Erdős-Deep Families of Arithmetic Progressions

by

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B.Sc., University of British Columbia (Okanagan), 2018

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ABSTRACT

Let $A \subseteq \mathbb{Z}_n$ with $|A| = k$ for some $k \in \mathbb{Z}^+$. We consider the metric space $(\mathbb{Z}_n, \delta)$ in which δ is the distance metric on $\mathbb{Z}_n$ defined as follows: for every $x, y \in \mathbb{Z}_n$, $\delta(x, y) = |x - y|_n$ where $|z|_n = \min(z, n - z)$ for $z \in \{0, \ldots, n-1\}$. We say that A is *Erdős-deep* if, for every $i \in \{1, 2, \ldots, k-1\}$, there is a positive number d_i satisfying

$$|\{\{x, y\} \subseteq A : \delta(x, y) = d_i\}| = i.$$

Erdős-deep sets in $\mathbb{Z}_n$ have been previously classified as translates of: $\{0, 1, 2, 4\}$ when $n = 6$; and, modular arithmetic progressions $\{0, g, 2g, \cdots, (k-1)g\} \subseteq \mathbb{Z}_n$ for some generator g and size k.

Erdős-deep sets have primarily been considered in metric spaces $(\mathbb{Z}_n, \delta)$ and $(\mathbb{R}^d, \|\cdot\|)$ for $d = 2$, but some exploration for $d > 2$ has been done as well.

We introduce the notion of an *Erdős-deep family.* Let $\mathcal{F} = \{A_1, A_2, \ldots, A_s\}$, where $A_1, \ldots, A_s \subseteq \mathbb{Z}_n$. Then we say $\mathcal{F}$ is Erdős-deep if for some $k \in \mathbb{Z}^+$, for every $i \in \{1, 2, \ldots, k-1\}$ there is exactly one positive number d_i satisfying

$$\sum_{j=1}^{s} |\{\{x, y\} \subseteq A_j : \delta(x, y) = d_i\}| = i,$$

and no such d_i for any $i \geq k$.

We provide a complete existence theorem for Erdős-deep pairs of arithmetic progressions $A_1, A_2 \subseteq \mathbb{Z}_n$ and also give a conjectured classification for Erdős-deep families of three arithmetic progressions. Using an identity on triangular numbers, we show a general construction for larger families whose size s is the square of an integer. This construction suggests the existence of Erdős-deep families often relies on such number-theoretic identities.

We define an extremal case of the Erdős-deep family in $(\mathbb{Z}_n, \delta)$ in which both the distances and multiplicities are in $\{1, \ldots, k-1\}$; such families are called Winograd families. We conjecture that Winograd families of arithmetic progressions do not exist in the metric space $(\mathbb{Z}, |\cdot|)$.

Erdős-deep sets in $(\mathbb{Z}_n, \delta)$ correspond to a class of interesting musical rhythms. We conclude this work with a variety of musical demonstrations and original compositions using Erdős-deep rhythm families as a creative constraint in composing multi-voiced rhythms.

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DEDICATION

To Lorraine Brock who reminds me to always hope, have a plan, and enjoy Spring.

Chapter 1
Introduction

1.1 Erdős-Deep Sets

There are a variety of interesting questions to ask about pairwise distances between points in a set. For instance, given a set of specified distances, does there exist a set of points in some metric space whose pairwise distances are equal to those specified? Even simple shapes in the plane can exhibit a diversity of distance patterns – for example: every distance between vertices in an equilateral triangle is the same; an isosceles triangle has distances where two are equal and one is unique; and, all three of the distances in scalene triangles are distinct.

1.1.1 Number of Distinct Distances

The general idea of controlling the distances between points in a set motivates a variety of specific questions to explore. It is interesting to ask whether some distances occur more or less frequently than others. A question that has received much attention since it was first stated in 1946 [7] is the Erdős distinct distances conjecture. The Erdős distinct distances conjecture states that every k-set in $\mathbb{R}^2$ has number of distinct distances $\Omega(\frac{k}{\sqrt{\log(k)}})$. Erdős proved the upper bound in [7] and the currently best known lower bound, proven by L. Guth and N. H. Katz in 2015 [11], is $\Omega(\frac{k}{\log(k)})$. In 2011, J. Garibaldi, A. Iosevich, and S. Senger wrote a monograph [10] surveying techniques and results related to the distinct distances conjecture.

1.1.2 Controlling Multiplicities

This thesis contains new results on problems related to a different Erdős problem about points in the plane, which we will introduce shortly. Additionally, the main structure we explore has connections to musical rhythms, which we discuss in Chapter 6.

The goal of the remainder of this section is to give background for our main result. We begin with the problem in the plane, and from this foundation, we build the context around which our related problem can be best understood. We conclude the section with the statement of our main theorem.

Definition 1.1 (Distance Multiset of a Set)**.** Let X be a subset of a metric space $\mathcal{M}$ with metric δ. Then the *distance multiset* of X is

$$\Delta(X) = \{\delta(x, y) : \{x, y\} \subseteq X, x \neq y\}.$$

Observe that there must be $\binom{|X|}{2}$ distances in $\Delta(X)$.

Example 1.2. Suppose X is the set of vertices of a unit square in $\mathbb{R}^2$. Then $\Delta(X) = \{1, 1, 1, 1, \sqrt{2}, \sqrt{2}\}$, which has $\binom{4}{2} = 6$ elements.

In what follows, we are interested in the multiplicities of the distances in $\Delta(X)$; note in Example 1.2 the multiplicities are 4 and 2.

Definition 1.3 (Multiplicities of Distances in $\Delta(X)$)**.** Let X be a subset of a metric space $\mathcal{M}$ with distance metric δ and suppose $d \in \Delta(X)$. Then the *multiplicity* of d in $\Delta(X)$ is $|\{\delta(x, y) : \{x, y\} \subseteq X, x \neq y, \delta(x, y) = d\}|$, which we denote as $m(d, X)$. We denote the set of distinct multiplicities in $\Delta(X)$ by $M(\Delta(X))$.

When no confusion arises, we use the exponential notation $\Delta(X) = \{d_1^{m_1}, \ldots, d_k^{m_k}\}$ to mean that distance d_i occurs exactly m_i times in $\Delta(X)$, where $d_1, \ldots, d_k$ are the distinct values that occur in $\Delta(X)$.

Definition 1.4 (Support of a Multiset)**.** Let $\Delta = \{d_1^{m_1}, \ldots, d_k^{m_k}\}$ be a multiset, then the *set of distinct distances*, or equivalently the *support*, of Δ is $S(\Delta) = \{d_1, \ldots, d_k\}$.

Definition 1.5 (Deep set)**.** A set $X \subseteq \mathcal{M}$ is *deep* if the distances in $S(\Delta(X))$ have distinct multiplicities in $\Delta(X)$.

Using the exponential notation for distance multiset in Example 1.2, we have $\Delta(X) = \{d_1^{m_1}, d_2^{m_2}\} = \{1^4, \sqrt{2}^2\}$, where $d_1 = 1$, $m_1 = 4$, $d_2 = \sqrt{2}$, $m_2 = 2$. Since $m_1 \neq m_2$, the vertices of the unit square in Example 1.2 form a deep set in $\mathbb{R}^2$. On the other hand, the set $X = \{(0, 0), (1, 0), (0, 1), (2, 0)\}$ is not deep because it has distinct distances 1, 2 $\sqrt{2}$, and $\sqrt{5}$, the last three of which all occur once. In particular, both 2 and $\sqrt{2}$ have multiplicity 1 in $\Delta(X)$, and this fact alone disqualifies X from being deep.

The term "deep" alludes to the property that the multiplicities of a set's distances are distinct. Intuitively, if several distinct distances have the same multiplicity, then they are less unique and less distinguishable from one another in that they occur the same number of times. The property of a set X being deep imposes no restriction on the values of the multiplicities in $\Delta(X)$; the only requirement is that the multiplicities are distinct. Additionally, there is no restriction on $S(\Delta(X))$. That is, a set X can be deep regardless of the values in $S(\Delta(X))$, and even if $|M(\Delta(X))|$ is relatively small with some multiplicities being very large. In the following section, we introduce the class of deep set that maximizes $|M(\Delta(X))|$.

1.1.3 Erdős Problem on the Plane

In 1982, Paul Erdős posed a question in American Mathematical Monthly [8]:

Question 1 (Distances with Specified Multiplicities)**.** *Does there exist a k-set X of points in the plane, no three on a line and no four on a circle, such that for every $i \in \{1, \ldots, k-1\}$ there is a distance between points in X that occurs exactly i times?*

Erdős asks for not only a deep k-set X in $\mathbb{R}^2$, but one whose distance multiset achieves the specified multiplicities $\{1, \ldots, k-1\}$. Note also that since X has k points, there are $1+2+\cdots+(k-1) = \frac{k(k-1)}{2} = \binom{k}{2}$ total distances. So in fact Erdős asks us to find a k-set X with exactly $k-1$ distinct distances; that is, a solution X to Question 1 satisfies $|S(\Delta(X))| = k-1$. In the distance multiset notation, we require there to be distances $d_1, \ldots, d_{k-1}$ with distinct multiplicities coming from the set $\{1, \ldots, k-1\}$, which means we want

$$\Delta(X) = \{d_1^1, d_2^2, \ldots, d_{k-1}^{k-1}\}. \tag{1.1}$$

Definition 1.6 (Erdős-Deep Set)**.** Let X be a deep k-set in a metric space with distance metric δ. If $\Delta(X)$ satisfies (1.1), then we call X an *Erdős-deep* set; in particular, this means that

$$\Delta(X) = \{\delta(x, y) : \{x, y\} \subseteq X, x \neq y\} = \{d_1^1, d_2^2, \ldots, d_{k-1}^{k-1}\}.$$

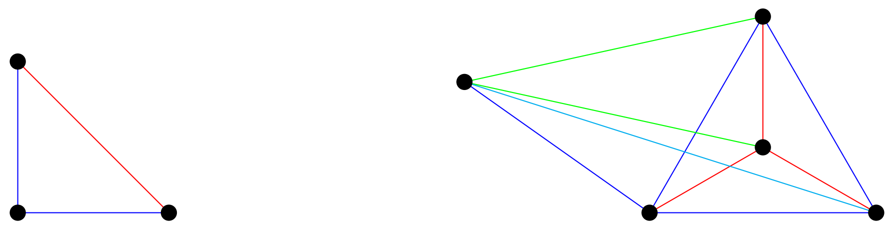
That is, there is some distance that occurs exactly once, another that occurs exactly twice, and so on, up to some distance that occurs exactly $k-1$ times.

Solutions to Question 1 are Erdős-deep sets in $\mathbb{R}^2$. Paul Erdős found Erdős-deep sets in $\mathbb{R}^2$ for $k \in \{3, 4, 6\}$; Carl Pomerance found a solution for when $k = 5$; and Ilona Palásti found solutions for $k \in \{7, 8\}$. See [9] for these solutions. However, no solutions for $k \geq 9$ have been found, and Erdős conjectured that for sufficiently large $k \geq k_0$, no solutions exist. We state this conjecture formally as follows:

Conjecture 1.7. *There exists a positive integer $k_0 \geq 9$ such that for every $k \geq k_0$, there does not exist an Erdős-deep set X in $(\mathbb{R}^2, \|\cdot\|)$ where $|X| = k$.*

Example 1.8 shows solutions to Question 1 for $k \in \{3, 5\}$.

Example 1.8. Figure 1.1 shows solutions of the Erdős problem for (a) $k = 3$ and (b) $k = 5$. The lines correspond to pairwise distances and the distinct colours correspond to distinct distance values.



(a) $k = 3$ (b) $k = 5$

Figure 1.1: Examples of Erdős-deep sets in $(\mathbb{R}^2, \|\cdot\|)$.

