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July 2021

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This article was originally published at:

<https://doi.org/10.3390/math9141706>

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Citation for this paper:

Wang, B., Srivastava, R., & Liu, J. (2021). A Certain Subclass of Multivalent Analytic Functions Defined by the  $q$ -Difference Operator Related to the Janowski Functions. *Mathematics*, 9(14), 1-16. <https://doi.org/10.3390/math9141706>.

Article

# A Certain Subclass of Multivalent Analytic Functions Defined by the $q$ -Difference Operator Related to the Janowski Functions

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**Abstract:** A class of  $p$ -valent analytic functions is introduced using the  $q$ -difference operator and the familiar Janowski functions. Several properties of functions in the class, such as the Fekete–Szegő inequality, coefficient estimates, necessary and sufficient conditions, distortion and growth theorems, radii of convexity and starlikeness, closure theorems and partial sums, are discussed in this paper.

**Keywords:** analytic functions;  $q$ -difference operator; univalent and multivalent functions; Janowski functions; Fekete–Szegő inequality; distortion theorem; radii of starlikeness and convexity; partial sum; closure theorems



**Citation:** Wang, B.; Srivastava, R.; Liu, J.-L. A Certain Subclass of Multivalent Analytic Functions Defined by the  $q$ -Difference Operator Related to the Janowski Functions. *Mathematics* **2021**, *9*, 1706. <https://doi.org/10.3390/math9141706>

Academic Editor: Abdelmejid Bayad

Received: 20 June 2021

Accepted: 8 July 2021

Published: 20 July 2021

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## 1. Introduction

The  $q$ -calculus is classical calculus without the concept of limit. In recent years,  $q$ -calculus has attracted great attention of scholars on account of its applications in the research field of physics and mathematics as, for example, in the study of quantum groups,  $q$ -deformed superalgebras, fractals and multifractal measures, optimal control problems and in chaotic dynamical systems. The application of  $q$ -calculus involving  $q$ -derivatives and  $q$ -integrals was initiated by Jackson [1,2]. Later, the  $q$ -derivative operator (or  $q$ -difference operator) was used to investigate the geometry of  $q$ -starlike functions for the first time in [3]. Moreover, Aral [4] and Anastassiou and Gal [5,6] generalized some complex operators which are known as the  $q$ -Picard and the  $q$ -Gauss–Weierstrass singular integral operators. Recently, Srivastava et al. [7] have written a series of articles [8–10] in which they combined the  $q$ -difference operator and the Janowski functions to define new function classes and studied their useful properties from different viewpoints. In addition, we choose to refer the interested reader to further developments on  $q$ -theory in [11–18]. In particular, in his recent survey-cum-expository review article, Srivastava [18] exposed the trivial and inconsequential developments in the literature in which known  $q$ -results are being routinely translated into the corresponding  $(p, q)$ -results by forcing an obviously redundant or superfluous parameter  $p$  into the known  $q$ -results.

Let  $A_p$  denote the class of  $p$ -valent analytic functions  $f(z)$  given by the following Taylor–Maclaurin series expansion:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

in the open unit disk  $D = \{z : |z| < 1\}$ . For  $p = 1$ , we write  $A := A_1$ . In the whole paper, we let  $\mathbb{N}$ ,  $\mathbb{C}$  and  $\mathbb{R}$  be the sets of positive integers, complex numbers and real numbers, respectively.

A function  $f(z) \in A_p$  is called to be a  $p$ -valent starlike function of order  $\delta$  and is written as  $f(z) \in S_p^*(\delta)$ , if it satisfies the following inequality:

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \delta \quad (0 \leq \delta < p) \quad (1)$$

for all  $z \in D$ .

A function  $f(z) \in A_p$  is known as a  $p$ -valent convex function of order  $\delta$  and is denoted by  $f(z) \in C_p(\delta)$ , if it meets the following condition:

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \quad (0 \leq \delta < p) \tag{2}$$

for all  $z \in D$ .

From (1) and (2), we have the following equivalence:

$$f(z) \in C_p(\delta) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\delta).$$

**Definition 1.** Let  $0 < q < 1$  and introduce the  $q$ -number  $[\lambda]_q$  by

$$[\lambda]_q = \begin{cases} \frac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Let  $[\lambda]_q \cdot [\lambda]_q := [\lambda]_q^2$ . In particular, when  $\lambda = 0$ , we have  $[0]_q = 0$ .

**Definition 2** (See [1,2]). Let  $0 < q < 1$ . Then the  $q$ -difference operator  $D_q$  of a function  $f(z)$  is given by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1 - q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

if  $f'(0)$  exists.

One can observe from Definition 2 that

$$\lim_{q \rightarrow 1^-} D_q f(z) = \lim_{q \rightarrow 1^-} \frac{f(qz) - f(z)}{(q - 1)z} = f'(z),$$

provided that  $f(z)$  is a differentiable function in a set of  $\mathbb{C}$ . Furthermore, for  $f(z) = z^p + \sum_{n=1}^\infty a_{p+n} z^{p+n}$ , one can see that

$$D_q f(z) = [p]_q z^{p-1} + \sum_{n=1}^\infty [p+n]_q a_{p+n} z^{p+n-1} \quad (z \neq 0),$$

where

$$[p]_q = \frac{1 - q^p}{1 - q} = 1 + q + q^2 + \dots + q^{p-1}.$$

A function  $f(z)$  belonging to  $A_p$  is called to be a  $p$ -valent  $q$ -starlike function of order  $\sigma$  and is written as  $f(z) \in S_{p,q}^*(\sigma)$ , if it meets the condition:

$$\operatorname{Re}\left(\frac{zD_q f(z)}{f(z)}\right) > \sigma \quad (0 \leq \sigma < [p]_q) \tag{3}$$

for all  $z \in D$ .

