principle (cf. [28–30]) and the S-equation of (11), we yield

$$0 \leqslant rS(x_1)\left(1 - \frac{S(x_1)}{K}\right) - \frac{\beta S(x_1) I(x_1)}{1 + \alpha I^2(x_1)} - \mu S(x_1)$$
$$\leqslant rS(x_1)\left(1 - \frac{S(x_1)}{K}\right),$$

so we have $S(x_1) \leqslant K$. By making use of the maximum principle and the I-equation of (11), one deduces

$$0 \leqslant \frac{\beta S(y_1) I(y_1)}{1 + \alpha I^2(y_1)} - (\gamma + d + \delta) I(y_1) - \frac{aeI(y_1)}{1 + beI(y_1)}$$
$$\leqslant \frac{\beta K}{2\sqrt{\alpha}} - (\gamma + d + \delta) I(y_1),$$

it gives that $I(y_1) \leqslant \frac{\beta K}{2\sqrt{\alpha}(\gamma+d+\delta)}$. Now we define $S(x_0) = \min_{x \in \overline{\Omega}} S(x)$, then by making use of the S-equation again, we get

$$rS(x_0)\left(1 - \frac{S(x_0)}{K}\right) - \frac{\beta S(x_0)}{2\sqrt{\alpha}} - \mu S(x_0)$$
$$\leqslant rS(x_0)\left(1 - \frac{S(x_0)}{K}\right) - \frac{\beta S(x_0) I(x_0)}{1 + \alpha I^2(x_0)} - \mu S(x_0) \leq 0,$$

this gives $S(x_0) \geqslant \frac{K[2\sqrt{\alpha}(r-\mu)-\beta]}{2r\sqrt{\alpha}}$. This ends the proof.

In what follows, let us denote $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$ and $\lim_{i\to\infty} \lambda_i = \infty$ by the complete set of eigenvalues of the operator $-\Delta$ with no-flux boundary conditions in Ω, and

$$\overline{S} = \frac{1}{|\Omega|} \int_\Omega S(x) dx, \overline{I} = \frac{1}{|\Omega|} \int_\Omega I(x) dx$$

by their averages over domain Ω. Then we have

$$\int_\Omega (S(x) - \overline{S}) dx = 0, \int_\Omega (I(x) - \overline{I}) dx = 0.$$

Denote by $\phi = S(x) - \overline{S}$ and $\psi = I(x) - \overline{I}$. We thus have $\int_\Omega \phi dx = 0$ and $\int_\Omega \psi dx = 0$ are true.

4.2. Some propositions

Proposition 1. Suppose that $d_S, d_I > 0$, then for any solution $(S(x), I(x))$ of system (11), we have

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla \phi|^2 dx \leqslant \frac{r^2 K^2 |\Omega| (1+\lambda_1)}{16 d_S^2 \lambda_1^2},$$
$$\int_\Omega \psi^2 dx + \int_\Omega |\nabla \psi|^2 dx \leqslant \frac{K^2 \beta^2 |\Omega| (1+\lambda_1)}{4\alpha d_I^2 \lambda_1^2}.$$

where λ_1 is the first non-zero eigenvalue of $-\Delta$ on Ω about zero-flux boundary conditions.

Proof. Multiply ϕ to the S-equation of (11) and note that the Cauchy–Schwarz inequality, one obtains

$$d_S \int_\Omega |\nabla \phi|^2 dx = \int_\Omega \phi \left[rS(x)\left(1 - \frac{S(x)}{K}\right) - \frac{\beta S(x) I(x)}{1 + \alpha I^2(x)} - \mu S(x) \right] dx$$
$$\leqslant \frac{rK}{4} \int_\Omega |\phi| dx$$
$$\leqslant \frac{rK\sqrt{|\Omega|}}{4} \left(\int_\Omega |\phi|^2 dx \right)^{\frac{1}{2}}.$$

