Generalisations of Irredundance in Graphs

by

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Abstract

The well studied class of irredundant vertex sets of a graph has been previously shown to be a special case of (a) a "Private Neighbor Cube" of eight classes of vertex subsets and (b) a family of sixty four classes of "generalised irredundant sets."

The thesis makes various advances in the theory of irredundance. More specifically:

(i) Nordhaus-Gaddum results for all the sixty-four classes of generalised irredundant sets are obtained.

(ii) Sharp lower bounds involving order and maximum degree are attained for two specific classes in the Private Neighbor Cube.

(iii) A new framework which includes both of the above generalisations and various concepts of domination, is proposed.

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Contents

List of Figures

List of Tables

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Chapter 1

Introduction

1.1 An Intuitive Approach to Irredundance

In this first section we will attempt to give the reader an intuitive feeling for the concepts of redundancy and irredundance. To accomplish this, we discuss two familiar situations involving the placement of queens on $n \times n$ chessboards and transmission in communication networks. We hope that these examples serve as motivation for the theory of irredundance.

Borrowing terminology from the game of chess we say that a queen placed on the chessboard *covers* (or *attacks*) all squares which are on the same row, column or diagonal. For our purposes a queen is assumed to cover her own square and we will consider an $n \times n$ chessboard.

Suppose that a set Q of queens has been placed on an $n \times n$ chessboard. A queen $q \in Q$ is called *redundant* in Q if every square covered by q is also covered by at least one other queen in the set Q. Otherwise, queen q is said to be *irredundant* in Q. Thus, q is irredundant in Q if q covers at least one square (called a *private neighbour* of q) which is not covered by any other queen in Q. We emphasise that the square occupied by q could be a private neighbour of q. A set Q of queens is called *irredundant* if every queen in Q is irredundant, or equivalently, if every queen in Q has a private neighbour.

We further illustrate this idea with Figure 1.1, in which there are two placements Q_1 and Q_2 of three queens (q_1, q_2 and q_3) on a 4×4 chessboard.

Suppose that rows and columns are numbered in the usual matrix fashion. Observe that for the placement Q_1 since every square covered by q_1 is also covered by q_2 or q_3, q_1 is redundant in Q_1. However, q_2 (respectively q_3) covers the square $(1, 1)$ (respectively $(4, 1)$) which is not covered by either q_1 or q_3 (respectively q_1 or q_2). Thus, queens q_2 and q_3 are irredundant in the set Q_1.

In the other placement Q_2, the queens q_1, q_2, q_3 cover squares $(4, 1)$, $(3, 2)$, $(2, 4)$ respectively, which are not covered by the other queens of Q_2. Thus in

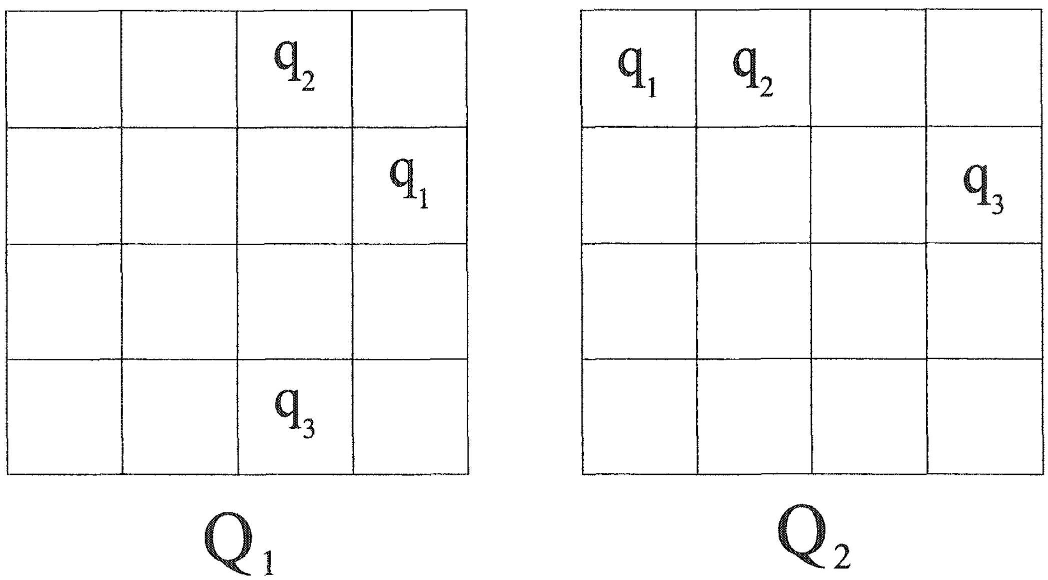

Figure 1.1: Two placements of queens.

Q_2 each of the three queens is irredundant and Q_2 is an irredundant set.

A natural question to ask is what is the maximum number of queens in an irredundant set on an $n \times n$ chessboard. This problem will be discussed further in Chapter 2.

Our other example involves communications networks consisting of a set of nodes together with links, which are pairs of nodes between which direct communication is possible. We say that node x is *accessible* from node y if $x = y$ or there is a link between x and y in the network. For example, the nodes may be processors in a computer network or people in some sociological situation. A node s is said to be *redundant* in a set S of nodes if any node accessible from s is also accessible from a node of $S-\{s\}$; otherwise, the node s is called *irredundant* in S. To further illustrate the concept of irredundance, consider the network N of Figure 1.2 and the sets of nodes $S_1 = \{1, 2, 3\}$ and $S_2 = \{3, 4, 5\}$.

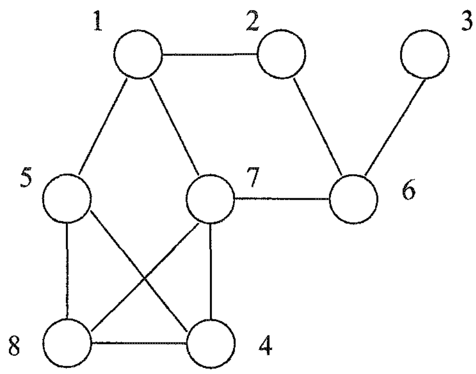

Figure 1.2: The communication network N

S_1:	1	1 2 $\underline{5}$ $\underline{7}$	S_2:	3	$\underline{3}$ $\underline{6}$
	2	1 2 6		4	4 5 $\underline{7}$ 8
	3	$\underline{3}$ 6		5	$\underline{1}$ 4 5 8

Table 1.1: Nodes accessible from S_1 and S_2 in N

Observe that the underlined nodes (see Table 1.1) in both examples appear on precisely one row. For example, for S_1, nodes 5 and 7 are underlined because they are both accessible by node 1, but are not accessible from nodes 2 and 3. Thus, 1 and 3 are irredundant in S_1, while 2 is redundant in S_1. The set S_2 contains only nodes which are irredundant in S_2. If transmitters are to be placed at some set S of nodes in a network, it might be desirable to choose S without redundant nodes because the totality of nodes accessible from the set of transmitters is unaffected by the removal of a transmitter positioned at a redundant node.

1.2 A Formal Look at Irredundance

The closed (open) neighbourhood of a vertex s of a simple graph $G = (V, E)$ is denoted by $N[s]$ $(N(s))$ and for a subset $S \subseteq V$, by $N[S] = \cup_{s \in S} N[s]$ $(N(S) = \cup_{s \in S} N(s))$.

A set S is *irredundant* if for every $s \in S$, $N[s] - N[S - \{s\}] \neq \emptyset$. Irredundant sets are sometimes called *CC-irredundant* since they are defined by the existence of a non-empty difference of two closed neighbourhoods.

The concept of irredundance in graphs was originally defined by Cockayne, Hedetniemi and Miller in [31] due to its relationship with the ideas of domination and independence, which have received much attention in the literature.

A set $S \subseteq V$ is a *dominating* set of G if $N[S] = V(G)$. The set S is *independent* if for every $s \in S$, $N(s) \cap S = \emptyset$, that is, no two vertices are adjacent. Due to its rich theory and diverse applications, domination in graphs has been the subject of more than 2000 papers since 1970. The reader is referred to the two volume collection on the topic by Haynes, Hedetniemi and Slater [60, 61] for an extensive bibliography.

Dominating sets and extremal independent sets are related by the following theorem, due to Berge [4].

Theorem 1.1 *([4]) (i) An independent set I is maximal independent if and only if it is dominating.*

(ii) If I is maximal independent, then it is minimal dominating.

Cockayne, Hedetniemi and Miller [31] and Bollobás and Cockayne [6] found a similar connection between extremal dominating and irredundant sets.

Theorem 1.2 *([6, 31]) (i) A dominating set D is minimal dominating if and only if it is irredundant.*

(ii) If D is minimal dominating, then it is maximal irredundant.

The *lower* and *upper independence numbers* ($i(G)$ and $\beta(G)$), *domination numbers* ($\gamma(G)$ and $\Gamma(G)$) and *irredundance numbers* ($ir(G)$ and $IR(G)$) are the smallest and largest cardinalities of a maximal independent, minimal dominating and maximal irredundant set, respectively. Theorems 1.1 and 1.2 imply the following chain of inequalities.

Proposition 1.3 *([31]) For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

This chain of inequalities has become known as the *domination chain* and has been the subject of well over 100 papers. The reader is referred to Haynes, Hedetniemi and Slater [60] for a comprehensive bibliography.

In this thesis we discuss generalisations of irredundant sets. Informally the basic ingredients of these generalisations are three properties which may make a vertex s important in a vertex subset S of a graph G. It will also help the intuition to replace the word "important" by "essential" or "non-redundant." Each property depends on the existence of one of the three types of *S-private neighbours* (S-pn) t for s, which we now formally define.

For $s \in S$, the vertex t is an:

(i) *S-self private neighbour* (S-spn) of s if $t = s$ and s is an isolated vertex of $G[S]$;

(ii) *S-internal private neighbour* (S-ipn) of s if $t \in S - \{s\}$ and $N(t) \cap S = \{s\}$;

(iii) *S-external private neighbour* (S-epn) of s if $t \in V - S$ and $N(t) \cap S = \{s\}$.

Observe that each such t is an element of $N[s] - N(S - \{s\})$ and that no $s \in S$ may have an S-pn of type (i) and an S-pn of type (ii).

The concept of private neighbours provides an alternative definition for irredundance. A vertex subset S of a graph is irredundant if and only if every vertex of S is either an S-spn or has an S-epn.

For $s \in S$, let $p(s,S)$, $q(s,S)$, $r(s,S)$ be Boolean variables which take the value 1 if and only if s has an S-pn of type (i), (ii), (iii), respectively. Whenever possible we use the abbreviations p, q, r for these variables.

Example 1.1. Consider the vertex subset $S = \{a, b, c, d\}$ of the graph G depicted in Figure 1.3. The S-pns of vertices of S are tabulated in Table 1.2

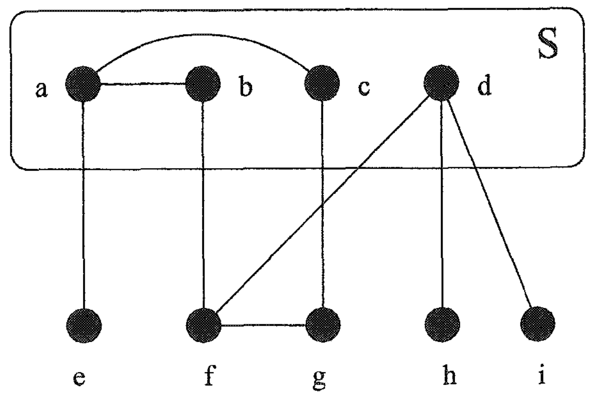

Figure 1.3: Graph G for Examples 1.1 and 1.3.

	type(i)	type(ii)	type(iii)
a		b,c	e
b			
c			g
d	d		h,i

Table 1.2: S-pns of vertices of Example 1.1.

1.3 The Private Neighbor Cube

The first generalisation of irredundance using private neighbours, known as the Private Neighbor Cube, was by Fellows, Fricke, Hedetniemi and Jacobs [53]. We change the wording of the definition here to be consistent with the notation used in the remainder of the thesis.

There are eight types of vertex subsets in the Private Neighbor Cube. Let $t = b_1b_2b_3$ be a binary sequence of three Boolean variables. Then we shall say that the vertex subset S of a graph G is of *type* t if for every $s \in S$,

$$(b_1 \wedge p(s,S)) \vee (b_2 \wedge r(s,S)) \vee (b_3 \wedge q(s,S))$$

is true.

Observe that a set S of type t (where $t = b_1b_2b_3$), is also a set of type t^*, where $t^* \in \{(b_1 \vee 0)(b_2 \vee 0)(b_3 \vee 1), (b_1 \vee 0)(b_2 \vee 1)(b_3 \vee 0), (b_1 \vee 1)(b_2 \vee 0)(b_3 \vee 0)\}$. For example, any type 001 set is also a type 011, type 101 and type 111 set.

We now explicitly define each of the eight types of sets in the Private Neighbor Cube.

Example 1.2.

Type 000. For any $s \in S$, $(0 \wedge p(s,S)) \vee (0 \wedge r(s,S)) \vee (0 \wedge q(s,S))$ is always false. Hence, S is the empty set.

Type 001. For any $s \in S$, $q(s,S)$ is true. Each vertex of S has an S-ipn. The sets correspond precisely to the *induced matchings* of G (see [12]). Induced matchings have also been called *strong matchings* (see [57]).

Type 010. For any $s \in S$, $r(s, S)$ is true. Each vertex of S has an S-epn. Such sets are called *open irredundant*. They were introduced in [46] and applied to broadcast networks. They are also known as *OC-irredundant sets* and have been studied in [7, 19, 20, 45, 49, 47, 53, 63].

Type 011. For any $s \in S$, $r(s, S) \vee q(s, S)$ is true. Each vertex of S has an S-ipn or has an S-epn. i.e. for each $s \in S$,

$$N(s) - N(S - \{s\}) \neq \emptyset.$$

Since there are two open neighbourhoods in this definition, these sets are called *open-open-*, *ooir-*, or *OO-irredundant* [45, 46].

Type 100. For any $s \in S$, $p(s, S)$ is true. Each vertex of S is an S-spn. These are precisely the independent sets of G.

Type 101. For any $s \in S$, $p(s, S) \vee q(s, S)$ is true. Each vertex of S is an S-spn or has an S-ipn. We notice that this implies that $\Delta(G[S]) \leq 1$. Such sets were first studied by Fink and Jacobson [54] and are called *1-dependent sets*.

Type 110. For any $s \in S$, $p(s, S) \vee r(s, S)$ is true. Each vertex of S is an S-spn or has an S-epn. These are precisely the irredundant sets of G.

Type 111. For any $s \in S$, $p(s, S) \vee r(s, S) \vee q(s, S)$ is true. Each vertex of S is either an S-spn, or has an S-ipn or an S-epn. This is the class of *closed-open-*, *coir-*, or *CO-irredundant* sets which are defined in [46] and studied in [32, 33, 41, 77].

These eight classes of sets are used to define the Private Neighbor Cube because of Figure 1.4. An arrow points from type t_1 to type t_2 if every t_1 set is also a t_2 set.

1.4 Generalised Irredundant Sets

The second extension of irredundance, and the main topic of this thesis, was first considered by Cockayne in [19]. The types of sets are collectively known

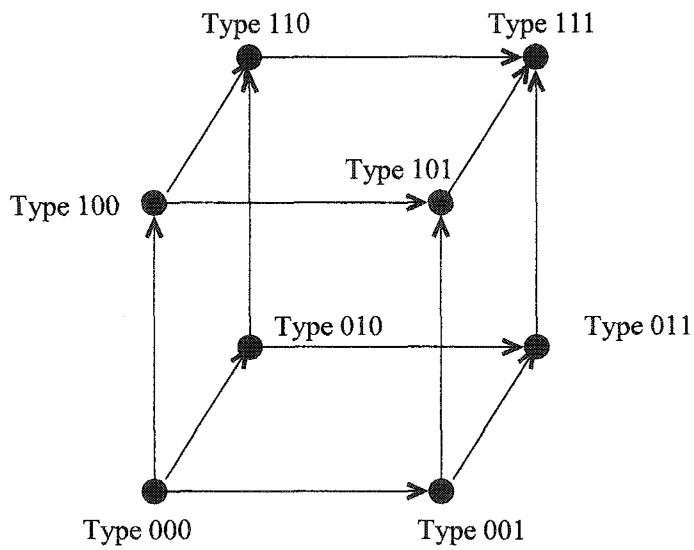

Figure 1.4: The Private Neighbor Cube

as generalised irredundant sets. We begin with a preliminary definition.

Let $S(s) = (p(s,S), q(s,S), r(s,S))$. Here S is a set and $s \in S$. So $S(s)$ is a triple which represents a type of vertex. This is distinct from the triples which represented types of sets previously discussed. Observe that for all s and S, $p(s,S) \wedge q(s,S) = 0$, i.e. the three Boolean variables are not independent and $S(s)$ is never $(1,1,0)$ or $(1,1,1)$.

Example 1.3. Consider the vertex subset $S = \{a,b,c,d\}$ of the graph G depicted in Figure 1.3. We observe,

$$S(a) = (0,1,1), \quad S(b) = (0,0,0), \quad S(c) = (0,0,1), \quad S(d) = (1,0,1)$$

We are now ready to define generalised irredundant sets. Let f be a

Boolean function of the three variables $p(s,S)$, $q(s,S)$, $r(s,S)$. A set $S \subseteq V$ is an *f-set* of G if for each $s \in S$

$$f\,(S\,(s)) = f(p(s,S), q(s,S), r(s,S)) = 1.$$

The function f may be viewed as a compound existence/non-existence property of the three types of S-pns. The class of all f-sets of G will be denoted by $\Omega_f(G)$ (abbreviated to Ω_f whenever possible). It is also called the *class of generalised irredundant sets defined by* f.

The rows of the truth table of f will be labelled $0, \ldots, 7$, so that the entry in row i is $f\,(p,q,r)$, where pqr is the binary representation of the integer i (e.g., $f\,(1,0,1)$ is the fifth entry in the table). Recall that for each $s \in S$, $S\,(s)$ is never equal to $(1,1,0)$ or $(1,1,1)$. We deduce:

(a) If the only 1's in the truth table for f occur in rows 6 or 7, then $\Omega_f = \emptyset$.

(b) If f' is formed from f by replacing the values in rows 6 and 7 by 0's, then $\Omega_{f'} = \Omega_f$.

Thus, we will only be concerned with the set $\mathcal{F}$ of 64 functions with 0's in rows 6 and 7. Two of these are in fact rather uninteresting since $f = 0$ gives $\Omega_f = \emptyset$, and the function g with 1's in all rows $0, 1, \ldots, 5$ has Ω_g equal to the class of all subsets of V.

The functions of $\mathcal{F}$ will be numbered (as in [23]) as follows. Let

$$a_0a_1a_2a_3a_4a_5$$

be the binary representation of i. Then f_i is defined to be the function with entries a_0, a_1, a_2, a_3, a_4 and a_5 in rows 0 through 5, respectively. Note that $\mathcal{F} = \{f_0, \ldots, f_{63}\}$.

For example, consider the function $p \vee r$. The truth table column is $0, 1, 0, 1, 1, 1, 0, 0$ and since $(23)_{10} = (010111)_2$, $p \vee r = f_{23}$. From Example 1.2, (type 110) we see that $\Omega_{f_{23}}(G)$ is precisely the set of all irredundant sets in G.

In Table 1.3 we show that each class of sets in the Private Neighbor Cube equals $\Omega_f(G)$ for some $f \in \mathcal{F}$.

1.5 An Overview

In Section 1.2 the parameters $i(G)$ and $\beta(G)$, $ir(G)$ and $IR(G)$ were defined for independent sets and irredundant sets, respectively. The natural analogues of these parameters for the class $\Omega_f(G)$ of generalised irredundant

Original Name	Private Neighbor Cube Triple	Generalised Irredundant Notation
$\emptyset$	000	f_0
Induced Matching	001	f_{12}
Open Irredundance	010	f_{21}
OO-irredundance	011	f_{29}
Independent	100	f_3
1-dependent	101	f_{15}
Irredundant	110	f_{23}
CO-irredundant	111	f_{31}

Table 1.3: Generalised irredundant notation for the Private Neighbor Cube.

sets are:

$q_f(G)$ and $Q_f(G)$, the smallest and largest cardinalities

of a maximal f-set of G.

In Chapter 2 we discuss some of the existing results concerning basic irredundance (f_{23}-sets), including those which are directly related to this thesis.

Chapter 3 is a survey of previous work on generalised irredundance (i.e. f-sets, where $f \in \mathcal{F}$).

The new work commences in Chapter 4 in which we determine Nordhaus-Gaddum bounds for $Q_f(G)$, for each of the 64 functions in $\mathcal{F}$. A Nordhaus-Gaddum result for a parameter $\eta(G)$ bounds $\eta(G)+\eta(\overline{G})$ or $\eta(G)\cdot\eta(\overline{G})$, where $\overline{G}$ is the complement of G, in terms of n, the order of G. These results have become known as Nordhaus-Gaddum bounds since Nordhaus and Gaddum [73] obtained the first such result concerning the chromatic number $\chi(G)$.

In Chapter 5 and Chapter 6 we consider the special cases of f_{31}-sets (i.e. CO-irredundant sets) and f_{21}-sets (i.e. OC-irredundant sets), and obtain lower bounds for $q_{31}(G)$ and $q_{21}(G)$ in terms of the order n and the maximum degree Δ.

Chapter 7 deals with a new (third) generalisation of irredundance, which includes the other two generalisations, as well as the concept of domination.

Some open problems are discussed in Chapter 8.

Chapter 2

Irredundance

In this chapter we look at the major results from the study of irredundance. In particular we touch on results relevant to the remainder of the thesis. Henceforth we will abbreviate any parameter $\lambda(G)$ or $\lambda(\overline{G})$ to λ and $\overline{\lambda}$ whenever the graph G is clear from the context.

2.1 A Maximality Condition

In light of Theorems 1.1 and 1.2 it is natural to consider a similar result for maximal irredundant sets. The maximality characterisation for irredundance involves external redundant vertex subsets, which were originally defined and generalised in [27]. We first define the following four vertex subsets, which we will use throughout the thesis. Given $s \in S \subseteq V$, let

$epn(s, S)$ be the set of all S-epns of s,

$ipn(s, S)$ be the set of all S-ipns of s,

$spn(s, S)$ be the set of all S-spns of s,

and $pn_{23}(x, X) = spn(x, X) \cup epn(x, X)$.

A set $S \subseteq V$ is an *external redundant set* (abbreviated *er-set*) if for all $v \in V - S$, there exists $w \in S \cup \{v\}$ such that $pn_{23}(w, S \cup \{v\}) = \emptyset$ and if $w \in S$, then $pn_{23}(w, S) \neq \emptyset$. The next theorem, proved in [27], gives an alternative definition of er-sets involving the set of vertices not dominated by S.

Theorem 2.1 ([27]) *Let $R = V - N[S]$. The set S is an external redundant set if and only if for all $v \in N[R]$, there exists $s_v \in S$ such that $\emptyset \neq pn_{23}(s_v, S) \subseteq N[v]$.*

Corollary 2.1.1 *If S is a dominating set of G, then S is external redundant.*

In [27], Theorem 2.1 is used to show the following result.

Theorem 2.2 ([27]) *(i) An irredundant set I is maximal irredundant if and only if it is an er-set.*

(ii) If I is maximal irredundant, then it is a minimal er-set.

This theorem can be used to augment the domination chain (Theorem 1.3). The following two graph parameters concerning er-sets were introduced in [28]. For any graph G let $er(G)$ and $ER(G)$ be the smallest and largest cardinalities of a minimal external redundant set in G. It follows from Theorem 2.2 that $er \leq ir$ and that $IR \leq ER$. In [28] examples are given to show that these inequalities may be strict. Thus we have:

$$er \leq ir \leq \gamma \leq i \leq \beta \leq \Gamma \leq IR \leq ER.$$

2.2 Equality in the Domination Chain

Much of the theory developed around Theorem 1.3 (the domination chain) considers the question of determining conditions under which some of the parameters in the domination chain are equal. In this section we only consider results of this type involving the parameters ir and IR; the reader is referred to [60, pp. 77-84] for other results. We first mention a fundamental result due to Cockayne and Mynhardt [38].

A sequence of six positive integers a, b, c, d, e, f is said to be a *domination sequence* if there exists a graph G such that, $ir = a$, $\gamma = b$, $i = c$, $\beta = d$, $\Gamma = e$ and $IR = f$. Cockayne and Mynhardt characterised the set of all possible domination sequences.

Theorem 2.3 ([38]) *A sequence a, b, c, d, e, f of positive integers is a domination sequence if and only if:*

(a) $a \leq b \leq c \leq d \leq e \leq f$,

(b) $a = 1$ *implies that* $c = 1$,

(c) $d = 1$ *implies that* $f = 1$, *and*

(d) $b \leq 2a - 1$.

Let $\phi(G)$ and $\lambda(G)$ be any two graph parameters. Then G is a (ϕ, λ)-graph if $\phi(G) = \lambda(G)$.

Many sufficient conditions for a graph to be a (β, IR)-graph have been produced in the literature. We list eighteen classes of (β, IR)-graphs. The definitions of these classes are omitted for brevity, but can be found in the cited references. Note that some classes listed are subclasses of others in the list.

1. Strongly perfect [17, 79]
2. Perfectly orderable graphs [13]
3. Peripheral graphs [71]
4. Chordal graphs and their complements [5]
5. Comparability graphs [5]
6. Permutation graphs [43]
7. Meyniel graphs [76]
8. Parity graphs [76]
9. Bipartite graphs [5]
10. Gallai graphs [76]
11. Certain $\mathcal{T}$-co-graphs [17]
12. Graphs with no induced P_4 [5]
13. Graphs with no odd cycle of length greater than three [76]
14. Interval graphs [56]
15. Block graphs [5]

16. Unicyclic graphs [78]

17. Circular arc graphs [57]

18. Upper bound graphs [18]

The problem of finding sufficient conditions for G to be a (ir, γ)-graph has also been well studied. The reader is referred to [60, pp. 77-78].

2.3 Bounds on ir and IR

In this section we consider bounds involving ir and (or) IR, together with a subset of the other parameters of the domination chain, maximum degree Δ, and minimum degree δ, and the order n, of G.

2.3.1 The Irredundance and Domination Numbers

One of the earliest results concerning irredundance was the following upper bound for γ in terms of ir, proved independently by Bollobás and Cockayne [6] and by Allan and Laskar [1].

Theorem 2.4 ([1, 6]) *For any graph G, $\gamma \leq 2ir - 1$.*

This was improved by Allan, Laskar and Hedetniemi [2] who noticed that the upper bound in Theorem 2.4 could be reduced by the number of isolated vertices in the induced subgraph of any irredundant set.

Theorem 2.5 ([2]) *Let S be a maximal irredundant set of G and suppose that $G[S]$ has k isolated vertices. Then $\gamma \leq 2ir - k - 1$.*

2.3.2 Upper Irredundance and Extremum Degrees

The first result of this section is a simple upper bound for IR first noticed by Favaron [50].

