EVALUATING THE EFFECTS OF RISK ON THE ECONOMICS OF JUVENILE SPACING AND COMMERCIAL THINNING

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by

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Abstract

The temporarily increased fire hazard which is believed to result from the process of thinning is included in a single-stand model for assessing the economic benefits of juvenile spacing. Formulas for the expected net present value and the land expectation value are given along with methods for determining the age of financial maturity and the optimal rotation age. A numerical example is given to illustrate the degree of loss due to the increased fire risk. The problem of commercial thinning when the risk of fire is present is addressed using continuous-time models. It is shown how, when the fire hazard is exogenous to the thinning activity, the problem reduces to one of deterministic optimal control with the discount rate adjusted upward by an amount equal to the fire hazard rate. In the case when the fire hazard increases whenever thinning is taking place, it is shown that in general the optimal thinning policy is qualitatively different from that which is optimal in the no-risk case and involves periods of thinning at the maximum rate interspersed with periods of no thinning activity.

1. Introduction

The economic benefits of the silvicultural practice of juvenile spacing of naturally regenerated stands have been well documented (see e.g., Nawitka Resource Consultants, 1987 and references therein; Brix, 1983 and Barclay and Brix, 1985). These benefits can include an increase in volume growth rate with a consequent reduction in rotation age, and an improvement in stem sizes resulting in increased value of timber produced. A 'down-side' consequence of spacing which has been recognized by foresters involved in protection, but which appears to have been largely ignored from the analytic point of view, is the potential increase in the fire hazard resulting from the spacing activity (Fuglem, Lawson and Hawkes, 1983). Large quantities of debris among spaced stands can become highly flammable during periods of dry weather. Thus as Fuglem et al. point out, "Expected gains in wood volumes can be negated by loss of projects and surrounding productive forest by fire."

Earlier results on the effects of the fire hazard on forest production (Martell, 1980, Van Wagner, 1983, Reed and Errico, 1985, 1986) suggest that the degree of this loss could be considerable. In view of this fact, it seems imperative that the increased fire risk be incorporated in any economic cost benefit analysis of juvenile spacing, either by including on the cost side additional expenditures involved in protecting a newly spaced stand, or through discounting the expected benefits in accordance with the increased fire risk.

It is the purpose of this paper to develop an analytic methodology which incorporates the increased fire hazard in the determination of the expected benefits of spacing. This will allow decision makers to determine whether the protection of spaced stands is cost-effective, and further whether the practice of spacing provides net economic benefits.

Another problem addressed in the paper is the question of how the risk of fire affects the optimal pattern of commercial thinning. The continuous–time model of Clark and de Pree (1979) is adapted to include a fire hazard. It is shown how, if the fire hazard depends only on the age of the stand, the problem can be reduced to one of deterministic optimal control and that the optimal thinning schedule is equivalent to that in a risk–free environment, but with a premium added to the discount rate; it involves thinning along a 'singular path', followed by a period of no thinning leading up to a final clear–cut harvest. If on the other hand the fire–hazard increases whenever thinning is taking place, the above situation no longer prevails. It is shown that in this case the optimal harvest pattern involves, in general, periodic bursts of thinning at the maximum rate, interspersed with periods of no thinning.

In Section 2 the methodology is developed for determining the expected net present value of a stand when juvenile spacing takes place. It is assumed that upon spacing, the fire hazard undergoes a sudden jump, but subsequently decreases smoothly to its initial (pre–spacing) level. The problem is addressed considering both a single rotation, and ongoing rotations under the Faustmann paradigm (see e.g., Samuelson, 1976.) Formulas for the expected net present value and the land expectation value are derived. In Section 3 the methods are applied to growth and yield data for coastal Douglas–fir.

In Section 4 the problem of commercial thinning in the presence of the risk of fire is considered.

2. Determination of the Age of Financial Maturity and the Net Present Value of a Spaced Stand.

Consider a naturally regenerated stand. The age of financial maturity and the net present value of the stand (with and without spacing) can be determined in terms of the following quantities:

$V_1(t)$ = per hectare value of lumber net of harvest costs, <u>for spaced stands</u> when harvested at age t years

$V_0(t)$ = per hectare value of lumber net of harvest costs <u>for unspaced stands</u>, when harvested at age t years[1]

t_s = age of spacing (years)

C_s = per hectare costs of spacing

δ = per annum instantaneous discount rate

$h(t)$ = instantaneous per annum fire hazard[2]

It is assumed that in the absence of spacing the fire hazard is constant

[1] $$h_0(t) \equiv \lambda,$$

say. When spacing takes place the hazard will typically experience a discontinuous 'jump' increase followed by a smooth decline, gradual at first but

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[1]Here and in the remainder of this section the subscript 0 is used to refer to unspaced stands, while the subscript 1 is used to refer to spaced stands.

[2]The hazard $h(t)$ is the instantaneous probability rate

$$h(t) = \lim_{\Delta \to 0} \{P[\text{stand destroyed in } (t,t+\Delta] \,|\, \text{alive at } t]/\Delta\}$$

A constant hazard $h(t) \equiv .01$, say, would correspond to, on average, 1 fire per one hundred years.

becoming steeper later on (P. Fuglem, <u>personal communication</u>). This could be modelled in a number of ways, but a simple model which captures the essence of this behaviour is to suppose that on top of the background hazard λ is superimposed an additional hazard shaped like the right–hand half of a normal probability density (see Fig. 1). Specifically we shall assume that for spaced stands the hazard follows:

[2] $$h_1(t) = \begin{cases} \lambda & t < t_s \\ \lambda + \rho e^{-\theta(t-t_s)^2} & t \geq t_3 \end{cases} .$$

The parameter ρ determines the magnitude of the initial jump increase, while θ determines how fast it decays.

