SPECTRUM PRESERVING LINEAR MAPS

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ABSTRACT

Let X and Y be Banach spaces. We show that a spectrum preserving surjective linear map ϕ from $\mathcal{B}(X)$ to $\mathcal{B}(Y)$ is either of the form $\phi(T) = ATA^{-1}$ for an isomorphism A of X onto Y or the form $\phi(T) = BT^*B^{-1}$ for an isomorphism B of X^* onto Y.

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Let X and Y be Banach spaces over the complex field and let $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ denote the algebras of all bounded linear operators on X and Y respectively. We consider linear maps ϕ from $\mathcal{B}(X)$ to $\mathcal{B}(Y)$ which preserve the spectrum; i.e., the spectrum $\sigma(T)$ of T equals $\sigma(\phi(T))$ for every $T \in \mathcal{B}(X)$. (No multiplicative or continuity properties of ϕ are assumed.) We show that if such a map ϕ is surjective, then it has one of the following forms.

(i) There exists an isomorphism $A: X \to Y$ such that $\phi(T) = ATA^{-1}$, for all $T \in \mathcal{B}(X)$;

(ii) there exists an isomorphism B from X^* (the dual of X) to Y such that $\phi(T) = BT^*B^{-1}$ for every $T \in \mathcal{B}(X)$.

In particular, such a mapping exists only if Y is isomorphic to X or to X^*.

We note that the injectivity of ϕ follows from the fact that it preserves the spectrum (Lemma 2). The mapping $\phi: \mathcal{B}(H) \to \mathcal{B}(H \oplus H)$ for a Hilbert space H, given by $\phi(T) = T \oplus T$ shows that the surjectivity assumption in our main result is needed even when $X = Y$.

The finite-dimensional case is related to a result of Marcus and Moyls [4] where they showed that a linear map ϕ on $M_n(\mathbb{C})$ which preserves eigenvalues and their multiplicity is of the form $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^tA^{-1}$. Our approach, specialized to finite-dimensional spaces, gives a proof different from [4].

We now fix some notation. The duality between a Banach space and its dual will be denoted by $\langle \cdot, \cdot \rangle$. For an $x \in X$ and an $f \in X^*$, we denote by $x \otimes f$ the rank one operator on X given by $u \to \langle u, f \rangle x$.

LEMMA 1. *Let* $A \in \mathcal{B}(X)$. *Then* $\sigma(T+A) \subseteq \sigma(T)$ *for every* $T \in \mathcal{B}(X)$ *if and only if* $A = 0$.

Proof. For the nontrivial implication, assume $A \neq 0$, and let x be a vector in X such that $Ax = y \neq 0$. It is easy to see that there exists an $f \in X^*$ such that $\langle x, f \rangle = 1$ and $\langle y, f \rangle \neq 0$. If $T = (x-y) \otimes f$, then $(T+A)x = x$, so $1 \in \sigma(T+A)$. But $\sigma(T) = \{0, \langle x-y, f \rangle\}$ and $\langle x-y, f \rangle = 1 - \langle y, f \rangle \neq 1$. Therefore, $\sigma(T+A) \nsubseteq \sigma(T)$. ■

LEMMA 2. *If* ϕ *is a spectrum preserving linear map from* $\mathcal{B}(X)$ *to* $\mathcal{B}(Y)$, *then* ϕ *is injective*.

Proof. If $\phi(A) = 0$, then $\sigma(T+A) = \sigma(\phi(T)) = \sigma(T)$ for every $T \in \mathcal{B}(X)$. By Lemma 1, $A = 0$. ■

In what follows, we will assume that ϕ is a surjective linear map preserving the spectrum.

LEMMA 3. $\phi(I) = I$.

Proof. Since ϕ is surjective, there exists an $S \in \mathcal{B}(X)$ such that $\phi(S) = I$. We have $\sigma(T+S-I) = \sigma(\phi(T-I)+I) = 1 + \sigma(T-I) = \sigma(T)$ for every $T \in \mathcal{B}(X)$. Therefore $S - I = 0$ by Lemma 1. ■

LEMMA 4. For $T \in \mathcal{B}(X)$, $x \in X$, $f \in X^*$ and $\lambda \notin \sigma(T)$, we have $\lambda \in \sigma(T + x \otimes f)$ if and only if $\langle (\lambda - T)^{-1} x, f \rangle = 1$.