A function  $f(z)$  belonging to  $A_p$  is referred to as a  $p$ -valent  $q$ -convex function of order  $\sigma$  and is written as  $f(z) \in C_{p,q}(\sigma)$ , if it meets the condition:

$$\operatorname{Re}\left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) > \sigma \quad (0 \leq \sigma < [p]_q) \tag{4}$$

for all  $z \in D$ .

From (3) and (4), it is not difficult to verify that

$$f(z) \in C_{p,q}(\sigma) \Leftrightarrow \frac{zD_q f(z)}{[p]_q} \in S_{p,q}^*(\sigma).$$

**Definition 3.** A function  $h(z)$ , analytic in  $D$  with  $h(0) = 1$ , is called to belong to  $P[A, B]$ , if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1),$$

equivalently we can write

$$\left| \frac{h(z) - 1}{A - Bh(z)} \right| < 1.$$

For analytic functions  $h(z)$  and  $p(z)$  ( $z \in D$ ), the function  $p(z)$  is said to subordinate to the function  $h(z)$  and written  $p(z) \prec h(z)$  ( $z \in D$ ), if there exists an analytic function  $w(z)$  ( $z \in D$ ) with  $w(0) = 0$  and  $|w(z)| < 1$  so that  $p(z) = h(w(z))$  ( $z \in D$ ). Suppose that  $h(z)$  is analytic univalent in  $D$ , then the following equivalence holds true:

$$p(z) \prec h(z) \quad (z \in D) \iff p(0) = h(0) \quad \text{and} \quad p(D) \subset h(D).$$

In  $q$ -calculus concept, we now define the following subclasses of  $A_p$  in connection with the  $q$ -difference operator  $D_q$ .

**Definition 4.** A function  $f(z)$  belonging to  $A_p$  is called to be in  $I_{p,q}(\alpha, A, B)$ , if it meets the condition

$$\frac{1}{1 - \alpha} \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right) \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad 0 \leq \alpha < 1, \quad q \in (0, 1),$$

or equivalently

$$\left| \frac{\frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} - (1 - \alpha)}{(1 - \alpha)A - B \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right)} \right| < 1. \tag{5}$$

We note that:

- (1) For  $\alpha = 0$ , we obtain  $S_{p,q}^*[A, B]$ , the family of  $p$ -valent  $q$ -starlike functions associated with Janowski function;
- (2) For  $\alpha = 0, A = 1$  and  $B = -1$ , we obtain  $S_{p,q}^*$ , the family of  $p$ -valent  $q$ -starlike functions;
- (3) For  $\alpha = 0, A = 1, B = -1$  and  $q \rightarrow 1^-$ , we have  $S_p^*$ , the family of  $p$ -valent starlike functions;
- (4) For  $\alpha = 0, A = 1, B = -1, q \rightarrow 1^-$  and  $p = 1$ , we obtain  $S^*$ , the family of starlike functions.

A well-known question in GFT is to discuss the functional composed of combinations of certain coefficients of functions. The class  $A_1$  is made up of functions of the form  $f(z) = z + a_2z^2 + a_3z^3 + \dots$  ( $z \in D$ ). The Fekete–Szegő functional describes a specific relationship between coefficient  $a_2$  and  $a_3$ , i.e.,  $|a_3 - \lambda a_2^2|, 0 \leq \lambda \leq 1$ . Fekete and Szegő [19] found that  $|a_3 - \lambda a_2^2|$  is bounded by  $1 + 2 \exp(-\lambda / (1 - \lambda))$  for  $0 \leq \lambda < 1$  and  $f(z) \in A_1$  and the bound is sharp for every  $\lambda$ . In particular, if we let  $f(z) \in A_1$  and  $\lambda = 1$ , then  $|a_3 - a_2^2| \leq 1$ . More recently, Srivastava et al. researched the Fekete–Szegő inequalities for several classes of  $q$ -convex and  $q$ -starlike functions in [20].

Let  $\Omega$  denote the family of functions of the form:

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \dots,$$

in  $D$  with  $|w(z)| < 1$ .

To derive the main results, we recall the following lemmas.

**Lemma 1** (References [21–23]). *Let  $w(z) \in \Omega$ . Then*

$$|w_2 - tw_1^2| \leq \max\{1, |t|\} = \begin{cases} -t & (t \leq -1) \\ 1 & (-1 \leq t \leq 1) \\ t & (t \geq 1). \end{cases}$$