Proposition 2.6 ([50]) *In any graph G with n vertices and minimum degree δ, $IR \leq n - \delta$.*

The extremal graphs for this inequality were also given by Favaron [50].

Theorem 2.7 ([50]) *For any graph G of order $n \geq 2$, $IR(G) = n(G) - \delta(G)$ if and only if $G \cong (K_p \times K_2) + F$, where F is any graph (perhaps the empty graph) satisfying $p \geq n(F) - \delta(F)$*

Proposition 2.8 ([50]) *If G is d-regular and satisfies $IR = n - d$, then $G \cong \overline{K_n}$ or $d \geq \frac{n}{2}$.*

Proposition 2.8 suggests that the upper bound $IR \leq n - \delta$ may be improved for regular graphs with low degree. The next result obtained by Henning and Slater [64] accomplishes this.

Proposition 2.9 ([64]) *If G has n vertices and is regular of degree d, then $IR \leq \min\{n-d, \frac{n}{2}\}$.*

Henning and Slater also characterise the graphs which attain this bound.

A upper bound of IR in terms of both the maximum and minimum degrees of a graph has recently been found by Bacsó and Favaron [3].

Theorem 2.10 ([3]) *Let G be a graph of order n, minimum degree δ and maximum degree $\Delta > 0$. Then*

$$IR \leq \frac{n}{1 + \frac{\delta}{\Delta}}.$$

2.3.3 Lower Irredundance and Extremum Degrees

It is easily shown that each of the lower parameters in the domination chain is bounded above by $n - \Delta$. Berge [4] originally proved this result for the domination number γ.

Proposition 2.11 *For any graph G with n vertices and maximum degree Δ*

$$ir \leq \gamma \leq i \leq n - \Delta.$$

A variety of results have been obtained by Domke, Dunbar and Markus [42] and Favaron and Mynhardt [52] concerning the inequalities of Proposition 2.11. In this chapter, we are only concerned with the inequality $ir \leq n - \Delta$ and when equality is achieved.

Let u be a vertex of G of degree Δ. For any $D \subseteq N(u)$, define $W_D = \{v \in N(u) | N[v] \subseteq N[D]\}$. We now state three properties the vertex u may have. The first two properties were originally given in [42] and the third in [52].

$P_1(u)$: $V - N[u]$ is independent.

$P_2(u)$: Every vertex of $N(u)$ has at most one neighbour in $V - N[u]$.

$P_3(u)$: For every $C \subseteq V - N[u]$, there does not exist a non-empty set $D \subseteq N(u)$ with $|D| \leq |C|$ such that the set $Y = D \cup (V - N[u] - (C \cup W_D))$ is a maximal irredundant set of G.

In [52], Favaron and Mynhardt use these properties to obtain a characterisation of the graphs for which $ir = n - \Delta$.

Theorem 2.12 ([52]) *Let G be a graph of order n and maximum degree Δ.*

(i) If $ir = n - \Delta$, then $P_1(u)$, $P_2(u)$ and $P_3(u)$ hold for every vertex u of degree Δ.

(ii) If $P_1(u)$, $P_2(u)$ and $P_3(u)$ hold for some vertex u of degree Δ, then $ir = n - \Delta$.

A far more detailed characterisation, which includes a procedure useful for determining whether the set Y described in $P_3(u)$ is a maximal irredundant set, is given in [52].

We now look at lower bounds on the size of the smallest maximal irredundant set of G (ir) in terms of Δ. The first result of this type was by Bollobás and Cockayne [7].

Theorem 2.13 ([7]) *For any graph G with $\Delta \geq 2$, $ir \geq \frac{n}{2\Delta-1}$.*

The extremal graphs for Theorem 2.13 were later characterised by Laskar and Pfaff [69]. Each graph in this characterisation has $\Delta = 2$ and is also a extremal graph of the following lower bound established by Cockayne and Mynhardt [36].

Theorem 2.14 ([36]) *For any graph G with, $\Delta \geq 2$,*

$$ir \geq \frac{2n}{3\Delta}.$$

All extremal graphs of the bound in Theorem 2.14 are characterised in [36].

2.3.4 Nordhaus-Gaddum Type Bounds

For a parameter λ of a graph G of order n, bounds of the form $\lambda + \overline{\lambda} \leq (\geq) f(n)$, and the form $\lambda \cdot \overline{\lambda} \leq (\geq) g(n)$, have become known as *Nordhaus-Gaddum bounds*, since Nordhaus and Gaddum [73] obtained the first such results concerning the chromatic number $\chi(G)$.

For the lower irredundance number inequalities of this type and their extremal graphs are identical to those of the lower domination number, which were proven by Jaeger and Payan [65] and Payan and Xuong [74].

The *corona* $H \bullet K_1$ *of a graph* H is that graph obtained by adding a leaf adjacent to each vertex of H.

Theorem 2.15 ([65, 74]) *For any graph G, of order n,*

(i) $ir + \overline{ir} \leq n + 1$, *with equality if and only if* $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$*;*

(ii) $ir \cdot \overline{ir} \leq n$, *with equality if and only if G is* K_n, $\overline{K_n}$, $K_3 \times K_3$, *each component of G (or $\overline{G}$) is C_4, or G (or $\overline{G}$) is the corona of some graph H.*

The results for the upper irredundance number were obtained by Cockayne and Mynhardt [40] and are summarised in Theorems 2.16 and 2.17.

Theorem 2.16 **([40])** *For any graph G, of order n,*

(i) $IR + \overline{IR} \leq n + 1$ *and*

(ii) $IR \cdot \overline{IR} \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil$.

Theorem 2.17 **([40])** *For any graph G, of order n,*

(i) $IR + \overline{IR} = n + 1$ *if and only if* $V = S \cup T$, *where* $|S \cap T| = 1$, $G[S]$ *is independent and* $G[T]$ *is complete.*

(ii) $IR \cdot \overline{IR} = \left\lceil \frac{n^2+2n}{4} \right\rceil$ *if and only if* $V = S \cup T$, *where* $|S \cap T| = 1$, S *is a set of* $\left\lfloor \frac{n+1}{2} \right\rfloor$ *independent vertices and* T *is a set of* $\left\lceil \frac{n+1}{2} \right\rceil$ *vertices such that* $G[T]$ *is complete.*

It should be noted that the upper bounds for $IR+\overline{IR}$ and $IR\cdot\overline{IR}$ given in Theorem 2.16 are precisely the same as those obtained for $\beta + \overline{\beta}$ and $\beta \cdot \overline{\beta}$ by Chartrand and Schuster [16]. These authors also give lower bounds for these quantities, which they note are unsatisfactory since the inequalities involve the classical Ramsey numbers, very few of which are known.

2.3.5 Other Bounds Involving Sums

In this section, we consider bounds on the sum of an irredundance parameter and another parameter from the domination chain. The first result is due to

Cockayne, Favaron, Payan and Thomason [25].

Proposition 2.18 ([25]) *For any graph G, of order n, having t isolated vertices,*

(i)	$\gamma + IR \leq n + t,$	*(iv)*	$ir + \beta \leq n + t$
(ii)	$\gamma + \Gamma \leq n + t$	*(v)*	$ir + \Gamma \leq n + t$
(iii)	$\gamma + \beta \leq n + t$	*(vi)*	$ir + IR \leq n + t$

We note that $K_{1,n}$ and C_4 each attain all six bounds found in Proposition 2.18. This bound can be sharpened for graphs with no isolated vertices and $\delta \geq 2$.

Theorem 2.19 ([25]) *For any graph G, of order n, having no isolated vertices,*

(i)	$\gamma + IR \leq n + \delta - 2$	*(iv)*	$ir + \beta \leq n + \delta - 2$
(ii)	$\gamma + \Gamma \leq n + \delta - 2$	*(v)*	$ir + \Gamma \leq n + \delta - 2$
(iii)	$\gamma + \beta \leq n + \delta - 2$	*(vi)*	$ir + IR \leq n + \delta - 2$

The parameter i is not involved in Proposition 2.18 and Theorem 2.19. An upper bound for the parameter sum $i + IR$ was found by Favaron [48] and independently by Cheng De Wang [81].

Theorem 2.20 ([48, 81]) *For any graph G of order n and minimum degree*

δ, *$i + IR \leq \min\{2n - 2\delta, 2n + 2\delta - 2\sqrt{2n\delta}\}$, the best bound being the first one if $n < 2\delta$ and the second one if $n > 2\delta$.*

Favaron also obtained the following theorem in [48].

Theorem 2.21 ([48]) *For any graph G of order n and minimum degree δ, $i + \sqrt{\delta IR} \leq \min\{n - \delta + 2\sqrt{\delta(n-\delta)}, n + \delta\}$.*

2.4 Criticality

A concept in graph theory, which has drawn much interest, is that of a critical set. Given a graph property P, a graph G can be considered in some sense to be critical with respect to P if G possesses property P, but, no proper induced subgraph, no proper induced spanning subgraph, or no proper induced spanning supergraph possesses property P. This notion can be useful in obtaining a deeper understanding of the property P.

Of more relevance to the theory of irredundance is the notion of criticality for a graph parameter. Given a graph parameter λ, a graph G can be considered to be λ-critical if either the deletion of a vertex or an edge or the addition of an edge will always raise or always lower the value of λ. Six kinds of criticality can be defined in this manner for a given λ. In this section we give an example of one of these and present results in the case when $\lambda = IR$.

2.4.1 IR-Critical Graphs

Let λ be any graph parameter. A graph G is called *λ-critical* if $\lambda(G - v) < \lambda(G)$, for each $v \in V(G)$.

Given a parameter λ in the domination chain, it is easy to see that each edgeless graph with more than one vertex is λ-critical. It turns out that these are all of the β-critical graphs.

Proposition 2.22 ([58, 59]) *The graph G is β-critical if and only if G is edgeless with more than one vertex.*

Gröbler and Mynhardt [58, 59] showed that the class of Γ-critical graphs is precisely the same as the class of IR-critical graphs. To characterise all of the Γ-critical and IR-critical graphs we first need to consider the concept of a one-to-one perfect matching. Let $G = (V, E)$ be a graph. A partition $\{S, T\}$ of V is called a *one-to-one perfect matching*, or *1-1 p.m.*, if every $s \in S$ is adjacent to exactly one $t \in T$ and every $t \in T$ is adjacent to exactly one $s \in S$. We note that if $\{S, T\}$ is a 1-1 p.m of G, then S (and T) is an irredundant dominating set of G.

Theorem 2.23 ([58, 59]) *For any connected graph G of order n, then the following statements are equivalent.*

1. *G is Γ-critical.*

2. *$n > 2$ and for every Γ-set S of G, $\{S, V - S\}$ is a 1-1 p.m. of G.*

3. *$\Gamma = \frac{n}{2}$ and no Γ-set S of G has any isolated vertices.*

4. *G is IR-critical.*

The follow three corollaries follow immediately.

Corollary 2.23.1 ([58, 59]) *For any IR-critical graph of order n, having k isolated vertices, each component of G is either K_1 or it has a 1-1 p.m. with more than two vertices. In this case $IR = \Gamma = \frac{n+k}{2}$.*

Corollary 2.23.2 ([58, 59]) *A graph G is Γ-critical if and only if it is IR-critical.*

Corollary 2.23.3 ([58, 59]) *For any connected graph G of order n, if G is IR-critical, then it has a 1-1 p.m., $\delta \geq 2$ and $\beta < \frac{n}{2}$.*

Along with Theorem 2.23, Gröbler and Mynhardt [58, 59] presented the following two propositions, giving sufficient conditions in terms of δ for a connected graph with a 1-1 p.m. to be IR-critical.

Proposition 2.24 ([58, 59]) *For any connected graph G of order n, if G has a 1-1 p.m. and $\delta \geq \lfloor \frac{n}{4} \rfloor + 2$, then G is IR-critical.*

Proposition 2.24 can be improved slightly if $\beta < \frac{n}{2}$.

Proposition 2.25 ([58, 59]) *For any connected graph G of order n, if G has a 1-1 p.m., $\delta \geq \lfloor \frac{n}{4} \rfloor + 1$. and $\beta < \frac{n}{2}$, then G is IR-critical.*

We finish with two more results by Gröbler and Mynhardt [58, 59] concerning regular graphs.

Proposition 2.26 ([58, 59]) *For any connected r-regular graph G of order n, if G has a 1-1 p.m. and $\beta < \Gamma = IR = \frac{n}{2}$, then G is IR-critical.*

Proposition 2.27 ([58, 59]) *If G is a graph with n vertices and a 1-1 p.m. $\{S, T\}$ such that $G[S]$ and $G[T]$ are connected r-regular graphs with*

$$\beta(G[S]) + \beta(G[T]) < \frac{n}{2},$$

then G is IR-critical.

2.5 Irredundant Ramsey Numbers

Let G_1, G_2, ... G_t be an arbitrary t-edge colouring of K_n, where for every $i \in \{1, 2, ..., t\}$, G_i is the spanning subgraph of K_n consisting of all edges coloured with colour i. Let $\mu(G)$ denote the number of vertices in a maximum clique of G. The *classical Ramsey number for graphs* (as opposed to hypergraphs) $r(q_1, q_2, ..., q_t)$ is the smallest value of n such that for all t-edge colourings of K_n, there is an $i \in \{1, 2, ..., t\}$ for which $\mu(G_i) \geq q_i$ (or equivalently $\beta(\overline{G_i}) \leq q_i$).

Since a clique of a graph corresponds to an independent set of vertices in the complement, the classical Ramsey numbers for graphs, which are usually define in terms of μ, can be defined in terms of β (as above). Further, since irredundance can be thought of as a generalisation of independence, it is natural to develop a theory of irredundant Ramsey numbers. This theory was first developed by Brewster, Cockayne and Mynhardt [8].

The *irredundant Ramsey number for graphs* $s(q_1, q_2, ..., q_t)$ is the smallest value of n such that for all t-edge colourings of K_n, there is an $i \in \{1, 2, ..., t\}$ for which $IR(\overline{G_i}) \geq q_i$. In the case where $t = 2$, the irredundant Ramsey number $s(p, q)$ is the smallest integer n such that for every graph G of order n, $\overline{IR} \geq p$ or $IR \geq q$. Since any independent set is irredundant, irredundant Ramsey numbers exist by Ramsey's theorem and satisfy $s(q_1, q_2, ..., q_t) \geq r(q_1, q_2, ..., q_t)$ for all $q_1, q_2, ..., q_t$.

Another generalisation of the classical Ramsey number, *the mixed Ram-*

sey number $t(p,q)$, was introduced by Cockayne, Hattingh, Kok and Mynhardt [29]. This is the smallest n such that for every graph G of order n, $\overline{IR} \geq p$ or $\beta \geq q$. It is easy to see that $s(p,q) \leq t(p,q) \leq r(p,q)$, for all values of p and q.

2.5.1 Bounds on $s(p,q)$ and $t(p,q)$

The difficulty of obtaining exact values for irredundant (and mixed) Ramsey numbers is comparable to that of the corresponding problem for classical Ramsey numbers. One important tool for determining irredundant Ramsey numbers is the following recurrence relation. It should be noted that the same recurrence relation holds for $r(p,q)$, the classical Ramsey numbers.

Proposition 2.28 ([8]) *For all integers* $p, q \geq 2$,

$$s(p,q) \leq s(p-1,q) + s(p,q-1),$$

$$t(p,q) \leq t(p-1,q) + t(p,q-1),$$

and strict inequality holds if both $s(p-1,q)$ *and* $s(p,q-1)$ *(or* $t(p-1,q)$ *and* $t(p,q-1)$*) are even.*

Using the probabilistic approach, Erdös [44] proved that

$$r(p,p) > 2^{p/2}p/e\sqrt{2},$$

for p sufficiently large. An adaption of this proof was given by Chen, Hattingh and Rousseau [14] to obtain an asymptotic bound for irredundant Ramsey numbers.

Theorem 2.29 ([14]) *For all sufficiently large p,*

$$s(p,p) > \sqrt{\frac{p}{3}}2^{\frac{p}{2}}.$$

We end with another bound from [14].

Theorem 2.30 ([14]) *For all $q \geq 1$,*

$$s(3,q) \leq t(3,q) \leq \frac{q^{\frac{2}{3}}\sqrt{10}}{2}.$$

2.5.2 Known Irredundant and Mixed Ramsey Values

Very few of the irredundant (and mixed) Ramsey numbers are known. It is easy to see that $s(1,q) = 1$ and $s(2,q) = 2$. Similarly, $t(1,q) = t(p,1) = 1$ and $t(2,q) = t(p,2) = 2$. In [39] and [37] it was shown that $s(3,3,3) = 13$. All other known results are summarised in Tables 2.1 and 2.2. The numbers in square brackets are references.

For more information on irredundant Ramsey numbers the reader is referred to [72].

	$p=$ 3	4	5	6	7
$q=$					
3	6[8]	8[8]	12[8]	15[9]	18[30, 15]
4		13[21]			

Table 2.1: Irredundant Ramsey numbers $s(p, q)$

	$p=$ 3	4	5	6
$q=$				
3	6[29]	9[29]	12[29]	$\leq$ 16[29]
4	8[29]	5[29]	13[29]	
5	13[29]			

Table 2.2: Mixed Ramsey numbers $t(p, q)$

2.6 Irredundance on Chessboards

A chessboard graph G is constructed from the moves of a chess piece P on an $n \times n$ chessboard as follows. Each vertex in G corresponds to a square on the chessboard and two vertices u and v are adjacent if P can attack u from v. Five types of graphs are formed this way: the Queens graph Q_n, the Kings graph K_n, the Rooks graph R_n, the Bishops graph B_n, and the

Knights graph N_n. The grid graph G_n is also considered a chessboard graph. The question posed in Section 1.1 regarding queens on a chessboard, may now be restated: what is $IR(Q_n)$?

2.6.1 The Queens Graph

The size of $IR(Q_n)$ and $ir(Q_n)$ is only known for small values of n. The results for $n = 1, 2, \ldots, 10$ are summarised in Table 2.3. The reader is referred to [62], for results which are not referenced below.

$n =$	1	2	3	4	5	6	7	8	9	10
$IR(Q_n)$	1	1	2	4	5	7	9	11	13[67]	15[67]
$ir(Q_n)$	1	1	1	2	3[11]	3[11]	4[66]	5[67]	5[67]	5[67]

Table 2.3: Irredundance numbers of Q_n, for small n.

For larger values of n, the value of $IR(Q_n)$ is bounded in Theorem 2.31. The upper bound is due to Burger, Mynhardt and Cockayne [10] and the lower bound is due to Kearse and Gibbons [66].

Theorem 2.31 ([10, 66]) *For $n \geq 6$,*

$$6n - O(n^{\frac{2}{3}}) \leq IR(Q_n) \leq \left\lfloor 6n + 6 - 8\sqrt{n + 1 + \sqrt{n}} \right\rfloor .$$

2.6.2 Other Chessboard Graphs

The irredundance numbers of other chessboard graphs have also been studied. Known values of these parameters are given in Table 2.4 and bounds for $IR(K_n)$ and $ir(K_N)$ are given in Theorem 2.32. Any result not referenced in Table 2.4, the reader is referred to [55].

$G =$	ir	IR
Kings: K_n	?	?
Rooks: R_n	n	$2n - 4$
Bishops: B_n	n	$4n - 14$
Knights: N_n	?	$\left\lceil \frac{n^2}{2} \right\rceil$ [25]
Grid: G_n	?	$\left\lceil \frac{n^2}{2} \right\rceil$ [25]

Table 2.4: Irredundance numbers for chessboard graphs.

Theorem 2.32 ([51]) *For any n,*

$$\left\lceil \frac{n^2}{9} \right\rceil \leq ir(K_n) \leq \left\lfloor \frac{(n+2)}{3} \right\rfloor^2$$

and for $n \geq 6$,

$$\left\lceil \frac{(n-2)^2}{3} \right\rceil + 3 \leq IR(K_n) \leq \frac{n^2}{3}.$$

The value of graph parameters (in the domination chain) on chessboards graphs has been the concentration of much research. The reader is referred to [62] for more details.

2.7 Complexity

Fellows, Fricke, Hedetniemi and Jacobs [53] and Laskar, Pfaff, Hedetniemi and Hedetniemi [70], showed that the decision questions, (i) does G have an irredundant set of size $\geq k$ (for a positive integer k)? and (ii) does G have a maximal irredundant set of size $\leq k$? are NP-complete.

Chapter 3

Generalised Irredundance

The subject of this chapter is the generalised irredundant sets which were defined in Section 1.4. Known results concerning generalised irredundant sets are surveyed. We begin with a basic observation due to Cockayne [19].

Recall that q_f (respectively Q_f) is the smallest (largest) cardinality of a maximal f-set of G. When $f = f_i$ we will sometimes write q_i (respectively Q_i) instead of q_{f_i} (respectively Q_{f_i}). For $f, g \in \mathcal{F}$, we write $f \Rightarrow g$ if f logically implies g.

Lemma 3.1 ([19]) *If $f, g \in \mathcal{F}$ and $f \Rightarrow g$, then $Q_f(G) \leq Q_g(G)$ for every graph G.*

3.1 When is Ω_f Hereditary?

A property P of vertex subsets is called *hereditary* if for every set $S \subseteq V$ with property P and every $T \subseteq S$, T has property P. We say that $f \in \mathcal{F}$ is *hereditary* or is a *hereditary function* if for any f-set, $S \subseteq V$, every subset of S is also an f-set.

In [19], Cockayne shows exactly which functions of $\mathcal{F}$ are hereditary. This result is given in Theorem 3.2; definitions of the hereditary functions of $\mathcal{F}$ can be found in Table 3.1. The structure of the twelve hereditary classes of generalised irredundant sets can be seen in Figure 3.1, where an arrow points from f_i to f_j if $f_i \Rightarrow f_j$.

Theorem 3.2 ([19]) *If for $f \in \mathcal{F}$, Ω_f is hereditary if and only if*

$$f \in \{f_1, f_3, f_5, f_7, f_9, f_{11}, f_{13}, f_{15}, f_{21}, f_{23}, f_{29}, f_{31}\}.$$

3.2 Ramsey Numbers for f-sets.

In Section 2.5 we discussed a new type of Ramsey number by replacing the word "independent" in the definition of classical Ramsey numbers with the word "irredundant." In [19], the word independent was replaced with f-set

Label	The function f	name (if any)
f_1	$p \wedge r$	
f_3	p	independent
f_5	$(p \vee q) \wedge r$	
f_7	$p \vee (q \wedge r)$	
f_9	$(p \wedge r) \vee (q \wedge \overline{r})$	
f_{11}	$p \vee (q \wedge \overline{r})$	
f_{13}	$(p \wedge r) \vee q$	
f_{15}	$p \vee q$	1-dependent
f_{21}	r	open irredundant
f_{23}	$p \vee r$	irredundant
f_{29}	$q \vee r$	OO-irredundant
f_{31}	$p \vee q \vee r$	CO-irredundant

Table 3.1: The hereditary functions of $\mathcal{F}$.

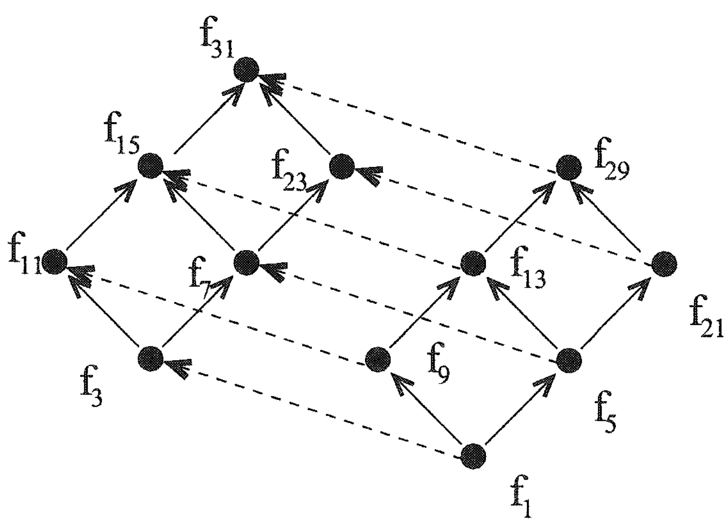

Figure 3.1: The structure of the hereditary functions of $\mathcal{F}$.

(where $f \in \mathcal{F}$). This gives a way of defining an f-Ramsey number (although they do not exist for all $f \in \mathcal{F}$).

Let $f \in \mathcal{F}$ and α be the t-tuple $(n_1, n_2, \ldots, n_t)$, where each n_i $(1 \leq i \leq t)$ is a positive integer. The *f-Ramsey number* $R_f(\alpha)$ is the smallest n_0 such that for all $n \geq n_0$ and every t-edge colouring $(G_1, G_2, \cdots, G_t)$ of K_n, there exists an $i \in \{1, \ldots, t\}$ such that $\overline{G_i}$ has an f-set of cardinality n_i. It is noted that for any permutation $\pi(\alpha)$ and $f \in \mathcal{F}$, $R_f(\alpha) = R_f(\pi(\alpha))$. For this reason, in the remainder of this chapter, we assume that $n_1 \geq n_2 \geq \ldots \geq n_t$.

3.2.1 Existence Results for f-Ramsey Numbers

It is not true that f-Ramsey numbers exist for every $f \in \mathcal{F}$. In fact, for some functions f, the existence depends on the parity of the n_i's.

Theorem 3.3 ([19]) *Suppose $f, g \in \mathcal{F}$, $f \Rightarrow g$ and the number $R_f(\alpha)$ exists. Then $R_g(\alpha)$ exists and $R_g(\alpha) \leq R_f(\alpha)$.*

Corollary 3.3.1 ([19]) *If $p \Rightarrow f \in \mathcal{F}$, then $R_f(\alpha)$ exists and $R_f(\alpha) \leq R_p(\alpha)$.*

This corollary establishes the existence of $R_{f_i}(\alpha)$ for $i \in \{4j+3 | 0 \leq j \leq 14\}$. The next theorem by Cockayne, Favaron, Gröbler, Mynhardt and Puech [23] establishes existence of $R_{f_i}(\alpha)$ for most α and all $i \in \{48 \ldots 62\}$.

Theorem 3.4 ([23]) *Let $f \in \mathcal{F}$ satisfy $f_{48} \Rightarrow f$ and $n_1, n_2 \geq 3$. Then $R_f(\alpha)$ exists.*

There is also a variety of theorems which give sufficient conditions on $f \in \mathcal{F}$ and α for $R_f(\alpha)$ not to exist. The reader is referred to [19] and [23] for more details.

3.2.2 Recurrence Inequalities

For $f \in \mathcal{F}$ and a given $\alpha = (n_1, \ldots, n_t)$, define

$$R^i = R_f(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_t).$$

It is well known that for $f = f_3 = p$ (classical Ramsey numbers),

$$R_f(\alpha) \leq \sum_{i=1}^{t} R^i. \tag{3.1}$$

Work done in [19] and [23] exhibit two subclasses of $f \in \mathcal{F}$ for which (3.1) holds. We omit the details and present only a list of those functions which satisfy the inequality.

Theorem 3.5 ([19, 23]) *If*

$$i \in \{3, 7, 15, 19, 23, 31, 48, 51, 52, 55, 60, 61\},$$

then the f_i-Ramsey numbers satisfy inequality (3.1).

