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Research Article

Statistical Λ-Convergence in Probabilistic Normed Spaces

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The main objective of the study was to understand the notion of Λ-convergence and to study the notion of probabilistic normed (PN) spaces. The study has also aimed to define the statistical Λ-convergence and statistical Λ-Cauchy in PN-spaces. The concepts of these approaches have been defined by some examples, which have demonstrated the concepts of statistical Λ-convergence and statistical Λ-Cauchy in PN-spaces. Previous studies have also been used to understand similar terminologies and notations for the extraction of outcomes.

1. Introduction

The notion of statistical metric spaces [1–3], called probabilistic metric spaces, was introduced by Menger [4]; it is an important generalization of metric spaces. The concept of probabilistic normed (PN) spaces [5] is a key generalization of the concept of normed spaces. The idea of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [6] individualistically. This concept has been studied by many researchers in different set-ups such as in normed linear spaces, in probabilistic normed spaces, in intuitionist fuzzy normed spaces, and in locally convex spaces. The theory of probabilistic metric spaces was studied by several authors (see, for details, [2, 7–10]). Karakus [11] studied the concept of statistical convergence in PN-spaces. It was subsequently carried out in the works due to [12–14]. To define the concept of Λ-convergence, statistical Λ-convergence, and statistical Λ-Cauchy in PN-spaces, the subsequent definitions are needed. The terminology and notations used in this paper are standard as in the recent works [1, 4, 15, 16].

Definition 1. Let M be the subset of set of N. The natural density, $\delta(M)$, is characterized by

$$\delta(M) = \lim_{n\to\infty}\frac{1}{n}\left|\{m \leqq n : m \in M\}\right|, \tag{1}$$

where $|\cdot|$ is the cardinality of the enclosed set. A number sequence $y = (y_m)$ is said to be statistically convergent to the number L if, for each $\epsilon > 0$, the set

$$M(\epsilon) = \{m \leqq n : |y_m - L| > \epsilon\} \tag{2}$$

has natural density zero; that is,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m \leqq n : |y_m - L| > \epsilon\}\right| = 0 \tag{3}$$

and it can be written as stat lim $y = L$.

Definition 2. A function $h : R \to R_{+}^{0}$ is a distribution function if it is nondecreasing and left-continuous with

$$\begin{gathered} \inf h(u) = 0 \quad u \in R, \\ \sup h(u) = 0 \quad u \in R. \end{gathered} \tag{4}$$

Here $D+$ is the set of all distribution functions such that $h(0) = 0$. If $a \in R_{+}^{0}$, then $H_a \in D+$, where

$$H_a(u) = \begin{Bmatrix} 0 & (u \leqq a) \\ 1 & (u > a) \end{Bmatrix}, \tag{5}$$

clearly for all $h \in D+$.

A t-norm is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an Abelian monoid with unit one and $m * n = k * l$ if $m = k$ and $n = l$ for all $k, l, m, n \in [0,1]$.

Definition 3. Let Y be a linear space of dimension ≥ 1. Suppose also that $*$ is a t-norm and $Y \to D+$. Then $\mathcal{G}$ is said to be a probabilistic norm and $(Y; \mathcal{G}, *)$ is referred to as a probabilistic normed space if it satisfies the following:

(i) $\mathcal{G}_y(0) = 0$ if y and z are linearly dependent, where $\mathcal{G}(y; z, t)$ is the value of $\mathcal{G}(y; z)$ at $t \in R$.

(ii) $\mathcal{G}_y(t) = 1$ for all $t > 0$ if and only if $y = 0$.

(iii) $\mathcal{G}_{\rho y}(t) = \mathcal{G}_y(t/|\rho|)$.

(iv) $\mathcal{G}_{y+z}(s+t) = \mathcal{G}_y(s) * \mathcal{G}_z(t)$ for all $y, z \in Y$ and $s, t \in R_0$.

