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**Precalculus Students' Problems in Understanding Variables,  
an Intervention, and its Effect**

by

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A Dissertation Submitted in Partial Fulfillment of the  
Requirements for the Degree of

**DOCTOR OF PHILOSOPHY**

in the Faculty of Education

We accept this dissertation as conforming  
to the required standard

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### ABSTRACT

The results of a qualitative analysis of 170 precalculus students' interpretations of mathematical variables, constituted the foundation for a teaching intervention in a precalculus course at the University of Victoria. Some serious misconceptions of variables were identified. The possible effects of the intervention were investigated in a retrospective analysis of students' mathematics course grades.

Students' interpretations of variables were extrapolated from their written explanations of answers to three algebra questions and from interview responses ( $N=17$ ). The subjects seldom interpreted variables as representing generalized sets of numbers or as co-variants. Their interpretations of variables were context-dependent, and generally inappropriate. In simplifications and equation solving most subjects appeared to use arbitrary rules to manipulate non-numeric symbols. When forced to consider numerical interpretations many described the variables as single numbers occurring in different instances. Some subjects appeared to substitute instances of variable use for the generalized number interpretation of variables, and patterns across instances for variable change. The interpretation of the variable as a single value in multiple instances can account for responses ranging from denial that variables change values to apparently correct descriptions of variable change. Some students interpreted letters as concrete objects or as units.

The intervention, which was incorporated into the researcher's precalculus course lectures, consisted of making explicit the contextual interpretations of mathematical variables as single, generalized, or co-varying numbers, and of expressions as actions or as variable objects. Student response to the intervention content was very positive.

The effect of the intervention was investigated quantitatively using log-linear models of the distributions of students' precalculus grades and, more important, their subsequent calculus grades. The models controlled for student changes over time, for instructor effects, and for differences in class composition based on students' year

classifications. For students continuing to calculus there was a possible association between the intervention and better calculus grades ( $N = 166$ ,  $p = 0.0008$ ) but the confound of year standing prevented conclusions being drawn for their precalculus grades. For the subjects who did not continue on to calculus ( $N=524$ ), there was no association between grade distributions and the experimental and control groups.

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## CHAPTER 1

### INTRODUCTION

The motivation for this research arose from observations of the extent and endurance of fundamental algebra errors made by precalculus students at the University of Victoria. Despite corrections throughout the term, the same errors appeared at the beginning and at the end of the course. On the final examinations, many problem solutions, such as those involving function composition, were set up correctly but then ruined by some 'simple' algebra error. Such a situation is frustrating for both students and instructors. This research was designed to investigate the source of these basic, repeated algebra errors made by precalculus students, and thereafter, to formulate and test an intervention.

#### Background to the Research

The experiences of instructors at the University of Victoria are not unique. Pinchback (1991) provided a description of precalculus errors based on her university classes in Arkansas, which fits with Matz' (1982) observations, and these in turn could describe precalculus classes at the University of Victoria. Similar errors were reported by Bell, Costello and Küchemann (1983) in their report prepared for the Cockcroft Committee of Inquiry into the Teaching of Mathematics in Schools in the United Kingdom and more recently, for example, in Poland by Sierpinska (1992) and in Israel by Vinner (1997). These errors are widespread, consistent and persistent, which makes it highly unlikely that they are either random or careless.

Concern over poor levels of understanding among students has led to the current reform movement in school mathematics teaching. It has been recognized that mathematical education too often emphasizes form over content, that is, correct performance over understanding. Students memorize rules and learn how to answer certain question types, but there is very little mathematical understanding in what they do. The major goal of the reforms is to establish student understanding as the primary aim of mathematics teaching and the associated pedagogy is based on the constructivist

theory that students will construct their own understanding. The teacher's role is thus to provide the resources and expectations which encourage students to develop their personal understanding.

Research at the elementary school level has identified a number of errors with identifiable causes rooted in students' confusion and misconceptions (e.g., Ashlock, 1998; Liedtke, 1996). Identifying causes for students' algebra errors is much more complex because older students have a longer history of mathematics learning, with many more opportunities to develop problems. Faulty arithmetic carries over into faulty algebra but, in addition, algebra carries its own potential pitfalls and problems (Herscovics & Linchevski, 1994).

Tall and Vinner (1981, quoted in Tall, 1991, p.7) described a concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." Precalculus students' concept images are constructed over years and based on many different experiences. The fact that such idiosyncratic cognitive structures can give rise to the same error patterns for many students in many places suggests very strongly that there are some common underlying constituents. One possible candidate for such a component of confusions and misconceptions is the algebraic variable (Leitzel, 1989). This possibility is strongly supported in recent research by White and Mitchelmore (1996) who identified inappropriate interpretations of variables as a major cause of calculus students' difficulties.

### Need for the Study

Mathematics education reforms may well ultimately be successful but in the interim there will continue to be many students arriving in senior high schools, colleges and universities, whose concept images are inadequate for the mathematical tasks at hand. It is also possible that such students will always be present, because rules are what they have chosen to extract from their classroom experiences. Thus for the immediate future and perhaps beyond, precalculus instructors can expect to continue to deal with students

for whom mathematics is a combination of rules and memorization, the antithesis of understanding and a poor foundation for further mathematics.

What is needed is some kind of bridge or intervention that will allow students to construct some of the elements of understanding that they have missed. Interventions must take into account three major points. First, they should build on students' good knowledge and address their weaknesses and misconceptions. This requires the instructor to know the nature and condition of students' understanding. Second, interventions should be practical. Reteaching the entire mathematics curriculum is not practical, nor is it desirable since it takes no account of the knowledge students already have. Third, students must be motivated to assimilate the intervention ideas and reconstruct their own concept images. The first two are within an instructor's control but for the third, the best that can be done is to create the conditions for learning and hope that students take up the challenge. It is the need for such an intervention for precalculus students at the University of Victoria that led to this research and frames the research questions.

### Research Questions

This research is an attempt to respond to precalculus students' difficulties in understanding mathematics. There are two parts. The first addresses the issue of precalculus students' mathematical knowledge and the second investigates the success or otherwise of one type of practical intervention. Specifically there are two questions:

1. What is the state of precalculus student understanding of mathematical variables?
2. Can an intervention consisting of explicit explanations of interpretations of symbols and basic concepts, delivered within the context of a regular precalculus course, improve students' mathematical understanding as indicated by their subsequent mathematical performance?

### Research Question 1

The variable was chosen as the primary focus of this research because it is fundamental to all algebra. If the implications and different meanings attached to variables are not understood, the student may misinterpret algebraic expressions and equations and be unable to express numerical connections and relationships correctly. Difficulties will inevitably follow. Much of the research reported in the late 1980s and early 1990s focused on the function and students' problems with the concept. The source of these problems could lie with the difficulty of the concept itself, but it is also possible that the problems are the symptoms of something deeper, a weakness in students' prior knowledge which makes the concept unattainable. The variable is a likely candidate for such a weakness.

The best comprehensive research into student understanding of variables dates back to the early 1980s with studies such Küchemann's (1981) algebra contribution to the British study, Concepts in Secondary Mathematics and Science Programme, or Wagner's (1981) research into functions and variables. In these studies, however, the target groups were beginning and early algebra learners. Student understanding of variables at the upper end of high school and early university levels has not been investigated extensively. Consequently, the first research question in this study addresses the state of precalculus students' understanding of mathematical variables.

### Research Question 2

The second question follows naturally from the first since an identification of weaknesses and errors in students' understanding creates a need to respond pedagogically. The realities of course structuring at the university mean that any response has to be contained within the existing course structure and content.

Variables at the precalculus level are mathematically interpreted as single numbers, as sets of numbers or generalized numbers, and as co-varying values. These interpretations also occur within expressions, which are themselves variously interpreted as calculations or as variable objects. These distinctions are not normally taught explicitly but are assumed to develop with time and in response to students' increasing

levels of mathematical experience. If development has not taken place and students are attempting to apply a lower level interpretation than is called for by the mathematical situation, it is not surprising that confusion and errors result.

The essence of the pedagogical response or intervention used in this study was to address directly the discrepancies observed between the understanding of variables necessary for precalculus mathematics, and the students' current understanding, as exposed in the first part of this research. By incorporating details of the specific symbol interpretations into the explanations provided in the lectures, students were provided with the information they needed within the existing course, without displacing any of the required course content. The second research question addresses the success or otherwise of this very direct intervention form.

### Methods and Hypothesis

The first question calls for a description of students' understandings and is therefore most appropriately approached using the methods of qualitative research. The subjects came from the University of Victoria's precalculus classes over a period of two years. Although some students were interviewed, the primary data are students' written explanations of responses on a three-question test administered at the beginning of precalculus courses, before any teaching had taken place. Each test question required a different interpretation of the variables involved, and it was hoped that the explanations would provide some insight into the state of students' understanding.

The main limitation associated with this part of the research is that the researcher designed, administered, and interpreted the results of the test. In response to the likelihood of bias, the reporting of the results contains much of the original raw data with photocopies of students' work, verbatim quotes, and interview transcript excerpts.

The second question was addressed through a quantitative analysis of students' grades. Subsequent calculus grades were taken as the primary response variable since the study of calculus requires the set and co-variant interpretations of variables to a much greater extent than is the case for precalculus. Precalculus grades, however, were also analyzed since many precalculus students do not continue to calculus, leaving precalculus

grades as the only available similar measures of mathematical performance. The entire data set was split, based on the presence or absence of calculus grades, into a calculus group and a non-calculus group.

This is a retrospective analysis and, as such, subject to confounding or nuisance variables. The explanatory variables consist of the instructor (*Instructor*), and the time (*Time*) of taking the precalculus course. Unfortunately precalculus class compositions were known to differ in terms of students' current academic year standings<sup>1</sup> (*Year*) and the level of students' high school mathematical experience (*High School*), creating two potential confounding variables. These were incorporated into the design as far as possible, although high school information was available for less than half the subjects.

The general null hypothesis for the quantitative analysis is that:

There is no difference in the distributions of either calculus or precalculus course grades between the experimental group of students who received the intervention, and the control groups consisting of students who did not and were either taught by the researcher prior to the introduction of the intervention or by other instructors.

With both calculus and precalculus grades as response variables and the limited availability of information on one of the nuisance variables, the quantitative investigation resolved itself into three main sections based on calculus or precalculus grades and the explanatory variables *Time*, *Instructor* and *Year*. These are displayed in the diagram in Figure 1. A fourth analysis using calculus grades as the response variable investigated the second potential confound, *High School*. Limited numbers of subjects precluded the use of both confounding variables in the same analysis. However, it turned out that students' high school background was not associated with the calculus grade distributions in the experimental and control groups and the factor was not investigated further.

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<sup>1</sup> The University of Victoria classifies regular undergraduate students by Year based on the number of units completed, as follows: Below 12 units – First year; 12 to 26.5 units – Second Year; 27 to 41.5 units – Third Year; 42 units or above – Fourth Year. 1.5 units is a normal one-semester course.

	Group	Response Variable	Explanatory Variables
1.	Calculus	Frequency in calculus grade categories	<i>Time:</i> <i>Time1, Time2</i> <i>Instructor:</i> <i>Researcher, Other</i> <i>Year:</i> <i>Y1, Y2, Y3+</i>
2.		Frequency in precalculus grade categories	
3.	Non-calculus	Frequency in precalculus grade categories	

**Figure 1.** The three main analyses used in the quantitative section of this study.

**Note:** The second possible confound *High School* is not shown. It was used only once, and replaced *Year* in an analysis of the calculus group's calculus grade frequencies.

### Report Outline

There are five chapters in this report. The next chapter, Chapter 2, contains a review of the relevant literature. Drawing heavily on Sfard's (1991, 1992, 1993) theory of reification and Kuchemann's (1981) description of students' levels of understanding of variables, the choice of variables as the research target is justified and the foundation laid for the content of the three written test questions. A discussion of the nature of mathematical knowledge provides the basis for the intervention material. Chapter 3 has two major sections. The first outlines the qualitative, and the second the quantitative, methods used in the research. The qualitative section includes the design, administration and interpretive principles for the written test, and a brief discussion of the methods used in developing the intervention. In the quantitative section the analytical approach is outlined, beginning with details of the factors, but omitting a description of the intervention treatment. The intervention, which is based on the results of the analysis of the written test responses, cannot be described until the results of the first part of the investigation are known. Following the factor descriptions, the preliminary exploratory data analysis and the subsequent development of statistical models of the data are

outlined. There are three sections describing the results of the study in Chapter 4. The first contains a description of students' understanding of variables derived primarily from their written explanations and with some reference to interview responses. The second section contains a brief outline of the intervention development and content. The third section of Chapter 4 contains the results of the quantitative analysis, which suggest an association between the precalculus intervention and students' success in calculus. In the fifth chapter a theory accounting for students' misinterpretations of variables is presented, together with a discussion of the likelihood that the intervention caused the observed differences in calculus grades. The consequences of the investigations for research and teaching are also considered, and in the concluding paragraphs the study and its results are summarized.

## CHAPTER 2

### REVIEW OF LITERATURE

The purpose of the literature review is to provide a theoretical framework and justification for the research questions. In addition to reporting related research there is also some extrapolation to justify the choice of test items and the content of the intervention.

The first section sets out the constructivist theoretical basis for this research, particularly as it relates to mathematics learning. Sfard's (1992) theory of reification is considered in the second section. Under this theory mathematical concepts are introduced first as actions or operations and later come to be reinterpreted as reified objects which can themselves be the subject of actions or operations. Mathematical knowledge thus described develops as an ever-increasing spiral, with understanding of any concept dependent on the understanding of earlier concepts.

Mathematical symbols are the main topic of the next two sections. Symbols are essential to algebra but are meaningless in themselves. Meaning must be supplied by the user and may be based on operations or reified concepts as required by the context. Difficulties occur when a student needs, but does not have, a reified concept, and the situation is made worse when, as often happens, the same symbol is used for both the action and the reified form. The various concepts and associations attached to a symbol combine to give each student an idiosyncratic concept image for the symbol, or, in Mason's (1987) terms, combine to make the symbols "palpable." These symbol images are part of imagistic mathematical processing, which, it is argued, constitutes doing mathematics.

Mathematics is not, formalist claims to the contrary, a matter of symbol manipulation according to set rules. Unfortunately such formalized mathematics is the experience of many students who learn the vocabulary and rules necessary for short-term classroom success. In such situations the symbols are treated as objects to be manipulated instead of being associated with mathematical meanings, and the resulting conceptual images were labeled by Vinner (1997) as pseudo-concepts.

The last two sections are concerned with the existing research on the variable in the context of high school algebra and precalculus. The primary categorization of variable use, which was developed by Kuchemann (1981) for students up to age 16, is extended to cover the higher level algebra of precalculus by including categories of variables as generalized sets of numbers and as co-varying quantities. Examples of variable use in specific contexts are also provided. These examples of ideal use contrast sharply with the results of research into students' understanding, which suggest considerable use of variables as objects or letters manipulated according to arbitrary rules.

The concluding paragraphs set up the two major parts of the research. The first part is an investigation into precalculus students' understanding of variables, and is necessary because, apart from investigations of ratio equations, much of the research into variables understanding has taken place at the lower end of high school. The second part of the research is a statistical analysis of the possible effects of an intervention at the precalculus level which focused on variables, variable use, and associated concepts.

### Theoretical Basis

The constructivist model of mathematics learning used in this study can be traced back to Aristotle, but owes its immediate roots in North America to Piaget's theory of learning. Constructivist theory asserts that all knowledge is achieved by the knower as the result of positive cognitive acts. Even apparently passive acts such as seeing and hearing are constructed as individuals exercise choice and discrimination in what they attend to, and what they ignore.

There are variations among constructivist theorists, particularly in relation to epistemological considerations. Radical constructivists like von Glasersfeld (1990) and Cobb, Yackel and Wood (1992) hold that knowledge is only what is constructed. There is no absolute truth, no reality which exists independently of humankind's minds. Each individual constructs his or her own knowledge and the agreements we perceive among different people's knowing are not reflections of some external reality, but are taken-as-shared meanings developed through communication and social interactions. Other theorists such as Noddings (1990) and Goldin (1990) accept the learning process

description but either deny or are not concerned with constructivism as epistemology. The general agreement is that learning is a process of adjustment and negotiation between learners and teachers, and all are engaged in active construction. The learner is trying to make sense of the teacher's acts, guidance and the activities provided, while the teacher is constructing his or her perception of what the student is doing, what meanings the student is developing, and whether or not the student has constructed something appropriate. Thus the learner and the environment are "co-implicative" (Steffe & Kieren, 1994). The environment in this sense includes both physical and social aspects and some writers, such as Cobb and Bauersfeld (1995), place greater emphasis on the social interactions.

If, as radical constructivists claim, mathematics does not exist outside of humankind's minds, the distinction between mathematics and how mathematicians think about it may be quite artificial. However, it is a useful one. From the perspective of the student, whether or not mathematics has existence outside of the mind is not relevant. The student is aware of a body of mathematical knowledge which exists in other people's minds. This knowledge is external to the student. He or she knows about it, and through communication, expects to learn it. In this sense mathematics can be said to exist separately from the student or learner and this is the sense that will be used henceforth.

In many areas of the world publications such as the *Curriculum and Evaluation Standards for School Mathematics*, (1989) and *Professional Standards for Teaching Mathematics* (1991) by the National Council of Teachers of Mathematics (NCTM) have signaled a serious attempt to change mathematics teaching to reflect the constructivist approach. However, Bauersfeld (1995) noted that much secondary school teaching still attempts to transport knowledge directly from teacher to student, ignoring the essential and active participation of the student. Students are thus often left without guidance or scrutiny as they construct meaning, resulting in rote-learned rules and other misconceptions. This problem is compounded when teachers themselves focus on the performance of symbol manipulation and the use of a narrow associated vocabulary, confirming students' belief that mathematics is nothing but meaningless rules. The consequences of such a situation can hardly be positive. Indeed, as Bauersfeld (1995)

observed, mathematics is the most ineffective school subject for many students, and the most disliked.

Teachers need to listen to students' ideas, and be able to frame subsequent questions, tasks and problems on the basis of those ideas (Fennema et al., 1996). In order to do this they need to have some understanding of the intellectual components of mathematical understanding and how these are developed. This need has led to an expanding body of theory and knowledge concerning the understanding of mathematics, the topic discussed in the next section.

### The Nature of High School Level Mathematical Knowledge

Mathematics is a highly structured, hierarchical body of knowledge. For example, in algebra, the use of variables to represent numbers is dependent, among other things, on there being a basis of knowledge of numbers. In trigonometry, the definitions of sine, cosine and tangent as ratios within similar right triangles, depend on the knowledge of ratios and the geometry of right triangles. For any given mathematical concept a hierarchical structure of prerequisite knowledge can be found. Such structures of prerequisite knowledge need not be unique, for although the hierarchical nature of mathematics is clear in the broad outline, at any level of detail a myriad of cross and back-connections becomes apparent. Slope, for example, can be introduced graphically as the gradient of a straight line or numerically as a pattern of change in a set of numbers. These are not mutually exclusive approaches since the eventual knowledge of slope and rate of change should include both components, but the sequencing within the hierarchy is not the same.

Learning theorists have long speculated how the human mind encompasses and organizes large interconnected bodies of knowledge. The theory put forward in the next section is specific to mathematics but could apply to any body of knowledge with large measures of abstraction.

### Development and Organization of Mathematical Knowledge: The Theory of Reification

Sfard's (1991) theory of reification is a refinement of the notion of chunking and is intended to account for the capabilities of humans for dealing with enormous quantities of interconnected information and knowledge. Kaput (1987) and Sfard (1991) separately described a process whereby concepts are first learned and used in an operational or procedural way, but in the end are transformed or incorporated into new mathematical objects which can themselves be operated upon. This is a qualitative change which Sfard (1991) labeled *reification*. For example, students can interpret an algebraic expression as a calculation when they perform the action of substituting values for the variables and calculating the result. This action interpretation is not appropriate when the expression is used as a numerical object such as a factor, a denominator, or an exponent.

The process of reification has been described by several other authors in a number of different ways. Dubinsky (1992) used the term *encapsulation* which he derived directly from Piaget's reflective abstraction, and Davis (1985) talked of frames which also included both object and action components. Observation of this duality of action and object in mathematics has not been confined to mathematics educators. For example, Hadamard (1945) inquired into the work of professional mathematicians and the resulting descriptions are of mathematical processes developing into mathematical objects. More recently, Davis and Hersh (1980) made similar claims for both the nature of mathematics and the way mathematicians think.

Sfard (1991) identified three stages in the development of a concept from action to object level: *interiorization*, *condensation* and *reification*. These three stages are compatible with the eight stages proposed by Pirie and Kieren (1994) in their model of the growth of understanding. A process has been interiorized once it can be thought about without actually having to be performed, which is the same sense of interiorize as was used by Piaget. Pirie and Kieren (1994) identified three substages to interiorizing wherein learners begin at a level of *primitive knowing* and work through a stage of practical experience or *image making*. *Image* as used here is the sense a student has of a concept and is not necessarily pictorial. Once this image is developed, described as the

stage of *image having*, the learner is free from the need for specific action (Pirie & Kieren, 1992).

The condensation period proceeds as learners combine more and more separate details into manageable wholes but the concept remains firmly attached to those details. In Pirie and Kieren's (1992) terms the stages are *property noticing*, *formalizing* and *observing*. The learner begins by being able to describe aspects or properties of his or her image and then generalizes or formalizes these. In this way the bases of general rules and patterns are abstracted from the learner's experiences. When these are combined at the observing level the learner is able to operate abstractly without needing to refer to specific examples. This is not, however, the final reified stage because the abstractions are still about, and hence tied to, the original operational processes.

Reification, or in Pirie and Kieren's (1992) terminology, structuring, is a qualitative shift when the entire concept becomes a single entity in its own right. As such it becomes an object of further study which Pirie and Kieren (1992) characterize as *inventising*. Several authors have observed that the reification step is difficult (e.g., Kaput, 1987; Sfard, 1991; Schwarz & Yerushalmy, 1992; Sfard & Linchevski, 1994). It does not develop as another stage in a sequence of increasing experience, rather it appears as a qualitative shift in thinking. By encapsulating many experiences, connections and relationships into one, reification provides for a simplification of the structure of mathematical knowledge. Without it algebra would become overwhelmingly detailed and complex.

Functions provide a good example of the intellectual demands of mathematics when considered in terms of the duality of processes and reified objects. Students have a great deal of difficulty with functions (Dubinsky, 1992; Eisenberg, 1992; Kaput, 1987), which Dubinsky (1992) suggested is due to the necessity of using both the action and object interpretations. Thus, for example,  $f(x) + g(x)$  is an addition of two mathematical objects, but to do the associated algebra requires students to "unpack" the objects and return to their action interpretation. Function composition provides a more complicated example of the dualistic interpretation where both the operational and object interpretations are intertwined. In the function composition notation  $f(g(x))$ , the expression  $g(x)$  is the input to the function  $f(x)$  and has noun or object status, while  $f(x)$  describes the action or operation

done. Students can, especially in examples involving numerical calculations, interpret function composition as sequential actions, but this is awkward and intimidating in its complexity when more than two functions are involved. If the reified object interpretation is used then the structure, even when several functions are involved, can be simplified to a single composed function and its input.

The sequence of interiorization, condensation and reification that carries learners from an operational to an object interpretation of a concept is repeated throughout mathematics. For any given concept, each learner has his or her own unique concept image which encompasses the entirety of the learner's experience with that concept (Tall, 1991). The concept image contains connections to other ideas and knowledge, emotional reactions, feelings of confidence or otherwise, as well as misunderstandings and misconceptions. It includes the processes and perhaps the reified object, and the objects upon which the processes act. It also includes the means by which these concepts are represented both externally and internally. The next sections discuss the relationship of mathematics to two representation forms: external symbols and internal images.

### Symbols, Symbol Systems and Mathematical Interpretation

Symbols play a fundamental role in all of mathematics. In some instances they are little more than a shorthand for frequently used words or phrases, but in others they carry large amounts of information and background meaning. They can imply exact definitions, which would be cumbersome if they had to be incorporated into a mathematical statement in words. For example, the expression  $\log x$  is a great deal shorter than *the power to which 10 must be raised to give x* and it also has implicit within it the restriction that  $x$  is strictly positive. The student of mathematics must read and use the symbols somewhat like written language, but in many cases what is read does not represent concrete, observable reality, but reified mathematical objects. Harel and Kaput (1991) suggested that the introduction of a perceptual presence in the form of a symbol may help with objectifying the associated concept. Whether this is true or not there is no doubt that the learner must integrate concepts and their associated symbols.

Kaput (1987) described three parts to a mathematical symbol system: a represented world or field of reference, a representing world or symbol scheme, and some connection or rule of correspondence between them. Although the represented world can be concrete reality, as mathematics becomes more advanced the represented world is more likely to be another symbol system. A similar tripartite model was described by Resnick, Cauzinille-Marmeche and Mathieu (1987) connecting situations involving quantities and relationships, mathematical formalism, and mathematical numbers and operations.

Kaput (1987) argued that a symbol scheme has both a set of symbols and a syntax or set of rules governing the correct use of the symbols. The numerals and operations of basic arithmetic form such a scheme. Certain combinations are possible under the system's syntax, while others are not. For example,  $4 + 3$  is possible, but  $4 3 +$  is not;  $4 - (5 - 2)$  can be replaced by  $4 - 5 + 2$ , but not by  $4 - 5 - 2$ . Hofstadter (1979) argued that although it is workable and operational, a symbol system is intrinsically meaningless. Concrete referents are not necessary for the operation of the system: that is, a student can work within the representing world without direct reference to the represented world.

Symbol meanings are supplied by the field of reference. In high school algebra the reference system is not immediate concrete reality but the complex structure of processes and reified objects built through and upon number (arithmetic) experiences. As students advance from actions to objects, either new symbols must be introduced or the existing symbols take on different meanings. Should a student fail to make the shift from action to object, he or she still makes some construction from what they are seeing and hearing. Sfard and Linchevski (1994) suggested that the student may substitute other objects such as pictures or even the symbols themselves with a consequent confusion between signifier and signified. When this happens all related subsequent learning is adversely affected.

For example, if  $x + 4$  and  $x + 3$  are always interpreted as addition instructions (actions) rather than number representations (objects), then the expression  $(x + 4)(x + 3)$  will be difficult to interpret as a product of factors. Students are faced with the symbol sequences  $(x + 4)$  and  $(x + 3)$  and a label *factor*, but without the reified concept of the expression as a number they cannot connect the algebra product with their existing knowledge of numerical products, (e.g., 7 and 6 are factors in  $7 \times 6$ ). When students are

unable to connect with a represented world, they have no option but to operate within the symbol scheme forming rules about symbol strings. Too many students are thus forced into the formalism of mathematics but, unlike the formalists among mathematicians, these students have no deep structure of understanding to fall back upon. Mathematics quickly becomes an ever-increasing set of symbol manipulations and attached rules (Gray & Tall, 1994; Harel & Kaput, 1992).

[Mathematics] has a spiraling complexity that more successful students compress by using symbols both as manipulable objects and as triggers to evoke mathematical processes. Meanwhile less successful students eventually become trapped in procedural cul-de-sacs as the subject – for them – grows ever more complex. (Gray & Tall, 1994, p. 138)

Clearly, using the same symbol for both action and object introduces ambiguity and uncertainty. However, representing each new object or process by a new symbol would be overwhelming. In addition a new symbol would not demonstrate the important connection between an action and its associated reified object. The price paid for manageability and the maintenance of connections is that ambiguities and overlapping uses exist. For example, “Evaluate  $2x - 3$  for  $x = 6$ ” uses the variable as a single number and the expression as a calculation. However, in the factoring statement  $2x^2 - x - 3 = (2x - 3)(x + 1)$ ,  $x$  represents any real number and the factor  $2x - 3$  is a number object.

A further complication occurs when symbols are used in more than one situation. Janvier, Girardon and Morand (1993) give several examples of *homonymies* where the same symbol represents different things and *synonymies* where there are multiple representations for the same objects. Clearly the dual interpretations of expressions as calculation actions and as numerical objects are homonymy examples but there are others, such as the fraction bar which can refer to a quotient, a fraction or to a ratio. These three related meanings encompass more than the simple duality of action and object. In other examples, the meanings are not always connected:  $\sin^2 x$  means the square of  $\sin x$  but  $\sin^{-1} x$  means the inverse function and not the reciprocal. The multiplicative inverse of  $x$  is  $x^{-1} = \frac{1}{x}$ , while the inverse of  $f(x)$  is  $f^{-1}(x)$  but  $f^{-1}(x) \neq \frac{1}{f(x)}$ .

Some notations have two unconnected meanings, as for example when  $(2, 5)$  can mean either the graph point or the interval with excluded endpoints.

To be successful students must learn to recognize the symbol, interpret it on the basis of its context and supply the appropriate conceptual meaning. The argument presented in the next section is that these connections between represented and representing worlds and the cognitive processes involved in doing mathematics occur through mental images.

### Symbols and Images

To this point, discussion has proceeded as if students make a direct connection between the formal symbols and the mathematics that they represent. However, this is not necessarily the case. Mason (1987) conjectured that while symbol use in mathematics might appear to operate at the syntactic or surface level, the confident student of mathematics is in fact operating at a deep structure level. For example, in a circle equation, the replacement of  $x^2 - 2x + 1 + y^2 = 1$  by  $(x - 1)^2 + y^2 = 1$  is achieved through recognition at the symbolic level of the surface pattern of the squared expression. However, there are many replacements possible under the symbol scheme. The choice of this particular replacement derives from the deeper conceptual knowledge that this equation form represents a circle, and further that the defining features of a circle, radius and centre, can be identified from the replacement equation.

Mason (1987) suggested that students make the symbols *palpable*. There is a mental object or image evoked by the symbols which students use or manipulate as they work. The source of the symbol image is the students' concept image for the symbol as well as the concept image for the wider context of the problem in which it is used. If the student is using only the surface structure, the image need not be any more complex than is necessary to make the symbols themselves palpable as letter objects. Typical of this level of operation are the phrases uttered by many teachers and students when solving linear equations that refer to moving or eliminating symbols in the equation. The variables and numerals are being given a temporary object status quite at odds with the deep structure meaning, yet entirely descriptive of the operation under way at the symbol manipulation level.

At the deep structure level the notion of palpability becomes much more complex. In the circle example in the previous paragraph the nature of the student's concept image for a circle must come into play as must the understanding of algebra. Palpability is the students' grasp of the procedures, and reified objects that underlie the particular use of the symbol and the symbols are "rich and substantial by virtue of algebraic experience" (Mason, 1987, p. 75). What makes them rich and substantial is the learner's ability to travel back into and use any part of that experiential structure, although that is not usually necessary unless something goes awry with the problem resolution.

The word palpable is useful in that it conveys a sense of image without specifying the image form and thus avoids a simplistic connection with pictorial images. Mathematical images are anything that can be imagined: visual, kinesthetic, auditory, to name the most common. The importance of students' images cannot be overstated if, as many authors claim, mathematical processing works through images (e.g. Davis & Hersh; 1980, Dreyfus; 1991, Goldin, 1987; Hadamard, 1945; Janvier, 1987; Kaput, 1987; Sfard, 1993; Tall, 1991). For example, Davis and Hersh (1980) relate intuition or the understanding of integers to images or mental representations.

[Intuition of integers] is the effect in the mind of certain experiences of activity and manipulation of concrete objects (at a later stage of marks on paper or even mental images). As a result of this experience, there is something (a trace, an effect) in the pupil's mind which is *his* representation of the integers. But his representation is equivalent to mine, in the sense that we both get the same answer to any question you ask – or if we get different answers, we can compare notes and figure out what's right. We do this, not because we have been taught a set of algebraic rules but because our mental pictures match each other (p. 398).

Clearly, for the multi-layered complexities of mathematical concepts a single image could never be sufficient. If an image is attached in some way to each stage in the development of a concept, then the overall concept image contains a multiplicity of images. The difficulty of observing the individual and internal process of image representation has led to several examples of personal introspection by, for example, Mason (1987) and Tall (1991), and reported introspections by Sfard (1993). These relatively recent observations echo those of Hadamard (1945) who surveyed the mathematicians of his time, asking them to describe their thought processes. He found that most of his respondents claimed that they did not think either in words or in

mathematical symbols. Their mental images were most frequently visual, but auditory and kinesthetic images were reported as well. Hadamard also noted that his own images and those reported to him were usually vague and not necessarily conscious. He provided the example shown in Figure 2, which displays his images in association with a partial proof that there is a no largest prime number.

Hadamard (1945) asserted that the vagueness of this mental picture or image is important in connecting conscious and unconscious thought because concentration on a precise image such as writing  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$  would interfere with the problem solving imagery he needed to formulate the proof. He saw symbol use as the first development of the precision necessary to communicate the proof and words as the last stage. In further support of his thesis he quoted several mathematicians including Einstein:

The words or language, as they are written or spoken, do not seem to play any role in my mechanisms of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be “voluntarily” reproduced and combined. (Einstein, in Hadamard, 1945, p.142)

STEPS IN THE PROOF	MY MENTAL PICTURE
I consider all primes from 2 to 11, say, 2, 3, 5, 7, 11.	I see a confused mass.
I form their product $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = N$	$N$ being rather a large number I imagine a point rather remote from the confused mass.
I increase that product by 1, $N+1$	I see a second point, a little beyond the first.
That number, if not prime, must admit of a prime divisor, which is the required number.	I see a place somewhere between the confused mass and the first point.
	(Hadamard, 1945, p.76)

**Figure 2.** Hadamard’s reported images in part of the proof that there is no largest prime.

From the foregoing discussion it is apparent that students engaged in mathematics are dealing with several different levels of representation at once. The symbol system itself is an overt or external representation, but its use requires internal representations or images which are more or less rich and palpable and more or less correct, depending on the individual. These inner workings can only be inferred from outside observations and much of recent mathematics education research has focussed on watching and listening to

students as they go about various mathematical tasks. This has led to a growing body of knowledge about how students learn mathematics, and also their errors, misconceptions and other difficulties faced in the process.

### Pseudo-concepts and Pseudo-analysis

The images described above are something above and beyond the symbols themselves and are a necessary part of comprehension. Without a palpable image students can only work within the symbol scheme, learning replacement and other syntactical rules devoid of any other deeper meaning. This undesirable situation is encouraged and given legitimacy from the student's perspective when the classroom focus is on the formal and symbolic part of mathematics, i.e., the production of correct answer forms. Thus although some students are forced, because of a lack of foundational understanding, to learn symbols and rules, others do so from choice in response to what they perceive to be the classroom expectations (Brown, 1996).

Vinner (1997) applied the terms pseudo-concept and pseudo-analysis to these impoverished images of symbols and superficial rules. Pseudo-concepts are usually shorter and less effortful to learn in the short term than are proper mathematical concepts, and are designed on the basis of the immediate goal of a correct answer. Vinner (1997) distinguished between pseudo-concepts and misconceptions in that the student believes a misconception. A pseudo-concept is neither believed nor disbelieved: it is merely the means to the end of correct performance. Successful pseudo-concepts look right and may sound right but are not based in mathematics. For example, many students correctly solve quadratic equations by factoring when the equation is given in the form  $ax^2 + bx + c = 0$ . The pseudo-concept underlying this apparently correct procedure has nothing to do with the zero-product rule. It is simply the template that each factor of  $ax^2 + bx + c$  be set equal to zero and the equation solved. Such an error becomes evident when the student sets each factor equal to  $k$  when the equation comes in the form  $ax^2 + bx + c = k$ . Uncovering pseudo-concepts requires setting problems which the students' rules do not cover. Asking for explanations of procedures is also effective (Vinner, 1997).

In high school algebra the variables, the operations, and “=” are among the most common symbols. If these are misunderstood or misused as is the case in pseudo conceptual thinking, then the algebra cannot be properly understood. The focus of the next section is on variables, their meaning within mathematics, the meanings students attach to them, and the consequences if these are incompatible.

### Variables in High School Algebra

Variables are everywhere in algebra. If they are not understood it is impossible for a student to understand algebra properly, and, unfortunately for students, there is no single interpretation of a variable that fits all situations. This section begins with a discussion of the different uses and interpretations of variables. The research cited is mostly from the early 1980s since more recent attention has shifted away from variables towards functions and their various representations. In addition, the studies are concerned mostly with early high school students and how their understanding of variables first develops. This provides a basis to interpret and extrapolate what might be present in older precalculus students. The framework for this interpretive endeavour is the theory of reification and the focus is on the role of variables in precalculus mathematics.

#### Variable Usage and Interpretation

Variables are a further example of a homonymy (Janvier et al., 1993). In this case the interpretations can be extended from the intended mathematical use to include the internal representations students construct for themselves. Küchemann (1981) identified six interpretations of letters in mathematics for children between the ages of 11 and 16. These are:

*Letter evaluated.*

This category applies to responses where the letter is assigned a numerical value from the outset.

*Letter not used.*

Here the children ignore the letter, or at best acknowledge its existence but without giving it a meaning.

*Letter used as an object.*

The letter is regarded as a shorthand for an object or as an object in its own right.

*Letter used as specific unknown.*

Children regard the letter as a specific but unknown number, and can operate upon it directly.

*Letter used as generalized number.*

The letter is seen as representing, or at least being able to take, several values rather than just one.

*Letter used as variable.*

The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

There are two problems with this categorization, both of which Küchemann (1981) himself acknowledges. The first concerns the letter as generalized number category. There is no distinction made here between the idea that a letter can take on several values in turn and the idea that a letter can represent a set of values in general. The former is the action of the letter taking on one value extended by allowing the action to be repeated a number of times. The latter is a reified concept where the letter is used to represent the elements of an entire set of numbers without any need to consider the substitution action. The second problem is that a systematic relationship between two sets of values can be interpreted as an observation of a static pattern, omitting the notion that the variables are co-varying (Clement, 1989). Sierpiska (1992) also noted the need, in connection with  $x$ ,  $y$  and functions, for students to be able to identify what changes as well as how it changes. Thus, in order to deal with the higher levels of abstraction encountered by older students in high school and first year university, two other categories should be added: letter used as a generalized set of numbers, and letters used as co-variants. There are, of course, more levels of representation, as for example, in linear algebra, but they are beyond the scope of this research.

Contexts and Variable Interpretations

The choice of variable interpretation depends on the context. An interpretation that is correct in one problem may cause an error in another. Alternatively, a wrong or

inadequate interpretation may still lead to a correct answer form. The next section contains examples of the levels of variable inherent in precalculus mathematics. The examples are drawn from three areas: tasks associated with *equals*, functions, and graphs. The coverage is not intended to be exhaustive but to provide an indication of the extent to which higher level concepts are needed and some of the probable consequences for students when that need is not met.

Tasks Associated with *Equals*. The symbol for equals (=) is a ubiquitous symbol in mathematics indicating, in high school algebra, an equality or identity relationship between two number representations. When = is used in the production of one expression from another by rewriting<sup>2</sup>, there is an action sense involved, which is compatible with the common interpretation students inherit from elementary school of = as a cue for action or answer finding (Kieran, 1981; Matz, 1982). However, once the equality is written the expressions must be interpreted as reified number representations if, for example, one is to be substituted for the other. For example, the equation  $\log_3 9 = 2$  can be created via a numerical action, but recognizing the equality of the two numbers and hence that they are interchangeable, is part of what is necessary to understand

$$3^{\log_3 9} = 9.$$

Variable usage depends on the context of the equation type. Conditional equations, which are true for a finite set of values, do not require the full generalized number interpretation. Students have only to accept that the variable can take on as many values as there are in the solution set, which, for linear equations, coincides with the variable as single number interpretation. However, it should be noted that a numerical concept for the variable is not in itself sufficient to enable students to solve equations in any but the most rudimentary way (Linchevski & Herscovics, 1993).

Equations resulting from rewriting algebra expressions use the variable in its generalized set of numbers sense. Expressions are equal only for the values of the variables for which the expressions are defined, a modifying statement which makes little sense without a generalized set interpretation for the variable. This domain statement is

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<sup>2</sup> Rewriting includes but is not confined to, simplification.

typically omitted with the intention of relaxing the rigour requirements and making the mathematics more accessible for students. However, this also allows students to avoid confronting the set interpretation for the variable. Identity equations such as statements of principles or trigonometric identities, are general statements about number connections and also require the generalized number set interpretation.

When students work with equations in equation-solving exercises, they can do so algorithmically, that is, by treating the variables at the surface syntactical level as symbols to be manipulated. For most of the equations experienced by high school students the solution makes sense if the variable is interpreted as representing one number or a small finite set of numbers. However, if the meaning of the equations themselves and the distinctions between types are to be understood then the expressions must be reified and the variables generalized.

Functions and Graphs. Much of the mathematics learned in high school is intended to lead up to, to develop, or to use the concept of functions. Yet the concept remains difficult and elusive for most students (Eisenberg, 1991; Herscovics, 1989). The discussion of the role of variables in function interpretation in the following paragraphs will be limited to examples using the typical algebraic function notation involving expressions in  $x$ ,  $y$ , and  $f(x)$ .

Evaluation exercises are an important part of the first introduction of functions, and for these there is no need for the student to understand the variable beyond its capability of taking on more than one value. The situation changes as soon as the concepts of domain and range are introduced since both are sets of numbers represented by the variables. Furthermore, if  $f(x)$  instead of  $y$  is to represent the range in the same way as  $x$  represents the domain then  $f(x)$ , as well as its associated expression, is a (reified) number object representing a set of numbers. In this situation the generalized number interpretation extends beyond the variables to include expressions in the variables. For example, finding the domain of  $f(x) = \sqrt{x-2}$  requires the student to interpret  $x-2$  as representing a set of values, describe that set as non-negative or greater than or equal to zero, set up the inequality  $x-2 \geq 0$ , and solve it for  $x$  representing a set of values. Finding the range requires recognition that the least possible value of a square root is 0

and that this value is possible for  $\sqrt{x-2}$ , or, alternatively, to recognize that the  $y$ -co-ordinates of the points on the graph are the values of the expression and read the range from the graph. All of these steps require either the variable or an expression, or both, to be interpreted as a set of numbers. Another example can be found in function inverses where understanding at any level beyond an undoing calculation action involves the reversal of the domain and range, and hence the generalized number interpretation for the variables.

The analogy of a function as input-output machine appears quite frequently in texts. The basis of this analogy of a function as a performing machine is that the function is a numerical action. However, if the input is described as  $x$  and the output as  $f(x)$  then the applicable interpretations are of  $f(x)$  as a reified object and both  $x$  and  $f(x)$  as representing generalized sets of numbers.

The difference quotient  $\frac{f(x+h)-f(x)}{h}$  (or  $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ ) has several levels of symbol interpretation, which may help explain why students find it such a major stumbling block. The functions  $f(x+h)$  and  $f(x)$  must both be interpreted as reified number representations since the binary operation of subtraction in the numerator requires two number objects. The expression  $x+h$  must also be a reified number representation since it is the argument to the function  $f$ . On the other hand, when  $x+h$  is substituted into the expression given for  $f$ , the function  $f$  is used in its algebraic action sense. In addition, since  $x$  and  $x+h$  both represent arguments to the function described as  $f(x)$ , the same letter  $x$  has different uses in the same problem. If students have any tendency to view the variable as one number then these overlapping uses are potentially very confusing.

A second, and perhaps greater, source of difficulty concerning the difference quotient is that it represents a rate of change of  $f(x)$  against  $x$ . As such, it requires the variables to be interpreted as numbers that change in relation one to the other, i.e., as co-variants. Students are introduced to slope or gradient in connection with linear functions relatively early in their high school careers. However, the context is usually graphical and the slope is the ratio of vertical to horizontal change along the line. The issue of co-varying  $y$ -values and  $x$ -values is seldom emphasized.

The most complete image for a function is its graph but students have great difficulty in interpreting graphs beyond the basic plotting of points (Kerslake, 1981, Dunham & Osborne, 1991). Creating a graph from a table of values requires no more than the application of a finite set of values for the dependent variable. Extending understanding of the graph to include intervening points requires a generalized set of values, and interpreting the graph as increasing, decreasing, and so on, requires the variables to be interpreted as changing values (Clement, 1989).

The examples and arguments presented demonstrate that variable interpretation in precalculus mathematics should be at least at the level of generalized numbers and sometimes at the true variable sense of changing values. Whether students' understanding is sufficient to meet these demands is open to question. The following section contains a review of the research pertaining to the development of students' understandings of variables.

### Students' Understanding of Variables

Students' actual, as distinct from possible, uses of letters in algebra have received some research attention at the high school level, particularly for the younger 12 to 16 year age groups. At the beginning stages of algebra learning students were observed to apply characteristics of the letters rather than the numbers the letters represented. Wagner (1981) reported that students assumed that a change of letter changed the algebra relationships, and further, that comparative values for the variables were related to their positions in the alphabet. Matz (1982) provided examples of many common algebra errors resulting from inappropriate generalizations of procedural or surface rules – that is, when the variables are treated as symbols to be manipulated, without reference to their numerical meaning.

Unfortunately, giving the variable a numerical meaning does not necessarily avoid confusion and errors. Two examples of errors based on numerical notation problems are that while  $2 \cdot n = 2n$  in algebraic notation,  $2 \cdot 4 \neq 24$  in arithmetic notation, and that while  $-4$  represents a negative number, it is not necessarily true that  $-x$  is negative (Matz, 1982). KÜchemann (1981) provided examples of inadequate variable interpretations from

interviews amplifying children's responses to the problem, "Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If  $b$  is the number of blue pencils bought and if  $r$  is the number of red pencils bought, what can you write down about  $b$  and  $r$ ?" The correct equation  $5b + 6r = 90$  can be seen as a relationship that is true for some specific but unknown numbers,  $b$  and  $r$ . It can also be seen as being true in a static way for a collection of number pairs. Students need not consider the idea that the variables can change, nor even the relationship between them, in order to produce the equation. These types of inadequacies in variable interpretation are not observable directly since they do not lead to overt errors.