3.2.3 Known Bounds and Values

The f-Ramsey numbers for six specific functions $f \in \mathcal{F}$ have been evaluated. In this section we list known values and bounds for f-Ramsey numbers, except for the classical Ramsey numbers (see for example [75]) and irredundant Ramsey numbers (see Section 2.5.2). The four remaining functions f for which f-Ramsey numbers have been studied are the f_{31}-(CO-irredundant), f_{15}- (1-dependent), f_{60}- ($\overline{p}$-) and f_{48}- ($(\overline{p} \wedge \overline{q})$-) Ramsey numbers.

Theorem 3.6 ([32, 41, 77]) *For $m \geq 3$, $R_{f_{31}}(3,m) = m$, $R_{f_{31}}(3,3,m) = 2m-1$, for m odd, and $2m-3$, for m even. Further, $R_{f_{31}}(4,4) = 6$, $R_{f_{31}}(4,5) = 8$, $R_{f_{31}}(4,6) = 11$, $R_{f_{31}}(4,7) = 14$, $R_{f_{31}}(5,5) \in \{14,15\}$, $R_{f_{31}}(3,3,4) = 6$, $R_{f_{31}}(3,4,4) = 8$, $R_{f_{31}}(4,4,4) = 11$ and $R_{f_{31}}(3,3,\ldots,3)$ (k-arguments) $= k+2$, if k is odd, and $k+1$, if k is even.*

Theorem 3.7 ([32]) *If $3 \leq n_i \leq 4$, for each $i = 1, \ldots t$, then*

$$R_{f_{31}}(n_1, n_2, \ldots, n_t) = R_{f_{15}}(n_1, n_2, \ldots, n_t).$$

Theorem 3.8 ([35]) *$R_{f_{15}}(4,5) = 9$, $R_{f_{15}}(4,6) = 11$, $R_{f_{15}}(4,7) = 16$, $R_{f_{15}}(4,8) = 17$ and $R_{f_{15}}(5,5) = 15$.*

Theorem 3.9 ([24]) *If $2 \leq l \leq m$, then $R_{f_{60}}(l,m) = m$.*

Theorem 3.10 ([24]) *(i)* $R_{f_{48}}(3,3) = 6$, $R_{f_{48}}(3,4) = 7$, $R_{f_{48}}(3,5) = 9$.

(ii) For $m \geq 6$, $R_{f_{48}}(3,m) = m+2$.

(iii) If $4 \leq l < m$, *then* $R_{f_{48}}(l,m) = m+1$.

(iv) For $m \geq 4$, $R_{f_{48}}(m,m) = m+2$.

3.3 Other Results on f-sets

In 1998, Simmons [77] provided an analogous result to Theorem 1.2 for CO-irredundant (f_{31}-sets) and total dominating sets.

Theorem 3.11 ([77]) *(i) A total dominating set* D *is minimal total dominating if and only if it is CO-irredundant.*

(ii) If a set D *is minimal total dominating, then it is maximal CO-irredundant.*

Sometimes we will write $COIR$ ($coir$) instead of $Q_{f_{31}}$ ($q_{f_{31}}$), and OIR (oir) instead of $Q_{f_{21}}$ ($q_{f_{21}}$). Nordhaus-Gaddum type results have been established for independent, CO-irredundant and open irredundant sets. We state the latter two here, since they will be used in Chapter 4.

Theorem 3.12 ([33]) *For a graph* G *of order* n, *(i)* $COIR + \overline{COIR} \leq n + 2$ *and*

(ii) $COIR \cdot \overline{COIR} \leq \frac{(n+2)^2}{2}$.

Theorem 3.13 ([20]) *For any graph G of order $n \geq 16$,*

$$OIR + \overline{OIR} \leq \left\lfloor \frac{3n}{4} \right\rfloor.$$

Further, if $n \geq 17$, then

$$OIR \cdot \overline{OIR} < \frac{9n^2}{64}.$$

NP-completeness results [53] have been established for each class of sets in the Private Neighbor Cube (see Table 1.3 for the corresponding Ω_f). In [57] it is shown that for any bipartite graph G, $COIR(G) = \beta^1(G)$(the maximum cardinality of a 1-dependent set of G). Farley and Shacham [46] have explored the relationships between OIR (oir) and parameters in the domination chain. A graph G is called *well-covered* if $\beta = i$. A well-covered graph G is called *stable well-covered* if the graph formed by the addition of any edge to G is also well-covered. King [68] showed a well-covered graph G is stable well-covered if and only if every vertex of every maximum independent set S of G has an S-epn. Equivalently, one could say a well-covered graph G is stable well-covered if and only if every maximum independent set is open irredundant.

Chapter 4

Nordhaus-Gaddum Bounds for Generalised Irredundant Sets

In this chapter we establish sharp Nordhaus-Gaddum bounds for the parameter Q_f, for each $f \in \mathcal{F}$. We will need Theorems 2.16, 3.12 and 3.13 from Chapters 2 and 3. Each of these theorems will be re-stated and re-numbered when they are needed. The bounds for the 63 non-zero values of i will be given in Theorems 4.2, 4.4, 4.6 and 4.12.

4.1 The Bounds

We first remind the reader of Lemma 3.1.

Lemma 4.1 *If $f_i \Longrightarrow f_j$, then for any graph G, $Q_i \leq Q_j$.*

Theorem 4.2 *If $i \geq 32$ and $n \geq 5$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) = 2n \qquad \textit{and} \qquad \max_G \left(Q_i \cdot \overline{Q}_i\right) = n^2,$$

and these bounds are sharp.

Proof. If $i \geq 32$, then $f_{32} \Longrightarrow f_i$, so that for all G (using Lemma 4.1) $Q_{32} \leq Q_i \leq n$ and $\overline{Q}_{32} \leq \overline{Q}_i \leq n$. Hence

$$Q_{32} + \overline{Q}_{32} \leq Q_i + \overline{Q}_i \leq 2n$$

and

$$Q_{32} \cdot \overline{Q}_{32} \leq Q_i \cdot \overline{Q}_i \leq n^2.$$

However, for $n \geq 5$, $Q_{32}(C_n) = Q_{32}\left(\overline{C}_n\right) = n$ and the result follows. ■

We next use the Nordhaus-Gaddum bounds for standard irredundant (i.e. f_{23}-) sets, obtained by Cockayne and Mynhardt [36] (originally stated in Theorem 2.16), to deduce the same bounds for other values of i.

Theorem 4.3 ([36]) *If* $n \geq 3$, *then for any graph* G,

$$Q_{23} + \overline{Q}_{23} \leq n+1 \qquad \text{and} \qquad Q_{23} \cdot \overline{Q}_{23} \leq \left\lceil \frac{n^2+2n}{4} \right\rceil,$$

and these bounds are sharp.

Theorem 4.4 *If* $n \geq 5$ *and* $i \in \{2,3,6,7,18,19,22,23\}$, *then*

$$\max_G \left(Q_i + \overline{Q}_i\right) = n+1 \qquad \text{and} \qquad \max_G \left(Q_i \cdot \overline{Q}_i\right) = \left\lceil \frac{n^2+2n}{4} \right\rceil,$$

and these bounds are sharp.

Proof. If $i \in \{2,3,6,7,18,19,22,23\}$, then $f_2 \implies f_i \implies f_{23}$. Hence, by Lemma 4.1 and Theorem 4.3,

$$Q_2 + \overline{Q}_2 \leq Q_i + \overline{Q}_i \leq Q_{23} + \overline{Q}_{23} \leq n+1,$$

and

$$Q_2 \cdot \overline{Q}_2 \leq Q_i \cdot \overline{Q}_i \leq Q_{23} \cdot \overline{Q}_{23} \leq \left\lceil \frac{n^2+2n}{4} \right\rceil.$$

Consider the graph H which consists of a set X of $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices, a set Y of $\left\lceil \frac{n+1}{2} \right\rceil$ vertices (where $X \cap Y = \{x\}$), and a set of edges such that X is independent, $H[Y]$ is complete and there is a matching joining the vertices of $X - \{x\}$ to $Y - \{x\}$. In the case where n is even, an edge is added between the vertex of Y which was not previously matched and any vertex of $X - \{x\}$.

Since each vertex of an f_2-set S is an S-spn and has no S-epn, it is easily seen that X and Y are f_2-sets of H and $\overline{H}$, respectively, and so $Q_2(H) \geq |X|$ and $Q_2(\overline{H}) \geq |Y|$. Hence, for H, all of the above inequalities are equalities and the result follows. ■

We now proceed in a similar manner using the bounds for CO-irredundant (i.e. f_{31}-) sets established by Cockayne, McCrea and Mynhardt [33].

Theorem 4.5 ([33]) *For any graph G of order n*

$$Q_{31} + \overline{Q}_{31} \leq n+2 \qquad \textit{and} \qquad Q_{31} \cdot \overline{Q}_{31} \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor,$$

and these bounds are sharp.

Theorem 4.6 *If $8 \leq i \leq 15$ or $24 \leq i \leq 31$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) \leq n+2, \qquad \max_G \left(Q_i \cdot \overline{Q}_i\right) \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor$$

and these bounds are sharp for $n \equiv 2 \pmod 4$, $n \geq 6$.

Proof. For any i satisfying $8 \leq i \leq 15$ or $24 \leq i \leq 31$, $f_8 \Longrightarrow f_i \Longrightarrow f_{31}$. Thus, by Lemma 4.1 and Theorem 4.5, for any graph G of order n,

$$Q_8 \cdot \overline{Q}_8 \leq Q_i \cdot \overline{Q}_i \leq Q_{31} \cdot \overline{Q}_{31} \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor$$

and

$$Q_8 + \overline{Q}_8 \leq Q_i + \overline{Q}_i \leq Q_{31} + \overline{Q}_{31} \leq n+2.$$

Thus, the bounds of the theorem are established. Now let $n \equiv 2 \pmod 4$ and $n \geq 6$. Let the graph H consist of vertex sets X and Y, where $|X| = |Y| = (n+2)/2$ and $|X \cap Y| = 2$. Add edges so that $H[X]$ and $\overline{H}[Y]$ are both isomorphic to $\left(\frac{n+2}{4}\right) K_2$ and add a matching from $X - Y$ to $Y - X$.

Since a subset S is an f_8-set if each vertex has an S-ipn and no S-epn, it is easily seen that X and Y are f_8-sets of H and $\overline{H}$, respectively. Therefore, the bounds are sharp for the graph H. ■

In order to find the bounds for the remaining values of i, it will be necessary to improve the following result of Cockayne [20] concerning open irredundant (i.e. f_{21}-) sets. Recall that a set S is an f_{21}-set if each $s \in S$ has an S-epn.

Theorem 4.7 ([20]) *For any graph G of order $n \geq 16$,*

$$Q_{21} + \overline{Q}_{21} \leq \left\lfloor \frac{3n}{4} \right\rfloor,$$

and this bound is sharp. Further if $n \geq 17$, then

$$Q_{21} \cdot \overline{Q}_{21} < \frac{9n^2}{64}.$$

We show that for $n \geq 32$, the second bound of Theorem 4.7 can be improved to $n^2/8$. This will be accomplished by more detailed analysis of the cases used in the proof of Theorem 4.7 given in [20]. Some of the details of our proof may be found in [20] but are repeated here for completeness.

Let X (Y) be open irredundant sets of $G\left(\overline{G}\right)$, $|X| = x$ and $|Y| = y$. Each $u \in X$ $(v \in Y)$ has an at least one X-epn in G (Y-epn in $\overline{G}$). Let $u_r(v_b)$ be any X-epn of u in G (Y-epn of v in $\overline{G}$). The edges of G (respectively $\overline{G}$) will be coloured red (blue). Occasionally u_r (v_b) will be called a *red epn* of u (*blue epn* of v). Let $X' = \{u_r | u \in X\}$. Then each edge of $\{uu_r | u \in X\}$ is red while all other edges joining X to X' are in the complement. Hence, the set $\{uu_r | u \in X\}$ induces a matching in G. Similarly, it can be seen that the set $\{vv_b | v \in Y\}$ induces a matching in $\overline{G}$. Note that the set X' is also an open irredundant set of G and u is an X'-epn of u_r in G. Let $Z = V - (X \cup X')$.

The principal result will follow immediately from three propositions, which are broken down into cases depending on the distribution of vertices of Y and its blue epns among the three sets X, X', Z.

The open irredundance property implies that both x and y are at most $n/2$. From this we deduce that $xy \leq \frac{n^2}{8}$ if x (or y) $\leq \frac{n}{4}$. Hence, it is sufficient to establish the propositions under the assumption that $x, y > \frac{n}{4}$, and we use this hypothesis in the proofs without further emphasis. We also repeatedly use the following obvious fact.

Lemma 4.8 *Let A be an open irredundant set in a graph G and $B \subseteq V(G)$. If each $u \in A \cap B$ has A-epn in B, then $|A \cap B| \leq |B|/2$.*

Proposition 4.9 *If $n \geq 32$ and $|Y \cap X| \geq 3$, then $xy \leq n^2/8$.*

Proof. Since $|Y \cap X| \geq 3$, for each $u \in Y \cap X$, $u_b \notin X'$. Hence, $u_b \in X \cup Z$. Define

$$X_1 = \{u \in Y \cap X | u_b \in X\},$$

$$X_2 = \{u \in Y \cap X | u_b \in Z\},$$

$$X_3 = X - (X_1 \cup X_2),$$

and for $i = 1, 2, 3$, let $|X_i| = x_i$.

For $w \in Y \cap Z$, $w_b \notin X_1 \cup X_2 \cup X'$, hence $w_b \in X_3 \cup Z$.

Case 1 $Y \cap X' = \phi$

Let $t = |\{w \in Y \cap Z | w_b \in X_3\}|$. Then by Lemma 4.8,

$$|\{w \in Y \cap Z | w_b \in Z\} \leq (n - 2x - x_2 - t)/2| \tag{4.1}$$

We will now give more detailed justification for (4.1). Similar explanations will be omitted in future cases of the propositions. Define

$$B = Z - (\{w \in Y \cap Z | w_b \in X_3\} \cup \{w_b \in Z | w \in X_2\})$$

(disjoint union)

Note that $|B| = (n - 2x - x_2 - t)$ and

$$\{w \in Y \cap Z | w_b \in Z\} = \{w \in Y \cap B | w_b \in B\}.$$

Then (4.1) follows by applying Lemma 4.8 with $A = Y$.

Now

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap Z| \\ &\leq x + (x_1 + x_2) + t + \left(\frac{n - 2x - x_2 - t}{2}\right) \\ &= x_1 + \frac{x_2}{2} + \frac{t}{2} + \frac{n}{2}. \end{aligned} \tag{4.2}$$

The blue epns in X_3 are distinct and so $x_3 \geq t + x_1$, i.e.

$$\frac{t}{2} \leq \frac{x_3}{2} - \frac{x_1}{2}. \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$x + y \leq \left(\frac{x_1 + x_2 + x_3}{2}\right) + \frac{n}{2} = \frac{x}{2} + \frac{n}{2}.$$

Therefore $y \leq \frac{n}{2} - \frac{x}{2}$ and $xy \leq \frac{nx}{2} - \frac{x^2}{2}$. By elementary calculus, xy attains a maximum of $\frac{n^2}{8}$ when $x = \frac{n}{2}$.

Case 2 $|Y \cap X'| \geq 2$.

In this case $x_1 = 0$, each $w \in Y \cap Z$ has $w_b \in Z$ and for each $w \in Y \cap X'$, $w_b \notin X'$, i.e. $w_b \in X_3 \cup Z$.

Subcase 2(a) $w \in Y \cap X'$ has $w_b \in X_3$.

This implies $|Y \cap X'| = 2$. Let $Y \cap X' = \{w, v\}$. Now

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\leq x + x_2 + 2 + \frac{(n - 2x - x_2 - \lambda)}{2}, \end{aligned}$$

where $\lambda = 1$ (respectively 0) if $v_b \in Z(X_3)$. Hence,

$$x + y \leq \frac{n}{2} + \frac{x_2}{2} - \frac{\lambda}{2} + 2. \tag{4.4}$$

By counting blue epns in X_3, we obtain $x_3 \geq 2 - \lambda$ and since $|Z| \geq x_2$, we deduce $x_2 \leq n - 2x$. Use of these gives

$$x_2 \leq n - 2(x_1 + x_2 + x_3) = n - 2(x_2 + x_3).$$

Therefore,

$$x_2 \leq \frac{n - 2x_3}{3} \leq \frac{n - 4 - 2\lambda}{3}. \tag{4.5}$$

From (4.4) and (4.5),

$$x + y \leq \frac{2n + 4}{3} - \frac{5\lambda}{6} \leq \frac{2n + 4}{3},$$

so that $xy \leq x\left(\frac{2n+4}{3} - x\right)$. Calculus shows that $xy \leq \left\lfloor \left(\frac{n+2}{3}\right)^2 \right\rfloor \leq \frac{n^2}{8}$ (for $n \geq 32$).

Subcase 2(b) Each $w \in Y \cap X'$ has $w_b \in Z$.

In this situation every $v \in Y$ has $v_b \in Z$. Therefore, $y \leq |Z| = n - 2x$ and $xy \leq nx - 2x^2$. The maximum of this for $x \in \left[\frac{n}{4}, \frac{n}{2}\right]$ is $\frac{n^2}{8}$.

Case 3 $|Y \cap X'| = \{v\}$.

Define λ as in subcase 2(a) and let $\mu\,(= 0 \text{ or } 1)$ be the number of vertices in $Y \cap Z$ with blue epns in X_3.

The set Z contains $\lambda + x_2$ blue epns of vertices in $Y \cap (X \cup X')$ and μ vertices of $Y \cap Z$ have blue epns in X_3. Hence, using Lemma 4.8 we obtain

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\leq x + (x_1 + x_2) + 1 + \mu + \left(\frac{n - 2x - \mu - x_2 - \lambda}{2}\right) \\ &= \frac{n}{2} + x_1 + \frac{x_2}{2} + \frac{(\mu - \lambda)}{2} + 1. \end{aligned} \tag{4.6}$$

By counting blue epns in X_3, we obtain $x_3 \geq (1 - \lambda) + x_1 + \mu$, and since $|Z| \geq x_2$, we have $x_2 \leq n - 2x$. Use of these gives

$$x_2 \leq n - 2\,(x_1 + x_2 + x_3)\,.$$

Hence,

$$\begin{aligned} x_2 &\le \frac{n-2(x_1+x_3)}{3} \\ &\le \frac{n-2x_1-2[(1-\lambda)+x_1+\mu]}{3} \\ &= \frac{n-4x_1-2-2(\mu-\lambda)}{3}. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$x+y \le \frac{2n+2}{3} + \frac{x_1}{3} + \frac{\mu-\lambda}{6}. \tag{4.8}$$

However, the hypothesis and the private neighbour property imply that

$$x_1 + \mu \le 1.$$

Hence, from (4.8) we deduce

$$x+y \le \frac{2n+3}{3} - \left(\frac{\lambda+\mu}{6}\right) \le \frac{2n+3}{3}.$$

Calculus shows that $xy \le \left(\frac{2n+3}{6}\right)^2 \le \frac{n^2}{8}$ (for $n \ge 32$). This completes the proof of Proposition 4.9.■

Proposition 4.10 *If* $n \ge 32$ *and* $|Y \cap X| \le 2$, *then* $xy \le n^2/8$.

Proof. Define $Y' = \{v_f | v \in Y\}$. If $|Y \cap X'|$ ($|Y' \cap X|$ or $|Y' \cap X'|$) > 2, then we may apply Proposition 4.9 to the open irredundant sets Y, X' (Y', X

or Y', X') of $\overline{G}$ and G and infer the result. Thus, we assume that $|Y \cap X'|$, $|Y' \cap X|$ and $|Y' \cap X'|$ are at most two. Then,

$$\begin{aligned} n &\geq |X| + |X'| + |Y| + |Y'| - |Y \cap X| - |Y' \cap X| - |Y \cap X'| - |Y' \cap X'| \\ &\geq 2x + 2y - 2 - 2 - 2 - 2. \end{aligned}$$

Hence, $x + y \leq \frac{n+8}{2}$ and therefore, by elementary calculus, $xy \leq \left(\frac{n+8}{4}\right)^2 \leq \frac{n^2}{8}$ (for $n \geq 32$). ■

The preceding propositions have established a bound for $Q_{21} \cdot \overline{Q}_{21}$.

Theorem 4.11 *If $n \geq 32$, then $Q_{21}\overline{Q}_{21} \leq n^2/8$.*

Proof. Immediate from Propositions 4.9 and 4.10. ■

We now use Theorems 4.7 and 4.11 to determine exact Nordhaus-Gaddum bounds for the remaining values of i.

Theorem 4.12 *If $n \geq 32$ and $i \in \{1, 4, 5, 16, 17, 20, 21\}$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) \leq \frac{3n}{4}, \qquad \max_G \left(Q_i \cdot \overline{Q}_i\right) \leq n^2/8,$$

and these bounds are sharp for infinitely many values of n.

Proof. For any $i \in \{1, 4, 5, 16, 17, 20, 21\}$,

$$\begin{aligned} f_1 &\Longrightarrow f_i \Longrightarrow f_{21}, \\ f_4 &\Longrightarrow f_i \Longrightarrow f_{21}, \end{aligned}$$

or

$$f_{16} \Longrightarrow f_i \Longrightarrow f_{21}.$$

Hence by Lemma 4.1, Theorems 4.7 and 4.11, for any G of order n

$$Q_j + \overline{Q}_j \leq Q_i + \overline{Q}_i \leq Q_{21} + \overline{Q}_{21} \leq \frac{3n}{4}$$

and

$$Q_j \cdot \overline{Q}_j \leq Q_i \cdot \overline{Q}_i \leq Q_{21} \cdot \overline{Q}_{21} \leq \frac{n^2}{8},$$

where $j \in \{1, 4, 16\}$. Thus, the bounds of the theorem are established. To show that they are attained it is sufficient to exhibit, for each $j \in \{1, 4, 16\}$, graphs satisfying

$$Q_j + \overline{Q}_j \geq \frac{3n}{4} \qquad \text{and} \qquad Q_j \cdot \overline{Q}_j \geq \frac{n^2}{8}.$$

In order to describe the three examples, we need the following definition. Let A, B be disjoint m-vertex subsets of a graph L. We say there is an *induced matching from A to B in L* if the bipartite subgraph of L defined by the set of edges between A and B is isomorphic to mK_2.

We form the n-vertex graph H as follows. Let $V(H) = X \cup Y \cup Y'$ (disjoint union), where $n \equiv 0 \pmod 4$, $n \geq 32$, $|X| = \frac{n}{2}$, $|Y| = |Y'| = \frac{n}{4}$ and $X' = Y \cup Y'$. Add edges so that there are induced matchings from X to X' in H and from Y to Y' in $\overline{H}$.

Each of the three examples will be formed by adding edges to H. For each of the three values of j it is easily checked that X and Y are f_j-sets of the constructed graph H^* and $\overline{H^*}$ respectively, so that H^* satisfies equality in the established bounds. In each case we remind the reader of the f_j-set definition.

$j = 1$ Subset S is an f_1-set if each $s \in S$ is an S-spn and has an S-epn. Form H^* from H by adding edges so that $H^*[Y]$ is complete.

$j = 4$ Subset S is an f_4-set if each $s \in S$ has both an S-ipn and an S-epn. In this case we require $n \equiv 0 \pmod 8$. Form H^* from H by adding edges so that $H^*[X]$ and $\overline{H^*}[Y]$ are isomorphic to $\frac{n}{4}K_2$ and $\frac{n}{8}K_2$, respectively.

$j = 16$ Subset S is an f_{16}-set if each $s \in S$ has an S-epn, has no S-ipn and is not an S-spn. Form H^* from H by adding edges so that $H^*[X]$ and $\overline{H^*}[Y]$ are isomorphic to $C_{\frac{n}{2}}$ and $C_{\frac{n}{4}}$, respectively.■

Chapter 5

A Lower Bound for the CO-Irredundance Number

In this chapter we establish a necessary and sufficient condition for a CO-irredundant set (i.e. f_{31}-set) of vertices of a graph to be maximal, and show that the smallest cardinality of a maximal CO-irredundant set (i.e. the parameter $coir = q_{31}$) in a graph of order n with maximum degree Δ is bounded below by $\frac{n}{2}$ for $\Delta = 2$, $\frac{4n}{13}$ for $\Delta = 3$, and $\frac{2n}{3\Delta-3}$ for $\Delta \geq 4$. This result is best possible and extremal graphs are characterised for $\Delta \geq 3$. Note that our result is the analog for CO-irredundance of Theorem 2.14.

5.1 Maximal CO-irredundant Sets

In this section we establish necessary and sufficient conditions for a CO-irredundant set to be maximal. Let $G = (V, E)$ be a graph. For any $x \in X \subseteq V$, define,

$$pn_{31}(x, X) = spn(x, X) \cup ipn(x, X) \cup epn(x, X).$$

Lemma 5.1 *Let $X \subseteq V$, $x \in X$ and $v \in V - X$.*

(i) $pn_{31}(v, X \cup \{v\}) = N[v] - N(X)$.

(ii) $pn_{31}(x, X \cup \{v\}) = pn_{31}(x, X) - N(v)$.

Proof. If $y \in pn_{31}(v, X \cup \{v\})$ then either $y \in spn(v, X \cup \{v\})$, $y \in ipn(v, X \cup \{v\})$ or $y \in epn(v, X \cup \{v\})$. In each case $y \in N[v]$ but $y \notin N(X)$, thus $y \in N[v] - N(X)$. If $y \in N[v] - N(X)$, then either $y = v$ and v is an $(X \cup \{v\})$-spn, or y is adjacent to v but no other vertex in $X \cup \{v\}$ (i.e. y is either an $(X \cup \{v\})$-ipn or an $(X \cup \{v\})$-epn). Thus, $y \in pn_{31}(v, X \cup \{v\})$. A similar proof may be used to prove part (ii) of the lemma. ■

Theorem 5.2 *Let X be a CO-irredundant set of G and $S = V - N(X)$. Then X is a maximal CO-irredundant set if and only if for every $v \in N[S] - X$, there exists an $x_v \in X$ such that $pn_{31}(x_v, X) \subseteq N(v)$.*

Proof. Let X be a maximal CO-irredundant set and suppose $v \in V - X$. Since $X \cup \{v\}$ is not CO-irredundant, there is an $x_v \in X \cup \{v\}$ such that $pn_{31}(x_v, X \cup \{v\}) = \emptyset$. If $x_v = v$, then by Lemma 5.1(i), $N[v] \subseteq N(X)$ and thus $v \notin N[S]$. Otherwise, $x_v \in X$ and by Lemma 5.1(ii), $pn_{31}(x_v, X) \subseteq N(v)$.

Conversely, suppose that X is not a maximal CO-irredundant set. Then there exists $v \in V - X$ such that $X \cup \{v\}$ is a CO-irredundant set. Since $pn_{31}(v, X \cup \{v\}) \neq \emptyset$, by Lemma 5.1(i), $N[v] - N(X) \neq \emptyset$ and thus $v \in N[S] - X$. However, for any $x \in X$, $pn_{31}(x, X \cup \{v\}) \neq \emptyset$, and so by Lemma 5.1(ii), $pn_{31}(x, X) \nsubseteq N(v)$. ■

Let $x \in X$ and $v \in N[S] - X$ (where $S = V - N(X)$). Theorem 5.2 motivates the following definition. For the remainder of this chapter, we will say v *annihilates* (or is an *annihilator* of) x, if $pn_{31}(x, X) \subseteq N(v)$.