With these assumptions it can be shown (see Appendix 1) that the expected discounted return net of costs over a single rotation, with harvest age T, is

[3] $$J_0(T) = V_0(T)\, e^{-(\lambda+\delta)(T-t_s)}$$

for unspaced stands; and is

[4] $$J_1(T) = V_1(T)\, \exp\left\{-(\lambda+\delta)(T-t_s) - \frac{\rho}{2}\,\mathrm{erf}(\sqrt{\theta}(T-t_s))\right\} - C_s$$

if the stand is spaced at age t_s. In the above formulas, δ is the <u>instantaneous discount rate</u> and $\mathrm{erf}(\cdot)$ is the <u>error function</u>[3]

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[3]Alternatively the expected net present value can be expressed in terms of the cumulative distribution function of a standard normal deviate,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}X^2}\, dX$$

as $$J_1(T) = V_1(T)\, \exp\left\{-\rho\sqrt{\frac{\pi}{\theta}}\,[\Phi(\sqrt{2\theta}(t-t_s))-\tfrac{1}{2}] - \lambda(t-t_s)\right\} - C_s .$$

[5] $$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\, dx$$

for which numerical values are readily available.

The ages T_0 and T_1 which maximize J_0 and J_1 respectively give the (single–rotation) ages of financial maturity for unspaced and spaced stands, while the corresponding values of J_0 and J_1 give the expected net present value of the stands of the two types at age t_s.

A comparison of $J_0(T_0)$ and $J_1(T_1)$ will determine whether spacing is worthwhile or not in the presence of an increased fire hazard due to the act of spacing.

To extend the above formulas to take into account ongoing rotations using the Faustmann paradigm, it is necessary to make some assumptions about future rotations. The simplest assumption is that a stand is naturally re–established following either a harvest or a fire, and that in subsequent rotations it follows the same value curve (V_0 or V_1).

Under these assumptions it can be shown (see Reed, 1987 for details of a similar derivation) that the expected present value net of harvest costs at time $t = t_s$ over an infinite sequence of rotations, and using harvest age T, is

[6] $$M_0(T) = \frac{J_0(T)}{\delta \int_0^T e^{-\delta t}\, S_0(t)\,dt} = \frac{(\lambda+\delta)e^{-(\lambda+\delta)(T-t_s)}\; V_0(T)}{\delta(1 - e^{-(\lambda+\delta)T})}$$

if no spacing is practiced in any rotation; and is

[7] $$M_1(T) = \frac{J_1(T)}{\delta \int_0^T e^{-\delta t}\, S_1(t)\,dt}$$

if spacing is practiced in every rotation when a stand reaches age t_s.[4]

In the above formulas $S_0(t)$ and $S_1(t)$ are the probabilities that unspaced and spaced stands survive until at t:

[8] $$S_0(t) = \exp\left\{-\int_0^t h_0(x)\,dx\right\} = e^{-\lambda t}$$

[9] $$S_1(t) = \exp\left\{-\int_0^t h_1(x)\,dx\right\} = \begin{cases} e^{-\lambda t} & , \ t < t_s \\ \exp\left\{-\frac{\rho}{2}\sqrt{\frac{\pi}{\theta}}\,\mathrm{erf}(\sqrt{\theta}(t-t_s)) - \lambda t\right\}, & t \geq t_s \end{cases}.$$

If there are costs C_p per hectare associated with re–establishment, the formulas [6] and [7] must be ammended. In this case the expected net present value of returns, starting with a bare site is

[10] $$L_0(T) = \frac{V_0(T)\ e^{-\delta T}\ S_0(T) - C_p}{\delta \int_0^T e^{-\delta t}\ S_0(t)\ dt} = \frac{(\lambda+\delta)\,[e^{-(\lambda+\delta)T}\ V_0(T) - C_p]}{\delta(1 - e^{-(\lambda+\delta)T})}$$

when there is no spacing, and

[11] $$L_1(T) = \frac{V_1(T)\ e^{-\delta}\ S_1(T) - C_p - C_s\ e^{-\delta t_s}\ S_1(t_s)}{\delta \int_0^T e^{-\delta t}\ S_1(t)\ ds}$$

[4]Note that it is possible that some rotations do not reach the spacing age t_s, being destroyed by fire (with underlying hazard = λ) at some earlier age.

when spacing takes place. The integral in the denominator cannot be performed analytically; however it can be performed numerically using standard methods of quadrature (see e.g. Atkinson, 1978).

Maximization of [10] and [11] over T will determine the optimal rotation ages. The corresponding maximum values of [10] and [11] determine the land expectation value of a bare site, given that spacing does not, or does take place. A comparison of these two quantities will determine the financial advantage (if any) of spacing.

3. Numerical Results

In this section the methods developed in the previous section are used in an example, using growth and yield data for coastal Douglas-fir [pseudotsuga menziesii (Mirb.) Franco] generated by the FANSY (Financial Analysis System) simulator used by the B.C. Forest Service. Table 1 gives estimated per hectare lumber values net of harvest and processing costs, for natural and spaced stands at various harvest ages. Spaced stands were assumed to be spaced at age 15 years corresponding approximately to spacing taking place at height 10m. The per hectare cost of spacing, C_s, was set at \$1000 and planting costs, C_p, were set to zero, it being assumed that after harvesting, stands are left to regenerate naturally. The background fire hazard λ was set at 0.005.

Three values of ρ, the parameter representing the jump in the hazard on spacing, ($\rho = 0.01$, 0.03 and 0.05), and two values for the parameter θ representing how fast the extra induced hazard decays ($\theta = 0.01$ and 0.1), were used in the calculations. With θ set at 0.1 the extra hazard has dropped by 50 percent after 2.6 years, and by 90 percent after 4.8 years. The corresponding times with θ set at 0.01 are 8.4 years and 15.2 years. This range seems to cover what one might reasonably expect for most species in most locations. To give some idea of the values of ρ used in the computations, the probability of destruction by fire in the year immediately following spacing has been calculated and is given in Table 2. These values should be compared with the probability 0.005 assumed for unspaced stands. The largest value (with $\rho = 0.05$ and $\theta = .01$) is 0.0534 or about a 1 in 20 chance, while the smallest value (with $\rho = .01$ and $\theta = 0.1$) is 0.0146 or about a 1 in 70 chance. This range of values should cover what one might expect to experience in practice.

Table 3 gives the single-rotation expected net present values (maxima of $J_0(T)$ and $J_1(T)$ in [3] and [4]; ages of financial maturity in parentheses) for spaced and unspaced stands. Table 4 gives the corresponding expected net present values with ongoing rotations (maxima of $M_0(T)$ and $M_1(T)$ in [6] and [7]) with optimal rotation ages in parentheses.