By a similar way, multiply ψ to the I-equation of (11) and use Cauchy–Schwarz inequality

$$d_A \int_\Omega |\nabla \psi|^2 dx = \int_\Omega \psi \left(\frac{\beta S(x) I(x)}{1+\alpha I^2(x)} - (\gamma + d + \delta) I(x) - \frac{aeI(x)}{1+beI(x)} \right) dx$$
$$\leqslant \frac{\beta K}{2\sqrt{\alpha}} \int_\Omega |\psi| dx$$
$$\leqslant \frac{\beta K \sqrt{|\Omega|}}{2\sqrt{\alpha}} \left(\int_\Omega |\psi|^2 dx \right)^{\frac{1}{2}}.$$

Note the Poincaré's inequality, they are

$$\int_\Omega \phi^2 dx \leqslant \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 dx, \ \int_\Omega \psi^2 dx \leqslant \frac{1}{\lambda_1} \int_\Omega |\nabla \psi|^2 dx,$$

we can yield

$$d_S \int_\Omega |\nabla \phi|^2 dx \leqslant \frac{rK\sqrt{|\Omega|}}{4} \left(\int_\Omega |\phi|^2 dx \right)^{\frac{1}{2}} \leqslant \frac{rK}{4} \sqrt{\frac{|\Omega|}{\lambda_1}} \left(\int_\Omega |\nabla \phi|^2 dx \right)^{\frac{1}{2}},$$
$$d_I \int_\Omega |\nabla \psi|^2 dx \leqslant \frac{\beta K \sqrt{|\Omega|}}{2\sqrt{\alpha}} \left(\int_\Omega |\psi|^2 dx \right)^{\frac{1}{2}} \leqslant \frac{\beta K}{2\sqrt{\alpha}} \sqrt{\frac{|\Omega|}{\lambda_1}} \left(\int_\Omega |\nabla \psi|^2 dx \right)^{\frac{1}{2}}.$$

These two inequalities show that

$$\int_\Omega |\nabla \phi|^2 dx \leqslant \frac{r^2 K^2 |\Omega|}{16 d_S^2 \lambda_1}, \ \int_\Omega |\nabla \psi|^2 dx \leqslant \frac{\beta^2 K^2 |\Omega|}{4\alpha d_I^2 \lambda_1}.$$

Consequently, one yields

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla \phi|^2 dx \leqslant \frac{r^2 K^2 |\Omega| (1+\lambda_1)}{16 d_S^2 \lambda_1^2},$$
$$\int_\Omega \psi^2 dx + \int_\Omega |\nabla \psi|^2 dx \leqslant \frac{K^2 \beta^2 |\Omega| (1+\lambda_1)}{4\alpha d_I^2 \lambda_1^2}.$$

The proof is completed.

Proposition 2. Suppose that $r > 2\mu, 0 < \beta \leqslant \sqrt{\alpha}(r - 2\mu)$ and $\frac{K^2 \beta^2}{4(\gamma+d+\delta)^2} < 1$, then one has

$$\frac{16\alpha(3d_I\lambda_1-4\beta K)(\gamma+d+\delta)^6 d_I\lambda_1}{\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2} \leqslant \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |\nabla\psi|^2 dx} \leqslant \frac{4\beta^2K^2}{3d_S^2\lambda_1^2},$$

and

$$\frac{16\alpha(3d_I\lambda_1-4\beta K)(\gamma+d+\delta)^6 d_I\lambda_1^2}{(1+\lambda_1)\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2} \leqslant \frac{\int_\Omega\left(|\nabla\phi|^2+\phi^2\right)dx}{\int_\Omega\left(|\nabla\psi|^2+\psi^2\right)dx}$$
$$\leqslant \frac{4\beta^2K^2(1+\lambda_1)}{3d_S^2\lambda_1^3},$$

where λ_1 is the first positive eigenvalue of $-\Delta$ on Ω about zero-flux boundary conditions.

Proof. Multiply the S-equation of (11) by ϕ, we get

$$0 = \int_\Omega\left[d_S\Delta S(x)+rS(x)\left(1-\tfrac{S(x)}{K}\right)-\tfrac{\beta S(x)I(x)}{1+\alpha I^2(x)}-\mu S(x)\right]\phi dx$$
$$= -d_S\int_\Omega|\nabla\phi|^2dx + r\int_\Omega S(x)\left(1-\tfrac{S(x)}{K}\right)\phi dx - \beta\int_\Omega \tfrac{S(x)I(x)}{1+\alpha I^2(x)}\phi dx - \mu\int_\Omega S(x)\phi dx$$
$$= -d_S\int_\Omega|\nabla\phi|^2dx + r\int_\Omega\left(1-\tfrac{S(x)+\overline{S}}{K}\right)\phi^2dx - \beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx$$
$$-\beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx - \mu\int_\Omega\phi^2dx.$$