5.2 The Bound

For a given n and Δ, let G be any edge-minimal graph with $n(G) = n$, $\Delta(G) \leq \Delta$ and *coir* a minimum. Let X be a maximal CO-irredundant set of G with $|X| = coir$. The set X induces the following partition of the vertex set:

$$Y_0 = \{x \in X \mid |N(x) \cap X| = 0\}$$

$$Y_1 = \{x \in X \mid |N(x) \cap X| = 1 \text{ and } x \text{ has an } X\text{-ipn}\}$$

$$Y_2 = \{x \in X \mid |N(x) \cap X| \geq 2 \text{ and } x \text{ has an } X\text{-ipn}\}$$

$$Z_1 = \{x \in X \mid |N(x) \cap X| = 1\} - Y_1$$

$$Z_2 = \{x \in X \mid |N(x) \cap X| \geq 2\} - Y_2$$

$$B = \bigcup_{x \in X} epn(x, X)$$

$$C = N(X) - (B \cup X)$$

$$R = V - N[X].$$

Let $|Y_i| = y_i$, for $i = 1, 2$. Notice that Y_0, Y_1, Y_2, Z_1 and Z_2 form a partition of X and that the set S defined in Theorem 5.2 is equal to $Y_0 \cup R$, so that $N[S] - X = N[R] \cup N(Y_0)$. We will use this fact, without mention, for the remainder of this chapter.

The following four preliminary results will be used in the proof of a lower bound for *coir*.

Lemma 5.3 *If $v \in B$ annihilates $x \in X$, then $x \notin N(v)$.*

Proof. This follows directly from the definition of B, the definition of the word annihilates and the fact that $v \notin N(v)$. ■

Lemma 5.4 *If $v \in R$ annihilates $x \in X$, then x has no X-spn and no X-ipn.*

Proof. If x is an X-spn or has an X-ipn, then v is adjacent to some y (possibly x) in X and so $v \notin R$, a contradiction. ■

The next result follows directly from the definitions of Y_2 and Z_1.

Lemma 5.5 *If $z \in Z_1$, then $N(z) \cap X = \{y\}$, where $y \in Y_2$.*

Lemma 5.6 *If $w \in B$ annihilates $y \in Y_2$ and $N(w) \cap X = \{z\}$, then z is the only X-ipn of y and $z \in Z_1$. Further, z is annihilated by some vertex of R. If, in addition, y has an X-epn, then z is annihilated by no more than $\Delta - 2$ vertices in R.*

Proof. From the definitions of sets B and Y_2, it is clear that $z \in Z_1$ and that z is the only X-ipn of y. Since w annihilates y, $w \in N[R] \cup N(Y_0)$ and thus w is adjacent to some $v \in R$. If v does not annihilate z, then v annihilates some other vertex of X and therefore has degree at least two. Consider $G^* = G - wv$. Clearly, $N_{G^*}[R] \cup N_{G^*}(Y_0) \subseteq N_G[R] \cup N_G(Y_0)$ and each vertex of $N_{G^*}[R] \cup N_{G^*}(Y_0)$ annihilates a vertex of X in G^*. Thus, X is a maximal CO-irredundant set of G^*, $coir(G^*) \leq coir(G)$ and G^* has fewer edges than G. This contradiction shows that v annihilates z.

If y has an X-epn, then w is adjacent to z and to the X-epns of y and, hence, to at most $\Delta - 2$ vertices of R. ■

Theorem 5.7 *For* $\Delta = 2$, $coir \geq \frac{n}{2}$, *for* $\Delta = 3$, $coir \geq \frac{4n}{13}$, *and for* $\Delta \geq 4$, $coir \geq \frac{2n}{3\Delta-3}$.

Proof. By Theorem 5.2 and Lemma 5.4, each vertex of R annihilates at least one vertex of $Z_1 \cup Z_2$. Let r_z be the number of vertices in R that annihilate $z \in Z_1 \cup Z_2$. Then

$$|R| \leq \sum_{z \in Z_1 \cup Z_2} r_z. \tag{5.1}$$

Define the following sets:

$A_1^* = \{z \in Z_1 | r_z \geq \Delta - 1\}$,

$A_1 = \{z \in Z_1 | 0 < r_z < \Delta - 1\}$,

$A_2 = \{z \in Z_2 | r_z > 0\}$,

$A_3^* = \{z \in Z_1 | r_z = 0 \text{ and } |N(z) \cap B| = \Delta - 1\}$,

$A_3 = \{z \in Z_1 | r_z = 0\} - A_3^*$, and

$A_4 = \{z \in Z_2 | r_z = 0\}$.

Let $|A_i| = a_i$, for $i \in \{1,2,3,4\}$, and $|A_i^*| = a_i^*$, for $i \in \{1,3\}$. It is clear that $Z_1 = A_1^* \cup A_1 \cup A_3^* \cup A_3$ (disjoint union) and $Z_2 = A_2 \cup A_4$ (disjoint union).

For each $z \in A_1^*$ and $w \in pn_{31}(z, X)$, w is adjacent to z and r_z vertices of R. Since $deg(w) \leq \Delta$, this implies

$$\sum_{z \in A_1^*} r_z = \sum_{z \in A_1^*} (\Delta - 1) = (\Delta - 1)a_1^*. \tag{5.2}$$

For each $z \in A_2$ and $w \in pn_{31}(z, X)$, w is adjacent to z, r_z vertices of R and at least one other vertex (as w annihilates some $y \in X (y \neq z)$ by Theorem 5.2 and Lemma 5.3). Thus,

$$\sum_{z \in A_2} r_z \leq \sum_{z \in A_2} (\Delta - 2) \leq (\Delta - 2)a_2. \tag{5.3}$$

Therefore, from (5.1), (5.2), (5.3) and the definition of A_1,

$$\begin{aligned} |R| &\leq \sum_{z \in Z_1 \cup Z_2} r_z \\ &\leq \sum_{z \in A_1^*} r_z + \sum_{z \in A_1} r_z + \sum_{z \in A_2} r_z + \sum_{z \in A_3^* \cup A_3 \cup A_4} r_z \\ &\leq (\Delta - 1)a_1^* + (\Delta - 2)(a_1 + a_2) + 0. \end{aligned} \tag{5.4}$$

Let

$B_1 = B \cap N(A_1^* \cup A_1 \cup A_2 \cup Y_0)$,

$B_2 = B \cap N(A_3^* \cup A_3 \cup A_4)$,

$B_3 = B \cap N(Y_2)$, and

$B_4 = B \cap N(Y_1)$.

Notice that B_1, B_2, B_3 and B_4 form a partition of B. Let b_z be the number of vertices of B_1 that annihilate $z \in Z_1 \cup Z_2 \cup Y_2$.

Since each element of B_1 is in $N[R] \cup N(Y_0)$, it annihilates some $z \in Z_1 \cup Z_2 \cup Y_2$, and thus

$$|B_1| \leq \sum_{z \in Z_1 \cup Z_2 \cup Y_2} b_z. \tag{5.5}$$

Now partition Y_2 into the following four sets:

$D = \{y \in Y_2 | b_y = 0\}$,

$D_0 = \{y \in Y_2 - D || N(y) \cap B| = 0\}$,

$D_1 = \{y \in Y_2 - D || N(y) \cap B| = 1\}$ and

$D_2 = \{y \in Y_2 - D || N(y) \cap B| \geq 2\}$.

Let $d_i = |D_i|$ and $d = |D|$.

For $z \in Z_1 \cap N(D_1 \cup D_2)$ let d_z be the number of X-epns of y, where $\{y\} = N(z) \cap (D_1 \cup D_2)$. If $z \in ((Z_1 \cup Z_2) - (Z_1 \cap N(D_1 \cup D_2)))$, let $d_z = 0$.

Suppose that $d_z \neq 0$. Then $z \in Z_1 \cap N(D_1 \cup D_2)$ and so $z \in Z_1$ is an X-ipn of $y \in (D_1 \cup D_2) \subseteq Y_2$. The definition of $D_1 \cup D_2$ implies that $b_y > 0$. Therefore, some $w \in B$ annihilates $y \in Y_2$, and so $N(w) \cap X = \{z\}$. By Lemma 5.6, $1 \leq r_z \leq \Delta - 2$, and we conclude that $z \in A_1$. Hence,

$$d_1 + 2d_2 \leq \sum_{z \in Z_1 \cup Z_2} d_z = \sum_{z \in A_1} d_z. \tag{5.6}$$

If $d_z \neq 0$, then the vertex w defined in the previous paragraph is adjacent to z, r_z vertices of R, b_z vertices of B_1 and d_z vertices of B_3. For $z \in Z_1 \cup Z_2$ with $d_z = 0$, any X-epn w of z has these adjacencies. Since $deg(w) \leq \Delta$, we conclude that $b_z \leq (\Delta - 1) - r_z - d_z$. Hence,

$$\sum_{z \in Z_1 \cup Z_2 - A_3^*} b_z \leq \sum_{z \in Z_1 \cup Z_2 - A_3^*} [(\Delta - 1) - r_z - d_z]. \tag{5.7}$$

If, in addition, $z \in A_3^*$ and $v \in B$ annihilates z, then v is adjacent to the $\Delta - 1$ X-epns of z and to some $y \in X$. Since $deg(v) \leq \Delta$, $v \notin N[R]$ and thus $v \in N(Y_0)$. This implies

$$\begin{aligned} \sum_{z \in A_3^*} b_z &\leq \min\left(\Delta y_0, \sum_{z \in A_3^*} [(\Delta - 1) - r_z - d_z]\right) \\ &= \min\left(\Delta y_0, (\Delta - 1)a_3^*\right). \end{aligned} \tag{5.8}$$

If $z \in Y_2$ and $b_z \neq 0$ then by Lemma 5.6, z has exactly one X-ipn, $w \in Z_1$. However, w is adjacent to z and at most $\Delta - 1$ other vertices. Thus $b_z \leq$

$(\Delta - 1)$. Hence, from inequalities (5.1), (5.5), (5.6), (5.7) and (5.8),

$$
\begin{aligned}
|B_1| &\leq \sum_{z \in Z_1 \cup Z_2 \cup Y_2} b_z \\
&= \sum_{z \in A_1^* \cup A_1 \cup A_2} b_z + \sum_{A_3 \cup A_4} b_z + \sum_{z \in A_3^*} b_z + \sum_{z \in Y_2} b_z \\
&\leq (\Delta - 1)(a_1^* + a_1 + a_2 + a_3 + a_4) - \sum_{z \in Z_1 \cup Z_2} r_z - \sum_{z \in A_1} d_z \\
&\quad + \min(\Delta y_0, (\Delta - 1)a_3^*) + (\Delta - 1)(d_0 + d_1 + d_2) \\
&\left.\begin{aligned}
&\leq (\Delta - 1)(a_1^* + a_1 + a_2 + a_3 + a_4 + d_0) + (\Delta - 2)d_1 \\
&+ (\Delta - 3)d_2 + \min(\Delta y_0, (\Delta - 1)a_3^*) - |R|.
\end{aligned}\right\} \qquad (5.9)
\end{aligned}
$$

Each vertex of A_4 is adjacent to at least two vertices in X and thus is adjacent to at most $\Delta - 2$ vertices in B. Hence,

$$
\begin{aligned}
|B_2| &= \sum_{z \in A_3^* \cup A_3 \cup A_4} |N(z) \cap B| \\
&= \sum_{z \in A_3^*} |N(z) \cap B| + \sum_{z \in A_3} |N(z) \cap B| + \sum_{z \in A_4} |N(z) \cap B| \\
&\leq (\Delta - 1)a_3^* + (\Delta - 2)a_3 + (\Delta - 2)a_4. \qquad (5.10)
\end{aligned}
$$

For $y \in Y_2$, let $k_y = |N(y) \cap Z_1|$ and let $l_y = |N(y) \cap (Y_2 \cup Z_2)|$. By Lemmas 5.5 and 5.6, $d_0 + d_1 + \sum_{y \in D \cup D_2} k_y = \sum_{y \in Y_2} k_y \geq |Z_1|$. If $z \in D_2$, then by Lemma 5.6, z is adjacent to at least one $y \in Y_2 \cup Z_2$. Thus, $\sum_{z \in D_2} l_z \geq d_2$.

It now follows that

$$\begin{aligned}|B_3| &\le \Delta|D \cup D_2| + |D_1| - \sum_{y\in D\cup D_2} k_y - \sum_{y\in D\cup D_2} l_y \\ &\le \Delta(d+d_2) + d_0 + 2d_1 - |Z_1| - d_2 \\ &= \Delta d + d_0 + 2d_1 + (\Delta-1)d_2 - (a_1^* + a_1 + a_3^* + a_3).\end{aligned} \tag{5.11}$$

Furthermore, $|B_4| \le (\Delta-1)|Y_1|$. Therefore, by inequalities (5.9), (5.10) and (5.11):

$$\left.\begin{aligned}|B| + |R| \le (\Delta-1)&(a_1^* + a_1 + a_2 + a_3 + a_4 + d_0) \\ &+ (\Delta-2)d_1 + (\Delta-3)d_2 + (\Delta-1)a_3^* \\ &+ (\Delta-2)a_3 + (\Delta-2)a_4 + \Delta d + d_0 \\ &+ 2d_1 + (\Delta-1)d_2 - (a_1^* + a_1 + a_3^* + a_3) \\ &+ (\Delta-1)|Y_1| + \min(\Delta y_0, (\Delta-1)a_3^*) \\ &= (\Delta-1)y_1 + (\Delta-2)a_1^* + (\Delta-2)a_1 \\ &+ (\Delta-1)a_2 + (\Delta-2)a_3^* + (2\Delta-4)a_3 \\ &+ (2\Delta-3)a_4 + \Delta d + \Delta d_0 + \Delta d_1 \\ &+ (2\Delta-4)d_2 + \min(\Delta y_0, (\Delta-1)a_3^*).\end{aligned}\right\} \tag{5.12}$$

The number η of edges incident with a vertex in C and a vertex in X, satisfies

$$2|C| \leq \eta \leq \Delta|Y_0| + (\Delta - 1)(|Y_1| + |Z_1|) + (\Delta - 2)(|Y_2| + |Z_2|) - |B|. \tag{5.13}$$

Therefore, by inequalities (5.4), (5.12) and (5.13),

$$\begin{aligned}
2n &= 2|Y_0| + 2|Y_1| + 2|Y_2| + 2|Z_1| + 2|Z_2| \\
&\quad + 2|B| + 2|R| + 2|C| \\
&\leq (\Delta + 2)|Y_0| + (\Delta + 1)(|Y_1| + |Z_1|) + \Delta(|Y_2| + |Z_2|) \\
&\quad + |B| + 2|R| && \text{(by (5.13))} \\
&\leq (\Delta + 2)y_0 + (\Delta + 1)y_1 + \Delta y_2 + 2\Delta a_1^* + (2\Delta - 1)a_1 \\
&\quad + (2\Delta - 2)a_2 + (\Delta + 1)a_3^* + (\Delta + 1)a_3 + \Delta a_4 && \text{(by (5.4))} \\
&\quad + |B| + |R| \\
&\leq (\Delta + 2)y_0 + 2\Delta y_1 + (3\Delta - 2)a_1^* + (3\Delta - 3)a_1 \\
&\quad + (3\Delta - 3)a_2 + (2\Delta - 1)a_3^* + (3\Delta - 3)a_3 \\
&\quad + (3\Delta - 3)a_4 + 2\Delta d + 2\Delta d_0 + 2\Delta d_1 + (3\Delta - 4)d_2 \\
&\quad + \min(\Delta y_0, (\Delta - 1)a_3^*) && \text{(by (5.12))}.
\end{aligned}$$

By re-ordering the terms on the right hand side, we obtain

$$\left.\begin{aligned} 2n \le 2\Delta d_0 &+ (3\Delta-2)a_1^* + (\Delta+2)y_0 + (2\Delta-1)a_3^* \\ &+ \min(\Delta y_0, (\Delta-1)a_3^*) + 2\Delta(d+d_1+y_1) \\ &+ (3\Delta-3)(a_1+a_2+a_3+a_4) + (3\Delta-4)d_2. \end{aligned}\right\} \tag{5.14}$$

Let $z \in A_1^*$ and $w \in pn_{31}(z, X)$. Then w is adjacent to z and to $\Delta - 1$ vertices in R. Since $w \in N[R]$, by Theorem 5.2 there exists a $y_w \in X$ such that $pn_{31}(y_w, X) \subseteq N(w)$. Clearly $pn_{31}(y_w, X) = \{z\}$ and by Lemmas 5.5 and 5.6, $y_w \in D_0$. This implies that y_w is adjacent to exactly one vertex of $Z_1 \cup Y_1$ (namely z) and thus $a_1^* \le d_0$. Let $x_1 = d_0 - a_1^*$ and $x_2 = 2a_1^*$. Then

$$\left.\begin{aligned} & x_1, x_2 \ge 0 \\ & x_1 + x_2 = a_1^* + d_0 \\ \text{and} \qquad & 2\Delta d_0 + (3\Delta-2)a_1^* = 2\Delta x_1 + (\tfrac{5}{2}\Delta - 1)x_2. \end{aligned}\right\} \tag{5.15}$$

From (5.14) and (5.15) we deduce

$$\left.\begin{aligned} 2n \le (\Delta+2)y_0 &+ (2\Delta-1)a_3^* + \min(\Delta y_0, (\Delta-1)a_3^*) \\ &+ (3\Delta-3)(a_1+a_2+a_3+a_4) + (3\Delta-4)d_2 \\ &+ 2\Delta(d+d_1+y_1+x_1) + (\tfrac{5}{2}\Delta-1)x_2. \end{aligned}\right\} \tag{5.16}$$

We now make further substitutions which depend on the minimum included in (5.16).

Case 1 If $\Delta y_0 > (\Delta - 1)a_3^*$, then let

$$x_3 = y_0 - \frac{\Delta - 1}{\Delta} a_3^* \text{ and } x_4 = \frac{2\Delta - 1}{\Delta} a_3^*.$$

Case 2 If $\Delta y_0 \leq (\Delta - 1)a_3^*$, then let

$$x_3 = a_3^* - \frac{\Delta}{\Delta - 1} y_0 \text{ and } x_4 = \frac{2\Delta - 1}{\Delta - 1} y_0.$$

Then

$$\left.\begin{array}{l} \text{(i) in both Cases 1 and 2, } x_3, x_4 \geq 0 \text{ and} \\ \quad x_3 + x_4 = a_3^* + y_0 \\ \text{and} \\ \text{(ii) } (\Delta + 2)y_0 + (2\Delta - 1)a_3^* + \min(\Delta y_0, (\Delta - 1)a_3^*) \\ \quad = \frac{(4\Delta^2 - \Delta - 2)}{2\Delta - 1} x_4 + \begin{cases} (\Delta + 2)x_3 & \text{(Case 1)} \\ (2\Delta - 1)x_3 & \text{(Case 2).} \end{cases} \end{array}\right\} \quad (5.17)$$

From (5.16) and (5.17) we obtain

$$\left.\begin{array}{l} 2n \leq (3\Delta - 3)(a_1 + a_2 + a_3 + a_4) + (3\Delta - 4)d_2 \\ \quad + 2\Delta(d + d_1 + y_1 + x_1) + (\frac{5}{2}\Delta - 1)x_2 \\ \quad + \max(\Delta + 2, 2\Delta - 1)\, x_3 + \frac{(4\Delta^2 - \Delta - 2)}{2\Delta - 1} x_4. \end{array}\right\} \quad (5.18)$$

Let $h(\Delta)$ be the largest coefficient on the right hand side of (5.18). Since $x_1 + x_2 + x_3 + x_4 = d_0 + a_1^* + a_3^* + y_0$ (by (5.15) and (5.17)), it follows from (5.18) that

$$\begin{aligned} 2n &\leq h(\Delta)[y_0 + y_1 + (d + d_0 + d_1 + d_2) \\ &\qquad + (a_1 + a_1^* + a_3 + a_3^*) + (a_2 + a_4)] \\ &= h(\Delta)\,(y_0 + y_1 + y_2 + z_1 + z_2) \end{aligned}$$

and therefore

$$2n \leq h(\Delta)|X|. \tag{5.19}$$

It is easily seen that

$$h(\Delta) = \begin{cases} 4 & \text{if } \Delta = 2 \\ \frac{13}{2} & \text{if } \Delta = 3 \\ 3\Delta - 3 & \text{if } \Delta \geq 4 \end{cases}$$

and so the result follows immediately from (5.19). ■

5.3 Extremal Graphs

For n even (respectively odd) let X be an $\lceil \frac{n}{2} \rceil$ vertex subset of C_n whose induced subgraph contains no edge (respectively one edge). Further, for n odd let X be an independent set of P_n of cardinality $\lceil \frac{n}{2} \rceil$.

In each case (by Theorem 5.2) X is a maximal CO-irredundant set and so C_n (and P_n for n odd) are extremal graphs for the bound (and its obvious improvement for n odd) of Theorem 5.7 in the case $\Delta = 2$.

Now suppose that H is an edge-minimal graph which attains the bound of Theorem 5.7 for some n and $\Delta \geq 3$ and let X be a maximal CO-irredundant set of H with $|X| = coir(G)$.

Lemma 5.8 *The partition of $V(H)$ induced by X (developed in Section 5.2) satisfies:*

(a) $D = D_1 = D_2 = Y_0 = Y_1 = A_3^* = \emptyset$.

(b) $|A_1^*| = |D_0|$.

(c) $A_1 = A_3 = A_4 = \emptyset$.

(d) *(i)* $|B| = |B_1| = (\Delta - 1)|A_1^*| + |A_2|$.

(ii) $2|C| = (\Delta - 2)|A_1^*| + (\Delta - 3)|A_2|$.

(iii) $|R| = (\Delta - 1)|A_1^*| + (\Delta - 2)|A_2|$.

(e) *Each $z \in A_1^*$ joins $\Delta - 1$ vertices of B and a vertex in D_0. Further, each member of $N(z) \cap B$ joins z and each member of a vertex subset $S \subseteq R$, where $|S| = \Delta - 1$.*

(f) *Each vertex of* D_0 *is adjacent to one vertex of* A_1^*, *one vertex of* $A_2 \cup D_0$ *and* $\Delta - 2$ *vertices in* C.

(g) *Each* $z \in A_2$ *joins one vertex of* B, w_z, *two vertices in* $A_2 \cup D_0$ *and* $\Delta - 3$ *vertices of* C. *Further,* w_z *joins* z, *one other vertex of* $N(A_2) \cap B$ *and* $\Delta - 2$ *vertices of* R.

(h) *Each vertex of* C *is adjacent to exactly two vertices of* $A_2 \cup D_0$.

(i) *Each vertex of* R *annihilates exactly one vertex of* X.

(j) *If* $\Delta = 3$, *then* $A_2 = \emptyset$ *and if* $\Delta \geq 5$, *then* $A_1^* = D_0 = \emptyset$.

Proof. Since H attains the bound, we have equality in all the inequalities used in the proof of Theorem 5.7. Therefore,

all variables in (5.18) with coefficients strictly less than $h(\Delta)$, are zero, (5.20)

i.e. for $\Delta \geq 3$,

$$d = d_1 = d_2 = y_1 = x_1 = x_3 = x_4 = 0.$$

Now $x_3 = x_4 = 0$ implies that $a_3^* = y_0 = 0$ and $d_0 - a_1^* = x_1 = 0$. Therefore, (a) and (b) are established.

Now (c) is shown to be true. If $\Delta = 3$, then (5.20) yields $a_1 = a_3 = a_4 = 0$. Therefore, we need only consider the case $\Delta \geq 4$.

Suppose (contrary to the statement) that $z \in A_4$. Equality in (5.7) implies that $b_z = \Delta - 1 \geq 1$, and thus there is a $w \in B_1$ which annihilates z. From (a), $y_0 = 0$ and so w is adjacent to some $y \in A_1 \cup A_1^* \cup A_2$. Equality in (5.2), (5.3) and (5.4) yield $r_y \geq \Delta - 2$. Since $w \in pn_{31}(y, X)$, it follows that w is adjacent to at least $\Delta - 2$ vertices of R. Further, from (5.10) we deduce $|N(z) \cap B| = \Delta - 2$. Since w annihilates z, this implies that w is adjacent to at least $\Delta - 2$ vertices in each of B and R and to y. Thus, $deg(w) \geq 2\Delta - 3 > \Delta$. This is a contradiction which shows that $A_4 = \emptyset$.

Suppose that $z \in A_1 \cup A_3$. By Lemma 5.5 and (a), z is adjacent to some $y \in Y_2 = D_0$. Using (b), we deduce that $A_1^* \neq \emptyset$. For each $v \in A_1^*$ choose $w_v \in pn_{31}(v, X)$. In view of Lemma 5.5 and (a), let $N(v) \cap D_0 = \{y_v\}$ and $D^* = \{y_v | v \in A_1^*\}$. Now since $w_v \in N[R]$, it annihilates some $u \in X$. By definition of A_1^*, w_v is adjacent only to v and to $\Delta - 1$ vertices of R. It follows that $u = y_v$ and by Lemma 5.6,

$$N(y_v) \cap Z_1 = \{v\}. \tag{5.21}$$

Equation (5.21) implies that $|D^*| \geq |A_1^*|$ and so from (b) we deduce that $D^* = D_0$. This equality and (5.21) show that y cannot exist. This contradiction proves that $A_1 \cup A_3 = \emptyset$. Thus, (c) is established.

Observe that (a) and (c) imply that $X = A_1^* \cup A_2 \cup D_0$. This fact will be used in the remainder of this proof without mention.

Equality in each of (5.4), (5.9), (5.12), (5.13) together with (a), (b) and (c) imply (d).

We now establish (e)-(h). From the definition of D_0 and Lemma 5.4 we deduce that for $y \in D_0$, $r_y = 0$. Thus, equality in (5.7) implies that $b_y = \Delta - 1$. Hence, by Lemma 5.6, y is adjacent to exactly one vertex z_y of A_1^*. Since $b_y = \Delta - 1$, z_y is adjacent to the $\Delta - 1$ annihilators of y in B and to y. It follows, from (b) and the definition of D_0, that each $z \in A_1^*$ is adjacent to $\Delta - 1$ vertices of B and one vertex of D_0, and that each $y \in D_0$ is adjacent to a vertex of A_1^* and a vertex of $A_2 \cup D_0$. It is easily seen that

$$|N(A_1^*) \cap B| = (\Delta - 1)|A_1^*|. \tag{5.22}$$

Equality in (5.2) implies that each $z \in A_1^*$ is annihilated by $\Delta - 1$ vertices in R. Thus, if $w \in N(z) \cap B$, then w is adjacent to z and to the $\Delta - 1$ vertices of R which annihilate z. Hence, (e) is established.