*For  $t > 1$  or  $t < -1$ , the equality occurs when  $w(z) = z$  or  $w(z) = ze^{i\theta}$ . For  $t \in (-1, 1)$ , the equality is true when  $w(z) = z^2$  or  $w(z) = z^2e^{i\theta}$ . For  $t = -1$ , the equality occurs when*

$$w(z) = \frac{(\eta + z)z}{\eta z + 1} \quad (0 \leq \eta \leq 1).$$

*For  $t = 1$ , the equality occurs when*

$$w(z) = -\frac{(\eta + z)z}{\eta z + 1} \quad (0 \leq \eta \leq 1).$$

*These above upper bounds are best possible, and they could be further extended. For  $t \in (-1, 1)$ :*

$$|w_2 - tw_1^2| + (t + 1)|w_1|^2 \leq 1 \quad (t \in (-1, 0]),$$

and

$$|w_2 - tw_1^2| + (1 - t)|w_1|^2 \leq 1 \quad (t \in (0, 1)).$$

**Lemma 2** (Reference [24]). *Let  $w(z) \in \Omega$ . Then for  $q_1, q_2 \in \mathbb{R}$ , we have the following sharp estimates:*

$$H(q_1, q_2) \geq |q_1w_1w_2 + q_2w_1^3 + w_3|,$$

where

$$H(q_1, q_2) = \begin{cases} 1 & (q_1, q_2) \in D_1 \cup D_2 \\ |q_2| & (q_1, q_2) \in \cup_{k=3}^7 D_k \\ \frac{2}{3}(1 + |q_1|) \left( \frac{1 + |q_1|}{3(1 + |q_1| + q_2)} \right)^{\frac{1}{2}} & (q_1, q_2) \in D_8 \cup D_9 \\ \frac{q_2}{3} \left( \frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left( \frac{q_1^2 - 4}{3(q_2 - 1)} \right)^{\frac{1}{2}} & (q_1, q_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\} \\ \frac{2}{3}(|q_1| - 1) \left( \frac{1 - |q_1|}{3(1 - |q_1| + q_2)} \right)^{\frac{1}{2}} & (q_1, q_2) \in D_{12}. \end{cases}$$

The above  $D_k$  ( $k = 1, 2, \dots, 12$ ) are given as the following:

$$\begin{aligned}
 D_1 &= \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, |q_2| \leq 1 \right\}, \\
 D_2 &= \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27}(1 + |q_1|)^3 - (1 + |q_1|) \leq q_2 \leq 1 \right\}, \\
 D_3 &= \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, q_2 \leq -1 \right\}, \\
 D_4 &= \left\{ (q_1, q_2) : |q_1| \geq \frac{1}{2}, q_2 \leq -\frac{2}{3}(1 + |q_1|) \right\}, \\
 D_5 &= \{ (q_1, q_2) : |q_1| \leq 2, q_2 \geq 1 \}, \\
 D_6 &= \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, q_2 \geq \frac{1}{12}(8 + q_1^2) \right\}, \\
 D_7 &= \left\{ (q_1, q_2) : |q_1| \geq 4, q_2 \geq \frac{2}{3}(|q_1| - 1) \right\}, \\
 D_8 &= \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, -\frac{2}{3}(1 + |q_1|) \leq q_2 \leq \frac{4}{27}(1 + |q_1|)^3 - (1 + |q_1|) \right\}, \\
 D_9 &= \left\{ (q_1, q_2) : |q_1| \geq 2, -\frac{2}{3}(1 + |q_1|) \leq q_2 \leq \frac{2|q_1|(1 + |q_1|)}{q_1^2 + 2|q_1| + 4} \right\}, \\
 D_{10} &= \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, \frac{2|q_1|(1 + |q_1|)}{4 + q_1^2 + 2|q_1|} \leq q_2 \leq \frac{1}{12}(8 + q_1^2) \right\}, \\
 D_{11} &= \left\{ (q_1, q_2) : |q_1| \geq 4, \frac{2|q_1|(1 + |q_1|)}{4 + q_1^2 + 2|q_1|} \leq q_2 \leq \frac{2|q_1|(|q_1| - 1)}{4 + q_1^2 - 2|q_1|} \right\}, \\
 D_{12} &= \left\{ (q_1, q_2) : |q_1| \geq 4, \frac{2|q_1|(|q_1| - 1)}{4 + q_1^2 - 2|q_1|} \leq q_2 \leq \frac{2}{3}(|q_1| - 1) \right\}.
 \end{aligned}$$

Unless otherwise stated, we assume the entire paper that

$$-1 \leq B < A \leq 1, 0 \leq \alpha < 1 \text{ and } q \in (0, 1).$$

In this paper, we shall study some geometric properties of functions belonging to  $I_{p,q}(\alpha, A, B)$  such as Fekete–Szegő inequality, necessary and sufficient conditions, distortion and growth theorems, coefficient estimates, radii of convexity and starlikeness, closure theorems and partial sums.

**2. Main Results**

**Theorem 1.** Let  $f(z) \in A_p$  and

$$0 \leq \alpha \leq \frac{[p + 1]_q [p]_q - [p]_q^2}{[p + 3]_q^2 - [p]_q^2}.$$

If  $f(z) \in I_{p,q}(\alpha, A, B)$  and  $\mu \in \mathbb{C}$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2} \max \left\{ 1, \left| B + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( \mu \frac{E_2}{E_1} - 1 \right) \right| \right\} \tag{6}$$

and

$$|a_{p+3}| \leq \frac{(1 - \alpha)(A - B)[p]_q^2}{E_3} H(q_1, q_2), \tag{7}$$

where

$$\begin{aligned}
 E_n &= [p+n]_q([p]_q - \alpha[p+n]_q) - (1-\alpha)[p]_q^2 \quad (n = 1, 2, 3), \\
 q_1 &= -2B + \frac{(1-\alpha)(A-B)[p]_q^2(E_1 + E_2)}{E_1 E_2}, \\
 q_2 &= B^2 - \left( \frac{(1-\alpha)(A-B)[p]_q^2}{E_1} \right)^2 \\
 &\quad + \frac{(1-\alpha)(A-B)[p]_q^2(E_1 + E_2)}{E_1 E_2} \left( -B + \frac{(1-\alpha)(A-B)[p]_q^2}{E_1} \right).
 \end{aligned}$$