The first thing that stands out from the results in Tables 3 and 4 is that spacing does not appear to be economically attractive even in the absence of an extra fire hazard. This conclusion, however, depends crucially on the growth and yield values of Table 1, and since no effort has been made here to validate these values, not too much weight should be placed on the conclusion, even though it agrees with that of Nawitka Resource Consultants (1987, p. 64).

The loss in the estimated single–rotation net present value of thinned stands due to the increased fire hazard varies between 3.7% (when $\rho = .01$, $\theta = .1$) and 48.8% (when $\rho = .05$ and $\theta = .01$) for a discount rate of 3 percent. The corresponding range of losses for a discount rate of 5 percent is from 10.2% ($\rho = .01$, $\theta = .1$) to 100% ($\rho = .05$, $\theta = .01$), since in this latter case the expected net present value is negative.

For ongoing rotations the estimated loss is slightly less, ranging from 2.4% to 39.4% for a discount rate of 3 percent, and from 9.5% to 100% for a discount rate of 5 percent.

These results indicate that with a discount rate of 5 percent or higher the loss in expected present value due to increased fire risk is likely to be quite significant (ranging from 10 percent to 100 percent in the examples given). For a lower discount rate the loss in expected present value is likely to be less serious but could still be significant in dry regions where the jump in hazard due to spacing is likely to be large. In situations where spacing is estimated to be of marginal economic benefit, the additional loss due to increased fire risk could well tip the scales against spacing projects.

It is difficult to estimate exactly what the increased fire hazard would be, but local foresters with experience on spacing projects could well estimate ranges for the parameters used, and thereby one could obtain a broad estimate of expected losses.

4. Commercial Thinning

The problem of determining an optimal pattern of thinning along with the optimal rotation age has been discussed by several authors. Amidon and Akin (1968) and Kilkki and Väisänen (1969) used discrete dynamic programming, while Clark and de Pree (1979) and Cawrse, Betters and Kent (1984) formulated the problem in continuous time and used the methods of optimal control theory to solve it. In this section we shall essentially use the formulation of Clark and de Pree, but include the additional possibility of the stand being destroyed by fire.

As in Section 2 we denote the hazard function by $h(t)$, the survivor function by $S(t)$, and the discount rate by δ. In addition let

$X(t) =$ volume of timber in stand at age t

$p(t) =$ the unit value (price) of timber at age t

$C_0 =$ the cost per unit volume of thinning

$r(t) =$ $p(t) - C_0$ = the unit net revenue from thinning at age t

$C_1 =$ the cost per unit volume of clear-cut harvesting

$q(t) =$ $p(t) - C_1$ = the unit net revenue from clear-cut harvesting at age t

$u(t) =$ the rate (volume per unit time) of thinning at age t

$T =$ the age of clear-cut harvesting

$S(t) =$ probability of stand surviving until age t

$y(t) =$ $-\ell n\, S(t)$

Following Clark and de Pree and Kilkki and Väisänen it will be assumed that the volume grows according to the differential equation

[12] $$\frac{dX}{dt} = g(t)\, F(X) - u(t)$$

with $X(0) = X_0$. The function $g(t)$ is assumed to be positive and decreasing; and $F(X)$ is assumed to be positive and concave with $F(0) = 0$. Thus the proportional growth rate $\dot{X}/X$ decreases both with age and volume.

If the thinning schedule $u(t)$ is followed over $0 \leq t \leq T$, with a clear-cut harvest occurring at age T, then the expected present value of net revenues earned over a single rotation is:

[13] $$J = \int_0^T e^{-\delta t} r(t)u(t)S(t)dt + e^{-\delta T} q(T)X(T)S(T)$$

The terms $S(t)$ and $S(T)$ are included because of the fact that the stand may be destroyed by fire at any time prior to T (see Reed, 1987). We seek a thinning schedule $u(t)$ and a clear-cut harvest age T to maximize [13]. It follows from the definition of $S(t)$ (see Appendix 1), that

[14] $$\frac{dy}{dt} = \frac{d}{dt}(-\ell n\, S(t)) = h(t).$$

Thus the optimization problem can be expressed as the following linear deterministic optimal control problem:

Maximize

[15] $$J = \int_0^T e^{-\delta t - y(t)} r(t)u(t)dt + e^{-\delta T - y(T)} q(T)X(T)$$

subject to:

[16] $$\frac{dX}{dt} = g(t)F(X) - u(t); \qquad X(0) = X_0$$

[17] $$\frac{dy}{dt} = h(t); \qquad y(0) = 0$$

over T and $u(t)$ $(0 \le t \le T)$ with $0 \le u(t) \le u_{max}$.

Here u_{max} is the maximum rate at which thinning can take place.

Note that the inclusion of the possibility of catastrophic destruction results in the inclusion of a second state variable, $y(t)$, in the control problem. This variable which is increasing in t, is related to the cumulative probability that the stand is destroyed by time t. However when the hazard depends only on the age of the stand (and not on the thinning activity $u(t)$, nor on the volume $X(t)$) the problem can readily be reduced to a variant of the no-risk problem addressed by Clark and de Pree [1979].

To see this let

[18] $$\Delta(t) = \delta + h(t)$$

be a "risk-adjusted" discount rate, variable in time. Then the discount factor $e^{-\delta t - y(t)}$ in the objective [15] can be expressed as $e^{-\int_0^t \Delta(s)ds}$, and so the problem becomes identical with that solved by Clark and de Pree, except for the substitution of a variable discount rate $\Delta(t)$, for the constant discount rate δ.

We can thus conclude that the optimal policy, in general, involves a period of no thinning followed by a period of thinning, in which the volume $X(t)$ is kept on a singular path, followed finally by a period of no thinning leading up to a clear-cut harvest.