By the definition of a CO-irredundant set, each vertex of A_2 has at least one X-epn. Together, (d), (5.22) and the definition of B imply that $|N(A_2) \cap B| = |A_2|$, and thus each vertex of A_2 is joined to exactly one vertex of B.

Hence, each vertex of A_2 is joined to two vertices of $A_2 \cup D_0$ and one vertex of B. Now we have accounted for Δ (respectively three, two) vertices adjacent to each vertex of A_1^* (respectively A_2, D_0). Therefore, from (d) and the definition of C, it follows that each vertex of A_1^*, A_2 and D_0 is adjacent to 0, $\Delta - 3$ and $\Delta - 2$ vertices of C, respectively, and each vertex of C is adjacent to two vertices of $D_0 \cup A_2$ (establishing (f) and (h)). For each $z \in A_2$ equality in (5.3) (respectively in (5.7)) implies that z is annihilated by $\Delta - 2$ vertices of R (respectively one vertex of B). Since each vertex of $N(A_1^*) \cap B$ has degree Δ, this implies, for each $y \in (N(A_2) \cap B)$, that y is joined to exactly one $z \in A_2$, one vertex of $B \cap A_2$ and the $\Delta - 2$ vertices of R which annihilate z. Thus, (g) is established.

Together, Theorem 5.2 and equality in (5.1) imply (i). Part (j) follows directly from (5.20). ■

Theorem 5.9 *Let $G = (V, E)$ be a graph with $\Delta(G) \geq 3$. Then G attains the bound established in Theorem 5.7 if and only if, for some minimum CO-irredundant set X of G, the partition of G induced by X (developed in Section 5.2) satisfies conditions (a)-(j) in Lemma 5.8 and*

(k) any edge uv in G, which is not required by conditions (a)-(j), is such that $\{u, v\}$ is a subset of C or R.

Proof. Let H be an edge-minimal spanning subgraph of G with maximum degree $\Delta(G)$ and $coir(H) = coir(G)$. Then, by Lemma 5.8, H has CO-irredundant set X with cardinality $coir(H)$ and the partition of V in H induced by X satisfies conditions (a)-(j). Thus each vertex of $X \cup B$ (in this partition) has degree Δ in G. It follows that the partition of V in G induced by X is the same partition of V in H induced by X and this partition satisfies conditions (a)-(i) in G. Condition (k) follows from Theorem 5.2 and the fact that no vertex of C annihilates a vertex of X.

Let G be a graph with CO-irredundant set X, whose partition of G induced by X satisfies conditions (a)-(k). Theorem 5.2 shows that X is a maximal CO-irredundant set. It is easy to check that $|X|$ attains the bound established in Theorem 5.7. ■

Chapter 6

Open Irredundance and Maximum Degree

This chapter is concerned with open irredundant sets (aliases f_{21}-sets, or OC- or open irredundant sets). The parameter of interest is $oir(G)$, the smallest cardinality of a maximal f_{21}-set. The main result is Theorem 6.13, a lower bound for the ratio $oir(G)/n$ involving the maximum degree Δ. This is the analog of Theorems 2.14 and 5.7 for open irredundance.

We restrict ourselves to isolate-free graphs since $oir(G)/n$ may be arbitrarily small if isolates are permitted. The case $\Delta = 1$ is trivial, where $oir(G) = \frac{n}{2}$, and if $\Delta = 2$, then each component of G is a path or a cycle and it is easy to see that $oir(G) \geq \frac{n}{3}$. Hence, we assume that $\Delta \geq 3$.

6.1 Maximal Open Irredundance

We first establish a necessary and sufficient condition for an f_{21}-set to be maximal. Recall from Section 1.4 that $X \subseteq V$ is an f_{21}-set if for all $u \in X$,

$$N(u) - N[X - \{u\}] = \operatorname{epn}(u, X) \neq \emptyset.$$

The following result is obvious, its proof is omitted.

Lemma 6.1 *Let $X \subseteq V$, $u \in X$ and $v \in V - X$. Then*

(i) $\operatorname{epn}(v, X \cup \{v\}) = N(v) - N[X]$, *and*

(ii) $\operatorname{epn}(u, X \cup \{v\}) = \operatorname{epn}(u, X) - N[v]$.

Our next proposition characterises the maximality of f_{21}-sets.

Proposition 6.2 *Let X be a f_{21}-set of G and $R = V - N[X]$. Then X is a maximal f_{21}-set if and only if*

$$\text{for each } v \in N(R) \text{ there exists } u_v \in X \text{ such that } \operatorname{epn}(u_v, X) \subseteq N[v]. \tag{6.1}$$

Proof. Suppose X is a maximal f_{21}-set and let $v \in N(R)$. Then v is adjacent to some $r \in R$. By maximality, $X \cup \{v\}$ is not an f_{21}-set. Now, by Lemma 1(i), $r \in N(v) - N[X]$ which implies that $\operatorname{epn}(v, X \cup \{v\}) \neq \emptyset$. Hence for some $u_v \in X$, $\operatorname{epn}(u_v, X \cup \{v\}) = \emptyset$. By Lemma 1$(ii)$, $\operatorname{epn}(u_v, X) \subseteq N[v]$ as required.

Conversely, suppose that condition (6.1) holds and let $v \in V - X$. If $v \notin N(R)$, then $N(v) \subseteq N[X]$ and so, by Lemma 1(i), $\operatorname{epn}(v, X \cup \{v\}) = \emptyset$ and $X \cup \{v\}$ is not an f_{21}-set. If $v \in N(R)$, then by (6.1), $\operatorname{epn}(u_v, X) \subseteq N[v]$ for some $u_v \in X$. By Lemma 1(ii), $\operatorname{epn}(u_v, X \cup \{v\}) = \emptyset$. Again $X \cup \{v\}$ is not an f_{21}-set and so X is a maximal f_{21}-set. ■

In view of Proposition 2 we make additional definitions. We emphasise that the following is a new definition of annihilation required for maximality of f_{21}-sets. It is therefore a different definition from that of Chapter 5.

If $\emptyset \neq \operatorname{epn}(u, X) \subseteq N[v]$, where $u \in X$ and $v \in V - X$, we say u is *annihilated* by v, and v *annihilates* u (or v is an *annihilator* of u), written $v \to u$. Further, if u is the only vertex of X which is annihilated by v we write $v \overset{*}{\to} u$.

6.2 Preliminary Results

Let $\mathcal{H}_\Delta$ be the set of all isolate-free graphs with maximum degree $\Delta \geq 3$. We will consider *pairs* (G, X), where X is a maximal f_{21}-set of a graph $G \in \mathcal{H}_\Delta$. Define

$$\lambda(G, X) = \frac{|X|}{|V|}.$$

Partition V into $X \cup B \cup C \cup R$ (disjoint union), where

$$B = \bigcup_{u \in X} \operatorname{epn}(u, X), \qquad |B| = b$$

$$C = \{u \in V - X : |N(u) \cap X| \geq 2\}, \qquad |C| = c$$

$$\widehat{C} = \{u \in C : u \text{ is an annihilator}\}, \qquad |\widehat{C}| = \widehat{c}$$

$$R = V - N[X], \qquad |R| = r.$$

When we require additional maximal f_{21}-sets, such as X'' and X^*, we define partitions $X'' \cup B'' \cup C'' \cup R''$ (disjoint union) and $X^* \cup B^* \cup C^* \cup R^*$ (disjoint union) analogously.

Let $|X| = x$. For each $u \in X$, let $B_u = \operatorname{epn}(u, X)$. For $i = 1, 2 \ldots, \Delta$, let $X_i = \{u \in X : |B_u| = i\}$, $|X_i| = x_i$, $Y_i = \cup_{u \in X_i} B_u$ and $|Y_i| = y_i$. Note that $y_i = ix_i$.

Finally let $k = \left\lfloor \frac{\Delta+1}{2} \right\rfloor$ and define $\overline{X} = \bigcup_{i=2}^{k} X_i$ and $\overline{Y} = \bigcup_{i=2}^{k} Y_i$.

We now define fifteen properties that the pair (G, X) may possess.

P1. X is independent and each $v \in X$ has degree Δ.

P2. The sets R and C are independent.

P3. Each vertex in R is adjacent to exactly one vertex in $B \cup C$.

P4. For each annihilator $v \in B \cup C$, $\deg(v) = \Delta$.

P5. For $v_i \in B_{u_i}$, where $i = 1, 2$ and $v_1 \neq v_2$ (possibly $u_1 = u_2$), $v_1 v_2 \in E(G)$ if and only if

$$v_1 \in N(R) \quad \text{and} \quad v_1 \overset{*}{\rightarrow} u_2, \quad \text{or} \quad v_2 \in N(R) \quad \text{and} \quad v_2 \overset{*}{\rightarrow} u_1. \quad (6.2)$$

P6. For each $v \in B_u$ and $w \in C$, $vw \in E(G)$ if and only if

$$w \in N(R) \quad \text{and} \quad w \overset{*}{\rightarrow} u. \quad (6.3)$$

P7. If $u \in X$ is annihilated by $v \in B \cup C$, then each $w \in B_u$ is an annihilator.

P8. Each $w \in B$ is an annihilator.

P9. If $\Delta = 3$, then $\widehat{c} = x_3 = 0$.

P10. If $w \in Y_i$ where $i \geq 2$, then $|N(w) \cap R| \geq \frac{\Delta - 1}{2}$.

P11. If $u \in \overline{X}$ and $w \in N(B_u) \cap B$, then $w \overset{*}{\rightarrow} u$.

P12. For $i = 1, 2$ let $v_i \in B_{u_i}$ where $u_i \in \overline{X}$, $v_1 \neq v_2$ (possibly $u_1 = u_2$).

Then $v_1 v_2 \in E$ if and only if

(a) for all $w \in B_{u_1}$, $N(w) \cap \overline{Y} = B_{u_2} - \{w\}$

and (b) for all $w \in B_{u_2}$, $N(w) \cap \overline{Y} = B_{u_1} - \{w\}$.

P13. Let $u, v \in \overline{X}$ and $w \in B_u$. If $w \rightarrow v$, then $u = v$.

P14. Let $w \in B_u$ where $u \in \overline{X}$. Then $w \rightarrow u$ or w annihilates a vertex of X_1.

P15. Each $u \in \overline{X}$ satisfies one of the following:

(a) u is annihilated by exactly $\frac{\Delta+1}{2}$ vertices of $(B \cup C) - Y_1$

(b) u is annihilated only by vertices of Y_1

(c) $u \in X_2$ and u is annihilated only by vertices of B_u

P16. If $w \in C$ and $w \rightarrow u$, then $u \in \overline{X}$.

For each $i = 1, 2, \ldots, 16$, define the parameter δ_i of G by

$$\delta_i(G, X) = \begin{cases} 1 & \text{if } (G, X) \text{ has property } \mathbf{Pi} \\ 0 & \text{otherwise.} \end{cases}$$

The remainder of the proof of the lower bound for oir/n has two principal parts. Firstly, lemmas will show that for each pair (G, X) there exists a pair (G', X') satisfying

$$\delta_i\,(G', X') = 1 \quad \text{for} \quad i = 1, 2, \ldots, 10 \quad \text{and} \quad i = 12, 13, \ldots, 16 \tag{6.4}$$

and

$$\lambda\,(G', X') \leq \lambda\,(G, X)\,.$$

It will then be sufficient to establish the lower bound for graphs $G \in \mathcal{H}_\Delta$ with a smallest maximal f_{21}-set X such that (G, X) satisfies (6.4). This second part of the argument is achieved in Theorem 6.13.

Lemma 6.3 *For each* $i = 1, 2, \ldots, 11$, *if* $\delta_j\,(G, X) = 1$ *for each* $j = 1, 2, \ldots,$ $i - 1$, *then there exists a pair* (G', X') *satisfying* $\delta_j\,(G', X') = 1$ *for each* $j = 1, 2, \ldots, i$ *and* $\lambda\,(G', X') \leq \lambda\,(G, X)$. *Moreover equality in this latter inequality is possible only if* G' *is a spanning subgraph of* G.

Proof. In this proof we repeatedly form a pair, say (G', X'), from a pair (G, X). The fact that X' is a maximal f_{21}-set of G' will follow from the properties of (G, X) and an application of Proposition 2 in the form:

The f_{21}-set X' is maximal if and only if each $v \in N\,(R')$ annihilates some vertex of X'.

1. Suppose that $\delta_1(G,X)=0$. Form G^* from G by deleting all edges of $G[X]$. Let G' be the graph obtained from two disjoint copies of G^* by joining, for every $v \in X$ with $\deg_{G^*}(v)=\Delta - j$ $(j>0)$, j new vertices to both copies of v in G^*. Now X', the union of the two copies of X is an f_{21}-set of G' and the new vertices are in C'. Each vertex of $N_{G'}(R')$ is an annihilator so X' is a maximal f_{21}-set of G' satisfying $\lambda(G',X') < \lambda(G,X)$ and $\delta(G',X')=1$.

2. Suppose $\delta_1(G,X)=1$ and $\delta_2(G,X)=0$. Let $v_1v_2 \in E(G)$, where $v_1, v_2 \in R$ or $v_1, v_2 \in C$; define $G''=G-v_1v_2$. Note that G'' is a spanning subgraph of G. Then $\delta_1(G'')=1$, hence $\Delta(G'')=\Delta$. If $v_i \in R$, then $v_i \in N_G(R)$ and hence, by Proposition 6.2, $\deg_G(v_i) \geq 2$, hence $\deg_{G''}(v_i) \geq 1$, for $i=1,2$, and so $G'' \in \mathcal{H}_\Delta$. If $v_i \in C$, for $i=1,2$, then obviously $G'' \in \mathcal{H}_\Delta$. Now $X''=X$ is an f_{21}-set of G'' and is maximal since $N_{G''}(R'') \subseteq N_G(R)$. Hence, $\lambda(G'',X'')=\lambda(G,X)$. By repeating the process for other (possible) edges in R or C, we obtain a spanning subgraph G' of G with the desired properties.

3. Suppose $\delta_1(G,X)=\delta_2(G,X)=1$ and $\delta_3(G,X)=0$. Since R is independent and G is isolate-free, each vertex in R is adjacent to at least one vertex in $B \cup C$. Suppose $v \in R$ is adjacent to distinct $u, u' \in B \cup C$. Construct G'' by deleting vu' and joining u' to a new vertex in R. Now $X'' = X$ is a maximal f_{21}-set of G'' and $\lambda(G'', X'') < \lambda(G, X)$. Repetition of the construction (if necessary) yields (G', X') with $\delta_i(G', X') = 1$, for $i = 1, 2, 3$.

4. Suppose that $\delta_i(G, X) = 1$, for $i = 1, 2, 3$, and $\delta_4(G, X) = 0$. Let $\deg(v) = \Delta - j$, $j > 0$, for some annihilator $v \in B \cup C$. Form G'' by joining v to j new vertices and let $X'' = X$. The new vertices are in R''. The set X'' is a maximal f_{21}-set in G'' with $\lambda(G'', X'') < \lambda(G, X)$. Repeat this construction, if necessary, for other vertices in $B \cup C$ to obtain (G', X') with the desired properties.

5. Suppose that $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 4$, and $\delta_5(G, X) = 0$. If condition (6.2) is satisfied, then $v_1v_2 \in E$ (by definition of annihilation). Therefore, suppose that $v_1v_2 \in E$ but (6.2) is violated. Let G'' be the graph obtained from G by deleting v_1v_2, and if v_i is an annihilator, joining v_i to a new vertex. Define $X'' = X$. Note that any new vertex is in R''.

   Since the two conditions ($v_1 \in N(R)$ and $v_1 \overset{*}{\to} u_2$) and ($v_2 \in N(R)$

and $v_2 \xrightarrow{*} u_1$) are both false,

$$X'' \text{ is a maximal } f_{21}\text{-set of } G'' \quad \text{and} \quad \lambda(G'', X'') \leq \lambda(G, X),$$

with equality only if G'' is a spanning subgraph of G. A pair (G', X') with the desired properties is obtained by repeating the construction.

6. Suppose that $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 5$, and $\delta_6(G, X) = 0$. If (6.3) holds then $vw \in E$ (by the definition of annihilation). Let $v \in B_u$, $w \in C$, $vw \in E$ and (6.3) be violated. Form G'' from G by deleting vw, and if w (respectively v) is an annihilator, joining w (respectively v) to a new vertex. Let $X'' = X$ and note that any new vertex (if it exists) is in R''. Since (6.3) is false, X'' is a maximal f_{23}-set of G'' satisfying $\lambda(G'', X'') \leq \lambda(G, X)$ with equality only if G'' is a spanning subgraph of G. Repetition of this construction produces the required pair (G', X').

7. Suppose that $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 6$, and $\delta_7(G) = 0$. If $B_u = \{b\}$, then $b \to u$. Hence, suppose that $|B_u| \geq 2$, $w \in B_u$ is not an annihilator, and $v \to u$. Let

$$D_w = \{s \in N(w) \cap (B \cup C) |\ s \text{ is an annihilator }\}.$$

By Proposition 6.2, $w \notin N(R)$ and $v \in D_w$. Form G^* from G as follows:

(i) For each $s \in D_w$, join s to a new vertex. Observe that at least one vertex has been added since $v \in D_w$.

(ii) Delete w.

Since $w \notin N_G(R)$, G^* has no isolates and since we added at least one vertex in the formation of G^*, G^* had at least n vertices. The graph G^* has maximum degree Δ (since G has property **P4** and v is an annihilator) and so $G^* \in \mathcal{H}_\Delta$. Each vertex of $X^* = X$ has an X^*-epn in G^*, $N_{G^*}(R^*) = N_G(R) \cup D_w$ and each annihilator in G is also an annihilator in G^*. Also by definition, each member of $D_w^* = D_w$ annihilates some vertex of X in G and, hence, some vertex of X^* in G^*. We conclude that X^* is a maximal f_{21}-set of G^* and $\lambda(G^*, X^*) \leq \lambda(G, X)$. It is easy to check that (G^*, X^*) satisfies **P2, P3, P5, P6**. From (i) in the construction of G^*, it follows that (G^*, X^*) also satisfies **P4**. Repeat this process as necessary to construct (G'', X''), which satisfies $\delta_i(G'', X'') = 1$, for $i = 2, 3, \ldots, 7$. As in **1**, construct (G', X') from two disjoint copies of (G'', X'') as follows. If for $u \in X'' = X^*$, $deg_{G''}(u) = \Delta - j$, join j new vertices to each copy of u in G'.

Let X' be the union of the two copies of X'' in G'. It is evident that $G' \in$

$\mathcal{H}_\Delta$, $\delta_i(G', X') = 1$, for $i = 1, 2, \ldots, 7$, and $\lambda(G', X') < \lambda(G'', X'') \leq \lambda(G, X)$, as desired.

8. Suppose $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 7$, and $\delta_8(G, X) = 0$. Then the set

$$W = \{u \in X | \text{there is a } w \in B_u \text{ which is not an annihilator}\}$$

is non-empty. Let $u \in W$. Then there is a $w_u \in B_u$ which is not an annihilator and $\Delta \geq |B_u| = m_u \geq 2$. By **P7**, u is not annihilated. Therefore, by **P5** and **P6**, $\deg(w_u) = 1$. Suppose $l_u \geq 0$ vertices $v_{u1}, v_{u2}, \ldots, v_{ul_u}$ in B_u are annihilators. By **P4**, $\deg(v_{uj}) = \Delta$ for each $j = 1, 2, \ldots, l_u$, and each v_{uj} is adjacent to at least one vertex in B; say v_{uj} is adjacent to s_{uj} vertices in $B \cup C$. Since u is not annihilated, these s_{uj} vertices belong to $B - B_u$ (by **P5** and **P6**). By **P4**, $|N(v_j) \cap R| = \Delta - s_j - 1$.

We first construct a set of pairs (G_i, X^i) for $i = 1, 2, \ldots, \Delta$. The graph G_1 is formed from G by processing each u as follows:

(i) Delete all vertices of $(B_u - \{w_u\})$ and all vertices of $\bigcup_{j=1}^{l_u} (N(v_{uj}) \cap R)$. Observe that by **P4**, $(m_u - 1) + \sum_{j=1}^{l_u} (\Delta - s_{uj} - 1)$ vertices have been deleted.

(ii) For each $j = 1, 2, \ldots, l_u$ and each $z \in N(v_{uj} \cap B)$, join z to a new vertex. Observe that $\sum_{j=1}^{l_u} s_{uj}$ vertices have been added.

Define $X^1 = X$. Using **P5**, **P7** and the fact that u is not annihilated in (G, X), we conclude that each vertex of $N(R_1)$ is an annihilator for (G_1, X^1), i.e. X^1 is a maximal f_{21}-set of G_1. Let u_1 and w_{u_1} be the vertices of G_1 corresponding to u and w_u, respectively, and note that $\deg_{G_1}(u_1) = \Delta - (m_u - 1)$.

For each $i = 2, 3, \ldots, \Delta$, let G_i be obtained from G by joining the vertex w_{u_i}, which corresponds to w_u, to $\Delta - 2$ new vertices and let $X^i = X$.

Next form the pair (G^*, X^*), where G^* is obtained from $G_1, G_2, \ldots, G_\Delta$ by joining w_{u_i} to w_{u_1}, for each $i = 2, 3, \ldots, \Delta$, and $X^* = \bigcup_{i=1}^{\Delta} X^i$. Each vertex of $N_{G^*}(R^*)$ is an annihilator for (G^*, X^*); in particular, each $w_{u_i} = 1, 2, \ldots, \Delta$ annihilates u_1. Thus, X^* is a maximal f_{21}-set of G^*.

Now $G^* \in \mathcal{H}_\Delta$, $|X^*| = \Delta x$ and the number of vertices in $V(G^*)$ is

$$\Delta n + \sum_{u \in W} \left[\sum_{j=1}^{l_u} (\Delta - s_{uj} - 1) - (m_u - 1) + \sum_{j=1}^{l_u} s_{uj} + (\Delta - 1)(\Delta - 2) \right]$$

$$= \Delta n + \sum_{u \in W} \left[(\Delta - 1)(\Delta - 2) - (m_u - 1) - l_u (\Delta - 1) + 2 \sum_{j=1}^{l_u} s_{uj} \right]$$

$$\geq \Delta n + \sum_{u \in W} \left[(\Delta - l_u - 1)(\Delta - 3) \right]$$

$$\geq \Delta n \qquad \text{since } l_u \leq \Delta - 1,\ \Delta \geq 3.$$

Hence, $\lambda(G^*, X^*) \leq \lambda(G, X)$. The construction of G^* ensures that $\delta_i(G^*) = 1$, for $i = 2, 3, \ldots, 8$.

Now form (as in **1**) G' by joining $m_u - 1$ new vertices to u_1 in each of two copies of G^* (for every $u \in W$) and let X' be the union of the two copies of X^*. Then (G', X') is the required pair.

9. Let $\Delta = 3$ and suppose $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 8$, and $\delta_9(G, X) = 0$. If $w \in \widehat{C}$, then w is adjacent to a vertex in B and, by **P6**, to a vertex in R. Since w is also adjacent to at least two vertices in X, it follows that $\deg(w) \geq 4$, a contradiction. Hence $\widehat{c} = 0$ and we may assume that $x_3 \neq 0$. Let $u \in X_3$ and consider $w_i \in B_u$, for $i = 1, 2, 3$. By **P8**, each w_i is an annihilator and, thus, is adjacent to a vertex in B. Hence, by **P5**, $w_i \in N(R)$. Then $w_i \to u_i \in X_1$ (otherwise $\deg(w_i) > 3$), where possibly $u_i = u_j$ for $i \neq j$; say $N(w_i) \cap N(u_i) = \{v_i\}$. Note that $v_i \to u_i$. Degree considerations also show that w_i is adjacent to exactly one vertex $z_i \in R$. Construct G^* by deleting w_3 and z_3 and (in addition to the edges incident with w_3) the edges $v_i w_i$, for $i = 1, 2$; then joining w_1 and w_2, and finally joining each v_i, for $i = 1, 2, 3$, to a new vertex s_i (where the s_i are all distinct). In G^*, w_1 and $w_2 \to u$ and so every vertex in $N(R^*)$ is an annihilator. Hence, X is a maximal open irredundant set of G^*, $\Delta(G^*) = 3$ and $|V(G^*)| > |V(G)|$, so that $\lambda(G^*, X) < \lambda(G, X)$. It is also easy to see that $\delta_i(G^*, X) = 1$, for $i = 2, 3, \ldots, 8$. Repeat this process as required to construct (G'', X'') such that $\delta_i(G'', X'') = 1$, for $i = 2, 3, \ldots, 9$. Now construct (G', X') as in (**1**) from two copies of G'' by joining j new vertices (in C') to u in each copy of G' for each u in X'' with $deg_{G''}(u) = \Delta - j$. Let X' be the union of the two copies of X'' in G'. Then $\delta_i(G', X') = 1$, for $i = 1, 2, \ldots, 9$.

10. Let $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 9$, and $\delta_{10}(G, X) = 0$. Suppose that $w \in B_u$, where $u \in X_i$ for $i \geq 2$ and $|N(w) \cap R| < \frac{\Delta - 1}{2}$. By **P4** and **P8**, w has at least $\frac{\Delta}{2}$ neighbours in $B \cup C$. Form G''' from G by

(i) deleting all vertices in $\{w\} \cup (N(w) \cap R)$ and

(ii) for each $v \in N(w) \cap (B \cup C)$, joining v to a new vertex.

Note that

$$\begin{aligned}|V(G''')| &= n - (1 + |N(w) \cap R|) + |N(w) \cap (B \cup C)| \\ &\geq n - \left(1 + \frac{\Delta - 2}{2}\right) + \frac{\Delta}{2} \\ &\geq n.\end{aligned}$$

Define $X''' = X$. Then X''' is a maximal f_{21}-set of G''', and $\lambda(G''', X''') \leq \lambda(G, X)$. Repeat this process as necessary to form (G'', X''), which satisfies $\delta_i(G'', X'') = 1$, for $i = 2, 3, \ldots, 10$. Form (G', X') from (G'', X'') as in **1** and notice that $\lambda(G', X') < \lambda(G'', X'')$.

11. Let $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 10$, and $\delta_{11}(G, X) = 0$. The set

$$W = \left\{u \in \overline{X} | \text{ for some } w \in N(B_u) \cap B,\ w \xrightarrow{*} u \text{ is false}\right\}$$

is nonempty. Suppose $u \in W$, $v \in B_u$, $w \in B$, $vw \in E(G)$ and $\left(w \xrightarrow{*} u\right)$ is false. By **P8**, $w \rightarrow z$ (where $z \neq u$).

Construct G^* from disjoint copies G_1 and G_2 of G, as follows. The vertex sets corresponding to X in G_i will be denoted by X^i, while the copies of u, v, w, C, R, etc. in G_i will be denoted by u_i, v_i, w_i, C_i, R_i, etc. This notation will also be used in subsequent lemmas. For each $u \in W$, perform the following three operations:

(i) Join u_2 to each vertex of $B_{u_1} - \{v_1\}$ and delete all edges between u_1 and $B_{u_1} - \{v_1\}$.

(ii) By definition of $\overline{X}$, $2|B_u| \leq \Delta + 1$. Hence, $|B_u| - 1 \leq \Delta - |B_u| = |N_G(u) \cap C|$.