Recently, in [17–19], some λ-sequence spaces were introduced and studied. In this paper, let $\lambda = (\lambda_j)_{j=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity; that is,

$$\lim_{j\to\infty} \lambda_j \longrightarrow \infty, \quad 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_j < \cdots. \tag{6}$$

In this case a sequence $y = (y_m)_{m=0}^{\infty}$ is λ-convergent to the number $L \in C$, which is known as λ-limit of y, if $\Lambda_n(y) \to L$ as $n \to \infty$, where

$$\Lambda_m(y) = \frac{1}{\lambda_m} \sum_{j=0}^{m} \left(\lambda_j - \lambda_{j-1}\right) y_j \quad (m \in \mathbb{N}). \tag{7}$$

Hence it says that $y = (y_m)_{m=0}^{\infty}$ is λ-convergent to the number L if and only if the sequence $\Lambda_n(y)$ is convergent to L. Here (and in the sequel), it will take the convention that any term with a negative subscript is equal to zero; for example, $\lambda - 1 = 0$ and $y - 1 = 0$.

The sets of all Λ-bounded, Λ-convergent, and Λ-null sequences l_{∞}^{Λ}, c^{Λ}, and c_0^{Λ}, respectively, are defined as follows:

$$\begin{aligned} l_{\infty}^{\Lambda} &= \left\{ y = (y_m)_{m=1}^{\infty} : {}^{\sup}_{m} \left|\Lambda_m(y)\right| < \infty \right\}, \\ c^{\Lambda} &= \left\{ y = (y_m)_{m=1}^{\infty} : \lim_{n\to\infty} \Lambda_m(y) \text{ exists} \right\}, \\ c_0^{\Lambda} &= \left\{ y = (y_m)_{m=1}^{\infty} : \lim_{m\to\infty} \Lambda_m(y) = 0 \right\}. \end{aligned} \tag{8}$$

In the present investigation, it is proposed to systematically study the idea of Λ-convergence in PN-spaces. In particular, statistical Λ-convergence and statistical Λ-Cauchy were investigated in PN-spaces and give some illustrative examples to demonstrate these concepts.

Definition 4. Let $(Y, \mathcal{G}, *)$ be a PN-space. A sequence $y = (y_m)$ is called convergent in $(Y; \mathcal{G}, *)$ or, simply, $\mathcal{G}$-convergent to ζ if, for every $\epsilon > 0$ and $\emptyset \in (0,1)$, there exists a positive integer $m_0 \ni \mathcal{G}_{y_m - \zeta}(\epsilon) > 1 - \emptyset$ whenever $m \subset m_0$ and it is written as

$$\mathcal{G} \lim_{m\to\infty} y_m = \zeta \tag{9}$$

and ζ is G limit of $y = (y_m)$.

Definition 5. Let $(Y; \mathcal{G}, *)$ be a PN-space. A sequence $y \subset (y_m)$ is called statistically convergent in $(Y; \mathcal{G}, *)$ or, simply, $\mathcal{G}$ (stat)-convergent to ζ if, for $\epsilon > 0$ and $\emptyset \in (0,1)$,

$$\delta\left(\left\{m \in \mathbb{N} : \mathcal{G}_{y_m - \varsigma}(\epsilon) \leqq 1 - \emptyset\right\}\right) = 0, \tag{10}$$

or equivalently

$$\delta\left(\left\{m \in \mathbb{N} : \mathcal{G}_{y_m - \varsigma}(\epsilon) > \emptyset\right\}\right) = 1 \tag{11}$$

and this case is stated by

$$\mathcal{G}\,(\text{stat}) \lim y = \zeta \tag{12}$$

and ζ is $\mathcal{G}$(stat) limit of y.

Definition 6. Let $(Y; \mathcal{G}, *)$ be a PN-space. A sequence $y = (y_m)$ is known to be statistically Cauchy in $(Y; \mathcal{G}, *)$ or, simply, $\mathcal{G}$ (stat)-Cauchy if, for $\epsilon > 0$, there exists a number $N = N(\epsilon) \ni$,

$$\delta\left(\left\{m \in \mathbb{N} : \mathcal{G}_{y_n - y_k}(\epsilon) \leqq 1 - \emptyset\right\}\right) = 0, \quad \forall n, k \geq N. \tag{13}$$

In addition to the above definitions, the following definitions are given.