The singular path is determined by the equation[5]

[19] $$g(t)F'(X) = \delta + h(t) - r'(t)/r(t)$$

This equation is a time-dependent version of the golden-rule of capital theory (see Clark, 1985, pp. 77–78). The left hand side represents the marginal timber volume productivity of the resource corresponding to a unit increase in stand volume, while the right hand side is the discount rate adjusted for the changing unit value of timber $(r'(t)/r(t))$ and for the presence of risk $(h(t))$. The clear-cut harvest age by the equation

[20] $$\frac{g(T)F(X(T))q(T) + X(T)q'(T)}{X(T)q(T)} = \delta + h(T).$$

This is a Wicksell-type condition (Wicksell, [1934]). The left-hand side is the proportional rate of change in clear-cut harvest revenue, $q(T)X(T)$, while the right-hand side is the discount rate adjusted for risk.

The effect of the presence of a fire hazard is the same as that of an increased discount rate — it results in faster optimal thinning and a lower harvest age.

Note that in order to determine the switching times when thinning starts and ends, numerical methods are required. Similar methods to those used in the no-risk problem (Clark and de Pree, 1979) can be employed. Also in order to solve the problem with ongoing rotations (Faustmann paradigm), numerical methods are required (see Appendix 2 for details of one possible method).

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[5]This equation is derived formally using the Pontryagin maximum principle in Appendix 2.

As an example the problem treated by Clark and de Pree (following Kilkki and Väisänen) for ongoing rotations was solved with the presence of fire risk. It is for a stand of Scotch Pine, for which the volume, $X(t)$, of usable wood per hectare is assumed to follow

$$\frac{dX}{dt} = 56.9t^{-1.23}Xe^{-.003X}.$$

The unit net revenues from thinning and clear-cut harvesting are

$$r(t) = .337t + 5.06 \qquad q(t) = 1.1r(t) \qquad (\mathrm{Fmk/m^3}).$$

A discount rate of $\delta = 0.02$, and a constant (age independent) hazard of $h(t) \equiv 0.013$ were assumed. Replanting costs were assumed to be $C_p = 800$ Fmk/ha.

Figure 2 shows the optimal trajectory for $X(t)$. It can be seen that there is no harvesting below age 57 years or above age 72 years. The optimal rotation age is 76 years. In the case of no fire risk Clark and de Pree report ages 85 and 90 years (rounded to nearest 5 years) for the final age of thinning and for rotation.

__Hazard dependent on thinning activity__.

As noted in the introduction the act of spacing or thinning is likely to increase the fire hazard. To incorporate an increased hazard of the half-normal or similar form (equation [2]) into the dynamic thinning model discussed above would lead to an intractable optimal control problem for a system with delay. However something of the flavour of the problem can be incorporated using a simpler model in which the fire hazard increases at times when thinning is taking place, and falls back to its background level as soon as thinning stops. This is equivalent to setting the decay parameter θ in [2] to a very large value. Thus we shall assume that

$$\text{hazard at time } t = h(t) + \rho\, H(u)$$

where H(u) is the step function

[21] $$H(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0. \end{cases}$$

The only change required in the optimal control problem above is that the state equation [16] be replaced by

[22] $$\frac{dy}{dt} = h(t) + \rho\, H(u).$$

This change, however, introduces a non-linearity into the problem — indeed the dynamic equaton is no longer continuous in u. Also the problem cannot be reduced to a single-state variable problem using a risk adjusted discount rate, because y(t) depends on thinning activity at previous times.

Nonetheless the maximum principle can still be used as a heuristic to determine the nature of the optimal thinning pattern.[6] It is shown in Appendix 3 that the optimal thinning is qualitatively different from that which prevails when the hazard depends only on stand age. It no longer involves singular control over a period. Rather any thinning that takes place is done at the maximum rate (u_{max}). In general, periods of thinning at the maximum rate alternate with periods of no thinning. The reason for this is fairly obvious. Since the act of thinning

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[6]While the simplest statement of the maximum principle assumes that the state equations are continuous in the control variables, its use here can be justified by appealing to the Hamilton–Jacobi–Bellman equation of dynamic programming, which results in the same solution as that provided by the maximum principle – see Appendix 3.

increases the fire hazard one wants to keep the time of thinning as short as possible. This is achieved by using the maximum rate of thinning for a short time, rather than using some intermediate rate of thinning over a longer time period. Thus optimally, thinning occurs in distinct pulses at the maximum rate. In Appendix 3 the conditions for switching from no thinning to maximum thinning are derived and interpreted.

Besides depending on the age of the stand and on whether or not thinning is taking place, the fire hazard may also depend on the volume density of the stand. For example a high volume stand of a given age may have a lower hazard than a low volume stand of similar age. For a high volume stand much of the forest floor will be in shade most of the time, and so the litter on the forest floor will tend to retain moisture longer than it would in the case of a low volume stand where the forest floor would receive more direct sunlight.

To model this we can assume that the hazard is of the form

$$\varphi(t,X) + \rho\, H(u)$$

where the function φ has partial derivative $\varphi_X \leq 0$. The dynamic equation for the state variable y becomes in this case

$$\frac{dy}{dt} = \varphi(t,X) + \rho\, H(u). \qquad [23]$$

It is easily demonstrated (see Appendix 3) that in this case as well the optimal thinning schedule involves alternating periods of zero thinning and thinning at the maximum rate.

5. Summary and Conclusions

Techniques have been developed for determining the effects of fire risk on the economics of juvenile spacing and commercial thinning. This is important since it is believed by fire–protection foresters that these activities may well increase the fire hazard significantly at least in the years immediately following the thinning activity.

In the case of juvenile spacing, it is assumed that the hazard jumps immediately spacing takes place, and then decays following a form like the right–hand half of a normal probability density. While there is no reason to believe that the hazard would be exactly of this form, it does capture the essential characteristics and is convenient from a mathematical point of view. Formulas are derived for the expected net present value and the land expectation value under these assumptions, and it is demonstrated how the age of financial maturity and optimal rotation age can be determined. In a numerical example using simulated growth and yield data for coastal Douglas–fir, it is seen that the presence of the extra fire hazard induced by spacing could easily reduce the net present value of a spaced stand by 10 percent or more.

In the case of commercial thinning, two distinct models in continuous time are analyzed. In the first, the fire hazard is assumed to be exogenous to the silvicultural activities, depending only on the age of the stand. In this case the effect upon the optimal thinning schedule and the net present value of the stand is equivalent to an upward adjustment of the discount rate by an amount equal to the hazard (for a non–constant hazard, this would result in a non–constant adjusted discount rate). The overall effect is an increase in the optimal thinning rate and a decrease in the optimal rotation period and in the net present value.