Choose $D \subseteq N_{G_2}(u_2) \cap C_2$ of cardinality $|B_u| - 1$.

Join u_1 to each vertex of D and delete all edges between u_2 and D.

(iii) By **P8, P4** and **P10**,

$$|N(v) \cap (B \cup C)| \leq |N(v) \cap R|,$$

so we may choose $R' \subseteq N_{G_1}(v_1) \cap R_1$ of cardinality $|N(v) \cap (B \cup C)|$.

Join v_1 to each vertex of $N_{G_2}(v_2) \cap (B_2 \cup C_2)$. Delete each edge between v_2 and $N_{G_2}(v_2) \cap (B_2 \cup C_2)$. Join v_2 to each of R'. Delete each edge between v_1 and R'.

Now $G^* \in \mathcal{H}_\Delta$. Let $X^* = X^1 \cup X^2$. Any vertex (except possibly v_2) which annihilated u_2 in (G_2, X^2), now annihilates u_1 in (G^*, X^*).

Form G^{**} from G^{*} by:

deleting edges v_1w_1, v_2w_2,

adding the edge v_1v_2, and

joining each of w_1, w_2 to a new vertex,

for each $u \in W$. Now $v_2 \to v_1$ and for $i = 1, 2,\ w_i \to z_i$ in (G^{**}, X^{*}), so that X^{*} is a maximal f_{21}-set of G^{**}.

Construct G'' from G^{**} by edge deletion and vertex addition (if necessary) as in **5** and **6** and let $X'' = X^{*}$. It is easy to check that (G'', X'') satisfies **P1**, **P2** ...,**P10** and $\lambda(G'', X'') < \lambda(G, X)$. Since $\left|B''_{u_2}\right| > |B_u|$, repeated application of this process (if necessary) yields (G', X') with the desired properties. ■

Lemma 6.4 *If* $\delta_i(G, X) = 1$, *for* $i = 1, 2, \ldots, 11$, *then* $\delta_{12}(G, X) = 1$.

Proof. (a) Let $v_1v_2 \in E$. From hypothesis and **P11**, it follows that $v_1 \overset{*}{\to} u_2$. Suppose there exists $w \in B_{u_1}$ satisfying $wv_3 \in E$, where $v_3 \in B_{u_3}$ and $u_3 \in \overline{X}$. By **P11**, $v_3 \overset{*}{\to} u_1$, which implies $v_1v_3 \in E$. Again by **P11**, $v_1 \overset{*}{\to} u_3$. Hence, $u_2 = u_3$ as asserted.

The proof of (b) is similar and the converse is obvious. ■

Lemma 6.5 *Let $\delta_i(G,X)=1$, for $i=1,2,\ldots,12$. Then there exists (G',X'), with $\delta_i(G',X')=1$ for $i=1,2,\ldots,9,10,12,13$ (note 11 is not present), and $\lambda(G',X')\leq\lambda(G,X)$, with equality only if $(G',X')=(G,X)$.*

Proof. Suppose that $\delta_{13}(G,X)=0$. Then, using **P11** and **P12**, there exists a non-empty set W of disjoint vertex subsets $\{u,v,w\}$ of G, each of which satisfies:

$\{u,v\}\subseteq\overline{X}$, $\quad w\in B_u \quad$ and $\quad w\rightarrow v$,

each vertex of B_u (respectively B_v) annihilates v (respectively u),

no vertex of B_u (respectively B_v) annihilates u (respectively v)

and if $\{u,v,w\}$ and $\{u',v',w'\}$ are in W, there are no edges between the sets $(B_u\cup B_v)$ and $(B_{u'}\cup B_{v'})$.

For $\{u,v,w\}\in W$ define

$$A_u=\{y\in B\cup C|y\rightarrow u\},$$

$$A_v=\{y\in B\cup C|y\rightarrow v\},$$

and $\quad A=A_u\cup A_v \quad$ (disjoint union by **P6** and **P11**).

We form G^* from two copies G_1,G_2 of G.

For each $\{u,v,w\}\in W$ perform the following constructions (i) - (v) for $i=1$ and $i=2$ and then perform constructions (vi) and (vii).

(i) Delete all edges between $\left(B_{u_j} \cup B_{v_j}\right)$ and $(B_j \cup C_j)$.

(ii) For each $z \in B_{u_j} - \{w_j\}$ (respectively $z \in B_{v_j}$) join z to $|A_u| - 1$ (respectively $|A_v| - 1$) new vertices.

(iii) Since $u \in \overline{X}$, $|N(w) \cap R| \geq \frac{\Delta-1}{2}$ (**P10**). Now, in G, $w \not\rightarrow u$ (**P11**), hence, by considering edges incident in G with w and using **P4, P8**, we obtain $|A_u| + |N(w) \cap R| + 1 \leq \Delta$, which gives $|A_u| \leq \frac{\Delta-1}{2}$. A similar argument shows that $|A_v| \leq \frac{\Delta-1}{2}$ and so for $j = 1$ and $j = 2$ $|A_v| \leq |N(w_j) \cap R_j|$. This facilitates the construction:

Delete $|A_v|$ vertices from $N(w_j) \cap R_j$.

(iv) Join v_j to each vertex of $B_{u_j} - \{w_j\}$ and delete all edges from u_j to $B_{u_j} - \{w_j\}$.

(v) Join w_j to each vertex of A_j.

(vi) Let D be a subset of $N_G(v) \cap C$ with cardinality $|B_u| - 1$. Such a D was shown to exist in the proof of Lemma 3, **11** (ii).

Delete all edges from v_1 to D_1 (respectively v_2 to D_2) and join u_2 to each vertex of D_1 (respectively, u_1 to each vertex of D_2).

(vii) Join each $z \in A_1 \cup A_2 - B_{u_1} - B_{u_2} - B_{v_1} - B_{v_2}$ to a new vertex.

Using Properties of (G, X), definition of W and Proposition 2, it is easy to check that $G^* \in \mathcal{H}_\Delta$ and $X^* = X^1 \cup X^2$ is a maximal f_{21}-set of G^*. The difference $n(G^*) - 2n(G)$ is given by

$$\sum_{\{u,v,w\} \in W} 2\left[(|B_u| - 1)(|A_u| - 1) + (|B_v|)(|A_v| - 1) - |A_v| + |A| - |B_u| - |B_v|\right]$$

$$= \sum_{\{u,v,w\} \in W} 2\left[|B_u||A_u| - 2|B_u| + |B_v||A_v| - 2|B_v| + 1\right]$$

$$= \sum_{\{u,v,w\} \in W} 2\left[|B_u|(|A_u| - 2) + |B_v|(|A_v| - 2) + 1\right] \geq 2|W|.$$

Hence $\lambda(G^*, X^*) < \lambda(G, X)$. Now form (G', X') from (G^*, X^*) using the constructions of **4**,**5** and **6** in the proof of Lemma 3. Then (G', X') has the desired properties. ■

Lemma 6.6 *If* $\delta_i(G, X) = 1$, *for* $i = 1, 2, \ldots, 10, 12, 13$, *then* $\delta_{14}(G, X) = 1$.

Proof. Suppose that $w \in B_u$, where $u \in \overline{X}$. By **P8**, w annihilates some $v \in X$. By **P10**, $u \in X_1 \cup \overline{X}$ and so by **P13**, $v \in X_1$ or $v = u$. ■

Lemma 6.7 *Let* $\delta_i(G, X) = 1$, *for* $i = 1, 2, \ldots, 10, 12, 13, 14$. *Then there exists* (G', X') *with* $\delta_i(G', X') = 1$, *for* $i = 1, 2, \ldots, 10, 12, 13, \ldots, 15$ *and* $\lambda(G', X') \leq \lambda(G, X)$, *with equality only if* $(G', X') = (G, X)$.

Proof. Suppose that $\delta_{15}(G, X) = 0$. Then there exists a non-empty set $Q \subseteq \overline{X}$ such that for each $u \in Q$, (a)(b)(c) of **P15** are all false. For $u \in Q$ and $w \in B_u$, **P14** asserts that $w \to u$ or w annihilates a vertex of X_1. For $u \in Q$, define $A_u = \{v \in (B \cup C) - Y_1 | v \to u\}$.

We form G^* from two copies G_1 and G_2 of G by processing each $u \in Q$ as detailed below. There are different cases which depend on properties of u.

Case 1 $w \to u$.

By **P12** each vertex of B_u annihilates u.

(i) For $j = 1, 2$, delete all edges from B_{u_j} to $(B_j \cup C_j)$.

(ii) For each $v \in (B_{u_1} \cup B_{u_2}) - \{w_1\}$, join v to $|A_u| - 2$ new vertices.

(iii) By **P10**, $|N(w) \cap R| \geq \lceil \frac{\Delta - 1}{2} \rceil = \lfloor \frac{\Delta}{2} \rfloor$, and since **P15**(a) is false, $|A_u| < \frac{\Delta+1}{2}$, which implies $|A_u| \leq \lfloor \frac{\Delta}{2} \rfloor$. Hence, $|A_u| \leq |N(w) \cap R|$, which allows the construction:

delete $|A_u|$ vertices from $N_{G_1}(w_1) \cap R_1$.

(iv) Join u_2 to each vertex of $(B_{u_1} - \{w_1\})$. Delete each edge from u_1 to $(B_{u_1} - \{w_1\})$.

(v) Let D be a subset of $N_{G_2}(u_2) \cap C_2$ of cardinality $|B_u| - 1$.

Join u_1 to each vertex of D and delete all edges from u_2 to D.

(vi) Join w_1 to each vertex of $(A_{u_1} \cup A_{u_2}) - \{w_1\}$.

(vii) Since **P15**(c) is false, if $u \in X_2 \cap Q$, then $A_u - B_u \neq \emptyset$. Hence, $|A_u| \geq 3$ and the following construction adds at least two new vertices.

If $u \in Q \cap X_2$, then for each $v \in A_u - B_u$ and $j = 1, 2$, join v_j to a new vertex.

We show that the change p_u in the number of vertices due to the processing of each $u \in Q$ is positive:

If $u \in X_i \cap Q$, where $i \geq 3$, then

$$p_u = (2i - 1)(|A_u| - 2) - |A_u|.$$

Since each $v \in B_u$ annihilates u, $|A_u| \geq i$ and so

$$p_u \geq 2i(i-3) + 2 \geq 2.$$

If $u \in X_2 \cap Q$, then

$$\begin{aligned} p_u &= 3(|A_u| - 2) - |A_u| + 2(|A_u| - |B_u|) \\ &= 4|A_u| - 10 \geq 2. \end{aligned}$$

Case 2 $w \not\twoheadrightarrow u$.

By **P12**, no vertex of B_u annihilates u and so by **P14** each vertex of B_u annihilates a vertex of X_1.

(i) For $j = 1, 2$, delete all edges between B_{u_j} and $(B_j \cup C_j) - Y_{1j}$ (Y_{1j} is the copy of Y_1 in G_j).

(ii) Join each $v \in (B_{u_1} - \{w_1\}) \cup B_{u_2}$ to $|A_u|$ new vertices.

(iii) Delete $|A_u| - 1$ vertices from $N_{G_1}(w_1) \cap R_1$.

(iv), (v) and (vi): same as Case 1.

(vii) Delete an edge from w_1 to Y_1.

The net increase in vertices by processing u is given by

$$p_u = (2i - 1)|A_u| - (|A_u| - 1)$$

$$= 2(i - 1)|A_u| + 1 \geq 1.$$

Using the hypothesis and Proposition 2, it is easily verified that $X^* = X^1 \cup X^2$ is a maximal f_{21}-set of G^*. The bounds for p_u show that $\lambda(G^*, X^*) < \lambda(G, X)$. The pair (G^*, X^*) satisfies **P1**, **P2**, and **P3**. Form (G', X') by following constructions **4**, **5** and **6** of Lemma 6.3. This pair has the required properties. ■

Lemma 6.8 *Let $\delta_i(G, X) = 1$, for $i = 1, 2, \ldots, 10, 12, 13, \ldots, 15$. Then there exists (G', X') with $\delta_i(G', X') = 1$, for $i = 1, 2, \ldots, 10, 12, 13, \ldots, 16$ and $\lambda(G', X') \leq \lambda(G, X)$, with equality only if $(G', X') = (G, X)$.*

Proof. Suppose that $\delta_{16}(G,X) = 0$. Then there exists $w \in \widehat{C}$ and $u \in X - \left(X_1 \cup \overline{X}\right) = \bigcup_{k+1}^{\Delta} X_i$ such that $w \rightarrow u$. Form $G'' \in \mathcal{H}_\Delta$ from G as follows:

(i) Delete each vertex of $N(w) \cap R$.

(ii) Join each vertex of $N(w) \cap B$ to a new vertex.

(iii) Delete all edges from w to B.

By **P2** and **P4**,

$$|N(w) \cap R| + |N(w) \cap X| + |N(w) \cap B| \leq \Delta.$$

Noting that $|N(w) \cap X| \geq 2$, we obtain

$$-|N(w) \cap R| \geq |N(w) \cap B| + 2 - \Delta.$$

Therefore,

$$\begin{aligned} n(G'') - n(G) &\geq 2|N(w) \cap B| + 2 - \Delta \\ &\geq 2\left(\left\lfloor \frac{\Delta+1}{2} \right\rfloor + 1\right) - \Delta + 2 > 0. \end{aligned} \tag{6.5}$$

By Proposition 2, $X'' = X$ is a maximal f_{21}-set of G'' and (6.5) shows that $\lambda(G'', X'') < \lambda(G, X)$. It is easily checked that (G'', X'') satisfies **P1**, **P2**,...,**P10** and (since $u \notin \overline{X}$) **P12**,**P13**,...,**P15**. Repeated application of this process (if necessary) yields (G', X') with the desired properties. ■

Because of the above lemmas, it is sufficient to establish the lower bounds for any pairs having properties **P1**,**P2**,...,**P10**, **P12**, **P13**,...,**P16**. Henceforth, (G, X) will denote a pair with these properties. We need a definition and further lemmas.

Recall $k = \left\lfloor \frac{\Delta+1}{2} \right\rfloor$. For $i = 2, 3, \ldots, k$, define

$$Z_i = \left\{ u \in X_i \,\middle|\, \frac{\Delta+1}{2} \text{ vertices of } (B \cup C) - Y_1 \text{ annihilate } u \right\}.$$

For all other values of i (i.e. $i = 1, k+1, k+2, \ldots, \Delta$) , let

$$Z_i = \emptyset.$$

Lemma 6.9 *If $u \in Z_i$ and $w \in B_u$, then $w \to u$.*

Proof. The hypothesis implies that $i \in \{2, 3, \ldots, k\}$. Let $A_u = \{v | v \to u\}$. By definition of annihilation,

$$A_u \subseteq N[w] - R. \tag{6.6}$$

Using (6.6), **P4**, **P8** and **P10** we obtain

$$\frac{\Delta+1}{2} = |A_u| \le |N[w] - R| = \Delta - |N[w] \cap R| \le \Delta - \left(\frac{\Delta-1}{2}\right) = \frac{\Delta+1}{2}.$$

Therefore, each inequality among these relations is an equality and so $|A_u| = |N[w] - R|$. By (6.6), $w \in N[w] - R = A_u$ as required. ■

Lemma 6.10 *If* $w \in \widehat{C} \cup \left(\bigcup_{i=3}^{\Delta} \bigcup_{u \in X_i - Z_i} B_u \right)$, *then* w *annihilates exactly one vertex.*

Proof. If $w \in \widehat{C}$, then the result follows directly from **P6**. Hence, we consider $w \in B_u$, where $u \in X_i - Z_i$ for $i \geq 3$. By **P8**, w annihilates at least one vertex, so suppose, contrary to the statement, that w annihilates distinct vertices u_1 and u_2. For $j = 1, 2$, let $v_j \in B_{u_j}$. Then for $j = 1, 2$, $v_j w \in E$ and since $w \overset{*}{\rightarrow} u_j$ is false, **P5** asserts that

$$v_j \overset{*}{\rightarrow} u. \tag{6.7}$$

There are now two cases depending on the value of i.

Case 1 $3 \leq i \leq k$.

Now $u \in \overline{X}$, **P15**(a) is false because $u \notin Z_i$ and **P15**(c) is false since $i \geq 3$. Therefore, **P15**(b) holds, i.e. u is only annihilated by vertices of Y_1. By (6.7) $v_1 \in Y_1$ and so $u_1 \in X_1$. Hence, v_1 annihilates both u and u_1, contrary to (6.7).

Case 2 $k + 1 \leq i \leq \Delta$.

By **P16** (respectively **P10**) no vertex of $\widehat{C}$ (respectively $B - Y_1$) annihilates u. Hence, u is only annihilated by vertices of Y_1 and we contradict (6.7), as in Case 1. ■

For the pair (G, X), let $H = G[B \cup C]$. The next result gives a lower bound for η, the degree sum of H. Let $|Z_i| = z_i$, for $i = 1, 2, \ldots, k$.

Lemma 6.11

$$\eta \geq 2x_2 + 2\widehat{c} + 2\sum_{i=3}^{\Delta} ix_i + (\Delta - 3)\, z_2 + \sum_{i=3}^{\Delta} \left(\Delta i - i^2 - \Delta - 1\right) z_i.$$

Proof. We partition $B \cup \widehat{C}$ into four sets W_1, W_2, W_3, Y_1, where:

$$W_1 = \bigcup_{v \in X_2 - Z_2} B_v,$$

$$W_2 = \bigcup_{i=2}^{k} \bigcup_{v \in Z_i} B_v,$$

$$W_3 = \left(\bigcup_{i=3}^{\Delta} \bigcup_{v \in X_i - Z_i} B_v \right) \cup \widehat{C},$$

and

$$Y_1 = \bigcup_{u \in X_1} B_u.$$

For $w \in W_1 \cup W_2 \cup W_3$, let $\eta(w) = \deg_H(w) + |N_H(w) \cap Y_1|$. We will find, for each $i = 1, 2, 3$, a lower bound for $\eta_i = \sum_{w \in W_i} \eta(w)$. Then $\sum_{i=1}^{3} \eta_i$ will give the required bound for η.

Case 1 $\quad w \in W_1$.

Let $w \in B_u$, where $u \in X_2 - Z_2$. By **P8**, $\deg_H(w) \geq 1$ and so

$$\eta_1 \geq |W_1| = 2\,(x_2 - z_2)\,. \tag{6.8}$$

Case 2 $w \in W_2$.

Let $w \in B_v$, where $v \in Z_i$ for $2 \leq i \leq k$. By Lemma 8, $w \to v$. By definition of Z_i, w is adjacent to the $\frac{\Delta-1}{2}$ other vertices which annihilate v. Hence,

$$\eta_2 \geq \sum_{i=2}^{\Delta} \sum_{v \in Z_i} \left(\frac{\Delta - 1}{2}\right) i = \sum_{i=2}^{\Delta} \left(\frac{\Delta - 1}{2}\right) i z_i. \tag{6.9}$$

Case 3 $w \in W_3$.

By **P8**, **P10**, **P16** and Lemma 10, $w \overset{*}{\to} u \in \overline{X} \cup X_1$. If $u \in \overline{X}$, then **P15**(b) and (c) are false and we conclude that **P15**(a) holds. Hence,

$$u \in Z_i \text{ for some } i \in 2, 3, \ldots, k \text{ or } u \in X_1. \tag{6.10}$$

By Lemma 9, each $u \in Z_i$ is annihilated by precisely $\left(\frac{\Delta+1}{2} - i\right)$ vertices of $\left(B \cup \widehat{C}\right) - (Y_1 \cup B_u)$. By **P13** none of these vertices is in B_v, where $v \in \overline{X}$. Hence, all of them are in $\bigcup_{i=k+1}^{\Delta} Y_i \cup \widehat{C} \subseteq W_3$. We emphasise:

$$\left.\begin{array}{l}\text{for } i = 2, 3, \ldots, \Delta, \text{ each } u \in Z_i \text{ is annihilated by precisely} \\ \left(\dfrac{\Delta+1}{2} - i\right) \text{ vertices of } W_3.\end{array}\right\} \tag{6.11}$$

By (6.11) if $M = \left\{w \in W_3 | w \to u \in \bigcup_{i=2}^{\Delta} Z_i\right\}$, then

$$\sum_{w \in M} \deg_H(w) \geq \sum_{i=2}^{\Delta} \left(\frac{\Delta+1}{2} - i\right) i z_i. \tag{6.12}$$

Further, by (6.11) and Lemma 6.10,

$$|M| = \sum_{i=2}^{\Delta} \left(\frac{\Delta+1}{2} - i\right) z_i. \tag{6.13}$$

Now each $w \in W_3 - M$ annihilates a vertex of X_1 (by (6.10)), hence $\eta(w) \geq 2$. Hence, by (6.13),

$$\sum_{w \in W_3 - M} \eta(w) \geq 2\left(|W_3| - \left|\widehat{C}_L\right| - |M|\right)$$
$$\geq 2\left(|W_3| - \sum_{i=k+1}^{\Delta} \widehat{C}_i - \sum_{i=2}^{\Delta}\left(\frac{\Delta+1}{2} - i\right) z_i\right) \qquad (6.14)$$

The sum of the right hand sides of (6.12) and (6.14) is a lower bound for η_3. We substitute

$$|W_3| = \sum_{i=3}^{\Delta} (x_i - z_i)\, i + \widehat{c}$$

into (6.14) and obtain

$$\eta_3 \geq \sum_{i=2}^{k}\left(\frac{\Delta+1}{2} - i\right) i z_i + 2\widehat{c} + 2\sum_{i=3}^{\Delta}(x_i - z_i)\, i - 2\sum_{i=2}^{k}\left(\frac{\Delta+1}{2} - i\right) z_i. \qquad (6.15)$$

This completes Case 3.

Therefore, by (6.8), (6.9) and (6.15),

$$\eta \geq \eta_1 + \eta_2 + \eta_3$$
$$\geq 2(x_2 - z_2) + \sum_{i=2}^{k}\left(\frac{\Delta-1}{2}\right) i z_i + \text{ (all terms of (6.15))}.$$
$$= 2x_2 + 2\widehat{c} + 2\sum_{i=3}^{\Delta} i x_i + (\Delta - 3) z_2 + \sum_{i=3}^{\Delta}\left(\Delta i - i^2 - \Delta - 1\right) z_i$$

as required. ■

Lemma 6.12 *If* $\Delta > 3$, *then*

$$\widehat{c} + \sum_{i=2}^{\Delta} y_i \leq (\Delta - 1)\, x_1 + 2\,(x_2 - z_2) + \sum_{i=2}^{k} \left(\frac{\Delta+1}{2}\right) z_i.$$

Proof. By **P8**, **P10** and **P16**, each $w \in \left(B \cup \widehat{C}\right) - Y_1$ annihilates $u \in X_1 \cup \overline{X}$. Since $w \notin Y_1$, **P15**(b) is false. Hence, u satisfies:

(1) $u \in X_1$, or

(2) $u \in Z_i$, for some $2 \leq i \leq k$, or

(3) $u \in X_2 - Z_2$ and u is only annihilated by vertices of B_u.

For each of these cases we obtain an upper bound for the number of vertices which annihilate u.

$$\text{If } u \in X_1, \text{ then } u \text{ is annihilated by at most } \Delta - 1 \text{ vertices of } \left(B \cup \widehat{C}\right) - Y_1. \tag{6.16}$$

$$\text{If } u \in Z_i \text{ for } 2 \leq i \leq k, \text{ then } u \text{ is annihilated by exactly } \frac{\Delta+1}{2} \text{ vertices of } \left(B \cup \widehat{C}\right) - Y_1. \tag{6.17}$$

$$\text{If } u \in X_2 - Z_2, \text{ then } u \text{ is annihilated by at most two vertices of } \left(B \cup \widehat{C}\right) - Y_1. \tag{6.18}$$

By (6.16), (6.17) and (6.18),

$$\widehat{c}+\sum_{i=2}^{\Delta} y_i = \left|\left(B\cup \widehat{C}\right) - Y_1\right| \leq (\Delta-1)\,x_1 + 2\,(x_2 - z_2) + \sum_{i=2}^{k} \binom{\Delta+1}{2} z_i$$

as required. ■

6.3 A Lower Bound for $oir(G)$

We now establish the lower bound for $oir(G)$. We remind the reader that the definitions of b, c, $\widehat{c}$, r, x, x_i, and y_i may be found on page 89 and that the definition of z_i may be found on page 113.

Theorem 6.13 *Let G have n vertices and maximum degree Δ. Then*

$$\frac{\text{oir}\,(G)}{n} \geq \begin{cases} \dfrac{2}{11} & \Delta = 3 \\ \dfrac{1}{8} & \Delta = 4 \\ \dfrac{(3\Delta-1)}{2\Delta^3 - 5\Delta^2 + 8\Delta - 1} & \Delta \geq 5. \end{cases}$$

Proof. A count of the edges from C to X yields

$$c \leq \sum_{i=1}^{\Delta} \binom{\Delta-i}{2} x_i. \tag{6.19}$$

Using (6.19) and the equality $b = \sum_{i=1}^{\Delta} i x_i$ we obtain

$$2\,(x+b+c) \leq \sum_{i=1}^{\Delta} (2+i+\Delta)\,x_i. \tag{6.20}$$

By **P4** and **P8**,

$$r \leq \sum_{i=1}^{\Delta} (\Delta - 1)\, y_i + (\Delta - 2)\, \widehat{c} - \eta.$$

Using the bound for η of Lemma 11 and substituting $y_i = ix_i$ we deduce

$$\begin{aligned} r \;\leq\; & (\Delta - 1)\, x_1 + 2\,(\Delta - 2)\, x_2 + \sum_{i=3}^{\Delta} (\Delta - 3)\, ix_i \\ & + (\Delta - 4)\, \widehat{c} - (\Delta - 3)\, z_2 + \sum_{i=3}^{k} \left(i^2 - \Delta i + \Delta + 1\right) z_i. \qquad (6.21) \end{aligned}$$

By (6.20) and (6.21)

$$\begin{aligned} 2n &= 2\,(x + b + c) + 2r \\ &\leq (3\Delta + 1)\, x_1 + (5\Delta - 4)\, x_2 + \sum_{i=3}^{\Delta} [(2\Delta - 5)\, i + (\Delta + 2)]\, x_i \\ &\quad + 2\,(\Delta - 4)\, \widehat{c} - 2\,(\Delta - 3)\, z_2 + 2 \sum_{i=3}^{k} \left(i^2 - \Delta i + \Delta + 1\right) z_i. \qquad (6.22) \end{aligned}$$

Case 1 $\Delta = 3$.

By **P9**, $\widehat{c} = x_3 = z_3 = 0$, hence by (6.22),

$$2n \leq 10x_1 + 11x_2 \leq 11\,(x_1 + x_2) = 11x.$$

Hence, $\frac{x}{n} \geq \frac{2}{11}$ as required.

Case 2 $\Delta = 4$.

Since $\frac{\Delta+1}{2}$ is non-integral, each $z_i = 0$. Hence from (6.22),

$$2n \leq 13x_1 + 16x_2 + 15x_3 + 18x_4. \qquad (6.23)$$

However by Lemma 12,

$$0 \leq \widehat{c} \leq 3x_1 + 2x_2 - (2x_2 + 3x_3 + 4x_4)$$

which yields $3x_3 + 4x_4 \leq 3x_1$.