Definition 7. A sequence $y = (y_m)_{m=1}^{\infty}$ is called statistically Λ-convergent to the number L if, for each $\epsilon > 0$, the set given by

$$M(\epsilon) = \left\{m \leqq n : \left|\Lambda_m(y) - L\right| \geqq \epsilon\right\} \tag{14}$$

has asymptotic density zero; that is,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{m \leqq n : \left|\Lambda_m(y) - L\right| \geqq \epsilon\right\}\right| = 0. \tag{15}$$

Definition 8. A sequence $y = (y_m)_{m=1}^{\infty}$ is known to be statistically Λ-Cauchy sequence if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon) \ni$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{m \leqq n \left|\Lambda_m(y) - \Lambda_N(y)\right| \geqq \epsilon\right\}\right| = 0. \tag{16}$$

Based on the previous definitions, the concept of Λ-convergence, statistical Λ-convergence, and statistical Λ-Cauchy in PN-spaces is defined.

Definition 9. Let $(Y; \mathcal{G}, *)$ be a PN-space. A sequence $y = (y_m)$ is said to be convergent in $(Y; \mathcal{G}, *)$ or, simply, $\mathcal{G}_{\Lambda}$-convergent to ζ if, for $\epsilon > 0$ and $\emptyset \in (0,1)$, there exists a positive integer $m_0 \ni \mathcal{G}_{\Lambda}(y) - \zeta(\epsilon) > 1 - \emptyset$ whenever $m = m_0$. In this case $\mathcal{G}_{\Lambda} \lim y_m = \zeta$ can be written and ζ is called $\mathcal{G}_{\Lambda}$ limit of the sequence $y = (y_m)$.

Definition 10. Let $(Y;\mathscr{G},*)$ be a PN-space. A sequence $y=(y_m)$ is known to be statistically Λ-convergent in $(Y;\mathscr{G},*)$ or, simply, $\mathscr{G}_{\Lambda}$(stat)-convergent if, for every $\epsilon>0$ and $\emptyset\in(0,1)$,

$$\delta\left(\left\{m\in\mathbb{N}:\mathscr{G}_{\Lambda_{m(y-\varsigma)}}(\epsilon)\leqq 1-\emptyset\right\}\right)=0. \tag{17}$$

Or,

$$\delta\left(\left\{m\in\mathbb{N}:\mathscr{G}_{\Lambda_{m(y-\varsigma)}}(\epsilon)\leqq 1-\emptyset\right\}\right)=1; \tag{18}$$

that is,

$$\lim_{n\to\infty}\frac{1}{n}\left(\left\{m\in\mathbb{N}:\left|\mathscr{G}_{\Lambda_{m(y-\varsigma)}}(\epsilon)\right|\leqq 1-\emptyset\right\}\right)=0; \tag{19}$$

since

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y=\varsigma \tag{20}$$

and ς is called $\mathscr{G}_{\Lambda}$(stat) limit of the sequence y.

Definition 11. Let $(Y;\mathscr{G},*)$ be a PN-space. A sequence $y=(y_m)$ is called statistically Λ-Cauchy in $(Y;\mathscr{G},*)$ or, simply, $\mathscr{G}_{\Lambda}$(stat)-Cauchy to ζ if, for $\epsilon>0$, there exists a number $N=N(\epsilon)\ni$,

$$\delta\left(\left\{m\in\mathbb{N}:\mathscr{G}_{\Lambda_n(y)-\Lambda_k(y)}(\epsilon)\leqq 1-\emptyset\right\}\right)=0,\quad \forall n,k\geq N. \tag{21}$$

2. Main Results

By making use of the definitions given in the preceding section, it is proposed here to systematically investigate the notion of statistical Λ-convergence and statistical Λ-Cauchy in PN-spaces and apply our findings to the problem of approximating positive linear operators.

Theorem 12. *Let $(Y;\mathscr{G};*)$ be a PN-space. If a sequence $y=(y_m)$ is $\mathscr{G}_{\Lambda}$(stat)-convergent, then the $\mathscr{G}_{\Lambda}$(stat) limit is unique.*

Proof. Suppose that

$$\begin{gathered}\mathscr{G}_{\Lambda}\,(\text{stat})\lim Y=\varsigma_1,\\ \mathscr{G}_{\Lambda}\,(\text{stat})\lim Y=\varsigma_2.\end{gathered} \tag{22}$$