In the previous example concerning red and blue pencils, the misconception of interpreting the variable as an object typically leads to the error  $b + r = 90$ . However, Küchemann (1981) also observed that the variable as object misconception does not always result in errors. For example  $2a + 5a = 7a$  and  $2a + 5b + a = 3a + 5b$  both have sense if the students' image of the letters is of objects, either as something like apples and bananas or as the letters  $a$  and  $b$ : two apples and five apples makes seven apples, and, two apples, five bananas and another apple makes three apples and five bananas. The interpretation is concrete and therefore initially easier than the proper abstraction and it gets the right answer – a good example of what Vinner (1997) later called a pseudo concept.

A similar object imagery for variables, but here leading to a wrong answer, was reported by Clement (1982) in his study using the students and professors problem. Students were asked to provide an equation representing the relationship "there are six students for every professor". Out of a class of first-year engineering students 27% gave the wrong equation and the most common error was to write  $6s = p$ . Based on interview data Clement suggested two possible reasons for this error. The first is that students simply match the word order in the problem to the letter order in the equation. The second approach is characterized as a static comparison where 6 is used as an adjective describing the noun  $s$ , and  $=$  symbolizes a correspondence or association instead of a numerical equivalence. The attached image is of six letters  $s$  and one letter  $p$ . In both cases the variables are used as objects. Herscovics (1989) and Kaput (1987)

characterized this error as the result of interference by natural language patterns. Another possible source suggested by Kaput (1989) is the confusion of variables and units of measure.

During interviews most of Clement's (1982) students altered their equations when faced with the numerical inconsistency of  $6s = p$ . If a student has a numerical interpretation of variables, the interference from language may be occasional, temporary and easily overridden. However, if the student sees the variable as an object then this interpretation and natural language are mutually supportive and the error more permanent.

This reversal error in ratio equations has been found in both high school and university level classes to be both common and persistent (e.g., Clement, 1982; Herscovicz, 1989; Kaput, 1987; Solloway, Lohead & Clement, 1982; Wyeth, 1991). The frequency of the error is increased if a more complicated ratio is used, or if the question is reversed and students are asked to provide a problem context for a ratio equation. Kaput (1987) also observed that provision of a visual image of objects in the given ratio increased the error rate. It seems that the variable as object interpretation is present in many students' concept images.

The meanings students construct for variables depend on their experiences. It has already been noted that the object interpretation is less abstract, and superficially easier, than the numerical interpretations. It is therefore an attractive interpretation and, in addition, can provide some limited success. Unfortunately this incorrect interpretation is unwittingly encouraged by the language used by teachers, texts and mathematicians. For example, in problems involving linear equations the represented world should be involved in setting up the equation but the solution procedure can take place within the symbol scheme. Teachers and texts describe the process of symbol manipulations using phrases such as combining like terms, collecting terms, and moving terms. If taken literally, these wordings are compatible with a letter as object interpretation, and the result is unintentional support for pseudo-concepts and pseudo-analyses.

In his discussion on pseudo-conceptual and pseudo-analytical behaviours Vinner (1997) points out that these are based on the recognition of superficial features and the recall of some connected rule. When some important feature is not recognized or the

recall connection is faulty errors result. These kinds of errors have been characterized and discussed by various researchers using terms such as mal-rules (Sleeman, 1984), cognitive obstacles (Herscovics, 1989) and pseudo-structural (Sfard & Linchevski, 1994). In all cases the student is reacting to the surface form and not to the deeper meaning, that is, they are treating the symbols as letters or objects. A non-numerical interpretation of a variable appears to be a key component of pseudo-conceptual behaviour.

One difficulty in identifying pseudo-concept usage is that in a well-practiced situation most students work at the surface symbol manipulation level, meaning that pseudo conceptual and true mathematical behaviour can have the same outward appearance. Sfard and Linchevski (1994) suggested that the most useful tools for separating the pseudo-conceptual from the conceptual are “singularities and things that happen at the fringes of mathematical definitions.” As an example they noted the difference between a system of equations with a single solution and a singular system with infinite solutions. The former can be solved with the variables interpreted as single numbers or pseudo-analytically where the algorithm halts when a line of the form  $x = n$  is reached. The less common singular system requires the variables to be interpreted as infinite sets. In most cases there is no initial indication in a problem that a singular system or an identity is involved. Faced with such an unexpected development a student with a rich concept image, including the ability to accept the variable as representing a set of numbers, has a basis to work out a reasonable response. However, the impoverished pseudo-analyst who sees the variable as a letter or object or possibly a single number, responds by trying to force the problem into the expected form by, for example, concluding  $x = n$  from  $n = n$  (Sfard & Linchevski, 1994).

A further complication in investigating students' ideas about a ubiquitous object such as a variable is that it is unlikely that students' ideas are consistent across different contexts. There is considerable evidence of inconsistencies in students' ideas about functions and graphs (e.g. Tall, 1990; Tirosh, 1990; Vinner, 1990) where, for example, students' formal definitions of a function often contradict the definitions used in practice (Vinner & Dreyfus, 1989). In some cases of inconsistency students either do not recognize the conflict or are not concerned by it. However, the compartmentalization of knowledge (Vinner, 1992) means that students may never have considered the conflicting

ideas together (Tirosh, 1990). Vinner (1997) suggested that compartmentalization is a typical feature of the pseudo-concept. If the notion of compartmentalization and consequent inconsistency is applied to variables, it is possible that, although students know that a variable represents a number or numbers, they may prefer to operate in practice as if the variable is a letter or object.

### Conclusion

The concept image a student constructs of any algebraic concept must have, as one component, the mathematical symbolism, which includes variables. Studies of early high school students have shown a number of difficulties concerning the interpretation of letters in algebra. At the more advanced algebra levels of senior high school and university, a precalculus student's understanding of variables should be at least at the level the generalized set of numbers and co-varying values. There are indications that students do not generally meet these requirements. For example, calculus students were observed by White and Mitchelmore (1996) to follow the pseudo-conceptual pattern of treating variables as manipulable letter objects. These researchers concluded that an "underdeveloped" concept of variables is a major source of students' difficulties in calculus. If this is true for calculus students, there is no reason to suppose that the situation is better for precalculus students.

An investigation into the state of precalculus students' understanding of variables constitutes the first part of this research. The intention is to develop a general overview of students' interpretations of variables, particularly their ability to interpret variables as generalized sets of numbers and as co-varying values.

If, as must be expected, student understanding turns out to be weak then the question arises as to whether any remediation is possible. The tenets of constructivism say that students construct their understanding from experience through thinking and communicating about that experience. The least appropriate classroom format is the university lecture where teaching by telling still reigns – yet that is the only form available for this research. However, encouragement is provided by Dubinsky (1992).

Our experience has been that when a student is presented with concepts that he or she is capable of understanding, when the constructions are possible for the student,

and if this capability is apparent to the student, then a natural drive to learn, to understand, to construct, is released and the level of effort and concentration on mathematical ideas leaves little to be desired. (p. 120)

If Dubinsky (1992) is correct then remediation within a large class lecture format is possible. His conditions for success can be met if the intervention is firmly grounded in students' prior knowledge of number relationships and algebra rules, and the meanings and uses of the symbols are explained explicitly. These are the basic premises for the intervention used in the second part of this research.

The next chapter contains descriptions of the methods used to investigate these two research questions. Qualitative methods are used for the initial description of students' understanding, while the effect of the intervention is analyzed statistically.

## CHAPTER 3

### METHOD

Over the past decade or so mathematics education research has moved away from both the behavioural theory that learning consists of sequences of operations and its associated positivistic research approach. Under this theory researchers attempted to fit education research into the pattern of classic quantitative experimentation and statistical inference by operationalizing constructs and attempting to design objective measures. A number of writers (e.g. Kilpatrick, 1992, Kieran, 1994, Steffe & Kieren, 1994) have pointed out the inadequacy of this learning model and the inappropriateness of positivistic research for many educational contexts. The shift has been towards constructivism and qualitative research methods. It is important, however, not to deny value to quantitative methods. Schoenfeld (1994) advocated an eclectic approach, matching method and associated analysis to the type and intent of a particular investigation.

This research has both qualitative and quantitative components. There are two qualitative components, the first being an analysis of students' understandings of variables and related mathematical symbols and vocabulary based on written explanations and interviews. In the second part some of the techniques of action research (Kemmis & McTaggart, 1988; Winter, 1989) were used to develop a teaching intervention for precalculus students based on the results of the initial descriptive analysis. The quantitative component consisted of a statistical analysis comparing the subsequent calculus performances of precalculus students who had received the teaching intervention to those who had not. Precalculus grades were also analyzed. The sequencing of the various parts of this research are shown in Figure 3.

Qualitative Research Components	Year	Term	Quantitative Research Components
First pilot	1993	Fall	Precalculus control (without intervention)
	1994	Spring	
		Summer	
Second Pilot		Fall	Precalculus control (without intervention)
	1995	Spring	
		Summer	
Written tests and interviews		Fall	Intervention development
	1996	Spring	
		Summer	
		Fall	Precalculus experimental & control (with & without intervention)
	1997	Spring	
		Summer	
		Fall	Precalculus experimental & control (with & without intervention)
	1998	Spring	
		Summer	
		Fall	

Calculus classes, supplying calculus grades for precalculus students

**Figure 3.** Sequencing of qualitative and quantitative research components.

## The Investigation of Students' Understanding of Variables

The first part of this description of the investigation into students' understanding of variables contains a general outline of the principles of qualitative research together with a discussion of its strengths and weaknesses. The two methods of data collection by written test and supplementary interviews are described next, including the development of the test and the purpose and limitations of each question. This section is completed with a description of the subjects, and of the test administration and interviews.

### Qualitative Research

By the time students reach precalculus they have had at least 11 years to construct their mathematical knowledge and it is evident to any teacher that there are serious deficiencies in this knowledge. Finding the sources of students' errors requires probing, analyzing and describing their thinking, that is, it requires a qualitative investigation. Bogdan and Biklen (1992) outlined five characteristics of qualitative research:

1. Qualitative research has the natural setting as the direct source of data and the researcher is the key instrument. Researchers tape-record, video-tape, take notes and interview their subjects in as a setting that is as natural for the subjects as possible.
2. Qualitative research is descriptive. Qualitative data are not numerical. The raw data consists of interview transcripts, field notes, video-tapes or recordings made by the researcher as well as other information sources such as personal documents, pictures, and environment descriptions. Qualitative data are a mass of detail.
3. Qualitative researchers are concerned with process rather than simply with outcomes or products. Researchers are interested in how students come to know or to do. The focus is on how students interpret or make sense of their perceptions, not whether these can be judged correct or otherwise.
4. Qualitative researchers tend to analyze their data inductively. Theory is developed from the bottom up. The researcher starts with an area of interest or some general question rather than a clear hypothesis. The collected data are reviewed or

explored and theories or further questions derived post-hoc. Thus the research design is known only in broad outline at the start.

5. Meaning is of essential concern to the qualitative approach.

Qualitative research in mathematics education is a relatively new field and there is no current consensus on how it should be applied or what constitutes good research. Schoenfeld (1994) noted that there are a number of unanswered questions about qualitative research concerning such things as evidence, robust versus fragile methods, the role of prediction, falsifiability, the linkage between theory and method, and validity and reliability. All of these are important issues but there is, as yet, no generally accepted approach to addressing them. Writers on qualitative research use terms such as credibility and trustworthiness, neutrality rather than objectivity, and extrapolation rather than generalization (Lancy, 1993).

The issue of credibility in this research was addressed using a combination of three different approaches: triangulation, logical argument and provision of original data. Triangulation is the confirmation of findings by alternative sources or methods (Brannen, 1992). In this study written explanations and interviews serve as alternative sources, as do pilots and replication at different times. Logical argument is necessary to justify choices where there is no precedent in other research, and is the means of addressing the concern of bias introduced by the close involvement of the researcher. The inclusion of significant quantities of raw data and background description in the reporting allows readers to form individual judgements which can then be matched against the researcher's conclusions (Lancy, 1993).

### The Written Test

The written test questions, which are the main source of data in this investigation, were tested in two pilots. Some interview data were also collected but the primary means of justifying the questions was through a logical analysis of their ability to generate responses at different levels.

Written response data are traditionally associated more with quantitative research. However, in this case the test was used as a large scale source of descriptive data and not

for scores or frequencies. The question requirements are therefore directed to that purpose. A variety of responses to a question are useful but full explanations of answers are more informative about students' thinking. Questions can be designed to elicit errors but must be, or appear to be, simple enough that students have sufficient confidence in their responses to give full explanations. Unfortunately, simple questions are susceptible to the use of pseudo-concepts which produce correct answers under a veneer of correct vocabulary. Setting more difficult questions can expose students' inadequate concepts but at the expense of simplicity and a consequent loss of explanations. Individual interviews may provide a better means of separating the pseudo-conceptual from the conceptual. However, they are time-consuming and often dependent on a volunteer sample which is not necessarily typical of the larger population.

In this research it was decided to use a written test, shown in Table 1, which allowed for many participants, supplemented by individual interviews for confirmation and amplification of students' written explanations. It was assumed that although every student's knowledge is a unique construction, certain patterns or connections could emerge from a test given to a large group of students. Two pilot tests and a number of interviews were used to establish the final form and content of the test. None of the questions produced responses free from interpretive uncertainty, but this is inevitable.

Each test question was analyzed by performing a sorting procedure that grouped students first by answers and then by similarity of explanations. A conceptualization was considered to be present provided at least one clear description could be found, together with other reasonably similar explanations. In some cases interview data were also available but some of the conceptualizations reported are based solely on the written data. Frequencies and percentages were found where deemed to be useful, but a large number of explanations could not be categorized unambiguously. Thus, although presence is reported, in most cases strength of presence could not be gauged.

Table 1

Questions on the Understanding of VariablesQuestion 1

Simplify  $\frac{12a}{-6a}$ . Explain and justify each step you took to arrive at your answer.

Question 2

Recall that solving an equation means that you are trying to find the values of the variable that satisfy the equation. The solution to the equation presented below is correct as far as it goes.

How would you complete the following equation solution? Give reasons.

$$\begin{aligned} \text{Solve for } x: \quad & 2(x - 3) + 7 = 2(x + 5) - 9 \\ & 2x - 6 + 7 = 2x + 10 - 9 \\ & 2x + 1 = 2x + 1 \\ & 2x = 2x \end{aligned}$$

Question 3

In this question you are given five verbal descriptions labeled by capital letters, and three equations. For each equation identify which of the verbal descriptions fit and explain your choices. If a description would be true for an equation write the letter of the description beside the equation. You can have more than one letter with each equation.

- A As  $p$  increases  $q$  increases      B As  $p$  decreases  $q$  increases  
 C  $p$  and  $q$  remain constant      D  $p$  and  $q$  are never negative  
 E None of the descriptions A, B, C or D.

Equation	Description(s) which apply.	Explanations
$p + q = 15$		
$p = 5q$		
$p + q = r$		

### Development of the Three Test Questions

The three test questions shown in Table 1 were chosen on the basis of the previously described criteria of simplicity and the ability to provide a context for students' explanations. Length was also an important factor since the test had to be completed within twenty minutes, the maximum time acceptable to all the instructors involved. The extent to which the questions facilitated students' demonstrations of their understanding was checked through interviews, all conducted within two weeks of administering the tests.

The first pilot test (See Appendix B) was given to 18 precalculus students, five of whom were interviewed, in the fall of 1994. It contained ten questions, only one of which, Question 1, survived without change in the final test. An attempt was also made in the first pilot to test students twice, once at the beginning, and once during the term. This failed due to the extent of students' recall of the first test. When recall was not a difficulty the questions were judged to be too dissimilar. The second pilot (See Appendix B), given to 116 precalculus and 98 calculus students in the fall of 1995, contained three questions. In the final test (See Table 1), Question 1 remained, Question 3 was revised slightly and Question 2 replaced completely.

All three questions were created specifically for this research. Each question was intended to cover a different variable interpretation: as numbers vs. objects, as representatives of a set of numbers, and as co-variants. In the following three subsections the logical justification of each question is presented and possible weaknesses identified.

#### Question 1

This question, asking students to explain their simplification of  $\frac{12a}{-6a}$  arose from classroom observations. The variable  $a$  is used in the generalized number sense in the expression. Students had to perform the specific division  $12 \div (-6)$  and apply the general rule that a number divided by itself is 1. If students used the full generalized number interpretation for  $a$  they could also be expected to observe that the expression simplifies to  $-2$  only if  $a$  is not 0. At the other extreme, substitution of a specific value will produce

a correct solution. This is a very familiar and common problem type, simplifiable with any numerical interpretation of the variable, yet precalculus students at the University of Victoria had been observed to have difficulty with it. Consequently it fitted the requirements of this research test very well. The question itself seemed sufficiently non-threatening that students were ready to explain their answers, but it also had a capacity for discrimination.

As must be expected with such a simple and common problem, there is a major difficulty with pseudo conceptualization and pseudo-analysis. However, this problem has one useful feature in this regard. Both numbers and variables require the same treatment, (division) but if students interpret the numbers and variables differently they are likely to use different vocabulary in their explanations.

### Question 2

This question does not have the same developmental background as Question 1. It was used to investigate students' understanding of variables as generalized numbers in the sense that they represent sets of numbers. Finding a suitable question that could discriminate between the more sophisticated generalization to a set of numbers and the lesser generalization to a few values proved to be difficult.

The first form of the question in the first pilot test, taken from Küchemann (1981), used a variable as a generalized number in a formula context. In its original form the problem had an incomplete polygon diagram with the accompanying question: "What is the perimeter (distance around) the shape below? Part of the shape is not drawn. There are  $n$  sides altogether, each of length 2." Half the students answered correctly but the interviews suggested that they were interpreting the formula as a pattern to be followed when "plugging in" single numbers without attending in any way to the domain of the formula. The generalization used in this context was as a pattern of single number use and did not extend consideration to the set to which those numbers belonged. This lower level interpretation was sufficient for the problem and consequently, a student's ability to use the higher level understanding was not tested.

The second form of Question 2 was based on range rather than domain. Students were given three formulas in the variable  $n$ , each of which produced numbers with factors differing by 3. These numbers were represented by  $x$ , and for any given value of  $n$  each formula produced a different value for  $x$ . In order to solve the problem students had to recognize that each formula would produce the same set of values for  $x$  and in the process interpret  $x$  as representing a set of numbers. The problem proved to be much too difficult for precalculus students. Most could not produce a coherent response, which made it impossible to identify whether the source of their confusion was related to the word problem context, the difficulty of the concept, or something else.

The final form of Question 2 is the solution of an identity equation where the variable  $x$  represents the elements of the set of real numbers. There is a confound from the context in that students need to understand that an equation can have more than one solution. An attempt was made to reduce the incidence of this effect by including an introductory statement reminding students of the meaning of equation solving, but it is doubtful whether the students paid much attention to it. If students failed to use the generalized interpretation, it was not possible to separate the effect of the equation solving context from the variable concept. The lack of a general number interpretation may have caused the student to expect only specific number solutions or, conversely, the expectation of specific solutions may have stopped the student from interpreting the variables as representing the elements of a set of numbers. This confounding effect was a major restriction on this question's value. However, all other simple questions considered had a similar problem. At least this question has the virtue that the confounding effect of equation solving understanding was also important in this research. Students provided informative explanations and useful data on both variables and equation solving was gathered.

### Question 3

Question 3 was used in an attempt to investigate students' understanding of variables as changing quantities. Its final form is slightly different from both the first and second piloted forms. The three equations and the options A, B, and C were unaltered

but the remaining options are the result of two transformations (see Appendix B for early test versions). The three equations  $p + q = 15$ ,  $p = 5q$ , and  $p + q = r$  each represent different uses of co-varying variables, and the purpose was to see if, or how, students would distinguish between them. In order to focus students' attention on the desired issue of changing values the problem was put into the multiple choice format shown in Table 1.

The equation  $p = 5q$  is in the form used by Clement (1982) in his problem concerning relative numbers of students and professors. Some of his subjects used an object interpretation for the variables, which in this example would be one  $p$  and five  $qs$ . In this research the basic equation was used without the word problem context and in the form  $p = 5q$  because of the potential confound of student response to algebra word problems. In addition, the students and professors problem and its attendant equation have made their way into algebra texts and students could have been familiar with that specific example. One difficulty with this simple function form of equation is that students could treat it merely as a template for a multiplication and respond on the basis of an image of the five times table. This potential confound was countered by including the equation  $p + q = 15$ , where the relationship is still functional and one-to-one but not in direct function form. However, students could still use a tabular image, in this case of numbers that sum to 15. The third equation, with three variables, removes the usefulness of the tabular image unless  $r$  is considered constant, but introduces the difficulty of describing the relationship between three variables.

The pilot tests gave some information about the efficacy of the option choices in Question 3. In some cases the information resulted in changes but in others the unwanted effects were considered manageable, if not desirable. Options A (As  $p$  increases  $q$  increases.) and B (As  $p$  decreases  $q$  increases.) are correct for  $p = 5q$  and  $p + q = 15$ , respectively. However, the wording of option B relating change in  $q$  to change in  $p$  was chosen to further counteract the interpretation of  $p = 5q$  as a simple calculation of  $p$  from  $q$ . No particular problems were associated with options A and B except for an occasional student who misread B as "As  $p$  decreases  $q$  decreases".

Option C was offered originally for those who might consider variables to represent single unchanging values. This interpretation was minimally evident in the pilot test results but some other interesting interpretations and uses of *constant* appeared which justify its

retention. No particular interpretive difficulties beyond the expected pseudo conceptualizations and pseudo-analyses were associated with any of options A, B, and C but this is not true for options D and E.

The common error that variables are positive because they have no negative sign was offered in option D but some students went further and interpreted D to mean that  $p$  and  $q$  could not be negative at the same time. However, only a few students used this interpretation and it did not seem to obscure other valuable information.

Option E was the correct choice for the three variable equation, but unfortunately it was also the option most chosen by students with no reasoned answer to give. The explanations clarified matters to some extent but there was still more uncertainty associated with this response than with any of the others. Nevertheless, despite these various problems Question 3 provided much useful information on student interpretations of the variables in the equations.

### Interviews

Interviews are not a primary data source in this research. They were used to supplement and confirm data collected from the tests and from classroom observations during the development of the intervention. All interviews were semi-structured, that is, a pre-determined set of questions provided the structure but once the student responded the subsequent course depended on the interviewer's ability to react to what the student said. The guidelines followed are those suggested by Ashlock (1986).

1. Be accepting.

The interviewer must not appear distant, but communicates interest and encouragement. The interviewee should understand that all responses are of interest to the interviewer. Whether answers are right or wrong is irrelevant. The interviewer should also be sensitive to the interviewee's emotional state.

2. Collect data - do not instruct.

Interviewers should avoid inadvertent verbal cues or communication by body language and tone of voice. Questions and comments must be neutral, otherwise student responses may be contaminated.

3. Be thorough.

The interviewer must be well prepared with planned questions and requests or activities, but must also be ready to redirect or depart from the initial plan to follow where the student leads.

#### 4. Look for patterns.

It is important for the interviewer to retain "critical awareness" ... The purpose of the interview is to establish a coherent picture of the subject's patterns of thought and not be sidetracked by careless slips. (pp. 13-14)

A major potential confound associated with all interview data is that the researcher is both the collection instrument and the interpreter of the data. Any interpretation of another persons' actions or words must always be contaminated to some extent by the observer's own constructed knowledge and beliefs. The most common counter-measure to the confounding threats associated with interviews and interpretations of a subject's words is to include transcripts and quotes in the reporting. In this instance the reporting includes many quotes from students' written tests but only a few interview quotes. These are clearly insufficient to demonstrate the competence or otherwise of the interviewer. For this purpose two complete transcripts are included in Appendix A.

The issues of bias and skewed reporting are not a major concern in this research. The intent is discovery and although other results were known, there was no prior expectation of specific findings. A greater concern is the possibility that too much is read into the data. This is countered by reporting multiple quotes with each finding.

#### Subjects Used in the Investigation of Students' Understanding of Variables

The subjects who wrote the three test questions were precalculus students from three extant classes, two spring term precalculus classes in January 1996 and one summer session class in May 1996. Registration in these classes at the start of the term is typically from 60 to 100 for the regular term and 30 to 50 for the summer term. Most students take precalculus as a prerequisite for other courses rather than for interest, and despite the Precalculus title, more than half the students continue to either a first year mathematics course other than calculus or to a statistics course in another department such as psychology or sociology.

Responses from students in calculus classes, who took part in the pilot test in the fall of 1995 were also used, primarily for comparison purposes. Two calculus classes<sup>3</sup> wrote the second pilot test in which the first and third questions are essentially the same as in the final test form.

### Administration of the Test and Associated Interviews

The test was administered during the first class of the term to avoid any instruction effects and all related interviews were conducted within two weeks of the test in order to minimize possible contamination from course work. Consequently, the responses should reflect students' long-term more than short-term understanding. Students were told that the test was intended to help the researcher understand how they thought about mathematics and was definitely not a pretest for entry to the course. Twenty minutes were allotted for the test.

In addition to the three mathematical questions, students were also asked about their background: age, gender, previous mathematics courses and feelings about the subject and the course. Permission was obtained from the University of Victoria's Human Research Ethics Committee to use all test responses as data unless the student explicitly denied consent. A statement that students' participation was assumed unless specifically disallowed appeared on the front page of the test, along with the form for withdrawing consent (See Appendix C). Seventeen interviews with volunteers were conducted after the pilot tests in the spring and fall of 1995 and after the final form of the test given in January and May, 1996.

### Development of the Intervention

The intervention was designed to address some of the problems uncovered in the initial analysis of students' understanding of variables, and its development followed the general pattern of action research. Action research is a type of qualitative enquiry that

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<sup>3</sup> Calculus for the Social and Biological Sciences (Math 102) and Calculus I (Math 100).

has recently gained in popularity in education. It was described by Winter (1989) as a set of methods for reflection and evaluation whereby professional practitioners engaged in a continuing process of change and improvement can justify and document their actions.

Kemmis and McTaggart (1988) described action research as a spiral:

In practice the process begins with a general idea that some kind of improvement or change is desirable.... Having decided on the field and made a preliminary reconnaissance, the action research group decides on a general plan of action. Breaking the general plan down into achievable steps, the action researchers settle on the first action step, a change in strategy which aims not only at improvement but a greater understanding about what it will be possible to achieve later as well. Before taking this first step, the action research group becomes more circumspect and devises a way of monitoring the effects of the first action step, the circumstances in which it occurs, and what the strategy begins to look like in practice. When it is possible to maintain the fact-finding by monitoring the action, the first step is taken. As a step is implemented, new data starts coming in, and the circumstances, action, and effects can be described and evaluated... This evaluation stage amounts to a fresh reconnaissance which can prepare the way for new planning. (p.8)

In the development of the intervention three steps formed a single turn of the spiral suggested by Kemmis and McTaggart (1988). The first involved the identification of contexts for particular forms of variable use. The primary method in this study was a logical examination or mind-experiment which took the form of a single question asked in different contexts: If variables are interpreted as objects, single numbers, generalized numbers, or varying quantities, what sense can be made of this particular problem or context? In this way various situations within the precalculus course were analyzed for their prerequisite variable knowledge requirements. The second step was to make the particular understanding explicit in the course and to point out likely consequences if students did not have or use the requisite knowledge. The third step was to evaluate what had been done and prepare for the next stage in the spiral. This step included analyses of students' test or examination answers, interview responses, anonymous course evaluations and ongoing observation of students' behaviour in the classroom and during drop-in tutorial sessions.

These three steps were repeated four times as the intervention took shape. The first sequence began in January, 1995 using a small volunteer group meeting weekly in a tutorial situation. Three class trials took place in courses beginning in September 1995,

January, 1996 and May 1996. By September 1996 the content and sequence of the intervention was established, and all four spring and fall term classes for the academic years 1996-97 and 1997-98 provided the experimental subjects for the quantitative research described in the next section. The chronological relationships between the classes used in the qualitative and in the quantitative parts of this research are shown in Figure 2, Chapter 3 (p. 34).

#### Investigation of Students' Grades in Calculus and Precalculus Courses

The quantitative part of this research is a retrospective analysis of students' mathematical performance after they had received the teaching intervention in precalculus. Entire classes of available subjects were used with no sampling, balancing of numbers, or random assignment of students to the experimental or control groups. The quantitative investigation is based on the assumption that if students' understanding of variables improved then this would have a positive effect on their subsequent calculus grades. Calculus grades were chosen as the primary focus of the study instead of precalculus grades because the calculus course makes more demands on students' understanding, and consequently, there is less opportunity for students to do well based on rule-learning and pseudo-concepts associated with variables. However, precalculus grades were also analyzed since they were the only available grades for those students who did not continue to calculus.

The analysis of precalculus students' grade data was carried out in S-Plus (1998) using the technique of generalized linear modeling. In addition to their grades, the students' instructor and date of taking precalculus were recorded. Using these two factors, students were classified into experimental and control groups. Students' year standing at the time of taking precalculus and, where available, their previous high school experience were also recorded.

#### Subjects Used in the Quantitative Investigation of Students' Grades

The University of Victoria precalculus course (Math 120) was offered first in the summer of 1993. The first subjects for the quantitative part of this research came from

the first regular term class in the fall of 1993. All available subjects from fall 1993 to spring 1998 were used with two exceptions. No subjects came from classes taught during fall 1995, spring 1996 or summer 1996, that is, during the period of the intervention development. In addition no subjects were included from summer term precalculus classes. Summer term classes are much smaller than regular term classes and the students are generally thought to be more motivated and serious. Since all but one of the summer school classes in the time period were taught by the researcher, the inclusion of these students was a potential source of bias and it was considered prudent to omit this whole group.

The experimental group came from four consecutive regular term precalculus classes who received the teaching intervention designed and taught by the researcher in the academic years 1996-97 and 1997-98. Three other groups were used to control for possible effects from the instructor and for possible changes in the subjects over time. The first control group consisted of students from four consecutive regular term precalculus classes taught by the researcher in the academic years 1993-94 and 1994-95, that is, prior to the introduction of the intervention. The gap of one full year between this control group and the experimental group is the developmental year for the intervention. The remaining control groups were made up of students who took the precalculus course from other instructors during the same two-year periods. These groups were all further split into two according to whether or not the students continued on to a calculus course or not.

An unsuccessful attempt was made to assign students randomly to different instructors in September 1995 during the developmental period. The process had to be done at the start of classes because there was no opportunity to assign students as they registered. Some students exercised their option to refuse to change sections, some transferred between sections and some joined the class late. It was estimated that between 10% and 20% of the eventual registration was not randomly assigned. Furthermore students found the process very disruptive and it appeared to increase their apprehension about the course. Overall it was decided that the cost to the students was too high given that full random assignment could not be achieved.

Four groups of students were eliminated from the data set. It was decided to drop subjects with a gap of more than three years between precalculus and calculus since such students may well have taken another precalculus course at college or elsewhere in the intervening years. This is not an issue for the other instructors' groups, but these subjects should not be included in the researcher's groups. In addition students who had failed precalculus but nevertheless continued to calculus were excluded because, like the previous group, their entry into calculus is not based on their precalculus course learning<sup>4</sup>. Students who audited either precalculus or calculus and students who had officially deferred their final examinations were also omitted. Those who did not complete the course were retained. In most cases these students do not bother to write the final because they have no expectation of passing, and can therefore be classified along with those who failed.

### Statistical Factors and Variables

The data collected for each subject consisted of the instructor, date or time of their precalculus class, their precalculus grade and, if relevant, subsequent calculus grade. Due to the lack of random assignment of students to classes, the possibility of confounding variables had to be considered.

The first potential confound is a student's year of study. Registration dates at the University of Victoria vary by students' year standing. The precalculus sections are filled sequentially, with the researcher's section filled first and a second section taught by a different instructor opened only after the waiting list for the first section is substantial. Consequently the other instructor's classes have fewer early registrants than do the researcher's classes. Information on students' grades, instructors, course dates and students' year standings are all available on the class grade lists. A second possible confound is the students' high school mathematical background, since there is

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<sup>4</sup> A student who has passed the provincial Math 12 examination or its equivalent at any time in the past is technically eligible for admission to calculus, regardless of any intervening failures.

considerable content overlap between the provincially examinable grade 12 mathematics course and precalculus. Regrettably, information on high school courses was recorded for less than half the subjects.

Students' subject area is another potential confounding factor which might result in differences between the experimental and control groups. Those who continued to calculus could be identified by their appearance in a calculus class but no other information pertaining to subject area was available through class lists. Each variable is described in the subsequent paragraphs.

### Instructor

*Instructor* is a dichotomous variable with levels designated *Researcher* and *Other*. Six other instructors were involved. Each of the two control groups with *Other* instructors had three instructors, and in both cases there was one with high school teaching experience and two with university teaching experience. There were no neophyte instructors.

### Time

The variable *Time* is also dichotomous with levels, *Time1* and *Time2*. *Time1* corresponds to the two year period prior to the introduction of the intervention, and *Time2* to the two years after. Specifically, *Time1* covers the regular fall and spring classes for the academic years 1993-1994 and 1994-1995, and *Time2* covers the same classes in the academic years 1996-1997 and 1997-1998. The academic year 1995-1996 was considered developmental for the intervention and is omitted rather than set as a third level for *Time*. During *Time2* there were always two sections of the precalculus class, one taught by the researcher, the other by a different instructor. All classes during *Time2* ran in the same late afternoon time slot. There were three other instructors in *Time2*, one of whom taught two sections. Only three classes are available from *Time1* for each of the *Instructor* categories because there were two terms when only one section of precalculus was run. As a result the group sizes are smaller in *Time1* than in *Time2*.

## Grade

Both precalculus and calculus grades were recorded but it is the calculus grades that are the primary interest, as an indicator of long term effects from the intervention.

Calculus grades were obtained for those students who took Math 102, designated as Calculus for the Social and Biological Sciences. Calculus I (Math 100), which is a more challenging course than Math 102, was not taken by enough precalculus students to provide an adequate data set for analysis. For the remainder of this report *calculus* refers to the Calculus for Social and Biological Sciences course, unless otherwise specified. The precalculus grades are from the course titled Precalculus (Math 120).

Both calculus and precalculus courses at the University of Victoria normally have failure rates between 15% and 30%, resulting in a high number of students who take the courses more than once. Students' first calculus grades were recorded and any subsequent grades were removed from the data set. For precalculus students the last grades were used, together with the associated instructor and time.

Students' final grades in both calculus and precalculus are a combination of term and examination marks. All students write a common final examination but term assessments vary between instructors. In order to balance this potential inequity the mathematics department's practice is to assign grades for each class on the basis of the distribution of grades achieved by that class on the common final examination. Thus, grade distribution differences between classes are not due to factors such as instructors' marking practices or choice of test content during the terms.

The University of Victoria grading system uses a 10-point scale running from 0 for an Incomplete or Fail through single increments for D, C, C+, B-, B, B+, A-, A, and A+ (9 points). In the Department of Mathematics and Statistics a fail is awarded for percentage scores between 0 and 50%, and an A+ for scores above 90%. Thus, despite its appearance, the grade scale from 0 to 9 is not an interval scale in the statistical sense. There were insufficient numbers of precalculus students continuing to calculus to support all ten grade levels and for much of the analysis the categories were collapsed to five, F, D, C, B, and A, or fewer. Collapsing of grade categories did not extend as far as the simple Pass and Fail dichotomy, which was considered to be too coarse to be of value.

The histograms in Figure 4 and Figure 5 show the grade distributions for the complete sets of precalculus and calculus (Math 102) classes for the periods, respectively, fall 1993 to summer 1998 and spring 1994 to fall 1998. The research subjects are subsets of the data sets used for these histograms and the distributions are not necessarily the same, but it is evident from both these figures that there can be no initial assumption of a normal distribution for either the precalculus or the calculus grade variable.

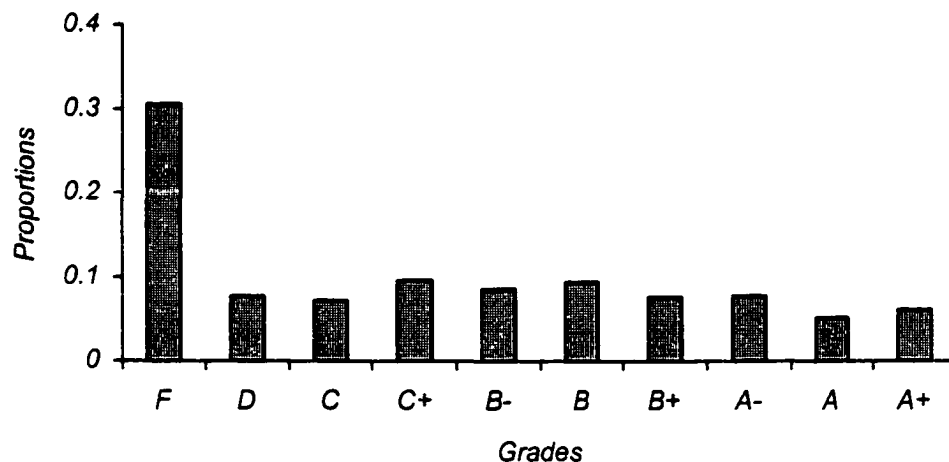
### Year

This variable records the student's year standing at the time of taking the precalculus course. Its potential as a confound is of particular concern because its effect is not clear. Students with later year standings can be expected to be more mature, but they will also be further removed from their high school mathematical experience. Students are classified by the university as first, second, third, or fourth year, or unclassified. The year levels were collapsed to first, second, and third or higher, and the unclassified group, who are usually mature or returning students, were added to the third year or higher group. This nuisance variable is necessary only because of the lack of randomisation in constructing the experimental and control groups, and its inclusion puts a strain on the size of the data sets. Accordingly, it was decided to use this variable only if there was evidence of different distributions based on *Year* in the experimental and control groups.

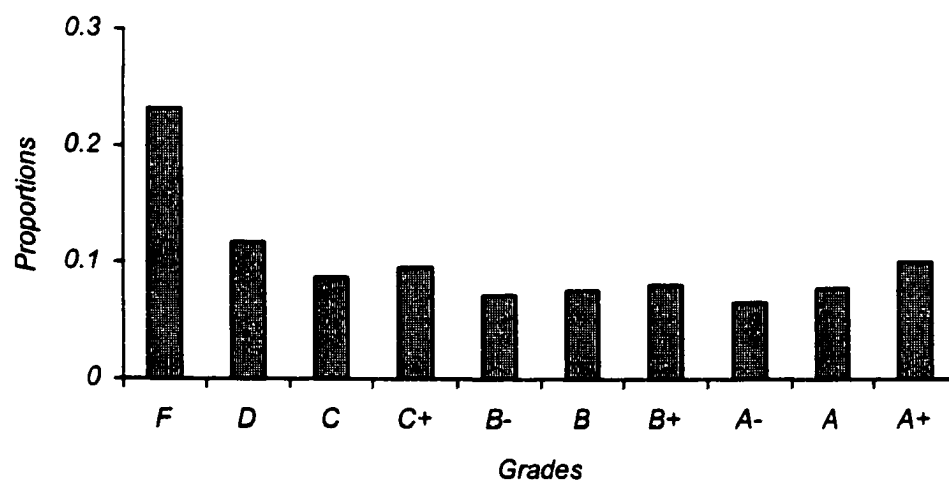
### High School

Background high school course information was unfortunately not available for most students. The *High School* variable is dichotomous with students' mathematics level prior to taking the precalculus course recorded as either G11 (Math 11 in British Columbia) or G12 (Math 12). Students who failed Math 12 could not be identified because this information is not recorded. Those who took a different, lower level course such as Survey Math 12 were included with the grade 11 group. Some information on high school mathematics failures was provided by students who wrote the pilot forms of the three-question tests and supplied background information on themselves. Among this

group approximately half of the precalculus students had taken and failed grade 12 mathematics.



**Figure 4.** Proportions in ten grade categories for entire set of precalculus classes (Math 120) at the University of Victoria, fall 1993 to summer 1998 ( $N=1105$ ).



**Figure 5.** Proportions in ten grade categories for entire set of calculus classes (Math 102) at the University of Victoria, spring 1994 to fall 1998 ( $N=2460$ ).

The *High School* variable was included because, like *Year*, it is a potential confound. Precalculus and high school grade 12 courses have a large overlap which could have given an advantage to students who have previously taken grade 12. Unfortunately with high school grades recorded for less than half the subjects, the subset is too small to be taken as the main data set. In addition, when *High School* is included as a variable the grade levels are reduced to three, making the analysis less sensitive.

### Statistical Analysis

The quantitative data analysis followed the four broad phases described by Cox and Snell (1981). These are:

- (i) Initial data manipulation, i.e. the assembling of the data in a form suitable for detailed analysis and the carrying out of checks on quality.
- (ii) Preliminary analysis, in which the intention is to clarify the general form of the data and to suggest the direction which a more elaborate analysis may take. Often this is best done by simple graphs and tables;
- (iii) Definitive analysis, which is intended to provide the basis of the conclusions;
- (iv) Presentation of conclusions in an accurate, concise and lucid form. Usually this leads to a subject-matter interpretation of the conclusions.

Two response variables were analyzed: frequencies of calculus grades and of precalculus grades. The full precalculus data set splits naturally into two subsets, the calculus group where both response variables are recorded, and the noncalculus group where only precalculus is available. The restricted availability of *High School* necessitated a further data subset, resulting ultimately in four data sets for analysis. These are: (a) the calculus group with response variable calculus grade frequencies, (b) a subset of the calculus group with *High School* available as an explanatory variable, (c) the calculus group with precalculus grade frequencies as response variable, and (d) the non-calculus group with precalculus grade frequencies as response variable. The analysis was approached similarly for each data set.

The first task in the analysis was to investigate the role of *Year* by a simple  $\chi^2$  test of association between *Year* and *Grade*, in the hope that *Year* might justifiably be omitted from the explanatory variables. The main analysis used the data in the form of contingency tables of frequencies in grade categories cross-classified by the appropriate explanatory variables. For each data subset a preliminary exploratory analysis was carried out primarily by a comparison of histograms drawn in Excel. Some use was also made of means and medians, which were calculated by substituting the scores 0 to 9 for the letter grades *F* to *A+*.

The statistical analyses were carried out using the technique of Generalized Linear Modeling (GLM) in the statistical package S-Plus. GLMs form a large class of statistical models that include as special cases the more traditional linear regression and analysis of variance models. Of particular value to this research is that GLMs include models for categorical response data.

Statistical modeling is based on the assumption that some of the variability in a given data set is systematic or explainable, and some is random noise. A model provides a summary of the systematic variation and an indication of the nature and size of the random effects (McCullagh & Nelder, 1992). In most cases the best model is parsimonious, that is, it “explains” the data adequately with fewest terms (Dobson, 1990).

GLMs have three components: a random component, a systematic component with a linear predictor, and a link function. The random component consists of the response variable and its associated probability distribution. Explanatory variables are identified in the systematic component and combine singly and in interactions to form the linear predictor. The final component is a link function which connects the random and systematic components (Agresti, 1996).

In all cases, frequencies in the categories of *Grade* are the response variables with *Instructor*, *Time* and the confound *Year*, as explanatory variables. In one analysis the second potential confound, *High School*, replaces *Year*. The marginal totals for the explanatory variables are considered fixed, since there was no random assignment of subjects into these categories. Fixing of the marginals has an effect on both the random and systematic components of the model. First, the natural choice for the probability

distribution is the Product Multinomial instead of the Multinomial, and second, the terms associated with these fixed values are required to be present in the final model. Within statistical packages the Poisson distribution is used for analysis purposes instead of the Multinomial distribution. The natural link function for Poisson models is the logarithm and the resulting model belongs to the group of log-linear models.

The model form described in the preceding paragraph is only one possibility out of many. There are, for example, other options for the link function. There are also other approaches to modeling this type of data. A traditional analysis approach would be through an ANOVA using the grade point values of 0 to 9 as scores. However, such data violates the ANOVA assumption of continuity, and the distributions observed in Figure 4 and Figure 5 violate the assumption of normality. Instead of relying on the robustness of ANOVA it was decided to use the techniques of generalized linear modeling where the assumptions coincide better with the data. A different alternative is the proportional odds model based on the odds ratios between the ordered grade categories. This model requires that the cumulative probabilities for each row in the contingency table do not cross (McCullagh & Nelder, 1992), a condition not met by the data in this study.

Model selection in this analysis began with the saturated model with all possible terms included. The final model was achieved through a trial and error process where terms contributing least to the overall fit were removed successively until the most parsimonious model with a reasonable fit was found. The fit of the model is demonstrated graphically by three plots:

1. A plot of response vs. fitted values, which should approximate to the straight line  $y = x$  if these are reasonably close.
2. Provided the model fits well and suitable distribution and link functions have been chosen there should be no association between the magnitude of the deviance residuals and the fitted values. A plot of the deviance residuals vs. the fitted values should therefore appear as a random cloud.
3. The size of deviance residuals and the normality of their distribution can be checked using a Normal quantile plot of the Pearson residuals.

This study is a retrospective investigation of possible association. Consequently once an adequate model has been found, its interpretation is based on the terms it

contains. If the minimal model is sufficient to explain the data then the null hypothesis of no association or interaction cannot be rejected. Each model is presented with its analysis of deviance table. The tables show the terms of the model, the reduction in deviance as each term is added to the model, and the significance of the reduction based on a  $\chi^2$  test. A description of the analysis of deviance table appears in Appendix E.