It is noticed that $1-\frac{S(x)+\overline{S}}{K}\leqslant 0$ is valid as $r>2\mu$ and $0<\beta\leqslant\sqrt{\alpha}(r-2\mu)$. As such, one can obtain

$$\beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx$$
$$= -d_S\int_\Omega|\nabla\phi|^2dx + r\int_\Omega\left(1-\tfrac{S(x)+\overline{S}}{K}\right)\phi^2dx$$
$$-\beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx - \mu\int_\Omega\phi^2dx.$$

We claim $\int_\Omega\phi\psi dx<0$ since $\frac{K^2\beta^2}{4(\gamma+d+\delta)^2}<1$. This implies

$$d_S\int_\Omega|\nabla\phi|^2dx = r\int_\Omega\left(1-\tfrac{S(x)+\overline{S}}{K}\right)\phi^2dx$$
$$-\beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx$$
$$-\beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx - \mu\int_\Omega\phi^2dx$$
$$\leqslant \beta K\int_\Omega|\phi\psi|dx,$$

it follows that

$$d_S\int_\Omega|\nabla\phi|^2dx \leqslant \beta K\int_\Omega|\phi\psi|dx$$
$$\leqslant \tfrac{d_S\lambda_1}{4}\int_\Omega\phi^2dx + \tfrac{\beta^2K^2}{d_S\lambda_1}\int_\Omega\psi^2dx$$
$$\leqslant \tfrac{d_S}{4}\int_\Omega|\nabla\phi|^2dx + \tfrac{\beta^2K^2}{d_S\lambda_1^2}\int_\Omega|\nabla\psi|^2dx.$$

As a result, one has

$$\frac{3d_S}{4}\int_\Omega|\nabla\phi|^2dx \leqslant \frac{\beta^2K^2}{d_S\lambda_1^2}\int_\Omega|\nabla\psi|^2dx. \tag{12}$$

Similarly, multiply the I-equation of (11) by ψ, we can deduce

$$0 = \int_\Omega\left(d_I\Delta I(x)+\tfrac{\beta S(x)I(x)}{1+\alpha I^2(x)}-(\gamma+d+\delta)I(x)-\tfrac{aeI(x)}{1+beI(x)}\right)\psi dx$$
$$= -d_I\int_\Omega|\nabla\psi|^2dx + \beta\int_\Omega\left(\tfrac{S(x)I(x)}{1+\alpha I^2(x)}-\tfrac{\overline{SI}}{1+\alpha\overline{I}^2}\right)\psi dx - (\gamma+d+\delta)\int_\Omega\psi^2dx$$
$$+\int_\Omega\left(\tfrac{ae\overline{I}}{1+be\overline{I}}-\tfrac{aeI(x)}{1+beI(x)}\right)\psi dx$$
$$= -d_I\int_\Omega|\nabla\psi|^2dx + \beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx + \beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\psi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx$$
$$-(\gamma+d+\delta)\int_\Omega\psi^2dx - ae\int_\Omega\tfrac{\psi^2}{(1+beI(x))\left(1+be\overline{I}\right)}$$
$$\leqslant -d_I\int_\Omega|\nabla\psi|^2dx + \beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx + \beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\psi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx.$$

It then follows that

$$d_I\int_\Omega|\nabla\psi|^2dx \leqslant \beta\int_\Omega \tfrac{I(x)\left(1+\alpha\overline{I}^2\right)\phi\psi}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx + \beta\int_\Omega \tfrac{\overline{S}\left(1-\alpha\overline{I}I(x)\right)\psi^2}{\left(1+\alpha I^2(x)\right)\left(1+\alpha\overline{I}^2(x)\right)}dx$$
$$\leqslant \tfrac{K^3\beta^3+4\beta K(\gamma+d+\delta)^2}{8\sqrt{\alpha}(\gamma+d+\delta)^3}\int_\Omega|\phi\psi|dx + \beta K\int_\Omega\psi^2dx$$
$$\leqslant \tfrac{K^3\beta^3+4\beta K(\gamma+d+\delta)^2}{8\sqrt{\alpha}(\gamma+d+\delta)^3}\int_\Omega|\phi\psi|dx + \tfrac{\beta K}{\lambda_1}\int_\Omega|\nabla\psi|^2dx.$$