**Proof of Theorem 1.** If  $f(z) \in I_{p,q}(\alpha, A, B)$ , by Definition 4, there is a function

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \dots \in \Omega,$$

such that

$$\frac{1}{1-\alpha} \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right) = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Since

$$\begin{aligned}
 \frac{1}{1-\alpha} \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right) &= 1 + \frac{E_1}{(1-\alpha)[p]_q^2} a_{p+1} z + \\
 &\quad \left( \frac{E_2}{(1-\alpha)[p]_q^2} a_{p+2} - \frac{E_1}{(1-\alpha)[p]_q^2} a_{p+1}^2 \right) z^2 + \\
 &\quad \left( \frac{E_3}{(1-\alpha)[p]_q^2} a_{p+3} + \frac{E_1}{(1-\alpha)[p]_q^2} a_{p+1}^3 - \right. \\
 &\quad \left. \frac{E_1 + E_2}{(1-\alpha)[p]_q^2} a_{p+1} a_{p+2} \right) z^3 + \dots
 \end{aligned}$$

and

$$\frac{1 + Aw(z)}{1 + Bw(z)} = 1 + (A-B)w_1z + (A-B)(w_2 - Bw_1^2)z^2 + (A-B)(w_3 - 2Bw_1w_2 + B^2w_1^3)z^3 + \dots,$$

we have

$$\begin{aligned}
 a_{p+1} &= \frac{(1-\alpha)(A-B)[p]_q^2 w_1}{E_1}, \\
 a_{p+2} &= \frac{(1-\alpha)(A-B)[p]_q^2}{E_2} \left[ w_2 + w_1^2 \left( -B + \frac{(1-\alpha)(A-B)[p]_q^2}{E_1} \right) \right], \\
 a_{p+3} &= \frac{(1-\alpha)(A-B)[p]_q^2}{E_3} \left\{ w_3 + \left( -2B + \frac{(1-\alpha)(A-B)[p]_q^2(E_1 + E_2)}{E_1 E_2} \right) w_1 w_2 + \right. \\
 &\quad \left[ B^2 - \left( \frac{(1-\alpha)(A-B)[p]_q^2}{E_1} \right)^2 + \frac{(1-\alpha)(A-B)[p]_q^2(E_1 + E_2)}{E_1 E_2} \right. \\
 &\quad \left. \left. \left[ -B + \left( \frac{(1-\alpha)(A-B)[p]_q^2}{E_1} \right) \right] \right] \right\}. \tag{8}
 \end{aligned}$$

Therefore, we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2} \{w_2 - v w_1^2\}, \tag{9}$$

where

$$v = B + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( \mu \frac{E_2}{E_1} - 1 \right).$$

By applying Lemmas 1 and 2 to (8) and (9), respectively, we obtain (6) and (7). Now the proof of the Theorem is completed.  $\square$

**Corollary 1.** Let  $f(z) \in A_p$  and  $0 \leq \alpha \leq \frac{[p+1]_q [p]_q - [p]_q^2}{[p+3]_q^2 - [p]_q^2}$ . If  $f(z) \in I_{p,q}(\alpha, A, B)$  and  $\mu \in \mathbb{R}$ , then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2} \left( -B + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( 1 - \mu \frac{E_2}{E_1} \right) \right) & (\mu \leq \sigma_1) \\ \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2} & (\sigma_1 \leq \mu \leq \sigma_2) \\ \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2} \left( B + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( \mu \frac{E_2}{E_1} - 1 \right) \right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 = \frac{[(-1 - B)E_1 + (1 - \alpha)(A - B)[p]_q^2]E_1}{(1 - \alpha)(A - B)[p]_q^2 E_2}$$

and

$$\sigma_2 = \frac{[(1 - B)E_1 + (1 - \alpha)(A - B)[p]_q^2]E_1}{(1 - \alpha)(A - B)[p]_q^2 E_2}.$$

Furthermore, let

$$\sigma_3 = \frac{\sigma_1 + \sigma_2}{2} = \frac{[-BE_1 + (1 - \alpha)(A - B)[p]_q^2]E_1}{(1 - \alpha)(A - B)[p]_q^2 E_2}.$$

If  $\sigma_1 < \mu \leq \sigma_3$ , then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{E_1^2}{(1 - \alpha)(A - B)[p]_q^2 E_2} \left[ (1 + B) + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( \mu \frac{E_2}{E_1} - 1 \right) \right] |a_{p+1}|^2 \leq \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2}.$$

If  $\sigma_3 < \mu < \sigma_2$ , then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{E_1^2}{(1 - \alpha)(A - B)[p]_q^2 E_2} \left[ (1 - B) + \frac{(1 - \alpha)(A - B)[p]_q^2}{E_1} \left( 1 - \mu \frac{E_2}{E_1} \right) \right] |a_{p+1}|^2 \leq \frac{(1 - \alpha)(A - B)[p]_q^2}{E_2}.$$

**Theorem 2.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . Additionally, let

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in A_p.$$

Then the function  $f(z)$  belongs to  $I_{p,q}(\alpha, A, B)$  if and only if

$$\sum_{n=1}^{\infty} \left( (1+B)[n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)(1+A)[p]_q^2 \right) |a_{p+n}| \leq (1-\alpha)(A-B)[p]_q^2. \tag{10}$$