For a given $\emptyset>0$, let $\rho\in(0,1)\ni$,

$$(1-\rho)*(1-\rho)>1-\emptyset. \tag{23}$$

Then, for any $\epsilon>0$,

$$\begin{aligned}M_{\mathscr{G},1}(\rho,\epsilon)&=\left\{m\in\mathbb{N}:\mathscr{G}_{\Lambda_m}(y)-\varsigma_1(\epsilon)\leqq(1-\rho)\right\},\\ M_{\mathscr{G},1}(\rho,\epsilon)&=\left\{m\in\mathbb{N}:\mathscr{G}_{\Lambda_m}(y)-\varsigma_2(\epsilon)\leqq(1-\rho)\right\};\end{aligned} \tag{24}$$

since

$$\begin{gathered}\mathscr{G}_{\Lambda}\,(\text{stat})\lim Y=\varsigma_1\\ \delta\left(M_{\mathscr{G},1}(\rho,\epsilon)\right)=0\end{gathered} \tag{25}$$

for $\epsilon>0$. Furthermore, by using

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y_m=\varsigma_2, \tag{26}$$

it has

$$\delta\left(M_{\mathscr{G},2}(\rho,\epsilon)\right)=0 \tag{27}$$

for all $\epsilon>0$.

Let

$$M_{\mathscr{G}(\rho,\epsilon)}=M_{\mathscr{G},1}(\rho,\epsilon)\cap M_{\mathscr{G},2}(\rho,\epsilon). \tag{28}$$

Clearly,

$$\delta\left(M_{\mathscr{G}}(\rho,\epsilon)\right)=0, \tag{29}$$

which implies that

$$\delta\left\{\mathbb{N}\setminus M_{\mathscr{G}}(\rho,\epsilon)\right\}=1. \tag{30}$$

For $n\in N\setminus MG(\rho,\epsilon)$, by using (2) and (3),

$$\begin{aligned}\mathscr{G}_{\varsigma_1-\varsigma_2}(\epsilon)&\geqq\mathscr{G}_{\Lambda_m(y)-\varsigma_1}\left(\frac{\epsilon}{2}\right)*\mathscr{G}_{\Lambda_{m(y)-\varsigma_2}}\left(\frac{\epsilon}{2}\right)\\ &\geqq(1-\rho)*(1-\rho)>1-\emptyset.\end{aligned} \tag{31}$$

Since $\emptyset>0$ is arbitrary, by using (5),

$$\mathscr{G}_{\varsigma_1-\varsigma_2}(\epsilon)=1 \tag{32}$$

$\forall\epsilon>0$. This implies $\zeta 1=\zeta 2$. Hence, $\mathscr{G}_{\Lambda}$ (stat) limit is unique. This establishes Theorem 12. □

Theorem 13. *Let $(Y;\mathscr{G};*)$ be a PN-space. If*

$$\mathscr{G}_{\wedge}\lim y=\varsigma, \tag{33}$$

then

$$\mathscr{G}_{\wedge}\,(\text{stat})\lim y=\varsigma, \tag{34}$$

but the converse is not necessarily true in general.

Proof. By hypothesis, for every $\emptyset\in(0,1)$ and $\epsilon>0$, there exists a positive integer m_0 such that

$$\mathscr{G}_{\wedge_{m(y)-\varsigma}}(\epsilon)>1-\emptyset \tag{35}$$

whenever $m\geqq m_0$. This guarantees that the set

$$\left\{m\in\mathbb{Z}^{+}:\mathscr{G}_{\wedge_m}(y)-\varsigma(\epsilon)\leqq 1-\emptyset\right\} \tag{36}$$

has at most finitely many terms. As every finite subset of the set N of positive integers has density zero,

$$\delta\left(\left\{m\in\mathbb{Z}^{+}:\mathscr{G}_{\Lambda_{m(y)-\varsigma}}(\epsilon)\leqq 1-\emptyset\right\}\right)=0, \tag{37}$$

which establishes Theorem 13. □

Example 14. This example would show that the converse of the assertion in Theorem 13 needs not be true in general. Let $(R, |\cdot|)$ be the space of real numbers with the usual norm. Let

$$a * b = ab,$$
$$\mathcal{G}_{\Lambda_m(y)}(u) = \frac{u}{u + \left|\Lambda_{m(y)}(u)\right|}, \tag{38}$$

where $u = 0$. Here, it is noted that $(R, \mathcal{G}, *)$ is a probabilistic normed space. If it takes a sequence $\Lambda_m(y)$ whose terms are

$$\Lambda_m(y) := \left\{\begin{matrix} 1 & \left(m = k^2;\ k \in z^+\right) \\ 0 & \text{(otherwise)} \end{matrix}\right\} \tag{39}$$

then, $\forall \emptyset \in (0,1)$ and for any $\epsilon > 0$, let

$$M_{m_0}(\emptyset, \epsilon) := \left\{m \leqq m_0 : \mathcal{G}_{\Lambda_m(y)}(\epsilon) \leqq 1 - \emptyset\right\}. \tag{40}$$