The interest in this analysis is not in prediction or in parameter values but in the question of which terms are necessary to the model and which are not. Of particular interest is the interaction term *Instructor:Time:Grade*, since this reflects differences in grade distributions between the experimental and control groups. The nature of the contribution of a given term was investigated by examining relevant data histograms.

### Summary

Quantitative analysis was used in this research to investigate possible effects from the teaching intervention. The main interest was in long-term effects which were assumed to be indicated by precalculus students' performance in subsequent calculus classes. Thus the primary data set has the frequency of calculus grades cross-classified by the explanatory variables *Instructor*, *Time*, and if necessary, *Year*, and *High School*. Short term-effects were investigated for both calculus and non-calculus students using precalculus grades with *Instructor*, *Time*, and *Year*, as explanatory variables. The analyses were carried out using the techniques of GLM (generalized linear modeling).

### Conclusion

The choice of methods used in this research was based on Schoenfeld's (1994) suggestion of an eclectic approach, matching analysis type to purpose. The investigation into students' conceptualization of variables was conducted qualitatively. Descriptions were based on students' written explanations of their answers to three questions with support from a small number of interviews. Short-term effects of the intervention could also have been investigated qualitatively through student evaluations and interviews. However, since effects from interventions can be transient, it was decided to look at a

long-term effect, defined in this case as the student's performance in a subsequent calculus class. The data available are the students' grades and hence the appropriate choice is quantitative analysis. Both the calculus and non-calculus groups' precalculus grades were also analyzed. The results of these qualitative and quantitative approaches are presented in the next chapter.

## CHAPTER 4

### RESULTS

This chapter begins with the results of the qualitative investigation. The first section is a descriptive interpretation of students' written explanations of their answers on the three-question test on the use of variables, supplemented by related interview responses. Second is a section containing the classroom observations collected during the development of the intervention. These are followed by a short description of the intervention. The results of the statistical analysis of the effects of the intervention are reported in the third section.

#### Understanding of Variables Extrapolated from Students' Written Explanations

The subjects for this part of the study came from three sections of precalculus mathematics at the University of Victoria. During the first class of the term, before any teaching had taken place, the students wrote a three-question test in which they were asked to explain their answers. These explanations were grouped by type and the results are presented in the following sections. Each question is considered separately and a general summary provided at the end. The written test data were also supplemented by the responses recorded during interviews.

A total of 225 precalculus students wrote the three question test. Of these, 55 withheld permission to use their tests as data, leaving a total of 170 participants, 123 from the two regular term classes and 47 from the summer term group. Several students chose not to answer some of the personal information questions resulting in some variations in the overall totals in the following paragraph.

Gender information was provided by 163 students, of whom 53 were males and 110 females. The 147 students who gave details of their previous math courses were classified as having taken either grade 11 or grade 12 or their equivalents. The split is 74 with grade 11 and 73 with grade 12. Most of these students are not successful mathematics students. Approximately half of those reporting Math 12 as a previous course, or one fourth of the entire group, had failed it, and the majority of grades

reported, whether from grade 11 or grade 12, were B or C. Many of the students lacked recent mathematical experience, a gap of 3 to 5 years being the norm for the grade 11 group. The experience of the grade 12 group was more up-to-date, but overall, less than half of the students had taken a math class in the previous 2 years.

Responses to two questions, “How do you feel about taking this course?” and “How do you feel about mathematics in general?” betrayed a pervasive anxiety about the subject and the course. Statements suggestive of enjoyment or confidence were infrequent. The general picture of these precalculus students includes poor mathematical results, a lack of recent experience, and low confidence.

There were nine videotaped interviews conducted after the pilot tests in the Spring and Fall of 1995, including three with calculus students. A further ten interviews, of which two were unusable due to technical difficulties, took place after the administration of the final form of the test in January and May, 1996.

### Question 1

Simplify  $\frac{12a}{-6a}$ . Explain and justify each step you took to arrive at your answer.

This apparently simple question had a success rate of 52% excluding the explanations. This figure is consistent across the three individual classes comprising the subject group, and is higher than the success rate for the same question in the second pilot test (46%). The calculus classes in the second pilot, by comparison, had success rates of 80% (Math 102) and 96% (Math 100). The error response of  $-2a$  was given by 30% of the subject group of precalculus students. A further 6% did not deal with  $a$  at all, leaving answers such as  $\frac{2a}{-1a}$ . The remaining 12% show a mixture of errors ranging from simple arithmetic slips to indications of major confusion involving, for example, the use of incorrectly applied exponent rules, or solving for  $a$ .

### Correct Answers

Figure 6 and Figure 7 show examples of short explanations given for correct answers by precalculus students. Neither student defines canceling, opening the

possibility that students' understanding is pseudo-conceptual and rule-based. The single line used to strike out both variables in Figure 2 is suggestive of a non-numerical interpretation of  $a$ .

-2      1. divide 12 by -6  
2. cancel the a's

Figure 6. Question 1 response showing different vocabulary for variables and numbers.

$\frac{12a}{-6a}$  the a's cancel out =  $\frac{12}{-6}$  simple division = (-2)

Figure 7. Question 1 response showing variable canceling suggestive of a letter interpretation.

Longer explanations like the one shown in Figure 8, where the concentration is on the 12 and the  $-6$ , and the  $a$ 's are mentioned only briefly, are also suggestive of a non-numerical interpretation for  $a$ . In addition, the language used to refer to 12 and  $-6$  is usually different from that used in relation to the  $a$ 's. The  $a$ 's are *canceled* (see Figure 6 or Figure 7) or, in Figure 8, *crossed out*, whereas the numbers are *divided*. In addition, the student in Figure 8 produced an entirely mechanical explanation of why the final answer is negative. This latter explanation with *crossed out* instead of *cancel* looks clearly pseudo-conceptual and it must be assumed that, without an explanation demonstrating numerical understanding, a number of the other explanations using *cancel* are also pseudo-conceptual.

The interview responses confirmed the prevalence of a non-numeric interpretation of ' $a$ '. Three of the students interviewed stated quite clearly that letters represent

numbers but could not provide a numerical explanation of canceling the  $a$ 's. By contrast, calculus students' definitions of *cancel*, like the example shown in Figure 9, were more numerically based. Clearly the 52% of subjects who gave the correct  $-2$  as an answer includes some, perhaps many, whose reasoning is fallacious.

$$\frac{2a}{-1a}$$

The number 6 goes into 12 two times and into itself once. I divided six by 12 to get the numerator of 2 and I divided 6 by -6 to get the denominator of -1.

$$\frac{2}{-1}$$

I got rid of the two  $a$ 's because they cross each other out.

$$\frac{-2}{1}$$

I ended up with this answer because I divided 2 by -1 and I divided -1 by -1 in order to have the negative sign in the numerator.

Figure 8. Question 1 response showing contrasting vocabulary for variable and number operations, and contrasting vocabulary suggestive of an object interpretation of  $a$ .

$$\frac{2/2a}{-6a} = -2$$

the  $\frac{2}{2}$  is equal to 1 as is the  $\frac{6}{6}$   
 gotten by seeing 12 as  $2 \times 6$ , so the  
 fraction can be seen as  $\frac{2}{-1} \cdot \frac{6}{6} \cdot \frac{2}{a}$  or  $\frac{2}{-1} \cdot 1 \cdot 1$   
 or just  $-2$  (the negative does not  
 matter whether it is on the top or bottom)

Figure 9. Question 1, good response from a calculus student using same vocabulary when referring to both variables and numbers, and a numerical explanation of canceling.

$$-\frac{12a}{-a} \text{ divide 6 into the numerator (12)}$$

$$= -2. \text{ multiply the fraction by } -a \text{ to remove the denominator}$$

Figure 10. Question 1 response showing the technical term misuse, “multiply by  $-a$ ”.

Technical terms were frequently misused. In Figure 10, for example, the student describes canceling as “multiply the fraction by  $-a$  to remove the denominator.” During an interview another student referred to subtracting the  $a$ 's. This student had crossed out  $a$  from an answer of  $-2a$  and the interviewer asked her about her change of mind.

S Well, I think at first I didn't .. I don't know .. I just thought it would be there.

And then I decided to subtract them and it wouldn't exist any more.

I OK, so tell me how you got the negative two then.

S Twelve divided by six is two and then the negative.

I OK, but you said subtract just now (*Unintelligible, both speak at once*) .. Is that what you meant?

S Oh .. umm.

I So that was just the  $a$ 's you subtracted?

S I guess I subtracted the  $a$ 's and divided the twelve by six.

Thinking like this, where multiplication, division and subtraction are jumbled together might also be symptomatic of a deeper confusion involving exponent rules. Other examples of exponent rule use were present but led to wrong answers.

Among the successful students there were other examples of problems with terminology. Common factors were referred to as common denominators, lowest common multiples, highest common multiples or just common multiples. *Divides* was used backwards, for example in the phrase “12 divides 6”. If the explanations are required to be both full (cancel is explained or the same terminology used for both

numbers and variables) and correct, the success rate for precalculus students on this problem falls to a mere 3%.

Wrong answer:  $-2a$

$$\frac{12a}{-6a} = -2a$$

Since the numerator & denominator in this expression are of the same type (ex  $2a^6$ ) then I can advance the expression by doing clean division to arrive at a new answer:  $-6$  goes into  $12$   $-2$  times.

Figure 11. Question 1 response referring to  $a$  as a type, suggesting a unit interpretation.

$$\frac{12a}{-6a} = -2a \quad \begin{array}{l} \text{12a and -6a are in} \\ \text{the same 'unit' so simplify} \end{array}$$

Figure 12. Question 1 response referring to the variable as a unit.

Simplify  $\frac{12a}{-6a}$ .

$-2a$

- ① common base
- ②  $12 \div (-6) = -2$
- ③ answer  $-2a$

Figure 13. Question 1 response showing confusion between canceling rules and the rules for combining like terms.

The most common error response was  $-2a$  (30% of respondents), and was explained by the students in a variety of ways, none of which is consistent with interpreting the variable numerically. Rather, the explanations fit well with the notion of the variable as an object, a unit, or a letter. The explanations in Figure 11 and Figure 12, where  $a$  is described as identifying the type or unit of number, clearly show an object or unit interpretation for the variable. There may also be some confusion with the rules for adding or subtracting like terms, as shown by the response in Figure 13. A few students were also quite explicit in their belief that  $a \div a = a$  (Figure 14). Understanding of the numerical nature of the variable may be present but is not being used by this group of students.

$-2a$  - 1.) The number twelve ~~is~~ (multiplied by  $a$ ) ~~and~~ is divided by the integer  $-6$  (multiplied by  $a$ ).

2.)  $12/-6$  is equal to negative 2.

3.) as any number divided by itself is one, then  $a/a = 1a$  or  $a$ .

4.) This is expressed in simplest form as  $-2a$ .

Figure 14. Question 1 response showing the belief that  $\frac{a}{a} = a$ .

### Other Wrong Answers

The students (6%) who left the simplification partially completed, e.g.,  $\frac{2a}{-1a}$  should also be considered as having made an error. Such a response is difficult to mark wrong in a test since it is a simplified form of the original expression. However, it also suggests

that the student either did not know what to do about the variables, or was trying to apply the rule for combining like terms under addition or subtraction. In either case the thinking of such students has more in common with the error group than the correct group. Most explanations for this group of responses do not mention the variable at all, but when  $a$  is mentioned, as in Figure 15, it is clear that leaving the variable unsimplified is intentional.

ms.  $-2a$ ;  $\frac{12a}{-6a}$ . I reduce the numerator 12, by the denominator 6 which equals to 2. The negative stays, & the constant variable stays

Figure 15. Question 1 response showing  $a$  intentionally not canceled.

### Interview data

The interview data added little new information on Question 1. Erroneous explanations were confirmed as erroneous and not as poorly worded explanations, while correct explanations from the precalculus group were generally rule based. Students appeared to know that  $a$  represents a number but their descriptions of its use were not numerical. Even when specifically directed in an interview, most were unable to provide a numerically based reason for the relationship  $a \div a = 1$ . It is known as a fact or a rule and nothing more. The interview responses support the contention that precalculus students rely on recalled rules and do not use numerical understanding when simplifying this expression, i.e., their understanding can be classified as pseudo-conceptual.

### Summary of Question 1 Responses

A conservative estimate is that less than half of the students entering the precalculus class can simplify the expression  $\frac{12a}{-6a}$  correctly and demonstrate an understanding of the numerical role of the variable. The explanations accompanying both correct and incorrect responses suggest that in the context of this simplification there is a widespread interpretation of the variable as a non-numerical letter, object or unit, not as a number.

Many students appeared to use pseudo-concepts, leading in some cases to correct answers but in others to a confusion of their rules.

### Question 2

Recall that solving an equation means that you are trying to find the values of the variable that satisfy the equation. The solution to the equation presented below is correct as far as it goes.

How would you complete the following equation solution? Give reasons.

$$\begin{aligned} \text{Solve for } x: \quad & 2(x - 3) + 7 = 2(x + 5) - 9 \\ & 2x - 6 + 7 = 2x + 10 - 9 \\ & 2x + 1 = 2x + 1 \\ & 2x = 2x \end{aligned}$$

Students' responses to Question 2 fall into four basic categories pertaining to variable interpretations. However, it should be noted that in this question the interpretation of the variable is inextricably linked to the meaning the student has for equation solving. The four categories, with approximate percentages based on forms of responses rather than the associated reasons, are:

- Correct Group (15%): Showing an understanding of equation solving and the ability to interpret  $x$  as representing a set of numbers.
- Error Group 1 (35%): Showing procedurally correct equation solving with no conclusion.
- Error Group 2 (13%): Showing procedurally correct equation solving but concluding with a single value for  $x$ .
- Error Group 3 (37%): Showing wrong algebra or logic leading to a single value for  $x$ .

#### Correct Group

The maximum possible percentage of subjects who could be categorized as able to view  $x$  as representing a set of numbers in Question 2 is 15%. This figure is reached by including any response in which a set of numbers is part of, or implied in, the description

of  $x$ . However, this definition accepts as correct statements involving *any number* or *anything*, and two responses that indicate only integers. A more stringent requirement that students specify the set of real numbers in some way would reduce the percentage correct to 2%.

The *any number* or *anything* responses are considered suspect for two reasons. The first is that students may be referring to the limited sets of integers or even whole numbers. It is possible that this common error can co-exist with an interpretation of the variable as a set, in which case the error lies in the student's understanding of number types, not in the interpretation of variables.

The second reason is of greater concern and is exemplified in Figure 16 where *any number* is specifically associated with *no solution*. A similar observation was made during two interviews where tone of voice and body language made it clear that words like "x can be any number" were being offered, not as a solution, but as a protest or complaint that the equation was believed to be unsolvable. Such students deduced correctly that  $x$  can represent any number, but did not accept this as a solution to the equation.

$$2(x + 3) + 7 = 2(x + 5) - 9$$

$$2x - 6 + 7 = 2x + 10 - 9$$

$$2x + 1 = 2x + 1$$

$$2x = 2x$$

(no solution)

*x* could equal any number since the answer is  $2x = 2x$  |  $x$  is a constant they will both always equal

Figure 16. Question 2 response showing a contradiction between describing  $x$  as any number and claiming that the equation has no solution.

### Error Group 1

Superficially correct algebra was used by 35% of the subjects to reach, but not progress beyond, a statement such as  $x = x$ ,  $1 = 1$ . Unfortunately most (44 of 58) gave no further clue to their thinking, but two possible explanations can be extracted from the 14

reasons provided and from the interview responses. The first explanation identifies those students for whom algebra consists of something to do. Both the written reasons, like those in Figure 17 and Figure 18 and the interviews suggest that these students were engaging in symbol manipulation, the performance of which is complete when a line consisting of two single symbols straddling an '=' is reached. The rules used to reach the final line are arbitrary and students have little numerical understanding of either the steps or the overall purpose and meaning of equation solving. The students who concluded with the statement "x equals itself" may also belong in this group.

A different interpretation of conclusions like  $x = x$ ,  $1 = 1$ , or  $0 = 0$  is that students were expecting a conclusion of the form  $x = n$ . They were disturbed when this was not the outcome and were unable to proceed further. This is apparently the problem for the students in Figure 16 and Figure 19. One student in an interview claimed that "The sole goal in high school was to find out what x is."

$$\begin{array}{l}
 2x + 1 = 2x + 1 \\
 \frac{2x}{2} = \frac{2x}{2} \\
 x = x
 \end{array}$$

- If you divide 2, x, or 2x from one side, don't you have to do it to the other side?

This would result in  $\begin{array}{l} x = x \\ 2 = 2 \end{array}$

- however,  $x = x$  by dividing by 2 or  $\emptyset$  would be the answer if solving for the variable

Figure 17. Question 2 response showing a student who requires the solution to take the form  $x = n$  and sees either no solution from  $2 = 2$ , or  $x$  as the solution from  $x = x$ .

*I would divide each side by 2 and leave  $x = x$ . It does seem pointless though.*

Figure 18. Question 2 response showing a focus on the process rather than the result.

$$\frac{2x}{2} = \frac{2x}{2}$$

$$x = x$$

The value of  $x$  is not definable because it exists on both sides of the equation. I divided both sides by two to further simplify but still cannot define the value of  $x$ . Need a real number on the opposite side of the equation to find value for  $x$ .

Figure 19. Question 2 response showing a student who expects a single value for the variable in the form  $x = n$ .

### Error Group 2

Thirteen per cent of students worked correctly but ended with a numerical statement about  $x$ , usually  $x = 1$  or  $x = 0$ . Two different interpretations are possible. Some students who had reached the form  $n = n$  followed by the conclusion  $x = n$ , may have identified their desired single value for  $x$  with one of the numbers appearing at the end of their work. Figure 20 shows an example of a student who used the number appearing in the final equation. There is no error visible apart from the faulty final conclusion.

A second, more optimistic interpretation of the group with no visible algebra error other than a single value conclusion, is that having failed to find a single value for the variable using algebra they resorted to guessing and testing. The response in Figure 21, where the linear equation solving method is described followed by the non-sequitur statement "However, 'x' has a value of one...", is suggestive of a final guess and test approach once the practiced equation solving technique failed to provide the expected result. There were also a few students who found a single value for  $x$  using guess and test alone without any intervening algebra.

$$\begin{array}{c} 2x+1 = 2x+1 \\ \textcircled{L} \frac{2x}{2x} = \frac{2x}{2x} \textcircled{R} \\ 1 = 1 \end{array}$$

Divide left side by right side  
to achieve the answer

Figure 20. Question 2 response showing a student using the available number as the single solution.

$$\begin{array}{c} \frac{2x}{2} = \frac{2x}{2} \\ x = x \\ x = 1 \end{array}$$

to solve "x", you must isolate "x" of one side of the equation. Dividing by 2 does this and you are left with "x" = "x". However, "x" has a value of one so "x" equals 1

Figure 21. Question 2 response with a final conclusion apparently achieved by trial and error.

### Error Group 3

The students in this group provided a variety of algebraic or logical errors. Some students simply eliminated one of the variables by crossing it out, e.g.  $1x = x$  or used an associated number (Figure 22) with or without an attempt at some form of verbal justification like the "variable isolator" in Figure 23. There were also a number of examples of algebraic errors such as the claim that  $\frac{0}{0} = 1$ , and procedural errors when students tried to rework the given solution steps. Most of the students who attempted to

solve the equation from the beginning instead of completing the given solution made some type of algebra error.

$$2x = 2x \left\{ \frac{2x}{2} = \frac{2x}{2} \right\}$$

~~not~~

$$x = x$$

$\therefore x = 1$  because  $x$  is the same as  $1x$ .

Figure 22. Question 2 response showing the use of an associated number as the final solution.

$$\frac{2x}{2} = \frac{2x}{2} \quad \text{divide to isolate the variable}$$

$$x = \frac{2x}{2}$$

$x = x$  so if  $\Rightarrow$  is the variable solution

$$x = 2$$

Figure 23. Question 2 response showing an attempt at justifying the use of an associated number as the single solution.

### Summary of Question 2 Responses

The great majority of students did not conclude that the variable represented a set of numbers in the context of an identity equation. They appeared to interpret  $x$  as a single number, or to have a fixed assumption that equation solutions are single numbers. If the single number was not forthcoming from their algebra manipulations, they either gave up or reformulated their algebra in order to finish with a single value. A few resorted to guess and test but most of these did not seem to test more than one value.

### Question 3

In this question you are given five verbal descriptions labeled by capital letters, and three equations. For each equation identify which of the verbal descriptions fit and explain your choices. If a description would be true for an equation write the letter of the description beside the equation. You can have more than one letter with each equation.

- A      As  $p$  increases  $q$  increases      B      As  $p$  decreases  $q$  increases  
 C       $p$  and  $q$  remain constant      D       $p$  and  $q$  are never negative  
 E      None of the descriptions A, B, C or D.

Equation	Descriptions which apply	Explanations
$p + q = 15$		
$p = 5q$		
$p + q = r$		

Examples of student work have been retyped as bulleted quotes in this section. The originals are not reproduced because several students did not keep their responses confined to the boxes provided, making clear reproduction difficult.

The primary focus in this analysis is on the first two equations:  $p + q = 15$  and  $p = 5q$ . The responses to  $p + q = r$  also provide information, but mainly in conjunction with responses to the first two equations. Table 2 shows the percentages in answer categories for the first two equations individually in the marginal totals and in various answer combinations in the table cells. The correct phrase options for both  $p + q = 15$  and  $p = 5q$  were chosen by 36% of the subjects writing the test, while 27% chose incorrect options for both. The equation  $p = 5q$ , with 62% of respondents choosing option A, appears at first sight to be more successfully answered than  $p + q = 15$ , where only 48% of respondents chose B. However, examination of the students' reasons, particularly but not exclusively, to  $p = 5q$ , turned up a number of correct answers backed up by wrong explanations. The 62% and 48% figures are not representative of students' recognition of the variable nature of  $p$  and  $q$ . Many of these students are not correct at all.

The largest marginal totals, other than the correct options, shown in Table 2 are for options D, C and E for the equation  $p + q = 15$  and options C and E for equation  $p = 5q$ . However, the numbers for both option D and option E are inflated. Some of the explanations attached to option D show the interpretation that  $p$  and  $q$  must not be negative at the same time, while option E was sometimes used when the student had no real answer to give but preferred not to leave a blank. Thus the numbers for both options D and E are increased for reasons that are not directly related to students' interpretations of the variables. This leaves option C as the single most important error grouping at 13% for question  $p + q = 15$  and 9% for question  $p = 5q$ . Option C was also a frequent choice (21%) for the third equation,  $p + q = r$ .

Table 2

Percentages of the Subject Group in Selected Letter Choice Categories for the Equations  
 $p + q = 15$  and  $p = 5q$ . ( $N = 170$ )

Equation $p = 5q$	Equation $p + q = 15$								
	<u>B</u>	BC	BA	C	D present	E	other	blank	
<u>A</u>	<u>36</u>	2	1	10	5	6	1	1	<u>62</u>
AC	3	1	0	0	0	1	0	0	5
AB	1	1	0	1	2	1	0	0	5
C	2	0	2	2	1	1	1	0	9
D present	2	1	0	0	1	0	1	0	3 <sup>a</sup>
E	1	1	1	0	3	1	0	2	8 <sup>b</sup>
other	2	0	1	1	1	1	2	0	6
blank	1	0	0	0	1	0	0	1	2
	<u>48</u>	6	5	13	14 <sup>a</sup>	10 <sup>b</sup>	4	3	100

Notes: Correct responses and percents correct are underlined.

<sup>a</sup> Includes the misinterpretation that both  $p$  and  $q$  be negative.

<sup>b</sup> Includes those who chose it because they saw no acceptable alternative.

Once options D and E are discounted and option C is noted, the most striking features of the distributions of answer combinations are firstly, the extreme variability

and secondly, the proliferation of small groups of answer combinations. This appears to be due to the presence of multiple errors. In addition to their interpretations of the variables, many students showed a lack of understanding of different number sets and computation, while others had problems with terminology.

Despite the difficulties and problems described in the preceding paragraphs, some clear response patterns of students' thinking do emerge. These are summarized in Table 3 and discussed in detail in the next sections. No percentages are given because the many unclassifiable responses make accurate categorization impossible.

Table 3

Summary of Students' Interpretation of Variables Identified from Question 3

---

Interpretation 1:	$p$ and $q$ vary continuously, one dependent on the other.
Interpretation 2:	$p$ and $q$ are collections of number pairs which show patterns of values for $p$ and $q$ .
Interpretation 3:	Values of $p$ and $q$ are compared with <i>increasing</i> and <i>decreasing</i> interpreted as <i>greater than</i> and <i>less than</i> respectively.
Interpretation 4:	$p$ and $q$ are objects, letters or units, not numbers.
Interpretation 5:	<i>constant</i> is interpreted to mean a consistent relationship.
Interpretation 6:	<i>constant</i> is interpreted to mean constrained or controlled by the presence of a constant such as 15. Variables take on different values but once assigned a value do not change.
Interpretation 7:	Variables are interpreted as a series of fixed values which do not change.

---

The calculus classes who answered slightly different forms of this question in the second pilot test were much less variable with 80% correct in Math 100 and 75% in Math 102, and considerably more consistency in the reasons proffered. As in the precalculus classes, the largest error group for both  $p + q = 15$  and  $p = 5q$  involved the use of option C.

Whether from tiredness at the end of the test, or intimidation by three variables, the responses to  $p + q = r$  are somewhat less informative than those for  $p + q = 15$  and  $p = 5q$ . There are many blanks and the reasons offered are often too brief to be interpreted with much confidence. The problem with the E response has already been noted. As with questions  $p + q = 15$  and  $p = 5q$  the total apparently correct exaggerates the true extent of good understanding.

### Interpretation 1

Students with this interpretation show a sense of continuous change and the dependency of one variable on the other. The clearest and best explanations usually involved rewriting the equations, together with an equivalent description to show  $q$  calculated from  $p$ :  $q = 15 - p$ , and  $q = p/5$ . Explanations like these were fairly common in pilot test papers from the calculus classes, but rare in the precalculus group. The modified equations alone without an explanation must be considered suspect because there are cases where they were accompanied by wrong explanations. Some examples of the more appropriate explanations available from the precalculus group follow.

- Option A for  $p = 5q$ :  $p =$  both 5 and  $q$  multiplied together. For example if  $p$  increased so would  $q$  since the 5 remains constant.
- Option A for  $p = 5q$ : They are directly proportional meaning they go up or down together
- Option B for  $p + q = 15$ :  $q = 15 - p$  Therefore as  $p$  increases  $q$  will decrease (and reverse)

The wording of option A fits the calculation of  $q$  from  $p$  and not  $p$  from  $q$  as the equation is written. Many ignored the direction of the wording and expressed their reasons in terms of calculating  $p$  from  $q$ . It is impossible in most cases to tell whether this was the result of the reversal being ignored, not noticed, or from an understanding of the relationships involved.

### Interpretation 2

In this interpretation students are seen describing the patterns apparent in sets of specific number pairs that satisfy the equations. They do not see the variables as being involved in continuous and dependent change, and their explanations mainly refer to the calculation and its corresponding result. Separating Interpretation 1 from Interpretation 2 is not easy. In general an explanation was assumed to belong to Interpretation 1 rather than Interpretation 2 if it contained some correct numerical reference beyond the basic pattern or calculation. Thus the following two examples which contain nothing beyond the laying out of values and a calculation statement are classed as the lower level Interpretation 2.

- Option B for  $p + q = 15$ :
 
$$p = 10 \quad q = 5 \quad = 15$$

$$p = 9 \quad q = 6 \quad = 15$$
 as  $p$  decreases  $q$  increases
- Option A for  $p = 5q$ : If the value for  $q$  goes up,  $p$  the answer has to increase as well.

Many descriptions, like the one below, are too ambiguous. The explanation starts out well but the reference to the total could be related to an addition total, or even to the interpretation of a letter as an object as described in Interpretation 4 (p.78).

- Option B for  $p + q = 15$ : Because if  $p + q$  has to  $= 15$  as one changes the other must as well so the total is still 15.

### Interpretation 3

The explanations associated with Interpretation 3 are couched in terms that indicate a static comparison of the relative sizes of  $p$  and  $q$ : *larger than* and *smaller than* being used instead of *increasing* and *decreasing*. This error leads to a correct answer choice if a second error of considering only positive numbers is committed. Note that in the first of the following two examples the equation is rewritten to show  $q$  calculated from  $p$ , but the accompanying explanation does not indicate that the variables are interpreted as co-variants.

- Option A for  $p = 5q$ :  $q = p/5$   $p$  must be larger than  $q$  for this to be true.
- Option A for  $p = 5q$ :  $p$  has to be always greater than  $q$  because it is multiplied by 5.

Ironically the slight improvement shown in the next example of considering both positive and negative numbers produced a response that identifies A,B as the description for  $p = 5q$ .

- Option A B for  $p = 5q$ : if  $p$  is negative  $q$  is less negative.

These students were not looking at changes in  $p$  and  $q$ , only at the relationship between the variables.

#### Interpretation 4

A group of responses were worded in ways that can be explained by the object interpretation for variables, which was observed by Clement (1982) in his investigation of the students and professors problem. The equation is interpreted as being 5 letters  $q$  for every  $p$ . Using this kind of symbol meaning and logic, increasing  $q$  would produce more letters  $q$ , possibly represented by terms such as  $6q$  or  $7q$ , and this is the image present in the next examples.

- Option A for  $p = 5q$ : By multiplying  $q$  by a larger number then  $p$  must increase.
- Option A for  $p = 5q$ : if  $p=5q$  then any increase of  $p$  would cause  $q$  to become greater than  $5q$ .
- Option A for  $p = 5q$ :  $q$  increases 5 times faster than  $p$ .

The following examples are explainable by this object interpretation if it is combined with the interpretation of *increases* as *greater than* and *decreases* as *less than*. Students may have interpreted the question in terms of comparing the number of  $p$ 's to the number of  $q$ 's and for  $p = 5q$  the image of increasing  $p$  and  $q$  could be:

$$p \leftrightarrow qqqqq, pp \leftrightarrow qqqqqqqqq$$

Clearly in this image, "As  $p$  increases  $q$  increases" is true.

- Option A for  $p = 5q$ :  $q$  is 5 times more than  $p$  therefore it increases.
- Option A for  $p = 5q$ : whatever  $p$  is  $q$  is 5 times the size.

While any one of these can be explained away individually as the result of poor or careless description skills, together they strongly suggest the presence of an object interpretation for the variables, and an allied misinterpretation of *increase* and *decrease*.

These misinterpretations may also underpin a variety of other explanations, such as:

- Option A for  $p = 5q$ :  $p$  can never be more than  $q$ .
- Option A for  $p = 5q$ : as  $p$  increases  $q$  adds up.

The presence of the object interpretation in responses to  $p = 5q$  raises the possibility that it may also be present in responses to  $p + q = 15$ . The object image for the addition would be of a number of  $p$ 's and a number of  $q$ 's to a total of 15. Unfortunately this wrong image for  $p + q = 15$  fits the correct response B (As  $p$  decreases  $q$  increases) as follows: Instead of the numerical examples  $p = 6$  and  $q = 9$ , and  $p = 5$   $q = 10$ , the image would be:

pppppp qqqqqqqqqq and ppppp qqqqqqqqqq

Thus this image of discrete objects, the image of specific number pairs, and the image of continuously changing variables can all be described with the same descriptors. The assignment of an explanation to one or other of the Interpretations 1, 2 or 4 must therefore often be made on a balance of probabilities or on the basis of an associated response. In the following two examples it is the combined responses to both  $p + q = 15$  and  $p = 5q$  that lead to the object interpretation being suggested for  $p + q = 15$ , as well as  $p = 5q$ .

- Option A for  $p = 5q$ :  $q$  would increase if  $p$  did because its value is 5 x that of  $p$ .  
Option B for  $p + q = 15$ :  $q$  would have to increase if  $p$  decreased to equal 15.
- Option A for  $p = 5q$ :  $q$  must increase by 5x to equal  $p$ .  
Option B for  $p + q = 15$ : the sum must equal 15. Therefore as one decreases the other must make up for it.

### Interpretation 5

The use of *constant* as meaning *consistent* was demonstrated very clearly by a group of five students (3%) who gave remarkably similar reasons for the response pattern, B to  $p + q = 15$  and AC to  $p = 5q$ . Four of the five also gave E for  $p + q = r$ . These students



For equation  $p + q = 15$

- C  $p$  and  $q$  can't change because the answer wouldn't remain 15
- C  $p$  and  $q$  remain constant because 15 is concrete. They add together to make 15.
- B if  $p$  decreases  $q$  must increase in order to compensate
- AB balance one another to equal 15 [Option A possibly misread.]
- C it's an adding equation

For equation  $p = 5q$

- A ex  $p = 10$   $q = 2$ ,  $p = 20$   $q = 4$
- A If  $p$  increases  $q$  has to increase because both sides = [sic] each other.
- C They must be constant because the 5 holds them to a certain value
- C remain constant with each other - when one number equals a number the other automatically equals a number
- A it's a  $\times$  (mult) equation

One of the three students who chose C for all three equations was interviewed, and as can be seen in the following transcript excerpt, provided a good demonstration of the ability to accept that variables can take on different values, while at the same time rejecting the notion that variables can change values.

I In what sense do you think that  $p$  and  $q$  remain constant?

S Well, because they represent like a set number and so they can't really change. You can't say  $p$  equals ten or five or something. So they're a set number.

I Mm hm. But does it always have to equal ten?

S No, it can equal whatever it would ... equal.

I Ok, but for .. OK, so what you mean is it doesn't ..

S It doesn't represent just anything. It represents a set number.

I OK, and then  $q$  would as well?

S A different number.

I Um. In that case let's take this one ( $p + q = 15$ ). You put down C. You could have put down more than one. Did you actually discount the others or ..

S No, I just didn't know what the ..

I OK, so it's not that you think it doesn't fit ..

S I just didn't know.

I OK, so for example B, what ..

S I didn't think any of them would hold just because they're different numbers so it shouldn't matter if one changes. It shouldn't necessarily mean that the other one is going to.

### Interpretation 7

The surface distinction between Interpretations 6 and 7 lies in different applications of the term *constant*. However, both groups of students describe the variables as taking on different values while denying that the variables change.

One group, 4% of subjects, chose option E (None of the descriptions A, B, C, or D) for equation  $p + q = 15$  along with option A for  $p = 5q$ , and C for  $p + q = r$ . These students, despite rejecting option B which deals with increasing and decreasing values, gave explanations reflecting an acceptance that the variables  $p$  and  $q$  could take on different values. This contradiction is apparent in the following examples.

- Option E for  $p + q = 15$ :  $p$  and  $q$  could be any numbers to equal 15
- Option E for  $p + q = 15$ : not constant as values can differ (ie 5,10 or 8,7). None apply.
- Option E for  $p + q = 15$ : there are a number of different variables for  $p$  and  $q$ .

It is particularly clear in the second of the preceding examples that the student is focusing on individual number pairs and not on any connection between the pairs.

The group who chose the option sequence E, A, and C are part of a much larger group, 28% of subjects, who chose option C for  $p + q = r$  either alone or in combination with other options. In addition to the E, A, and C group a second larger sub-group (7% of subjects) who chose B, A, and C can be identified. The explanations seem to suggest that the variables are not seen as constant when different values can be found for them. However, if 15 is replaced by  $r$  in equation  $p + q = r$  then  $p$  and  $q$  become constant because it is no longer possible to specify values for them. The following examples are given as combinations to show the distinctions made between the two addition equations.

- Option B for  $p + q = 15$ : If  $p$  were negative each time it decreased  $q$  would have to increase to keep the # at 15.  
Option C for  $p + q = r$ : Since  $r$  is constant
- Option B for  $p + q = 15$ : as  $p$  gets larger  $q$  gets smaller  
Option C for  $p + q = r$ : there is nothing done to  $p$  and  $q$ .
- Option E for  $p + q = 15$ :  $p$  and  $q$  could be any numbers to = 15  
Option C for  $p + q = r$ : must be constant because  $r$  is constant and  $p$  and  $q$  must equal  $r$ .

The sense that the use of the variable  $r$ , instead of a specific number, holds the other variables constant is also present in the following explanations.

- Option C for  $p + q = r$ : Neither  $p$  or  $q$  can change as they are equal to  $r$ .
- Option C for  $p + q = r$ : Since  $r$  is constant.
- Option C for  $p + q = r$ : the variables are set.

Out of the 28% of students who chose option C for equation  $p + q = r$  many show similar response patterns to those already discussed but with the addition of an extra error such as the inclusion of option D ( $p$  and  $q$  are never negative). This group also includes some of those identified as holding the variable as object interpretation (Interpretation 4, p. 78).

There is also a group who considered  $r$  to be a constant in the same sense that 15 is a constant. Approximately 9% of the subjects chose the same option for  $p + q = 15$  and  $p + q = r$ , combined with explanations that clearly identified  $r$  with the number 15. In some cases this response pattern may suggest a shaky grasp of the concept of variable change, but in others it may reflect the understanding that  $r$  must be held constant if changes in  $p$  and  $q$  are to be considered.

### Summary of Question 3 Responses

Interpretation 1 is indicative of good understanding of variables and continuous connected change. Students classed as demonstrating Interpretation 2 showed some

sense of connected variable change, but only as a series of discrete number pairs. Distinguishing between interpretations 1 and 2 based on students' explanations is difficult. However, the interview responses suggest that Interpretation 2 is the more common. Interpretations 3 and 4 are cases where the right answer choices may be associated with wrong reasons. In Interpretation 3 students are comparing  $p$  and  $q$  and misusing the terms *increasing* and *decreasing*. The variable is seen as an object, letter or unit in Interpretation 4. Interpretations 3 and 4 are clearly identifiable from some written reasons but may underlie apparently correct reasons as well. Option C ( $p$  and  $q$  remain constant) is connected to Interpretations 5 and 6 but in different ways. *Constant* is applied to the consistent relationship between  $p$  and  $q$  in Interpretation 5. Students with Interpretation 6 seem to view variables as a series of fixed values which cannot change because they are *constrained* or *controlled* by the presence of the 5 or the 15. The variable as a fixed value is also typical of the thinking described in Interpretation 7, but in this case the variables are only described as constant in the context of the third equation,  $p + q = r$ .

There is ample evidence of compartmentalization based on the form of the equations since very few students demonstrate a consistent or universal interpretation of the variables across all the questions. Students deal with additions and multiplication equations differently and make statements which, to the informed observer, are contradictory.

The interpretations described are all present but their incidence levels are unknown. This is partly due to a lack of sufficient discriminatory detail in the written explanations, and partly due to students' use of different interpretations in different contexts.

#### Summary of Students' Difficulties Extrapolated from the Written Test

The three questions in the test of variable understanding each provided a different context for interpreting variables and all uncovered weaknesses in students' understanding. Analysis of this type is far from precise. Students can produce rote-learned as well as apparently correct responses, which are based on a foundation of error. Alternatively careless reading and poor communication skills can mask good levels of understanding. Multiple errors also made classification difficult. Despite these

difficulties and uncertainties a number of conclusions can be drawn based on the combined results of all three questions.

Very few students were nominally correct on all three questions and even fewer gave both correct answers and satisfactory explanations. There was a great deal of evidence of misconceptions and pseudo-concepts, the former being believed by the student while the latter is simply a means to the end of producing a correct answer (Vinner, 1997). It was often impossible to distinguish between the two. However, the weight of evidence suggests that many, perhaps most, correct answers were pseudo-conceptual in nature and some may have been based on actual misconceptions.

There were several cases where students clearly interpreted variables as non-numerical objects or as units in both Question 1 and Question 3. Some of these may have been examples of pseudo-concepts, which are intrinsically non-numerical, but some could also be ascribed to the misconception of the variable as representing a physical object. One student used the term *unit* in reference to the variable in Question 1, while others used phrases such as *cross out* instead of *cancel*. In Question 3, several students gave explanations compatible with an image of  $5q$  as five separate letters. Unfortunately this kind of image gives rise to the same responses as a numerical image to the addition equation  $p + q = 15$ , leading to the possibility that this object interpretation is more prevalent than the evidence indicates. It is certainly clear that non-numerical pseudo-concepts are widespread.

When numerical values were assigned to the variables it was almost always as single numbers. Very few students identified a set as the solution to the identity equation in Question 2. Some went so far as to identify the set and then reject it in favour of a single number solution. Unfortunately it was not possible to distinguish between students who presented a single number solution because they assumed that the variable must have a single value, and those who assumed that the equation required just one solution.

The single valued variable also appeared in the responses to Question 3. Many students listed number pairs for the equations but in some cases went on to deny that the variables could increase or decrease, on the basis that if either variable changed the equation would cease to be true. Others compared the values of the different variables, i.e.,  $p$  compared to  $q$ , and responded in terms of greater than and less than, instead of

increasing and decreasing. Among the students who identified the correct descriptions of increasing and decreasing variable values there were indications that they did so on the basis of the patterns in a series of number pairs or calculations and not on any sense of continuous or dynamic change. Thus for both wrong answers and some right answers students appeared to focus on discrete number pairs, that is, on single values for each of the variables.

Further support for the presence of pseudo-concepts appeared in students' evident preference for manipulation action. Symbol manipulation dominated the responses to Questions 1 and 2, and students seemed most comfortable with equation  $p = 5q$  in Question 3 where one variable can be directly calculated from the other.

Compartmentalization was also much in evidence. Neither good conceptions nor misconceptions were applied consistently to all three equations. In addition the concepts of variables students used often appeared to be different in the two equation forms.

Overall, students' understanding can be described as very poor. Many students did not appear to understand the basic vocabulary, symbolism, and conventions used in the mathematics classroom. Furthermore, they were not necessarily aware that they lacked understanding. This situation does not provide any kind of foundation for future progress in mathematical learning and development.

### Other Observations of Precalculus Students' Knowledge

Among the many mistakes precalculus students were observed to make during classes, tutorials and interviews, there exist some common elements and features. These are described first and then illustrated through interview excerpts and other reported observations. The interviews used in this section were conducted following either the mid-term or the final examination in the fall 1995 term. Classroom and tutorial observations are primarily from the academic years 1994 -1995 and 1995 – 1996.

## Observed Problems

### Emphasis on Doing vs. Reflective Thought

A typical approach for students was to start into symbol manipulations immediately without any kind of reflective pause. These students also did not want to look back. They pressed forward, and tried to fix any problems that arose immediately without any reflection, backward reference, or check. They also found it difficult when the problem solution required the use of a previous result or some reference back to an earlier stage. These students reacted to the first cue they saw, which made the choice of the correct solution method a matter of chance.

### Connecting Words and Algebra

Many students could explain or describe a solution verbally, but had great difficulty putting their words into algebra symbols. The reverse was also true. Students could perform the algebra correctly but were unable to explain the purpose and meaning of what they were doing. The tools were there, in the sense that they knew the rules, but they did not know when or where to use them.

### Vocabulary Misuse

The students used many words incorrectly. In some cases this was faulty recall or carelessness, but in others it appears that they were confusing terms because they did not understand the distinctions. Most noticeable was the use of *equation* instead of *expression*. *Simplify* and *solve* were seen as distinct in terms of their definitions but students often confused the algebraic processes, turning a simplification into an equation solving exercise. They also tended to read algebraic symbols sequentially, or even phonetically: for example, reading both  $x^2$  and  $x_2$  as “x two”. Many students also had difficulty using and interpreting the symbols  $<$  and  $>$ .

### Algebra Seen as Symbol Manipulation

Terms and symbols were described by their physical position in relation to other terms or symbols, not by their numerical or algebraic relationship. Algebra procedures

were described by the movement of symbols and not by the underlying numerical meaning or logic. The pseudo-concepts and pseudo-analysis described by Vinner (1997) seemed widespread.

### Expressions Not Seen as Numerical Objects

Students could interpret expressions as calculations, but not as numerical objects. Sometimes students appeared to work with expressions as single objects but when pressed by the demands of the problem or by questioning it became apparent that the objects were non-numerical arbitrary sequences of symbols used within the larger context of symbol manipulation rules.

### Interview Excerpts Demonstrating the Observations

The first interview excerpt is from a student who confuses simplifying and solving, who works forward from her current position rather than looking back, and who works from rules. The student was asked to simplify the expression and find the domain of  $g(f(x))$  when  $f(x) = \sqrt{x-1}$  and  $g(x) = x^2 + 1$ . The initial substitution is done correctly but then the simplification is converted to equation solving (Figure 24). The single number conclusion is assumed to be wrong based on her experience that domains are sets, and instead of looking back, she takes the single value and converts it into a set.

#### Interview Excerpt 1.

$$\begin{array}{l}
 (\sqrt{x-1})^2 + 1 \\
 -1^2 = \sqrt{x-1} \\
 +1^2 = x-1+1 \\
 2 = x
 \end{array}$$

**Figure 24.** From Interview Excerpt 1: Simplification converted to equation solving.

- S (After writing down to  $2 = x$ ) No that doesn't make sense.
- I Why does that not make sense?
- S Well, the domain can't just be one number. The domain has to be a set of numbers. So  $x$  would equal ...  $x$  is greater than or equal to 2.

The next example shows another situation where a student converts a simplification exercise into an equation solution. The student's written solution from a mid-term test is shown in Figure 25, and the excerpt is from a subsequent interview. This student demonstrates the quick non-reflective response as well as vocabulary confusion. He appears to know the distinction between simplify and solve but does not apply this knowledge without prompting.