Proof. If $\langle (\lambda - T)^{-1} x, f \rangle = 1$, then

$$(T + x \otimes f)(\lambda - T)^{-1} x = T(\lambda - T)^{-1} x + x = \lambda(\lambda - T)^{-1} x,$$

and so λ is an eigenvalue of $T + x \otimes f$. Conversely, if $\lambda \in \sigma(T + x \otimes f)$, then by a variant of the Fredholm alternative, λ is an eigenvalue of $T + x \otimes f$ and so there exists a nonzero vector $u \in X$ such that $(T + x \otimes f)u = \lambda u$. Therefore $u = \langle u, f \rangle (\lambda - T)^{-1} x$. It follows that $\langle (\lambda - T)^{-1} x, f \rangle = 1$. ■

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators.

THEOREM 1. Let $A \in \mathcal{B}(X)$, $A \neq 0$. The following conditions are equivalent.

(i) A has rank 1

(ii) $\sigma(T+A) \cap \sigma(T+cA) \subseteq \sigma(T)$ for every $T \in \mathcal{B}(X)$ and every $c \neq 1$.

Proof. If A has rank 1, then $A = x \otimes f$ for an $x \in X$ and an $f \in X^*$. Let $T \in \mathcal{B}(X)$ and let $\lambda \notin \sigma(T)$. By Lemma 4, $\lambda \in \sigma(T+cA)$ if and only if $c\langle (\lambda - T)^{-1} x, f \rangle = 1$. Therefore λ cannot belong to $\sigma(T+cA)$ for two distinct values of c. This proves the implication (i) ⇒ (ii).

To prove the reverse implication, assume that rank $A \geq 2$. We will show that condition (ii) is not satisfied. We begin by considering the case where A is a scalar αI, $\alpha \neq 0$. Let T be an operator with $\sigma(T) = \{0, \alpha\}$. It follows that

$$\sigma(T+A) \cap \sigma(T+2A) = \{2\alpha\} \not\subseteq \sigma(T).$$

We now consider the case where A is not a scalar, in addition to the assumption that rank $A \geq 2$. We will construct a nilpotent operator N with $N^3 = 0$ and a scalar $c \neq 1$ such that $\sigma(N+A) \cap \sigma(N+cA)$ contains a nonzero scalar. We first consider the case where there exists a vector u in X such that u, Au, A^2u are linearly independent. Let U be the linear span of $\{u, Au, A^2u\}$ and let V be a (closed) complement of U in X. Define an operator N on X by

$$\begin{aligned} Nu &= u - Au, \\ NAu &= Au - 2A^2u, \\ NA^2u &= -u/2 + 3Au/2 - 2A^2u \\ Nv &= 0 \quad \text{for} \quad v \in V. \end{aligned}$$

Therefore $N \in \mathcal{B}(X)$, $N^3 = 0$, $(N+A)u = u$ and $(N+2A)Au = Au$, consequently $1 \in \sigma(N+A) \cap \sigma(N+2A)$.

Next we consider the case where for every $x \in X$, the vectors x, Ax, A^2x are linearly dependent. We will show that A satisfies a quadratic polynomial equation $p(A) = 0$. Since A is not a scalar, there exists a vector u_1 in X such that u_1 and Au_1 are linearly independent. Therefore the minimal polynomial p of u_1, i.e., the monic generator of the ideal $\{r\colon r(A)u_1 = 0\}$, is quadratic. Now let $x \in X$ and consider the restriction of A to the invariant subspace $U_1 = \operatorname{span}\{u_1, Au_1, x, Ax\}$. Let q be the minimal polynomial of $A|U_1$. By a standard result in linear algebra, there exists a vector $u \in U_1$ such that q is also the minimal polynomial of u and so by our assumption, $\deg q \leq 2$. On the other hand, p divides q, so $q = p$ and $p(A|U_1) = 0$; in particular, $p(A)x = 0$. Since x is arbitrary, we have $p(A) = 0$.