**Proof of Theorem 2.** Assuming that the inequality (10) holds true, we need to show the inequality (5). Now we have

$$\begin{aligned} & \left| \frac{\frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} - (1-\alpha)}{(1-\alpha)A - B \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right)} \right| \\ &= \left| \frac{[p]_q z D_q f(z) - \alpha z D_q(z D_q f(z)) - (1-\alpha)[p]_q^2 f(z)}{(1-\alpha)A [p]_q^2 f(z) - B([p]_q z D_q f(z) - \alpha z D_q(z D_q f(z)))} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} ([n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)[p]_q^2) |a_{n+p}| z^{n+p}}{(A-B)(1-\alpha)[p]_q^2 z^p - \sum_{n=1}^{\infty} (B[n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)A[p]_q^2) |a_{n+p}| z^{n+p}} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} ([n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)[p]_q^2) |a_{n+p}| z^n}{(A-B)(1-\alpha)[p]_q^2 - \sum_{n=1}^{\infty} (B[n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)A[p]_q^2) |a_{n+p}| z^n} \right| < 1, \end{aligned}$$

which shows that the function  $f(z)$  belongs to  $I_{p,q}(\alpha, A, B)$ .

On the other hand, we let the function  $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B)$ . Then from (5), one can see that

$$\begin{aligned} & \left| \frac{\frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} - (1-\alpha)}{(1-\alpha)A - B \left( \frac{zD_q f(z)}{[p]_q f(z)} - \alpha \frac{zD_q(zD_q f(z))}{[p]_q^2 f(z)} \right)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} ([p+n]_q(\alpha[p+n]_q - [p]_q) + (1-\alpha)[p]_q^2) |a_{p+n}| z^n}{(1-\alpha)(A-B)[p]_q^2 - \sum_{n=1}^{\infty} (B[p+n]_q(\alpha[p+n]_q - [p]_q) + (1-\alpha)A[p]_q^2) |a_{p+n}| z^n} \right| \\ &< 1. \tag{11} \end{aligned}$$

The inequality (11) is correct for  $z \in D$ . By choosing  $z = \text{Re}z \rightarrow 1$ , we obtain (10). Thus, the Theorem is proved.  $\square$

**Corollary 2.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If  $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B)$ , then

$$|a_{n+p}| \leq \frac{(A-B)(1-\alpha)[p]_q^2}{(1+B)[n+p]_q(\alpha[n+p]_q - [p]_q) + (1-\alpha)(1+A)[p]_q^2} \quad (n = 1, 2, \dots).$$

The result is best possible for  $f(z)$  defined as

$$f(z) = z^p - \frac{(A-B)(1-\alpha)[p]_q^2}{(1+B)[n+p]_q(\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2} z^{n+p} \quad (n = 1, 2, \dots).$$

**Theorem 3.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B),$$

then, for  $|z| = r < 1$ ,

$$r^p - \tau_1 r^{p+1} \leq |f(z)| \leq r^p + \tau_1 r^{p+1},$$

where

$$\tau_1 = \frac{(A - B)(1 - \alpha)[p]_q^2}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}.$$

The bounds are best possible for  $f(z)$  given as

$$f(z) = z^p - \frac{(A - B)(1 - \alpha)[p]_q^2}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2} z^{1+p}.$$

**Proof of Theorem 3.** Let

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}.$$

Then, by applying the triangle inequality, we have

$$|f(z)| = \left| z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \right| \leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}| |z|^{p+n}.$$

Since  $|z| = r < 1$ , we can see that  $r^{p+n} \leq r^{p+1}$ . Thus, we have

$$|f(z)| \leq r^p + r^{p+1} \sum_{n=1}^{\infty} |a_{p+n}| \tag{12}$$

and

$$|f(z)| \geq r^p - r^{p+1} \sum_{n=1}^{\infty} |a_{p+n}|. \tag{13}$$

Considering  $f(z) \in I_{p,q}(\alpha, A, B)$ , we know from Theorem 2 that

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( (1 + B)[n + p]_q(\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right) |a_{n+p}| \\ & \leq (A - B)(1 - \alpha)[p]_q^2. \end{aligned}$$

Since the sequence  $\left\{ (1 + B)[n + p]_q(\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right\}$  is increasing regarding  $n$  ( $n \geq 1$ ), we have

$$\begin{aligned} & \left( (1 + B)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right) \sum_{n=1}^{\infty} |a_{n+p}| \\ & \leq \sum_{n=1}^{\infty} \left( (B + 1)[n + p]_q(\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right) |a_{n+p}|. \end{aligned}$$

Hence by transitivity we obtain

$$\left( (B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right) \sum_{n=1}^{\infty} |a_{n+p}| \leq (A - B)(1 - \alpha)[p]_q^2,$$

which implies that

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{(A - B)(1 - \alpha)[p]_q^2}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}. \tag{14}$$

Substituting (14) into (12) and (13), we obtain the required results. The proof of Theorem 3 is completed.  $\square$

**Theorem 4.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$  and  $\frac{(1+A)[p]_q([p+1]_q - [p]_q) - [p+1]_q^2}{[p+1]_q^2} < B < A \leq 1$ . If  $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B)$ , then, for  $|z| = r < 1$ , we have

$$[p]_q r^{p-1} - \tau_2 r^p \leq |D_q f(z)| \leq [p]_q r^{p-1} + \tau_2 r^p,$$

where

$$\tau_2 = \frac{(A - B)(1 - \alpha)[p]_q^2 [1 + p]_q}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}.$$

The results are best possible for the following function

$$f(z) = z^p - \frac{(A - B)(1 - \alpha)[p]_q^2}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2} z^{1+p}.$$