Interview Excerpt 2

If  $f(x) = \sqrt{x^2 - 1}$ ,  $g(x) = x^2 + 1$ , and  $h(x) = \frac{1}{x}$ , find and simplify:

(a)  $g(x)h(x)$

$$(x^2 + 1) \left( \frac{1}{x} \right)$$

$$\frac{x^2 + 1}{x} = \frac{x^2}{x} + \frac{1}{x} \left( \frac{x^2}{x} \right)^x = \left( \frac{-1}{x} \right)^x$$

~~$x^2 = -1$~~   $x = i$

Figure 25. From Interview Excerpt 2: Simplification converted to equation solving.

- I See if you can remember and tell me what you did.
- S All right. Yes. Um. See I shouldn't have ... no uh 1 over  $x$ . So it's basically just multiplying the two functions, right?
- I Yes.
- S Well.

- I So that's right , that's right, that's right and that's right [*Points in turn* ] This is where you went wrong [*Points to start of equation form*].
- S Oh. .. times  $x$ , .. cancel.  $x$  squared equals negative 1. Wait. .. Negative one over  $x$ . OK I can't quite ... this is ..
- I No, not quite. Read again what you are being asked to do.
- S Find and simplify. .. oh *Both speak at once*
- I You go ..
- S  $g$  of  $x$ ,  $h$  of  $x$  ..  $x$  squared equals negative 1 over  $x$  ..
- I That's the problem. When you are asked to simplify something what should your answer .. what do you expect your answer to look like?
- S An  $x$  equals.
- I Ah.
- S I don't know. *Laughs* When I see numbers as well as the  $x$  I just automatically think solve. It doesn't say solve. Oh, jeez. OK.

The third example is a student showing both a desire to do something and the treatment of the symbols as objects on a page. This student used the term *instinct* several times during the interview, and it was clear that in most instances she reacted to the mathematical problem without any reflective pause. The exchange begins when the student is asked how she describes the function expression  $f(x) = x^2 + 9x + 20$  to herself.

### Interview Excerpt 3

- S Now this side doesn't confuse me at all. (*Points to the expression.*) If it were just  $x$  squared plus nine  $x$  plus twenty equals zero and it said solve for  $x$ , that's fine. You know I can figure that out, but as soon as that  $f$  of  $x$  is in front of it – I'm thinking – 'cos I'm always thinking that there's got to be a value – you're substituting a value for  $x$ . But that doesn't give you one, and so it's ... I can't explain why that confuses me, it just does.
- I OK. Well of course some of the questions you do. Of course you do substitute a value.
- S Right, right. I think it for every single one, and that's my first instinct – to do

that.

This same student also explained very clearly her need for the variable to be a single value when solving an equation (Interview Excerpt 4). The context is the solving of an identity equation and the algebra has been completed correctly to the point  $0 = 0$ . The student describes the variable as capable of taking on many (infinite) values but as a series of single instances which she can write down. There is no suggestion that she sees the variable as a representative of a set of numbers.

#### Interview Excerpt 4

I So the equation comes down to being just 0 equals 0. So what does that tell you about  $x$ ?

S It can be any real number?

I Yeah. Is that a comfortable idea for you?

S Not especially. Because I really don't care what  $x$  could be I want to know what it is. And all the domain and the range and all that, I don't like that idea at all. Just because I can't see it. I've got to see something to understand it.

I You want  $x$  to be able to be identified as a particular number?

S Yeah.

I Not just representative of a set?

S Yeah. Or it could be any positive number, that bothers me. I understand it but I don't ...

I It doesn't feel comfortable?

S Yeah, it doesn't feel right.

I So essentially what you're saying is that your basic understanding of the variables is as numbers. And that's how you like to be able to see them. You want to say, well, all right I'm using  $x$  but I could always use a specific number.

S Yeah, exactly, and that's comfortable.

I And so the idea of  $x$  as possibly representing a general set of numbers...

- S Yeah. It makes me just want to go and pick out every, the equation for every ... for every value
- I Which is a little bit of a problem! [*Laughs*]
- S Or I don't really believe that it works. [*Laughs*]
- I Yes, it would be, because if you have one of these infinite sets ...
- S I only have 2 hours to write it down is the problem!

The responses from the student in the next example demonstrate some symbol and technical vocabulary confusion; how prior (wrong) expectations of how an interval should look took precedence over the meaning of the symbols; and at the end a misuse of the word *equation*. Some confusion over the role of  $x$  and  $y$  in a graph is also shown, but unfortunately the interviewer did not pursue the point after the student's comment on switching  $x$  and  $y$  that "...it doesn't really matter." Her dependence on rules is evident throughout. The student had been trying unsuccessfully to find the range of a function using algebra when the interviewer interrupted by sketching the graph of the function  $f(x) = x^2 - 2x - 3$  and the exchange proceeded as follows.

#### Interview Excerpt 5

- I OK, now, with the graph in front of you .. do you ever use the graph for finding domain and range?
- S Yeah. A bit.
- I So how would you do that from the graph?
- S From the graph. Well, you can tell that it can't go any lower than negative 4. Because that's going up so that's one of the values .. but every other number ..it does. [*Points to right of vertex*]
- I So what are we talking about here, domain or range?
- S Oh, range.
- I OK. Do you want to write that?
- S The range would be .. negative 4 is greater than  $x$ , ..  $x$  is .. [*Writes  $-4 > x$* .  
*Pauses*]

- I So tell me again in words what ..
- S Well, basically that it can't be .. the range ..there's no  $y$ -values less than negative 4 ..but all other  $y$ -values are possible.
- I OK. So your problem is how to write that down.
- S Yeah. So that's .. you know that .. oh, that would be  $y$  instead of  $x$  [*Changes  $x$  to  $y$* ] or it doesn't really matter [*Pause*] Hm.
- I What's wrong?
- S *mutter* ..There's something missing here [*Points to space after  $-4 > x$* ] and I can't figure out how to do it. 4 would be anything .. then  $y$  would be greater, greater than negative 4 .. [*Starts to erase interval* ] but that wouldn't work either ..
- I Don't erase it. Go down here and try again.
- S OK. We know we have negative 4 and it has to be anything greater than negative 4, right [*Writes -4*].
- I OK.
- S We have  $x$  or  $y$  whatever ..  $x$  [*Writes  $x$* ] ..  $x$  is greater than [*Inserts  $<$  between  $-4$  &  $x$ , writes symbol right to left*] .. is that right, you can do it that way?
- I OK.
- S You have to finish off the equation with something here but I don't know exactly what you would put.

The next two interview excerpts involve students working with the difference quotient,  $\frac{f(x+h)-f(x)}{h}$ . Neither student sees  $(x+h)$  as a numerical object. Their descriptions are all about symbol manipulations. The first seems to look at the symbols individually throughout and has developed a substitution sequence that works for linear and quadratic functions with single occurrences of the variable, that is of the form  $ax + b$  or  $ax^2 + c$ , and does not work in the example used, where  $f(x) = x^2 - 3x + 2$ . The sequence seems to be that she takes terms in order and by the presence or absence of a variable. Thus, since  $f(x+h)$  appears first in the difference quotient she replaces  $x$  in the first term of the function ( $x^2$ ) with  $(x+h)$ .  $f(x)$  has a simple  $x$  so no replacement is needed and  $-3x$  is written down. Both  $f(x+h)$  and  $f(x)$  have been dealt with leaving no further

action for the +2, which is simply written down. She also has an idiosyncratic way of judging the correctness of her answers.

### Interview Excerpt 6

- S ... I was trying to put this [*Points to  $f(x)$  in the problem statement*] in there [*Points to the difference quotient*] That 's exactly what I'm doing but in a different way .. [*Laughs*] I don't know what I'm saying ..
- I It sounds a little bit, .. part of it is that you're trying to put this into this [*Points from function to quotient*]
- S Yeah but, I think, separately. I wasn't thinking of this [ $f(x)$ ] as a whole ... you know ... just like a ... or this [*Points to difference quotient*] I was looking at this [*difference quotient*] as one number - the whole thing together.
- I All of it? and trying to put that with that? [*Points from difference quotient to function*]. .. All in one?
- S And just squish it in. Yeah, exactly. And that's what I always did. Once in a while I'd get it right. You can tell when you're getting it right because other things cancel out. ...
- [Later in the same interview]
- I [*Referring only to  $f(x+h)$* ] What you said was that you thought you might do this. [*Writes  $(x+h)^2 - 3x + 2$* ]
- S Right, because I'm looking at that and put that in there [*Points first to  $f(x+h)$ , then at  $f(x)$  in the problem statement*], put that one in there [*Points at  $f(x)$  in difference quotient, then at  $-3x$  in the problem statement*] and then there's nothing left over for the 2 so it's left on its own. And that's what I would do all the time.
- I Yes, and then you see what happens, and there's quite a few examples in the book. If you do a function like this: [*Writes  $9x^2 - 1$* ] You put your  $x$  plus  $h$  squared and then minus 1.
- S Right.

The second student also describes a symbol manipulation sequence for dealing with the difference quotient as well as some of the difficulties he experienced in developing his sequence. His sequence, unlike the previous example, leads to a correct answer in general, rather than in a few specific cases. Here the function is  $f(x) = x^2 - 5x + 1$

### Interview Excerpt 7

S I substituted eh ..  $x$  plus  $h$  into the equation as  $x$ . And then I had to subtract em .. Oh I remember ..I substituted this .. and then .. I subtracted this. But because I subtracted it I changed this was minus  $x$  squared, plus  $5x$  because it's like it's multiplied by negative 1 and then minus 1.

I OK.

[Later in the same interview the interviewer asks if the student ever had problems with the difference quotient such as writing the common error form

$$\frac{x^2 - 5x + 1 + h - (x^2 - 5x + 1)}{h}$$

. The student agrees, although it is not clear that he is referring to this particular error form.]

S Yeah, that's what threw me more than anything was that I saw the  $x$  plus  $h$  .. and it's sort of weird because I'm taking this [ $x+h$  in  $\frac{f(x+h)-f(x)}{h}$ ] and substituting it into here [ $f(x)$  in problem statement] and then bringing it back into this equation [*Difference quotient*] .. in a sense ..

I Mm hm.

S And that's where I made my goof most of the time if I was doing .. is that I had to go from here [*Difference quotient*] to here [ $f(x)$  in problem statement] and then back to here [*Difference quotient*] by bringing .. after bringing this to here [*Points from  $x+h$  to problem statement*] then having to bring this [ $f(x)$  in problem statement] back over to here [ $f(x)$  in difference quotient] That's the way I see it anyways.

### Classroom Observations

Observations of students' work throughout the precalculus course were compatible with the test and interview results and replicated the difficulties and misunderstandings reported by the other researchers. A major area of difficulty for students, which is not included in this research, is graphing. However, some interviews did include graph questions and these, together with observations of students' assignment work and responses during tutorials, make it clear that precalculus students show most of the difficulties and limited understanding reported elsewhere in the literature. Students could plot points and read points off a graph but did not make the connection that for function graphs, the  $y$ -coordinate represents a function value, i.e. they could not connect the pattern of change inherent in the graph, with the pattern of  $y$ -coordinate or function value change. Consequently, they were unable to use graphs as a source of information about the functions involved. Some used the vocabulary of *increasing* and *decreasing* but this appeared to be based on the physical appearance of the graph rather than on the behaviour of the variables.

The physical appearance of a graph is of no help if students do not see an increasing line as an up-slope from left to right. A small experiment was performed asking a class to draw an up-sloping line before any discussion of graphs was begun. Most drew lines sloping up from left to right, but some drew them sloping up from right to left. For such students the internal image does not match the left-right convention.

A different manifestation of the rush to do something was observed during a small group tutorial for students having difficulty with problems involving finding linear function equations in two variables from point and slope information. Students were shown the problem below and asked to describe their initial reactions.

Find the equation of the line parallel to  $3x - 2y - 4 = 0$  and passing through  $(2, -1)$ .

Sketch the graph showing all intercepts.

Four of the nine students reported looking first at the algebra and the numbers for a clue as to how to begin. Key words were also frequently mentioned. During the subsequent discussion several students revealed that their normal response to word problems was to look first at the numbers and other mathematical symbols, or for key words as cues for

action. They made no attempt to understand the problem context or the numerical relationships involved.

### Summary: Interview Responses and Classroom Observations

Students entering precalculus appear to have poor understanding of some fundamental mathematics concepts and such understanding as they have is often not applied. They lack confidence and rely on manipulating symbols according to rules learned within narrow contexts. Some of their sets of rules are actually wrong but produce correct answers. Lack of understanding of variables in particular, either as a misconception (variable as letter or object) or as an inappropriate application (e.g., variable as single number applied in a context requiring variable change) is common.

There are also difficulties with the contexts in which variables appear. Equations and expressions are confused and misinterpreted, and students make little sense out of graphs. Problems are solved by recalling a template. The lack of confidence, the weak understanding of fundamental concepts and the reliance on rules are all interconnected and seem to push the student into an ever-increasing lack of understanding and dependency on rules. The intervention described in the next section is an attempt to break this undesirable cycle.

### The Intervention

The intervention has been used in the researcher's classes since the beginning of its development in the fall of 1995. It is intended to give students an opportunity to break the cycle of lack of understanding, decreasing confidence and rule use. The outline of the intervention given in this report is relatively brief since its content detail is not central to the research.

Due to the constraints imposed by time and large class sizes, the intervention is no more than an adaptation or adjustment of normal lecture content and form. The two to three week review period at the beginning of the course was used to present the intervention and develop vocabulary and references required throughout the course.

## Intervention Content

The major review topics were: Expressions as numbers; the Distributive Principle and its applications; Factoring; Equivalent Fractions and Canceling; Simplifications vs. Equations vs. Proofs; Equation Solving Methods; Variables; and Graphs. The first sections were covered in the given sequence, but the course topic of functions was introduced briefly before the graph section of the intervention to allow the use of function notation in connection with the graphs. Other typical precalculus review topics such as absolute values, radicals and exponents were dealt with through examples within the seven sections. The ideas developed during the review formed the basis of explanations throughout the rest of the course and some examples of these are listed after the main description. A set of Introductory Notes was prepared to accompany the review section of the course in order to give students back-up and extra practice with the ideas presented in the lectures. Students were asked to buy these notes at cost and most did so. The first pages of the introductory notes are supplied in Appendix D.

### Expressions as Numbers

Students were introduced to the structure of expressions and the distinction between procedural and reified interpretations through the notion of different ways of writing a number, e.g., writing 18 as  $2 + 16$ ,  $3 \times 6$ ,  $\sqrt{324}$ ,  $18 + (25\pi)^{17} - (25\pi)^{17}$ , and so on. Numerical examples were used before introducing algebra. Six students were asked during interviews after the final exam in the fall of 1995 if any one part of the intervention content stood out for them. Two could not identify any particular part but the remaining four all mentioned the idea that expressions can represent single numbers.

### The Distributive Principle

The form  $ab + ac = a(b + c)$  was introduced through the elementary school technique of using known multiplication facts to find unknown products. Examples of common factors ranged from single variables to exponential, radical and trigonometric expressions, and students were encouraged to identify the structure of expressions within expressions by substituting single capitalized variables. The rule for multiplying

expressions by “multiplying all terms in one bracket by all terms in the other” was derived and recommended as a replacement for FOIL.

### Factoring

The meaning of factoring as the rewriting of a number or expression as a product was emphasized. The conventional limitation that numerical factors are integers and algebraic factors have integer coefficients was noted and some non-integer examples provided. Examples generally used expressions rather than single variables.

### Equivalent Fractions and Canceling

The concrete and numerical basis of equivalent fractions was introduced and related to algebraic fractions. Multiplication and division by  $1$  in the form  $\frac{A}{A}$  were emphasized including examples of rationalization and simplification of algebraic fractions. Students were encouraged to show the  $1$ 's resulting from the division by common factors as well as crossing out the factors. Pseudo-conceptual rules such as term-by-term canceling were used as negative examples.

### Simplifications vs. Equations vs. Proofs and Equation Solving

The different uses of  $=$  in actions or operations, in rewriting, in conditional equations and in identities and proofs were compared. The purpose and meaning of equation solving was reviewed, particularly as a means to answer questions about numbers. Examples of problems were drawn from the regular course content, including finding  $x$ -intercepts and vertical asymptotes. Extraneous roots were mentioned in connection with the logic of equation solving and equivalent equations. Linear and quadratic equation-solving methods were contrasted. Time was also spent on the misleading vocabulary of symbol manipulation and the importance of taking such phrases as “collect all the variables” metaphorically and not literally.

## Variables

The three interpretations of variables as single numbers, sets of numbers and as co-varying values were presented with examples. A small self-test, which included the example  $\frac{12a}{-6a}$ , was given.

## Graphs

The  $y$ -coordinate as the expression or function value was emphasized. Some consideration was given to a graph as the representation of the solution set of a two variable equation, but greater time was spent on the connection between variables as co-varying values and interpreting graphs in terms of changing numbers. Slope was discussed in terms of rate-of-change and examples were chosen expressly to disrupt the notion that the physical shape of the graph represents the numerical change patterns.

## Applications

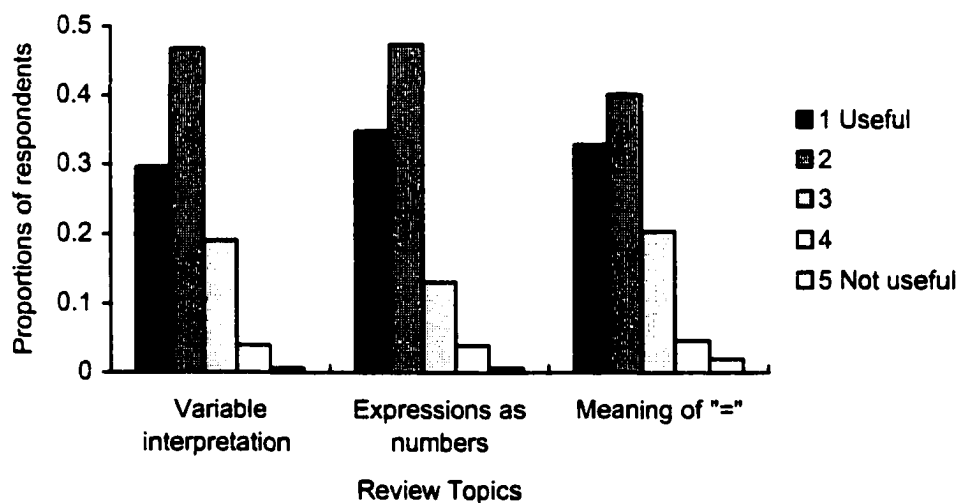
As examples arose throughout the course, the particular application of the review material was described on the assumption that the review alone, without follow-up, would not be sufficient to change students' mathematical thinking overall. The following examples are those given particular attention.

- Function composition where the argument function must be seen as a reified number object, i.e., as the input number.
- Domain and Range as sets represented by the variables and consequently requiring the generalized or set interpretation for the variables.
- The difference quotient as representing the average rate of change of the function value or  $y$  compared to  $x$ . Thus  $x$  and  $y$  are interpreted as co-varying values. In addition the expressions in the quotient must all be interpreted as numerical objects:  $(x+h)$  as the function argument,  $f(x+h)$  and  $f(x)$  as the terms in the difference  $f(x+h) - f(x)$ . The interchangeability of two forms of the same number must also be understood when  $f(x+h)$  and  $f(x)$  are replaced by their expression forms.
- Discussion of graph behaviour in relation to vertical and horizontal asymptotes and trigonometric periods. This requires the variables to be interpreted as changing

values. This material also requires students to remember that  $y$ -coordinates are not just calculated using function expressions (action), they also represent the expression (variables as sets of numbers and expressions reified as representing numbers).

### Student Response to the Intervention

As part of the anonymous evaluation process at the end of the course, students were asked to comment on various aspects of the course content. On a 5-point scale ranging from useful (1) to not useful (5) they were asked to rate how helpful they found the coverage of the ideas of variables as numbers, expressions as numbers, and the meaning of  $=$ , in the three different contexts of rational functions, logarithms and trigonometry. The results of the Spring 1997 term are summarized in the histogram in Figure 26 and indicate that the great majority of the students found the material on variables, expressions and “equals” to be useful. Written comments were also invited. These confirmed the positive responses and several students also reported a new understanding of mathematics.



**Figure 26.** Proportions of subjects (Precalculus class taught by researcher in spring, 1997) in five opinion categories for the three review topics ( $n=52$ ).

### Intervention Summary

This method of teaching the course is not innovative in the sense that material is still presented in the traditional lecture format. The mathematical approaches to topics are not unusual. They are, however, chosen to provide the best opportunity for including the fundamental descriptions of what the symbols mean as well as the underlying logic. It was thought that course coverage would be affected by the extra time needed to incorporate the intervention, but this turned out not to be the case. There seem to be two reasons. The first is that once the vocabulary conventions were established during the review, the reminders used during the course could often be very brief. The second is that, because this approach encouraged the identification of underlying similarities, the number of examples needed was reduced.

Students' positive responses to the course are undoubtedly encouraging but this does not provide any information on mathematical learning effects. Improvements over the short term can be investigated by looking at the precalculus class grades. However, since the focus of the intervention is on fundamental understandings, it was decided that it was more important to investigate possible long-term effects by analyzing grades from students' subsequent calculus classes. The next chapter section contains the results of the statistical analyses of students' precalculus and calculus grades.

#### Quantitative Analysis of Students' Precalculus and Subsequent Calculus Grades

The quantitative part of the research is a retrospective study, using entire available populations from extant classes, analyzed statistically. There is an experimental group of students who received the intervention in their precalculus classes and three control groups who did not. The first of the control groups, included to control for an instructor effect, consists of students taught by the researcher prior to the introduction of the intervention. The possibility of an effect from change in student population over time led to the inclusion of the other two control groups, both taught by other instructors, one matching the time of the first control group and one matching the time of the experimental group. The chronology of these various groups is shown in Figure 3 in

Chapter 3 (p.34). There was no random assignment of students to either the experimental or the control groups. Consequently, differences in the composition of these groups are potential confounds. Students' *Year* standings and their *High School* mathematical backgrounds are two possible sources of difference for which some data were available.

There were a total of 736 subjects in the regular term classes in the 1993-94, 1994-95, 1996-97, and 1997-98 academic years. Of these, approximately one fourth subsequently took Math 102, Calculus for the Social and Biological Sciences, and are designated as calculus students ( $N=166$ ). Students who took other calculus courses were removed from the data set and the remainder ( $N=524$ ) were labeled as non-calculus, although this group includes a small unknown number of students who had not yet taken their calculus course. The data recorded for each subject are:

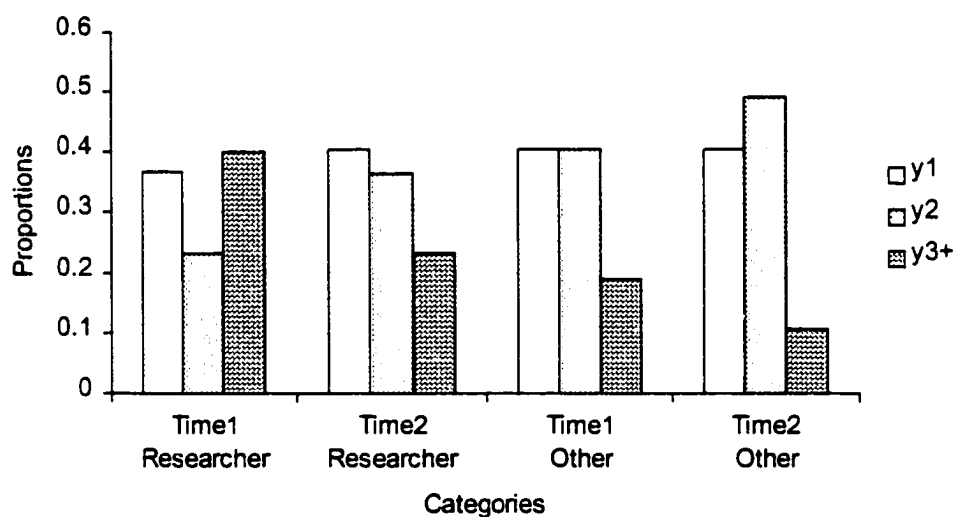
<u>Factor</u>	<u>Levels</u>
<i>Instructor</i> (precalculus):	<i>Researcher, Other.</i>
<i>Time</i> (of taking precalculus):	<i>Time1</i> (before the intervention) and <i>Time2</i> (coincident with the intervention)
<i>Year</i> (standing at time of precalculus):	<i>Y1, Y2, and Y3+</i> (respectively first, second, and third or higher).
<i>High School</i> (if recorded):	<i>G12</i> (provincially examinable Math 12) and <i>G11</i> (basic university entrance requirement).
<i>Grade</i> (calculus or precalculus):	<i>N</i> (incomplete), <i>F, D, C, C+, B-, B, B+, A-, A, A+</i> , collapsed to <i>F, D or C, B, and A</i> , or further.

Separate contingency tables were constructed from the raw data, with *Grade* frequencies classified by the explanatory variables, *Instructor* and *Time*, and either *Year*, or, for those with recorded high school information, *High School*.

Both the *Year* and *High School* factors are potential confounds because they affect the compositions of the experimental and control groups. *Year* data is available for all subjects, but the limited number of students with *High School* information recorded restricts the extent to which this variable can be investigated or accommodated. In addition *High School* data were only collected for the calculus group of students because the analysis of the non-calculus group's data was terminated due to the confounding

effects from *Year*. Initial investigations of the *Year* variable are reported in the next paragraphs, but consideration of *High School* is left until the analysis of the calculus group's calculus grades.

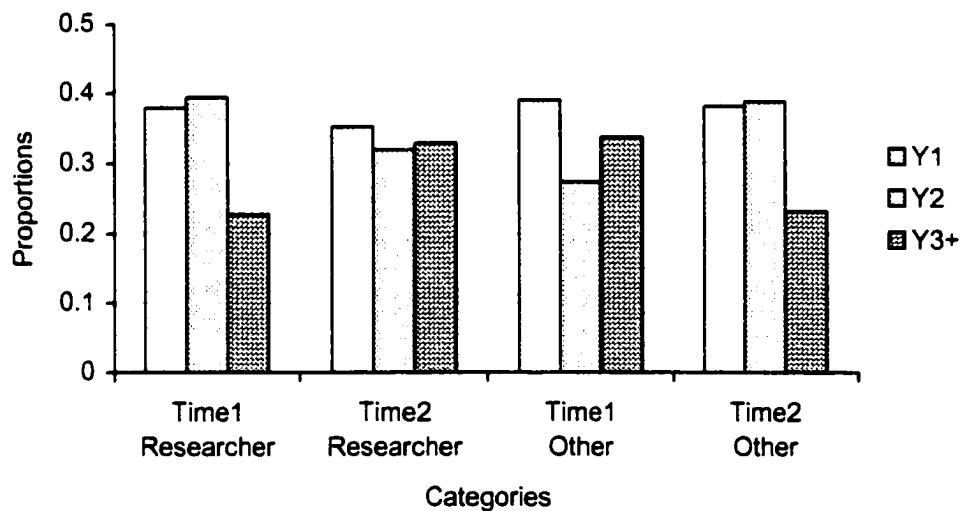
Figure 27 shows the variations in *Year* standings at the time of taking precalculus for the set of students continuing on to calculus. The relative proportions of students with different year standings appear to vary between the groups and these visual differences were supported statistically with a  $\chi^2$  test ( $N = 166, p = 0.1$ ). For the non-calculus students differences in proportions shown in Figure 28 are less apparent, and this is reflected statistically by the higher  $p$ -value from a  $\chi^2$  test of the frequencies in each *Year* category ( $N = 524, p = 0.25$ ).



**Figure 27.** Calculus students: proportions by year standing when taking precalculus for experimental (*Time2 Researcher*,  $n=52$ ) and control groups (*Time1 Researcher*,  $n=30$ ; *Time1 Other*,  $n=37$ ; *Time2 Other*,  $n=47$ ).

In addition to the differences in proportions of different levels of *Year*, the calculus and non-calculus students also show differences in the relationships between students' year standings and the grades achieved in precalculus. The results of  $\chi^2$  tests using the

data from the group of calculus students suggest that there is no overall association between year standings and students' grades in either calculus or precalculus ( $N = 166$ ,  $p = 0.92$ ,  $p = 0.59$ , respectively). For the non-calculus students the reverse is true since a  $\chi^2$  test of this group's data produces a low p-value ( $N = 524$ ,  $p = 0.03$ ). Because of this difference between the calculus and non-calculus groups, and because a simple two-way test of association takes no account of any possible interactions between the *Year* and the other explanatory variables, it was decided that there was insufficient justification for removing *Year* from the analysis.



**Figure 28.** Non-calculus students: proportions by year standing when taking precalculus, for experimental (*Time2 Researcher*,  $n=216$ ) and control groups (*Time1 Researcher*,  $n=66$ ; *Time1 Other*,  $n=95$ ; *Time2 Other*,  $n=147$ ).

For each data set an exploratory analysis of the contingency table data was carried out graphically through a series of histograms. Means and medians for the various categories were also found. Subsequent statistical analyses used the Generalized Linear Modeling package in S-Plus (1998). Marginal and some interaction totals in the contingency tables are considered to be fixed since this is a retrospective study and

students could not be randomly assigned to either *Time* or *Instructor*, and *Year* could not be balanced. In such a situation the distribution function for the cell counts is the Product Multinomial. Modeling in S-Plus for the Product Multinomial distribution is done using the Poisson distribution function with required terms in the model corresponding to the fixed marginal totals. Comparisons between hierarchically related models were carried out using the Likelihood Ratio Test. Analysis of deviance tables are presented for the final parsimonious models achieved by successive removal of terms from the saturated models. A general description of the components of an analysis of deviance table and the tables for the saturated models with all main effects and interaction terms included can be found in Appendix E.

Although the intervention took place in the precalculus course, the ideas and concepts promoted are not truly tested until the students face the challenge of calculus. Consequently it is the calculus students' grades that are the main focus in this analysis. Precalculus grades are also of interest but it is the calculus grades that are analyzed first.

Four models are presented in all. In the first model the frequency at each calculus grade level is the response variable, with explanatory variables *Instructor*, *Time* and *Year*, and in the second model, using the reduced data set, *High School* replaces *Year*. The full set of calculus subjects are used for the third model, but the response variable is the frequency in precalculus grade levels. The final model uses the non-calculus group's precalculus grades. Only the first model appears reasonably free of the confounding effect of *Year* and it suggests that there is a difference between the experimental and the control groups. For each group a preliminary exploratory analysis is reported followed by a statistical model of the data which provides an analytical test of the observations from the exploratory analysis.

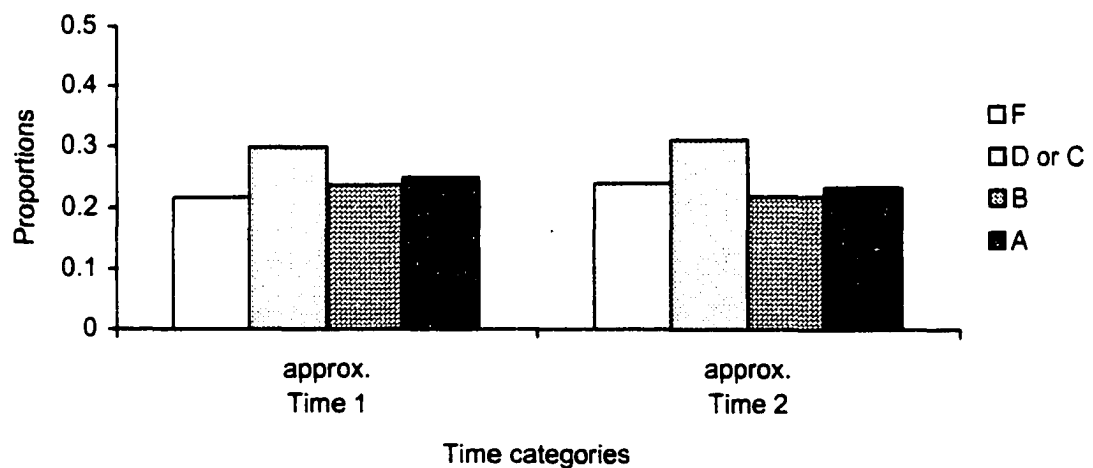
#### Calculus Group: Analysis of Calculus Grade Frequencies

The first of two models of calculus grade frequencies for precalculus students continuing to calculus has *Instructor*, *Time* and *Year* as explanatory variables, and the second model has *Year* replaced by the *High School* variable, another possible confound. Unfortunately, since less than half the subjects have high school courses recorded the

numbers were not sufficient in the second model for good analysis, even with *Grade* collapsed to three levels and *Year* omitted.

### Calculus Group: Exploratory Analysis of Calculus Grade Frequencies

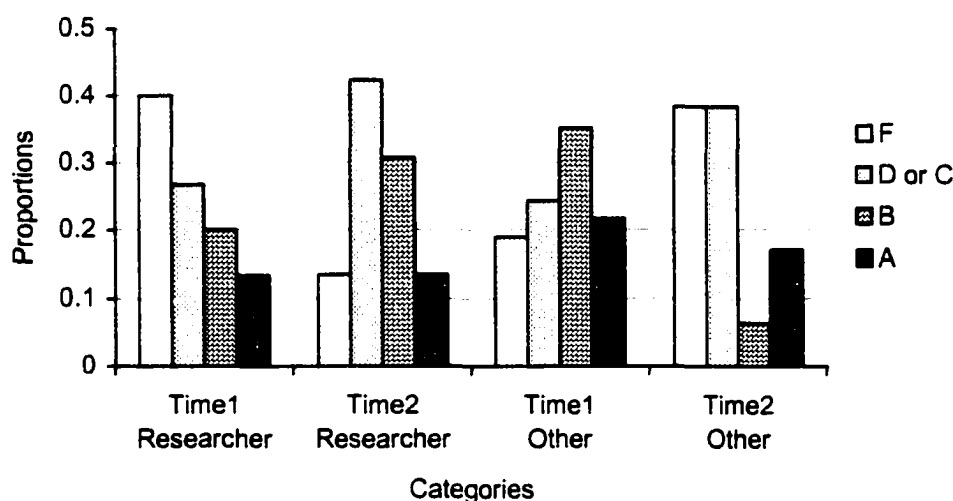
The grade distributions for calculus classes in general are presented first to provide a background context for the calculus data from the precalculus subjects. Figure 29 shows the distribution of proportions of different grades for all Math 102 (Calculus for the Social and Biological Sciences) calculus classes over the period of the research ( $N=2460$ ). These have been split to match approximately the calculus classes taken by precalculus students at the time of the intervention (*Time2*) and earlier (*Time1*). A  $\chi^2$  test indicates that the slight difference in performance between *Time1* and *Time2* apparent in the increases in proportions of Fails and D or C grades, with concomitant decreases of B and A grade proportions, is significant at  $p = 0.05$ . Thus the general trend of calculus grades over the research time period is for slightly decreasing performance levels.



**Figure 29.** Proportions of grades for all calculus classes split to match timing for precalculus *Time1*, ( $n=1303$ ) and *Time2* ( $n=1157$ ) classes.

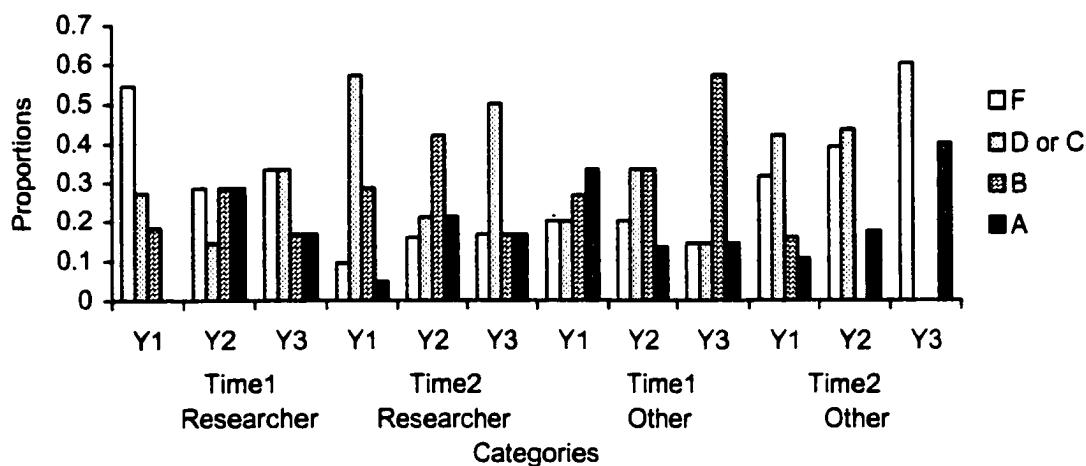
The histogram in Figure 30 of the proportions of calculus grades obtained by the subject population shows an improvement over time for the *Researcher's* group with

increasing proportions of passing grades and decreasing F grades. The *Other* instructor's group shows a contrary trend with increasing proportions of F and D grades and decreasing proportions of B and A grades. The overall suggestion is that among students from the researcher's precalculus classes the calculus grades improved over time while the reverse is true for students from *Other* instructors' precalculus classes. Moreover, this increase for the *Researcher* group occurred while the general trend for calculus grades was slightly decreasing (see Figure 29).



**Figure 30.** Calculus group: calculus grade proportions classified by *Time* and *Instructor*. *Researcher Time2* is the experimental group (In order,  $n=30$ ,  $n=52$ ,  $n=37$ ,  $n=47$ ).

The histogram becomes more complicated, and difficult to interpret, once the variable *Year* is introduced (Figure 31). In addition it should be noted that there are a number of small cell counts, indicating that caution should be used in interpreting the results of this analysis. The first year students show a pattern of changes similar to the overall trends apparent in Figure 30, but for second and third year students the changes are less clear-cut, with mixed patterns of increases and decreases in the passing grades. However, all years show decreasing failure rates for the *Researcher* category over *Time* compared to increasing rates for *Other* category of *Instructor*.



**Figure 31.** Calculus group: calculus grade proportions classified by *Instructor*, *Time* and *Year*. *Researcher Time2* is the experimental group (In order,  $n=11$ ,  $n=7$ ,  $n=12$ ,  $n=21$ ,  $n=19$ ,  $n=12$ ,  $n=15$ ,  $n=15$ ,  $n=7$ ,  $n=19$ ,  $n=23$ ,  $n=5$ ).

Table 4 shows the means and medians for the various combinations of *Instructor*, *Time* and *Year*. With one exception these support the evidence from the histograms that the *Researcher's* groups improve over time while the *Other* instructors' groups do not. The exception is the third and higher year students in the *Researcher* group, where the mean declines slightly, and the median stays constant. Thus the distribution of means for different years does not appear to be the same for the various categories of *Instructor* and *Time*, which supports the need to include the variable *Year* in the analysis. In addition, the differences between some means and associated medians reflect the non-normal distributions in the data. Specifically, the high failure rates in both precalculus and calculus place large proportions of subjects in the left tails of the distributions, a distinctly non-normal characteristic.

Table 4

Means and medians of calculus groups' calculus grades<sup>1</sup> classified by *Instructor with Time* and *Year*, and *Instructor with Time*

<i>Instructor</i>	<i>Time</i>	<i>Year Standing</i>	<i>With Year</i>		<i>Without Year</i>	
			Mean <sup>1</sup>	Median <sup>1</sup>	Mean <sup>1</sup>	Median <sup>1</sup>
<i>Researcher</i>	<i>Time1</i>	Y1	1.18	0	2.57	1.5
		Y2	4.00	4		
		Y3	3.00	2		
<i>Researcher</i>	<i>Time2</i>	Y1	2.76	2	3.29	2.5
		Y2	4.21	5		
		Y3	2.75	2		
<i>Other</i>	<i>Time1</i>	Y1	4.40	5	3.84	4
		Y2	3.27	3		
		Y3	3.86	4		
<i>Other</i>	<i>Time2</i>	Y1	2.42	1	2.53	1.5
		Y2	2.39	1		
		Y3	3.60	0		

Note<sup>1</sup>. Letter grades F to A+ converted to numerical scores 0 to 9.

The highest grades appear in the control group taught by other instructors prior to the intervention, raising a concern that the resulting gap in initial performance between the researcher's and the other instructor's groups might exaggerate the contrasting patterns of change for the two levels of *Instructor*. There were six precalculus classes in *Time1*, three in each of the two academic years 1993-1994 and 1994-1995. Calculus failure rates for students taking precalculus in the earlier year were 0%, 0% and 1% compared to 42%, 45% and 58% for those from the later year. The reason for this difference is not clear, but the researcher had two classes with high failure rates whereas the other instructors had only one. Clearly, this accounts for a good part of the difference between the *Researcher* and *Other* levels of *Instructor* at *Time1*. If data are restricted to the times when both the researcher and another instructor have a class the difference between

*Researcher* and *Other* is much reduced but still slightly favours the *Other* level of *Instructor*. However, the concomitant reduction in numbers at *Time1* produces too many low cell counts for good analysis when the factor *Year* is introduced. In addition the precalculus grades. Furthermore, the enhanced calculus performance for the *Other* instructor's students is not demonstrated in the precalculus grade distributions. These two reasons, together with the lack of an explanation for the variability of calculus failure rates in *Time1*, indicate that all the data should be retained. However, any conclusions should be checked using a data set limited to classes where both the researcher and another instructor taught a section.

#### Calculus Group: Model for Calculus Grade Frequency

Following the requirements laid out by Dobson (1990) all main effects and interactions of the unrandomized variables are included in the minimal models for the response variable, *Frequency* (of grade levels). If a minimal model is sufficient to fit the data then the null hypothesis of no difference between the experimental and the control groups is supported. All models are presented along with an analysis of deviance table which provides information similar to an analysis of variance. Appendix E contains a more detailed description.

The model and analysis of deviance presented in Table 5 has frequency of calculus grades as the response variable and *Instructor*, *Time* and *Year* as explanatory variables. *Grade* has four levels, F, D or C, B, and A, with a few small cell counts. The saturated and minimal models are as follows:

Saturated             $Frequency \sim Grade * Instructor * Time * Year^5$

Minimal              $Frequency \sim Instructor * Time * Year + Grade$

The terms left for optional inclusion are the interactions involving *Grade*.

The saturated model with zero deviance can be found in Appendix E. The minimal model, which consists of the main effects and interactions of *Instructor*, *Time* and *Year*, is not satisfactory (*Residual deviance* = 35.64, *Df* = 22). As shown by the graphs in Figure 32

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<sup>5</sup> The symbol \* denotes all main effects and interactions of the variables involved.

the final parsimonious model (Table 5), which was developed by successive removal of terms from the saturated model, fits well, with all standardized residuals lying between  $-2$  and  $2$ . The scatter plot of deviance residuals versus fitted values is sufficiently random, while the plots of observed frequencies versus fitted values and standardized Pearson residuals versus quantiles of standard normal are reasonably linear.

The final model (*Residual deviance* = 20.73, *Df* = 21) shown in Table 5 is the most parsimonious. Three interaction terms involving *Grade* are necessary, with the most important term being the interaction *Instructor:Time:Grade* ( $p = 0.0008$ ). The presence of this term indicates that the grade distributions vary among the experimental and control groups. The second interaction term of *Instructor:Year:Grade* suggests that there may be some combined effect of *Instructor* and students' *Year* standing on students' calculus grades ( $p = 0.07$ ). Although necessary for fit, the third interaction term, *Time:Grade* does not contribute to the model in any major way ( $p = 0.21$ ). The sources of the two significant interactions are investigated graphically in Figure 33 and Figure 34.

A coarser model was developed using the same data but with A and B grade frequencies combined in order to reduce the number of small cell counts. The only change from the reported model is that the *Time:Grade* interaction term is no longer needed. The two three-way interactions of *Instructor:Time:Grade* and *Instructor:Year:Grade* retain their relative levels of importance.