Erdős-deep sets in the plane are also known in the literature as crescent configurations. See D. Burt, E. Goldstein, S. Manski, S. J. Miller, E. A. Palsson, and H. Suh [2] for a recent introduction to the problem of characterizing crescent configurations, and a construction showing that, for every k, there exists for some dimension d, a set $X \subseteq \mathbb{R}^d$, $|X| = k$, that is Erdős-deep.

We now pose the generalized form of Erdős' Question 1 that applies to an arbitrary metric space.

Question 2 (Erdős Question for Arbitrary Metric Space)**.** *Let $\mathcal{M}$ be a metric space and $k \in \mathbb{Z}^+$. Does there exist $X \subseteq \mathcal{M}$ such that $|X| = k$ and X is Erdős-deep in $\mathcal{M}$?*

1.1.4 Erdős-Deep Sets on the Line and Circle

Why are sets with 3 points on a line or 4 points on a circle omitted in Question 1? It is possible to arrange k points on a line so that they are evenly distributed with distance d_1 between consecutive points. Here distance d_1 occurs $k-1$ times, $2d_1$ occurs $k-2$ times and in general, id_1 occurs $k-i$ times for all $i \in \{1, \ldots, k-1\}$. Similarly, we can distribute k points with equal angular spacing along a semi-circle and get distances with the required multiplicities. Since these cases of Erdős-deep sets are relatively simple to characterize, we consider them less interesting. This simplicity also motivates some authors to view the semi-circle pattern as canonical; hence some use the term "crescent configurations" to refer to Erdős-deep sets.

If we restrict our set X to a subset of equally spaced points that span the entire circumference of a circle, there is a more interesting class of Erdős-deep sets on the circle. Suppose there are n such equally spaced points around the circle, then there is an infinite class of arithmetic progression subsets of this discrete circle, which are Erdős-deep in $(\mathbb{R}^2, \|\cdot\|)$. We define an arithmetic progression formally as follows:

Definition 1.9 (Modular Arithmetic Progression)**.** Let $A \subseteq \mathbb{Z}_n$ and $n, g, k, a \in \mathbb{Z}^+$. Then the set $A = a + \{0, g, \ldots, (k-1)g\}$ is called a *modular arithmetic progression* of size k with generator g, or equivalently, a *k-AP with generator g*. We call a the *translation index* of A. We occasionally denote A by $a + AP(g, k, n)$.

Remark 1.10. We briefly remark on the cases of primary interest in Definition 1.9. Since we are only concerned with the distances between elements in a set, and both $\Delta(AP(g,k,n)) = \Delta(AP(n-g,k,n))$ and $\Delta(AP(g,k,n)) = \Delta(a + AP(g,k,n))$, we assume without loss of generality that $g \leq \lfloor n/2 \rfloor$ and $a = 0$. We also assume that $k \leq \lfloor \frac{n}{2\gcd(n,g)} \rfloor$ because otherwise $\Delta(AP(g,k,n))$ becomes harder to characterize; thus we omit consideration of these relatively large APs when $k > \lfloor \frac{n}{2\gcd(n,g)} \rfloor$.

Let A be a k-AP with generator g. If we set X to be the set of remainders after performing integer division on each element of A by n, then, as we will see in more detail later in this chapter, X is an Erdős-deep set in $(\mathbb{R}^2, \|\cdot\|)$. These Erdős-deep APs in $(\mathbb{R}^2, \|\cdot\|)$ correspond to Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$, where both $\mathbb{Z}_n$ and $|\cdot|_n$ will be defined formally in Definitions 1.16 and 1.17, respectively.

1.2 Erdős-Deep Families

The structures of interest in this thesis are set families consisting of subsets of a metric space $\mathcal{M}$. We denote such a set family of subsets $A_1, \ldots, A_s \subseteq \mathcal{M}$ by $\mathcal{F} = \{A_1, \ldots, A_s\}$, and we call $A_1, \ldots, A_s$ the *constituent sets* of $\mathcal{F}$.

Definition 1.11 (Distance Multiset of a Family)**.** Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a family of subsets of a metric space $\mathcal{M}$. Then the *distance multiset of* $\mathcal{F}$ is denoted by $\Delta(\mathcal{F})$ where $\Delta(\mathcal{F}) = \bigcup_{j=1}^{s} \Delta(A_j)$. To denote the multiplicity of some distance $d \in \Delta(\mathcal{F})$ we use the notation $m(d, \mathcal{F})$.

We now state the formal definition of an Erdős-deep family in a metric space $\mathcal{M}$.

Definition 1.12 (Erdős-deep Family)**.** Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a family of subsets of a metric space $\mathcal{M}$ with distance metric δ. Then $\mathcal{F}$ is *Erdős-deep* if there exists a $k \in \mathbb{Z}^+$ such that for every $i \in \{1, 2, \ldots, k-1\}$, there is a positive d_i satisfying

$$\sum_{j=1}^{s} |\{\{x, y\} \subseteq A_j : \delta(x, y) = d_i\}| = i.$$

In other words, $\mathcal{F}$ is an Erdős-deep family if $\Delta(\mathcal{F})$ satisfies Condition (1.1) for some positive integer k.

Observation 1 (Distance Equation)**.** Observe that if $\mathcal{F} = \{A_1, \ldots, A_s\}$ is an Erdős-deep family for some $k \in \mathbb{Z}^+$, then all of the distances within each distance multiset of the constituent sets in $\mathcal{F}$ must sum to $\binom{k}{2}$. That is, we require the following equation to hold:

$$\binom{k}{2} = \sum_{i=1}^{s} \binom{|A_i|}{2}.$$

This thesis focuses on Erdős-deep families in $(\mathbb{Z}_n, |\cdot|_n)$ wherein the constituent sets are also Erdős-deep. This is because, Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$ are essentially always arithmetic progression, which are elegant and convenient structures to work with. So, it is this serendipitous connection between Erdős-deep sets in $\mathbb{Z}_n$ and arithmetic progressions that contributes to why we think it is natural to study Erdős-deep families of Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$. But given that Erdős-deep families could exist in a variety of metric spaces, it may be less natural in other cases to require Erdős-deep constituent sets.

The following question generalizes Question 2 to one about the existence of an Erdős-deep family in a general metric space.

Question 3 (Erdős-Deep Family Existence in General Metric Space)**.** *Let $\mathcal{M}$ be a metric space and $s \in \mathbb{N}$. For which tuples $(k_1, \ldots, k_s) \in \mathbb{N}^s$ satisfying $k_1 \geq \cdots \geq k_s \geq 3$ does there exist an Erdős-deep family $\mathcal{F} = \{A_1, \ldots, A_s\}$ where for every $i \in \{1, \ldots, s\}$, $A_i \subseteq \mathcal{M}$ and $|A_i| = k_i$?*

Note that any family $\mathcal{F}$ can be made Erdős-deep by merely including the appropriate collection of 2-sets. So, to deter this type of solution, we insist that each constituent set in an Erdős-deep family has size at least 3.

In Theorem 3.14 stated at the end of this chapter, we answer Question 3 in the case when $\mathcal{M} = (\mathbb{Z}_n, |\cdot|_n)$, $s = 2$, and $A_1, \ldots, A_s$ are assumed to be arithmetic progressions. Additionally, in Conjecture 4.5, we propose a potential answer, based on computational results, to Question 3 in the same case addressed in Theorem 3.14, but for $s = 3$.

To illustrate the generality of Question 3, we return briefly to $\mathbb{R}^2$ and discuss the case when $\mathcal{M} = (\mathbb{R}^2, \|\cdot\|)$. Perhaps the simplest example of an Erdős-deep family in $(\mathbb{R}^2, \|\cdot\|)$, shown in Example 1.13, occurs when $s = 2$ and $k_1 = k_2 = 3$.

Example 1.13. Figure 1.2 shows a solution to Question 3 when $\mathcal{M} = (\mathbb{R}^2, \|\cdot\|)$, $s = 2$, and $k_1 = k_2 = 3$. Each of the two sets are the vertices of isosceles triangles scaled such that the legs of the larger one have the same length as the hypotenuse of the smaller one. Supposing the distances coloured red are 1, the union of distances within each triangle point set is $\{\sqrt{2}^3, 1^2, 2^1\}$.

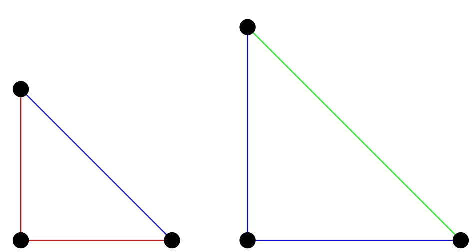


Figure 1.2: An Erdős-deep family in $(\mathbb{R}^2, \|\cdot\|)$ of two sets, each with size 3.

Recall Conjecture 1.7, in which Erdős hypothesized a constant upper bound on the size of an Erdős-deep set in $(\mathbb{R}^2, \|\cdot\|)$. If Conjecture 1.7 is true whereby there exists some $k_0 \in \mathbb{Z}^+$ such that no set X satisfying $|X| \geq k_0$ can be Erdős-deep, then this would imply a major limitation on the existence of Erdős-deep families in $(\mathbb{R}^2, \|\cdot\|)$ in which all constituent sets are Erdős-deep. That is, each constituent set would need to be smaller than k_0. However, this limitation implies that the number of tuples $(k_1, \ldots, k_s) \in \mathbb{N}^s$ for which there exists an Erdős-deep family as described in Question 3 would be finite, which means a complete classification of these tuples that permit the existence of an Erdős-deep family in $(\mathbb{R}^2, \|\cdot\|)$ would likely be possible to determine.

However, fully answering Question 3 when $\mathcal{M} = (\mathbb{R}^2, \|\cdot\|)$ wherein no restrictions are imposed on the constituent sets of an Erdős-deep family, is potentially a very difficult problem. But if Conjecture 1.7 is true, then it would be interesting to determine whether it generalizes as follows:

Conjecture 1.14. *Let $s \in \mathbb{Z}^+$ such that $s \geq 2$. Then there exists $k_{0,s} \in \mathbb{Z}^+$ such that no Erdős-deep family $\mathcal{F} = \{A_1, \ldots, A_s\}$ can exist where for some $i \in \{1, \ldots, s\}$, $|A_i| \geq k_0$.*

It may be the case that Conjecture 1.14 holds for only some $s \geq 2$, but it would be interesting to know whether Conjecture 1.7 generalizes in this way.

Before we shift our focus specifically to the metric space $(\mathbb{Z}_n, |\cdot|_n)$, we mention another example of an Erdős-deep family in $(\mathbb{R}^2, \|\cdot\|)$.

Example 1.15. Figure 1.3 is a solution to Question 3 when $\mathcal{M} = (\mathbb{R}^2, \|\cdot\|)$ and $s = 4$. The distances are coloured as follows (cyan ~ 1, red ~ 2, orange $\sim \sqrt{2}$, blue $\sim 2\sqrt{2}$, green $\sim \sqrt{5}$, and pink $\sim 2\sqrt{5}$). The union of pairwise distances within each point set is $\{1^6, 2^5, \sqrt{2}^4, (2\sqrt{2})^3, \sqrt{5}^2, (2\sqrt{5})^1\}$.

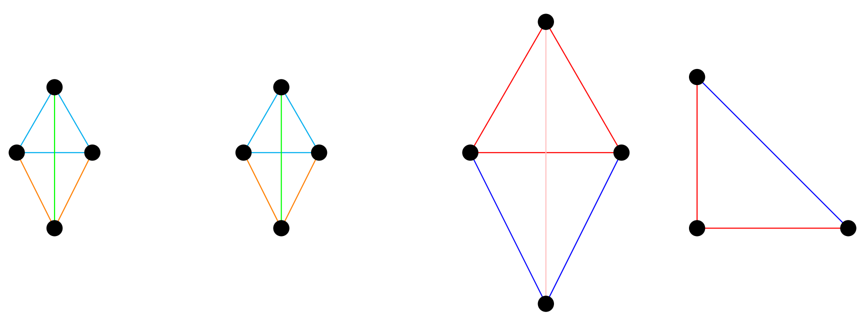


Figure 1.3: This figure shows an Erdős-deep family of 4 sets in $(\mathbb{R}^2, \|\cdot\|)$.