**Proof of Theorem 4.** Let

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}.$$

Then, from Definition 2, we can write

$$D_q f(z) = [p]_q z^{p-1} - \sum_{n=1}^{\infty} [p + n]_q |a_{p+n}| z^{p+n-1}.$$

By applying the triangle inequality, we obtain

$$|D_q f(z)| = \left| [p]_q z^{p-1} - \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| z^{n+p-1} \right| \leq [p]_q |z|^{p-1} + \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| |z|^{n+p-1}.$$

Furthermore, we have

$$|D_q f(z)| \leq r^{p-1} [p]_q + r^p \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| \tag{15}$$

and

$$|D_q f(z)| \geq r^{p-1} [p]_q - r^p \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}|. \tag{16}$$

Because of the function  $f(z)$  belonging to the class  $I_{p,q}(\alpha, A, B)$ , we find from Theorem 2 that

$$\sum_{n=1}^{\infty} \left( (B + 1)(\alpha[n + p]_q - [p]_q) + \frac{(1 + A)(1 - \alpha)[p]_q^2}{[n + p]_q} \right) [n + p]_q |a_{n+p}| \leq (A - B)(1 - \alpha)[p]_q^2.$$

As we know that

$$\left\{ (1 + B)(\alpha[p + n]_q - [p]_q) + \frac{(1 - \alpha)(1 + A)[p]_q^2}{[p + n]_q} \right\}$$

is an increasing sequence regarding  $n$  ( $n \geq 1$ ), so

$$\begin{aligned} & \left( (B + 1)(\alpha[1 + p]_q - [p]_q) + \frac{(1 + A)(1 - \alpha)[p]_q^2}{[1 + p]_q} \right) \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| \\ & \leq \sum_{n=1}^{\infty} \left( (B + 1)(\alpha[n + p]_q - [p]_q) + \frac{(1 + A)(1 - \alpha)[p]_q^2}{[n + p]_q} \right) [n + p]_q |a_{n+p}|. \end{aligned}$$

Thus, by transitivity, we have

$$\left( (B + 1)(\alpha[1 + p]_q - [p]_q) + \frac{(1 + A)(1 - \alpha)[p]_q^2}{[1 + p]_q} \right) \sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| \leq (A - B)(1 - \alpha)[p]_q^2,$$

which implies that

$$\sum_{n=1}^{\infty} [n + p]_q |a_{n+p}| \leq \frac{(A - B)(1 - \alpha)[p]_q^2 [1 + p]_q}{(B + 1)[1 + p]_q(\alpha[1 + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}. \tag{17}$$

Now, by putting (17) in (15) and (16), we complete the proof of Theorem 4.  $\square$

**Theorem 5.** Let

$$\frac{[p]_q}{[p + 1]_q} \leq \alpha < 1 \quad (-1 \leq B_2 < 0 < B_1 \leq 1).$$

Additionally, let  $0 \leq \delta < p$ . If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_1$ ,  $f(z)$  is  $p$ -valent starlike function of order  $\delta$ , where

$$r_1 = \min \left\{ \inf_{n \geq 1} \left( \frac{(p - \delta)(B_1 - B_2) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{((p - \delta)(B_1 - B_2) + n(1 - B_2))(A - B)(1 - \alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Proof of Theorem 5.** Let  $f(z) \in I_{p,q}(\alpha, A, B)$ . In order to prove  $f(z) \in S_p^*(\delta)$ , we need to show that

$$\frac{zf'(z) - \delta f(z)}{p - \delta} \prec \frac{1 + B_1z}{1 + B_2z}, \quad (0 \leq \delta < p, -1 \leq B_2 < 0 < B_1 \leq 1).$$

The subordination above is equivalent to  $\left| \frac{zf'(z) - pf(z)}{-B_2zf'(z) + (pB_1 - \delta(B_1 - B_2))f(z)} \right| < 1$ . After some calculations and simplifications, we obtain

$$\sum_{n=1}^{\infty} \frac{(p - \delta)(B_1 - B_2) + n(1 - B_2)}{(p - \delta)(B_1 - B_2)} |a_{p+n}| |z|^n < 1. \tag{18}$$

From the inequality (10), we can obviously find that

$$\sum_{n=1}^{\infty} \frac{(B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}{(A - B)(1 - \alpha)[p]_q^2} |a_{n+p}| < 1.$$

Inequality (18) can be seen to be true if it satisfies the following inequality:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(p - \delta)(B_1 - B_2) + n(1 - B_2)}{(p - \delta)(B_1 - B_2)} |a_{p+n}| |z|^n \\ & < \sum_{n=1}^{\infty} \frac{(1 + B)[p + n]_q (\alpha[p + n]_q - [p]_q) + (1 - \alpha)(1 + A)[p]_q^2}{(1 - \alpha)(A - B)[p]_q^2} |a_{p+n}|. \end{aligned}$$

The above inequality indicates that

$$|z|^n < \frac{(p - \delta)(B_1 - B_2) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{((p - \delta)(B_1 - B_2) + n(1 - B_2))(A - B)(1 - \alpha)[p]_q^2}$$

or

$$|z| < \left( \frac{(p - \delta)(B_1 - B_2) \left( (1 + B)[p + n]_q (\alpha[p + n]_q - [p]_q) + (1 - \alpha)(1 + A)[p]_q^2 \right)}{((p - \delta)(B_1 - B_2) + n(1 - B_2))(1 - \alpha)(A - B)[p]_q^2} \right)^{\frac{1}{n}}.$$

Let

$$r_1 = \min \left\{ \inf_{n \geq 1} \left( \frac{(p - \delta)(B_1 - B_2) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{((p - \delta)(B_1 - B_2) + n(1 - B_2))(A - B)(1 - \alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\},$$

then we obtain the required result. The proof of Theorem 5 is completed.