Table 5

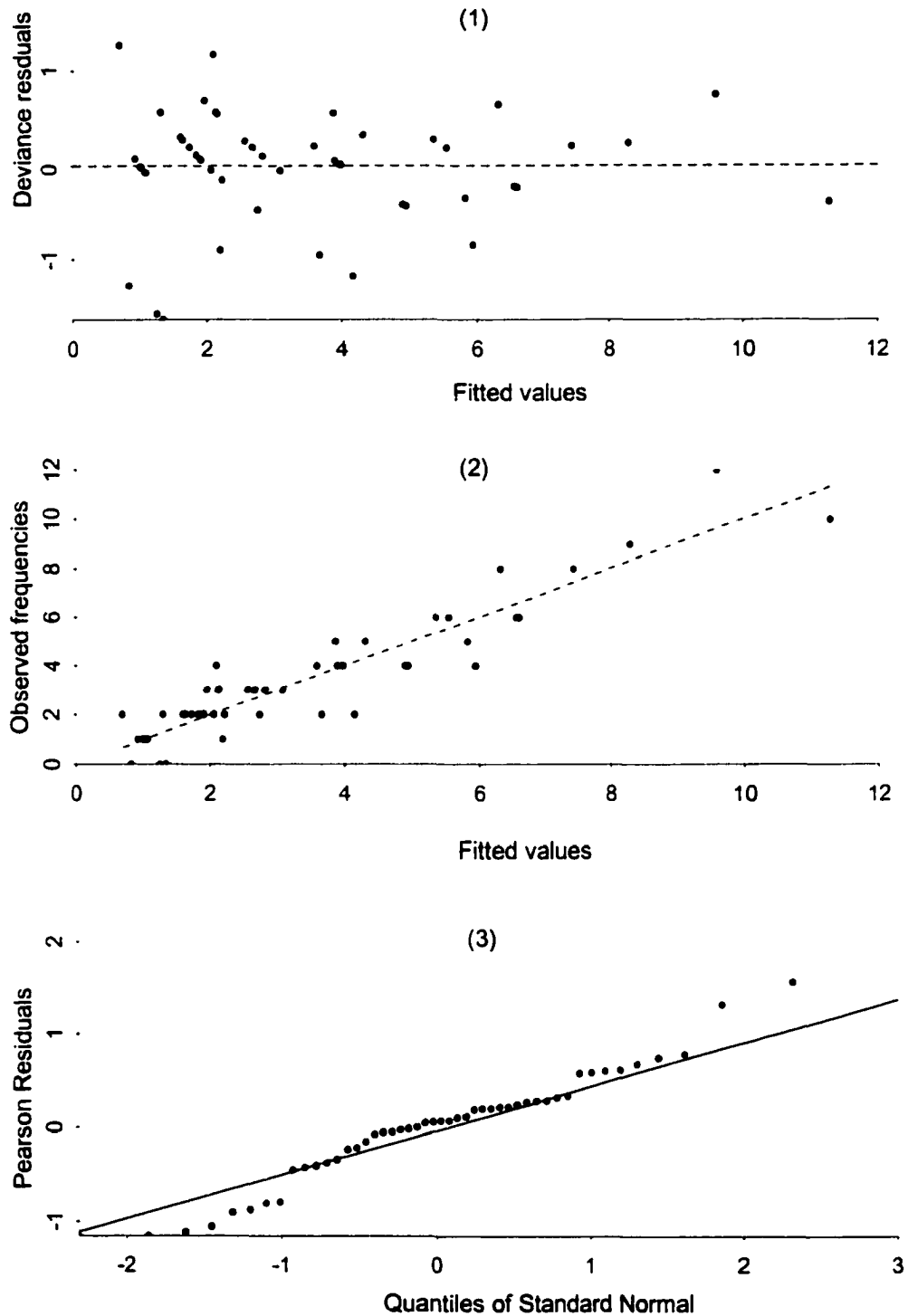
Calculus Group: Calculus Group: Model for Calculus Grade Frequencies Classified by  
*Instructor, Time and Year*

*Frequency ~ Instructor\*Time\*Year + Grade + Time:Grade + Instructor:Time:Grade  
 + Instructor:Year:Grade*

Model Terms	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
<i>NULL</i>			47	93.95	
<i>Instructor</i> <sup>1</sup>	1	0.02	46	93.92	0.88
<i>Time</i> <sup>1</sup>	1	6.21	45	87.71	0.01
<i>Year</i> <sup>1</sup>	2	10.94	43	76.77	0.00
<i>Grade</i> <sup>1</sup>	3	11.42	40	65.35	0.01
<i>Instructor:Time</i> <sup>1</sup>	1	0.96	39	64.39	0.33
<i>Instructor:Year</i> <sup>1</sup>	2	6.38	37	58.01	0.04
<i>Time:Year</i> <sup>1</sup>	2	4.19	35	53.83	0.12
<i>Time:Grade</i>	3	4.50	32	49.33	0.21
<i>Instructor:Time:Year</i> <sup>1</sup>	2	0.08	30	49.25	0.96
<i>Instructor:Time:Grade</i>	3	16.85	27	32.40	0.00
<i>Instructor:Year:Grade</i>	6	11.67	21	20.73	0.07

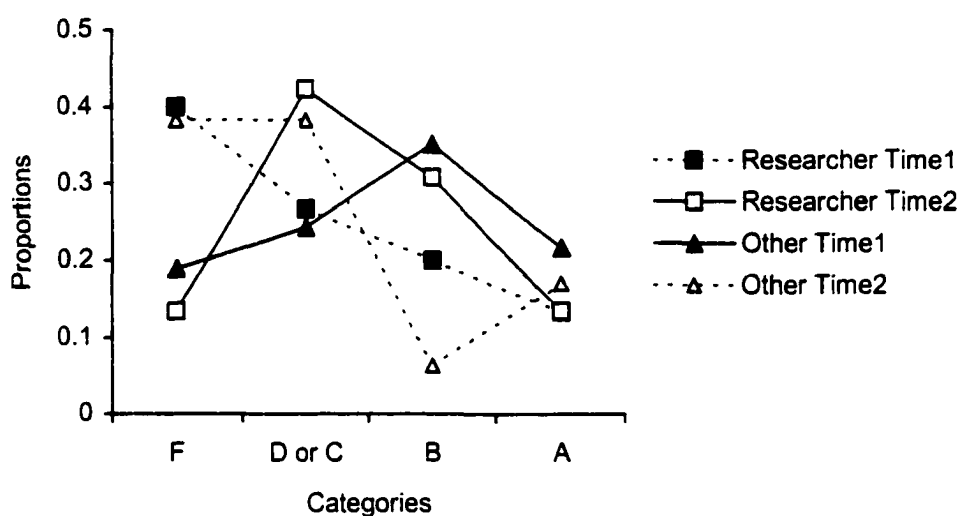
Note. <sup>1</sup> Required terms of the minimal model.

(For an explanation of the structure of the analysis of deviance table, see Appendix E.)



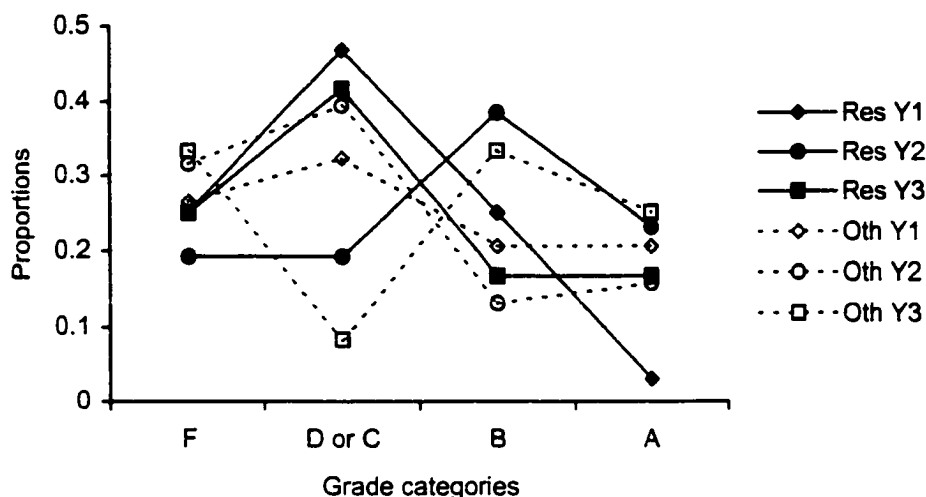
**Figure 32.** Graphs showing the fit of the calculus group, calculus grades model (*Year* included): (1) Deviance residuals versus fitted values, (2) Observed frequencies versus fitted values, (3) Standardized Pearson residuals versus quantiles of standard normal.

The line graphs presented in Figure 33 highlight the nature of the interaction term *Instructor:Time:Grade*, with solid and broken lines used to emphasize two pairs of similar distributions. These similar distributions belong to opposing combinations of factors and levels: that is, *Researcher Time2* is similar to *Other Time1*, and *Researcher Time1* is similar to *Other Time2*. Insofar as the *Other* instructors' groups represent the performance trend independent of the intervention, the performance of the experimental group is the reverse of what would be expected under the null hypothesis of no association.



**Figure 33.** Line graph showing the inverted trends in *Grade* distribution resulting in similar distributions for *Researcher Time2* and *Other Time1*, and for *Researcher Time1* and *Other Time2*.

The interaction between *Instructor*, *Year* and *Grade* is examined in the line graph in Figure 34 in which the proportions for the *Grade* categories calculated within each combination of *Instructor* and *Year* are shown as connected lines. If the pairs of lines for each of the three categories of *Year* are traced, the source of the interaction effect becomes apparent, since each of these pairs of lines crosses at least once. This is a complicated interaction, but does not involve the experimental and control groups.



**Figure 34.** Calculus group: Line graph of the distributions of proportions at each *Grade* level for each combination of *Researcher* and *Year*.

A model was also developed without *Year* and with only matched precalculus classes in *Time1*. Although the *Other* groups at *Time1* performed relatively less well in this smaller data set than in the full data set, they still showed a slightly lower proportion of F grades and higher proportions of A and B grades than the *Researcher's* groups, and the inverted trends of the interaction observed in the full data set were still present. This model remained consistent with the main model from the full data set, since the only term necessary beyond the minimal model is the significant interaction *Instructor:Time:Grade* ( $p=0.002$ ).

#### Calculus Group, subset with *High School*: Analysis of Calculus Grade Frequencies

The possibility of an effect from students' high school mathematics level was investigated using the subset of subjects whose high school record was available ( $N=73$ ). With less than half the number of subjects, analysis of the data using the *High School* factor is affected by low population size. Reduction of the grade levels from four to three by grouping the grades, F, D or C, B or A helps but there are still some low cell counts.

Calculus Group subset with High School: Exploratory Analysis for Calculus Grade Data

It appears from a comparison of Figure 30 and Figure 35 that the grade distributions are similar in the full calculus group and in its subset for which high school data is available. In addition,  $\chi^2$  tests suggest that the distributions of students with grade 11 and grade 12 backgrounds within the experimental and control categories are not different ( $p = 0.42$ ), and furthermore that there is no overall association between the *High School* and *Grade* factors ( $p = 0.43$ ).

The proportions of the three grade categories for all the different combinations of *Instructor*, *Time* and *High School* level are shown in Figure 36. The most striking feature of this set of histograms is the performance of the students with only grade 11 from high school. In all but the experimental group, the grade 11 group's failure rate is less than that of the matching grade 12 group. Furthermore, the only groups with a failure rate of 50% or more are grade 12 groups (*Researcher* at *Time*1, and *Other* instructors at *Time* 2) while the groups showing the highest proportion of grades as B or A are all grade 11. This suggests a possible effect related to students' high school backgrounds, although not perhaps in the expected direction.

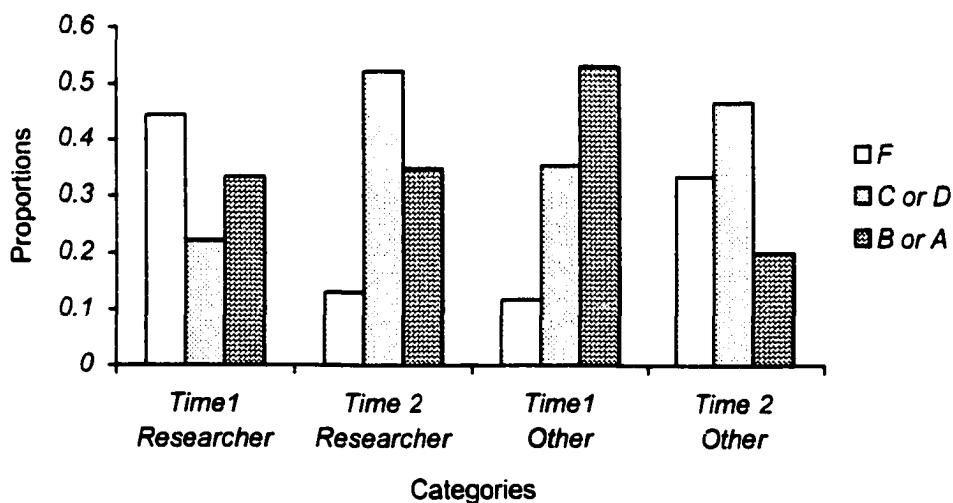
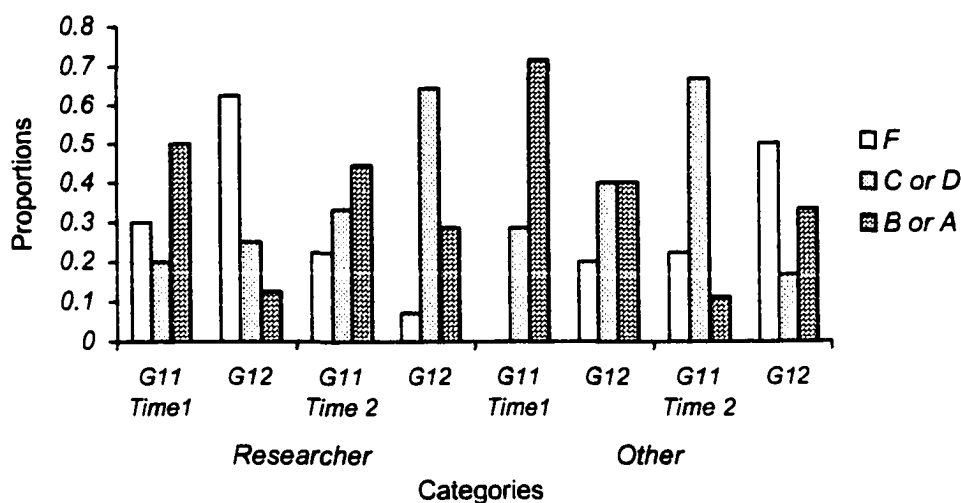


Figure 35. Calculus group, subset with *High School*: calculus grade proportions classified by *Instructor* and *Time* (In order,  $n=19$ ,  $n=22$ ,  $n=16$ ,  $n=16$ ).



**Figure 36.** Calculus group, subset with *High School*: proportions in calculus grade categories classified by *High School*, *Time* and *Instructor* (In order,  $n=10$ ,  $n=9$ ,  $n=8$ ,  $n=14$ ,  $n=7$ ,  $n=9$ ,  $n=10$ ,  $n=6$ ).

#### Calculus Group subset with *High School*: Model for Calculus Grade Frequency

All main effects and interactions of the three explanatory variables (*Instructor*, *Time* and *High School*) are required terms in this model since any effects between and among these variables are products of the populations used. Table 6 contains the final form of the calculus grade frequencies model including the *High School* factor. The graphs in Figure 37 show a good overall fit for the model, with small, randomly spread deviance residuals, and a sufficiently straight quantile plot. The interaction of *Instructor*, *Time* and *Grade*, without the *High School* variable remains the most important term at  $p = 0.03$ . However, the interaction *Time:High.Sch:Grade* at  $p = 0.1$  also requires some attention.

The *Grade* distributions classified by *Time* and *High School*, without *Instructor*, are presented in Figure 38. At *Time2* the grade 11 and grade 12 groups look similarly distributed, but at *Time1* the distribution patterns for grade 11 and grade 12 are almost mirror images of each other, with neither looking like the distributions at *Time2*. These

distributions reflect the observation in the exploratory analysis that students with a grade 11 background do as well or better than those with grade 12. However, this interaction does not involve the *Instructor* factor.

Table 6

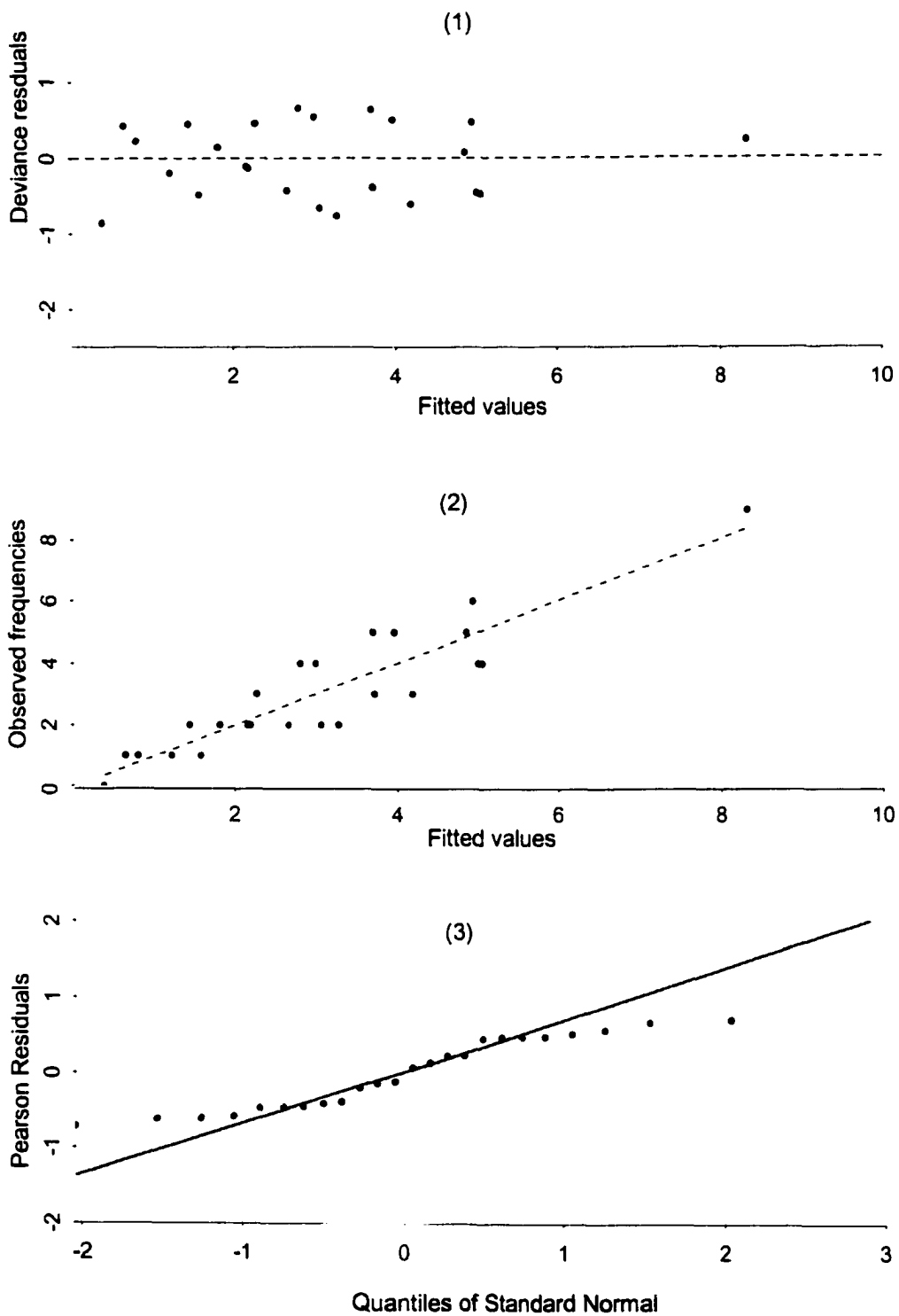
Calculus Group, subset with *High School*: Calculus Grade Frequencies Model

*Frequency* ~ *Instructor\*Time\*High.Sch* + *Grade+Time:Grade* + *Instructor:Time:Grade*  
+ *Instructor:High.Sch:Grade* + *Time:High.Sch:Grade*

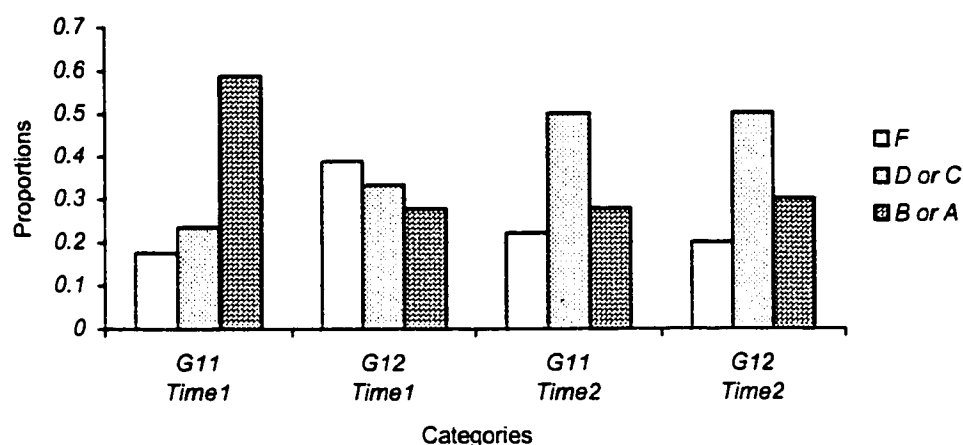
	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
<i>Null</i>			23	30.47	
<i>Instructor</i> <sup>1</sup>	1	1.09	22	29.36	0.29
<i>Time</i> <sup>1</sup>	1	0.12	21	29.23	0.73
<i>High.Sch</i> <sup>1</sup>	1	0.12	20	29.11	0.73
<i>Grade</i> <sup>1</sup>	2	2.71	18	26.34	0.25
<i>Instructor:Time</i> <sup>1</sup>	1	0.63	17	25.73	0.43
<i>Instructor:High.Sch</i> <sup>1</sup>	1	0.10	16	25.63	0.76
<i>Time:High.Sch</i> <sup>1</sup>	1	0.01	15	25.63	0.94
<i>Time:Grade</i>	2	3.56	13	22.07	0.17
<i>Instructor:Time:High.Sch</i> <sup>1</sup>	1	2.23	12	19.84	0.14
<i>Instructor:Time:Grade</i>	2	7.00	10	12.84	0.03
<i>Instructor:High.Sch:Grade</i>	2	2.62	8	10.22	0.27
<i>Time:High.Sch:Grade</i>	2	4.68	6	5.54	0.10

Note. <sup>1</sup> Required terms of the minimal model

(For an explanation of the structure of the analysis of deviance table, see Appendix E.)



**Figure 37.** Graphs showing the fit of the calculus group subset with *High School*, calculus grades model: (1) Deviance residuals versus fitted values, (2) Observed frequencies versus fitted values, (3) Standardized Pearson residuals vs. quantiles of standard normal.



**Figure 38.** Calculus group, subset with *High School*: calculus grade proportions classified by *Time* (In order,  $n=17$ ,  $n=18$ ,  $n=18$ ,  $n=20$ ).

It appears from this second model for the frequencies of students' calculus grades that the variable *High School* does have some effect, but only in combination with *Time* and *Grade*. The interaction between *Instructor*, *Time* and *Grade* is the most important term, as is also the case in the first calculus grades model with the *Year* variable included.

The apparent effect from *High School* background is interesting, particularly in relation to the performance of students without grade 12 mathematics. However, since the factor *High School* does not appear in the model in conjunction with the *Instructor* factor it is unlikely that *High School* is an underlying variable explaining the better performance of the experimental group in the calculus grades model. This model does not supercede the model developed for calculus grade frequencies using *Instructor*, *Time* and *Year* as the explanatory variables.

### Calculus Group: Analysis of Precalculus Grade Frequencies

The model for the calculus group's precalculus grade frequencies was developed with the basic explanatory variables of *Instructor* and *Time*, and the potential confound variable, *Year*. Factors for this model are the same as those used in the calculus group's

calculus grades model, but the analysis results are somewhat different. In particular, it turns out that the variable *Year* acts as a confound in this data set. The exploratory analysis and the related statistical model are described in the next sections.

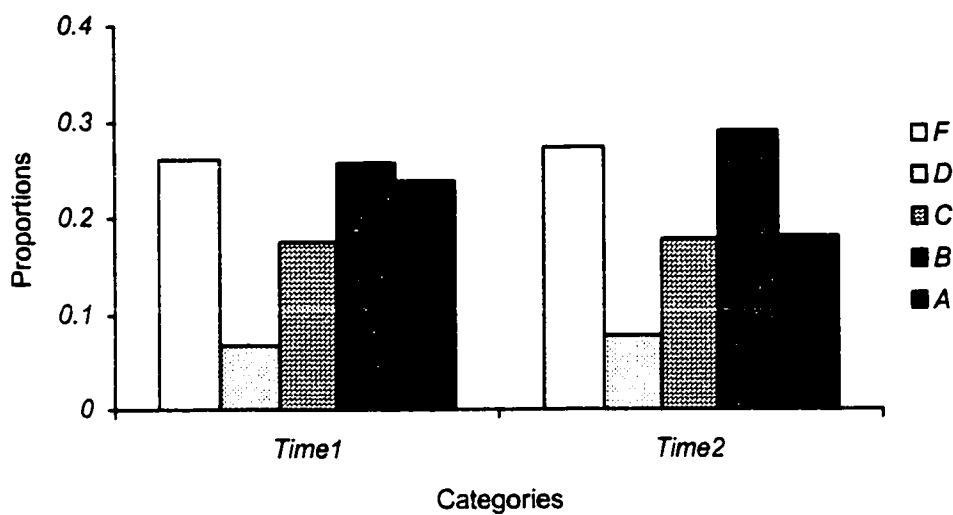
### Calculus Group: Exploratory Analysis of Precalculus Grade Frequencies

The grade distributions for the entire precalculus data set split between *Time1* and *Time2* are shown in Figure 39. There is a small overall decrease in performance with an increase in the proportions of F and D grades and a decrease in B and A grades.

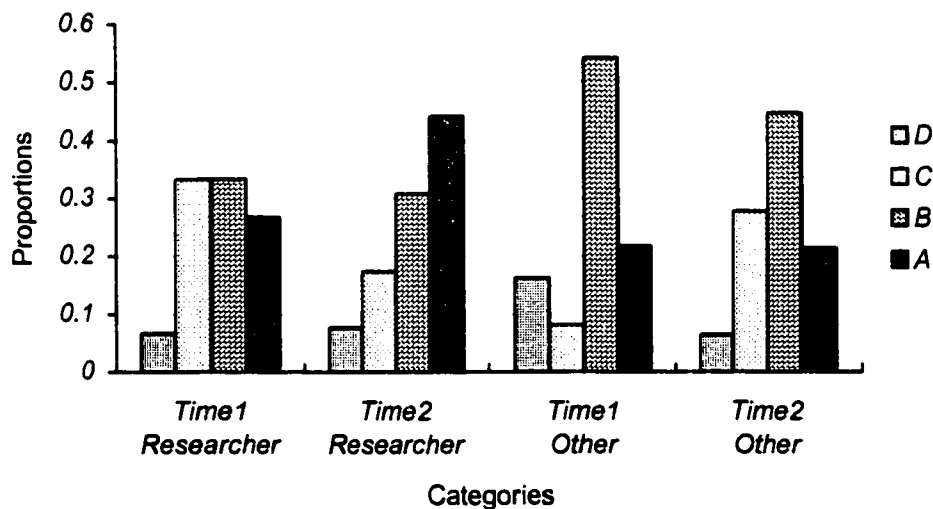
The changes in proportions of precalculus grades over *Time* and by *Instructor* for the calculus group alone are shown in Figure 40. There are no F grades since the population consists only of those precalculus students who continued into calculus and, particularly for the *Researcher Time2* and *Other Time1* combinations, the distributions look different from the overall distributions in Figure 39.

The *Researcher* groups appear to show an improvement in precalculus grade distributions between *Time1* and *Time2*, with the proportions of D and B grades remaining similar, while C grades decrease and A grades increase. At *Time2* the A grades form the highest proportion. The *Other* groups show a decrease over time for D grades and an increase for C grades. This improvement is, however, countered by a decrease in B grades while the A grade proportions are unchanged. At *Time2* B grades form the highest proportion for the *Other* instructors. There is little, if any, improvement apparent in the grade distributions of the *Other* category of *Instructor*.

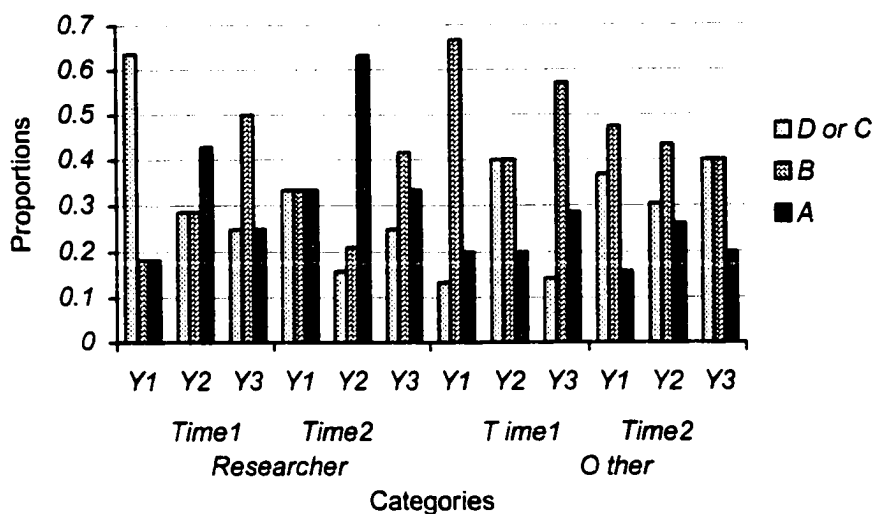
In the histogram in Figure 41 the factor *Grade* is collapsed to three levels in order to accommodate the extra variable, *Year*. The changes between *Time1* and *Time2* are not the same for each level of *Year*. For the researcher the first and second year groups show improvement over time with decreases in the proportions of D or C grades and increases in the proportions of A grades. The trends for the *Other* instructors vary with the year standing of the students, with the second year groups showing an improvement over time while the first and third year groups show decreasing proportions of passing grades and increasing failure rates. The students in the *Researcher* category appear to show a degree of improvement absent in the *Other* instructors category, but there is also some variation within these categories associated with the factor *Year*.



**Figure 39.** Proportions of precalculus grades for the entire set of precalculus classes from 1993 to 1998 split by Time (In order,  $n=240$ ,  $n=605$ ).



**Figure 40.** Calculus group: precalculus grade proportions classified by *Instructor* and *Time* (In order,  $n=30$ ,  $n=52$ ,  $n=37$ ,  $n=47$ ).



**Figure 41.** Calculus group: precalculus grade proportions classified by *Instructor*, *Time* and *Year* (In order,  $n=11$ ,  $n=7$ ,  $n=12$ ,  $n=21$ ,  $n=19$ ,  $n=12$ ,  $n=15$ ,  $n=15$ ,  $n=7$ ,  $n=19$ ,  $n=23$ ,  $n=5$ ).

The changes observed in the histograms are also apparent in the variations among the means and medians shown in Table 7. From *Time1* to *Time2* the overall means and medians for *Researcher* increase from 4.63 and 4 to, respectively, 5.60 and 6, while the *Other* instructor's numbers are only slightly increasing, 4.68 and 5 vs. 4.79 and 5. When broken down by *Year*, however, we see that the *Other* instructors' small increase is composed of decreases in means for first and third year students and an increase for second year students. Meanwhile, for the *Researcher*, the means for all levels of *Year* increase over *Time*. The overall impression is that over time the *Researcher* groups show an improvement while the *Other* instructor's groups are more variable, depending on *Year*, but in general do slightly less well.

Table 7

Calculus group: Precalculus Grade Means<sup>1</sup> and Medians for Groups Classified by Instructor and Time, with and without Year.

<i>Instructor</i>	<i>Time</i>	<i>Year</i>	Mean <sup>1</sup>	Median	Mean <sup>1</sup> without Year	Median without Year
<i>Researcher</i>	<i>Time1</i>	<i>Y1</i>	3.55	2	4.63	4
		<i>Y2</i>	5.71	6		
		<i>Y3</i>	5.00	4.5		
	<i>Time2</i>	<i>Y1</i>	4.86	5	5.60	6
		<i>Y2</i>	6.68	7		
		<i>Y3</i>	5.17	6		
<i>Other</i>	<i>Time1</i>	<i>Y1</i>	5.00	5	4.68	5
		<i>Y2</i>	4.07	4		
		<i>Y3</i>	5.29	6		
	<i>Time2</i>	<i>Y1</i>	4.47	5	4.79	5
		<i>Y2</i>	5.09	5		
		<i>Y3</i>	4.60	5		

Note<sup>1</sup>. Letter grades F to A+ converted to numerical scores 0 to 9.

In general, based on a  $\chi^2$  test, there appears to be no association between *Grade* and *Year* ( $p = 0.58$ ). However, the distributions of proportions within each combination of *Instructor*, *Time* and *Year* vary a great deal, as do the magnitudes of the changes. The evidence from these exploratory techniques suggests, but not strongly, that among the calculus group the experimental group performed better in precalculus than did the control groups. However, there is also evidence of interference from the *Year* factor. The statistical model presented in the next section does not support the initial exploratory observations of differences between the experimental and the control groups, primarily due to a confounding effect from *Year*.

#### Calculus Group: Model for Precalculus Grade Frequencies

The explanatory variables are the same in this model as in the first calculus grades model but the response variable is now the frequency of grades achieved in precalculus.

Two models were planned, of which the first was completed and used the full data set with *Instructor*, *Time* and *Year* as explanatory variables. The second model would have been developed using the subset of subjects with high school records, with *Year* replaced by *High School*, as was done for the calculus grades. However, the results of the first model made the second redundant.

The model for precalculus *Grade* frequencies with explanatory variables *Instructor*, *Time* and *Year* is presented in Table 8. The first two graphs in Figure 42 show the desired random cloud and linear relationships. The third graph is somewhat curvilinear but overall the fit of the model to the data is satisfactory. Required terms in the model are, as before, all main effects and interactions involving the non-randomized and unbalanced explanatory variables but the other terms in this model differ markedly from those in the calculus grades model. Apart from the required model terms, the most important term, at  $p = 0.06$ , is the four-way interaction of *Instructor*, *Time*, *Year* and *Grade*. It appears that for students' precalculus grades the confounding effect of their year standing at the time of taking precalculus cannot be separated from the other explanatory variables. The main source for this interaction is likely to be the *Other* category of *Instructor* since it was observed that the directions of changes in grade frequencies for the *Other* groups varied depending on the *Year*. Because the possible effect of *Year* may differ between the *Researcher* and *Other* groups, a confounding effect from *Year* in this data cannot be discounted.

The analysis of data including high school course information was not performed. If *Year* is a confounding effect for the calculus group's precalculus grade data set, as it appears to be from the preceding model, then the results of any further analysis that does not include *Year* cannot be usefully interpreted. Since the data set with High School is too small to sustain the introduction of another variable, further analysis of students' precalculus grades was abandoned, and attention was turned to non-calculus students' precalculus grades.

Table 8

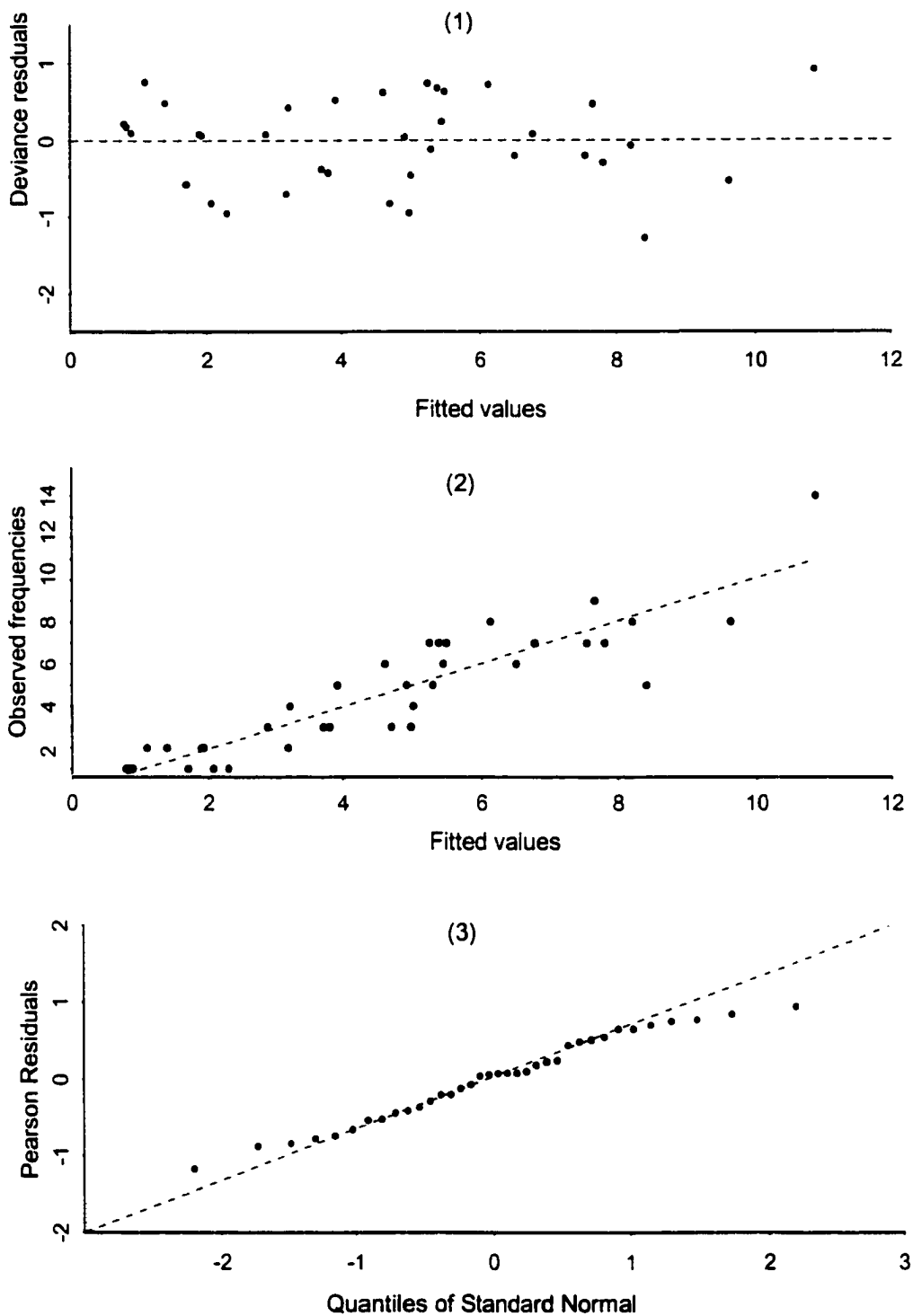
Calculus group: Model for Precalculus Grade Frequencies Classified by Instructor, Time and Year

*Frequency ~ Instructor\*Time\*Year + Grade + Instructor:Grade + Instructor:Year:Grade + Instructor:Time:Year:Grade*

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
NULL			35	67.14	
<i>Instructor</i> <sup>1</sup>	1	0.02	34	67.12	0.88
<i>Time</i> <sup>1</sup>	1	6.21	33	60.91	0.01
<i>Year</i> <sup>1</sup>	2	10.94	31	49.97	0.00
<i>Grade</i> <sup>1</sup>	2	5.79	29	44.18	0.06
<i>Instructor:Time</i> <sup>1</sup>	1	0.96	28	43.22	0.33
<i>Instructor:Year</i> <sup>1</sup>	2	6.38	26	36.84	0.04
<i>Time:Year</i> <sup>1</sup>	2	4.19	24	32.66	0.12
<i>Instructor:Grade</i>	2	4.58	22	28.08	0.10
<i>Instructor:Time:Year</i> <sup>1</sup>	2	0.08	20	28.00	0.96
<i>Instructor:Year:Grade</i>	4	7.60	16	20.40	0.11
<i>Instructor:Time:Year:Grade</i>	4	8.95	12	11.45	0.06

Note. <sup>1</sup> Required terms of the minimal model.

(For an explanation of the structure of the analysis of deviance table, see Appendix E.)



**Figure 42.** Graphs showing the fit of the calculus group, precalculus grades model: (1) Deviance residuals versus fitted values, (2) Observed frequencies versus fitted values, (3) Standardized Pearson residuals versus quantiles of standard normal.

### Non-calculus Group: Analysis of Precalculus Grade Frequencies

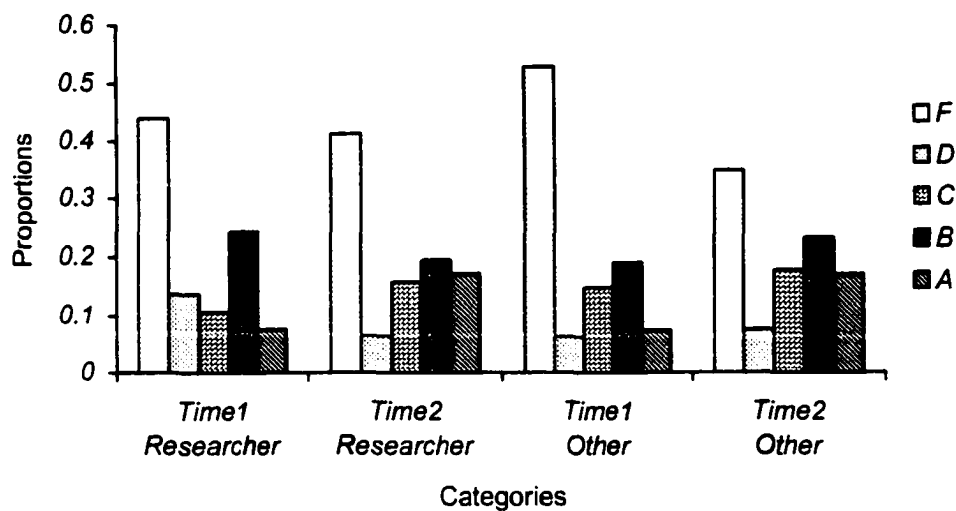
There are more non-calculus than calculus students, and the numbers are sufficiently large that *Grade* was maintained at five classification levels. It is recognized that the non-calculus groups contain a small proportion of calculus students who were classified as non-calculus only because they had not taken their calculus course at the time the data was collected. Students who continued into calculus courses other than the one used in this research could be identified and were dropped from the data set.

The results for this non-calculus subset of precalculus students are quite different from the results for the calculus students. No evidence is present of improvements over time or that the *Researcher* and *Other* groups differ in any way except in association with *Year*.

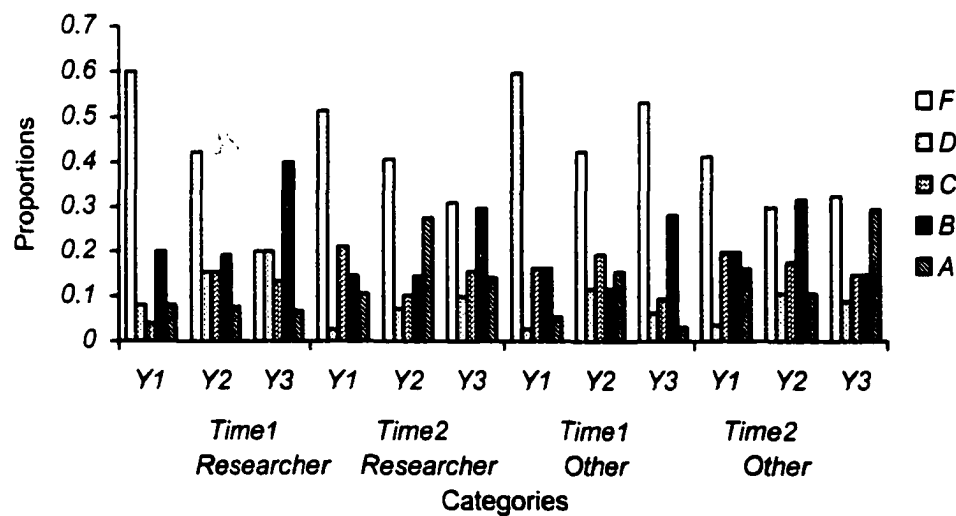
### Non-calculus Group: Exploratory Analysis of Precalculus Grade Frequencies

The histogram in Figure 43 shows the proportions at different grade levels over *Time* for the *Researcher* and *Other* instructor's groups. These show considerable similarity among themselves. Both *Instructor* groups show some improvement over time with the *Other* instructors showing the greater improvement. Clearly the decrease in F grades for the *Other* instructors is more dramatic than for the *Researcher* and the changes in the balance between low and high grade proportions probably also favours the *Other* instructors variable.

When the variable *Year* is included (Figure 44) some variation within these overall changes becomes apparent. Many of the grade distributions shown in the three-variable histograms classified by *Year* as well as *Instructor* and *Time* are quite similar to the overall distributions without the *Year* variable, and both the *Researcher's* and *Other* instructor's first and second year subjects continue to show improvements. However, for the *Researcher's* third year groups there are contradictory trends with a decrease in F grades, but varying changes among the passing grades. These trends look roughly balanced, whereas the *Other* instructors' third year subjects show an improvement over *Time*.



**Figure 43.** Non-calculus group: proportions at different precalculus grade levels classified by *Instructor* and *Time* (In order,  $n=66$ ,  $n=216$ ,  $n=95$ ,  $n=147$ ).



**Figure 44.** Non-calculus group: proportions at different precalculus grade levels classified by *Instructor*, *Time*, and *Year*. (In order,  $n=25$ ,  $n=26$ ,  $n=15$ ,  $n=76$ ,  $n=69$ ,  $n=71$ ,  $n=37$ ,  $n=26$ ,  $n=32$ ,  $n=56$ ,  $n=57$ ,  $n=34$ ).

Consideration of the means and medians shown in Table 9 supports the pattern of a greater improvement over *Time* for the *Other* instructors' groups compared to the *Researcher*'s subjects. Furthermore, much of this difference lies in the third and higher year groups, where the *Researcher* shows a slight decrease in the mean between *Time1* and *Time2*, while the means for the *Other* variable have their largest increase. These observations suggest that there are differences in grade distributions between the experimental (*Researcher, Time2*) and control groups that depend on the variable *Year*, and hence that there is a confounding effect from *Year*.

Table 9

Non-calculus group: Precalculus Grade Means<sup>1</sup> and Medians for Groups classified by Instructor and Time, with and without Year

<i>Instructor</i>	<i>Time</i>	<i>Year</i>	Mean <sup>1</sup>	Median	Mean <sup>1</sup> without Year	Median without Year
<i>Researcher</i>	<i>Time1</i>	Y1	1.84	0	2.23	1
		Y2	2.12	1		
		Y3	3.07	3		
<i>Researcher</i>	<i>Time2</i>	Y1	2.13	0	2.75	2
		Y2	3.19	2		
		Y3	3.00	3		
<i>Other</i>	<i>Time1</i>	Y1	1.92	0	1.97	0
		Y2	2.24	1		
		Y3	1.81	0		
<i>Other</i>	<i>Time2</i>	Y1	2.86	3	3.06	3
		Y2	3.04	3		
		Y3	3.44	3		

Note<sup>1</sup>. Letter grades F to A+ converted to numerical scores 0 to 9.