Consequently, if $0<4\beta K<3\lambda_1 d_I$ is valid, one has

$$\tfrac{d_I\lambda_1-\beta K}{\lambda_1}\int_\Omega|\nabla\psi|^2dx \leqslant \tfrac{K^3\beta^3+4\beta K(\gamma+d+\delta)^2}{8\sqrt{\alpha}(\gamma+d+\delta)^3}\int_\Omega|\phi\psi|dx$$
$$\leqslant \tfrac{d_I\lambda_1}{4}\int_\Omega\psi^2dx + \tfrac{\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2}{64\alpha(\gamma+d+\delta)^6 d_I\lambda_1}\int_\Omega|\phi|^2dx$$
$$\leqslant \tfrac{d_I}{4}\int_\Omega|\nabla\psi|^2dx + \tfrac{\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2}{64\alpha(\gamma+d+\delta)^6 d_I\lambda_1^2}\int_\Omega|\nabla\phi|^2dx.$$

Henceforth

$$\frac{3d_I\lambda_1-4\beta K}{4\lambda_1}\int_\Omega|\nabla\psi|^2dx \leqslant \frac{\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2}{64\alpha(\gamma+d+\delta)^6 d_I\lambda_1^2} \times \int_\Omega|\nabla\phi|^2dx. \tag{13}$$

As a result, benefitting from (12) and (13), we have

$$\frac{16\alpha(3d_I\lambda_1-4\beta K)(\gamma+d+\delta)^6 d_I\lambda_1}{\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2} \leqslant \frac{\int_\Omega|\nabla\phi|^2dx}{\int_\Omega|\nabla\psi|^2dx} \leqslant \frac{4\beta^2K^2}{3d_S^2\lambda_1^2}.$$

Furthermore, we note that

$$\int_\Omega\left(|\nabla\phi|^2+\phi^2\right)dx \leqslant \frac{1+\lambda_1}{\lambda_1}\int_\Omega\nabla|\phi|^2x,\quad \int_\Omega\left(|\nabla\psi|^2+\psi^2\right)dx \leqslant \frac{1+\lambda_1}{\lambda_1}\int_\Omega\nabla|\psi|^2x.$$

We, therefore, obtain

$$\frac{\int_\Omega\left(|\nabla\phi|^2+\phi^2\right)dx}{\int_\Omega\left(|\nabla\psi|^2+\psi^2\right)dx} \leqslant \frac{(1+\lambda_1)\int_\Omega|\nabla\phi|^2dx}{\lambda_1\int_\Omega\left(|\nabla\psi|^2+\psi^2\right)dx}$$
$$\leqslant \frac{(1+\lambda_1)\int_\Omega|\nabla\phi|^2dx}{\lambda_1\int_\Omega|\nabla\psi|^2dx} \leqslant \frac{4\beta^2K^2(1+\lambda_1)}{3d_S^2\lambda_1^3},$$

and

$$\frac{\int_\Omega\left(|\nabla\phi|^2+\phi^2\right)dx}{\int_\Omega\left(|\nabla\psi|^2+\psi^2\right)dx} \geqslant \frac{\lambda_1\int_\Omega\left(|\nabla\phi|^2+\phi^2\right)dx}{(1+\lambda_1)\int_\Omega|\nabla\psi|^2dx} \geqslant \frac{\lambda_1\int_\Omega|\nabla\phi|^2dx}{(1+\lambda_1)\int_\Omega|\nabla\psi|^2dx}$$
$$\geqslant \frac{16\alpha(3d_I\lambda_1-4\beta K)(\gamma+d+\delta)^6 d_I\lambda_1^2}{(1+\lambda_1)\left[K^3\beta^3+4\beta K(\gamma+d+\delta)^2\right]^2}.$$

The proof is finished.