In the second case considered, it is assumed that the fire hazard increases by a fixed amount whenever thinning is taking place, but returns to its background level as soon as thinning ceases. It is shown how in this case the optimal policy is qualitatively different from that which is optimal in the no-risk case (or in the case of an exogenously determined hazard). Rather than thinning along a singular path, the optimal policy now in general involves pulses of thinning at the maximum feasible rate interspersed with periods of no thinning. This is closer to what happens in practice with commercial thinning, although the reasons for this are probably due to economic factors rather than the presence of risk. The model captures some of the features of how thinning might affect the fire hazard although, unlike the juvenile spacing model, it does not allow the hazard to persist after the thinning activity ceases. To handle such a model in continuous time would be extremely difficult, although it could possibly be handled numerically in discrete time using stochastic dynamic programming provided the extra induced hazard does not persist for more than one period (e.g., decade). We speculate that the optimal policy would still involve periods of thinning at the maximum rate, although possibly with fewer pulses than occur in the model described herein.

Finally it should be noted that the only form of risk discussed in this paper is the risk of catastrophic loss through fire. Risks associated with future uncertainty in lumber prices and in volume growth have been ignored. The problem of determining the optimal harvest rule and the land expectation value in the face of such uncertainty has been addressed elsewhere (Reed and Clarke [1990], Clarke and Reed [1989], Brock, Rothschild and Stiglitz [1988]). To adapt these results to the problem of juvenile spacing should present no theoretical difficulty. To decide whether spacing is justified a comparison of expected net present values with and without spacing can be made as discussed in Section 2. The inclusion of price and growth stochasticity in the model for commercial thinning however would lead to a difficult problem in stochastic optimal control.

APPENDIX 1

Derivation of the Expected Net Present Value of Spaced and Unspaced Stands

For unspaced stands the hazard is given by (see [1]),

[A1.1] $$h_0(t) \equiv \lambda$$

while for spaced stands it is given by (see [2])

[A1.2] $$h_1(t) = \begin{cases} \lambda & , \quad t < t_s \\ \lambda + \rho e^{-\theta(t-t_s)^2}, & t \geq t_3 \end{cases} .$$

Related to any hazard function $h(t)$ is the <u>survivor function</u>

[A1.3] $$S(t) = \exp\left\{\int_0^t h(z)\, dz\right\}$$

which determines the probability that the stand survives from establishment (at age 0) to age t. Given that the stand is alive at age $t = t_s$, the conditional probability that it survives to age t $(> t_s)$ is

[A1.4] $$S(t|t_s) = S(t)/S(t_s) = \exp\left\{\int_{t_s}^t h(z)\, dz\right\}$$

For an unspaced stand this conditional survival probability is:

[A1.5] $$S_0(t|t_s) = e^{-\lambda(t-t_s)}$$

while for a spaced stand it is:

[A1.6] $$S_1(t|t_s) = \exp\left\{-\int_{t_s}^{t} (\lambda + \rho e^{-\theta(z-t_s)^2}) dz\right\}$$

which can be expressed in terms of the error function, erf(·) (see [5]) as:

[A1.7] $$S_1(t|t_s) = \exp\left\{-\frac{\rho}{2}\sqrt{\frac{\pi}{\theta}}\, \mathrm{erf}(\sqrt{\theta}(t-t_s)) - \lambda(t-t_s))\right\}/$$

For a stand alive at age $t = t_s$, the expected present value of the net return per hectare, with a harvest age T is given by

[A1.8] $$J_0(T) = V_0(T)e^{-\delta(T-t_s)}\, S_1(t|t_s) - C_s$$

$$= V_1(T)\ \exp\left\{-(\lambda+\delta)(T-t_s) - \frac{\rho}{2}\sqrt{\frac{\pi}{\theta}}\, \mathrm{erf}(\sqrt{\theta}(T-t_s))\right\} - C_s$$

if the stand is spaced at age t_s. Equations [A1.8] and [A1.9] are the formulas [3] and [4] given in Section 2 for expected net present value.

APPENDIX 2

Determination of the Optimal Rotation Period and the Land Expectation Value for the Thinning Model of Section 4.

For a single rotation, the optimal thinning schedule, and clear–cut harvest age are determined by solving the optimal control problem described in Section 4 by [15]–[17]. In the case of ongoing rotations this problem must be modified to take into account the fact that after a clear–cut harvest or after a fire, the land still has a value which can be imputed from the revenues earned by future rotations that can be grown upon it. Denote this land expectation value by L and the costs of re–establishment of a stand by C_p.

The net revenue earned in the first rotation using thinning schedule $u(t)$ and clear–cut harvest age T is a random variable with value

[A2.1] $$\int_0^Z e^{-\delta t} r(t)u(t)dt + e^{-\delta Z}(L-C_p)$$

if the stand is destroyed by fire at age $Z < T$, and value

[A2.2] $$\int_0^T e^{-\delta t} r(t)u(t)dt + e^{-\delta T}(q(T)X(T) + L - C_p)$$

if the stand survives to be harvested at age T. The expected value of this random variable is

[A2.3] $$\int_0^T e^{-\delta t-y(t)} [r(t)u(t) - \delta L] + e^{-\delta T-y(T)} q(T)X(T) + L$$

(see Reed [1987] for details of this calculation in a different problem). If the rotation age T and the thinning schedule $u(t)$, $0 \leq t \leq T$ are chosen to maximize [A2.3], then the resulting value of [A2.3] is exactly equal to the land expectation value L minus the cost of re-establishment C_p. Thus we have the following equation:

$$[A2.4] \qquad \max\left\{\int_0^T e^{-\delta t - y(t)}[r(t)u(t) - \delta L]\,dt + e^{-\delta T - y(T)}q(T)X(T)\right\} - C_p = 0$$

where the maximization is over T and $u(t)$.

This equation for L can be solved in an iterative way. For a fixed value of L the optimal T and $u(t)$ can be found using the maximum principle and the corresponding value of the left-hand side of [A2.4] can be computed. Proceeding iteratively (e.g. using the Secant Method (see e.g. Atkinson [1978])), one can find the value of L which solves [A2.4] along with the corresponding optimal T and $u(t)$. Thus the land expectation value, the optimal rotation age and the optimal thinning schedule can be found.