Notice in both Examples 1.8 and 1.15 that multiple copies of the same shape can be in the same family, and some of these copies are scaled in just the right way so that the family is Erdős-deep.

1.2.1 Erdős-Deep Families of Erdős-Deep Sets in $\mathbb{Z}_n$

We now focus our attention primarily to the metric space $(\mathbb{Z}_n, |\cdot|_n)$. We begin by stating the general question we address in this thesis.

Question 4 (Erdős-Deep Families of APs in $\mathbb{Z}_n$)**.** *Let $s \in \mathbb{Z}^+$ and $\mathcal{F} = \{AP(g_i, k_i, n) : i \in \{1, \dots, s\}\}$ such that $k_j \leq \lfloor \frac{n}{2\gcd(g_j, n)} \rfloor + 1$. Then for which tuples $(n, k_1, \dots, k_s)$ is it the case that $\mathcal{F}$ could be an Erdős-deep family?*

Note that Question 4 is a particular case of Question 3 when $\mathcal{M} = (\mathbb{Z}_n, |\cdot|_n)$ in which we require each set to be an AP that is not too large, and we also want to know for which moduli n, a tuple $(k_1, \dots, k_s) \in \mathbb{N}^s$ permits the existence of an Erdős-deep family. As we will see in Theorem 3.14, the existence of an Erdős-deep family in $(\mathbb{Z}_n, |\cdot|_n)$ often depends on the modulus, and not just the sizes of the constituent sets.

We now provide formal definitions of both the set $\mathbb{Z}_n$ and the metric $|\cdot|_n$.

Definition 1.16 (Integers Modulo n: $\mathbb{Z}_n$)**.** We say that all integers with remainder r when divided by n are congruent to r modulo n, and equivalent under the congruence relation modulo n. Such a set is called a *congruence class modulo* n, and is formally denoted $[r]_n$. The set of all congruence classes modulo n, $\{[0]_n, [1]_n, [2]_n, \dots, [n-1]_n\}$ is denoted $\mathbb{Z}_n$ and is called the *integers modulo* n, where the smallest positive element in each congruence class is chosen as representative. For simplicity, we often write $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$.

Distances in $\mathbb{Z}_n$ are calculated differently from the Euclidean metric in $\mathbb{R}^2$ that we've considered so far. We now formally define the distance metric we use throughout this thesis in $\mathbb{Z}_n$.

Definition 1.17 (Distance Metric in $\mathbb{Z}_n$; $|\cdot|_n$)**.** For $z \in \{0, 1, \dots, n-1\}$, let $|z|_n = \min(z, n-z)$. We extend this definition to any integer by reducing (mod n) to a least residue in $\{0, 1, \dots, n-1\}$, and then measuring this least residue. Then given $x, y \in \mathbb{Z}_n$, the distance between x and y is $\delta(x, y) = |x - y|_n$.

For example, in $\mathbb{Z}_6$ we have $\delta(1, 3) = |1 - 3|_6 = |-2|_6 = 2$. In $\mathbb{Z}_{12}$, we have $\delta(11, 2) = 3$, agreeing with the usual notion of distance in the 'clock arithmetic' of $\mathbb{Z}_{12}$.

Given this distance metric $\delta(x, y) = |x - y|_n$ in $\mathbb{Z}_n$ from Definition 1.17, we have that the distance multiset of a set $A \subseteq \mathbb{Z}_n$ is given by

$$\Delta(A) = \{|x - y|_n : \{x, y\} \subseteq A, x \neq y\}.$$

In 2009, E. D. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G. T. Toussaint, T. Winograd, and D. R. Wood showed in [4] the following:

Theorem 1.18. *Aside from translates of* $\{0, 1, 2, 4\}$ *when* $n = 6$*, all translates of APs of the form* $AP(g, k, n)$ *satisfying* $k \leq \lfloor \frac{n/\gcd(n,g)}{2} \rfloor + 1$ *are the only subsets of* $\mathbb{Z}_n$ *that are Erdős-deep in* $(\mathbb{Z}_n, |\cdot|_n)$.

For the exceptional case $\{0, 1, 2, 4\}$, observe that when $n = 6$, any translation of $\{0, 1, 2, 4\}$ has distance multiset $\{|1 - 0|, |2 - 1|, |4 - 2|, |2 - 0|, |4 - 1|, |0 - 2|\} = \{1, 1, 2, 2, 3, 2\} = \{3^1, 1^2, 2^3\}$, which satisfies Condition 1.1, implying that the set is Erdős-deep by Definition 1.6.

A primary objective of this thesis is to begin to answer Question 3 in the case when $\mathcal{M} = (\mathbb{Z}_n, |\cdot|_n)$ and where each constituent set in Erdős-deep families are Erdős-deep and therefore arithmetic progressions.

Note that no translate of the exceptional $\{0, 1, 2, 4\}_6$ set can be a constituent set in an Erdős-deep family because the only possible distance values are exactly $\{1, 2, 3\}$ when $n = 6$. These distances 1, 2, and 3 already occur in the set, so including more distances from the other sets will necessarily prevent the family from being Erdős-deep. So, indeed Erdős-deep sets in Erdős-deep families must be arithmetic progressions.

Observation 2 notes a crucial relationship between the Erdős-deep APs in $\mathbb{Z}_n$ and their distances.

Observation 2. Let $A = AP(g, k, n)$ such that $k \leq \lfloor \frac{n}{2\gcd n,g} \rfloor + 1$. Then the distances in $\Delta(A)$ will always be the minimum of either $\pm jg \pmod{n}$ where $j \in \{1, \ldots, k - 1\}$. So, given $x \in \mathbb{Z}_n$, let $|x|_n = \min(x, n - x)$ where x is assumed in $\{0, 1, \ldots, n - 1\}$. Then the distances within A are of the form $|jg|_n$. Observe that there is a one-to-one correspondence between the distinct distances in $\Delta(A)$ and the $k - 1$ multiples of g.

We mention a couple of examples of Erdős-deep families of APs in $(\mathbb{Z}_n, |\cdot|_n)$:

Example 1.19. The simplest example of an Erdős-deep family of APs is the pair $\mathcal{F} = \{\{0, 1, 2\}, \{0, 2, 4\}\}$, which is Erdős-deep for all $n \geq 7$. Observe that $\Delta(\mathcal{F}) = \{2^3, 1^2, 3^1\}$ when $n = 7$.

Example 1.20. Another example of an Erdős-deep family of APs in $\mathbb{Z}_n$ is

$$\{\{0, 1, 2, 3\}, \{0, 1, 2, 3\}, \{0, 4, 8, 12\}, \{0, 4, 8\}\},$$

which has multiset union $\{1^6, 4^5, 2^4, 8^3, 3^2, 12^1\}$. This example only works for all $n \geq 17$ except for $n = 20$, because in this case, $|3 \cdot 4|_n = |12|_n = -12 \pmod{20} \equiv 8 \pmod{20}$, and so 8 has multiplicity 4, which is the same multiplicity as distance 2. In Chapter 4, we will show a general construction, of which the above is an example, for solutions to Question 4 when s is a square and n is sufficiently large.

1.2.2 Winograd Families

Up until this point, we have only been interested in the multiplicities of distances, and not the distance values themselves. We briefly introduce a sub-class of Erdős-deep families of arithmetic progressions in $\mathbb{Z}_n$ whereby the $k-1$ distinct distances must additionally have values $1, \ldots, k-1$ for some k. We call these Winograd families because they generalize the notion of a Winograd set in $\mathbb{Z}_n$, which is an Erdős-deep set with distinct distances $1, \ldots, k-1$. We discuss Winograd families in Chapter 5. We define the Winograd set and family formally as follows:

Definition 5.1 (Winograd-Deep Family)**.** Let $\mathcal{M}$ be a metric space. Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be an Erdős-deep family in $\mathcal{M}$ where $|M(\mathcal{F})| = |S(\Delta(X))| = k-1$ for some $k \in \mathbb{Z}^+$. Then we call $\mathcal{F}$ *Winograd-deep* if $S(\Delta(\mathcal{F}) = \{1, \ldots, k-1\}$.

In Chapter 5, we discuss Winograd families in $(\mathbb{Z}_n, |\cdot|_n)$ with the additional restriction that all constituent sets are Erdős-deep (and therefore APs). While the constituent sets of Winograd families need not be Erdős-deep, we exclusively study Winograd families that have APs as constituent sets. So, throughout this thesis, we will always assume that the sets in a Winograd family are APs. Also, by *Winograd set*, we mean the set in a Winograd family when $s = 1$. We now show examples of a Winograd set and family.

Example 1.21. An example of a Winograd set in $\mathbb{Z}_{12}$ is $X = \{0, 5, 10, 3, 8, 1, 6\}$ because $\Delta(X) = \{5^6, 2^5, 3^4, 4^3, 1^2, 6^1\}$.

Example 1.22. An example of a Winograd family in $\mathbb{Z}_{17}$ is

$$\mathcal{F} = \{\{0, 1, 2, 3, 4, 5\}, \{0, 3, 6, 9, 12, 15\}, \{0, 8, 16, 7\}\}.$$

because $\Delta(A_1) = \{1^5, 2^4, 3^3, 4^2, 5^1\}$, $\Delta(A_2) = \{3^5, 6^4, 8^3, 5^2, 2^1\}$, and $\Delta(A_3) = \{8^3, 1^2, 7^1\}$. Thus $\Delta(\mathcal{F}) = \{1^7, 2^5, 3^8, 4^2, 5^3, 6^4, 7^1, 8^6\}$, where $k = 9$ in this case.

1.2.3 Erdős-Deep Families of Sets Satisfying Property P

It is easier to find Erdős-deep families in $\mathcal{M}$ in which we impose no restrictions on the constituent sets than if we do impose restrictions. For example, throughout this thesis, we insist that the constituent sets of Erdős-deep families are themselves Erdős-deep, but it is much easier to find Erdős-deep families in $(\mathbb{Z}_n, |\cdot|_n)$ of unrestricted sets. Example 1.23 shows a Erdős-deep family of such unrestricted sets.

Example 1.23. Let $n = 23$. Then the family

$$\mathcal{F} = \{\{0, 2, 11, 14, 17\}, \{1, 4, 5, 8, 11, 14, 15, 17, 18, 21\}\}$$

is Erdős-deep but the constituent sets are not. Note that

$$\Delta(\mathcal{F}) = \{8^1, 2^2, 1^3, 11^4, 4^5, 9^6, 7^7, 6^8, 10^9, 3^{10}\}.$$

Because Erdős-deep sets in $\mathbb{Z}_n$ are APs, it is especially interesting to restrict the constituent sets of Erdő-deep families to being Erdős-deep. Similarly, in other metric spaces, it may be interesting to insist that the constituent sets of Erdős-deep families satisfy some special property P. Given this hypothesis, we pose the following variation of Question 3:

Question 5. *Let $\mathcal{M}$ be a metric space and $s \in \mathbb{N}$ and P be some property that a subset of $\mathcal{M}$ can satisfy. For which tuples $(k_1, \ldots, k_s) \in \mathbb{N}^s$ satisfying $k_1 \geq \cdots \geq k_s \geq 3$ does there exist an Erdős-deep family $\mathcal{F} = \{A_1, \ldots, A_s\}$ where for every $i \in \{1, \ldots, s\}$, $A_i \subseteq \mathcal{M}$, $|A_i| = k_i$ and A_i satisfies property P?*

Question 5 is a variation of Question 3 that emphasizes the connection between specified properties of constituent sets and the Erdős-deep property of the family. For example, Question 4 is a version of Question 5 where we specify the constituent sets satisfy property P, where P means satisfying the Erdős-deep property. We ask Question 4 primarily because Erdős-deep sets in $\mathbb{Z}_n$ are also APs, which are themselves elegant sets; but it may be interesting in general to try to answer Question 5 when P is the property of being Erdős-deep.