Hence, from (6.23),

$$\begin{aligned} 2n &\leq 13x_1 + 16x_2 + 12x_3 + 14x_4 + (3x_3 + 4x_4) \\ &\leq 16x_1 + 16x_2 + 12x_3 + 14x_4 \\ &\leq 16x, \end{aligned}$$

or $\frac{x}{n} \geq \frac{1}{8}$.

Case 3 $\Delta \geq 5$.

Let μ denote the relation (6.22) multiplied by $(3\Delta - 1)$. The term involving $\widehat{c}$ in μ is

$$\left(6\Delta^2 - 26\Delta + 8\right)\widehat{c} = \left(2\Delta^2 - 4\Delta - 6\right)\widehat{c} + 2\left(2\Delta^2 - 11\Delta + 7\right)\widehat{c}. \tag{6.24}$$

The coefficients of $\widehat{c}$ on the right hand side of (6.24) are positive for $\Delta \geq 5$. Therefore, we can apply the upper bound for $\widehat{c}$ given in Lemma 12 (respectively given in (6.19)) to the first term (respectively, second term) of (6.24) and obtain

$$\begin{aligned}&\left(6\Delta^2-26\Delta+8\right)\widehat{c}\leq\\&\quad\left(2\Delta^2-4\Delta-6\right)\left[\left(\Delta-1\right)x_1+2\left(x_2-z_2\right)+\sum_{i=2}^{k}\binom{\Delta+1}{2}z_i-\sum_{i=2}^{\Delta}y_i\right]\\&\quad+2\left(2\Delta^2-11\Delta+7\right)\sum_{i=1}^{\Delta}\binom{\Delta-i}{2}x_i.\end{aligned}$$

Using this inequality in μ, substituting $y_i = ix_1$ and simplifying we deduce

$$\left.\begin{aligned}2\left(3n-1\right)n\ &\leq\left(4\Delta^3-10\Delta^2+16\Delta-2\right)x_1\\&+\left(2\Delta^3+12\Delta-10\right)x_2+\left(\Delta^3-11\Delta^2+23\Delta+3\right)z_2\\&+\textstyle\sum_{i=3}^{\Delta}\left[\left(2\Delta^2-2\Delta+4\right)i+\left(2\Delta^3-8\Delta^2+12\Delta-2\right)\right]x_i\\&+\textstyle\sum_{i=3}^{k}\left[\left(6\Delta-2\right)i^2-\left(6\Delta^2-2\Delta\right)i+\left(\Delta^3+5\Delta^2-\Delta-5\right)\right]z_i.\end{aligned}\right\}\quad(6.25)$$

Let $f\left(\Delta\right)=4\Delta^3-10\Delta^2+16\Delta-2$. We use the inequality $0\leq z_i\leq x_i$ in (6.25) and observe that this implies $a_1x_i+a_2z_i\leq\max\left\{a_1,a_1+a_2\right\}x_i$, for any real numbers a_1 and a_2. Therefore, the right hand side of (6.25) is at most

$$\left.\begin{aligned}&f\left(\Delta\right)x_1+\max\left\{2\Delta^3+12\Delta-10,3\Delta^3-11\Delta^2+35\Delta-7\right\}x_2\\&+\textstyle\sum_{i=3}^{k}\max\{\left(2\Delta^2-2\Delta+4\right)i+\left(2\Delta^3-8\Delta^2+12\Delta-2\right),\\&\qquad\left(6\Delta-2\right)i^2-4i\left(\Delta^2-1\right)+\left(3\Delta^3-3\Delta^2+11\Delta-7\right)\}x_i\\&+\textstyle\sum_{i=k+1}^{\Delta}\left\{\left(2\Delta^2-2\Delta+4\right)i+\left(2\Delta^3-8\Delta^2+12\Delta-2\right)\right\}x_i.\end{aligned}\right\}\quad(6.26)$$

We determine the largest coefficient in (6.26).

(i) It is easily seen that the coefficient of x_2 is less than $f(\Delta)$, for $\Delta \geq 5$.

(ii) For $\Delta \geq 5$, $(2\Delta^2 - 2\Delta + 4)\, i + (2\Delta^3 - 8\Delta^2 + 12\Delta - 2)$ attains its maximum when $i = \Delta$ (since the term in each bracket is positive). This maximum is equal to $f(\Delta)$.

(iii) Let $g(i, \Delta) = (6\Delta - 2)\, i^2 - 4i\,(\Delta^2 - 1) + (3\Delta^3 - 3\Delta^2 + 11\Delta - 7)$. Now $g(i, \Delta)$ is a minimum when $i = \frac{(\Delta^2 - 1)}{3\Delta - 1} < k$. Hence,

$$\begin{aligned}\max_{3 \leq i \leq k} g(i, \Delta) &= \max\{g(3, \Delta), g(k, \Delta)\} \\ &< \max\left\{g(3, \Delta), g\left(\frac{\Delta + 1}{2}, \Delta\right)\right\}.\end{aligned}$$

By (ii) and (iii), the largest coefficient of x_i in (6.26), where $i \in \{3, 4, \ldots, k\}$, is bounded above by

$$\max\left\{f(\Delta), 3\Delta^3 - 15\Delta^2 + 65\Delta - 13, \frac{1}{2}\left(5\Delta^3 - 5\Delta^2 + 27\Delta - 11\right)\right\}.$$

Now

$$\begin{aligned}&f(\Delta) - (3\Delta^3 - 15\Delta^2 + 65\Delta - 13) \\ &\qquad = \Delta\left(\Delta^2 + 5\Delta\right) - 49\Delta + 11 \\ &\qquad \geq 50\Delta - 49\Delta + 11 > 0 \qquad (\text{since } \Delta \geq 5).\end{aligned}$$

Moreover,

$$2\left[f(\Delta) - \frac{1}{2}\left(5\Delta^3 - 5\Delta^2 + 27\Delta - 11\right)\right] = 3\Delta^2(\Delta - 5) + 5\Delta + 7 > 0.$$

By (i), (ii) and (iii) we see that all coefficients in (6.26) are bounded above by $f(\Delta)$, hence

$$2n(3\Delta - 1) \leq f(\Delta)\, x,$$

$$\text{i.e.} \quad \frac{x}{n} \geq \frac{2(3\Delta - 1)}{4\Delta^3 - 10\Delta^2 + 16\Delta - 2}$$

as required. ■

6.4 Some Extremal Graphs

We now give three examples of extremal graphs for the bound given in Theorem 6.13.

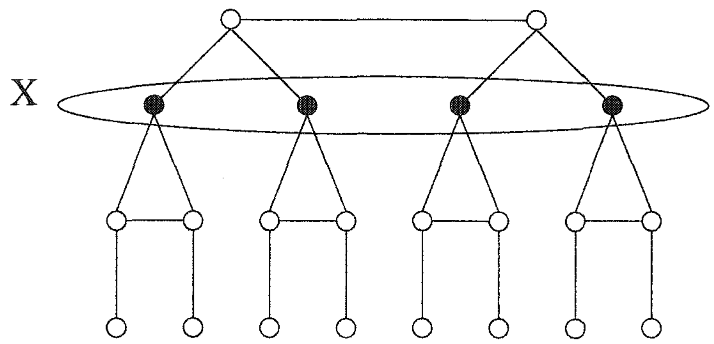

Figure 6.1: A graph with $n = 22$, $\Delta = 3$ and $oir = 4$.

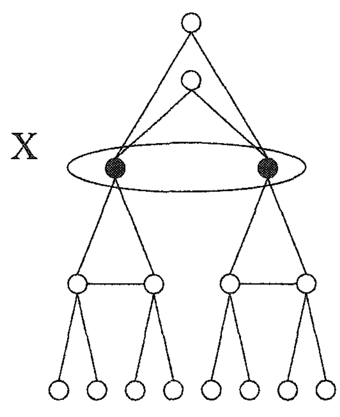

Figure 6.2: A graph with $n = 16$, $\Delta = 4$ and $oir = 2$.

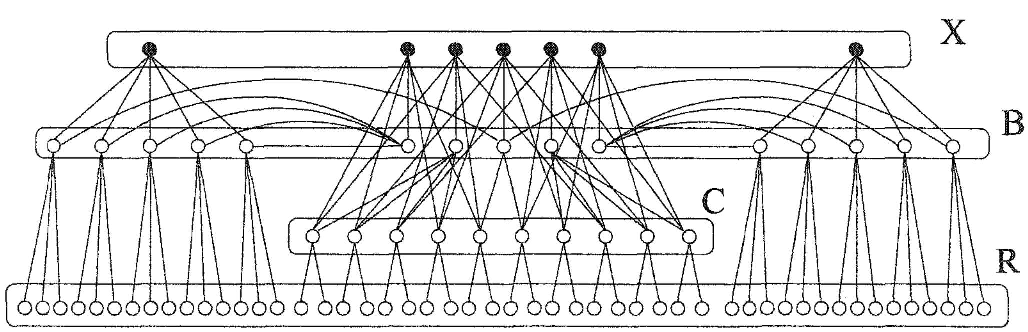

Figure 6.3: A graph with $n = 82$, $\Delta = 5$ and $oir = 7$.

Chapter 7

A More General Framework

In this chapter we greatly extend the concept of generalised irredundance. The work was motivated by the desire to embed not only independence and irredundance, but also domination, in a more general framework. The work presented is merely an introduction to the topic, with only a few basic results included. It is anticipated that a rich theory may be developed.

Let $\mathcal{G}$ be the set of all ordered pairs $g = (h, k)$ of 4-variable Boolean functions. For each of the 2^{32} elements g of $\mathcal{G}$, we will define vertex subsets of a graph G called g-collections. The class of all *g-collections* will be denoted by $\Phi_g(G)$. It will be seen that each class $\Omega_f(G)$ of generalised irredundant sets will equal $\Phi_g(G)$ for a suitable choice of g. In addition, for example, special values of g will make $\Phi_g(G)$, equal to the classes of dominating sets, minimal dominating sets, independent dominating sets or total dominating sets of G.

7.1 Basic Definitions

We now proceed to develop the definition of g-collections. This definition resembles that of generalised irredundant sets but has two key differences. Firstly, the Boolean variables involve both private neighbours and non-private neighbours (defined below). Secondly, in our new definition, vertices of both the set itself and its complement are subject to neighbourhood restrictions.

Let $x \in X \subseteq V(G)$. Recall that for $x \in X$, we define the vertex y to be an X-internal private neighbour (X-ipn) (respectively, an X-external private neighbour (X-epn)) of x if $y \in X - \{x\}$ (respectively, $y \in V - X$) and $N(y) \cap X = \{x\}$. The vertex $y \in X - \{x\}$ (respectively, $y \in V - X$) is an *X-internal non-private neighbour* of x (X-inpn) (respectively, *X-external non-private neighbour* (X-enpn)) if $\{x\}$ is a proper subset of $N(y) \cap X$.

It should be noted that x is an X-spn if and only if x has no X-ipn and no X-inpn. Thus, a set $X \subseteq V$ in which no vertex has an internal neighbour is an independent set.

For additional motivation of private neighbours, suppose troops are to be placed at the vertices of X of a graph, which models some terrain where soldiers may move along edges to either attack adjacent vertices or to reinforce them. If, for example, some $x \in X$ has an X-ipn y, then the removal of x from X would mean that y is vulnerable to attack from $V - X$. Thus, in some sense, x is an essential vertex in X. If x has an X-enpn y, then y could be considered easier to attack from X.

Each vertex of X will be categorised based on the type of neighbours it has (or types of neighbours it doesn't have). Classes of sets will be formed by insisting that only certain categories of vertices be allowed in X and in $V - X$. Because of the importance of the complement to this framework, we will denote $V - X$ by $\overline{X}$ for the remainder of this chapter.

We will need the two Boolean functions $q(x, X, G)$ and $r(x, X, G)$, which were defined in Section 1.4. In addition, we need the Boolean variables defined by:

$$q_1(x, X, G) = \begin{cases} 1 & \text{if } x \text{ has an } X\text{-inpn in } G \\ 0 & \text{otherwise} \end{cases}$$

$$r_1(x, X, G) = \begin{cases} 1 & \text{if } x \text{ has an } X\text{-enpn in } G \\ 0 & \text{otherwise.} \end{cases}$$

These functions will be denoted $q(x, X)$, $q_1(x, X)$, $r(x, X)$, and $r_1(x, X)$ or q, q_1, r, r_1, when the graph G, set X and vertex x are clear from the context.

Example 7.1. The values of q, q_1, r and r_1 for each vertex of the vertex subset X of the graph shown in Figure 7.1, are recorded in Table 7.1.

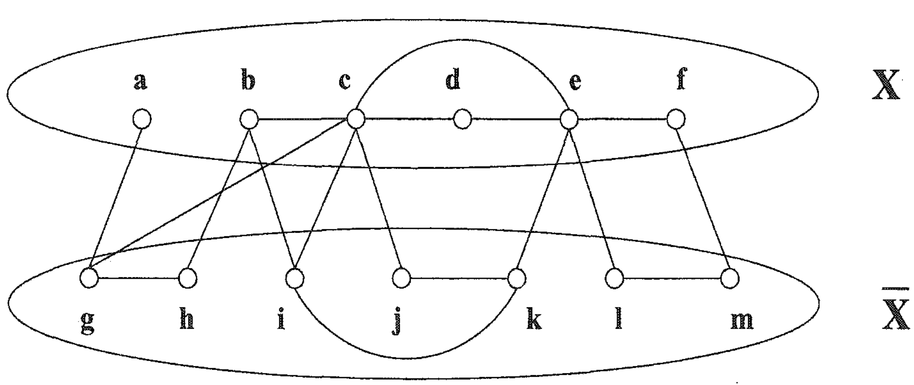

Figure 7.1: The graph G for Examples 7.1 and 7.2.

Let h be a Boolean valued function of the four variables q, q_1, r and r_1. The truth table of h will be given with the variables in this order and lexicographically with 0 preceding 1. An example of this order (called standard) is shown in Table 7.2.

We emphasise that $h = h(x, X, G) = h(q(x, X), q_1(x, X), r(x, X), r_1(x, X))$. Thus, h assigns 0 or 1 to each $x \in X$ and, hence, may be regarded as a com-

Vertex	a	b	c	d	e	f
$q(x, X)$	0	0	1	0	1	0
$q_1(x, X)$	0	1	1	1	1	1
$r(x, X)$	0	1	1	0	1	1
$r_1(x, X)$	1	1	1	0	0	0

Table 7.1: Results of Example 7.1.

pound existence/non-existence property of neighbour types of x relative to X.

Since sixteen row truth tables are cumbersome, we introduce an alternative method of specifying of 4-variable Boolean functions.

The ordered pair of values of q and q_1 in the truth table of h will be represented by the decimal equivalent from the set $\{0, 1, 2, 3\}$ of the binary number qq_1 (e.g. $q = 1$, $q_1 = 1$ is represented by the number 3, since $(3)_{10} = (11)_2$).

Now fix $i \in \{0, 1, 2, 3\}$ and suppose that the entries (in standard order) of the truth table column of h in the four rows with $(qq_1)_2 = (i)_{10}$, are c_0, c_1, c_2, c_3. Let $(a_i)_{10} = (c_3c_2c_1c_0)_2$. Then $0 \leq a_i \leq 15$ and (the truth table of) h is completely specified by the four-tuple $[a_0, a_1, a_2, a_3]$. In fact (somewhat imprecisely), we will write $h = [a_0, a_1, a_2, a_3]$. Square brackets are used here to avoid subsequent ambiguity.

Example 7.2. The function h defined by Table 7.2 is specified by the 4-tuple $[5, 12, 0, 9]$.

A vertex subset X is called an *h-ensemble* if $h(x, X, G) = 1$ for every $x \in X$. We now present the definition of the central concept of this chapter.

Let $g = (h, k)$ be an ordered pair of the Boolean functions of variables q, q_1, r and r_1 and let $\mathcal{G}$ be the set of all possible ordered pairs g. The set $X \subseteq V$ is a *g-collection* if X is an h-ensemble and $\overline{X}$ is a k-ensemble. The class of all g-collections of G will be denoted $\Phi_g(G)$ and will be abbreviated to Φ_g whenever possible.

If $h = [a_0, a_1, a_2, a_3]$ and $k = [b_0, b_1, b_2, b_3]$, then we write

$$g = [a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3]$$

and if $a_0 = a_1 = a_2 = a_3 = a$ (respectively, $b_0 = b_1 = b_2 = b_3 = b$) we use the abbreviation $g = [a; b_0, b_1, b_2, b_3]$ (respectively, $g = [a_0, a_1, a_2, a_3; b]$). If $a_0 = a_1 = a_2 = a_3 = a$ and $b_0 = b_1 = b_2 = b_3 = b$ then we will write $g = [a; b]$. The Boolean function which is identically one will be denoted by 1.

Observe that the function pair $g = (h, 1)$ imposes no restriction on the vertices of $\overline{X}$. Moreover, the Boolean variable p used in the definition of

	q	q_1	r	r_1	h	
$i=0$	0	0	0	0	1	$a_0 = (5)_{10} = (0101)_2$
	0	0	0	1	0	
	0	0	1	0	1	
	0	0	1	1	0	
$i=1$	0	1	0	0	0	$a_1 = (12)_{10} = (1100)_2$
	0	1	0	1	0	
	0	1	1	0	1	
	0	1	1	1	1	
$i=2$	1	0	0	0	0	$a_2 = (0)_{10} = (0000)_2$
	1	0	0	1	0	
	1	0	1	0	0	
	1	0	1	1	0	
$i=3$	1	1	0	0	1	$a_3 = (9)_{10} = (1001)_2$
	1	1	0	1	0	
	1	1	1	0	0	
	1	1	1	1	1	

Table 7.2: Boolean function specification.

generalised irredundant sets (see Section 1.4) satisfies $p \Leftrightarrow \overline{q} \wedge \overline{q_1}$. It therefore follows that for each f used to define generalised irredundant sets, $\Omega_f(G) = \Phi_g(G)$, for all graphs G and the appropriate choice of the pair g. Specific instances of this equality are included in the following examples.

Example 7.3.

(i) The function $g = (\overline{q} \wedge \overline{q_1}, 1) = [15, 0, 0, 0; 15]$. Each vertex of a g-collection X has no X-ipn and no X-inpn. Thus, $\Phi_g(G)$ is precisely the class of independent sets of G.

(ii) The function $g = (q \wedge \overline{q_1}, 1) = [0, 0, 15, 0; 15]$. Each vertex of a g-collection X has an X-ipn and no X-inpn. Thus, $\Phi_g(G)$ is precisely the class of induced matchings (or strong matchings) of G.

(iii) The function $g = ((\overline{q} \wedge \overline{q_1}) \vee r, 1) = [15, 12, 12, 12; 15]$. Each vertex of a g-collection X either is an X-spn or has an X-epn. Thus, $\Phi_g(G)$ is precisely the class of irredundant sets of G.

(iv) The function $g = (r, 1) = [12; 15]$. Each vertex of a g-collection X has an X-epn. Thus, $\Phi_g(G)$ is precisely the class of open irredundant (or OC-irredundant) sets of G.

(v) The function $g = ((\overline{q} \wedge \overline{q_1}) \vee q \vee r, 1) = [15, 12, 15, 15; 15]$. Each vertex of a g-collection X either is an X-spn, has an X-ipn or has an X-epn. Thus, $\Phi_g(G)$ is precisely the class of CO-irredundant sets of G.

(vi) The function $g = (q \vee r, 1) = [12, 12, 15, 15; 15]$. Each vertex of a g-collection X has an X-ipn or has an X-epn. Thus, $\Phi_g(G)$ is precisely the class of OO-irredundant sets of G.

(vii) The function $g = (q \vee (\overline{q} \wedge \overline{q_1}), 1) = [15, 0, 15, 0; 15]$. Each vertex of a g-collection X is an X-spn or has an X-ipn. Thus, $\Phi_g(G)$ is precisely the class of 1-dependent sets of G.

(viii) The function $g = (r \vee r_1, 1) = [14; 15]$. Each vertex of a g-collection X has an X-epn or an X-enpn. Thus, for each vertex v of X, $N[v] \not\subseteq X$. Thus, $\Phi_g(G)$ is precisely the class of enclaveless sets of G (see [60] for references). Enclaveless sets are also known as non-blocker sets. The complement of an enclaveless set is a dominating set.

(ix) The function $g = (1, r \vee r_1) = [15; 14]$. Each vertex in the complement of a g-collection X either has an $\overline{X}$-epn or an $\overline{X}$-enpn. Thus, $\Phi_g(G)$ is precisely the class of all dominating sets of G.

(x) The function $g = (\overline{q} \wedge \overline{q_1}, r \vee r_1) = [15, 0, 0, 0; 14]$. Each vertex of a g-collection X has no X-ipn and no X-inpn and each vertex of $\overline{X}$ has either an $\overline{X}$-epn or an $\overline{X}$-enpn. Thus, $\Phi_g(G)$ is precisely the class of independent dominating sets of G.

(xi) The function $g = (q \vee q_1, r \vee r_1) = [0, 15, 15, 15; 14]$. Each vertex of a g-collection X has an X-ipn or an X-inpn and each vertex of $\overline{X}$ has either an $\overline{X}$-epn or an $\overline{X}$-enpn. Thus, $\Phi_g(G)$ is precisely the class of open (or total) dominating sets of G.

(xii) The function $g = (\overline{q} \wedge \overline{q_1} \wedge \overline{r_1}, 1) = [5, 0, 0, 0; 15]$. Each vertex of a g-

collection X is an X-spn and has no X-enpns. Thus, $\Phi_g(G)$ is precisely the class of all 2-packings of G.

7.2 Fundamental Results

In this section fundamental properties of the new framework are explored. For the purposes of this section and Section 7.3, let

$$f = (f_1, f_2) = [a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3] \text{ and}$$

$$g = (g_1, g_2) = [c_0, c_1, c_2, c_3; d_0, d_1, d_2, d_3].$$

We begin with a proposition about a vertex set and its complement.

Proposition 7.1 *If X is an f-collection, then $\overline{X}$ is an h-collection, where $h = (f_2, f_1)$.*

Proof. If X is an f-collection then X is an f_1-ensemble and $\overline{X}$ is an f_2-ensemble. Hence, $\overline{X}$ is an h-collection. ∎

We shall write $f \Rightarrow g$ if for every graph G, vertex subset $X \subseteq V(G)$ and vertex $v \in V(G)$,

$$f_1(v, X, G) \Rightarrow g_1(v, X, G) \qquad \text{if } v \in X \text{ and}$$

$$f_2(v, \overline{X}, G) \Rightarrow g_2(v, \overline{X}, G) \qquad \text{if } v \in \overline{X}.$$

Proposition 7.2 *If $f \Rightarrow g$, then for any graph G, $\Phi_f(G) \subseteq \Phi_g(G)$.*

Proof. Suppose that the vertex subset $X \in \Phi_f(G)$. Then X is an f_1-ensemble and $\overline{X}$ is an f_2-ensemble. Thus, for every $x \in X$, $1 = f_1(x, X) \Rightarrow g_1(x, X)$ and for every $y \in \overline{X}$, $1 = f_2(y, \overline{X}) \Rightarrow g_2(y, \overline{X})$. Therefore, X is a g_1-ensemble, $\overline{X}$ is a g_2-ensemble and, hence, X is a g-collection. ■

We now present two definitions which will be used in a characterisation of f-collections and also to explore the order induced on $\mathcal{G}$ by the relation $\Rightarrow$.

If u and v are Boolean values, $(uv)_2$ will denote the binary integer with digits (in order) u and v. For a graph G and $x \in X \subseteq V$, let

$$(i)_{10} = (q(x, X)q_1(x, X))_2 \quad \text{and} \quad (k)_{10} = (r(x, X)r_1(x, X))_2,$$

and use these values to define:

$$X_I(x) = i \quad \text{and} \quad X_E(x) = 2^k.$$

Consider the partial order $\preceq$ on $\{0, 1, 2, ..., 15\}$ for which $a \preceq b$ if and only if for each i, the i^{th} digit in the binary expansion of a is less than or equal to that of b. For example, $10 \preceq 14$ but $2 \not\preceq 13$. The Hasse diagram for $\preceq$ is displayed in Figure 7.2. It should be noted that $\preceq$ on $\{0, 1, 2, ..., 15\}$ is isomorphic to the subset lattice on $\{0, 1, 2, 3\}$.

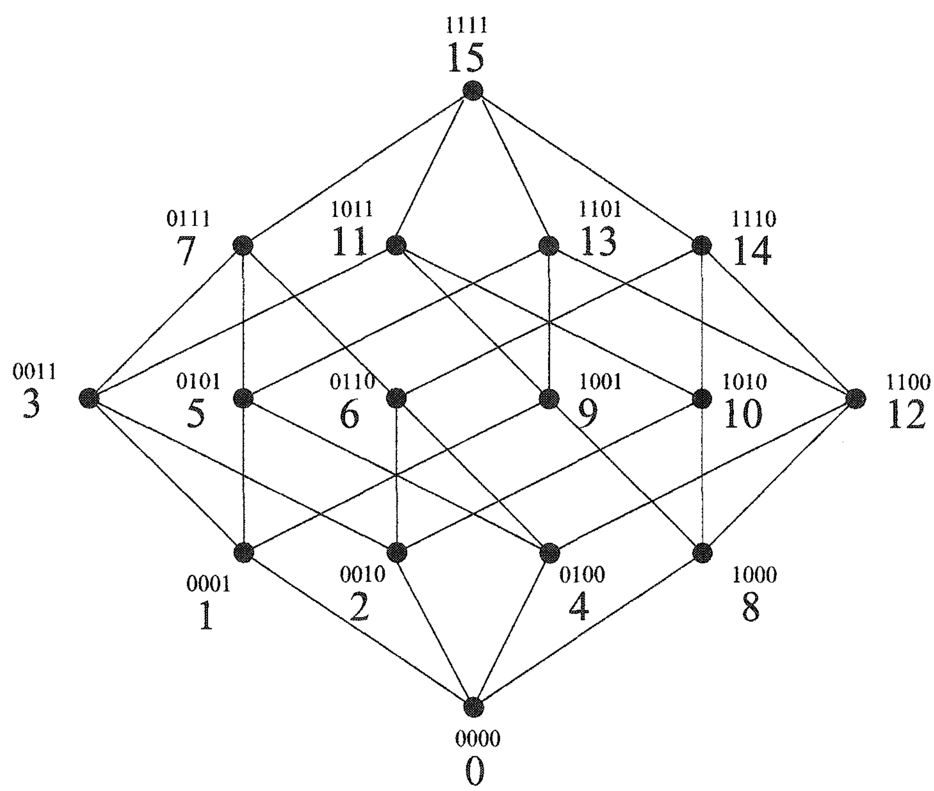

Figure 7.2: The Hasse diagram for $\preceq$.

We now characterise an f-collection in terms of the functions X_I and X_E and the integers $a_0, \ldots, a_3$, $b_0, \ldots, b_3$.