APPENDIX 3

Determination of Optimal Harvest Policies in the Presence of Fire Risk

We shall consider only the single rotation problem. Extension to the case of ongoing rotations can be achieved in the fashion described in Appendix 1.

In the simplest case when the fire hazard depends only on the age of the stand, the problem of determining the optimal commercial thinning schedule and clear-cut harvest age can be expressed as (see [15]–[17] in Section 4):

Find T and u(t) $(0 \le t \le T)$ to maximize

[A3.1] $$J = \int_0^T e^{-\delta t - y(t)}\, r(t)u(t)dt + e^{-\delta T - y(T)}\, q(T)X(T)$$

subject to

[A3.2] $$\frac{dX}{dt} = g(t)F(X) - u(t)$$

[A3.3] $$\frac{dy}{dt} = h(t),$$

and

[A3.4] $$0 \le u(t) \le u_{max}.$$

This problem can be solved using the Pontryagin maximum principle (see e.g. Kamien and Schwartz [1981]). We first define the current-value Hamiltonian:

[A3.5] $$\mathscr{H}(t) = e^{-y(t)}\, r(t)u(t) + \mu_1(t)(g(t)F(X) - u(t)) + \mu_2(t)\, h(t)$$

where μ_1 and μ_2 are costate variables (shadow prices) satisfying the costate equations

[A3.6] $$\frac{d\mu_1}{dt} = \delta\mu_1 - \mu_1 g(t)F(X)$$

[A3.7] $$\frac{d\mu_2}{dt} = \delta\mu_2 - e^{-y(t)}\ r(t)u(t)$$

with terminal (transversality) conditions

[A3.8] $$\mu_1(T) = e^{-y(T)}\ q(T)$$

[A3.9] $$\mu_2(T) = -e^{-y(T)}\ q(T)X(T).$$

Necessary conditions for the thinning schedule $u(t)$ to be optimal are that $u(t)$ maximizes $\mathscr{H}$ at each t, $0 \leq t \leq T$. In addition, a necessary condition for the clear-cut harvest age to be optimal is:

[A3.10] $$\mathscr{H}(T) + e^{\delta T} \frac{\partial}{\partial T}\left[e^{-\delta T - y(T)}\ q(T)X(T)\right] = 0.$$

Since the unit cost of clear-cut harvesting is assumed to be lower than that of thinning $(C_1 < C_0)$ it follows that $r(T) < q(T)$. From this it follows that it could never be optimal to thin right up to the clear cut harvest age, T. Thus $u(T) = 0$ is optimal. Using this in [A3.5] along with [A3.8], [A3.9] and [A3.10] results in the following Wicksell-type condition for the optimal clear-cut harvest age T^*

$$[A3.11] \qquad \frac{g(T^*)F(X(T^*))q(T^*) + X(T^*)q'(T^*)}{q(T^*)X(T^*)} = \delta + h(T^*)$$

which is equation [20] in Section 4.

To determine the optimal thinning pattern over $[0,T^*]$, we note that the Hamiltonian is linear in u

$$[A3.12] \qquad \mathscr{H}(t) = [e^{-y(t)}r(t) - \mu_1(t)]u(t) + (\text{terms not involving } u).$$

Thus $\mathscr{H}(t)$ will be maximized by

$$u^*(t) = \begin{cases} 0 & \text{if } e^{-y(t)}r(t) - \mu_1(t) < 0 \\ u_{max} & \text{if } e^{-y(t)}r(t) - \mu_1(t) > 0 \end{cases}$$

("bang-bang control"). The only other possibility is that the switching function

$$[A3.13] \qquad \sigma(t) = e^{-y(t)}r(t) - \mu_1(t)$$

be identically equal to zero over an interval. In this case $u^*(t)$ will be that value of the control required to keep $X(t)$ on a trajectory for which $\sigma(t) \equiv 0$ (singular control). By setting $\sigma(t) \equiv 0 \equiv \frac{d\sigma}{dt}$ and solving one obtains the following equation for the singular path

$$[A3.14] \qquad g(t)F'(X) = \delta + h(t) - \frac{r'(t)}{r(t)}$$

which is equation [19] in Section 4. The synthesis of controls (zero thinning followed by thinning along the singular path followed by zero thinning) is similar to that presented in Clark and de Pree.

We turn now to the case when the hazard increases whenever thinning takes place. In this case the state equation [A3.3] is replaced by

[A3.15] $$\frac{dy}{dt} = h(t) + \rho H(u)$$

where $H(u)$ is the step function [25].

The current-value Hamiltonian is

[A3.16] $$\mathscr{H}(t) = e^{-y(t)} r(t)u(t) + \mu_1(t)(g(t)F(X) - u(t)) + \mu_2(t)(h(t) + \rho H(u))$$

$$= [e^{-y(t)} r(t) - \mu_1(t)]\ u(t) + \rho\mu_2(t)H(u)$$

$$+ \text{(terms not involving } u).$$

The determination of the optimal clear-cut harvest time T^* follows exactly as before (T^* is still given by equation [24]). However the optimal thinning pattern is different in this case since the Hamiltonian while linear in u for $0 < u \leq u_{max}$, has a discontinuity at $u = 0$. It follows that it will be maximized at either $u = 0$ or $u = u_{max}$ according as to whether

[A3.17] $$[e^{-y(t)} r(t) - \mu_1(t)] u_{max} + \rho\mu_2(t)$$

is less than or greater than zero. In the case of equality both values maximize $\mathscr{H}$. Note that it is impossible for $\mathscr{H}$ to be maximized at an interior value (in the open interval $(0, u_{max})$) of u. Thus singular control cannot be optimal, the optimal control being purely "bang-bang". Thus the optimal thinning schedule involves periods of no thinning, alternating with periods of thinning at the maximum rate.