We now consider four subcases according as $p(t) = (t-\alpha)(t-\beta)$ or $(t-\alpha)^2$ or $t(t-\alpha)$ or t^2 where $\alpha \neq 0 \neq \beta \neq \alpha$. By the standard decomposition of algebraic operators and since rank $A \geq 2$ and A is not a scalar, we see that A has a finite-dimensional invariant subspace W such that $A|W$ has a matrix representation

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

respectively. We consider a complement Z of W in X and an operator N such that $N|Z = 0$ and $N|W$ has matrix representation

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ \alpha^2 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 2\alpha \\ 0 & -2\alpha & -2\alpha \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

respectively. Let c be $\alpha\beta^{-1}$ or 4 or 2 or 2 respectively. Then $N^2 = 0$ and $\sigma(N+A) \cap \sigma(N+cA)$ includes a nonzero scalar, namely α or 2α or 2α or $\sqrt{2}$ respectively. ∎

The proof of the above theorem establishes the following result.

COROLLARY 1. Let $A \in \mathcal{B}(X)$. The following conditions are equivalent.

(i) A has rank 1 or is a scalar.

(ii) $\sigma(N+A) \cap \sigma(N+cA) \subseteq \{0\}$ for every nilpotent operator N satisfying $N^3 = 0$ and every scalar $c \neq 1$.

In infinite dimensional spaces, this result can be refined as follows.

COROLLARY 2. *If* X *is infinite-dimensional, then* rank $A \leq 1$ *if and only if* $\sigma(N+A) \cap \sigma(N+cA) = \{0\}$ *for every nilpotent operator* N *with* $N^3 = 0$ *and every scalar* $c \neq 1$. *Furthermore,* A *is a nonzero scalar if and only if the above intersection is empty*.

Proof. This follows from the fact that for a nilpotent N and a compact K, the essential spectrum of $N + K$ is $\{0\}$ and so the spectrum includes 0. ∎

Remark. In Corollaries 1 and 2 we may replace N by a quasi-nilpotent operator.

We now return to our study of the mapping ϕ.

LEMMA 5. *If* $R \in \mathcal{B}(X)$ *has rank one, then* $\phi(R)$ *has rank* 1.

Proof. This follows easily from Theorem 1 and Lemma 2 since ϕ preserves the spectrum. ∎

We are now ready to prove our main theorem.

THEOREM 2. *If* $\phi: \mathcal{B}(X) \to \mathcal{B}(Y)$ *is a spectrum preserving surjective linear mapping, then either*

(i) *there is a bounded invertible operator* $A: X \to Y$ *such that* $\phi(T) = ATA^{-1}$ *for every* $T \in \mathcal{B}(X)$;

or

(ii) *there is a bounded invertible operator* $B: X^* \to Y$ *such that* $\phi(T) = BT^*B^{-1}$ *for every* $T \in \mathcal{B}(X)$.

Proof. For every nonzero $x \in X$ and every nonzero $f \in X^*$ consider the sets $L_x = \{x \otimes h : h \in X^*\}$ and $R_f = \{u \otimes f : u \in X\}$. Each of L_x and R_f is a linear subspace of $\mathcal{B}(X)$ consisting of rank one operators and is maximal among such spaces. It follows that for every x, $\phi(L_x)$ is either an L_y for some $y \in Y$ or an R_g for some $g \in Y^*$. Furthermore, we cannot have $\phi(L_u) = L_y$ and $\phi(L_v) = R_g$ simultaneously for some u and v in X since $L_y \cap R_g$ is a one-dimensional space while $L_u \cap L_v$ has dimension 0 or $\dim X^*$. So we have two cases.