As in the other exploratory analyses the general association between *Year* and *Grade* was checked using a  $\chi^2$  test of association. Here again the non-calculus group differed from the calculus group with the  $\chi^2$  test suggesting a strong association ( $N = 524$ ,  $p = 0.003$ ) between *Year* and *Grade*. This result, together with the observations from the

histograms, suggests that *Year* has a confounding effect on the non-calculus students' precalculus results, an effect which is also shown in the statistical model described in the next section.

#### Non-calculus Group: Models for Precalculus Grade Frequencies

The graphs in Figure 45 show that the model presented in Table 10 fits quite well except for one non-conforming point. As before, the explanatory variables are *Instructor*, *Time*, *Year* and *Grade*, with *Grade (Frequency)* as the response variable. This model has two terms with low  $p$ -values beyond the minimal model consisting of *Grade* and all main effects and interactions of *Instructor*, *Time* and *Year*. These are the two-way interactions *Time:Grade* ( $p = 0.01$ ), and *Year:Grade* ( $p = 0.01$ ). The three-way interaction, *Instructor:Year:Grade* ( $p = 0.15$ ) is less important.

Table 10

#### Non-calculus group: Model for Precalculus Grade Frequencies Classified by *Instructor*, *Time* and *Year*

*Frequency* – *Instructor\*Time\*Year* + *Grade* + *Time:Grade* + *Year:Grade* + *Instructor:Year:Grade*

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
NULL			59	343.28	
<i>Instructor</i> <sup>1</sup>	1	3.06	58	340.23	0.08
<i>Time</i> <sup>1</sup>	1	79.92	57	260.30	0.00
<i>Year</i> <sup>1</sup>	2	5.21	55	255.10	0.07
<i>Grade</i> <sup>1</sup>	4	163.18	51	91.92	0.00
<i>Instructor:Time</i> <sup>1</sup>	1	15.39	50	76.53	0.00
<i>Instructor:Year</i> <sup>1</sup>	2	0.72	48	75.80	0.70
<i>Time:Year</i> <sup>1</sup>	2	0.33	46	75.47	0.85
<i>Time:Grade</i>	4	12.84	42	62.63	0.01
<i>Year:Grade</i>	8	21.23	34	41.41	0.01
<i>Instructor:Time:Year</i> <sup>1</sup>	2	6.97	32	34.44	0.03
<i>Instructor:Year:Grade</i>	8	12.05	24	22.39	0.15

Note. <sup>1</sup> Required terms of the minimal model

(For an explanation of the structure of the analysis of deviance table, see Appendix E.)

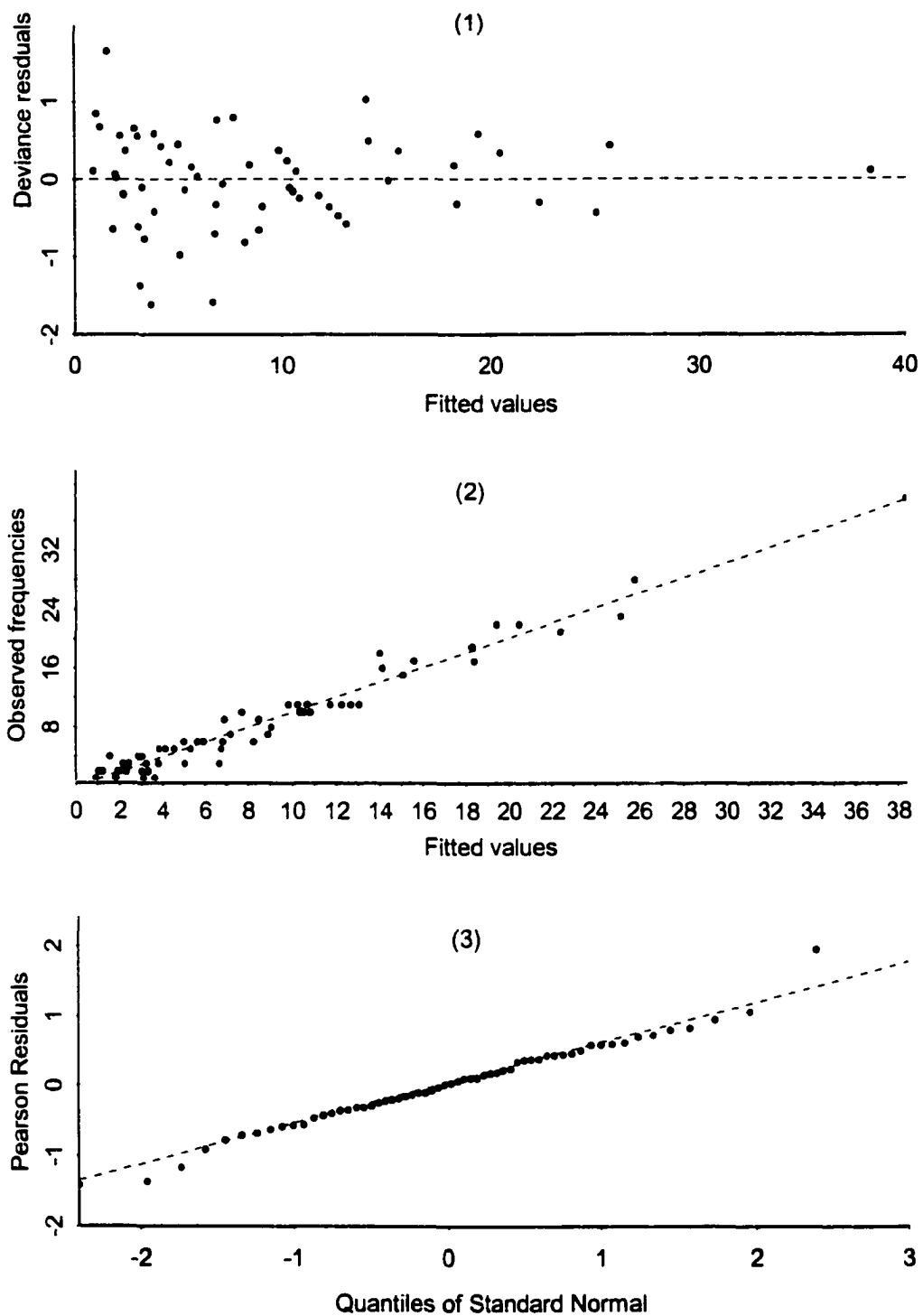


Figure 45. Graphs showing the fit of the non-calculus group, precalculus grades model:

- (1) Deviance residuals versus fitted values, (2) Observed frequencies versus fitted values, (3) Standardized Pearson residuals versus quantiles of standard normal.

Interpreting this model presents some challenges. It is, however, clear that with no interaction between *Instructor*, *Time* and *Grade*, with or without *Year*, there is no suggestion of any difference in *Grade* for the experimental compared to the control groups. This is consistent with the grade distribution similarities for the experimental and control groups noted in the exploratory analysis. The possible contributions of *Year* and the various other interaction terms look interesting, but these issues are not germane to the present investigation.

As in the preceding analyses of precalculus grade frequencies, the precalculus grades model using the *High School* variable was omitted. In this instance there are sufficient numbers in the data set to include both *High School* and *Year* variables. However, given the confounding effect of *Year*, it was considered that the inclusion of *High School* would not add much useful information.

#### Summary of Quantitative Analysis Results

For the precalculus students who continue on to calculus there is evidence that an improvement in calculus performance among students from the researcher's classes may have occurred after the intervention was introduced. This improvement stands in contrast to the difference between the control groups taught by *Other* instructors where the performance level dropped from above the researcher's groups to below, and is also contrary to a slight general downward trend for all calculus classes over the time period. The confounding variable *Year* (students' year standing at the time of taking precalculus) could not justifiably be dropped but did not affect these conclusions. The model developed with the *High School* factor instead of *Year* was somewhat inconclusive but did not contradict these results.

When the response variable is changed to precalculus grades the results are somewhat different. The variable *Year* has a confounding effect for both calculus and non-calculus groups. The calculus group appears from the exploratory analysis to show improved performance for the experimental group compared to the control groups, but the model indicates that this apparent difference cannot be separated from the effects of *Year*. The confounding effect of *Year* also holds for the non-calculus groups. In addition,

the indications from the exploratory analysis are that, if a difference does exist between the groups of the non-calculus data set, it is the *Other* instructors' groups who show a slight improvement over *Time*. Given that *Year* is an acknowledged confound in relation to precalculus grade frequencies, the investigation did not proceed further.

### Conclusion

Precalculus students demonstrated many weak or wrong interpretations of mathematical variables in the contexts of the questions set in the first part of the research. There is also a strong suspicion that many of their correct answers derived from pseudo-conceptual rather than conceptual thinking, since many of the problems experienced by the students were associated with the non-numerical interpretations of the variable typical of pseudo-concepts. There was also considerable evidence of rule-based algebra or pseudo-analysis. Students could state that variables represented numbers but there were indications that, for at least a few students, the numbers meant numbers of letters. Students also demonstrated confusion over the meanings of associated vocabulary, and related concepts, such as expressions and equation solving, were not understood at the reified object level.

There was no evidence that students had, in general, developed concepts of a variable either as a generalized number or in the true sense of a varying number. They made many errors and even their correct responses were couched in terms that were judged to be more suggestive of rote-learned rules than conceptual understanding. It will be argued in the next chapter that the limited interpretation of variables as single numbers may underlie most of the responses, both correct and incorrect.

The intervention designed to help the precalculus students improve their interpretation of basic algebra symbols required the instructor to explain explicitly the interpretations in use for variables and expressions, and to distinguish between action and reified concepts, although not necessarily in those terms. The ideas were presented in the normal review period at the beginning of the course and applied throughout the term as an integral part of the explanations presented in the lectures.

The effect of the intervention was tested retrospectively by comparing students' performances as indicated by their course grades. The group given the intervention in their precalculus class performed better in calculus than the group taught by the researcher prior to the intervention, and this improvement was the reverse of the trend for students from precalculus classes taught by other instructors. Previous high school experience was shown to be an unlikely alternative explanation for these results and the confound variable *Year* did not affect these conclusions.

However, *Year* did have a confounding effect when precalculus grade frequencies were investigated. This meant that no useful conclusions could be drawn in relation to possible effects from the intervention on precalculus grades. Nevertheless the precalculus grade frequency models did produce some interesting results in relation to possible differences between the calculus students and the non-calculus students. Although entangled with interactions involving *Year*, the calculus group's precalculus grades did show a difference between the experimental and the control groups. However, for the non-calculus group, the results were strikingly different. None of the model terms involved a distinction between the experimental and control groups. There appeared to be a general improvement over time with the *Other* instructors showing the greater change, but in the statistical model the complications from the confounding effect of *Year* made further interpretation difficult.

Since this is a retrospective study, conclusions concerning cause and effect must be approached cautiously. It may be that the intervention caused the differences observed for the calculus students, but it is also possible that some other unconsidered factor is responsible. The likelihood that the intervention is a cause of the observed differences is discussed in the next chapter.

## CHAPTER 5

### DISCUSSION

This study consists of two major parts. The first is an investigation into university precalculus students' entry-level understanding of variables, using qualitative research methods. Based on the results of this section it is postulated that the numerical interpretation of variables used by most precalculus students is restricted to single numbers. Students see the variable as a fixed number with changes occurring between instances of variable use. For the second component, a teaching intervention for precalculus students was designed to address some of the issues identified from the data collected in the first part of the study. Using quantitative methods the possible effect of this intervention was investigated through a retrospective analysis of students' subsequent calculus grades. Although sequentially connected, the two parts raise different concerns and questions, and are therefore discussed separately. The implications for teaching and for further research are considered last, followed by the final conclusion.

#### Qualitative Investigation into Students' Understanding of Variables

The investigation of students' understandings of variables used qualitative methods and has the weaknesses of such research. The effectiveness of efforts to overcome these inherent flaws is discussed first, followed by a theoretical description of student understanding derived from the results of this study.

#### Limitations of the Qualitative Research Component

The major limitations of the research into students' understandings of variables are subjectivity and bias, since the researcher is the designer of the tests, the interviewer, and the interpreter of the results. Triangulation, or identification from multiple sources, and the presentation of original data are the primary means built into the study to control the subjectivity problem.

Students' responses were taken fairly literally with little allowance made for their mistakes in reading the problems or in expressing their answers. Undoubtedly some of those counted as displaying a particular error were guilty of no more than a slip of the pen, but the various interpretations discussed are all based on something more than a single response example. In each case there exists a core of response choices and explanations, including interview details, which display the given characteristic, and a larger surrounding group which are less definite but display some common feature with the core group. Thus, although any single response may be misleading or have been wrongly classified, the weight of evidence is that the interpretations described are present.

A slightly different problem arises from the extent of uncertainty about student responses. Many appeared correct but lacked explanatory detail such as definitions of technical terms. As a result it was frequently impossible to distinguish between an overly brief statement of an understood concept and the recitation of a pseudo-concept. This did not affect the identification of the presence of errors and misconceptions, but it made it impossible to gauge their extent.

When interpretations of written data could not be cross-checked in interviews this was reported, and in all cases examples of original quotes were provided. The extent to which these measures have succeeded in offsetting the inevitable problems of subjectivity must ultimately be judged by the reader.

#### General Observations on the Results of the Variable Test

The three problems in the test used to gather the basic data for this investigation are not difficult. No specialized knowledge is required beyond the solution of equations and the interpretation of variables as single numbers, generalized numbers or sets, and co-varying numbers. Yet the success rate was not high, and among those who gave correct responses there were unquestionably some who did so from a basis of rote-learned rules and not from understanding.

Much of the uncertainty about the true extent of students' misapprehensions is due to the many responses that used correct vocabulary without any indication of their

authors' understanding of the words. The best case scenario is that students did indeed understand and assumed that there was no need to explain further. There are two reasons to reject this best-case option for most of the students. The first is that in the pilot stage of the research there were calculus students who gave explanations of words such as cancel, that is, they did not assume that such explanations were unnecessary. If, in terms of understanding, there are precalculus students similar to these calculus students, it is not reasonable that only the calculus students would offer explanations. A much more likely possibility is that the precalculus students were unable to explain because they lacked the level of understanding held by the calculus students. The second reason to reject the best-case scenario is that few students gave correct responses to all three questions. If a student gave a dubious response to one question and made errors elsewhere it seems more likely that the dubious response derived from poor mathematical understanding rather than inadequate communication skills.

The worst-case scenario is that even precalculus students' best responses were not grounded in understanding but in rote rules or pseudo-concepts, and that virtually all precalculus students have inadequate or even wrong conceptualizations of mathematical variables. This pessimistic point of view has some support from both personal observations of the errors made by precalculus students, and from other reports in the literature describing precalculus and calculus students' inadequate mathematical concepts (e.g., Pinchback, 1991; White & Mitchelmore, 1996). The truth probably lies between these two extremes, but seems likely to be closer to the worst-case scenario of a widespread lack of understanding.

In what follows relatively little is said about those students who responded fully and correctly. These students are a minority of the study's subjects and are not the primary focus. It is postulated that among those whose responses were wrong or in some way inadequate there is a general form of understanding of the variable, which is capable of explaining most of the observations. This interpretation of precalculus students' understanding of variables is developed in the next section, followed by a discussion of its role in relation to the results of the study.

### Precalculus Students' Inadequate Understanding of Variables

The explanation of precalculus students' understanding proposed in this chapter is that there are three basic forms: correct, variables as single, non-varying numbers, and variables as non-numerical objects. In all cases a clearly identifiable core group could be described. However, there were also wider groups where the responses could not be directly classified as belonging to one of the three basic descriptions, but which, nevertheless, could be explained starting with one of the three as a basic premise.

It is certainly clear that students do not enter precalculus with the concepts of variables necessary to respond satisfactorily to the three test questions. Students do not, however, lack a concept image for variables. Each student has his or her individually constructed understanding. The possibility that these constructed understandings might have some common developmental features led to a reconsideration of the reported interpretations, resulting in a suggestion of two sources for students' errors: the variable interpreted as an unchanging number, and as a non-numerical object.

The limitations of the object interpretation should encourage students to reformulate their conception, but many may simply develop a pseudo-concept. This latter response is made more likely by the proximity of the object interpretation to the non-numerical use of variables in pseudo-concepts. The difference is that in the object interpretation the student believes that the variable represents a physical object, whereas in a pseudo-concept the letters are simply used and not given any meaning. In this study there were clear examples of students using an object interpretation in their explanations in Questions 1 and 3, and there may be others hidden by a pseudo-conceptual veneer. However, neither pseudo-concepts nor the object interpretation will be discussed further. This is not to deny the seriousness of the fact that they were found, but the study did not add anything new on the subject.

The second incorrect source for students' interpretation of variables is that although they recognize that variables take on different values, they do not recognize that variables change value. This apparent paradox can be explained if students view the variable as taking a single fixed value in a series of separate occurrences or instances. It is argued in the next section that this underdeveloped variable concept is manifested in many of the

observed errors and, in a more developed or superficially improved form, in many of the correct responses as well.

### Students' Interpretation of Variables as Invariant Numbers

The interpretation of a variable as a single number is compatible with students' early experiences with letters in mathematics, which are mostly concerned with formulas or expressions and with solving simple linear equations. Students are accustomed to replacing letters with different numbers, and to solving a series of equations, all of which use the same variable but have different solutions. Clearly, in performing these tasks they recognize that a letter can take on different values. However, in these early algebra contexts the letters take on only one value in each problem or instance. Students may, therefore, be coming out of their early algebra experience with the belief that when variables are used they always represent single numbers. When a series of different numbers are used it is not the letters that change value, it is the instance or occurrence that changes. Thus, for example, the instruction to evaluate an expression at  $x = 1, 2$  or  $3$  is interpreted as three separate instances, each of which has a new value for the letter. The variable,  $x$ , does not change, but represents a single number in a series of three separate instances.

Students can accommodate the use of different values for a variable which is seen as a single number by reinterpreting each value as a separate instance, but will face some difficulties with a proper generalization such as the use of variables to demonstrate some universal number property. It is possible, however, to develop the appearance of describing variable change, not by describing the variables directly but by considering a series of instances and describing the pattern of change between instances. Thus students are able to provide apparently adequate descriptions of variable change without letting go of their fundamental identification of variables with single numbers.

The test of this fundamental explanation of students' responses, that students see variables in a series of separate instances wherein the variable does not change, is its ability to explain the observations in the study. In the next section it is argued that the

interpretation of the variable as a single value in multiple instances explains many of the observations.

### Observed Responses Explained on the Basis of Single-Valued Variables

Question 3 responses are the most interesting of the three test questions because, although the equations are simple, the question context does not allow students to reproduce learned algorithms or pseudo-conceptual responses to the extent evident in Questions 1 and 2. Consequently, the following discussion focuses primarily on the results of Question 3.

There were seven interpretations listed as a result of analyzing students' responses to Question 3. Interpretations 1 and 4, the correct and object interpretations, are not pertinent to this discussion. Interpretation 5, where students misinterpreted *constant* as referring to rate of change can be included with Interpretation 2, where students used correct vocabulary but appeared to focus on a series of static pairs of values for  $p$  and  $q$ . This last interpretation and the others remaining can all be explained through the interpretation of the variable as a single fixed number in multiple instances.

The hypothesized single-value interpretation of the variable is derived from the explanations given by some of the group of students classified as having either the sixth or seventh interpretations. For at least one of the equations  $p + q = 15$  and  $p = 5q$  there were students who explicitly rejected options A (As  $p$  increases  $q$  increases) and B (As  $p$  decreases  $q$  increases), which describe the variables as changing. Some chose option E (None of the options A, B, C, or D) and some chose option C ( $p$  and  $q$  are constant), but their reasoning is essentially the same: that a change in either variable causes the equation to be wrong. If the focus is exclusively on the instance, that is, on the particular equation and a particular number pair, then nothing is changing and it is logical to apply the constant description. On the other hand, if students pay attention to several instances then they have to consider different values for the variable. Their single number interpretation causes them to reject both of the change descriptions, and their recognition that the variable takes on different values also causes them to reject the constant option. There remains option E, which was chosen by several students.

A number of students chose both C, describing the variables as constant, and A or B, describing the variables as increasing or decreasing. These choices are contradictory only if they refer to the same thing. However, under the misconception proposed here, variables have two facets. The choice of the constant option C fits with the variable's intrinsic unchanging nature, while the choice of one of the change options A or B fits with the patterns students are able to observe in the instances of the variable's use. Whether a student chooses all, some, or none of C, A or B depends on how the two facets are balanced.

For many students the awareness of different instances appears to be tied to the presence of the 15 or the 5, which provides them with a starting point to identify specific number pairs. In the three-variable equation this reference point is missing. Students cannot generate their own values for  $r$ . To do so would require understanding that there is a set of values available, which is incompatible with the view of a variable as a single number. Thus they have no means of finding separate instances, and, consequently, no evidence of change. A logical consequence of such thinking in relation to  $p + q = r$  would be, as was observed, the choice of the *constant* option C and a description of, not only  $p$  and  $q$  as constant but  $r$  also.

It is interesting to note that although both  $p + q = 15$  and  $p = 5q$  occasioned responses where the variables were described in some way as unchanging, these seldom occurred together. It is possible that the two equation types are associated with different interpretations of the variable. The multiplication was answered correctly more often than the addition, probably because it is easier to see  $p$  and  $q$  as co-varying in the direct action of multiplication, than in the addition. Variables in the addition can be identified as passive pairs of addends, each of which constitutes an instance or an individual number pair. This explanation appears to be contradicted by the presence of responses with the reversed answer pattern, that is, changing values of the variables identified in  $p + q = 15$ , but not in  $p = 5q$ . However, there is no contradiction if the variables in  $p + q = 15$  are interpreted as physical objects and not as co-variants. It has already been noted that the interpretation of variables as objects gives rise to the correct option choice for the addition equation. There is also support for the object interpretation for  $p + q = 15$  in the associated explanations where the language used contains words compatible with a

non-numerical interpretation, such as *compensate* or *balance*. If students interpret the variables as objects in  $p + q = 15$  and as single numbers in  $p = 5q$ , the multiplication continues to be associated with the higher level interpretation.

Interpretation 3, where students appeared to confuse *increasing* and *decreasing* with *greater than* and *less than*, can also be explained on the basis of a single-value interpretation for the variables. It has already been argued that students with this interpretation cope with the idea of variable change by substituting change between instances. If students do not make this substitution, relying only on the individual instances and not on the differences between them, their focus is on the number pairs alone. When directed by a question to consider change the only place to look is within the number pairs themselves. For example, in  $p = 5q$ , if  $p$  is taken as larger than  $q$ , the increase is from  $q$  to  $p$ . The option whose wording refers to  $q$  increasing is option A (As  $p$  increases  $q$  increases). Thus students make the correct choice, but explain it by comparing  $p$  to  $q$ . The result is that they appear to be confusing *increasing* and *decreasing* with *greater than* and *less than*, as was evident in the third interpretation.

In situations requiring interpretations of variables as generalized numbers or as co-variants, students can use an adapted form of the single number in multiple instances misconception of variables. They may be considering the patterns apparent in many instances of variable use as they would appear in the columns of a table of values. In doing so students need not abandon their single value interpretation if each number pair or row entry in the table is taken as a separate instance. It was observed in the description of Interpretation 2 that some students explained correct option choices for the equations  $p + q = 15$  and  $p = 5q$  in ways suggestive of a description of the patterns in the columns of a table of values. For example, some students simply displayed examples of values for  $p$  and  $q$ , most often involving only whole numbers, without making any reference to connected or dynamic changing values.

The difference between considering patterns of instances as described in the previous paragraph, and genuine variation lies primarily in the presence or absence of a requirement that the variable represent a single number. The true concept and the misconception will lead to many similar responses, but there are certain limits to the misconception. For example, instances are discrete. Thus change, in the context of the

misconception, is in discrete steps rather than continuous. Consequently the concept of continuous variation or continuous measurement is likely to be difficult for these students.

In addition, students may show an excessive preference for the integers, to the exclusion of the full set of real numbers. Thinking of variables as single numbers is compatible with the discrete nature of the integers, since there are both next instances and next numbers. Examples given by students in response to Question 3 were almost exclusively integers, and there were also responses to the identity equation in Question 2 where “any number” was identified with the set of integers.

An equation with a single solution is an example of an instance, and finding the solution means identifying the instance. Thus, for students who rely on instances to deal with different values of the variable, the identity equation violates their fundamental understanding. The equation does not yield a single instance and the student, as a result, is lost. Such students may have difficulty following the logic of finding a value of the variable that makes the equation true. For them, the equation is not a statement but an instance of variable use and consequently neither true nor false.

A single value interpretation of the variable is also not compatible with the correct use of variables in simplification problems such as Question 1, since these require the variable to be interpreted as a generalized set of numbers. A student may know that any number divided by itself is 1, but to state this fact algebraically as  $\frac{a}{a} = 1$  requires  $a$  to represent a generalized number and not specific numbers. Such a generalization goes beyond the scope of single number and multiple instances, since values are neither being given nor being found for the variable. Students whose interpretation of variables precludes generalization have little option but to resort to rote rules and pseudo-concepts, where the variable is treated as a non-numerical symbol. They are able, provided their recall is not faulty, to produce correct answers but are unable to explain their method with any reference to numbers. This is precisely the situation observed in the responses to Question 1.

## Summary

The source of the theory that students are interpreting variables as single fixed values in a series of separate instances lies with the students who explicitly stated that the variable could take on different values, and also stated that the variable did not change values. It is argued that the single value and multiple instance interpretation for variables explains many of the observations in this study. When variables are required to be interpreted as a set or as co-variants, students may look at the patterns across instances and use these. If students do not consider the multiple instances and remain focused on the individual values, questions about changing values are answered as if the change was from one of the variables to the other, or else change is denied absolutely.

Many of the written responses which constituted the data for this part of the study are ambiguous, which makes estimation of the extent of the problems impossible. Compartmentalization and the resulting lack of consistency in students' responses, and the presence of pseudo-concepts both contribute to the uncertainty surrounding this data. The theory proposed here, that students think of variables as single numbers in changing instances, provides an explanation of students' thinking that bridges the gap between the interpretation of variables as objects and as generalized numbers or co-variants. The use of variables as representing single values is an action, and the incorporation of the notion of multiple instances may be a precursor to the reified concept.

### Effectiveness of the Intervention: Quantitative Analysis of Students' Subsequent Calculus Grades

Most mathematics teaching proceeds on the assumption that students understand the technical terms and specialized vocabulary they use. The results of the first section of the research suggest that this can be a false assumption. In the intervention, the context-dependent shifts in meaning of algebraic symbols were explained explicitly to students in the review section of the precalculus course, and also as examples arose throughout the term. These explanations of meaning included the use of variables as single numbers, generalized numbers, or varying numbers; the use of expressions as actions or numerical objects; and the use of graphs as collections of points, or as visualizations of the patterns

of changing variable values. The intent of this approach was to present a foundation upon which students could reconstruct their thinking about variables and associated concepts.

Testing for the success of this approach focused on long-term performance effects, since short-term success can be had through the diligent rote learning of a new set of rules rather than conceptual change. Calculus grades were used as the measure of long-term performance, based on the argument that if students' thinking had changed for the better it should affect their performance in calculus classes where a good mathematical foundation is critical. Precalculus grades were also investigated as the only measure available for the many precalculus students who did not continue to calculus. The experimental design and its limitations are considered first, followed by a discussion of the results.

#### Limitations of the Experimental Design

The quantitative analysis section of the study is a retrospective analysis of students' grades. No random assignment of students to the experimental groups and control groups could be carried out, which raises the possibility of an unintentional effect from some other associated factor. The comparison is between the experimental group of students who received the intervention from the researcher, and three control groups, one consisting of students taught by the researcher prior to the introduction of the intervention, and two composed of classes from the same time periods as the researcher's classes but taught by other instructors. This basic design controlled for possible effects from changes in the student populations over time, and from differences between instructors. However, retrospective studies are never free from the possibility of confounding factors.

Two potential confounding factors affecting the composition of the research groups were identified: (a) students' year classification by the university, and (b) students' previous mathematical experience as indicated by their last mathematics course in high school or its equivalent. Students' year standings turned out to have a considerable confounding effect, especially in the precalculus grades models. Testing for the effect of

high school background was confined to the calculus group and omitted the *Year* factor, since the data set is too small to allow concurrent testing of both confounding factors. The results of this part of the analysis should be treated with caution but suggest that high school background did not affect students' calculus grades. The high school variable was not tested further because the identification of the *Year* confound in relation to the precalculus grade data rendered any further investigation profitless.

In addition to its presence as a confound, the variable *Year* presents a further limitation because the nature of its effect is far from clear. Certainly, maturity is an issue, but although first year students may be more likely to do poorly due to immaturity, they could also benefit from more recent exposure to mathematics. Maturity may benefit third and higher year students, but they are further removed from their high school mathematics and may also contain a disproportionate number of weak, confident students who have been putting off taking the course. These competing possibilities make the effect of students' year standings on grades very complex and the factor's contribution to the statistical model obscure.

The population is insufficient to support a full analysis using all factors at all levels. Such an analysis would have been more satisfactory than the separation into subsets of the data and the collapsing of levels that was necessary. Nevertheless, the statistical models fit well and are compatible with the exploratory observations. Only the calculus group's calculus grades model is sufficiently free of the confounding variables to allow conclusions to be drawn. However, this is the most important group in the study since calculus grades are the best available measure of the intervention effects. The precalculus data analyses do contribute one issue that merits discussion: the difference in results between the calculus and the non-calculus groups.

### Interpretation of the Quantitative Analysis Results

In this section the major point for discussion is the extent to which the experimental group's higher calculus grades can be related to the intervention. The precalculus grade results are also considered in relation to the confounding effect of the *Year* factor and the

final topic is the difference between the calculus and non-calculus students' precalculus grade models.

For students who continued to calculus the experimental group performed better than the control group from the same time period, and better than the control group consisting of students from the researcher's earlier precalculus classes. A statistical association can be claimed between the experimental group and higher calculus grades despite the presence of the confound *Year*. The most significant term in the calculus grades model for the calculus group is the three-way interaction between *Instructor*, *Time* and *Grade* (See Table 5, p.113). The sources of this interaction are a decreasing failure rate and increasing proportions of higher grades over time for the researcher's groups, with a reversed pattern for the other instructors' groups (See Figure 30, p.108; Figure 33 p. 115). In addition, this improvement between the researcher's groups runs counter to the slight downward trend in overall calculus grades over the same time period.

As with all retrospective studies, the possibility of unforeseen and uncontrolled factors precludes conclusions of cause and effect. However, it is appropriate to consider the likelihood that the intervention is the cause of the observed association between the experimental group and higher calculus grades.

There are two obvious explanations, other than the intervention, for the observed differences between the experimental and the control groups. Both relate to the *Instructor* variable. The first is that the changes associated with the researcher are connected to increased teaching skill, and the second, that the teaching of the other instructors was less effective in the second time period. Neither is likely for several reasons.

Since the bulk of student learning takes place outside the classroom as they study and practice, instructor differences probably have to be quite dramatic before they affect course outcomes in any significant way. The researcher was already an experienced teacher at the start of the study, and students' end-of-term course evaluations did not change over the period of the study, suggesting that the level of teaching skill remained constant. There is a greater possibility for instructor difference among the six other instructors. However, if an effect existed in relation to less effective instructors it should affect all students, that is, both the calculus and the non-calculus groups, and also be apparent in the more immediate precalculus grades. Neither situation can be observed.

For *Other* instructors the precalculus grades for first and third year students may show a decline but this is balanced by the second year students whose results show improvement (See Figure 41, p. 124). Overall, when not broken down by *Year*, there is no change between the first and second time periods. Among the non-calculus groups there is a general improvement, whether looked at as a whole, or broken down by *Year* (See Figure 43, p. 130; Figure 44, p. 130). Thus the precalculus results for both the calculus and non-calculus groups do not support the possibility that there was a decline in teaching effectiveness for the *Other* instructors. The intervention, therefore, remains a likely cause for the observed superior performance of the experimental group of calculus students.

Precalculus grades were not expected to provide a good measure of students' understanding of the intervention content. All precalculus classes write a common final examination worth 60% of the course grade, with the grade distribution for each class dependent on the grade distribution achieved by that class on the common final. This procedure balances out differences in term marks between classes. The questions on the common final have to be equally accessible to all students which means that, since other instructors did not incorporate the intervention into their classes, there were no questions that tested this material directly. The ability of students to use pseudo-concepts and pseudo-analysis successfully probably accounts for much of the success in precalculus classes, which makes the precalculus examination a poor test for any effects from the intervention. Models based on precalculus grades must therefore be interpreted with caution.

The confound *Year* is difficult to discuss because the possible effect of a student's year standing on his or her grade is unpredictable. Maturity, recency of mathematical experience and attitude towards precalculus could all be associated with *Year*, sometimes in contradictory ways. In addition students' year standing may be associated with their degree program. Science students tend to take their mathematics early in their program since both precalculus and calculus are first year courses. Some non-science students take precalculus as a prerequisite for a first year mathematics course (Finite Mathematics) but for many it is required for entry into a third year statistics course, and may therefore tend to be taken later in their academic career. How these various influences might

interact could constitute a further study. For the present, however, no useful discussion of the effect of *Year* is possible.

There are two issues connected to differences between the statistical models. The first is the difference between the calculus group's calculus and precalculus grades models (See Table 5, p. 113; Table 7, p. 125). These are based on data from the same subjects and should therefore be similar. The calculus grades model differentiates very strongly between the experimental and the control groups, but the precalculus grades model does so only in conjunction with the confound *Year*. This discrepancy between the models may be because success in precalculus does not depend on understanding variables to the same extent as does success in calculus where understanding necessitates the interpretation of variables as co-variants and as sets. Precalculus grades may also be more susceptible than calculus grades to the unidentified effect of *Year*. Thus, although the precalculus model does not directly support an association between the intervention and performance, neither does it oppose such an effect.

The second difference between models is between the calculus and the model for the non-calculus group. In the latter model, the optional terms with a  $p$ -value at 0.1 or better, are *Time:Grade* and *Year:Grade*, neither of which includes the *Instructor* factor (See Table 9, p. 131), implying that there is no differentiation in this model between the experimental and control groups. This stands in sharp contrast to the models for the calculus group, and suggests very strongly that if there was an effect from the intervention, it affected the calculus group and not the non-calculus group.

The possibility that, if an effect from the intervention exists, it is confined to calculus students can be explained by differences in motivation. Students who propose to take calculus enter precalculus knowing that the ideas and knowledge they gain will be directly applicable in their subsequent calculus course. This may be less true for the non-calculus students whose subsequent course is typically statistics, which they are aware does not have the same close relationship to precalculus. In addition, the calculus group is composed predominantly of science students who may have more skill and interest in mathematics, as well as more opportunity for experience with variables, graphs, and manipulations of scientific formulae, than the non-science and non-calculus students.

Such a difference in prior knowledge may make the intervention content more accessible to most of the calculus students.

It is also possible that the intervention had no effect on understanding, but is still associated with improved calculus performance. Students' self-reports subsequent to the intervention indicated that many believed that they had a better understanding of their mathematics. Such a belief, even if unfounded, could have the effect of increasing confidence, which could in turn, result in improved performance.

### Implications for Teaching and Learning

The results from this research raise several questions about precalculus students' learning and understanding of the concept of variables. The nature of students' knowledge of variables, as inferred from students' responses in this study, is often limited to single numbers, with some modifications designed to accommodate the higher level set and co-variant interpretations of variables. Students also appear to use an object interpretation for the variables. Although further research is needed to refine this description, there is no doubt that a problem exists with students' understanding of variables. An attempted remedy was included in this study and may have yielded some success. Taken together, students' lack of understanding of variables and the apparent effects of the intervention have implications for both the past and future learning for these students.

Studies such as this reinforce the need for mathematics educators to focus on students' understanding of mathematics. In the specific case of precalculus courses the goal is preparation for calculus. The most basic concepts in calculus depend on generalizations about variables and expressions, and on the notion of continuous change. Students whose foundation of variable understanding is limited by a single-valued variable are likely to resort to pseudo-conceptualizations and pseudo-analysis when faced with mathematical ideas dependent on continuous variables. Consequently, it should be recognized that a precalculus course that does not address the issue of understanding of variables is not preparing students adequately for calculus.

Precalculus students at the University of Victoria have met or exceeded the university entrance requirement of grade 11 mathematics. The evidence of a widespread and fundamental lack of understanding of variables among these students has implications for all of high school mathematics, and suggests that the variable and its interpretations be given a much more central role in classroom discourse.

Part of classroom discourse is mathematical communication. The results of this study, and others, have shown that students use incorrect or inappropriate interpretations of fundamental mathematical terms and symbols. Clearly, instructors and teachers should be aware of students' interpretations, but they should also be aware that their own words may be misinterpreted. The success of the intervention used in this study needs to be replicated and confirmed, but it is clear that students liked the approach of explicit explanations. At the very least, the detailing of specific meanings should have helped students avoid misinterpreting the instructor's words. This is not intended to imply that the direct explanation approach is universally appropriate. There are many situations, such as when a concept is developing, where direct telling can interfere with students' individual concept constructions. The suggestion based on this research is that there should be on-going classroom attention to the detailed, specific, and context-dependent interpretations of mathematical terms and symbols.

Mathematics educators need to look critically at the opportunity afforded by many standard presentations for the development of misunderstanding. For example, the concrete or object image may be unintentionally encouraged in the classroom by the use of concrete examples. When a student looks at a picture of two pears and three apples and a question about costs, does he or she focus on the costs or the objects? What is the thinking behind an associated expression such as  $2p + 3a$ ? It seems inevitable that some, perhaps many, students will focus on the objects, and the question then is whether or not there is feedback in the system to expose the error.

Another example can be found in the development of an algebraic relationship from a series of single numerical examples, or the demonstration of an error by the substitution of single numbers. These common and perfectly acceptable practices may serve to reinforce a student's interpretation of variables as single numbers. A teacher is

presenting examples of a generalization, while the student sees individual instances of variable values.

The use of metaphorical language presents another opportunity for confusion. For example, speaking of *taking* and *collecting*  $x$  during algebraic manipulations is very convenient, but teachers need to be sure that students do not take the words literally. Such language could be taken by students as confirmation of an object interpretation. Teachers need to examine their language and presentations for this kind of ambiguity.

The potential for misconstruction and misinterpretation is present throughout the mathematics curriculum. Improvements in the sequencing or content of the curriculum may reduce, but are unlikely to stop, the kind of subtle error uncovered in this study. These errors may develop when students misconstrue what they see or hear, but in such a way that correct answers are still forthcoming with neither the teacher nor the student being aware of a problem. The solution lies in the development of situations where teachers and students air and examine their interpretations, that is, on wide and deep-ranging classroom discourse.

Teaching of the kind implied in the previous paragraph makes huge demands on teachers. With respect to variables, teachers need to be specifically aware of the situations where the single value, set and co-variant interpretations of a variable are called for, and to recognize which kinds of thinking are developmentally sound and which are not. They also need some means of identifying their students' thinking. Further, they must be able to encourage students whose understanding is developing successfully, and to redirect students who are heading towards a misconception or a pseudo-concept. This kind of teaching requires a body of knowledge about mathematical thinking and its development, which is largely lacking at present. There are implications here for those who direct and conduct research and for those responsible for teacher training and professional development.

### Implications for Research

The first and most important research issue related to this study is the need for replication. Students' understanding of variables needs to be investigated elsewhere and

with a wider range of subjects. Replication of the intervention using other instructors, subjects and investigation methods, is also important. In particular, the effect of the intervention should be tracked using qualitative methods or, if quantitative methods are used, a more sensitive measure of students' knowledge than course grades should be found.

There should also be research specifically directed at a better and less hypothetical, description of the single-value interpretation of variables. The description advanced in this report may be confirmed or replaced, but in either case the result should lead to better knowledge of students' understanding.

Research should also be undertaken into the sequential development of variable understanding. This information would provide a basis for curriculum design and a foundation for interventions when the development goes astray. The possibility that the calculus and non-calculus students in this study responded differently to the intervention suggests that the conditions necessary for developmental progress may be quite specific.

The mathematical variable has not been given much attention in recent years perhaps because, although mentioned by individual contributors, it was not included as part of the NCTM research agenda (Wagner & Kieran, 1989). The results of the present study suggest that the omission of the variable was an oversight that should be remedied. The variable is a fundamental part of mathematical understanding about which too little is known.

### Conclusion

This research began as an attempt to improve the effectiveness of the researcher's precalculus teaching at the University of Victoria. The first part, using qualitative research methods, was intended to provide background information on students' understanding, and to form the basis for developing an intervention. The second part consisted of a quantitative investigation of the effectiveness of the intervention.

The initial information on variables came from Küchemann's (1981) contribution to a study of British school children between 11 and 16 years old. His report identified a series of levels for students' understanding of variables, which are still used by current

researchers. However, these levels were derived from younger high school students and were incomplete when applied to senior high school or university populations and the mathematics studied at that level.

The precalculus students' interpretations of variables identified in this study begin with the erroneous variable as object. The object in this interpretation can take the form of a concrete object such as an apple, a non-numerical abstract object such as a letter, or as a non-numerical mathematical object such as a unit. The simplest numerical interpretation is as a single number, but as these precalculus students interpreted it, the variable was not precluded from taking on different values. It was reasoned that this apparent contradiction could be explained if students considered each assignment of a value as a separate instance. The instances change, but not the variables. Starting from this premise most of the students' responses in this study could be explained. Very few students clearly and successfully explained variables in terms of a set or as a generalized number. Explanations of the variables as co-varying quantities were also infrequent.

The intervention designed on the basis of these first results took the direct approach of explaining explicitly, the normally implicit meanings and interpretations of the variables and other algebraic symbols. Students generally responded very positively to this extension of normal lecture content. This was success of a kind, but the possible effect on students' mathematical performance was also investigated.

The second part of the study was a retrospective analysis of students' calculus and precalculus grades. Without random assignment of subjects the results of the analysis cannot be used to claim cause and effect, but there did appear to be an association between the intervention and improved calculus performance. A similar analysis of the same group's precalculus grades was less successful due to some confounding effects. Confounds were also a problem for the analysis of the precalculus grades for students who did not continue to calculus. However, the difference between the statistical models for the calculus and the non-calculus groups suggest that, while some kind of association was present between the intervention and the performance of the calculus group, no such connection existed for the non-calculus group. There was no qualitative investigation of changes in students' thinking after the intervention, but it can be speculated that

variations in motivation and mathematical disposition resulted in different responses to the intervention.

It was not expected that this study would lead to a definitive or finished description of precalculus students' understanding of variables, but that it would provide a preliminary view from which further questions and research directions could be developed. In this the study has succeeded. The suggestion that precalculus student generally understand variables as single numbers and not as representing sets or co-varying quantities, should be tested and refined. Research is also needed into the qualitative effects of the intervention.

This study adds to the growing body of knowledge of students' understanding of mathematics. As has been observed elsewhere, there is a notable discrepancy between the mathematics precalculus students are supposed to have learned and the actual state of their knowledge. The direct intervention used here may not be applicable outside of lecture rooms, but within those confines it appeared to have some success. As one student remarked: "I don't think I'm going to pass this course, but at least now I know what it is that I don't know."

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## APPENDIX A

## Interview Examples

Both interviews began with some talk designed to put the student at ease but the transcripts start with the first direct question. The written materials referred to in the interviews are shown in figures preceding the relevant interview sections

Interview Transcript 1

This interview followed the pretest, and took place in January 1996. The subject was female, between 20 and 24 years old, a psychology major, and her last mathematics course was grade 11 in 1991 where she achieved a C grade. When asked about her feelings about studying mathematics she wrote: "Horrid. I hate math."

Figure A1 and Figure A2 show her written responses to the test Questions 1 and 2 respectively.

Simplify  $\frac{12a}{-6a}$ . Explain and justify each step you took to arrive at your answer.



Figure A1. Interview example 1 Question 1 response

- I Nice answer – what happened to the explanation? What did you score out there?
- S The a.
- I So why did you think it might be in and why did you take it out?
- S Well, I think at first I didn't, I don't know, I just thought that it would be there and then I decided to subtract them and then it wouldn't exist any more.
- I OK. So tell me how you got the negative two then.
- S Twelve divided by six is two [*exaggerated patience*] and then the negative.
- I OK, but you said subtract just now ...
- S Oh....

- I So that was just the a's you subtracted? Is that what you meant?  
 S I guess I subtracted the a's and divided the twelve by six.  
 I OK

$$\begin{array}{l}
 2(x-3) + 7 = 2(x+5) - 9 \\
 2x - 6 + 7 = 2x + 10 - 9 \\
 2x + 1 = 2x + 1 \\
 2x = 2x
 \end{array}$$

~~$2x + 7 = 2x + 10 - 9$~~   
 $2x - 6 + 7 = 2x + 10 - 9$   
 $2x + 1 = 2x + 1$   
 $2x = 2x$   
 ~~$2x = 2x$~~   
 ~~$2x = 2x$~~   
 ~~$2x = 2x$~~   
 $2x - 2x = 0$   
 $x = 0$   
 ~~$2x = 2x$~~

Figure A2. Interview example 2: Question 2 response.