The switching function [A3.17] has an economic interpretation. Multiplying by $e^{y(t)}$ it becomes

$$[A3.18] \qquad [r(t) - e^{y(t)}\mu_1(t)]\, u_{max} + \rho e^{y(t)}\mu_2(t).$$

Given that the stand is alive at age t, $e^{y(t)}\mu_1(t)$ is the user cost of harvesting a single unit of timber. The direct revenue from such a harvest is $r(t)$. The costate variable $\mu_2(t)$ is the user cost associated with a unit increase in y, i.e.

$$[A3.19] \qquad \mu_2(t) = \frac{\partial J^*}{\partial y} = -e^{-y}\frac{\partial J^*}{\partial S}$$

where J^* is the optimal present value of the resource at time t. Thus $\rho e^{y(t)}\mu_2(t) = -\rho\frac{\partial J^*}{\partial S}$, which, given the stand is alive at time t, is equal to $-\rho J^* = -[(\rho+h(t))J^* - h(t)J^*]$. This is the increase in the expected loss of revenue resulting from a fire if the hazard is increased from $h(t)$ to $h(t)+\rho$, by the act of thinning. Thus [A3.18] represents the value of harvesting at time t at the maximum rate, u_{max}, net of the user cost and net of the increase in expected loss due to fire if harvesting takes place. Harvesting takes place if and only if this quantity is positive.

To obtain the switching times between periods of no harvesting and periods of harvesting at the maximum rate numerical methods are required.

In the above application of the maximum principle, we have skipped over the fact that the state equation [A3.15] is not continuous in the control variable u.

Rather we have used the maximum principle as a heuristic, hoping that it will give the correct answer. It can be shown formally that it does by an appeal to the methods of dynamic programming (e.g. Kamien and Schwartz [1981, p. 238]). We have used the maximum principle rather than dynamic programming to show how the thinning problem with risk relates to the no-risk problem treated by Clark and de Pree.

Finally in the case when the hazard depends on stand volume as well as age and thinning activity with

$$\frac{dy}{dt} = \varphi(t,X) + \rho H(u)$$

the Hamiltonian is as [A3.16] plus a term $\mu_2(t)\varphi(t,X)$. Since this additional term does not depend on u it follows that as before the optimal rate of thinning $u^*(t)$ is either zero or u_{max} according as to whether the switching function is less or greater than zero.

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Age	Unspaced stands	Spaced stands
45	–1472	4743
55	6881	11,965
65	16,799	21,324
75	26,676	30,633
85	35,074	39,714
95	43,836	47,885
105	48,205	56,110

Table 1.

Per hectare lumber values (net of harvesting and processing costs) in Canadian dollars for natural (unspaced) and spaced stands of coastal Douglas–fir.

		ρ		
		0.01	0.03	0.05
θ	0.1	0.0146	0.0335	0.0520
	0.01	0.0149	0.0343	0.0534

Table 2.

Probability of destruction in the year immediately following spacing, using the hazard model [2], for various values of the parameters θ and ρ and with $\lambda = 0.005$. These values should be compared with the probability 0.0050 assumed for unspaced stands.

Discount rate (%)	Unspaced stands	Spaced stands						
		ρ=0	ρ=.01		ρ=.03		ρ=.05	
			θ=.01	θ=.1	θ=.01	θ=0.1	θ=.01	θ=0.1
3	3227(73)	2751(75)	2433(75)	2648(75)	1875(75)	2449(75)	1408(65)	2261(75)
5	1047(65)	363(65)	248(65)	326(65)	45(65)	253(65)	0	185(65)

Table 3.

Single rotation expected net present values ($ per ha) for unspaced stands and for spaced stands at age 15 under various fire–hazard scenarios. The age of financial maturity in years is given in parentheses.

Discount rate (%)	Unspaced stands	Spaced stands						
		ρ=0	ρ=.01		ρ=.03		ρ=.05	
			θ=.01	θ=.1	θ=.01	θ=0.1	θ=.01	θ=0.1
3	4108(75)	3518(65)	3228(65)	3432(65)	2665(65)	3262(65)	2130(65)	3093(65)
5	1215(65)	411(65)	288(65)	382(65)	55(65)	296(65)	0	220(65)

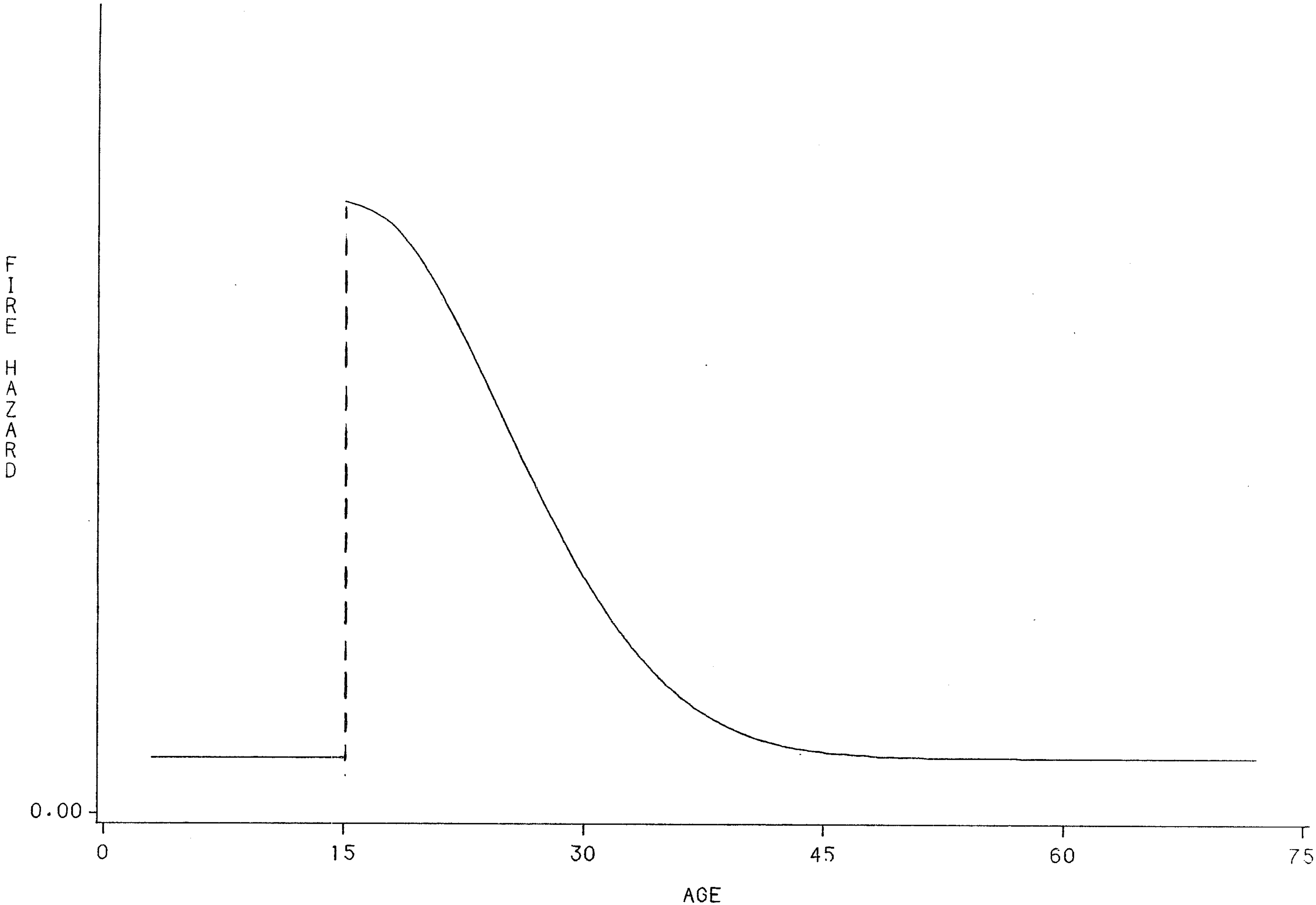
Table 4.