Case 1: For every $x \in X$, there exists a $y \in Y$ such that $\phi(L_x) = L_y$, so $\phi(x \otimes f) = y \otimes g$. The mapping $f \to g$ is linear, so $g = C_x f$ for a linear transformation $C_x : X^* \to Y^*$. We will show that the space $\{C_x : x \in X\}$ has dimension 1. If this is not the case, then there exist $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and two linearly independent transformations C_1, C_2 such that $\phi(x_1 \otimes f) = y_1 \otimes C_1 f$ and $\phi(x_2 \otimes f) = y_2 \otimes C_2 f$ for every f. It follows that $y_1 \otimes C_1 f + y_2 \otimes C_2 f = \phi((x_1 + x_2) \otimes f)$ and so has rank 1 for every f. Since C_1 and C_2 are linearly independent we must have that y_1 and y_2 are linearly dependent, and so $L_{y_1} = L_{y_2}$, implying that $L_{x_1} = L_{x_2}$ and so x_1 and x_2 are linearly dependent. However, in this case, we get that C_1 and C_2 are linearly dependent which is a contradiction. This establishes the fact that $\dim\{C_x : x \in X\} = 1$ and so by absorbing a constant in the first term of the tensor product, we have one linear transformation $C : X^* \to Y^*$ such that $\phi(x \otimes f) = y \otimes Cf$. Now the mapping $x \to y$ is linear, and we have $\phi(x \otimes f) = Ax \otimes Cf$ where A is a linear transformation from X to Y. Furthermore, both A and C are bijections since ϕ is bijective.

Now let T be an arbitrary operator on X, then

$$\phi(T + x \otimes f) = \phi(T) + Ax \otimes Cf.$$

Let λ be a complex number with $\lambda \notin \sigma(T)$. By Lemma 4, we have that $\langle(\lambda-T)^{-1} x, f\rangle = 1$ if and only if $\langle(\lambda-\phi(T))^{-1} Ax, Cf\rangle = 1$, and so, by linearity, we have

$$\langle(\lambda-T)^{-1} x, f\rangle = \langle(\lambda-\phi(T))^{-1} Ax, Cf\rangle$$

for every $x \in X$, $f \in X^*$ and $\lambda \notin \sigma(T)$. By the closed graph theorem, we can easily establish the fact that A and C are bounded. Replacing λ with $\frac{1}{z}$, we get

$$\langle(1-zT)^{-1} A^{-1} y, f\rangle = \langle(1-z\phi(T))^{-1} y, Cf\rangle$$

for every nonzero complex number z in some neighbourhood $\{z: |z| < \delta\}$ of 0. Each side of the above equation is analytic in $\{z: 0 < |z| < \delta\}$ with a removeable singularity at 0. Taking the limit as $z \to 0$, we get $\langle A^{-1} y, f\rangle = \langle y, Cf\rangle$. Taking the derivative at $z = 0$, we get $\langle TA^{-1} y, f\rangle = \langle\phi(T)y, Cf\rangle$. Therefore $\langle TA^{-1} y, f\rangle = \langle A^{-1} \phi(T)y, f\rangle$ and we get $\phi(T) = ATA^{-1}$.

Case 2: For every $x \in X$, there exists a $g \in Y^*$ such that $\phi(L_x) = R_g$. By a proof similar to the above, we get an isomorphism B from X^* onto Y such that $\phi(T) = BT^*B^{-1}$ for every $T \in \mathcal{B}(X)$. ■

COROLLARY 3. If $\phi: \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective linear map, then ϕ preserves the spectrum if and only if it is an algebra-isomorphism or an anti-isomorphism.

The following corollary is well-known [2].

COROLLARY 4. Every automorphism of the algebra $\mathcal{B}(X)$ is inner.

REMARKS. 1. If ϕ takes the form (ii) of Theorem 2, the surjectivity of ϕ implies that every operator on X^* is a dual of an operator on X. This, in turn, implies that X is reflexive. Since Y must be isomorphic to X^*, it follows that Y too is reflexive.

2. When $X = Y = H$, a Hilbert space, case (ii) of Theorem 2 takes the form $\phi(T) = CT^tC^{-1}$ where $C \in \mathcal{B}(H)$ and T^t denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.

3. Linear maps satisfying the weaker condition of preserving invertibility have been characterized in the finite-dimensional case in [5]. Other results on such maps between commutative Banach algebras are in [3] and also in [1] for certain maps between C*-algebras.

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