4.3. Nonexistence of the positive steady states

Theorem 4.2. Suppose that $d_S, d_I > 0$ hold, then system (11) has no nonconstant steady state as $d_S > d_S^*$ and $d_I > d_I^*$, where

$$d_S^* = \frac{1}{\lambda_1}\left(\frac{K^3\beta^3 + 8r(\gamma+d+\delta)^2}{8(\gamma+d+\delta)^2} + \frac{\beta}{4\sqrt{\alpha}}\right),$$
$$d_I^* = \frac{1}{\lambda_1}\left(\frac{K^3\beta^3}{8(\gamma+d+\delta)^2} + \frac{\beta(1+4K\sqrt{\alpha})}{4\sqrt{\alpha}}\right),$$

and λ_1 is the first non-zero eigenvalue of $-\Delta$ on Ω with respect to no-flux boundary conditions.

Proof. Multiply the S-equation of (11) by ϕ, we have

$$\begin{aligned}
d_S \textstyle\int_\Omega |\nabla\phi|^2 dx &= \textstyle\int_\Omega \left[rS(x)\left(1-\frac{S(x)}{K}\right) - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right]\phi dx \\
&= \textstyle\int_\Omega \left[rS(x)\left(1-\frac{S(x)}{K}\right) - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right]\phi dx \\
&\quad - \textstyle\int_\Omega \left[r\overline{S}\left(1-\frac{\overline{S}}{K}\right) - \frac{\beta\overline{S}\overline{I}}{1+\alpha\overline{I}^2} - \mu\overline{S}\right]\phi dx \\
&= r\textstyle\int_\Omega S(x)\left(1-\frac{S(x)}{K}\right)\phi dx - \int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\phi dx - \mu\int_\Omega S(x)\phi dx \\
&= S_1 + S_2 + S_3,
\end{aligned}$$

where

$$S_1 = r\int_\Omega S(x)\left(1-\frac{S(x)}{K}\right)\phi dx,\ S_2 = -\int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\phi dx,\ S_3 = -\mu\int_\Omega S(x)\phi dx.$$

Then we have

$$\begin{aligned}
&S_1 - \textstyle\int_\Omega r\overline{S}\left(1-\frac{\overline{S}}{K}\right)\phi dx \\
&= r\textstyle\int_\Omega \left[S(x)\left(1-\frac{S(x)}{K}\right) - \overline{S}\left(1-\frac{\overline{S}}{K}\right)\right]\phi dx, \\
&= r\textstyle\int_\Omega \left(1-\frac{S(x)+\overline{S}}{K}\right)\phi^2 dx \\
&\leqslant r\textstyle\int_\Omega \phi^2 dx,
\end{aligned}$$

$$\begin{aligned}
&S_2 + \textstyle\int_\Omega \frac{\beta\overline{S}\overline{I}}{1+\alpha\overline{I}^2}\phi dx \\
&= \beta\textstyle\int_\Omega \left(\frac{\overline{S}\overline{I}}{1+\alpha\overline{I}^2} - \frac{S(x)I(x)}{1+\alpha I^2(x)}\right)\phi dx \\
&= \textstyle\int_\Omega \frac{\beta\left(\alpha I\overline{I}\overline{S} - S(x)\right)\phi\psi}{\left(1+\alpha\overline{I}^2\right)\left(1+\alpha I^2\right)} dx - \int_\Omega \frac{\beta\overline{I}\left(1+\alpha I\overline{I}\right)}{\left(1+\alpha\overline{I}^2\right)\left(1+\alpha I^2\right)}\phi^2 dx \\
&\leqslant \tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2}\textstyle\int_\Omega \phi^2 dx + \tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2}\int_\Omega \psi^2 dx,
\end{aligned}$$

and

$$S_3 + \mu\int_\Omega \overline{S}\phi dx = -\mu\int_\Omega \phi^2 dx.$$

Hence

$$\begin{aligned}
d_S \textstyle\int_\Omega |\nabla\phi|^2 dx &\leqslant \tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2}\textstyle\int_\Omega \phi^2 dx \\
&\quad + \tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2}\textstyle\int_\Omega \psi^2 dx + r\int_\Omega \phi^2 dx - \mu\int_\Omega \phi^2 dx \\
&\leqslant \tfrac{K^3\beta^3 + 8r(\gamma+d+\delta)^2}{8(\gamma+d+\delta)^2}\textstyle\int_\Omega \phi^2 dx + \tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2}\int_\Omega \psi^2 dx.
\end{aligned} \tag{14}$$