With the aid of the method in the proof of Theorem 5, we also obtain the following theorems for the classes  $C_p(\delta)$ ,  $S_{p,q}^*(\sigma)$  and  $C_{p,q}(\sigma)$ , respectively.  $\square$

**Theorem 6.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1, -1 \leq B_2 < 0 < B_1 \leq 1$  and  $0 \leq \delta < p$ . If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_2$ ,  $f(z)$  is  $p$ -valent convex function of order  $\delta$ , where

$$r_2 = \min \left\{ \inf_{n \geq 1} \left( \frac{p(p-\delta)(B_1 - B_2) \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right)}{(n+p) \left( (p-\delta)(B_1 - B_2) + n(1-B_2) \right) (A-B)(1-\alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Theorem 7.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1, -1 \leq B_2 < 0 < B_1 \leq 1$  and  $0 \leq \sigma < [p]_q$ . If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_3$ ,  $f(z)$  is  $p$ -valent  $q$ -starlike function of order  $\sigma$ , where

$$r_3 = \min \left\{ \inf_{n \geq 1} \left( \frac{([p]_q - \sigma)(B_1 - B_2) \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right)}{([n+p]_q(1-B_2) - [p]_q(1-B_1) - \sigma(B_1 - B_2)) (A-B)(1-\alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Theorem 8.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1, 0 \leq \sigma < [p]_q$  and  $-1 \leq B_2 < 0 < B_1 \leq 1$ . If the function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_4$ ,  $f(z)$  is  $p$ -valent  $q$ -convex function of order  $\sigma$ , where

$$r_4 = \min \left\{ \inf_{n \geq 1} \left( \frac{[p]_q([p]_q - \sigma)(B_1 - B_2) \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right)}{[n+p]_q([n+p]_q(1-B_2) - [p]_q(1-B_1) - \sigma(B_1 - B_2)) (A-B)(1-\alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

Taking  $B_1 = 1$  and  $B_2 = -1$  in Theorems 5 to 8, we obtain the following corollaries, respectively.

**Corollary 3.** Let  $0 \leq \delta < p$  and  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If the function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_5$ ,  $f(z)$  is  $p$ -valent starlike function of order  $\delta$ , where

$$r_5 = \min \left\{ \inf_{n \geq 1} \left( \frac{(p-\delta) \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right)}{(A-B)(1-\alpha)(n+p-\delta)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Corollary 4.** Let  $0 \leq \delta < p$  and  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If the function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_6$ ,  $f(z)$  is  $p$ -valent convex function of order  $\delta$ , where

$$r_6 = \min \left\{ \inf_{n \geq 1} \left( \frac{p(p - \delta) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{(A - B)(1 - \alpha)(n + p)(n + p - \delta)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Corollary 5.** Let  $0 \leq \sigma < [p]_q$  and  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If the function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_7$ ,  $f(z)$  is  $p$ -valent  $q$ -starlike function of order  $\sigma$ , where

$$r_7 = \min \left\{ \inf_{n \geq 1} \left( \frac{([p]_q - \sigma) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{([n + p]_q - \sigma)(A - B)(1 - \alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

**Corollary 6.** Let  $0 \leq \sigma < [p]_q$  and  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If the function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in I_{p,q}(\alpha, A, B),$$

then, for  $0 < |z| < r_8$ ,  $f(z)$  is  $p$ -valent  $q$ -convex function of order  $\sigma$ , where

$$r_8 = \min \left\{ \inf_{n \geq 1} \left( \frac{[p]_q([p]_q - \sigma) \left( (B + 1)[n + p]_q (\alpha[n + p]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2 \right)}{[n + p]_q([n + p]_q - \sigma)(A - B)(1 - \alpha)[p]_q^2} \right)^{\frac{1}{n}}, 1 \right\}.$$

Next, we will study the ratio of a function

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}$$

to its sequence of partial sums given by

$$f_k(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \quad (\forall z \in D).$$

**Theorem 9.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n} \in I_{p,q}(\alpha, A, B),$$

then

$$\operatorname{Re} \left( \frac{f(z)}{f_k(z)} \right) > 1 - \frac{1}{\varphi_{k+1}} \quad (z \in D) \tag{19}$$

and

$$\operatorname{Re} \left( \frac{f_k(z)}{f(z)} \right) > \frac{\varphi_{k+1}}{1 + \varphi_{k+1}} \quad (z \in D), \tag{20}$$

where

$$\varphi_{k+1} = \frac{(B + 1)[1 + p + k]_q (\alpha[1 + p + k]_q - [p]_q) + (1 + A)(1 - \alpha)[p]_q^2}{(A - B)(1 - \alpha)[p]_q^2}. \tag{21}$$

**Proof of Theorem 9.** To prove (19), we set

$$\varphi_{k+1} \left[ \frac{f(z)}{f_k(z)} - \left( 1 - \frac{1}{\varphi_{k+1}} \right) \right] = \frac{1 - \sum_{n=1}^k |a_{p+n}| z^n - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| z^n}{1 - \sum_{n=1}^k |a_{p+n}| z^n} = \frac{1 + w(z)}{1 - w(z)}.$$