1.2.4 Relationship to Other Families Characterized by Distance Patterns

We now place Erdős-deep families in $(\mathbb{Z}_n, |\cdot|_n)$ within the larger context of other set families that are characterized by distances. We begin by showing how Erdős-deep families of Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$ are an extremal family out of other families of Erdős-deep sets. Then we note how Erdős-deep families compare with the well studied "difference families".

General AP Families

Recall in Theorem 1.18 that essentially all Erdős-deep sets in $\mathbb{Z}_n$ are APs. Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a family of Erdős-deep sets in $\mathbb{Z}_n$ such that there exists a $k \in \mathbb{Z}^+$ satisfying $\binom{k}{2} = \sum_{i=1}^{s} \binom{|A_i|}{2}$. Erdős-deep families in $(\mathbb{Z}_n, |\cdot|_n)$ of Erdős-deep sets are extremal in that they maximize $|M(\Delta(\mathcal{F}))|$ as a function of $|S(\Delta(\mathcal{F}))|$. There are many other such families $\mathcal{F}$ of Erdős-deep sets in $\mathbb{Z}_n$ that do not satisfy $|M(\Delta(\mathcal{F}))| = |S(\Delta(\mathcal{F}))| = k-1$. In Figure 1.4, we show a scatter plot that maps each family $\mathcal{F}$ of Erdős-deep sets when $(n, s) = (101, 2)$ to a point $\left(\frac{|M(\Delta(\mathcal{F}))|}{k-1}, \frac{|S(\Delta(\mathcal{F}))|}{k-1}\right)$. There are similar scatter plots for each case (n, s), and we will call this type of scatter plot an (n, s)-plot. Observe that the point $(1, 1)$ corresponds to Erdős-deep pairs of Erdős-deep sets. So, we can visualize Theorem 3.14 as describing exactly which pairs of Erdős-deep sets in $\mathbb{Z}_n$ correspond to the point $(1, 1)$ in a scatter plot corresponding to any case $(n,s) = (n,2)$, where n is arbitrary. It would be interesting to classify the other regions of these (n,s)-plots. For instance the line corresponding to the case when $|M(\Delta(\mathcal{F}))| = |S(\Delta(\mathcal{F}))|$ would be especially interesting to classify since this case generalizes the Erdős-deep case corresponding to the point $(1,1)$.

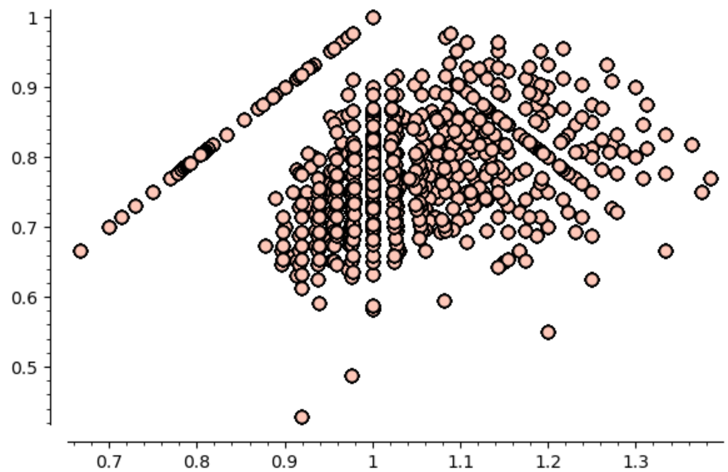


Figure 1.4: A scatter plot of points for each pair of Erdős-deep sets when $(n,s) = (101,2)$. The vertical axis is $\frac{|M(\Delta(\mathcal{F}))|}{k-1}$ and the horizontal axis is $\frac{|S(\Delta(\mathcal{F}))|}{k-1}$.

Difference Families

We mention another type of family of sets $\mathcal{F}$ in $\mathbb{Z}_n$ that specifies the support $S(\Delta(\mathcal{F}))$ of $\mathcal{F}$. Now we impose the requirement that all possible distances in $1,\ldots,\lfloor n/2 \rfloor$ must have positive multiplicity in $\Delta(\mathcal{F})$. Such a family is called a difference family, and it is defined formally as follows:

Definition 1.24 (Difference Family)**.** A family $\mathcal{F} = \{A_1, A_2, \ldots, A_s\}$, where each A_i is a k-subset of $\mathbb{Z}_n$, is an (n,k,λ)-*difference family* if the multiset of differences occurring within each set in $\mathcal{F}$ produces every nonzero element in $\mathbb{Z}_n$ exactly λ times each; that is, $\mathcal{F}$ is a difference family if $\cup_{i=1}^{s} \Delta(A_i) = \{x^\lambda : x \in \mathbb{Z}_n \setminus \{0\}\}$.

Note that since only the minimal differences between every pair of points are considered by the definition of $|\cdot|_n$, we include only the differences $1,\ldots,\lfloor n/2 \rfloor$ in $\Delta(\mathcal{F})$, which are what we have been calling "distances". Observe that difference families contrast Erdős-deep families because all distances with positive positive multiplicity must have equal multiplicity rather than distinct multiplicity.

For a thorough introduction to difference families, see Douglas Stinson's book on Combinatorial Designs [14].

Below is an example of a $(13,3,1)$-difference family:

Example 1.25. Let $A_1 = \{1,3,9\}_{13}$ and $A_2 = \{2,5,6\}_{13}$. Then $\{A_1, A_2\}$ is a $(13,3,1)$-difference family because

$$\begin{aligned}\Delta(A_1) &= \{1-3, 3-1, 1-9, 9-1, 3-9, 9-3\} \\ &= \{11, 2, 5, 8, 7, 6\}, \\ \Delta(A_2) &= \{2-5, 5-2, 2-6, 6-2, 5-6, 6-5\} \\ &= \{10, 3, 9, 4, 12, 1\},\end{aligned}$$

and $\Delta(A_1) \cup \Delta(A_2) = \mathbb{Z}_{13} \setminus \{0\}$.

Example 1.26. This example shows a $(19,4,2)$-difference family. Observe that λ can be larger than 1.

$$\mathcal{F} = \{\{0,1,7,11\}, \{0,2,3,14\}, \{0,4,6,9\}\}.$$

1.2.5 Musical Motivation

We discuss in Chapter 6 the application of Erdős-deep families of arithmetic progressions in $\mathbb{Z}_n$ as a system of rhythms where each arithmetic progression corresponds to the rhythm of a particular voice in a musical phrase. E. D. Demaine, F. Gomez-Martin, H. Meijer, D. Rappaport, P. Taslakian, G. T. Toussaint, T. Winograd, and D. R. Wood in [4], and G. T. Toussaint in [16] reference Erdős-deep and deep sets in $\mathbb{Z}_n$, respectively, as being particularly interesting as musical rhythms given their distance multiplicity patterns. We present musical demonstrations and compositions by the author using Erdős-deep families of Erdős-deep sets along with analysis of this music.

1.3 Main Results and Outline

In Chapters 2 and 3, we answer Question 4 thereby proving the classification of Erdős-deep families of Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$ when $s = 2$. Our theorem is as follows:

Theorem 3.14. Let $\mathcal{F} = \{A_1, A_2\}$ be a pair of arithmetic progressions in $\mathbb{Z}_n$ such that $3 \le k_2 \le k_1 \le \lfloor \frac{n}{2\gcd(n,g_1)} \rfloor + 1$, $\gcd(n, g_1, g_2) = 1$, and $g_2 \le \lfloor n/2 \rfloor$. Then $\mathcal{F}$ is an Erdős-deep pair if and only if

1. $k_1 = k_2 = 3$ and $g_2 \in \{2g_1, \frac{n-g_1}{2}\}$ for all $n \ge 7$, or
2. $(n, k_1, k_2) \in \{(13,6,4), (19,7,6), (31,11,9)\}$.

In Chapter 4, we discuss the case when $s > 2$. We show a bound on s as a function of the constituent set sizes, a construction of Erdős-deep families of APs for when s is a square, and we propose a conjectured classification for $s = 3$. We state the latter conjecture here:

Conjecture 4.5. The only tuples (k, k_1, k_2, k_3) that permit an Erdős-deep family of three arithmetic progressions in $\mathbb{Z}_n$ with sizes $3 \leq k_3 \leq k_2 \leq k_1 \leq \lfloor \frac{n}{2\gcd(n,g_1)} \rfloor + 1$ are of the form

1. $(k, k_1, k_2, k_3) = (6, 4, 4, 3)$ for infinitely many $n \geq 13$, and $(k, k_1, k_2, k_3) = (7, 6, 3, 3)$ for infinitely many $n \geq 15$;
2. the remaining tuples exist for only finitely many n, and they are:

$$\{(8, 6, 5, 3), (9, 6, 6, 4), (10, 6, 6, 6), (10, 7, 7, 3,), (10, 9, 4, 3), (11, 8, 7, 4), (12, 8, 8, 5), (12, 10, 6, 4), (13, 12, 4, 4), (14, 13, 5, 3), (15, 13, 7, 4), (17, 16, 5, 4), (22, 21, 6, 4)\}.$$

Chapters 3 and 4 form the basis for a paper by P. J. Dukes and the author [5] to be submitted for publication.

Chapter 5 includes theoretical and computational results and conjectures about Winograd families. We propose a conjecture that, unlike the Erdős-deep case, Winograd families cannot exist in $(\mathbb{Z}, |\cdot|)$.

We conclude the thesis with a chapter on the musical application of Erdős-deep families of Erdős-deep sets to rhythms in the form of demonstrations and original compositions. Audio for these demonstrations and compositions can be found at the following GitHub repository containing the supplementary material for this thesis: `www.github.com/taogaede/MScThesis`.

Chapter 2
Preliminaries

2.1 Elementary Number Theory

Our work pertains to combinatorial structures based on arithmetic in $\mathbb{Z}_n$, so we briefly recall basic notions of elementary number theory, which are relevant to this thesis. For a complete introduction to elementary number theory, consult D. M. Burton's book (see [3]) on the subject.

In the following Lemma and throughout the thesis, we use floor and ceiling functions. For any real number x, the *floor* of x, denoted $\lfloor x \rfloor$, is the largest integer less than x; and the *ceiling* of x, denoted $\lceil x \rceil$, is the smallest integer greater than x.

The division algorithm is a formal statement about integer division with remainders; and we present it in a form that specifies the quotient because we will use this quotient in Chapter 3.

Lemma 2.1 (Division Algorithm)**.** *Let a and b be positive integers. Then $a = \lfloor \frac{a}{b} \rfloor b + r$ where $0 \leq r < b$.*

The integer r in Lemma 2.1 is called the remainder when a is divided by b. If the integers x and y have the same remainder when divided by n, then we write $x \equiv y \pmod{n}$. That is, $x \equiv y \pmod{n}$ if and only if n divides $x - y$.

A very significant property of some elements of $\mathbb{Z}_n$ is that sometimes for a pair $a, b \in \mathbb{Z}_n$ it is the case that $ab + \ell n = 1$ for some $\ell \in \mathbb{Z}$. When this equation holds for a and b with modulus n, we say that a and b are *multiplicative inverses* of one another and each are *coprime* with n. In terms of congruence classes, this equation is equivalent to $ab \equiv 1 \pmod{n}$.

Recall that for any $a \in \mathbb{R} \setminus \{0\}$, the multiplicative inverse of a is the unique $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$. Then for $a, b \in \mathbb{Z}_n$, $ab \equiv 1 \pmod{n}$ is the modular arithmetic version of this property and here a is the unique multiplicative inverse of b (and vice versa) in $\mathbb{Z}_n$. Unlike real numbers, it is not always the case that a non-zero element of $\mathbb{Z}_n$ has a multiplicative inverse. Recall that an element $a \in \mathbb{Z}_n$ has a multiplicative inverse if and only if a and n are coprime. When an element of $\mathbb{Z}_n$ has a multiplicative inverse, it is called a unit.