Since

$$\begin{aligned} M_{m_0}(\emptyset, \epsilon) &= \left\{m \leqq m_0 : \frac{u}{u + \left|\Lambda_m(y)\right|} \leqq 1 - \emptyset\right\} \\ &= \left\{m \leqq m_0 : \left|\Lambda_m(y)\right| \geqq \frac{\emptyset u}{1 - \emptyset} > 0\right\} \\ &= \left\{m \leqq m_0 : \left|\Lambda_m(y) = 1\right|\right\} \\ &= \left\{m \leqq m_0 : m = k,\ k \in z^+\right\}, \end{aligned} \tag{41}$$

it gets

$$\begin{aligned} \frac{1}{m_0}\left|M_{m_0}(\emptyset, \epsilon)\right| &\leqq \frac{1}{m_0}\left|\left\{m \leqq m_0 : m = k^2,\ k \in z^+\right\}\right| \\ &\leqq \frac{\sqrt{m_0}}{m_0} \end{aligned} \tag{42}$$

which implies that

$$\lim_{m_0 \to \infty} \frac{1}{m_0}\left|M_{m_0}(\emptyset, \epsilon)\right| = 0. \tag{43}$$

Hence, by Definition 7,

$$\mathcal{G}_{\Lambda_{(y)}}(\text{stat})\lim y = 0. \tag{44}$$

Nevertheless, as the sequence $(\Lambda_m(y))$ shown in (39) is not convergent in the space $(R, |\cdot|)$, by Remark 1 of [11], it is clear that the sequence $(\Lambda_m(y))$ is not convergent with respect to the probabilistic norm.

Theorem 15. *Let $(Y; \mathcal{G}; *)$ be a PN-space. If*

$$\begin{aligned} &\mathcal{G}_\Lambda(\text{stat})\lim y = \varsigma_1, \\ &\mathcal{G}_\Lambda(\text{stat})\lim z = \varsigma_2 \end{aligned} \tag{45}$$

then

$$\begin{aligned} &\text{(i)}\quad \mathcal{G}_\Lambda(\text{stat})\lim(y \pm z) = (\varsigma_1 \pm \varsigma_2), \\ &\text{(ii)}\quad \mathcal{G}_\Lambda(\text{stat})\lim \sigma y = \sigma\varsigma_1 \quad (\sigma \in \mathbb{R}). \end{aligned} \tag{46}$$

Proof. (i) Let

$$\begin{aligned} &\mathcal{G}_\Lambda(\text{stat})\lim y = \varsigma_1, \\ &\mathcal{G}_\Lambda(\text{stat})\lim y = \varsigma_2, \quad \epsilon > 0. \end{aligned} \tag{47}$$

Also let $\emptyset \in (0,1)$. Choose $\rho \in (0,1)$ such that

$$(1 - \rho) * (1 - \rho) > 1 - \emptyset. \tag{48}$$

Then,

$$\begin{aligned} M_{\mathcal{G},1}(\rho, \epsilon) &= \left\{m_0 \in \mathbb{N} : \mathcal{G}_{\Lambda_m(y)-\varsigma_1}(\epsilon) \leqq 1 - \rho\right\}, \\ M_{\mathcal{G},2}(\rho, \epsilon) &= \left\{m_0 \in \mathbb{N} : \mathcal{G}_{\Lambda_m(y)-\varsigma_2}(\epsilon) \leqq 1 - \rho\right\}. \end{aligned} \tag{49}$$

Since

$$\mathcal{G}_\Lambda(\text{stat})\lim y = \varsigma_1 \tag{50}$$

it has

$$\delta\left\{M_{\mathcal{G},1}(\rho, \epsilon)\right\} = 0 \tag{51}$$

for all $\epsilon > 0$. Furthermore, by using

$$\mathcal{G}_\Lambda(\text{stat})\lim z = \varsigma_2 \tag{52}$$

get

$$\delta\left\{M_{\mathcal{G},2}(\rho, \epsilon)\right\} = 0 \tag{53}$$

$\forall \epsilon > 0$. Let

$$M_{\mathcal{G}}(\rho, \epsilon) = M_{\mathcal{G},1}(\rho, \epsilon) \cap M_{\mathcal{G},2}(\rho, \epsilon). \tag{54}$$