- I What were you doing here? Just checking that ...?  
 S I didn't know what the question wanted me to do that was the thing. I didn't know what I was supposed to be doing. See ... It had it all there and then it had the answer. I didn't know what to do.  
 I Well you've actually got another answer down here. (*Laughs*) I didn't write  $x=0$  you did! That's actually what I was looking for – just to finish it. So now,  $x$  equal to  $0$  – that's coming from this statement here [*Points to  $2x - 2x = 0$* ].  
 S Took that subtracted it ...  
 I So when you subtract  $2x - 2x$  you get  $x$ ?  
 S No, you get  $0$ .  
 I Mm. That's not quite what you wrote there.  
 S What? .. would I just .. equal to  $0$ , without the  $x$ ?  
 I You could do that. That's one thing to do. Is there another way you could answer the question?  
 S  $x$  equal to  $x$ ?  
 I Mm.  
 S By dividing those two ..

- I Would that be finished – would that be alright?
- S *No response*
- I What sort of an answer would you expect to get for this?
- S I don't know.
- I I mean would you think  $x$  equals  $x$  is the sort of answer you would get or is  $x$  equals  $0$  more like what you would expect or ..
- S Um. Good question. I don't know. [*laughs*]
- I When you think about a variable like  $x$  what kind of .. or do you have any sort of image, thoughts, or is it just  $x$ ?
- S It's just  $x$  unless it's given a number like  $x$  equals  $6$  or something. Then it would be  $6$ .
- I But you don't think of  $x$  in any particular sense?
- S No, it's just a letter.
- I If I asked you what  $x$  is doing there what would you say?
- S I guess it's pretending to be a number.
- I [*Writes  $3a + 2b$ .*] If I asked you to explain to someone who doesn't know anything about algebra – they know about number but not about algebra – what does this mean?
- S The  $a$  is being substituted for a different number. You times the  $a$  by  $3$  and the  $b$  by  $2$  and you add them. If you had numbers.
- I If you had numbers. And if you don't have numbers what does it mean?
- S You can't do anything.
- I What about this one? [*Writes  $3 + 2b$ .*]
- S Um. Well, the  $b$  belongs to the  $2$  because it's being multiplied so you can't do anything with the  $3$  because it's not the same. It doesn't have  $b$ . It's different numbers.
- [*In the test the student wrote C for all three equations in Question 3. with no explanations*].
- I You put C's all the way for question 3. In what sense do you think that  $p$  and  $q$  are constant?
- S Well, because they represent like a set number and so they can't really change. You can't say  $p$  equals  $10$  or equals  $5$  or something. So they're a set number.
- I Mm hm. But does it always have to equal  $10$ ?
- S No, it can equal whatever it would .. equal.
- I OK, but for .. OK so what you mean is it doesn't sort of ..
- S It doesn't represent just anything, it represents a set number.
- I OK, and then  $q$  would as well?
- S A different number

- I Um .. in that case let's take the one  $p+q=15$ . You put down C. You could have put down more than one. Did you actually discount the others or ..
- S No, I just didn't know what the ..
- I OK so it's not that you think it doesn't fit ..
- S I just didn't know.
- I OK. So for example, number B – as  $p$  increases  $q$  decreases ...
- S I didn't think any of these would hold just because they're different numbers so it shouldn't matter if one changes. It shouldn't necessarily mean that the other one's going to.
- I Oh, OK. So if one changes it doesn't necessarily mean that the other one is going to .. and this is implying that they're both changing?
- S Yeah.
- I OK, what about  $p$  and  $q$  are never negative?
- S I just thought that they could be. I mean if you want to make  $p$  negative 3 it could be.
- I You'd just have to make  $q$  ..
- S To equal 15 .. *mumble* .. negative 5.
- I OK, what about this one ( $p+q=r$ ). It's quite like  $p + q = 15$  except that instead of a number we've got a variable. Does that make a difference do you think?
- S No, except for you wouldn't be able to just pick numbers to fit that equation. Like this one - cause you don't have a clue what it's supposed to be like.
- I So just anything?
- S Yes.
- I OK.

Interview Transcript 2

This interview took place after a mid-term test in May 1996. The first questions referred to a prepared problem but thereafter the questions are from the test. This student is female, between 20 and 24 years old, a Sociology major, and had achieved a D grade in Algebra 12 two years previously. She was very articulate and with considerable metacognitive ability.

$$f(x) = x^2 + 9x + 20 \quad g(x) = x + 5$$

$$f(4) = 4^2 + 9(4) + 20$$

$$f(x) + g(x) =$$

$$f(x) \cdot g(x) = (fg)(x)$$

$$f \circ g(x) \quad f \circ g(x) \quad \text{composition}$$

$$f(g(x)) \quad f(x+5) \quad (x+5)^2 + 9(x+5) + 20$$

$$f(x-1)$$

Figure A3. Interview example 2: Prepared function problem and subsequent questions.

- I Functions. What I want to get at is some sort of sense of how you think about functions. So, I guess, the first question I would ask is - just take  $f(x)$ , leave  $g(x)$  for the moment. Now, you've seen that [*Points to  $f(x) = x$* ] a lot of times. How would you describe it to yourself?
- S Umm This side doesn't confuse me at all. If this were just  $x$  squared plus  $9x$  plus  $20$  equals  $0$  and it said solve for  $x$ , that's fine. You know I can figure that out, but as soon as that  $f$  of  $x$  is in front of it - I'm thinking 'cos I'm always thinking that there's got to be a value - you're substituting a value in for  $x$ . But that doesn't give you one, and so it's .. I can't explain why that confuses me. it just does.
- I OK. Well of course some of the questions ...
- S I can't explain.
- I Some of the questions you do, of course you do substitute a value
- S Right, right, I think it's for every single one, and that's my first instinct to do that.
- I So something - if you were asked to do that, say *Writes  $f(4)$*  That's no problem for you?
- S That's no problem at all. There's a number there, but it's because there's no number on the left side. That's why it confuses me, I think.
- I Can you just do that one. And tell me how you do it
- S Oh, OK. Just substitute this in .. is that right?
- I Yes.
- S OK. You want me to just go ahead and do it?
- I No. I assume you can calculate .. I was looking for what you said: you said, substitute this in. That's the words you use to yourself?
- S Yeah, yeah, am I using the wrong words?
- I No, no, I mean this is not right/wrong. I'm trying to find out - I know the words I use, but I'm trying to find out what students use. Because if they use the words I use then that's fine, but if they don't, maybe I can improve the words I use.
- S OK.
- I That's the sort of thing that I'm after with this. And you don't find any of that out by looking at what people write.
- S Yeah, that's true.
- I So that's what all this is about. OK, so if I took something like: suppose we wrote  $f$  of  $x$  plus  $g$  of  $x$  .. OK now: that was the first of the things that we did that actually used function notation ..
- S Right.

I Uh, what's your reaction?

S *Unintelligible*. My first reaction is a little bit of doubting myself, cause I think OK, so that would be  $x$  squared plus  $9x$  plus  $20$  plus  $x$  plus  $5$ . Is that right?

*Unintelligible, both talking at once.*

I OK. Now a word of warning . This one is right but I'm not going to keep telling you you're right, because sometimes if you're wrong I don't want to tell you that because I want to find out a little bit more.

S OK, OK.

I Now what I will do at the end of the interview, if there is something wrong we'll go back and we'll sort it out ... But I kind of, I want to get a response, and if I tell you that you're wrong you'll change direction, whereas I may want to know a little bit more about where this is coming from.

S Yes, OK.

I So. Yes, you're right there. Now, the question is: do you just know to use the  $x$  squared plus  $9x$  plus twenty here?

S Yes, because it says  $f$  of  $x$  equals and so I use that and just substitute it in.

I So this is again, this is a place where you ...

S Yeah, just substituting in ...*[Laughs]*

I Can you give me there another verb you might use if I said I don't want you to use "substitute for". What verb would you choose that would be closest to .. ?

S Replacing.

I OK right. OK. Now what about this then?

S OK.

I *[Writes  $f(x) + g(x)$ ]*

S I would put this expression *[points to  $f(x)$ ]* in brackets plus this  $x$  plus  $5$  in brackets *[points to  $g(x)$ ]*.

I And ..*[Writes  $f(x)g(x)$ ]* What about this then?

S I would put this expression *[Points to  $f(x)$ ]* in brackets times  $x$  plus  $5$  in brackets.

I And then this one. *[Writes  $f \circ h(x)$ ]* You know what that little circle means?

S  $f$  of  $h$  of  $x$ .

I Yep. Oh. sorry, it should be  $g$  of  $x$ . *[Scribbles out and rewrites correctly,  $f \circ g(x)$ .]*

S Oh *[laughs]* I see where we are going now. OK. That means  $f$  of  $x$  times  $g$  of  $x$ . Which is exactly the same as this. *[Points to  $f(x)g(x)$ ]*

- I OK.
- S Is that right? I'm trying to think to myself. No it's not ... it means  $f$  times  $g$  of  $x$ .
- I OK what about this one? [*Writes  $f(g(x))$* ] That's another notation we use.
- S That is the same as that. [*Points to  $f \circ g(x)$* ] [*Hesitates*] Yeah.
- I So you're seeing that as multiplication.
- S Yeah.
- I That is wrong.
- S OK. [*Laughs*] I could tell by your voice.
- I This one is multiplication. [*Points to  $f(x)g(x)$* ] The book also uses the notation  $fg$  of  $x$  [*Writes  $= fg(x)$  beside  $f(x)g(x)$* ] which I don't like but that's the same as there [*Points to  $f(x)g(x)$* ]. This [*Points to  $f \circ g(x)$* ] is composition.
- S Yeah that's where I get lost .. that wee little symbol in there and I just look at it and automatically just think multiplication.
- I What would happen if I wrote that [*Writes  $f(x-1)$* ]? How would you see that?
- S Hm. [*Pause*] OK, my first instinct is to see it as this expression [*Points to  $f(x)$* ] times  $x-1$ . Then my second [*Laughs*] - what would be my second - instinct would be ...  $f$  of  $x$ , which is this expression [*Points to  $x^2 + 9x + 20$* ], minus 1.
- I I have a question here for you.
- S OK.
- I With this one [*Points to  $f(4)$* ] you had no problem with substituting 4 and just putting it in for the  $x$ .
- S Right, I had no problem with that.
- I There is a connection between the 4 and the  $x-1$
- S OK so instead of substituting the  $x$  for 4 you substitute for  $x$  minus 1. So it would be  $x$  minus 1 squared plus 9 times  $x$  minus 1 plus 20. See, it's just - if there's a different -see, if there's something else - if there's more than one number in that bracket that just throws me off. If there's just one number, 4 or 9 or 3 in the bracket I'm fine with that. But as soon as you give me another equation or expression then that just - I look at it - OK, see you later! - and I get nervous and ...
- I And you try things ...
- S Yes.
- I So, your first thought was multiplication.
- S Yeah.

- I How do you read that by the way? To yourself? What words do you use? I know what I say, but ..
- S I know I'm supposed to say  $f$  of  $x$  but I don't know what that means. That means nothing to me at all.
- I OK so when you read it in to yourself you just say  $f x$ ?
- S I say  $f$  of  $x$  but in confusion because I don't know what I'm saying to myself.
- I OK, so the words are OK but you don't know what they mean.
- S Yeah, right, I look at that and say - well - so every time you put  $f$  and something different between the brackets, that's what you substitute  $x$  for?
- I Hm.
- S Yeah, and I don't know. That's what I told myself, but then when I see things like this [*Points to  $f(g(x))$* ] I think OK, now what - because I know that I'm supposed to do something with this and with this [*Points to  $f(x)$  and  $g(x)$* ], but I don't know what that is.
- I *Unintelligible*
- S Exactly, so I just try all different combinations.
- I So Your first thought was to take the expressions and multiply.
- S Right.
- I I can certainly see where the multiplication idea comes from.
- Unintelligible with laughter.*
- I Well, if I wrote [*Writes  $ax$* ]  $ax$  there's multiplication. [*Puts in the brackets  $a(x)$* ]
- S Yeah. Same thing.
- I It's multiplication. But that [*Points to  $a(x)$* ] could also mean a function.
- S Yeah right.
- I And you've got that ambiguity in the notation. So that's my guess of where the multiplication comes from.
- S *Unintelligible.*
- I And you're right it does look like it. [*Crosses out the  $a(x)$* ] But it's not, unfortunately [*Laughs*] Now, your next thought was to do, you said, to take that expression and then subtract 1 from it.
- S Yeah. Cos I'm thinking  $f$  of  $x$  - OK, there's the  $f$  of  $x$  which is this [*Points to  $f(x)$* ] and then I just subtract 1. But I know that's wrong, but that's my instinct to do that.
- I Well, you're doing things in sequence - OK, there's  $f$  of  $x$  [*Points to  $f(x)$* ], - there we go - minus 1.

- S Yeah.
- I That one I quite understand as well.
- S OK
- I So then the question is your last option. It wouldn't be your favourite one clearly, but you did tell me that you would take the  $x$  minus 1 and take it squared plus 9 times it plus 20.
- S Right.
- I Can you tell me - can you identify where that idea came from?
- S By looking at this [*Points to  $f(4)$* ]
- I So it was my pointing up to the 4?
- S Yes.
- I So how are the 4 and the  $x-1$  related then, that lets you ..? Because you made the connection. What is the relationship?
- S Just I, I just saw the 4 here in brackets here and that means to substitute in. So this is in brackets so I figure I do the same thing.
- I So you substitute  $x-1$  as, as a number.
- S Yeah, right, yes.
- I *Unintelligible*. Which is in fact what you are supposed to do.
- S OK
- I But that comes down the list of things you try.
- S Oh yeah, yeah. [*Laughs*] It takes me a while. And sometimes I never get it. Sometimes I'm looking at it. And then I end up giving up and say OK, I can't do that.
- I Yes, well when you've got two possible passes before you get to the right one ..
- S Yeah, exactly.
- I All right now, having said what we've said let's come back to this one. [*Points to  $f(g(x))$* ]
- S OK. [*Laughs*]
- I We've established it's not multiplication. So ..
- S Yeah OK. I think I get it. I think you take  $g$  of  $x$ , which I'm looking at as one number like I did with the  $x-1$  and the 4.  $g$  of  $x$  is  $x$  plus 5. So I take, it's like looking at - if it said  $f$ ,  $x$  plus 5 in brackets [*Pointing with pen*]
- I Why don't you write it down?
- S OK. It's replacing the  $g$  of  $x$ , and then I write the equation out  $x$  plus 5 squared plus 9 times  $x$  plus 5 plus 20. [*Writes  $x + 5^2 + 9(x + 5) + 20$* ]

- I Now there is a problem with what you wrote - not what you said. What would your next line be? Let's just see if it's a problem with notation or what.
- S OK My next line would be - oh, OK, I see what I did. I forgot to put brackets in. Just a little mechanical ...
- I Yes, I thought that was it, but I just wanted to clarify. [*Points to  $f \circ g(x)$* ] And this one is just a notation for that.
- S So they mean the same thing.
- I Yes you are quite right that those two mean the same thing, but they don't mean multiplication.
- S Yeah, what I don't understand is why, if they mean the same thing, why do they need two different ways.

*Brief digression to discuss notation*

Handwritten work on lined paper:

$$f(4) = 6 \quad g(7) = 4$$

$$\text{find } f \circ g(7) = f(g(7)) = f(4) = 6$$

$$g^{-1}(4)$$

There are also some scribbles and a circled '4' in the work.

Figure A4. Interview example 2: Function composition question.

- I Let's go to this one. This is one of the test questions. And I gave you this information.  
[*Writes out the problem.*]
- S I find this easier now, so just do the first one first? OK I would just rewrite this [*Writes  $f(g(7))$* ] [*Pause, points with pen at top*] I don't know. There's too many numbers here, there's the 4 and the 6 and the 7 and the 4.
- I Why are you looking up here? It's not wrong, I'm just wondering what you're looking for?

- S Because - an equation or something that I can - that I can do something with. I mean, I've looked at it, and I just don't see anything to do. Like the last one there was an equation and an expression that I could do something with. But this - I don't know. This just completely defeats me. *[Laughs]*
- I In this, you want to be able to do something? *[Points to  $f(g(7))$ ]* When you say do something with, do you mean add, subtract, multiply divide sort of thing?
- S Or like with a value, substituting values. That's my first thought.
- I Right now, what would you be looking to substitute in here? What might you do? Obviously you don't know, but could you find something to substitute?
- S Well, I look at  $g$  of 7 and I see it's 4. So, oh, OK so this is just the same thing. *[Writes  $f(4)$ ]* I just do that and then  $f$  of 4 is 6.
- I There you go.
- S And that's it, that's the answer?
- I That's it.
- S Oh, OK. A lot of the time I think there's more expected and I just don't think it would be that .. I just read too much into it.
- I Yeah, *Unintelligible* and then you're not sure if you're looking at the right thing.
- S And math makes me nervous.
- I Yeah.
- S You want me to do the next one?
- I Sure.
- S So that's the inverse *[Pause]* I don't know. I just draw the line with one like that.
- I OK. When I said, the first phrasing I used the function does, the inverse undoes. So in those terms  $g$  does to 7 and the answer is 4.  $g$  inverse will undo so you start with 4 and you undo, so you get back
- S 7, and then that's the answer.
- I That's it. You remember me saying about does and undoes, and inverses. So does that sort of make sense about doing something to numbers and you get an answer. And when you undo it you start with the answer and get back.
- S I don't think I'm really doing anything with functions. I think that's why. And, when you say that, I'm thinking, I'm doing what? You haven't really done anything.
- I Well, you know, how  $g$  gets from 7 to 4 is not explained here.

- S Right, I think that's what bothers me. [*Laughs*] I don't see an equation. If you wrote an equation for me and that was what it was then I could say Oh, OK. If it was logarithms or something I could say - I understood that perfectly. I understood that more than I've understood anything in this class. Which surprises me, but they don't bother me a bit. But things like this *Unintelligible* 4, OK, and its 7? Well, OK, if you say so. I just don't understand, why.
- I You like to have something you can do. None of this seeing a pattern?
- S Yeah, yeah.
- I OK. What about this. [*Writes  $f \circ f^{-1}(5)$* ]. Now with this one there is nothing to do.
- S Right
- I It's a matter of figuring out what's going on with the function. If you knew what to do you would figure out  $f$  of 5. If you had an expression you could work out what  $f$  of 5 was.
- S OK
- I And you would get an answer
- S Yes
- I You then put that answer into  $f$  inverse.
- S Yeah.
- I That's the sequencing from the composition. So then what, what's your answer going to be?
- S It would be 5 wouldn't it, because that's what you started with.
- I Mm hm.
- S So that's the answer to that. OK, see that makes sense. That makes sense when you explain it that way.
- I So for you it sounds to me like when you're not given an expression for  $f$  or whatever the function is called, and you need to think through something like this. You really need to think of it in terms of: if I knew how to calculate, this is what I would start with and there's my answer. What happens to my answer? So you have to really think of it in terms of calculation. Even if you don't know exactly what the calculation is you know something about what the answer is.
- S Yeah that .. *Unintelligible*
- I I have another question for you. Remember on the pretest, and actually it's come up since then. Suppose we had an equation like this. [*Writes  $2x - 4 = 2(x-2)$* ], and I asked you to solve for  $x$ . So, you say you like solving for things.

$$\begin{array}{l}
 2x - 4 = 2(x - 2) \\
 2x - 4 = 2x - 4 \\
 2x - 2x = -4 + 4 \\
 0 = 0
 \end{array}$$

Figure A5. Interview example 2: Work on an identity equation.

S [Writes down to  $0 = 0$ ]

I OK, now, you are solving an equation so that's not the answer.

S OK [tone of voice says OK, if you say so]

[Both laugh]

I You have reduced the equation ...

S Right.

I To its simplest form  $0 = 0$ . And that's definitely equivalent to that one [Points to the original], but it doesn't ... See, if you are solving an equation what are you doing?

S Finding a value for  $x$ .

I You're finding a value for  $x$ .

S Right.

I So the equation comes down to being just  $0$  equals  $0$ . So what does that tell you about  $x$ ?

S It can be any real number?

I Yeah. Is that a comfortable idea for you?

S Not especially. Because I really don't care what  $x$  could be I want to know what it is. And all the domain and the range and all that, I don't like that idea at all. Just because I can't see it. I've got to see something to understand it.

I You want  $x$  to be able to be identified as a particular number?

S Yeah.

I Not just representative of a set?

S Yeah. Or it could be any positive number, that bothers me. I understand it but I don't ...

I It doesn't feel comfortable?

S Yeah, it doesn't feel right.

- I So essentially what you're saying is that your basic understanding of the variables is as numbers. And that's how you like to be able to see them. You want to say, well, all right I'm using  $x$  but I could always use a specific number.
- S Yeah, exactly, and that's comfortable.
- I And so the idea of  $x$  as possibly representing a general set of numbers...
- S Yeah. It makes me just want to go and pick out every, the equation for every ... for every value.
- I Which is a little bit of a problem! [Laughs]
- S Or I don't really believe that it works. [Laughs]
- I Yes, it would be, because if you have one of these infinite sets ...
- S I only have 2 hours to write it down is the problem!
- I 2 years wouldn't do it! ..
- [Both laugh]

Handwritten work on lined paper showing the derivation of the difference quotient for the function  $f(x) = x^2 - 3x + 2$ .

The work includes the formula for the difference quotient:

$$\frac{f(x+h) - f(x)}{h}$$

and the function definition:

$$f(x) = x^2 - 3x + 2$$

The student then expands the numerator:

$$(x^2 - 3x + 2) - (x^2 - 3x + 2)$$

and further expands the terms:

$$(x+h)^2 - 3(x+h) + 2 - (x^2 - 3x + 2)$$

Figure A6. Interview example 2: Work on difference quotient.

- I OK, lets slip back still to functions. [Puts difference quotient problem down.]
- S I hate this stuff. It's just ...I mean when I start doing it and i get the hang of it I'm fine, but if out of the blue you just put it in front of me, then I just ... Oh my God ...
- I Mm hm, so you start with this sort of ..
- S Yeah. And that's terrible because then it's so negative.

I Yeah, but it's also very common.

[Unintelligible exchange.]

I You have this kind of response. I know this. OK, so, ... so you've panicked. [Both laugh]

S Yeah. But then, but then because we've been working on functions already... So, but if this were just to show up after working on  $x$  and  $y$  intercepts and everything that would throw me, but .. that's OK. So, do you want me to write it out?

I Sure.

S OK. So, OK,  $f$  of  $x$  plus  $h$  is .. See, I look at it and I think OK I know that something is going with something, but I don't know which way it goes. Does  $x$  plus  $h$  have it in for all the values of  $x$  over here, [Points to  $f(x)$ ] or do you put .. you put  $x$  plus  $h$  squared minus  $3x$  plus  $2$ , or ... See I'm confused, I don't know which way to go. So my first instinct would be to [Writes  $(x^2 - 3x + 2) - {}$ ]. That doesn't make any sense. I don't know, I'm lost [Scribbles out the last bracket]. ..

I OK. In fact this is an application of what we were doing over here. [Points to first sheets involving composition and evaluation] But of course, instead of being a single one, ...

S Right, there's more. We've got two of them and it's all in an expression ..

I Right lets leave that for a moment.

S OK.

I One of the things I've suggested that students do is this: Instead of having the  $x$  we change it to input. [Writes  $f(\text{input}) = \text{input}^2 - 3 \cdot \text{input} + 2$ ]. Remember that?

S Yeah, right.

I We change the  $x$  to input. Does that help?

S [Pause] Not really. What is the input? Is it  $x+h$ , or is it  $x$ , or is it just  $h$  on its own? I just see these different, separate numbers and I don't know which one to put in.

I OK. Yes, now ...[Pauses and murmurs. Covers all in the quotient except  $f(x+h)$ ]  $f$  of  $x$  plus  $h$  that's fine. Could you do that?

S Just that one alone? Yes, [unintelligible] [Writes  $(x + h)^2 - 3(x + h) + 2$ ].

I So what about  $f$  of  $x$  then? You said it was just that ..

S Oh, OK. That would just be, just be minus  $x$  squared minus  $3x$  plus  $2$ . [Completes correctly]

I Mm hm

S Over  $h$ . And then when you multiply it all out and all the  $h$ 's cancel themselves out. OK. I like it when everything cancels out and then I know that that's my .. OK, OK. That makes sense to me now. I was trying to put this in there [Points to  $f(x)$  first and then difference

*quotient second*] That 's exactly what I'm doing but in a different way .. [*Laughs*] I don't know what I'm saying ..

- I It sounds a little bit, .. part of it is that you're trying to put this into this [*Points from function to quotient*]
- S Yeah but, I think, separately. I wasn't thinking of this [ $f(x)$ ] as a whole, you know just like a ... or this [*Points to difference quotient*] I was looking at this [*difference quotient*] as is one number the whole thing together.
- I All of it? and trying to put that with that . [*Points from difference quotient to function*]. All in one?
- S And just squish it in. Yeah, exactly. And that's what I always did. Once in a while I'd get it right. You can tell when you're getting it right because other things cancel out.
- I There are some functions that I think - I'm trying to get a handle on just what it is that people go wrong .. Part of it is that some functions, with the wrong thinking can actually - you came every close to it with what you said once - that I know is a mistake. With some functions it's a mistake that doesn't happen because the particular place it would happen isn't there. What you said was that you thought you might do [*Writes  $(x+h)^2 - 3x + 2$* ]  $x$  plus  $h$  squared minus  $3x$  plus  $2$ .
- S Right, because I'm looking at that and put that in there [*Points first to  $f(x+h)$ , then at  $f(x)$* ], put that one in there [*Points at  $f(x)$  (or just  $x$ ) in difference quotient, then at  $-3x$* ] and then there's nothing left over for the  $2$  so it's left on its own. And that's what I would do all the time.
- I Yes, and then you see what happens, and there's quite a few examples in the book, if you do a function like this,  $9x$  squared minus  $1$ . You put your  $x$  plus  $h$  squared and then minus  $1$ .
- S Right.
- I It's right, it turns out to be right. So the place for the mistake to happen is not there. Which is why the questions I ask always have that - because I want to catch that.

Interview concluded with some informal discussion.

## APPENDIX B

### Pilot Test 1

There are two forms of the first pilot test because it was tried both at the beginning of, and during the term. Students wrote the two tests in a randomly chosen sequence. Questions 2, 3, 4, 5, and 9 are based on Küchemann's (1981) report. In one version of the test, the original question is used and in the other an attempt has been made to match it. Each test consists of a mix of original and matching questions. Question 1 is taken from research by Kaput and Sims-Knight (1983).

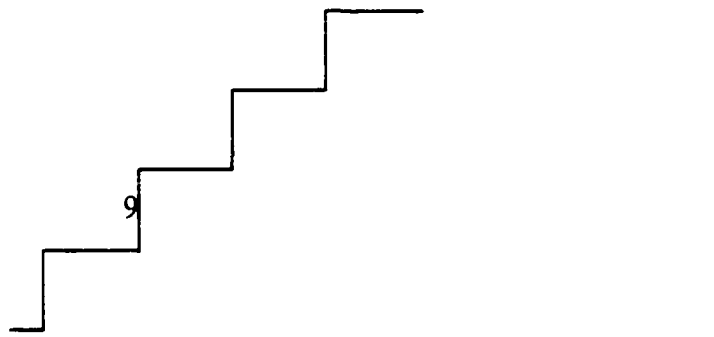
Variables Test Version 1

1. At Vallapart motors the equation  $5B = 4R$  describes the relationship which exists between  $B$ , the number of blue cars produced and  $R$  the number of red cars produced.

Next to each of the following statements place a T if the statement follows from the equation, an F if the statement contradicts the equation, and a U if there is no certain connection.

- \_\_\_ a. There are 5 blue cars produced for every 4 red cars
- \_\_\_ b. The ratio of red to blue cars is 5 to 4.
- \_\_\_ c. More blue cars are produced than red cars.
- \_\_\_ d. The same number of blue and red cars are produced, but the blue cars are bigger.
- \_\_\_ e.  $5B$  is larger than  $4R$ .
- \_\_\_ f. The equation  $\frac{B}{R} = \frac{4}{5}$  expresses the same relationship between the number of blue cars and the number of red cars as the original equation.

2. What is the height of the staircase below? Part of the staircase is not drawn. There are  $n$  steps altogether, each of distance 9.



3. Which is larger,  $2n$  or  $n+2$ ? Explain or show how you arrived at your answer.

4. 
$$2a + b + c = 2a + m + c$$

Is this statement true? Always / Never / sometimes when \_\_\_\_\_

5. 
$$3x = 3y$$

Is this statement true? Always / Never / sometimes when \_\_\_\_\_

6. Explain what the mathematical symbols in  $3a + 2b$  mean.

7. Simplify  $5 - 2b$ . Explain your answer.

8. Simplify  $\frac{12a}{-6a}$ . Explain your answer.

9. The area of a rectangle is given by the formula *length x width*. Suppose you are given the following problem.

In a rectangle the ratio of length to width is 19 : 5. If the area is 583.751 square metres, what are the dimensions of the rectangle?

You need only do the first step in solving such a problem, which is to set up the diagram below. Since you don't know the dimensions you will have to use a letter or letters.



10. As far as possible match the following verbal descriptions and equations. If a description would be true for the equation write the letter of the description beside the equation.

Descriptions:

A as  $r$  increases  $s$  increases

E  $r$  is never negative

B as  $r$  decreases  $s$  increases

F  $s$  is never negative

C  $r$  and  $s$  remain constant

G  $t$  is never negative

D when  $r$  is 1  $s$  is 3

Equations	Description letters
$r + s = 13$	
$r = 3s$	
$r + s = t$	

Variables Test Version 2

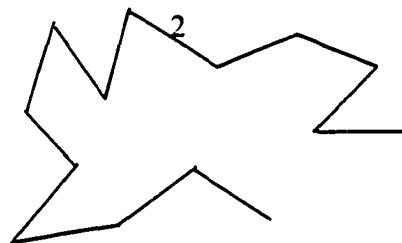
1. At Fasto Coffee Bar the equation  $5N = 4C$  describes the relationship which exists between  $N$ , the number of Nanaimo bars produced and  $C$  the number of coffee cakes produced.

Next to each of the following statements place a T if the statement follows from the equation, an F if the statement contradicts the equation, and a U if there is no certain connection.

- \_\_\_ a. There are 5 Nanaimo bars produced for every 4 coffee cakes
- \_\_\_ b. The ratio of Nanaimo bars to coffee cakes is 5 to 4.
- \_\_\_ c. More Nanaimo bars are produced than coffee cakes.
- \_\_\_ d. The same number of Nanaimo bars and coffee cakes are produced, but the Nanaimo bars are bigger.
- \_\_\_ e.  $5N$  is larger than  $4C$ .
- \_\_\_ f. The equation  $\frac{N}{C} = \frac{4}{5}$  expresses the same relationship between the number of Nanaimo bars and the number of coffee cakes as the original equation.

2. What is the perimeter (distance around) the shape below? Part of the shape is not drawn.

There are  $n$  sides altogether, each of length 2.



3. Which is larger,  $2n$  or  $n^2$ ? Explain or show how you arrived at your answer.

4  $3a = 3b$

Is this statement true? Always / Never / Sometimes when \_\_\_\_\_

5.  $x + y + z = x + p + z$

Is this statement true? Always / Never / sometimes when \_\_\_\_\_

6. Explain what the mathematical symbols in  $5a + 3b$  mean.

7. Simplify  $6-4b$ . Explain your answer.

8. Simplify  $\frac{15b}{-3b}$ . Explain your answer.

9. The area of a rectangle is given by the formula *length x width*. Suppose you are given the following problem.

In a rectangle the ratio of length to width is 17 : 3. If the area is 369.298 square metres what are the dimensions of the rectangle?

You need only do the first step in solving such a problem, which is to set up the diagram below.

Since you don't know the dimensions you will have to use a letter or letters.



10. As far as possible match the following verbal descriptions and equations. If a description would be true for the equation write the letter of the description beside the equation.

Descriptions:

A as  $a$  increases  $b$  increases

E  $a$  is never negative

B as  $a$  decreases  $b$  increases

F  $b$  is never negative

C  $a$  and  $b$  remain constant

G  $c$  is never negative

D when  $a$  is 1  $b$  is 5

Equations	Description letters
$a + b = 15$	
$a = 5b$	
$a + b = c$	

Pilot Test 2

1 Simplify  $\frac{12a}{-6a}$ . Explain and justify each step you took to arrive at your answer.

2. **The story.** A teacher wanted a collection of numbers with factors that differed by 3. He started checking numbers and found three examples: 4 (with factors 4 and 1), 10 (with factors 5 and 2) and 28 with factors 4 and 7. Checking numbers seemed to be very slow so he went and asked his mathematical girlfriend if there was a shorter way to find more such numbers. She gave him the following response..

"If  $x$  is a number with factors that differ by three, then  $x$  can be found by many different formulas based on integer values of  $n$ . For example,

*formula 1:*  $x = n(n + 3)$       *formula 2:*  $x = n(n - 3)$       *formula 3:*  $x = (n-1)(n+2)$

The teacher was confused so he went and asked three friends about the response.

Fred said "Surely there's more than three numbers like that!"

Ted said "The formulas can't all be right!"

Ned said "She's right!"

**The question.** Which one of Fred, Ted, Ned, or none of them is right? Justify or explain your choice.

3. As far as possible connect the following verbal descriptions and equations. If a description would be true for an equation write the letter of the description beside the equation. You can have more than one letter with each equation.

Descriptions:

A As  $a$  increases  $b$  increases

B As  $a$  decreases  $b$  increases

C  $a$  and  $b$  remain constant

D When  $a$  is 1  $b$  is 5

E  $a$  is never negative

F  $b$  is never negative

G None of the above descriptions. (Explain)

Equation	Description letters
$a + b = 15$	
$a = 5b$	
$a + b = c$	

## APPENDIX C

Withdrawal of Consent Form from Front Page of Written Test**Withdrawal of consent to allow access to written course work.****Complete only if you do NOT wish to participate in Margaret Wyeth's research**

1996

You should know that your responses to written questions may form part of the data for the research being conducted by Margaret Wyeth into student understanding of mathematical variables, graphs and functions. All information will be kept confidential and stored in a locked room. Your anonymity will be protected since no names will be attached to any published results and stored data will be identified by code numbers alone. If you do NOT consent to allow your responses to form part of M. Wyeth's research data you should fill out the form below. Giving or withholding consent will not affect your grade or standing in the class in any way.

I, [Print name] \_\_\_\_\_, Student number \_\_\_\_\_,  
do NOT consent to have my written work used as part of the data for research being  
conducted by Margaret Wyeth.

Signed  
\_\_\_\_\_

## APPENDIX D

First pages of Introductory Notes to Accompany the Intervention Lecture Content**MATH 120 INTRODUCTORY NOTES**

These notes and exercises are intended to supplement your text. The exercises are intended to make you think about the meaning of what you write, familiarize you with my terminology, and to prepare you for mathematics coming later in the term.

**SECTION 1****Names for Numbers and Expressions as Numbers****1.1 Numerical background**

A number like 18 can be written in many different ways: e.g.  $4+14$ ,  $218-200$ ,  $15.4+2.6$ ,

$$3 \times 6, 324^{1/2}, 2 \times 3^2, 4^2 + 3 - 1, 18 + 3456789 - 3456789, (27-9) \cdot \frac{(347^5 - 347^4)}{347^4 (346)}$$

A common response among students at this point is that  $4+14$  is an addition,  $3 \times 6$  is a multiplication, etc.. These numerical expressions have two interpretations: as calculations and as single numbers. Which interpretation we use depends on the context. Thus 18 can be written as an addition, subtraction, multiplication, division, square, square root, etc., or indeed any combination of numbers and arithmetic operations that calculates out to 18.

We could say the same thing in the language of algebra by saying that 18 can be written in the form  $a+b$ ,  $a-b$ ,  $axb$ ,  $a \div b$ ,  $a^2$ ,  $a^{1/2}$ , . . . , or more complex algebraic expressions. Using the dual interpretation idea we can consider  $a+b$  as both an addition and a number.

**Example 1.1 - 1** Writing numbers in different forms

Write the number 35 in progressively more complicated ways.

	<u>Verbal description</u>
$35 = 5 \times 7$	multiplication of 5 and 7
$= (2 + 3) \times (12 - 5)$	multiplication of the addition of 2 and 3, and the difference between 12 and 5
$= (\sqrt{4} - (-\sqrt{9})) \times ((29-25+8) - (45/9))$	multiplication of the result of subtracting the negative square root of 9 from the positive square root of 4, and the result of subtracting the division of 9 into 45 from the result of subtracting 25 from 29 and adding 8.

We can see how using words to describe what is being done becomes horrible and even ambiguous very quickly, whereas the mathematical notation remains clear. Nevertheless, the words are very important in connecting our thoughts and plans to the abstract mathematics.

The middle number expression,  $(2 + 3) \times (12 - 5)$ , is a simple example of how our interpretation of the numbers must vary depending on context. Consider  $2 + 3$ . In the larger expression  $(2 + 3)$  represents a number. It is one of two factors, the other being  $(12 - 5)$ . It is typical that expressions are enclosed in brackets when they are used to represent numbers. However, when we compute or simplify we focus on the computation interpretation and perform the addition  $2 + 3$  by recall of facts or possibly by counting on fingers! Here we are interpreting  $2 + 3$  as a mathematical action or procedure.

- ◆ Numbers can be written in many different ways, provided the different expressions ultimately represent the same number.
- ◆ Numerical expressions can be treated as single numbers (that is, as the numbers they represent), or as mathematical actions (Usually called calculations or computations).
- ◆ A number representation is composed of other numbers.

Example 1.1 - 2. Compute  $75 - (5 - 23)$

In multi-step numerical computations we switch between the number and the procedure interpretations without being particularly conscious of it.

Following the BEDMAS rule for the order of operations we know to compute the bracketed expression first. What we are actually doing is performing the mathematical action  $5 - 23$  in order to find the number to be subtracted from 75. Implicit in this is the recognition that  $(5 - 23)$  represents a number and that this number is to be subtracted from 75.

To make sense of algebra, the distinction between procedure and number is important. Many people adjust their interpretation unconsciously: an analogy from reading is knowing from the context whether to pronounce *read* like *reed* or like *red*.

The students I see in Math 120 typically focus on the procedure (action or computation) interpretation and have difficulty understanding algebra when the expression as number interpretation is needed.

## 1.2 Algebra

In this section what was shown in 1.1 for specific numbers is extended to the general case involving variables and algebra expressions. A non-specific number  $x$  can be written in many different ways:

Indeed,  $x$  can be any combination of numbers and arithmetic operations that :

(i) we are told equals  $x$ :  $x = a + b$

or (ii) we can show simplifies to  $x$ :  $x = (3x - 2x) \cdot \frac{(347^5 - 347^4)}{347^4(346)}$

Example 1.2 - 1 Different ways of writing a number  $x$ 

$$\begin{aligned}
 x &= p \cdot q && \text{We are told or given that } x = pq \\
 &= (a + b) \cdot (c - d) && \text{Given that } p = a+b \text{ and } q = c-d \text{ then } x \text{ can} \\
 &&& \text{be rewritten as the product of a sum and a} \\
 &&& \text{difference.} \\
 &= \left( \left( a^3 \right)^{\frac{1}{3}} + \left( b^{\frac{1}{5}} \right)^5 \right) \cdot [-(2c-3c) - d(\sin 90^\circ)] && a, b, c, d \text{ rewritten.} \\
 &&& \text{etc.}
 \end{aligned}$$

Example 1.2 - 2 Finding different numbers in the number  $((m^2 + n - 1) - (t/v))$ 

Single symbols:  $m, n, -1, t, v$

Combinations:  $m^2, 1/v, t/v, n-1, m^2 + n, m^2 + n - 1, m^2 - 1, m^2 - t/v, n - 1 - t/v, -1 - t/v$ , etc.

If we rewrite the original number then we can find more,

$$\text{e.g. } \frac{v(m^2 + n - 1) - t}{v} \text{ results in } vm^2, vn, \text{ etc.}$$

Notes

- When we treat algebraic expressions as single numbers, the signal that this is what we are doing is usually that the expression has brackets (or something like a radical that implies brackets) around it.
- We calculate in order to check that a numerical expression represents a given number. For the general case (algebra) we use substitution and simplification. If substitution is used there must be somewhere a statement of equality that allows the substitution. This statement can be worked out or recalled from memory; e.g., replacing  $3a \times 4a = 12a^2$ . Alternatively, the substitution statement can be given; e.g., if  $3a - 1 = x$ , then  $2x - 5 = 2(3a - 1) - 5$ .
- A number representation is composed of other numbers. We can find other numbers within a representation. If a particular number is not used, we can change the representation so that it is used. This happens quite frequently in algebra manipulations.

Do not forget that variables and expressions represent numbers and therefore number rules apply. Although they may end up looking a little different on the paper, basic algebra manipulations are not different from basic number rules.

**Exercises for 1.1 & 1.2** Numbers, expressions and variables, and number representation.

Perform rewriting exercises like those in 1.1 & 1.2

## 1. Different ways of writing 35

$$\begin{aligned}
 35 &= && 5 && \cdot && 7 \\
 &= && (2 + 3) && \cdot && (12 - 5) \\
 &= && (\sqrt{4} - (-\sqrt{9})) && \cdot && ((29-25+8) - (45/9)) \text{ etc.}
 \end{aligned}$$

a. Continue with at least two more rewritings of these numbers. Then use your calculator to check that the mess you have in front of you still represents 35.

b. Write 35 so that it:

(i) is a product of a division and a subtraction, (ii) includes the number 487,

(iii) includes the number  $(333 - 222)$ , (iv) includes the number  $(\frac{5^{-36}}{36^{-5}})$ .

c. Take any of the expressions representing 35 and try to identify all of the numbers used in it.

## 2. Rewriting a variable

a. Write  $x$  so that it (i) is a product of an addition and a subtraction, (ii) includes the number 487, (iii) includes the number  $(333 - 222)$ , (iv) includes the number  $(\frac{5^{-36}}{36^{-5}})$

b. Take any of the expressions representing  $x$  and try to identify the numbers used in it.

---

**SECTION 2****Common Problem Areas: "-", brackets, canceling.**

The intention in this section is to look at places in algebra where students frequently make mistakes. By returning to the numerical basis for algebra and relating back to students' number knowledge I hope to provide a basis for understanding why these errors are wrong and give students a reason for the rule so that such mistakes can be avoided in the future.

**SECTION 2.1****Meanings for "-"**

The symbol "-" has two different meanings: (a) the operation *subtract* and (b) the description *negative or opposite*. Unfortunately the word *minus* is used for both meanings!

**Example 2.1 - 1.**

5 - (-4) can be said as: *five subtract negative four*    *five minus negative four*  
*five subtract minus four*    *five minus minus four*

The specific meaning of "-" can only be inferred from the context.

["+ " also has two meanings, "plus" or "add", and "positive", but this duplication does not seem to cause as many errors as the duplication of meaning for "-".]

Students are frequently taught the rule: *two like signs make a positive, two unlike signs make a negative*. This is a good rule as long as it is remembered that the rule works differently for addition and subtraction than it does for multiplication and division.

In addition and subtraction the two like signs are in fact an operation and a number type:

*add a positive*                      or                      *subtract a negative.*

In multiplication and division two like signs means that the numbers being multiplied or divided are either both positive or both negative.

It is not necessary to change the words you use for "+" and "-". Just be sure you are clear about the distinction between the operation and the number type.

**Example 2.1 - 2**

Compare         $-4 - 7$                        $-4 - (-7)$                        $(-4) \times (-7)$

$-4 - 7$  This means negative four subtract seven. The solution is  $-11$ .

$-4 - (-7)$  This means negative four subtract negative seven. Subtracting negative seven is the same as adding seven, so this becomes negative four add seven ( $-4 + 7$ ). The solution is 3.

$(-4) \times (-7)$  This means negative four times negative seven. The solution is 28.

---

**Exercises 2.1**

Try to avoid doing these exercises mechanically. Where possible think about the why the computations work as they do, and relate your work to the number line.

1. Explain why  $-4-5$  is  $-9$  not  $+9$ . Demonstrate on the number line.
2. Explain why  $a - b$  is the negative of  $b - a$ .
3. How do you recognize the negative of a number if the number is an expression?
4. Only a few numbers are represented in the following list. Find them without using your calculator. This is not an exercise in hitting buttons! Many common errors related to notation are represented here. [For a review of exponents and radicals see your course text ]

$26 - 17$	$-3 \times 3$	$\frac{27}{-3}$	$-1 \times -9$
$-3 + 6$	$-54/6$	$(-1)(-3 - 6)$	$\frac{27 - 3}{9 - 1}$
$\sqrt{81}$	$[(-3)(-27)]^{1/2}$	$(-2 - 5 - 6 + 5 + 7 - 6 - 2)$	$[(-3)(5) - (2)(-3)]$
$[(-3)(-5) - (-3)(-2)]$	$(27) \div (-3) \div (-3)$	$18 \div ((-2)(-3))$	$-[-6+3-(-9)+(-\sqrt{9})]$
$((-36)(-\frac{1}{4}) + (3)(-12+9))$	$(-3)^2$	$-3^2$	$-\sqrt{81}$
$9 \cdot \frac{x - y}{y - x}$		$(x^2 - 2x + 1) \frac{3}{(x - 1)^2}$	
$\frac{9a - 9b}{a + b} \cdot \frac{(a + b)^2}{a^2 - b^2}$		$(x^2 - 2x + 1) \frac{3^2}{(1 - x)^2}$	
$(x^2 - 2x + 1) \frac{3^2}{(1 - x)^2}$		$\frac{3(a^2 - b^2)c^{-2}}{3^{-1}(a + b)} \cdot \frac{c^2a + bc^2}{(a - b)(a + b)}$	
$a^{-2}b^3a^4b(3a^{-1}b^{-2})^2$		$-a^2(-b)^3a^4(-b)(3a^{-1}b^{-2})^2$	

**Section 2.2 Distributive Principle****2.2.1 Numerical background**

It will be useful in this section to think of multiplication as "groups of", e.g.  $5 \times 6$  is 5 groups of 6, or if it is more convenient, as 6 groups of 5 since  $5 \times 6$  and  $6 \times 5$  represent the same quantity.