In what follows, multiply the I-equation of (11) by ψ, we have

$$\begin{aligned}
d_I \textstyle\int_\Omega |\nabla\psi|^2 dx &= \textstyle\int_\Omega \left(\frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - (\gamma+d+\delta)I(x) - \frac{aeI(x)}{1+beI(x)}\right)\psi dx \\
&= \textstyle\int_\Omega \left(\frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - (\gamma+d+\delta)I(x) - \frac{aeI(x)}{1+beI(x)}\right)\psi dx \\
&\quad - \textstyle\int_\Omega \left(\frac{\beta\overline{S}\overline{I}}{1+\alpha\overline{I}^2} - (\gamma+d+\delta)\overline{I} - \frac{ae\overline{I}}{1+be\overline{I}}\right)\psi dx \\
&= \textstyle\int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\psi dx - \int_\Omega(\gamma+d+\delta)I(x)\psi dx \\
&\quad - \textstyle\int_\Omega \frac{aeI(x)}{1+beI(x)}\psi dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 = \int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\psi dx,\ I_2 = -\int_\Omega(\gamma+d+\delta)I(x)\psi dx\ I_3 = -\int_\Omega \frac{aeI(x)}{1+beI(x)}\psi dx.$$

Henceforth, one yields

$$\begin{aligned}
&I_1 - \textstyle\int_\Omega \frac{\beta\overline{S}\overline{I}}{1+\alpha\overline{I}^2}\psi dx \\
&= \beta\textstyle\int_\Omega \left(\frac{S(x)I(x)}{1+\alpha I^2(x)} - \frac{\overline{S}\overline{I}}{1+\alpha\overline{I}^2}\right)\psi dx \\
&= \textstyle\int_\Omega \frac{\beta I(x)\phi\psi}{1+\alpha I^2(x)} dx + \int_\Omega \frac{\beta\overline{S}\left(1-\alpha I(x)\overline{I}\right)}{\left(1+\alpha\overline{I}^2\right)\left(1+\alpha I^2\right)}\psi^2 dx \\
&\leqslant \tfrac{\beta}{2\sqrt{\alpha}}\textstyle\int_\Omega |\phi\psi| dx + \beta K\int_\Omega \psi^2 dx \\
&\leqslant \tfrac{\beta}{4\sqrt{\alpha}}\textstyle\int_\Omega \phi^2 dx + \tfrac{\beta(1+4K\sqrt{\alpha})}{4\sqrt{\alpha}}\int_\Omega \psi^2 dx,
\end{aligned}$$

$$I_2 + \int_\Omega(\gamma+d+\delta)\overline{I}\psi dx = -(\gamma+d+\delta)\int_\Omega \psi^2 dx,$$

and

$$I_3 + \int_\Omega \frac{ae\overline{I}}{1+be\overline{I}}\psi dx = -\int_\Omega \frac{ae}{\left(1+be\overline{I}\right)(1+beI(x))}\psi^2 dx.$$