Then

$$\delta\left\{M_{\mathcal{G},1}(\rho, \epsilon)\right\} = 0, \tag{55}$$

which implies

$$\delta\left\{\mathbb{N} \setminus M_{\mathcal{G}}(\rho, \epsilon)\right\} = 1. \tag{56}$$

If

$$m_0 = \mathbb{N} \setminus M_{\mathcal{G}}(\rho, \epsilon), \tag{57}$$

then

$$\begin{aligned} &\mathcal{G}_{\Lambda_m(y)-\varsigma_1}(\epsilon) + \Lambda_{m(z)-\varsigma_2}(\epsilon) \\ &\quad \geqq \mathcal{G}_{\Lambda_m(y)-\varsigma_1}\left(\frac{\epsilon}{2}\right) + \mathcal{G}_{\Lambda_m(y)-}\left(\frac{\epsilon}{2}\right) \\ &\quad + \mathcal{G}_{\Lambda_m(z)-\varsigma_2}\left(\frac{\epsilon}{2}\right) > (1 - \rho) * (1 - \rho) > 1 - \emptyset. \end{aligned} \tag{58}$$

This shows that

$$\delta\left(\left\{m_0 \in \mathbb{N} : \mathcal{G}_{\Lambda_m(y)-\varsigma_1+\Lambda_m(z)-\varsigma_2}(\epsilon) \leqq 1 - \emptyset\right\}\right) = 0. \tag{59}$$

So

$$\mathcal{G}_\Lambda(\text{stat})\lim(y \pm z) = \varsigma_1 \pm \varsigma_2. \tag{60}$$

(ii) Let

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y = \varsigma \tag{61}$$

and suppose that $\emptyset \in (0,1)$ and $\epsilon > 0$. Firstly, consider $\sigma = 0$. Then,

$$\mathscr{G}_{0\Lambda_m(y)-0\cdot\varsigma}(\epsilon) = \mathscr{G}(\epsilon) = 1 > 1 - \emptyset. \tag{62}$$

So

$$\mathscr{G}_{0\cdot\Lambda_m}(y) = 0. \tag{63}$$

Now let $\sigma \in \mathbb{R}$ $(\sigma \neq 0)$. Since

$$\mathscr{G}_{\Lambda_m(\text{stat})}\lim y = \varsigma, \tag{64}$$

it follows from Theorem 13 that

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y = \varsigma. \tag{65}$$

If

$$M_{\mathscr{G}}(\rho,\epsilon) \coloneqq \left\{m_0 \in \mathbb{N} : \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) \leqq 1 - \emptyset\right\}, \tag{66}$$

then

$$\delta\left\{M_{\mathscr{G}}(\rho,\epsilon)\right\} = 0 \tag{67}$$

$\forall\epsilon > 0$. In this case,

$$\delta\left\{\mathbb{N} \setminus M_{\mathscr{G}}(\rho,\epsilon)\right\} = 1. \tag{68}$$

If $m_0 \in N \setminus M_{\mathscr{G}}(\rho,\epsilon)$, then

$$\begin{aligned}\mathscr{G}_{\Lambda_m(y)-\rho\varsigma}(\epsilon) &= \mathscr{G}_{\Lambda_m(y)-\varsigma}\left(\frac{\epsilon}{|\sigma|}\right)\\ &\geqq \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) * \mathscr{G}_0\left(\frac{\epsilon}{|\sigma|} - \epsilon\right)\\ &= \mathscr{G}_{\Lambda_m(y)-}\varsigma(\epsilon) * 1 = \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon)\\ &= 1 - \emptyset\end{aligned} \tag{69}$$

for $\sigma \in \mathbb{R}$ $(\sigma \neq 0)$.