Children in elementary school are often taught to go from known facts to find unknown multiplications based on groups. For example, if you know your multiplications for 2, 3, 4, and 5, you can use these to work out  $7 \times 8$  by interpreting the 7 as  $(2+5)$  and finding  $2 \times 8 + 5 \times 8$ .

[7 groups of 8 is the same as 2 groups of 8 added to (or joined with) 5 groups of 8 ]

We use this idea of "groups of" a lot in basic computations.

e.g. instead of  $9 \times 6$  we can think of 9 as the subtraction  $(10-1)$ , and hence we get 10 groups of 6 subtract 1 group of 6. [Say this out loud - it works better!]

The procedure works the other way too. We can find  $4 \times 27 + 6 \times 27$  quickly if we think of it as 4 groups of 27 plus 6 groups of 27, making 10 groups of 27 or 270.

$$\text{Thus: } 4 \times 27 + 6 \times 27 = (4+6) \times 27 = 10 \times 27.$$

### 2.2.2 Algebra

The formal name for the manipulations described in the previous section is the

**Distributive Principle.** The general forms are

$$a(b+c) = ab + ac \quad \text{or} \quad (b+c)a = ba + ca \quad \text{and} \quad a(b-c) = ab - ac \quad \text{or} \quad (b-c)a = ba - ca.$$

If we remember that the letters can represent negative numbers as well as positive, that subtraction is the same as adding the negative or opposite of a number and that multiplication is commutative (order does not matter), then we can state the distributive principle with a single equation:

$$a(b+c) = ab + ac \quad \text{the other forms listed above are basic algebra variants of this equation.}$$

We use the Distributive a lot in algebra. Common descriptions are *multiplying out the bracket*, or *taking out a common factor*.

#### Examples 2.2 - 1 Using the Distributive Principle in Algebra

- ◆ This example uses  $(y - 2)$  as a single number along with  $b$  and  $c$ .

$$\begin{aligned} (y - 2)(b + c) &= (y - 2)b + (y - 2)c && \text{Uses } (y - 2) \text{ as a single number, multiplied by } b \text{ and then by } c. \\ &= yb - 2b + yc - 2c && \text{Uses single numbers } y, 2, b \text{ \& } c \text{ in the distributive procedure.} \end{aligned}$$

The previous example used the distributive to multiply out brackets. The next example uses the disdistributive to take out a common factor.

- ◆ This example uses  $(x - 2)$  as a single number along with  $b$ , and  $c$

$$(x - 2)b + (x - 2)c = (x - 2)(b+c) \quad \text{Uses } (x - 2) \text{ as a single number: the common factor in two numbers written as products.}$$

- ◆ This example uses  $(x - 2)$  as a single number along with  $(x+7)$  and  $x^5$

$$\begin{aligned} (x - 2)(x + 7) + (x - 2)x^5 &= (x - 2)[(x+7) + x^5] && (x - 2) \text{ is a common factor in two products:} \\ &&& \text{the product } (x - 2)(x + 7) \text{ and the product } (x - 2)x^5 \\ &= (x - 2)[x+7 + x^5] && \text{the numbers } (x+7) \text{ and } x^5 \text{ are combined to make the} \\ &&& \text{single number } [x+7 + x^5], \text{ which is multiplied with} \\ &&& \text{the number } (x - 2) \end{aligned}$$

- ◆ This example uses  $(x + 3)$  as a single number along with  $(x - 7)$  and  $y^5$ . Identify the procedures that use these numbers.

$$\begin{aligned}
 (x + 3 - y^5)(x - 7 + y^5) &= [(x + 3) - y^5][(x - 7) + y^5] && \text{Brackets } (x + 3) \text{ \& } (x - 7) \text{ for} \\
 &= [(x + 3) - y^5](x - 7) + [(x + 3) - y^5]y^5 && \text{use as single numbers.} \\
 &= (x+3)(x-7) - y^5(x - 7) + (x + 3)y^5 - y^5y^5 && \text{Uses } [(x + 3) - y^5] \text{ as a single} \\
 &\text{etc.} && \text{number, distributed over the} \\
 & && \text{sum } (x - 7) + y^5 \\
 & && \text{Uses } (x + 3), (x - 7) \text{ and } y^5 \text{ as} \\
 & && \text{single numbers. Once again} \\
 & && \text{the action is distributive.}
 \end{aligned}$$

If you are not sure how we got this far emphasize the use of expressions as single numbers by overwriting the expressions  $(x + 3)$ ,  $(x - 7)$  and  $y^5$  with the letters,  $A$ ,  $B$ , and  $C$  as follows:

$$\begin{aligned}
 &= (A - C)(B + C) \\
 &= (A - C)B + (A - C)C \\
 &= AB - CB + AC - CC
 \end{aligned}$$

Using a single letter is a useful technique if you tend to feel overwhelmed when there are lots of symbols. In formal terms what is being done is the **substitution**,  $A = (x + 3)$ ,  $B = (x - 7)$ , and  $C = y^5$ . If you use this method of substituting single letters for expressions be sure to remember to substitute back to the original variables!

Using a single letter is a useful technique if you tend to feel overwhelmed when there are lots of symbols. In formal terms what is being done is the **substitution**,  $A = (x + 3)$ ,  $B = (x - 7)$ , and  $C = y^5$ . If you use this method of substituting single letters for expressions be sure to remember to substitute back to the original variables!

### The FOIL method of multiplying and a better alternative

Many students are taught to multiply expressions with two terms by the FOIL method. This is actually just an application of the Distributive Principle.

FOIL stands for the order in which terms are multiplied and stands for First Outer, Inner, Last. The order does not actually matter and the main purpose of teaching FOIL is to make sure that all necessary steps are done. Unfortunately FOIL is limited to expressions with two terms.

### An Alternative to FOIL

An equally easy, but more flexible rule than FOIL is to **multiply every term in one bracket by every term in the other bracket**.

Based on the Distributive we can see how this works to produce the same result as FOIL.

$$\text{To multiply } (a + b)(c + d) = a(c+d) + b(c+d)$$

Think of  $(c+d)$  as a single number that is to be multiplied by the addition  $(a+b)$ .

$$= ac + ad + bc + bd. \quad \text{Rewrite using the distributive.}$$

Notice that this fits the rule of multiplying everything in the first bracket by everything in the second. We are multiplying  $a$  and  $b$  from the first bracket by the second bracket terms,  $(c+d)$ .

**Example:** Multiplying expressions with more terms than a binomial (i.e., more than two terms)

$$\begin{aligned} \text{Multiply } (3x - 4y + 5)(6x + 7y - 8) \\ = 18x^2 + 21xy - 24x - 24xy - 28y^2 + 32y + 30x + 35y - 40 \end{aligned}$$

The terms  $18x^2 + 21xy - 24x$  result from multiplying  $3x$  by every term in the second bracket  $(6x + 7y - 8)$ , the next three terms,  $-24xy - 28y^2 + 32y$  result from multiplying  $-4y$  by every term in  $(6x + 7y - 8)$  and the last three,  $+30x + 35y - 40$  result from multiplying  $+5$  by every term in  $(6x + 7y - 8)$ . Thus to multiply  $(3x - 4y + 5)$  by  $(6x + 7y - 8)$  we have multiplied everything in the first bracket by everything in the second.

$$= 18x^2 - 3xy + 6x - 28y^2 + 67y - 40 \quad \text{Finally, combine like terms.}$$

### Exercises 2.2.

1. Multiply out using the Distributive principle: (Specify which expressions you are treating as single numbers as you work.)

- i.  $(x + y + 5)(x + y)$       ii.  $(x + y - 4)(x - 1)$       iii.  $(x + y - 4)(x - y - 1)$  I  
 iv.  $(3 - x - x^2)(5 + x)$       v.  $(x - 2)(x^2 + x - 1)$       vi.  $(x^2 + 5x - 6)(x^2 - 3x - 4)$   
 vii.  $(1 - \sin x)(1 + \sin x)$       viii.  $(3\sin x - 2)(\sin x + 1)$

$\sin x$  is an abbreviated form of the trigonometric function *sine of the number*  $x$ . In grade 10 or 11 trigonometry you should have learned that there are six trigonometric functions (*sine*, *cosine*, *tangent*, *cosecant*, *secant* and *cotangent*), each assigning a special ratio to an angle: e.g.

$\sin 30^\circ = \frac{1}{2}$ ,  $\sin 90^\circ = 1$ ,  $\sin 16^\circ = .2756$ ,  $\cos 30^\circ = \frac{\sqrt{3}}{2}$ , etc.. When we do trigonometry we will use a more general definition for the functions and real numbers rather than angles. What you need for the moment is to know that  $\sin x$  represents a number in the same way that any algebra expression represents a number.

2. (a) Take out the common factor or factors: (Specify which expressions you are treating as single numbers as you work.)

- i.  $3(x + y) - b(x + y)$       ii.  $(x - 2)(x + 1) - (x - 2)5$       iii.  $(x - 2)8 + (x + 1)(x - 2) + (x - 2)$

(b)  $3 - 3x + x^2 - x^3$  Treat  $3 - 3x$  and  $x^2 - x^3$  as numbers and rewrite each as a product.

Observe that you now have a common factor. Use it to factor the expression completely. This is the process called factoring by grouping.

(c) Factor (rewrite as a product):

- i.  $8 - 2x + 4x^2 - x^3$       ii.  $3x^3 - 2x + 3x^2 - 2$       iii.  $6x^3 + 3x^2 - 2x - 1$       iv.  $\sin^2 x \cos^2 x - \sin^2 x$   
 v.  $1\sqrt{x-1}(x+2) - \sqrt{x-1}(2x-3)$       vi.  $\sqrt[5]{x-4}(2x+1) + (x-4)^{\frac{1}{5}}(2x-1)$

3. Multiply out using any method:

- i.  $(x - 1)(x + 2)(x - 3)$

A similar numerical example is to multiply  $4x7x8$ . The numbers are multiplied in pairs and there is no need to stick to the given order. It makes no difference to the final product if we multiply  $4x7$  and then  $28x8$ , or  $7x8$  first and then  $4x56$ , or  $4x8$  first and then  $7x32$ . In this algebra example you

should multiply two factors, and then the result by the third factor. The order of factors is your own choice.

ii  $(a - b)^4$  [Rewrite as  $(a - b)^2(a - b)^2$ , expand or multiply out each  $(a - b)^2$  and multiply the resulting expressions.]

iii.  $(a - b)^2(a - 2b)(3a + 5)$     iv.  $(\cos x - 1)(\sin x - 1)(\cos x + 1)(\sin x + 1)$

4. i. Find the expression with exactly three factors  $(a - 3)$ ,  $(a + 5)$ , and  $(a + 3)$ .  
 ii. Find the expression with exactly four factors 2,  $(x - 1)$ ,  $(x + 4)$ , and  $(x^2 - 2x + 3)$ .

5. Substituting.

For each of the following expressions make the indicated substitutions and expand.

(a)  $x^2 - 1$                       (b)  $x^2 - 5x + 12$                       (c)  $(x^2 - 3)(x^2 + 3)$ .

i Substitution  $x = a - 2$ ,                      ii Substitution  $x = 1 - y$

iii. What would the expression be, if, before calculating we are required to add 2 to  $x$ ?

### Section 2.3 Factoring

**Factoring means rewriting a number (expression) as a product.**

#### **Numerical background**

##### Examples 2.3-1

- ◆ Factor 18. Ans.  $2 \times 9$ , or  $3 \times 6$  (partially factored), or  $2 \times 3 \times 3$  (completely factored). These products are ones that we do not have to find - we know them.
- ◆ Factor 209. Ans.  $19 \times 11$ . Most people would not know that 209 is  $19 \times 11$ , so we have to work to find two numbers that multiply together to give 209. The process is mainly trial and error but if we use what we know about numbers and multiplication patterns we can avoid pointless trials.

For example, we know that there will be no even factors and neither will a factor end in a 5. We try 3. It does not work so now we also know not to bother trying any multiples of 3, i.e. 6, 9, 12 .... We try 7 and find it does not work, but when we try 11 it works, and the other factor is 19. 11 and 19 are both prime which means that there are no other factors.

#### **Algebra**

As with the factoring of specific numbers, factoring algebra expressions is sometimes a matter of recognition and recall and sometimes a matter of trial and error.

If you completed the exercises for 2.2 you have already worked with common factors and factoring by grouping. Recognizing a common factor in algebra is something like knowing 5 is a factor because the number ends in 5. Factoring by grouping involves some trial and error in the terms you choose to group and some recognition where you find common factors in the grouped terms.

For Math 120 you are expected to have an efficient method for factoring quadratic and similar expressions. A quadratic expression has the general form  $ax^2 + bx + c$ . The method taught in most high schools is the  $ac$ -method - for a review see your text. For an alternative method see the Appendix at the end of these notes.

### Factoring quadratics by recognition of forms

There are some quadratic forms that you should be able to recognize quickly and easily. These are the perfect squares and the difference of squares.

$$\text{Perfect square: } (A + B)^2 = A^2 + 2AB + B^2$$

A two term expression squared is one number squared plus twice the product plus the other number squared.

$$\text{Difference of squares: } (A + B)(A - B) = A^2 - B^2$$

The sum of two numbers times their difference is the same as the first squared minus the second squared.

In examples 2.3 - 2 and 2.3 - 3 we look at factoring expressions with underlying quadratic form. The key is to get past the mess and recognize the underlying form of the expression. This often means creating or identifying a bracketed expression that is being used as a single number.

#### Example 2.3 - 2

◆ Factor  $(x^2 + 1)^{2/3} - 2(x^2 + 1)^{1/3} + 1$

This is a quadratic in the number  $(x^2 + 1)^{1/3}$ . If needed review rational exponents in your text.

This can be seen quite clearly if we substitute  $(x^2 + 1)^{1/3} = A$

$$A^2 - 2A + 1$$

You should recognize this as a perfect square, and write down its factorization immediately as

$$(A-1)^2$$

If you can recognize this form without the substitution, go straight to the factors. The substitution can be a useful intermediate step but it is not a required one.

The final result is:

$$((x^2 + 1)^{1/3} - 1)^2$$

#### Example 2.3 - 3

Recognizing when expressions are being treated as numbers and hence the useful substitutions to make.

<u>Expression</u>	<u>Number expression</u>	<u>Form with A substituted</u>
$x^4 - 7x^2 - 8$	$x^2$	$A^2 - 7A - 8$
$3^{2x} - 5 \cdot 3^x + 6$	$3^x$	$A^2 - 5A + 6$
$(x+5)^4 - (x+5)^3 - 2(x+5)^2$	$(x+5)$	$A^4 - A^3 - 2A^2$
		Common factor $A^2$ for $A^2(A^2 - A - 2)$
$2\sin^2x - \sin x - 1$	$\sin x$	$2A^2 - A - 1$
$e^{2x} - e^x - 2$	$e^x$	$A^2 - A - 2$

Notes: •  $\sin x$  represents a number (a trigonometric ratio). Instead of writing the square  $(\sin x)^2$  we write  $\sin^2x$ .  
•  $e$  represents a number (2.718281828459...). Like  $\pi$  it is an irrational constant.

### Exercises 2.3

Factoring skills will be needed throughout the course. The times allotted on tests and exams are based on the assumption that you can do this kind of basic manipulation quickly.

Factor

i.  $(2x-3)^2 - (x-5)^2$     ii.  $(x+2)^4 - 2(x+2)^2 + 1$     iii.  $(x-5)^{2/3} + 6(x-5)^{1/3} + 9$   
 iv.  $\cos^2 x - \sin^2 x$     v.  $4\sin^2 x - 4\sin x + 1$     vi.  $e^{2x} - 10e^x + 25$

2. Factor

i.  $(2x-3)^2 - (2x-3) - 2$     ii.  $(x+2)^4 - 2(x+2)^2 - 3$     iii.  $(x-5)^{2/3} - 5(x-5)^{1/3} - 6$   
 iv.  $2\cos^2 x - \cos x$     v.  $3\sin^2 x - 2\sin x - 1$     vi.  $e^{2x} - 3e^x - 4$   
 vii.  $\sin^4 x - 1$     viii.  $e^{3x}x^2 - e^{3x}x - 2x + 2$     ix.  $3\cos x \sin^2 x - \sin^2 x \cos^2 x - 10\sin^2 x$

### Section 2.4 Equivalent fractions and Canceling

#### Numerical background



If the big rectangle is one, then the amount shaded is the fraction  $\frac{9}{24}$

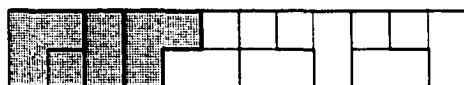
We can show equivalent fractions by keeping the amount shaded the same and changing the partitions.

If we divide every partition into equal parts we get 48 parts with 18 shaded parts.



Equivalent fraction calculation:  $\frac{9 \times 2}{24 \times 2} = \frac{18}{48}$

If we group the partitions by threes, we get 8 parts with 3 shaded parts.



Equivalent fraction calculations:  $\frac{9 \div 3}{24 \div 3} = \frac{3}{8}$

or  $\frac{9 \times 1/3}{24 \times 1/3} = \frac{3}{8}$  or  $\frac{\text{One third of } 9}{\text{One third of } 24}$

Note that the number or quantity has not changed, but is being written in different ways.

Equivalent fractions: Abstract explanation

Multiplying by 1 does not change a number. Thus we can rewrite any fraction by multiplying by 1 in the form  $a/a$ . i.e. where numerator and denominator are the same<sup>6</sup>. Instead of the drawings above we could argue as follows:

First equivalent:  $9/24$  is multiplied by 1 in the form  $2/2$  to give  $9 \times 2 / 24 \times 2 = 18/48$

Second equivalent  $9/24$  is multiplied by 1 in the form  $1/3$  to give  $\frac{9 \times (1/3)}{24 \times (1/3)} = \frac{3}{8}$

Alternatively: divide the numerator and the denominator by 3 to give  $\frac{9 \div 3}{24 \div 3} = \frac{3}{8}$

**Algebra**

We can rewrite numbers of the form  $\frac{A}{B}$  by multiplying or dividing by 1 in the form  $\frac{C}{C}$

giving us  $\frac{A}{B} = \frac{AC}{BC}$  or  $\frac{A}{B} = \frac{A \div C}{B \div C}$  Remember that  $A$ ,  $B$  &  $C$  can also be expressions.

Note: Rewriting fractions or rational expressions like this works only with

multiplication or division.  $\frac{A+C}{B+C}$  does NOT equal  $\frac{A}{B}$  and  $\frac{A-C}{B-C}$  does NOT equal  $\frac{A}{B}$

**Canceling**

Canceling is based on the ideas of equivalent fractions but extended to include any kind of number.

Example 2.4 - 1 A numerical example

Simplify  $\frac{315}{90}$

For canceling we see that the numerator and denominator have a common factor. We divide both numbers by 5, replacing the 315 with 63, and 90 with 18.

$$\frac{315}{90} = \frac{63}{18}$$

The numerical justification for this is

$$\frac{315}{90} = \frac{63 \times 5}{18 \times 5} = \frac{63 \times 5 \div 5}{18 \times 5 \div 5} = \frac{63 \times 1}{18 \times 1} = \frac{63}{18}$$

In words: (1) **write numerator and denominators as products.**  
 (2) **divide numerator and denominator by a factor common to both products.**

---

<sup>6</sup> Except for  $0/0$  which is undefined.

There are other ways to describe what is done with numbers and canceling, but the version in the box connects directly to what we do to simplify algebra expressions by canceling.

**Example 2.4-2** Using the equivalent fraction ideas on algebra expressions (rational expressions).

• Simplify 
$$\frac{x^3 - 6x^2 + 11x - 6}{x^2 + x - 12}$$
 Numerator and denominator both written as products.

$$= \frac{(x-1)(x-3)(x-2)}{(x+4)(x-3)}$$

Divide top (numerator) and bottom (denominator) by the number  $(x-3)$ .

[If you prefer, you can think of multiplying both numerator and denominator by  $1/(x-3)$ .] Generally we do not bother to write out all the steps; we just strike out the matching factors and replace them with 1.

$$\text{Thus } \frac{(x-1)\cancel{(x-3)}(x-2)}{(x+4)\cancel{(x-3)}_1} = \frac{(x-1)(x-2)}{(x+4)}$$

**Notice that canceling can only work if the numerator and denominator are considered to be single numbers that can be written as PRODUCTS, and the numbers to be canceled are factors of those products.**

In  $\frac{(x-1)(x-2)(x-3)}{(x+4)(x-3)}$  the numbers  $x, -1, 2, 4$  etc. are not available for canceling

because they are not factors of the products in the numerator and the denominator. We could divide numerator and denominator by any one of  $x, -1, 2, 4$  etc., or all of them if we felt like it. Or multiply the numerator and denominator by the reciprocals,  $1/x, -1, 1/2, 1/4$ . The resulting expressions would represent the same number providing the algebra was done correctly, but we would not have produced anything useful.

The following examples show some of the places where canceling appears in the course.

**Example 2.4-3** Simplifying a Rational Expression

• 
$$\frac{x^2 - 3x + 2}{x - 1}$$
 Numerator and denominator rewritten as products.

$$= \frac{(x-2)(x-1)}{(x-1)}$$

Divide numerator and denominator [Cancel] by the number  $(x-1)$

$$= x - 2$$

---

### A note about domains

The two expressions  $\frac{x^2 - 3x + 2}{x - 1}$  and  $x - 2$  represent the same number with one important difference. We can substitute any real number for  $x$  in  $x - 2$ , but the same is not

true for  $\frac{x^2 - 3x + 2}{x - 1}$ . The value 1 cannot be assigned to  $x$  in this expression since it results in division by 0, which is never possible.

The formal way of saying this is that the two expressions represent the same number but the DOMAIN is not quite the same. For  $x - 2$  the domain consists of all real numbers, but for  $\frac{x^2 - 3x + 2}{x - 1}$  the domain is all real numbers except  $x = 1$ , written  $\mathcal{R}, x \neq 1$ .

#### Example 2.4-4

$$\begin{aligned} \bullet (2) \quad \frac{m^2 - 7m - 8}{m^2 - 16m + 64} &= \frac{(m - 8)(m + 1)}{(m - 8)^2} && \text{Write numerator and denominator as products} \\ &= \frac{m + 1}{m - 8} && \text{Divide numerator and denominator by } m - 8 \end{aligned}$$

In this example the two expressions  $\frac{m^2 - 7m - 8}{m^2 - 16m + 64}$  and  $\frac{m + 1}{m - 8}$  represent the same number and both have the same domain (All Real numbers except 8, written  $\mathcal{R}, m \neq 8$ ).

Example 2.4-5 The result of using the formula for solving quadratics that do not factor nicely.

$$\bullet \text{ Solve } x^2 - 6x - 4 = 0$$

$$\text{Using the formula we get } x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-4)}}{2(1)} = \frac{6 \pm \sqrt{52}}{2} = \frac{6 \pm 2\sqrt{13}}{2}$$

At this point many students make the mistake of canceling the two 2's. The numerator is not written as a product with 2 as a factor. Before we can cancel the denominator 2 we

$$\text{either must get a factor of 2 in the numerator as well. } \frac{6 \pm 2\sqrt{13}}{2} = \frac{2(3 \pm \sqrt{13})}{2} = 3 \pm \sqrt{13}$$

or

$$\text{divide numerator \& denominator by 2. } \frac{(6 \pm 2\sqrt{13}) \cdot \cancel{2}}{\cancel{2} \cdot 2} = \frac{(6 \pm 2\sqrt{13}) \div 2}{2 \div 2} = \frac{(3 \pm \sqrt{13})}{1} = 3 \pm \sqrt{13}$$

#### Example 2.4-6

$$\bullet \text{ Rationalize the denominator of } \frac{2 + \sqrt{3}}{4 - \sqrt{3}}$$

Rationalizing the denominator (or occasionally the numerator) means finding a way of writing the number without any radicals in the denominator. In problems like this one we use the theory of equivalent fractions, not to cancel, but to find a different way of writing the number.

We need to find a number which will multiply with the denominator and give a number without a radical in it. For sums or differences involving radicals we can find such a number based on the difference of squares factoring.

Recall the difference of squares factorization:  $(A - B)(A + B) = A^2 - B^2$

If our denominator is a subtraction, we want to multiply it by its conjugate, the matching addition and vice versa for an addition. However, remember that we want to change the form but not the number itself. Multiplying only the denominator would change the number. To keep the number unchanged in value we must multiply by 1. We proceed as follows.

$$\frac{2 + \sqrt{3}}{4 - \sqrt{3}}$$

To remove the radicals in the denominator we need to multiply the difference  $4 - \sqrt{3}$  by the sum  $4 + \sqrt{3}$ .

We cannot simply multiply because that would change the number.

$$= \frac{2 + \sqrt{3}}{4 - \sqrt{3}} \cdot \frac{4 + \sqrt{3}}{4 + \sqrt{3}}$$

Two ways to explain: EITHER Multiply by 1 in the form  $\frac{4 + \sqrt{3}}{4 + \sqrt{3}}$ .

OR Write the equivalent "fraction" by multiplying numerator and denominator by  $4 + \sqrt{3}$ .

$$= \frac{8 + 2\sqrt{3} + 4\sqrt{3} + 3}{16 - 3}$$

Numerator and denominator are rewritten

$$= \frac{11 + 6\sqrt{3}}{13}$$

No further simplification is possible here.

## SECTION 3

## Simplification, Equations and Proofs

### Section 3.1 Simplifications

Recall: Numbers can be written directly and specifically, for example . They can be written as calculations, for example,  $4 \cdot 3$ ,  $\sqrt{144}$ ,  $(15 - 3)$ , etc.. Unspecified or general numbers can be written as variables, e.g.  $x$ ,  $y$ ,  $a$ , or as expressions involving variables, e.g.  $(x-3)$ ,  $57y$ ,  $(x-3y)^2$ , etc..

There are infinitely many ways of rewriting numbers, most of which are not useful. The useful ones are the ones we concentrate on in algebra.

#### **Simplification and rewriting are the same thing.**

Simplification should mean rewriting to get a simpler form, but sometimes the new version does not look much simpler than the old one.

e.g. Consider the two forms  $x^2 - 6x - 7$  and  $(x-7)(x+1)$ . I prefer the word "rewrite" since it does not imply any judgments about what is simpler and what is more complicated.

**Example 3.1 - 1.** Simplifications (we expect the result to be simpler than the original)

◆ Simplify	$\frac{1}{b-2} + \frac{(a-2)^2}{a(b-2) - 2(b-2)}$ $= \frac{1}{b-2} + \frac{(a-2)^2}{(b-2)(a-2)}$ $= \frac{1}{b-2} + \frac{(a-2)}{(b-2)}$ $= \frac{1+(a-2)}{b-2}$ $= \frac{a-1}{b-2}$	<p>See footnote <sup>7</sup></p> <p>Factor right expression denominator.</p> <p>Rewritten right expression has same denominator as left term.</p> <p>Combine numerators over a common denominator.</p> <p>Rewrite numerator.</p>
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### Section 3.2 Equations and Meanings for “=”

The word *equation* is becoming thoroughly misused. Many people use the word *equation* when they mean *expression*. A single expression can be thought of as a generalized number, representing all the numbers that can be created using the expression as a formula. An equation is a statement that two expressions are equal, or, put another way, that two expressions represent the same number.

**“=” is a verb meaning “is the same quantity or number as”**  
**An equation is a statement that two numbers (expressions) are equal.**

For example, the equation  $x^2 - 5x = x + 7$  means that the number represented by the expression  $x^2 - 5x$ , and the number represented by the expression  $x + 7$ , are the same number.

The symbol “=” is used frequently in mathematics in a variety of contexts. Two contexts which students sometimes confuse are as follows:

**(1) Statements of equality that are true for all (or almost all) values of the variable. (Identities)**

This use of “=” is the one we have already used between successive ways of rewriting an expression.

We can write  $x^2 - 6x - 7 = (x-7)(x+1)$  because one expression is a rewritten version of the other and therefore they both must represent the same generalized number. In this case the two expressions always represent the same number no matter what value is assigned to the variable  $x$ . There is no difference in the domains of the two expressions.

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<sup>7</sup>If you have problems following the addition of two rational expressions check your knowledge of fraction addition. If that is shaky do some review.

**(2) Statements of equality that are true for a limited number of values of the variables. (Conditional Equations)**

The second use of "=" occurs when the expressions are not rewritten versions of each other. In our equation example,  $x^2 - 5x = x + 7$ , the two expressions do not always represent the same generalized number. In fact, for most values of  $x$  these expressions would not be equal.

**Solving an equation means finding the value or values of the variables that make the statement of equality true.**

Vocabulary: The term **Root** of an equation is often used in preference to "solution".

Example 3.2 - 1 Solving a linear<sup>8</sup> equation.

- ♦ Solve  $2x+1 = x-7$

$2x+1$  and  $x-7$  are two different expressions. The equation statement is untrue for most values of the variable. Solving the equation means finding the value of  $x$  that will make this equation (or statement) true.

We solve the equation by treating the equality statement as true. Then we keep finding new statements that are true until we have a simple statement about the value of the variable. If the new statement or equation is derived in such a way that both the new and old statements are true for the same values of the variable, then the equations are equivalent. Derivation steps that are completely reversible always result in equivalent equations.

Thus we have:

$2x + 1$	$= x - 7$	Original equation.
$2x + 1 - 1$	$= x - 7 - 1$	A new (equivalent) equation found by subtracting 1 from both expressions. Equivalent because subtracting 1 can be reversed by adding 1.
$2x$	$= x - 8$	Rewriting both expressions. No change in the equation.
$2x - x$	$= x - 8 - x$	An equivalent equation derived by subtracting $x$ from both expressions.
$x$	$= -8$	Equivalent because subtracting $x$ can be reversed by adding $x$ . Rewriting both expressions.

This last equation is a direct statement about the value of  $x$ . This is the solution or root.

Equivalent equations

The steps used in solving linear equations: adding, subtracting, multiplying or dividing using the same number on both sides, are all reversible. Consequently the last equation

<sup>8</sup> Linear means that the variable has maximum exponent 1. Recall  $x^1 = x$  and  $x^0 = 1$

in the above series is equivalent to the first and the same value of the variable makes both equations true, that is,  $x = -8$ .

### Example 3.2 - 2 Solving a quadratic equation

- ◆ Solve  $5x - 4 = x^2 + 2$  for  $x$ . [Find the roots of  $5x - 4 = x^2 + 2$ ]

We note that although this is not in classic quadratic form the terms are those of a quadratic. Find an equivalent equation with an expression in standard quadratic form, equal to 0.

$5x - 4 = x^2 + 2$	Equation: statement that these two numbers are the same.
$5x - 4 + 4 = x^2 + 2 + 4$	Adding 4 to both the numbers gives us a new pair of equal numbers.
$5x = x^2 + 6$	Rewrite the two numbers.
$5x - 5x = x^2 + 6 - 5x$	Subtracting $5x$ from both numbers gives us a new pair of equal numbers.
$0 = x^2 - 5x + 6$	Rewrite the left number.
$x^2 - 5x + 6 = 0$	Make the equality statement the other way round.

We have reached this stage by a sequence of true statements, each one derived from the one before using simple arithmetic steps which are all reversible. The values of  $x$  that make the last statement true will be the same values that made the first statement true. We need the equation in this form so that we can use the Zero-Product Rule.

**The Zero-Product Rule: If a product equals zero, at least one of the factors must equal zero.**

$$(x-2)(x-3) = 0 \quad \text{Rewrite (factor) the left number.}$$

The statement  $(x-2)(x-3) = 0$  says that the product of two numbers,  $(x-2)$  and  $(x-3)$ , is 0. Based on the zero-product rule either number can be zero. Thus either a value of  $x$  that makes  $(x-2)$  equal to 0 or a value of  $x$  that makes  $(x-3)$  equal to 0, will make the product equal to 0.

$$(x-2) = 0 \text{ when } x = 2 \quad \text{and} \quad (x-3) = 0 \text{ when } x = 3$$

We can write the solution to an equation in several ways:

$$\text{e.g. } x = 2, x = 3: \quad x = 2, 3: \quad x \in \{2, 3\}$$

**Remember:** (i) The whole thing is like an argument, "If statement 1 is true, then statement 2 is true, and if statement 2 is true then statement 3 is true." This goes on until we can make a final statement about the values of  $x$ .

(ii) When solving equations each new line in the solution is derived from the previous line by one of two processes.

(1) Rewriting the numbers (simplifying). This does not change the equation.

(2) Replacing with a new equation (statement of equality). The new statement is derived from, or is a consequence of, the previous statement. Ideally the new equation is equivalent to the old one. Occasionally this is not true and the solutions to the new equation may be different.

Most of the equations students have to solve have straightforward solutions. The derived equations are equivalent equations and the values of the variable that make the first equation statement true are the same values that make all the subsequent statements true. The solutions that are found are the solutions to the equation. Students are told to check them, but this is for accuracy only.

However, there are equations that are not so nice and tidy and where checking may be essential. The solution may not be a neat list of numbers. In other cases there is a neat list but not every number in the list is a solution of the equation. There may even be no solution at all.

Example 3.2 - 3 An equation that turns out to be true for all Real numbers. [An identity]

♦ Solve  $3(x-4) - 5 = x + 2(x-8) - 1$

Remember that we are looking for the value or values of  $x$  that make the statement of equality true.

Solution:  $3(x-4) - 5 = x + 2(x-8) - 1$

The original equation or statement of equality.

$$3x - 12 - 5 = x + 2x - 16 - 1$$

Rewrite both expressions.

$$3x - 17 = 3x - 17$$

Rewrite both expressions

We observe that the expressions (numbers) are exactly the same, which happens when the expressions in the original equation are just rewritten versions of each other. This means that the two expressions must always represent the same number, regardless of the specific values of  $x$ . Put another way,  $x$  can be any number because the equality statement is always true. [An identity.]

The solution is therefore that  $x$  represents any real number, written  $x \in \mathcal{R}$

If you did something different with the equation or did not notice that  $3x - 17$  appears on both sides of the equation and continued with replacement statements you would get  $3x = 3x$ , or  $17 = 17$ , or  $0 = 0$ , or some such statement. These equations are all equivalent to the original and all are true regardless of the value of  $x$ . This is the same conclusion that we reached from  $3x - 17 = 3x - 17$ . Going as far as  $3x = 3x$ , or  $17 = 17$ , etc., is not wrong but entails unnecessary work.

Examples of equation solutions involving equations that are not equivalent

In the examples we have used up to now, if a new equation was derived it was always an equivalent equation because we used the normal operations of  $+$ ,  $-$ ,  $\times$ ,  $\div$ . These are all reversible with the exception of multiplication or division by  $0$ . Squaring both sides of an equation is a non-reversible step, and we will meet another in the chapter on logarithms.

Example 3.2 - 4 An equation without roots (i.e., no solution). [Inconsistent equation]

♦ Solve  $\frac{x-1}{x-5} = \frac{4}{x-5}$

The correct first step is to multiply both sides of this equation by  $x - 5$ , giving us a simple equation to solve. Unfortunately, although we cannot tell at this stage, we are multiplying both sides by  $0$ .

$$x - 1 = 4$$

$$x = 5$$

This last line is not the solution, as we see if we plug it back in: we get undefined numbers with denominators of 0. The solution process produced the impossible situation of division by 0. We conclude therefore that this equation can never be a true statement. It has no solution.<sup>9</sup>

**Example 3.2 - 5** Too many solutions.

- ◆ (i) Solve  $\frac{y^2 - 4}{y + 3} = 2 - \frac{y - 2}{y + 3}$  Note that  $y = -3$  is not a possible solution

The first step is to rewrite so that both sides have an expression with the same denominator

$$\frac{y^2 - 4}{y + 3} = \frac{2(y + 3)}{y + 3} - \frac{y - 2}{y + 3}$$

2 has been rewritten as  $\frac{2(y + 3)}{y + 3}$

$$\frac{y^2 - 4}{y + 3} = \frac{2(y + 3) - (y - 2)}{y + 3}$$

Right side has been rewritten as a single expression with a denominator.

The equation is now in a form where we can easily eliminate denominators by multiplying both sides by the common denominator,  $y + 3$ .

$$\frac{y^2 - 4}{y + 3}(y + 3) = \frac{2(y + 3) - (y - 2)}{y + 3}(y + 3)$$

Another opportunity to note that  $y \neq -3$ .

$$y^2 - 4 = 2(y + 3) - (y - 2)$$

With some rewriting etc., we solve this quadratic equation to get  $y = 4$ ,  $y = -3$  as solutions. When these are checked we find that  $y = 4$  satisfies, and we already know that  $y = -3$  is not possible.  $y = -3$  is called an extraneous solution. Extraneous solutions appear later in the course, especially in Ch. 5.

- ◆ (ii) Solve  $x = \sqrt{x + 2}$   
 $x^2 = (x + 2)$  Square both sides.

Squaring both sides is not reversible because squaring both sides of the different equation  $x = -\sqrt{x + 2}$  also produces  $x^2 = x + 2$ . If we try to go backwards from  $x^2 = (x + 2)$  we will have to choose between  $x = \sqrt{x + 2}$  and  $x = -\sqrt{x + 2}$ .

$x^2 = (x + 2)$  is a quadratic equation with solutions  $x = 2$ ,  $x = -1$

Since we used a non-reversible step we must check. the equality statement is true if  $x = 2$ , but  $x = -1$  gives the impossible statement  $-1 = 1$ .

Note that the solution  $x = -1$  satisfies the other equation  $x = -\sqrt{x + 2}$ .

<sup>9</sup>When solving an equation we would not normally multiply or divide both sides by 0, because when we are using numbers it is quite obvious using 0 will not help us find a solution.

The algebraic reasons for this are:

(a) Division by 0 is always impossible.

(b) Multiplication by 0 is possible but the resulting equation is  $0 = 0$ . This is a true statement but will not be an equivalent equation. Multiplication by 0 cannot be reversed since the reverse process of division by 0 is undefined.

When we are using numbers we naturally avoid multiplying or dividing by 0. In more complex equations we may multiply or divide by expressions only to find out later that the expression we used was actually equal to 0.

**Remember:**

(1) At this stage the two things to be wary of in deriving replacement equations are:

- (a) multiplying or dividing by an expression,      (b) squaring both sides.

When there are expressions in denominators make a note before you start of excluded or impossible values for  $x$ .

(2) If you multiply or divide both sides of an equation by an expression make a note that this expression cannot be  $0$ .

(3) If non-equivalent replacement equations are used, a check of the possible solutions is an essential part of the solution process.

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## APPENDIX E

Description of Analysis of Deviance Tables

Analysis of deviance performs the same function as analysis of variance but is based on maximum likelihoods. The residual deviance column indicates the extent to which the model composed of terms up to that point fails to fit the data. Specifically, it is defined to be  $2L_S - 2L_M$  where  $L$  denotes the log-likelihood function of a model and  $M$  and  $S$  identify the current model and the saturated model, respectively. The saturated model has one parameter for every observation, and describes the data completely. The model  $M$  has fewer parameters and is not a perfect fit, but if it describes the data well then its log-likelihood,  $L_M$ , will be relatively close to  $L_S$ .

Df stands for degrees of freedom. As is normal, the degrees of freedom for a factor are one less than the number of levels, which is the same as the number of non-redundant parameters. The residual degrees of freedom (Residual Df) is the difference between the total number of observations and the number of non-redundant parameters.

The residual deviance statistic has a  $\chi^2$  distribution with an expected value equal to its degrees of freedom. Thus a first indication of the fit of a model is that the residual degrees of freedom should equal or exceed the residual deviance. In addition the appropriate probability statistic used in the last column of the table is  $\chi^2$  rather than  $F$ .

The first two columns in the tables are, respectively, the degrees of freedom and the deviance associated with the particular term in the model.

Analysis of Deviance Tables for the Saturated Models in the Statistical Analysis

Table E1

Calculus Group: Saturated Model for Calculus Grade Frequencies Classified by  
Instructor, Time and Year

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
<i>NULL</i>			47	93.95	
<i>Instructor</i>	1	0.02	46	93.92	0.88
<i>Time</i>	1	6.21	45	87.71	0.01
<i>Year</i>	2	10.94	43	76.77	0.00
<i>Grade</i>	3	11.42	40	65.35	0.01
<i>Instructor:Time</i>	1	0.96	39	64.39	0.33
<i>Instructor:Year</i>	2	6.38	37	58.01	0.04
<i>Time:Year</i>	2	4.19	35	53.83	0.12
<i>Instructor:Grade</i>	3	2.84	32	50.99	0.42
<i>Time:Grade</i>	3	4.51	29	46.48	0.21
<i>Year:Grade</i>	6	2.46	23	44.02	0.87
<i>Instructor:Time:Year</i>	2	0.09	21	43.94	0.96
<i>Instructor:Time:Grade</i>	3	16.64	18	27.30	0.00
<i>Instructor:Year:Grade</i>	6	14.36	12	12.94	0.03
<i>Time:Year:Grade</i>	6	6.16	6	6.78	0.41
<i>Instructor:Time:Year:Grade</i>	6	6.78	0	0.00	0.34

Table E2

Calculus Group subset with High School: Saturated Model for Calculus Grade  
Frequencies Classified by Instructor, Time and High School

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
<i>NULL</i>			23	30.47	
<i>Instructor</i>	1	1.11	22	29.36	0.29
<i>Time</i>	1	0.12	21	29.23	0.73
<i>High School</i>	1	0.12	20	29.11	0.73
<i>Grade</i>	2	2.77	18	26.34	0.25
<i>Instructor:Time</i>	1	0.61	17	25.73	0.43
<i>Instructor:High School</i>	1	0.10	16	25.63	0.76
<i>Time:High School</i>	1	0.01	15	25.63	0.94
<i>Instructor:Grade</i>	2	0.25	13	25.38	0.88
<i>Time:Grade</i>	2	3.63	11	21.75	0.16
<i>High School:Grade</i>	2	1.66	9	20.08	0.44
<i>Instructor:Time:High School</i>	1	2.27	8	17.82	0.13
<i>Instructor:Time:Grade</i>	2	8.50	6	9.31	0.01
<i>Instructor:High School:Grade</i>	2	2.71	4	6.60	0.26
<i>Time:High School:Grade</i>	2	5.33	2	1.27	0.07
<i>Instructor:Time:High School:Grade</i>	2	1.27	0	0.00	0.53

Table E3

**Calculus Group: Saturated Model for Precalculus Grade Frequencies Classified by  
Instructor, Time and Year**

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
<i>NULL</i>			35	67.14	
<i>Instructor</i>	1	0.02	34	67.12	0.88
<i>Time</i>	1	6.21	33	60.91	0.01
<i>Year</i>	2	10.94	31	49.97	0.00
<i>Grade</i>	2	5.79	29	44.18	0.06
<i>Instructor:Time</i>	1	0.96	28	43.22	0.33
<i>Instructor:Year</i>	2	6.38	26	36.84	0.04
<i>Time:Year</i>	2	4.19	24	32.66	0.12
<i>Instructor:Grade</i>	2	4.58	22	28.08	0.10
<i>Time:Grade</i>	2	1.81	20	26.27	0.40
<i>Year:Grade</i>	4	5.29	16	20.98	0.26
<i>Instructor:Time:Year</i>	2	0.08	14	20.90	0.96
<i>Instructor:Time:Grade</i>	2	4.05	12	16.85	0.13
<i>Instructor:Year:Grade</i>	4	7.24	8	9.60	0.12
<i>Time:Year:Grade</i>	4	4.12	4	5.48	0.39
<i>Instructor:Time:Year:Grade</i>	4	5.48	0	0.00	0.24

Table E4

Non-calculus Group: Saturated Model for Precalculus Grade Frequencies Classified by  
Instructor, Time and Year

	Df	Deviance	Residual Df	Residual Deviance	Pr(Chi)
NULL			59	343.28	
<i>Instructor</i>	1	3.06	58	340.23	0.08
<i>Time</i>	1	79.92	57	260.30	0.00
<i>Year</i>	2	5.21	55	255.10	0.07
<i>Grade</i>	4	163.18	51	91.92	0.00
<i>Instructor:Time</i>	1	15.39	50	76.53	0.00
<i>Instructor:Year</i>	2	0.72	48	75.80	0.70
<i>Time:Year</i>	2	0.33	46	75.47	0.85
<i>Instructor:Grade</i>	4	0.86	42	74.61	0.93
<i>Time:Grade</i>	4	13.10	38	61.51	0.01
<i>Year:Grade</i>	8	21.21	30	40.30	0.01
<i>Instructor:Time:Year</i>	2	6.96	28	33.34	0.03
<i>Instructor:Time:Grade</i>	4	5.44	24	27.91	0.25
<i>Instructor:Year:Grade</i>	8	11.63	16	16.27	0.17
<i>Time:Year:Grade</i>	8	7.02	8	9.25	0.53
<i>Instructor:Time:Year:Grade</i>	8	9.25	0	0.00	0.32