To determine whether some element $a \in \mathbb{Z}_n$ is a unit, it suffices to determine whether a and n are coprime, which we show how to do next. Suppose we are given some $a \in \mathbb{Z}_n$ and we want to test whether a is a unit. This amounts to determining whether there exists some $b \in \mathbb{Z}_n$ such that $ab + \ell n = 1$ for some ℓ. The way to solve this problem is to calculate the *greatest common divisor* of a and n, denoted $\gcd(a, n)$. Recall that $\gcd(a, n) = 1$ if and only if a is a unit in $\mathbb{Z}_n$.

A number related to the greatest common divisor of a pair of integers a and n is the *lowest common multiple*, denoted $\operatorname{lcm}(a, n)$. The $\operatorname{lcm}(a, n)$ can be defined as

$$\operatorname{lcm}(a, n) = \frac{an}{\gcd(a, n)}.$$

2.2 Erdős-Deep Families

In this section, we introduce notation used throughout the thesis and prove some fundamental lemmas about Erdős-deep families of arithmetic progressions in $(\mathbb{Z}_n, |\cdot|_n)$.

Our main goal is to answer Question 4 when $s = 2$, but we provide some partial results to address the case when $s > 2$. Throughout this Chapter, we work in the metric space $(\mathbb{Z}_n, |\cdot|_n)$ and we assume that the constituent sets in any family under consideration are Erdős-deep. Recall by Theorem 1.18 and Definition 1.12 that all Erdős-deep sets in $\mathbb{Z}_n$, which are part of Erdős-deep families, must be arithmetic progressions. We denote $\mathcal{F} = \{A_1, \ldots, A_s\}$ to be a family of arithmetic progressions, where $|A_i| = k_i$ and g_i is the generator of A_i such that $k_i \leq \lfloor \frac{n}{2\gcd(n, g_i)} \rfloor + 1$ and $g_i \leq \lfloor n/2 \rfloor$. We assume without loss of generality that $k_1 \geq \cdots \geq k_s \geq 3$. From now on, by "Erdős-deep family" or "Erdős-deep famly of APs", we mean an Erdős-deep family of Erdős-deep sets in $(\mathbb{Z}_n, |\cdot|_n)$.

We begin with some basic results on Erdős-deep families of arbitrary size s. We then provide preliminary results on Erdős-deep pairs, and these will support our main discussion on pairs in Chapter 3.

2.2.1 Basic Observations and Lemmas

Observation 3 and Lemma 2.2 below are basic facts about Erdős-deep families that are used ubiquitously throughout this thesis. Lemma 2.5 shows the importance of units in $\mathbb{Z}_n$ when comparing distance multisets. Lemma 2.6 uses facts about the multiplicities of distances in a constituent set A_i of an Erdős-deep family $\mathcal{F}$ to provide a necessary condition for $\mathcal{F}$ being Erdős-deep in terms of k_i, g_i, k, and k_1.

Observation 3. Let $k_1 \geq \cdots \geq k_s \geq 3$. If a family of sets with sizes $k_1, \ldots, k_s$ form an Erdős-deep family, then the equation $\binom{k_1}{2} + \cdots + \binom{k_s}{2} = \binom{k}{2}$ holds for some $k \in \mathbb{Z}^+$.

Lemma 2.2. *Let $k > k_1 \geq \cdots \geq k_s$ be positive integers and define $t := k - k_1$. If $\binom{k_1}{2} + \cdots + \binom{k_s}{2} = \binom{k}{2}$, then $\sum_{i=2}^{s} k_i(k_i - 1) = 2k_1 t + t(t-1) = 2kt - t(t+1)$.*

Proof. Since $\sum_{i=2}^{s} k_i(k_i - 1) = k(k-1) - k_1(k_1 - 1)$ and $t = k - k_1$, we can get the two expressions involving t by rearranging after either substituting $k_1 := k - t$ or $k := k_1 + t$, respectively. □

Remark 2.3. We remark that when $s = 2$, Lemma 2.2 implies the following parameterization of k_1 in terms of k_2 and t:

$$k_1 = \frac{k_2(k_2 - 1)}{2t} - \frac{t-1}{2}.$$

This parameterization in Remark 2.3 will be used frequently in Chapter 3.

Lemma 2.4. *Let $k, k_1, \ldots, k_s$ be integers satisfying $k_1 \geq \cdots \geq k_s \geq 3$ and $\binom{k}{2} = \sum_{i=1}^{s} \binom{k_i}{2}$. Then $t \geq 1$.*

Proof. Suppose $t < 1$, then $k \leq k_1$. Since $\binom{k}{2} = \binom{k_1}{2} + \sum_{i=2}^{s} \binom{k_i}{2}$, we require that $\sum_{i=1}^{s} \binom{k_i}{2} \leq 0$, which contradicts $k_i \geq 3$. □

Lemma 2.5. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ where $A_i = \{a_i, a_i + g_i, \ldots, a_i + (k_i - 1)g_i\}$ and $a_i \in \mathbb{Z}_n$. Let $u \in \mathbb{Z}_n$ such that $\gcd(u, n) = 1$. Let $\mathcal{F}' = \{A'_1, \ldots, A'_s\}$ where $A'_i = \{a_i u, a_i u + g_i u, \ldots, a_i g_1^{-1} + (k_i - 1) g_i u\}$. Then $\Delta(\mathcal{F})$ and $\Delta(\mathcal{F}')$ are equal under the bijection $x \mapsto ux \pmod{n}$.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be as defined in the statement. Since $\gcd(u, n) = 1$, u is a unit in $\mathbb{Z}_n$ and so u^{-1} exists. It is sufficient to show that equal distances in $\Delta(\mathcal{F})$ are equal in $\Delta(\mathcal{F}')$ under the bijection of u multiplication. Let i, j, m_i, m_j be integers such that $1 \leq i < j \leq s$, $m_i \in [k_i - 1]$, and $m_j \in [k_j - 1]$. Suppose $x \equiv a_i + m_i g_i \pmod{n}$ and $y \equiv a_j + m_j g_j \pmod{n}$. Then $|x|_n = |y|_n \Leftrightarrow x \equiv \pm y \pmod{n} \Leftrightarrow xu \equiv \pm yu \pmod{n} \Leftrightarrow |xu|_n = |yu|_n$, as desired. □

Lemma 2.6. *Suppose A_1 has generator 1. If $\mathcal{F} = \{A_1, \ldots, A_s\}$ is an Erdős-deep family, then for every $i \in \{2, \ldots, s\}$, $k_i - 1 - g_i < t$.*

Proof. By Observation 2, we have that $m(g_i, \Delta(A_i)) = k_i - 1$. Note that since A_1 has generator 1,

$$m(g_i, \Delta(A_1)) = \begin{cases} k_1 - g_i, & \text{if } g_i < k_1; \\ 0, & \text{if } g_i \geq k_1. \end{cases}$$

Thus $m(g_i, \Delta(\mathcal{F})) \geq k_i - 1 + k_1 - g_i$. But since $\mathcal{F}$ is Erdős-deep, for every $d \in \Delta(\mathcal{F})$, $m(d) < k$. Therefore we have $k_i - 1 + k_1 - g_i < k \Leftrightarrow k_i - 1 - g_i < k - k_1 = t$, as desired. □

2.2.2 Preliminary Results on Erdős-Deep Pairs in $(\mathbb{Z}_n, |\cdot|_n)$

Let $\mathcal{F} = \{A_1, A_2\}$ be an Erdős-deep pair. We call $S(\Delta(A_2) \cap \Delta(A_1))$ the set of *shared distances*, and $S(\Delta(A_2) \setminus \Delta(A_1))$ the set of *unshared distances in* $\Delta(A_2)$. Observe that $|S(\Delta(\mathcal{F}))| - |S(\Delta(A_1))| = (k-1) - (k_1 - 1) = k - k_1$ is the number of unshared distances in $\Delta(A_2)$, and we use the parameter $t := k - k_1$ to denote this number. Then the number of shared distances is $|S(\Delta(A_2))| - t = k_2 - 1 - t$. These two numbers t and $k_2 - 1 - t$ come up many times throughout Chapters 2 and 3.

Lemma 2.7 provides convenient bounds on t and $k_2 - 1 - t$. The proof of Lemma 2.8 uses Lemma 2.7, and Proposition 2.9 shows that without loss of generality, we may assume that Erdős-deep pair $\{A_1, A_2\}$ satisfies $g_1 = 1$ unless $k_1 = k_2 = 3$. Then, Proposition 2.10 classifies Erdős-deep pairs in the case when $k_1 = k_2 = 3$.

Lemma 2.7. *We have $t \le \frac{3}{7}(k_2 - 1)$ unless $k_1 = k_2 = 3$.*

Proof. Suppose $t > \frac{3}{7}(k_2 - 1)$. Then, using Lemma 2.2, we have $k_2(k_2 - 1) = 2k_1 t + t(t-1)$, which implies

$$k_2 > \frac{6}{7}k_1 + \frac{3}{7}(t-1). \tag{2.1}$$

When $k_1 \ge k_2 + 1$, this gives

$$k_2 > \frac{6}{7}k_2 + \frac{9}{49}(k_2 - 1) + \frac{3}{7} = \frac{51k_2 + 12}{49},$$

a contradiction. When $k_1 = k_2$, (2.1) becomes

$$k_2 > \frac{6}{7}k_2 + \frac{9}{49}(k_2 - 1) - \frac{3}{7} = \frac{51k_2 - 30}{49}.$$

This forces $k_2 < 15$. It is straightforward to check that there are no integer solutions to $2k_2(k_2 - 1) = k(k-1)$ with $3 < k_2 < 15$. It follows that $k_1 = k_2 \le 3$. □

We show in Lemma 2.8 that if $k_1 \ge k_2 > 3$, then $\gcd(g_1, n) = 1$. This fact will be used to prove Proposition 2.9, which says that without loss of generality, we can assume that A_1 in an Erdős-deep pair is an interval in this case. This fact will be fundamental to the arguments in Chapter 3.

Lemma 2.8. *Let $\mathcal{F} = \{A_1, A_2\}$ be an Erdős-deep pair where $A_i = \{0, g_i, \ldots, (k_i - 1)g_i\}$. If $k_1 \ge k_2 > 3$, then $\gcd(n, g_1) = 1$.*

Proof. Since $\mathcal{F}$ is an Erdős-deep pair, there are $k_2 - 1 - t$ shared distinct distances of the form $|ig_1|_n = |jg_2|_n = d$. Note that $|ig_1|_n = |jg_2|_n = d \Leftrightarrow ig_1 = \pm jg_2 = \pm d \Leftrightarrow d \in \Delta(A_1) \cap \Delta(A_2) \Rightarrow \pm d \in A_1 \cap A_2$, so $|A_1 \cap A_2| \ge k_2 - 1 - t$.

We assume without loss of generality that $\gcd(g_1, g_2, n) = 1$. Let $h = \gcd(g_1, g_2)$. Then since $\gcd(g_1, g_2, n) = 1$, we must have $\gcd(h, n) = 1$. Therefore $[h^{-1}]_n$ exists in $\mathbb{Z}_n$, which implies by Lemma 2.5, that $\{AP(g_1, k_1, n),\ AP(g_2, k_2, n)\}$ is Erdős-deep if and only if $\{AP(|g_1h^{-1}|_n, k_1, n), AP(|g_2h^{-1}|_n, k_2, n)\}$ is Erdős-deep. Since $\gcd(\frac{g_1}{h}, \frac{g_2}{h}) = 1$, we may assume that $h = 1$.