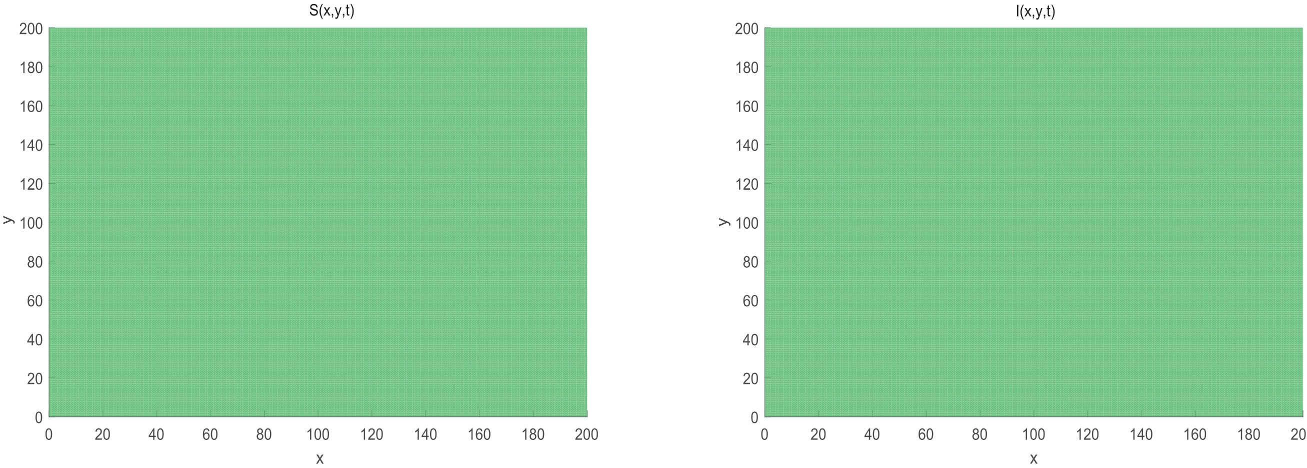
Therefore, one can obtain

$$d_I\int_\Omega |\nabla\psi|^2 dx \leqslant \frac{\beta}{4\sqrt{\alpha}}\int_\Omega \phi^2 dx + \frac{\beta(1+4K\sqrt{\alpha})}{4\sqrt{\alpha}}\int_\Omega \psi^2 dx. \tag{15}$$

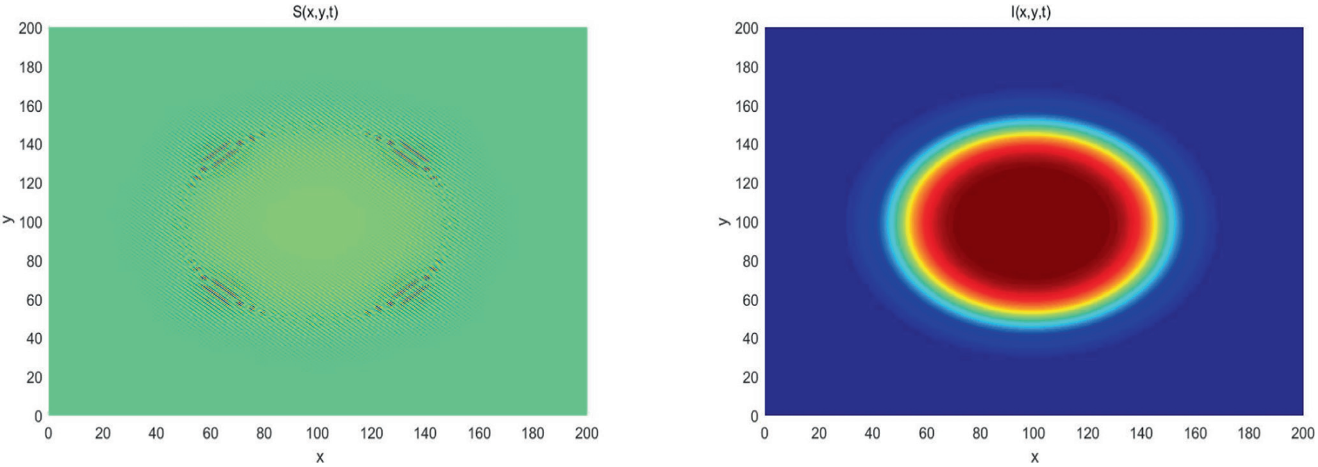
Combine (14) with (15), one yields

$$\begin{aligned}
&d_S\textstyle\int_\Omega |\nabla\phi|^2 dx + d_I\int_\Omega |\nabla\psi|^2 dx \\
&\leqslant \left(\tfrac{K^3\beta^3+8r(\gamma+d+\delta)^2}{8(\gamma+d+\delta)^2} + \tfrac{\beta}{4\sqrt{\alpha}}\right)\textstyle\int_\Omega \phi^2 dx \\
&\quad + \left(\tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2} + \tfrac{\beta(1+4K\sqrt{\alpha})}{4\sqrt{\alpha}}\right)\textstyle\int_\Omega \psi^2 dx \\
&\leqslant \tfrac{1}{\lambda_1}\left(\tfrac{K^3\beta^3+8r(\gamma+d+\delta)^2}{8(\gamma+d+\delta)^2} + \tfrac{\beta}{4\sqrt{\alpha}}\right)\textstyle\int_\Omega |\nabla\phi|^2 dx \\
&\quad + \tfrac{1}{\lambda_1}\left(\tfrac{K^3\beta^3}{8(\gamma+d+\delta)^2} + \tfrac{\beta(1+4K\sqrt{\alpha})}{4\sqrt{\alpha}}\right)\textstyle\int_\Omega |\nabla\psi|^2 dx.
\end{aligned}$$