Theorem 7.3 *A vertex set X of a graph G is an f-collection if and only if*

P1: for every $x \in X$, $X_E(x) \preceq a_i$, where $i = X_I(x)$ and

P2: for every $y \in \overline{X}$, $\overline{X}_E(y) \preceq b_j$, where $j = \overline{X}_I(y)$.

Proof. The set X is an f-collection if and only if

Q1: X is an f_1-ensemble and

Q2: $\overline{X}$ is an f_2-ensemble.

Let $x \in X$ and suppose $X_I(x) = i$ and $\log_2(X_E(x)) = k$. By the definition of $\preceq$, $X_E(x) \preceq a_i$ if and only if λ, the digit in $(a_i)_2$ corresponding to 2^k, is one. However, by the definitions of $X_I(x)$ and $X_E(x)$, $\lambda = 1$ if and only if $f_1(x, X) = 1$. Therefore, P1 if and only if Q1.

A similar proof shows that P2 if and only if Q2. ■

Proposition 7.4 *If for each $i \in \{0, 1, 2, 3\}$, $a_i \preceq c_i$ and $b_i \preceq d_i$, then $f \Rightarrow g$.*

Proof. Suppose that $f \not\Rightarrow g$. Then there exists a graph G, $X \subseteq V$ and $v \in V$ such that either $v \in X$, $f_1(v, X) = 1$ and $g_1(v, X) = 0$, or $v \in \overline{X}$, $f_2(v, X) = 1$ and $g_2(v, X) = 0$. By Proposition 7.1, we may assume $v \in X$, $g_1(v, X) = 0$ and $f_1(v, X) = 1$. From Theorem 7.3 we deduce that $X_E(v) \preceq a_i$, where

$i = X_I(v)$. However, $a_i \preceq c_i$ and so it follows that $X_E(v) \preceq c_i$. Hence, by the proof of Theorem 7.3, $g_1(v, X) = 1$. This contradicts the statement that $g_1(v, X) = 0$, which shows that $f \Rightarrow g$. ■

7.3 The Equation $\Phi_h = \Phi_f \cap \Phi_g$

In this section we use the notation of Section 7.2 for $f, g \in \mathcal{G}$ and, in addition, $h \in \mathcal{G}$, where

$$h = (h_1, h_2) = [s_0, s_1, s_2, s_3; t_0, t_1, t_2, t_3].$$

We will prove two sufficient conditions for f, g, h to satisfy the statement

$$\text{for any graph } G,\ \Phi_h = \Phi_f \cap \Phi_g \tag{7.1}$$

which we observe is equivalent to the statement

> for any graph G, $S \subseteq V$ is an h-collection if and only if S is both an f-collection and a g -collection.

We will use this equivalence without further notice.

Example 7.3 (i), (ix), (x) and Theorem 1.1 show that $f = [15, 0, 0, 0; 15]$ (independent sets), $g = [15; 14]$ (dominating sets) and $h = [15, 0, 0, 0; 14]$, (independent dominating or maximal independent sets) satisfy (7.1).

It will be seen that for fixed $h \in \mathcal{G}$, the functions $f, g \in \mathcal{G}$ are not unique. We will illustrate this phenomenon in later examples which include new characterisations of maximal independent and minimal dominating sets.

To establish the first condition we need additional notation. For $a, b \in \{0, 1, 2, ..., 15\}$, $a \wedge b$ is the greatest lower bound of a and b with respect to the partial order $\preceq$. Further for $f, g \in \mathcal{G}$ let

$$f \wedge g = [a_0 \wedge c_0, a_1 \wedge c_1, a_2 \wedge c_2, a_3 \wedge c_3; b_0 \wedge d_0, b_1 \wedge d_1, b_2 \wedge d_2, b_3 \wedge d_3].$$

Theorem 7.5 *If $h = f \wedge g$, then for any graph G, $\Phi_h = \Phi_f \cap \Phi_g$.*

Proof.

$$\begin{aligned} X \text{ is an } h_1\text{-ensemble } &\Leftrightarrow \text{for every } x \in X \text{ with } X_I(x) = i,\ X_E(x) \preceq a_i \wedge c_i \\ &\Leftrightarrow \text{for every } x \in X \text{ with } X_I(x) = i,\ X_E(x) \preceq a_i \\ &\quad\ \text{and } X_E(x) \preceq c_i \\ &\Leftrightarrow X \text{ is both an } f_1\text{-ensemble and a } g_1\text{-ensemble.} \end{aligned}$$

Similarly, $\overline{X}$ is an h_2-ensemble if and only if $\overline{X}$ is both an f_2-ensemble and a g_2-ensemble. Hence, (7.1) holds as required. ■

Theorem 7.6 *For each $i = 0, 1, 2, 3$, let f, g, h satisfy:*

$$s_i \preceq a_i \quad , \quad t_i \preceq b_i \tag{7.2}$$

$$\text{and} \quad s_i \preceq c_i \preceq 15 - a_i + s_i \quad , \quad t_i \preceq d_i \preceq 15 - b_i + t_i. \tag{7.3}$$

Then for any graph G, $\Phi_h = \Phi_f \cap \Phi_g$.

Proof. By Theorem 7.5, it suffices to show that $h = f \wedge g$.

Let $i \in \{0, \ldots 3\}$. Since $s_i \preceq a_i$ and $s_i \preceq c_i$, $s_i \preceq a_i \wedge c_i$. We now show that $a_i \wedge c_i \preceq s_i$. For $j \in \{0, \ldots, 15\}$, let (m, j) be the binary digit corresponding to 2^m in $(j)_2$.

Suppose that $(m, a_i \wedge c_i) = 1$. Then

$$(m, a_i) = (m, c_i) = 1. \tag{7.4}$$

But $c_i \preceq 15 - a_i + s_i$ and so from (7.4),

$$(m, 15 - a_i + s_i) = 1. \tag{7.5}$$

If $m = 0$, (7.4) and (7.5) imply $(m, s_i) = 1$.

If $1 \leq m \leq 3$, then $s_i \preceq a_i$, implies that $(m-1, 15 - a_i)$ and $(m-1, s_i)$ are not both 1. Therefore (7.4) and (7.5) also imply $(m, s_i) = 1$.

We have proved that for any $m \in \{0, 1, 2, 3\}$, $(m, a_i \wedge c_i) = 1$ implies that $(m, s_i) = 1$, i.e. $a_i \wedge c_i \preceq s_i$. Therefore $a_i \wedge c_i = s_i$. A similar proof gives $b_i \wedge d_i = t_i$ and so $h = f \wedge g$. ■

Given $h \in \mathcal{G}$, Theorem 7.6 may be used to find $f, g \in \mathcal{G}$ such that (7.1) is satisfied. We emphasise that such f, g are not necessarily unique. In fact given h we may choose any f satisfying conditions (7.2) and then select g such that (7.3) holds, to obtain solutions to (7.1).

Example 7.4. Alternative characterisations of maximal independent sets.

Let $h = [15, 0, 0, 0; 14]$ (h-collections are maximal independent sets).

First choose $f = [15, 0, 0, 0; 15]$ (f-collections are independent sets).

Observe that (7.2) is satisfied. Let $\mathcal{H}_1$ be the set of all $g \in \mathcal{G}$ such that

$$c_0 = 15, \quad 0 \leq c_i \leq 15 \text{ for } i = 1, 2, 3 \quad \text{and} \quad d_i = 14 \text{ for } i = 0, 1, 2, 3.$$

It is easy to check that (7.3) is satisfied for each $g \in \mathcal{H}_1$. Hence, by Theorem 7.6, we have the following corollary.

Corollary 7.6.1 *For any graph G and any $g \in \mathcal{H}_1$,*

X is a maximal independent set if and only

if X is independent and X is a g-collection.

Secondly, choose $f = [15; 14]$ (f-collections are dominating sets).

Observe that (7.2) is satisfied. Now let $\mathcal{H}_2$ be the set of all $g \in \mathcal{G}$ such that

$$c_0 = 15, \quad c_i = 0 \text{ for } i = 1, 2, 3 \quad \text{and} \quad d_i \in \{14, 15\} \text{ for } i = 0, 1, 2, 3.$$

Then (7.3) is satisfied for each $g \in \mathcal{H}_2$, hence, by Theorem 7.6,

Corollary 7.6.2 *For any graph G and any pair $g \in \mathcal{H}_2$,*

X is a maximal independent set if and only

if X is dominating and X is a g-collection.

In fact, the following stronger result is implied by Theorem 7.5.

Corollary 7.6.3 *For any graph G, and pairs $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$,*

X is a maximal independent set if and only

if X is both an f-collection and a g-collection.

We emphasise that other characterisations of maximal independent sets may be found with other choices of f.

Example 7.5. Alternative characterisations of minimal dominating sets.

Recall that Theorem 1.2 states that a vertex set of G is minimal dominating if and only it is dominating and irredundant. Hence, by Example 7.3 (iii), (ix) and Theorem 7.5, if

$$h = [15 \wedge 15, 12 \wedge 15, 12 \wedge 15, 12 \wedge 15; 15 \wedge 14] = [15, 12, 12, 12; 14],$$

then Φ_h is precisely the class of all minimal dominating sets of G.

Firstly, let $f = [15, 12, 12, 12; 15]$. Then (7.2) is satisfied. Let $\mathcal{H}_3$ be the set of all $g \in \mathcal{G}$ such that

$$c_0 = 15, \quad 12 \leq c_i \leq 15, \text{ for } i = 1,2,3, \quad \text{and} \quad d_i = 14, \text{ for } i = 0,1,2,3.$$

It is easy to check that each $g \in \mathcal{H}_3$ satisfies (7.3) and hence, by Theorem 7.6,

Corollary 7.6.4 *For any graph G and any pair $g \in \mathcal{H}_3$,*

X is a minimal dominating set if and only

if X is irredundant and X is a g-collection.

Secondly choose $f = [15; 14]$. Then (7.2) is satisfied. Let $\mathcal{H}_4$ be the set of all $g \in \mathcal{G}$ such that

$$c_0 = 15, \quad c_i = 12, \text{ for } i = 1,2,3, \quad \text{and} \quad d_i \in \{14, 15\}, \text{ for } i = 0,1,2,3.$$

Then (7.3) holds for each $g \in \mathcal{H}_4$ and hence, by Theorem 7.6,

Corollary 7.6.5 *For any graph G and any pair $g \in \mathcal{H}_4$,*

X is a minimal dominating set if and only

if X is dominating and X is a g-collection.

The following result is implied by Theorem 7.5.

Corollary 7.6.6 *For any graph G, and pairs $f \in \mathcal{H}_3$ and $g \in \mathcal{H}_4$,*

X is a minimal dominating set if and only

if X is both an f-collection and a g-collection.

Example 7.6.

Theorem 3.11 states that a total dominating set X of G is minimal if and only if X is a CO-irredundant set. Hence, by Example 7.3 (v), (xi) and Theorem 7.5, if $h = [0, 12, 15, 15; 14]$, then Φ_h is precisely the class of all minimal total dominating sets. New characterisations of these sets may be obtained by the technique demonstrated in the previous examples. We note that if $f = [0, 15, 15, 15; 14]$ and $g = [12, 12, 15, 15; 14]$ (the pair corresponding to OO-irredundance), then f, g, h satisfy (7.2) and (7.3). Hence, we have a special case of Theorem 7.6.

Corollary 7.6.7 *For any graph G, a total dominating set X is minimal if and only if it is also an OO-irredundant set.*

Yet another characterisation of minimal total dominating sets is obtained with $f = [14; 15]$ (domination) and $g = [0, 12, 15, 15; d_0, d_1, d_2, d_3]$, where $d_i \in \{14, 15\}$, for $i = 0, 1, 2, 3$. By Theorem 7.6 we have

Corollary 7.6.8 *For any graph G, X is a minimal total dominating set if and only X is dominating and X is a g-collection.*

Example 7.7.

Recall (Section 3.4) that King [68] showed that a graph is stable well-covered if and only if every maximal independent set is open irredundant. Let f and g be the functions in $\mathcal{G}$ corresponding to maximal independence and open irredundance respectively (i.e. $f = [15, 0, 0, 0; 14]$ and $g = [12; 15]$) and let

$$h = [15 \wedge 12, 0 \wedge 12, 0 \wedge 12, 0 \wedge 12; 14 \wedge 15] = [12, 0, 0, 0; 14].$$

By Theorem 7.5, a graph is stable well-covered if and only if $\Phi_f = \Phi_h$.

Chapter 8

Future Research

For most $f \in \mathcal{F}$, very little is known about the structure of Ω_f. We list some areas of future research, using the theories of irredundant and independent sets as a guide.

1. Lemma 3.1 states that for $f, g \in \mathcal{F}$, if $f \Rightarrow g$, then $Q_f \leq Q_g$. Are there other sufficient conditions for $Q_f \leq Q_g$? Are there sufficient conditions for $q_f \leq q_g$? Are there other relations between the parameters Q_f, Q_g, q_f, q_g?

2. For various functions $f \in \mathcal{F}$ and various classes of graphs, what are the values of Q_f and/or q_f?

3. In [53], Fellows, Fricke, Hedetniemi and Jacobs found parameter complexity results for each type of set in the Private Neighbor Cube. Determine similar results for the other generalised irredundant sets.

4. Determine parameter complexity results for any generalised irredundant set for a specific class of graphs.

5. Find the graphs for which $q_f = Q_f$.

6. Discover bounds for q_f, Q_f involving other graph parameters, such as the bounds presented in Chapters 5 and 6. For any such bound, characterise the extremal graphs.

7. Further calculations of f-Ramsey numbers.

8. Exploring analogues of colourings and chromatic numbers. A *proper f colouring* of G is defined to be a partition of V into f-sets and the *f-chromatic number* is the smallest order of a proper f colouring.

9. What is the behaviour of q_f, Q_f under edge addition or deletion? vertex addition or deletion? graph products? Explore the different notions of criticality.

A similar list could be made for the framework introduced in Chapter 7. However, there are four fundamental results which should first be considered.

1. For $f, g \in \mathcal{G}$, Proposition 7.2 states that if $f \Rightarrow g$, then for every graph $\Phi_f \subseteq \Phi_g$. Does this hold under different conditions?

2. Find necessary and sufficient conditions which ensure that $\Phi_f = \Phi_g$ for all graphs.

3. The notion of a minimal dominating set has received much attention in the literature, however, the notion of a maximal dominating set is uninteresting. Similarly, maximal independent and irredundant sets are of interest, and minimal independent and irredundant sets are trivial. For which $g \in \mathcal{G}$ does it make sense to study minimal g-collections? maximal g-collections? both?

4. Determine which classes of g-collections are hereditary and super-hereditary.

Bibliography

[1] R. B. Allan, R. C. Laskar, On domination and some related concepts in graph theory, *Congr. Numer.* **21** (1978), 43-58.

[2] R. B. Allan, R. C. Laskar, S. T. Hedetniemi, A note on total domination, *Discrete Math.* **49** (1984), 7-13.

[3] G. Bacsó, O. Favaron, Independence, irredundance, degrees and chromatic number in graphs, *Discrete Math.* **259** (2002), no. 1-3, 257–262.

[4] C. Berge, *Theory of Graphs and its Applications.* Methuen, London, 1962.

[5] C. Berge, P. Duchet, Strongly perfect graphs, *Ann. Discrete Math.* **21** (1984), 57-61.

[6] B. Bollobás, E. J. Cockayne, Graph theoretic parameters concerning domination, independence and irredundance, *J. Graph Theory* **3** (1979), 241-250.

[7] B. Bollobás, E. J. Cockayne, The irredundance number and maximum degree of a graph, *Discrete Math.* **69** (1984), 197-199.

[8] R. C. Brewster, E. J. Cockayne, C. M. Mynhardt, Irredundant Ramsey numbers for graphs, *J. Graph Theory* **13** (1989), 283-290.

[9] R. C. Brewster, C. M. Mynhardt, E. J. Cockayne, The irredundant Ramsey number $s(3,6)$, *Quaestiones Math.* **13** (1990), 141-157.

[10] A. P. Burger, E. J. Cockayne, C. M. Mynhardt, Domination numbers for the queen's graph, *Bull. Inst. Combin. Appl.* **10** (1994), 73-82.

[11] A. P. Burger, C. M. Mynhardt, Small irredundance numbers for queens graphs, papers in honour of E. J. Cockayne, *J. Combin. Math. Combin. Comput.* **33** (2000), 33-43.

[12] K. Cameron, Induced matchings, *Discrete Appl. Math.* **24** (1989), 97-102.

[13] V. Chvátal, Perfectly ordered graphs, *Ann. Discrete Math.* **21** (1984), 63-65.

[14] G. Chen, J. H. Hattingh, C. C. Rousseau, Asymptotic bounds for irredundant and mixed Ramsey numbers, *J. Graph Theory* **17** (1993), 193-206.

[15] G. Chen, C. C. Rousseau, The irredundant Ramsey number $s(3,7)$, *J. Graph Theory* **19** (1995), 263-270.

[16] G. Chartrand, S. Schuster, On the independence number of complementary graphs, *Trans. New York Acad. Sci., Ser. II*, **36** (1974), 247-251.

[17] G. Cheston, G. H. Fricke, Classes of graphs for which upper fractional domination equals independence, upper domination and upper irredundance, *Discrete Appl. Math.* **55** (1994), 241-258.

[18] G. Cheston, O. Hare, S. T. Hedetniemi, R. C. Laskar. Simplicial graphs, *Congr. Num.* **67** (1988), 105-113.

[19] E. J. Cockayne, Generalized irredundance in graphs: hereditary properties and Ramsey numbers, *J. Combin. Math. Combin. Comput.* **31** (1999), 15-31.

[20] E. J. Cockayne, Nordhaus-Gaddum results for open irredundance, *J. Combin. Math. Combin. Comput*, to appear.

[21] E. J. Cockayne, G. Exoo, J. H. Hattingh, C. M. Mynhardt, The irredundance Ramsey number $s(4,4)$, *Utilitas Math.* **41** (1992), 119-128.

[22] E. J. Cockayne, O. Favaron, C. M. Mynhardt, Open irredundance and maximum degree in graphs, unpublished manuscript, 2002.

[23] E. J. Cockayne, O. Favaron, P. J. P. Gröbler, C. M. Mynhardt, J. Puech, Ramsey properties of generalised irredundant sets in graphs, *Discrete Math.* **231** (2001), 123-134.

[24] E. J. Cockayne, O. Favaron, P. J. P. Gröbler, C. M. Mynhardt, J. Puech, Generalised Ramsey numbers with respect to classes of graphs, *Ars Combin.* **59** (2001), 279–288.

[25] E. J. Cockayne, O. Favaron, C. Payan, A. G. Thomason. Contributions to the theory of domination, independence and irredundance in graphs, *Discrete Math.* **33** (1981), 249-258.

[26] E. J. Cockayne, S. Finbow, Generalised irredundance in graphs: Nordhaus-Gaddum bounds. *Discussiones. Math*, to appear.

[27] E. J. Cockayne, P. J. P. Gröbler, S. T. Hedetniemi, A. A. McRae, What makes an irredundant set maximal?, *J. Combin. Math. Combin. Comput.* **25** (1997), 213-223.

[28] E. J. Cockayne, J. H. Hattingh, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, Using maximality and minimality conditions to construct inequality chains, *Discrete Math.* **176** (1997), no.1-3, 43-61.

[29] E. J. Cockayne, J. H. Hattingh, J. Kok, C. M. Mynhardt, Mixed Ramsey numbers and irredundant Turán numbers for graphs, *Ars Combin.* **29C** (1990), 57–68.

[30] E. J. Cockayne, J. H. Hattingh, C. M. Mynhardt, The irredundant Ramsey number $s(3,7)$, *Utilitas Math.* **39** (1991), 145-160.

[31] E. J. Cockayne, S. T. Hedetniemi, D. J. Miller, Properties of hereditary hypergraphs and middle graphs, *Canad. Math. Bull.* **21** (1978), 461-468.

[32] E. J. Cockayne, G. MacGillvray, J. Simmons, CO-irredundant Ramsey numbers for graphs, *J. Graph Theory* **34** (2000), 258-268.

[33] E. J. Cockayne, D. McCrea, C. M. Mynhardt, Nordhaus-Gaddum results for CO-irredundance in graphs, *Discrete Math.* **211** (2000), 209-215.

[34] E. J. Cockayne, C. M. Mynhardt, *Irredundance in graphs*, unpublished manuscript, 2002.

[35] E. J. Cockayne, C. M. Mynhardt, On 1-dependent Ramsey numbers for graphs, *Discussiones. Math.* **19** (1999), no.1, 93-110.

[36] E. J. Cockayne, C. M. Mynhardt, Irredundance and maximum degree in graphs, *Combin. Prob. Comput.* **6** (1997), 153-157.

[37] E. J. Cockayne, C. M. Mynhardt, The irredundant Ramsey number $s(3,3,3) = 13$, *J. Graph Theory* **18** (1994), 595-604.

[38] E. J. Cockayne, C. M. Mynhardt, The sequence of upper and lower domination, independence and irredundance numbers of a graph, *Discrete Math.* **122** (1993), 89–102.

[39] E. J. Cockayne, C. M. Mynhardt, On the irredundant Ramsey number $s(3,3,3)$, *Ars. Combin.* **29C** (1990), 189-202.

[40] E. J. Cockayne, C. M. Mynhardt, On the product of upper irredundance numbers of a graph and its complement, *Discrete Math.* **76** (1988), 117-121.

[41] E. J. Cockayne, C. M. Mynhardt, J. Simmons, The CO-irredundant Ramsey number t(4,7), *Utilitas Math.* **57** (2000), 193-209.

[42] G. S. Domke, J. E. Dunbar, L. Markus, Gallai-type theorems and domination parameters, *Discrete Math.* **167/168** (1997), 237-248.

[43] B. Dushnik, E. W. Miller, Partially ordered sets, *Amer. J. Math.* **63** (1941), 600-610.

[44] P. Erdös , Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[45] A. M. Farley, A. Proskurowski, Computing the maximum order of an open irredundant set in a tree, *Congr. Numer.* **41** (1984), 219-228.

[46] A. M. Farley, N. Shacham, Senders in broadcast networks: open irredundancy in graphs, *Congr. Numer.* **38** (1983), 47-57.

[47] O. Favaron, A note on the irredundance number after vertex deletion, Discrete Math. **121** (1993), 51-54.

[48] O. Favaron, Two relations between the parameters of independence and irredundance, *Discrete Math.* **121** (1993), 51-54.

[49] O. Favaron, A note on the open irredundance in a graph, *Congr. Numer.* **66** (1988), 316-318.

[50] O. Favaron, Stability, domination and irredundance in a graph, *J. Graph Theory* **10** (1986), 429-438.

[51] O. Favaron, G. H. Fricke, D. Pritikin, J. Puech, Irredundance and domination in kings graphs, Discrete Math. **262** (2003), 131-147.

[52] O. Favaron, C. M. Mynhardt, On equality in an upper bound for domination parameters of graphs, *J. Graph Theory* **24** (1997), 221-231.

[53] M. R. Fellows, G. H. Fricke, S. T. Hedetniemi, D. Jacobs, The private neighbor cube, SIAM J. Discrete Math. (1994), 41-47.

[54] J. F. Fink, M. S. Jacobson, n-domination in graphs. *Graph theory with applications to algorithms and computer science* (Kalamazoo, Mich., 1984), 283-300, Wiley-Intersci. Publ., Wiley, New York, 1985.

[55] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, C. K. Wallis, M. S. Jacobson, W. W. Martin, W. D. Weakley, Combinatorial problems on chessboards: a brief survey. *Graph Theory, Combinatorics, and Applications* **1** (1995), 507-528.

[56] P. C. Gillmore, A. J. Hoffman, A characterisation of comparability graphs and of interval graphs, *Canad. J. Math.* **16** (1979), 47-56.

[57] M. C. Golumbic, R. C. Laskar, Irredundancy in circular arc graphs, *Discrete Appl. Math.* **44** (1993), 79-89.

[58] P. J. P. Gröbler, *Critical Concepts in Domination, Independence and Irredundance in Graphs*, Ph.D. Thesis, University of South Africa, 1998.

[59] P. J. P. Gröbler, C. M. Mynhardt, Vertex criticality for upper domination and irredundance, *J. Graph Theory* **37** (2001), 205-212.

[60] T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[61] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds). *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[62] S. M. Hedetniemi, S. T. Hedetniemi, R. Reynolds, Combinatorial problems on chessboards: II. In T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds). *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998, 133-162.

[63] S. T. Hedetniemi, D. P. Jacobs, R. C. Laskar, Inequalities involving the rank of a graph, *J. Combin. Math. Combin. Comput.* **6** (1989), 173-176.

[64] S. T. Henning, P. J. Slater, Inequalities relating domination parameters in cubic graphs, *Discrete Math.* **158** (1996), 87-98.

[65] F. Jaeger, C. Payan, Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple, *C. R. Acad. Sci. Paris* **274** (1972), 728-730.

[66] M. D. Kearse, P. B. Gibbons, A new lower bound on upper irredundance in the queens' graph, *Discrete Math.* **256** (2002), no. 1-2, 225-242.

[67] M. D. Kearse, P. B. Gibbons, Computational methods and new results for chessboard problems, *Australas. J. Combin.* **23** (2001), 253-284.

[68] E. L. C. King. Characterizing and Comparing Some Subclasses of Well-Covered Graphs. Ph.D. Dissertation, Department of Mathematics, Vanderbilt University (2002).

[69] R. C. Laskar, J. Pfaff, Domination and irredundance in split graphs, Technical Report 430, Dept. Mathematical Sciences, Clemson Univ., 1983.

[70] R. C. Laskar, J. Pfaff, S. M. Hedetniemi, S. T. Hedetniemi, On the algorithmic complexity of total domination, *SIAM J. Algebraic Discrete Methods* **5** (1984), 420-425.

[71] J. Lehel, Peripheral graphs, *Congr. Numer.* **59** (1987), 179-184.

[72] C. M. Mynhardt, Irredundant Ramsey numbers for graphs: a survey, *Congr. Numer.* **86** (1992), 65-79.

[73] E.A. Nordhaus, J.W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956), 175–177.

[74] C. Payan, N. H. Xuong, Domination-balanced graphs, *J. Graph Theory* **6** (1982), 23-32.

[75] S. P. Radziszowski, Small Ramsey numbers, *Electronic J. Combin.* **1** (1994), DS1.

[76] G. Ravindra, Meyniel's graphs are strongly perfect, *Ann. Discrete Math.* **21** (1984), 145-148.

[77] J. Simmons, CO-irredundant Ramsey numbers for graphs, Master's Thesis, Departement of Mathematics and Statistics, University of Victoria, 1998.

[78] J. Topp, Domination, independence and irredundance in graphs, *Dissertationes Math. (Rozprawy Mat.)* **342** (1995).

[79] H. B. Walikar, B. D. Acharya, Domination critical graphs. *Nat. Acad. Sci. Lett.* **2** (1979), 70-72.

[80] H. B. Walikar, B. D. Acharya, E. Sampathkumar. Recent developments in the theory of domination in graphs. In *MRI Lecture Notes in Math., Mahta Research Instit., Allahabad,* volume 1, 1979.

[81] C. Wang, On the sum of two parameters concerning independence and irredundance in a graph, *Discrete Math.* **69** (1988), 199-202.