After some simplifications, we have

$$w(z) = \frac{-\varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| z^n}{2 - 2 \sum_{n=1}^k |a_{p+n}| z^n - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| z^n}$$

and

$$|w(z)| \leq \frac{\varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|}{2 - 2 \sum_{n=1}^k |a_{p+n}| - \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}|}.$$

Thus, we find that  $|w(z)| < 1$  if and only if

$$\sum_{n=1}^k |a_{p+n}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| < 1. \tag{22}$$

From (10) we have

$$\sum_{n=1}^{\infty} \varphi_n |a_{n+p}| < 1.$$

It is not difficult to see that the sequence  $\{\varphi_n\}$  is increasing regarding  $n$ . Additionally, we can see that  $\varphi_n \geq 1$  ( $n = 1, 2, \dots$ ). Thus, we obtain

$$\sum_{n=1}^k |a_{n+p}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{n+p}| \leq \sum_{n=1}^k \varphi_n |a_{n+p}| + \sum_{n=k+1}^{\infty} \varphi_n |a_{n+p}| = \sum_{n=1}^{\infty} \varphi_n |a_{n+p}| < 1.$$

Thus, the inequality (22) is true. This proves (19).

Next, in order to prove the inequality (20), we consider

$$(1 + \varphi_{k+1}) \left[ \frac{f_k(z)}{f(z)} - \frac{\varphi_{k+1}}{1 + \varphi_{k+1}} \right] = \frac{1 - \sum_{n=1}^k |a_{p+n}| z^n + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| z^n}{1 - \sum_{n=1}^{\infty} |a_{p+n}| z^n} = \frac{1 + w(z)}{1 - w(z)}.$$

By simple calculating, we find that

$$w(z) = \frac{(1 + \varphi_{k+1}) \sum_{n=k+1}^{\infty} |a_{n+p}| z^n}{2 + (\varphi_{k+1} - 1) \sum_{n=k+1}^{\infty} |a_{n+p}| z^n - 2 \sum_{n=1}^k |a_{n+p}| z^n}$$

and

$$|w(z)| \leq \frac{(1 + \varphi_{k+1}) \sum_{n=k+1}^{\infty} |a_{n+p}|}{2 - (\varphi_{k+1} - 1) \sum_{n=k+1}^{\infty} |a_{n+p}| - 2 \sum_{n=1}^k |a_{n+p}|}.$$

Thus, we obtain  $|w(z)| < 1$  if it satisfies the following condition:

$$\sum_{n=1}^k |a_{p+n}| + \varphi_{k+1} \sum_{n=k+1}^{\infty} |a_{p+n}| < 1.$$

The remaining part of the proof is similar to that of (19) and we omit it. Now we complete The proof of Theorem 9.  $\square$

**Theorem 10.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If

$$f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}| z^{p+n} \in I_{p,q}(\alpha, A, B) \quad (j = 1, 2),$$

then, for  $0 \leq \lambda \leq 1$ , the function

$$H_\lambda(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) \in I_{p,q}(\alpha, A, B).$$

**Proof of Theorem 10.** For  $0 \leq \lambda \leq 1$ , the function  $H_\lambda(z)$  can be written as follows:

$$H_\lambda(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) = z^p - \sum_{n=1}^{\infty} (\lambda |a_{n+p,1}| + (1 - \lambda) |a_{n+p,2}|) z^{n+p}.$$

For functions  $f_1(z), f_2(z) \in I_{p,q}(\alpha, A, B)$ , by Theorem 2, we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right) (\lambda |a_{n+p,1}| + (1-\lambda) |a_{n+p,2}|) \\ &= \lambda \sum_{n=1}^{\infty} \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right) |a_{n+p,1}| \\ &+ (1-\lambda) \sum_{n=1}^{\infty} \left( (B+1)[n+p]_q (\alpha[n+p]_q - [p]_q) + (1+A)(1-\alpha)[p]_q^2 \right) |a_{n+p,2}| \\ &\leq \lambda (A-B)(1-\alpha)[p]_q^2 + (A-B)(1-\alpha)(1-\lambda)[p]_q^2 \\ &= (A-B)(1-\alpha)[p]_q^2. \end{aligned}$$

This indicates that the function  $H_\lambda(z)$  belongs to  $I_{p,q}(\alpha, A, B)$ . Now we complete the proof of Theorem 10.  $\square$

**Corollary 7.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$  and

$$f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}| z^{p+n} \in I_{p,q}(\alpha, A, B) \quad (j = 1, 2, \dots, l).$$

Then  $F(z) = \sum_{j=1}^l \lambda_j f_j(z)$  ( $\lambda_j \geq 0$  and  $\sum_{j=1}^l \lambda_j = 1$ ) also belongs to  $I_{p,q}(\alpha, A, B)$ .

**Theorem 11.** Let  $\frac{[p]_q}{[p+1]_q} \leq \alpha < 1$ . If  $f_j(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n,j}| z^{p+n}$  belongs to  $I_{p,q}(\alpha, A, B)$  ( $j = 1, 2$ ), then  $Q_m(z) = \frac{(1-m)f_1(z) + (1+m)f_2(z)}{2}$  ( $-1 \leq m \leq 1$ ) also belongs to  $I_{p,q}(\alpha, A, B)$ .

**Proof of Theorem 11.** Theorem 11 is proved using the similar arguments as in Theorem 10.  $\square$

**Author Contributions:** Writing—original draft, B.W.; Writing—review & editing, R.S. and J.-L.L. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We would like to thank the reviewers for careful checking and suggestions.

**Conflicts of Interest:** The authors declare no conflict of interest.

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