Let $g' = \text{lcm}(g_1, g_2)$. Observe that for some positive integer $1 \leq k' \leq \lceil \frac{g_2(k_2-1)}{g'} \rceil$, we have $A_1 \cap A_2 \subseteq AP(g', k', n)$. Since $h = 1$, $g' = g_1g_2$. Suppose for a contradiction that $\gcd(g_1, n) > 1$, then there does not exist $[g_1^{-1}]_n$ such that $[g'g_1^{-1}]_n = [g_2]_n$. By Lemma 2.5, for any unit $u \in \mathbb{Z}_n$, $\Delta(AP(g', k', n))$ and $\Delta(AP(g'u, k', n))$ are isomorphic under the bijection $x \to xu \pmod n$. But since $g' = g_1g_2$ and $\gcd(g_1, n) > 1$, $g_1 \neq 1$ and there does not exist a unit $u \in \mathbb{Z}_n$ such that $[g'u]_n = [g_2]_n$. Thus $g' \geq 2g_2$. Therefore $|A_1 \cap A_2| = k' \leq \lceil \frac{g_2(k_2-1)}{g'} \rceil \leq \lceil \frac{k_2-1}{2} \rceil$. But then since $k_1 \geq k_2 > 3$, by Lemma 2.7 we have

$$\frac{4(k_2-1)}{7} \leq k_2 - 1 - t \leq |A_1 \cap A_2| \leq \left\lceil \frac{k_2-1}{2} \right\rceil,$$

which is a contradiction. So, $\gcd(g_1, n) = 1$, as desired. □

Proposition 2.9 (A_1 can be Assumed to be an Interval)**.** *Let $k_1 \geq k_2 > 3$. Then the pair $\{AP(g_1,\ k_1,\ n),\ AP(g_2,\ k_2,\ n)\}$ is Erdős-deep if and only if $\{AP(1,\ k_1,\ n),\ AP(g_1^{-1}g_2,\ k_2,\ n)\}$ is Erdős-deep.*

Proof. Since $k_1 \geq k_2 > 3$, by Lemma 2.8 we have $\gcd(g_1, n) = 1$, which means that g_1 is a unit, and so the result follows by Lemma 2.5. □

Proposition 2.10 classifies Erdős-deep pairs in the case when $k_1 = k_2 = 3$. We complete the classification of Erdős-deep pairs for when $k_1 \geq k_2 > 3$ in Chapter 3.

Proposition 2.10. *Let $\mathcal{F} = \{A_1, A_2\}$ be a pair of APs in $\mathbb{Z}_n$ satisfying $A_1 = \{0, g_1, 2g_1\}$, $A_2 = \{0, g_2, 2g_2\}$, and $g_1, g_2 \leq \lfloor n/2 \rfloor$. Then $\mathcal{F}$ is Erdős-deep if and only if either $g_2 = 2g_1$ or $g_2 = \frac{n-g_1}{2}$ where $n \geq 7$.*

Proof. Suppose $\mathcal{F}$ forms an Erdős-deep pair satisfying $k_1 = k_2 = 3$. Then $|S(\Delta(\mathcal{F}))| = k_2 - 1 - t = 2 - 1 = 1$.

We show that either $g_2 = 2g_1$ or $g_2 = \frac{n-g_1}{2}$. Since $k_1 = k_2 = 3$, it follows that $g_1(k_1 - 1) < n$ and $g_2(k_2 - 1) < n$ hold. We have that the possible distance values in $\Delta(\mathcal{F})$ are $g_1, g_2, 2g_1, 2g_2, n - 2g_1$, and $n - 2g_2$. Note we cannot have $n - g_1$ or $n - g_2$ because then $2g_1$ or $2g_2$ would need to be greater than n for g_1 or g_2 to be larger than $\lfloor n/2 \rfloor$. We must have g_1 and g_2 as distinct distances and exactly one other distinct distance, otherwise there will be more than $k - 1 = 3$ of the required distance values. Suppose without loss of generality that $g_2 \geq g_1$, so $2g_1 \leq 2g_2$. Additionally, suppose for a contradiction that neither $g_2 = 2g_1$ nor $g_1 = n - 2g_2$. Then $2g_1$ is the third distance and it must hold that $|2g_2|_n = 2g_2 = n - 2g_1$. But then $2g_2$ occurs once in $\Delta(A_1)$ and once in $\Delta(A_2)$, but g_1 and g_2 each occur twice overall so all distances have equal multiplicity, a contradiction. So $n - 2g_2 = g_1$ or $g_2 = 2g_1$. Note we require $n \geq 7$ so that there are at least 3 distinct distances, and $n \neq 6$ because since the sets have size three, $g_1, g_2 \in \{1, 2\}$ and distance 2 occurs four times in the pair $\{0, 1, 2\}$, and $\{0, 2, 4\}$ when $n = 6$.

The converse follows from a calculation. Observe that $\{\{0,\, g_1,\, 2g_1\},\, \{0,\, 2g_1,\, 4g_1\}\}$ and $\{\{0,\, g_1,\, 2g_1)\},\, \{0,\, \frac{n-g_1}{2},\, n - g_1\}\}$ have distance multisets $\{g_1^2,\, (2g_1)^3,\, (4g_1)^1\}$ and $\{(g_1)^3,\, (\frac{n-g_1}{2})^2,\, (2g_1)^1\}$, respectively. Recall from Definition 1.17 that $|n-g_1|_n = g_1$. □

Lemma 2.11 is a statement that will be used to prove Proposition 3.6 in Chapter 3. We state and prove it here because it applies to all Erdős-deep pairs of APs.

Lemma 2.11. *If $\mathcal{F} = \{A_1, A_2\}$ is an Erdős-deep pair satisfying $k_1 \geq k_2 > 3$, then $g_2(k_2 - 1) > n - k_1$.*

Proof. Suppose for a contradiction that $g_2(k_2 - 1) \leq n - k_1$. By Lemma 2.8, it follows that $\gcd(g_1, n) = 1$. Since $g_2(k_2 - 1) \leq n - k_1$, the shared distances for the Erdős-deep pair are precisely the first $k_2 - 1 - t$ multiples of g_2 that intersect A_1, namely $g_2, 2g_2, \ldots, \lfloor (k_1 - 1)/g_2 \rfloor g_2$.

Observe that we require there to be a shared distance with multiplicity m for every $m \in \{k_1, \ldots, k - 1\}$, otherwise there would be some multiplicity in $\{1, \ldots, k - 1\}$ that does not occur in $\Delta(\mathcal{F})$, which would contradict the Erdős-deep property of $\mathcal{F}$.

We also require that $g_2 \geq 2$ since otherwise $A_2 \subset A_1$ and the two distances with largest multiplicity in $\Delta(\mathcal{F})$ would be $g_2 = g_1 = 1$ and $2g_2 = 2g_1 = 2$, but the multiplicities of 1 and 2 would be $k_1 - 1 + k_2 - 1$ and $k_1 - 2 + k_2 - 2$, respectively, which are not consecutive, a contradiction if $\mathcal{F}$ is Erdős-deep.

In general, since only shared distances can have multiplicity at least k_1 in $\Delta(\mathcal{F})$, and $g_2 \geq 2$, no two of these shared distances can have consecutive total multiplicities at least k_1. Let $m_i = m(ig_2, \mathcal{F})$. Observe that $m_i = k_1 - ig_2 + k_2 - i$. Since $g_2 \geq 2$, no distance jg_2 with $j \in \{1, \ldots, k_2 - 1\} \setminus \{i\}$ can satisfy $m_j = m_i \pm 1$. But we require there to be a shared distance with multiplicity m for every $m \in \{k_1, \ldots, k - 1\}$ in $\Delta(\mathcal{F})$; since there are no shared distances with consecutive multiplicities in $\{k_1, \ldots, k - 1\}$, at most one distance can have multiplicity at least k_1 in $\Delta(\mathcal{F})$. Therefore $t = 1$.

Since $t = 1$ and $m_i = (k_2 - i) + (k_1 - ig_2)$, the distance with highest multiplicity must be g_2, implying that $k_2 - 1 + k_1 - g_2 = k_1 \Leftrightarrow g_2 = k_2 - 1$. Since $k_2 > 3$, $t = 1$, and $g_2(k_2 - 1) \leq n - k_1$, we have that for every $j \in \{1, \ldots, k_2 - 2\}$, jg_2 must be shared and $(k_2 - 1)g_2$ must be the sole unshared distance in $\Delta(A_2)$. Then $m_{k_2-1} = k_2 - (k_2 - 1) + k_1 - (k_2 - 1)g_2$, which we require to be equal to $k_1 - jg_2$, otherwise there would be no distance with this multiplicity. But when $m_{k_2-1} = k_1 - jg_2$, we have $k_2 - (k_2 - 1) + k_1 - (k_2 - 1)g_2 = k_1 - jg_2 \Leftrightarrow 1 = (k_2 - 1 - j)g_2 \Leftrightarrow g_2(k_2 - 1) = jg_2 + 1$, a contradiction since $g_2 \geq 2$. Therefore $t < 1$, but this contradicts Lemma 2.4, and so $g_2(k_2 - 1) > n - k_1$. □

2.3 Useful Visualizations for Erdős-Deep Pairs

There are many ways to visualize Erdős-deep pairs, and we conclude this chapter with a discussion on two such visualizations, which each illustrate important aspects of the structure of Erdős-deep pairs of arithmetic progressions.

2.3.1 Histogram of Multiplicities

We can use a histogram to model the distances in a distance multiset. Each bar corresponds to a possible distance value in $\{1, \ldots, \lfloor n/2 \rfloor\}$ and the height of the bar corresponds to the multiplicity of the distance. Following Figure 2.1 below, given a pair of arithmetic progressions $\{A_1, A_2\}$, we say that the blue bars correspond to multiplicities of distances from $\Delta(A_2)$ while the orange bars correspond to those of $\Delta(A_1)$.

The histogram model provides a useful way to visualize the gaps between distances in an Erdős-deep pair $\mathcal{F} = \{A_1, A_2\}$, which will be the focus of Section 3.4. Recall from Proposition 2.9 that we can assume that A_1 is an interval. So, observe that every distance $d \geq k_1$ has multiplicity only from $\Delta(A_2)$ on the right hand side of the histogram. Since $k_1 \geq k_2$, d has multiplicity less than k_1. But since every distance must have distinct multiplicity, there must be some distance $d' < d$ with equal multiplicity from $\Delta(A_1)$. Thus for $\mathcal{F}$ to be Erdős-deep, d' must have some multiplicity in $\Delta(A_2)$, thereby making d' shared distance.

Observe in Figure 2.1, that when $g_2 = 4$, the histogram corresponds to a valid Erdős-deep pair because each positive distance multiplicity is distinct and less than k. Also note that since $g_1 = 1$, the orange bars always follow a staircase-like pattern.