This demonstrates that

$$\delta\left(\left\{m_0 \in \mathbb{N} : \mathscr{G}_{\sigma\Lambda_m(y)-\sigma\varsigma}(\epsilon) \leqq 1 - \emptyset\right\}\right) = 0, \tag{70}$$

so

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim \sigma y = \sigma\varsigma, \tag{71}$$

thereby completing the proof of Theorem 15. □

Theorem 16. *Let* $(Y;\mathscr{G};*)$ *be a PN-space. Then*

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y = \varsigma, \tag{72}$$

if and only if there exists a set

$$m = \left\{m_1 < m_2 < m_3 < \cdots < m_k < \cdots\right\} \subseteq \mathbb{N} \tag{73}$$

with

$$\delta(M) = 1 \tag{74}$$

such that

$$\mathscr{G}\lim \Lambda_{m_n}(y) = \varsigma. \tag{75}$$

Proof. In order to prove the necessity part, first assume that

$$\mathscr{G}_{\Lambda}\,(\text{stat})\lim y = \varsigma. \tag{76}$$

Now, for every $\epsilon > 0$ and $j \in N$, let

$$\begin{gathered}M(j,\epsilon) : \left\{m \in \mathbb{N} : \mathscr{G}_{\Lambda_m(y-\varsigma)}(\epsilon) \leqq 1 - \frac{1}{j}\right\},\\ K(j,\epsilon) \coloneqq \left\{m \in \mathbb{N} : \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) > 1 - \frac{1}{j}\right\}.\end{gathered} \tag{77}$$

Then

$$\delta(M(j,\epsilon)) = 0, \tag{78}$$

$$K(1,\epsilon) \supset K(2,\epsilon) \supset \cdots \supset K(I,\epsilon) \supset K(i+1,\epsilon) \supset \cdots \tag{79}$$

It has

$$\delta(K(j,\epsilon)) = 1 \quad (j \in \mathbb{N}). \tag{80}$$

Now we prove that, for $m \in K(j,\epsilon)$, the sequence (y_m) is $\mathscr{G}_{\Lambda}$(stat)-convergent to ζ. Suppose, on the contrary, that the sequence (y_m) is not $\mathscr{G}_{\Lambda}$(stat)-convergent to ζ. Therefore, there exits $\rho > 0$ such that the set

$$\left\{m \in \mathbb{N} : \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) \leqq 1 - \rho\right\} \tag{81}$$

has infinitely many terms. Let

$$\begin{gathered}K(\rho,\epsilon) \coloneqq \left\{m \in \mathbb{N} : \mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) > 1 - \rho\right\},\\ \rho > \frac{1}{j} \quad (j \in \mathbb{N}).\end{gathered} \tag{82}$$

Then

$$\delta(K(p,\epsilon)) = 0, \tag{83}$$

so that, by using (79),

$$K(j,\epsilon) \subset K(\rho,\epsilon). \tag{84}$$

Hence

$$\delta(K(\rho,\epsilon)) = 0, \tag{85}$$

which contradicts with (80). Consequently, the sequence (y_m) is $\mathscr{G}_{\Lambda}$(stat)-convergent to ζ.

Next, to prove the sufficiency part, it is assumed that there exists a subset

$$\begin{aligned}M &= \left\{m_1 < m_2 < m_3 < \cdots < m_k < \cdots\right\}\\ &\subseteq \mathbb{N},\\ \delta(M) &= 1,\\ \mathscr{G}\lim_{n\to\infty}\Lambda_{m_n(y)} &= \varsigma.\end{aligned} \tag{86}$$

Then, for every $\rho \in (0,1)$ and $\epsilon > 0$,

$$\mathscr{G}_{\Lambda_m(y)-\varsigma}(\epsilon) > 1 - \rho \quad (m \in \mathbb{N}). \tag{87}$$

Now

$$K(\rho,\epsilon) := \left\{m \in \mathbb{N} : \mathcal{G}_{\Lambda_m(y)-\varsigma}(\epsilon) \leqq 1-\rho\right\} \subseteq \mathbb{N} - \left\{M_{N+1}, M_{N+2}, \ldots\right\}. \tag{88}$$

Therefore,

$$\delta\left(K(p,\epsilon)\right) \leqq 1-1 = 0. \tag{89}$$

Hence

$$\mathcal{G}_{\Lambda}(\text{stat}) \lim y = \varsigma. \tag{90}$$

□

Theorem 17. *A sequence* $y = (y_m)$ *in a PN-space* $(Y; \mathcal{G}; *)$ *is* $\mathcal{G}_\Lambda$(stat)*-convergent if and only if it is* $\mathcal{G}_\Lambda$ *(stat)-Cauchy.*