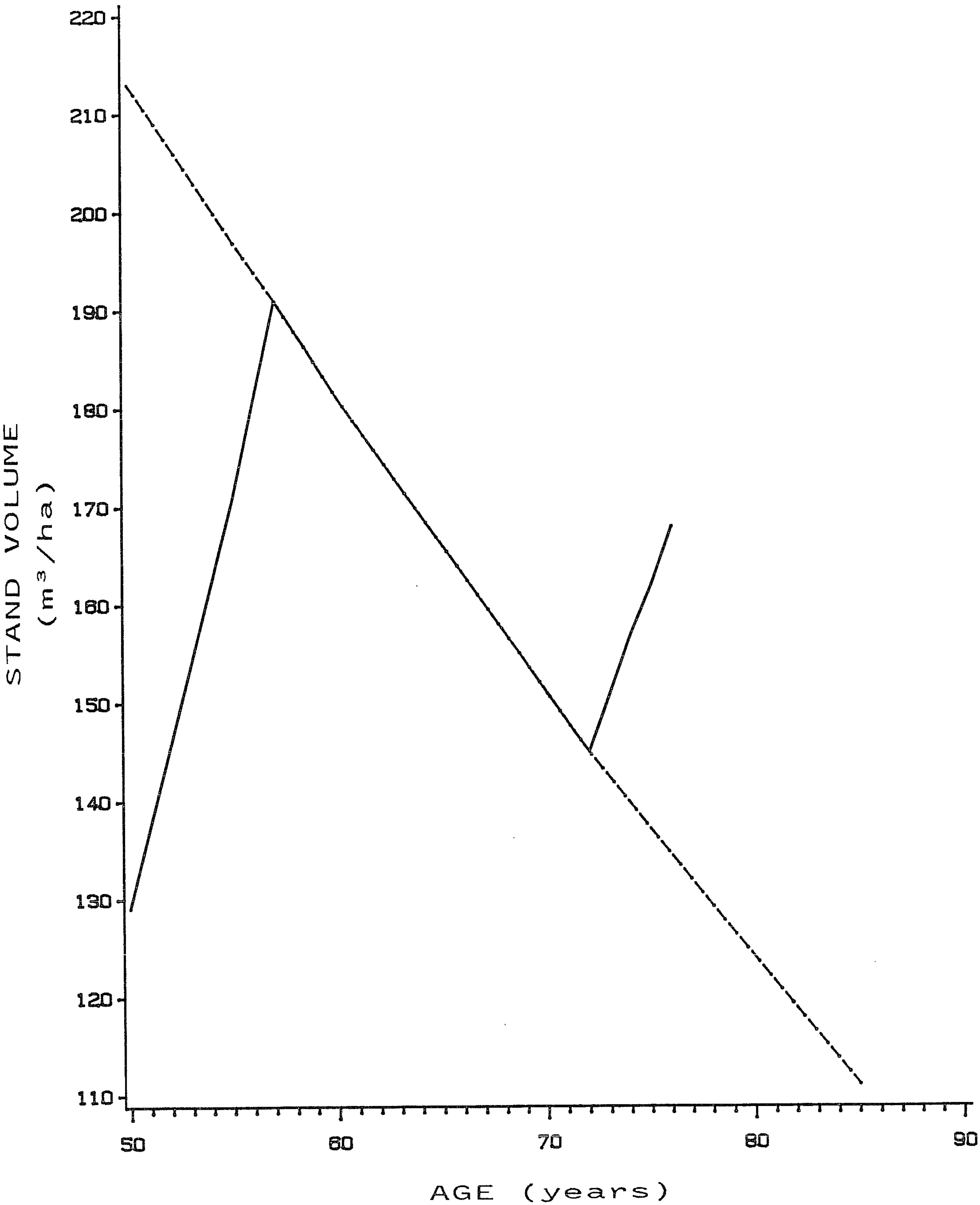
Expected net present values ($ per ha) over infinite rotations for unspaced and spaced stands at age 15, under various fire–hazard scenarios. The optimal (Faustmann) rotation age is given in parentheses.

Figure Captions

Figure 1 The assumed form for the fire hazard for a stand spaced at age $t_s = 15$ yrs.

Figure 2 Optimal trajectory for timber volume in the example of Section 4. Thinning takes place along the singular path between ages 58 and 73 years.


FIRE HAZARD
0.00
0
15
30
45
60
75
AGE



220
210
200
190
180
170
160
150
140
130
120
110
STAND VOLUME
(m³/ha)
50
60
70
80
90
AGE (years)