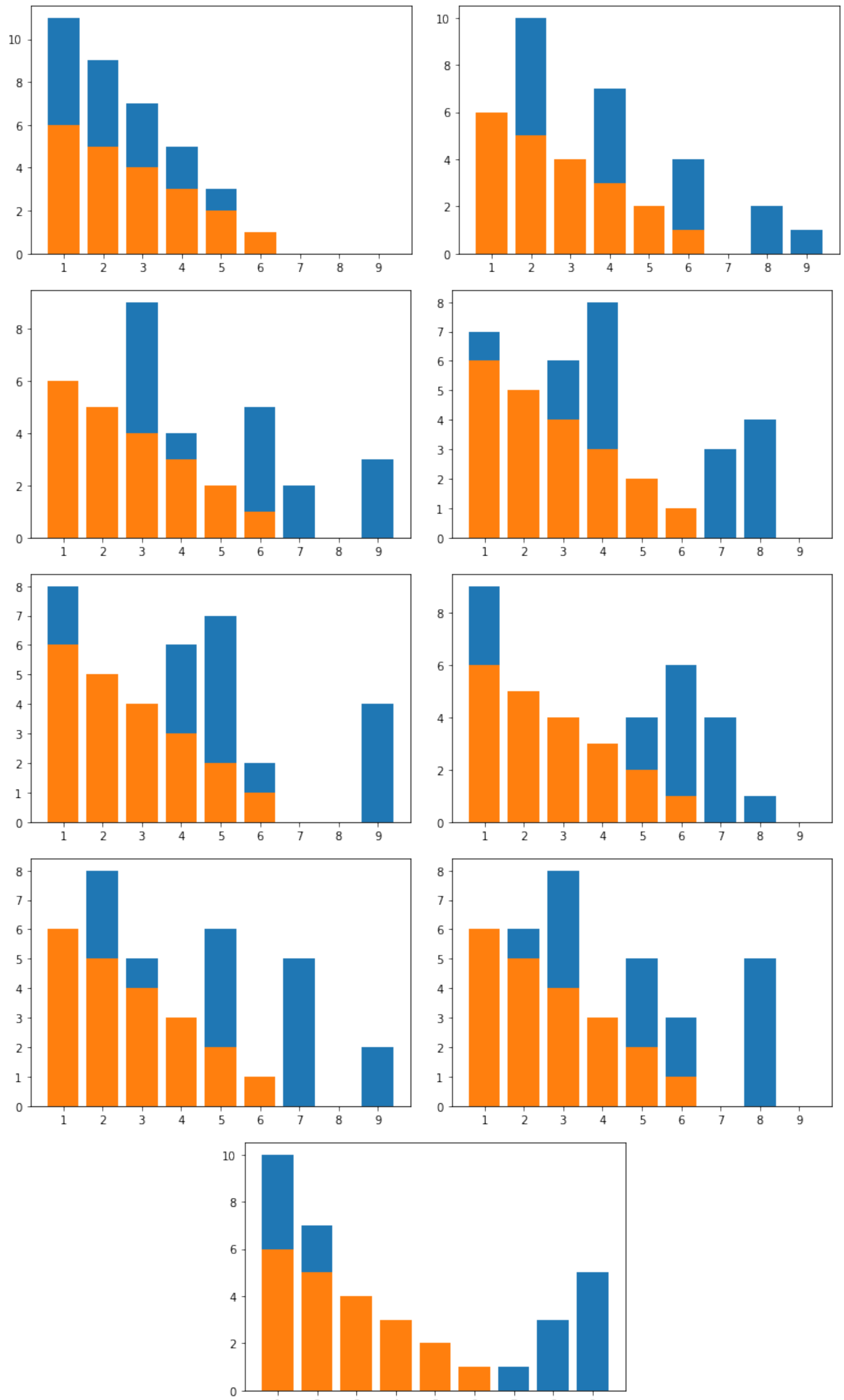


Figure 2.1: This figure shows multiplicity histograms of AP pairs $\{A_1, A_2\}$ where $\Delta(A_1) = \{1^{k_1-1}, 2^{k_1-2}, \ldots, (k_1-1)^1\}$ and $\Delta(A_2) = \{|g_2|_n^{k_2-1}, |2g_2|_n^{k_1-2}, \ldots, |(k_2-1)g_2|_n^1\}$. The blue bars correspond to $\Delta(A_2)$ and orange bars similarly to $\Delta(A_1)$. The histograms are ordered from top left to bottom right by increasing $g_2 \in \{1, \ldots, \lfloor n/2 \rfloor\}$. In this example, $(n, k, k_1, k_2) = (19, 9, 7, 6)$.

2.3.2 Ball in $\mathbb{Z}_n$

The next visualization gives a useful technique for accounting for which distances are shared or unshared between $\Delta(A_1)$ and $\Delta(A_2)$. Since the distances in $\Delta(A_2)$ are $\{|g_2|_n, |2g_2|_n, \ldots, |(k_2-1)g_2|_n\}$, and those in $\Delta(A_1)$ are $\{1, \ldots, k_1-1\}$, we can account for the shared distances by determining when A_2 intersects $B = \{\pm 1, \pm 2, \ldots, \pm(k_1 - 1)\}$. We call B the ball centred at 0 with radius k_1. In Section 3.2, we focus our attention on the relative number of shared distances and those only in $\Delta(A_2)$. When B is large relative to n, there will be more shared distances, and alternatively, when B is small, more of the distances in $\Delta(A_2)$ will be unshared. We use n/k_1 as a measure of how big B is relative to $\mathbb{Z}_n$, and we define $\beta := n/k_1$.

Figure 2.2 shows examples of the k_1-ball visualization. The example on the left corresponds to the Erdős-deep pair $(n, t, k_1, k_2, g_2) = (19, 2, 7, 6, 4)$. On the right, the tuple is $(31, 2, 7, 6, 13)$. Observe that since k_1 is larger relative to n in this latter case, more elements of A_2 will intersect B. In this case, note that the number of unshared distinct distances in $\Delta(A_2)$ should be $t = 2$ in an Erdős-deep pair. That is, we require $|\Delta(A_2) \setminus B| = t$. The example on the right is not an Erdős-deep pair because $|\Delta(A_2) \setminus B| = 3$. We show in Section 3.5, that $(n, t, k_1, k_2, g_2) = (31, 2, 11, 9, 13)$ permits an Erdős-deep pair.

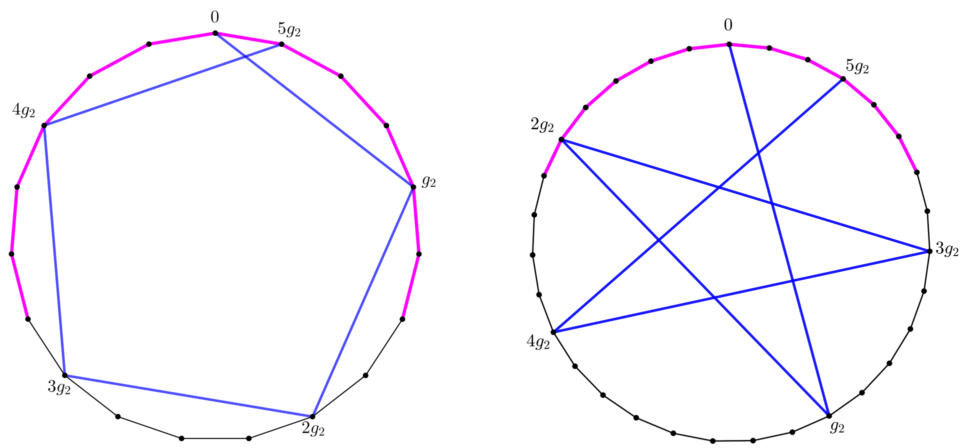


Figure 2.2: This figure shows two examples of how an AP $A_2 = \{0, g_2, \ldots, (k_2-1)g_2\}$ and ball $B = \{\pm 1, \pm 2, \ldots, \pm(k_1 - 1)\}$ intersect in $\mathbb{Z}_n$. The blue path shows the progression A_2 around $\mathbb{Z}_n$. The pink lines indicate the region of points in $\mathbb{Z}_n$ covered by B.

Chapter 3
Classification of Erdős-Deep Pairs of Arithmetic Progressions

We let $\mathcal{F} = \{A_1, A_2\}$ be an Erdős-deep family where for each $i \in \{1, 2\}$, $A_i = AP(g_i, k_i, n)$ such that $g_i \leq \lfloor n/2 \rfloor$ and $k_i \leq \lfloor \frac{n}{2 \gcd g_i, n} \rfloor$.

In Proposition 2.10 we classified Erdős-deep pairs in the case when $k_1 = k_2 = 3$. Recall from Proposition 2.9 that we may assume without loss of generality that A_1 is an interval when $k_1 \geq k_2 > 3$. The goal of this chapter is to finish the proof of Theorem 3.14 by classifying when $\mathcal{F}$ is an Erdős-deep pair in the case when $k_1 \geq k_2 > 3$.

We restate Theorem 3.14 presently, and we will state and prove it in the final section of this chapter.

Theorem 3.14. Let $\mathcal{F} = \{A_1, A_2\}$ be a pair of arithmetic progressions in $\mathbb{Z}_n$ such that $3 \leq k_2 \leq k_1 \leq \lfloor \frac{n}{2 \gcd(n, g_1)} \rfloor + 1$, $\gcd(n, g_1, g_2) = 1$, and $g_2 \leq \lfloor n/2 \rfloor$. Then $\mathcal{F}$ is an Erdős-deep pair if and only if

1. $k_1 = k_2 = 3$ and $g_2 \in \{2g_1, \frac{n-g_1}{2}\}$ for all $n \geq 7$, or
2. $(n, k_1, k_2) \in \{(13, 6, 4), (19, 7, 6), (31, 11, 9)\}$.

3.1 Proof Outline

Our approach is to prove several necessary conditions on the parameters n, k_1, k_2, t such that the number of tuples (n, k_1, k_2) that permit the existence of an Erdős-deep family is finite and sufficiently small to be searched by a computer. We can sort these necessary conditions into three categories:

C 1. The number of shared distances must be larger than the number of unshared distances.

C 2. Every distinct distance must occur with total multiplicity at most $k - 1$.

C 3. Every distinct distance must occur with distinct multiplicity.

Recall we established lemmas in Section 2.2.1 that are used throughout the argument in this chapter. Let $\beta = n/k_1$. We show in Section 3.2 that when $n/k_1 \geq 6$, Condition 1 fails. Section 3.3 establishes that both $n \geq \frac{(k_2-t)^2}{4}$ and $t \leq 5\beta - \frac{3}{4}$ hold, otherwise Condition 2 fails. In Section 3.4, we prove using a combinatorial argument to contradict Condition 3 that $n < 18k_2 + 36$. With these necessary conditions on $\mathcal{F}$ in place, it is sufficient to prove Theorem 3.14 in the case $k_1 \geq k_2 > 3$ by searching the resulting finite space of tuples (n, k_1, k_2). So, the argument concludes with Section 3.5, which proves Theorem 3.14, discusses the resultant finite number of tuples to be searched, the methodology of the computer search of this space, and a summary of the results.

We summarize the conditions on the parameters n, β, k_2, t in Table 3.1, wherein we assume $2 \leq \beta < 6$ due to the argument in Section 3.2.

$2 \leq \beta < 6$	$n > 18k_2 + 36$	$n \leq 18k_2 + 36$
$n < \frac{(k_2-t)^2}{4}$	Sections 3.3 and 3.4 (C 2 & C 3 fail)	Section 3.3 (C 2 fails)
$n \geq \frac{(k_2-t)^2}{4}$	Section 3.4 (C 3 fails)	Section 3.5 (Search)

Table 3.1: This table shows the four parameter regions, given the result of Section 3.2 (which shows $2 \leq \beta < 6$ by applying Condition 1), and which sections of this chapter address each region. Note that the Sections 3.3 and 3.4 arguments overlap in the top left cell.

3.2 Condition 1: Number of Distinct Distances

Let $\mathcal{F} = \{A_1, A_2\}$ be an Erdős-deep pair where $A_1 = AP(g_1, k_1, n)$ and $A_2 = AP(g_2, k_2, n)$. Recall from Proposition 2.9 that A_1 is an interval. By Observation 2, there is a one-to-one correspondence between the positive multiples of g_2 and $S(\Delta(A_2))$. Since A_1 is an interval, all shared distances correspond to multiples of g_2 that are in the set $\{n-(k_1-1), \ldots, n-1\} \cup \{1, \ldots, k_1-1\}$. Alternatively, all unshared distances in $\Delta(A_2)$ correspond to multiples of g_2 outside this set. Let $r \in \mathbb{Z}^+$ such that $r \leq \lfloor n/2 \rfloor$. We introduce the notation $B_r(0)$ to denote the subset of $\mathbb{Z}_n$ $\{n-(r-1), \ldots, n-1\} \cup \{1, \ldots, r-1\}$, and we call $B_r(0)$ the ball with radius r centred at 0. Since we never consider balls centred anywhere except 0, we denote $B_r(0)$ simply as B_r. We use the parameter $\beta := n/r$, which describes the size of B_r relative to n.

In this section we prove Proposition 3.1, which we use to show, under certain conditions, that the number of shared distances in $S(\Delta(A_2))$ is at most the number of unshared distances in $S(\Delta(A_2))$. We show in Proposition 3.6 that this inequality between the number of shared and unshared distances in $S(\Delta(A_2))$ contradicts $\mathcal{F}$ being

Erdős-deep when $r = k_1$, $\beta > 6$, and $k_1 \geq k_2 > 3$.

This section begins with three lemmas that apply generally to modular APs. Then the remainder of the section applies these lemmas to the classification of Erdős-deep pairs.