Proof. Let the sequence y be a $\mathcal{G}_\Lambda$ (stat)-convergent to ζ in PN-space; that is,

$$\mathcal{G}_{\Lambda}(\text{stat}) \lim y = \zeta. \tag{91}$$

Then, $\forall \epsilon > 0$ and $\rho \in (0,1)$,

$$\delta\left(\left\{m \in \mathbb{N} : \mathcal{G}_{\Lambda_m(y)-\zeta}(\epsilon) \leqq 1-\rho\right\}\right) = 0. \tag{92}$$

Select a number $N = N(\boldsymbol{\epsilon})$,

$$\mathcal{G}_{\Lambda_m(y)-\zeta}(\boldsymbol{\epsilon}) \leqq 1-\rho. \tag{93}$$

Now let

$$\begin{aligned} A(\rho,\boldsymbol{\epsilon}) &= \left\{m \in \mathbb{N} : \mathcal{G}_{\Lambda_M(y)-\Lambda_N(y)}(\epsilon) \leqq 1-\rho\right\}, \\ B(\rho,\boldsymbol{\epsilon}) &= \left\{m \in \mathbb{N} : \mathcal{G}_{\Lambda_M(y)-\zeta}(\epsilon) \leqq 1-\rho\right\} \\ C(\rho,\boldsymbol{\epsilon}) &= \left\{m = N \in \mathbb{N} : \mathcal{G}_{\Lambda_N(y)-\zeta}(\epsilon) \leqq 1-\rho\right\}. \end{aligned} \tag{94}$$

Then

$$A(\rho,\boldsymbol{\epsilon}) \subseteq B(\rho,\boldsymbol{\epsilon}) \cup C(\rho,\boldsymbol{\epsilon}). \tag{95}$$

Therefore,

$$\delta\left(A(\rho,\boldsymbol{\epsilon})\right) \leqq \delta\left(B(\rho,\boldsymbol{\epsilon})\right) + \delta\left(C(\rho,\boldsymbol{\epsilon})\right). \tag{96}$$

Hence, y is statistically Λ-Cauchy.

Conversely, let y be a statistically Λ-Cauchy sequence, but not statistically Λ-convergent. There exists N such that the set $A\ (\rho,\boldsymbol{\epsilon})$ has natural density zero. Therefore, the set

$$E(\rho,\boldsymbol{\epsilon}) = \left\{M \in \mathbb{N} : \mathcal{G}_{\Lambda_M(y)-\Lambda_N(y)}(\epsilon) > 1-\rho\right\} \tag{97}$$

has natural density 1; that is,

$$\left(E(\rho,\boldsymbol{\epsilon})\right) = 1. \tag{98}$$

Particularly, it can be expressed as

$$\mathcal{G}_{\Lambda_M(y)-\Lambda_N(y)}(\boldsymbol{\epsilon}) \leqq 2\mathcal{G}_{\Lambda_M(y)-\zeta} < \boldsymbol{\epsilon} \tag{99}$$

if

$$\mathcal{G}_{\Lambda_M(y)-\zeta} < \frac{\epsilon}{2}. \tag{100}$$

Since y is not statistically Λ-convergent, the set $(\rho,\boldsymbol{\epsilon})$ has natural density 1; that is,

$$\delta\left(\left\{M \in \mathbb{N} : \mathcal{G}_{\Lambda_M(y)-\zeta}(\epsilon) > 1-\rho\right\}\right) = 0. \tag{101}$$

Hence, by (10), we get

$$\delta\left(\left\{M \in \mathbb{N} : \mathcal{G}_{\Lambda_M(y)-\Lambda_N(y)}(\epsilon) > 1-\rho\right\}\right) = 0. \tag{102}$$

This is the contradiction that the set $(\rho,\boldsymbol{\epsilon})$ has natural density 1. Therefore, the sequence y is statistically Λ-convergent. □

3. Conclusion

This paper has used the notion of Λ-convergence and studied it in the context of probabilistic normed (PN) spaces. Statistical Λ-convergence and statistical Λ-Cauchy were defined in PN-spaces and gave some illustrative examples to demonstrate these concepts. Λ-convergence can be used to study the Korovkin type approximation theorems in PN-spaces (see, for example, [1] and the references cited therein).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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