Let



(a) $d_S = 2.5, d_I = 0.2$ (b) $d_S = 2.5, d_I = 0.2$



(c) $d_S = 0.04, d_I = 1.25$ (d) $d_S = 0.04, d_I = 1.25$

Fig. 1 The influence of the diffusion rates d_S and d_I on the nonconstant steady state of the SI model (1). Other parameters are $\beta = 0.31, \mu = 0.2, d = 0.1, \delta = 0.3, \gamma = 0.3, e = 0.4, a = 4, b = 6.3, K = 5.8, \alpha = 0.25, r = 1.53$.

$$d_S^* = \frac{1}{\lambda_1}\left(\frac{K^3\beta^3 + 8r(\gamma + d + \delta)^2}{8(\gamma + d + \delta)^2} + \frac{\beta}{4\sqrt{\alpha}}\right),$$

$$d_I^* = \frac{1}{\lambda_1}\left(\frac{K^3\beta^3}{8(\gamma + d + \delta)^2} + \frac{\beta(1 + 4K\sqrt{\alpha})}{4\sqrt{\alpha}}\right).$$

Consequently, if hypothesises $d_S > d_S^*$ and $d_I > d_I^*$ are valid, then $\nabla\phi = \nabla\psi = 0$.This implies the solution $(S(x), I(x))$ must be a constant solution of system (11). We end the proof.

To perform the effect of the diffusion rates d_S and d_I on the nonconstant steady states, we give the following example to verify this aspect. The bounded region we take here is $\Omega = [0, 200] \times [0, 200]$ in 2D space.

Example We take the parameters $\beta = 0.31, \mu = 0.2, d = 0.1, \delta = 0.3, \gamma = 0.3, e = 0.4,$ $a = 4, b = 6.3, K = 5.8,$ $\alpha = 0.25,$ $r = 1.53$. For the diffusion rates of the susceptible and infected individuals, one chooses $d_S = 2.5$ and $d_I = 0.2$. Then our numerical experiments illustrate that there is no nonconstant steady state in the model (1), see Fig. 1(a). However, when one takes $d_S = 0.04$ and $d_I = 1.25$, we can observe that the diffusive SI model (1) possesses the nonconstant steady state, see Fig. 1(b). Obviously, the diffusion rates d_S and d_I of the susceptible and infected individuals will affect the existence of the nonconstant steady states.

5. Conclusions

This paper investigates a diffusive susceptible-infected model with the saturated treatment, the non-monotonic incidence rate, and the logistic growth under the homogeneous Neumann boundary conditions. Firstly, the global existence and the uniform boundedness of the solution $(S(x,t), I(x,t))$ are given, see Theorem 2.1 and Theorem 2.2, respectively. The obtained results illustrate that the susceptible and infected individuals will long-term coexist in a bounded domain. Then, by using some Lyapunov functions, we establish the global stability of the disease-free equilibrium and the endemic equilibrium, see Theorems 3.1–3.3, respectively. To explore the dynamics of the steady states of the spatiotemporal SI model (1), we consider its elliptic form system, i.e., the system (11). By virtue of the maximum principle, the Poincaré's inequality and the Cauchy–Schwarz inequality, one gives a priori estimates, some properties and the nonexistence of the positive steady states, see Theorem 4.1, Proposition 1, Proposition 2 and Theorem 4.2 for details. Especially, the theoretical result performed in Theorem 4.2 shows that the elliptic system (11) has no nonconstant steady states when the diffusion rates d_S and d_I of the susceptible individuals and infected individuals exceed their critical thresholds, respectively. This implies that diseases will not perform spatial transmission characteristics under certain conditions from the point of view of disease transmission. These results may help us understand the dynamic behaviors of the diffusive SI model (1) in a bounded interval.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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