We introduce Lemma 3.3, which is used to account for the number of elements in A_2 that intersect a ball centred at 0 in $\mathbb{Z}_n$. We prove 3.5, which states that $g_2 < 2k_1$ if A_2 forms an Erdős-deep pair with A_1, using a similar approach to our proof of Lemma 3.3. Lemma 3.5 will be used in the search discussed in Section 3.5.

We note that Lemmas 3.2, 3.3, 3.4 form a self-contained result on general APs.

3.2.1 General Result on Arithmetic Progressions in $\mathbb{Z}_n$

We prove and use Lemmas 3.2, 3.3, 3.4 prove Proposition 3.1, which is a general result on APs in $\mathbb{Z}_n$. We remark that Proposition 3.1 is equivalent to Lemma 3.4; and it is stated presently with less abstract notation than Lemma 3.4 for the convenience of the reader who does not wish to scrutinize the details of this subsection.

Proposition 3.1. *Let $n, r, g,$ and k be positive integers satisfying $n \geq 6r$ and $\frac{n}{6(k-1)} < g < n - 2r + 1$. Let $A, B \subseteq \mathbb{Z}_n$ such that $A = \{0, g, \ldots, (k-1)g\}$ and $B_r = \{n - (r - 1), \ldots, n - 1\} \cup \{1, \ldots, r - 1\}$. Then $k - 1 \leq 2|A \setminus B_r|$.*

Proof. The statement is equivalent to Lemma 3.4, which is proven below. □

We begin with the basic Lemma 3.2, which is about the maximum and minimum size of the intersection between an interval L and arithmetic progression $A = \{0, g, \ldots, (k-1)g\}$ wherein both L and A are subsets of $\mathbb{Z}$. We use Lemma 3.2, which is a result about an AP intersecting an interval in $\mathbb{Z}$, to prove Lemma 3.3, which is a result about an AP intersecting intervals in $\mathbb{Z}_n$.

Lemma 3.2 (AP Intersecting Interval in $\mathbb{Z}$)**.** *Let $a, b, k, \ell \in \mathbb{Z}$ and $A, L \subseteq \mathbb{Z}$ such that $A = \{a, a + g, \ldots, a + (k - 1)g\}$ and $L = \{b, b + 1, \ldots, b + (\ell - 1)\}$. Then $\lfloor \frac{\ell}{g} \rfloor \leq |A \cap L| \leq \lceil \frac{\ell}{g} \rceil$.*

Proof. Suppose without loss of generality that $a = 0$, which means that $A = \{0, g, 2g, \ldots, (k - 1)g\}$.

For the upper bound, note that since all elements in $A \cap L$ must be nonnegative multiples of g, it suffices, in the maximally intersecting case, to consider both $\min(L)$ and $\max(L)$ to be multiples of g. In particular, we may assume that $b = 0$ and so $\min(L) = 0$ and $\max(L) = (x - 1)g$ for some integer $x \in [k]$, which means that $\ell = (x-1)g+1$. Thus we have $|A \cap L| \leq (x-1)+1$, where at most $x-1$ positive multiples of g and 0 are all in L. By the division algorithm (Lemma 2.1), for some integer $r \in \{1, \ldots, g-1\}$, we have $(x-1)g+1 = \lfloor \frac{(x-1)g+1}{g} \rfloor \cdot g + r = \lfloor \frac{\ell}{g} \rfloor g + r$. But since $r \neq 0$, we have $(x-1)g+1 = \lfloor \frac{\ell}{g} \rfloor g+r \Leftrightarrow x-1+1/g = \lfloor \frac{\ell}{g} \rfloor + r/g \Leftrightarrow x = \lfloor \frac{\ell}{g} \rfloor + 1 + \frac{r-1}{g} = \lceil \frac{\ell}{g} \rceil + \frac{r-1}{g}$. Since x is an integer and $\frac{r-1}{g} < 1$, it follows that $|A \cap L| \le x = \lceil \frac{\ell}{g} \rceil$.

Now we show the lower bound. In the minimally intersecting case, we may assume that $\min(L) = 1$, and $\max(L) = (x-1)g - 1$ for some $x \in [k]$. Thus $\ell = (x-1)g - 1$ and $|A \cap L| \ge (x-2)$ where at least $x-2$ of the multiples of g are in L. So we have $\frac{\ell}{g} = (x-1) - \frac{1}{g}$. Since g does not divide ℓ, it follows that $\lfloor \frac{\ell}{g} \rfloor < \frac{\ell}{g} = x - 1 - \frac{1}{g}$ because $\lfloor \frac{\ell}{g} \rfloor$ is an integer, and so $\lfloor \frac{\ell}{g} \rfloor \le x - 2 \le |A \cap L|$. □

Let $A \subset \mathbb{Z}_n$ where $A = AP(g,k,n)$. We define a function f, which we use to show a lower bound on the number of elements in A that do not intersect with B_r minus the number that do. Define $f(\beta, g, r) := \lfloor \frac{\beta r - 2r + 1}{g} \rfloor - \lceil \frac{2r-1}{g} \rceil$.

In Lemma 3.3, we partition the elements of $A \setminus \{0\}$ into what we call *passes around* $\mathbb{Z}_n$. For an integer $i \ge 1$, we say that the *i-th pass of A around $\mathbb{Z}_n$*, call it P_i, is the subset of A such that for every $x \in P_i$, there exists a $q \in \mathbb{Z}$ where $r \le q < n + r$, satisfying $x = in + q$. We define P_0 to be the subset of A whose elements are all less than r. The integer parameter p is used to denote the *final pass* whereby P_p contains the element $g(k-1)$. Observe that $p = \lceil \frac{g(k-1)-(r-1)}{n} \rceil$ because $\lceil \frac{g(k-1)}{n} \rceil$ is the number of times A crosses 1. We say that p is the *number of passes of A around* $\mathbb{Z}_n$, or the *number of times A crosses* r, where we say that A *crosses* r when there are consecutive multiples of g, jg and $(j+1)g$, and integers $q_j, q_{j+1} \in \{r, \ldots, n+r-1\}$ such that $jg = in + q_j$ and $(j+1)g = (i+1)n + q_{j+1}$.

Lemma 3.3 (AP Intersecting Intervals in $\mathbb{Z}_n$)**.** *Let $A \subset \mathbb{Z}_n$ where $A = AP(g,k,n)$ and let $\beta = n/r$. Then*

$$|A \setminus B_r| - |A \cap B_r| \ge (p-1) \cdot f(\beta, g, r) - \left\lfloor \frac{r-1}{g} \right\rfloor.$$

Proof. We partition $A \setminus \{0\}$ into passes around $\mathbb{Z}_n$, which are as follows:

$$P_0 = \{ig : 0 < i \le \lfloor (r-1)/g \rfloor\},$$
$$P_j = \left\{ ig : \left\lfloor \frac{(j-1)n + r - 1}{g} \right\rfloor < i \le \left\lfloor \frac{jn + r - 1}{g} \right\rfloor \right\} \; \forall j \in [1, p-1],$$
$$P_p = \left\{ ig : \left\lfloor \frac{(p-1)n + r - 1}{g} \right\rfloor < i \le k - 1 \right\}.$$

In the case when $j \in [1, p-1]$, we can apply Lemma 3.2 as follows. Since both $B_r - r$ and $(\mathbb{Z}_n \setminus (B_r - r))$ are contained in $[\min(P_j - r), \max(P_j - r)]$, we have that $|P_j \cap B_r| \le \lceil \frac{2r-1}{g} \rceil$ and $|P_j \setminus B_r| \ge \lfloor \frac{n-(2r-1)}{g} \rfloor = \left\lfloor \frac{(\beta - 2 + \frac{1}{r})r}{g} \right\rfloor$. Therefore we have that $|P_j \setminus B_r| - |P_j \cap B_r| \ge f(\beta, g, r)$.

Note that $P_0 \subset B$ and $|P_0| = \lfloor (r-1)/g \rfloor$, so again by Lemma 3.2, $|P_0 \setminus B_r| - |P_0 \cap B_r| = -\lfloor (r-1)/g \rfloor$.

For P_p, in the case that $g(k-1) \pmod{n} \in B$, it follows by Lemma 3.2 that $|P_p \cap B_r| \leq \lceil \frac{2r-1}{g} \rceil$, and so $|P_p \setminus B_r| - |P_p \cap B_r| \geq f(\beta, g, r)$. However, if $g(k-1) \pmod{n} \in \mathbb{Z}_n \setminus B_r$, then $|P_p \cap B_r| = 0$. But $|P_p \setminus B_r|$ may not be at least $\left\lfloor \frac{(\beta - 2 + \frac{1}{r})r}{g} \right\rfloor$, which means we can only say in this case that $|P_p \setminus B_r| - |P_p \cap B_r| \geq 0$. So in the arbitrary case, we can assume only that $|P_p \setminus B_r| - |P_p \cap B_r| \geq 0$.

Altogether we have

$$\begin{aligned} |A \setminus B_r| - |A \cap B_r| &= \sum_{j=0}^{p} |P_j \setminus B_r| - |P_j \cap B_r| \\ &\geq (p-1) \cdot f(\beta, g, r) - \lfloor (r-1)/g \rfloor. \end{aligned}$$

□

Lemma 3.4. *Let $A \subset \mathbb{Z}_n$ where $A = AP(g,k,n)$. If $\beta \geq 6$ and $\frac{n}{6(k-1)} < g < n - 2r + 1$, then $k - 1 \leq 2|A \setminus B_r|$.*

Proof. Suppose for a contradiction that $k-1 > 2|A \setminus B_r|$. Note $k-1 = |A \setminus B_r| + |A \cap B_r|$. By Lemma 3.3 we have $|A \setminus B_r| - |A \cap B_r| = 2|A \setminus B_r| - (k-1) \geq (p-1) \cdot f(\beta, g, r) - \lfloor \frac{r-1}{g} \rfloor$.

Applying the hypothesis that $2 < \frac{k-1}{|A \setminus B_r|}$ to the LHS of the inequality and the definition of f to the RHS gives

$$0 > (p-1) \left\lfloor \frac{(\beta-2)r+1}{g} \right\rfloor - (p-1) \left\lceil \frac{2r-1}{g} \right\rceil - \left\lfloor \frac{r-1}{g} \right\rfloor. \tag{3.1}$$

Recall $p = \lceil \frac{g(k-1)-(r-1)}{n} \rceil$. By hypothesis, $g(k-1) > \frac{n}{6}$ and $r \leq \frac{n}{6}$, so $p \geq 1$.

Suppose $p \geq 2$. Recall by assumption that $g < n - 2r + 1$, and also that $\beta \geq 6$, therefore we can divide both sides of (3.1) by $\left\lfloor \frac{(\beta-2)r+1}{g} \right\rfloor$. Similarly, since $p \geq 2$, we can divide by $p - 1$. Thus we have

$$0 > 1 - \frac{\lceil \frac{2r-1}{g} \rceil}{\lfloor \frac{\beta r - (2r-1)}{g} \rfloor} - \frac{\lfloor \frac{r-1}{g} \rfloor}{(p-1) \lfloor \frac{\beta r - (2r-1)}{g} \rfloor}. \tag{3.2}$$

When $g \geq r$, (3.2) implies that $1 \leq \frac{2}{\lfloor \beta/2 - 1 \rfloor}$, which is at most 1 when $\beta \geq 6$. Now when $\frac{n}{6(k-1)} < g < r$, observe that if there is some number $x \geq 2$, then $\frac{\lceil x \rceil}{\lfloor 2x \rfloor} \leq 0.75$. Thus as $\beta \geq 6$, we have

$$0 > 1 - \frac{\lceil \frac{2r-1}{g} \rceil}{\lfloor \frac{4r+1}{g} \rfloor} - \frac{\lfloor \frac{r}{g} \rfloor}{(p-1) \lfloor \frac{4r+1}{g} \rfloor} \geq 1 - 0.75 - 0.25,$$

a contradiction. Therefore we have that $k - 1 \leq 2|A \setminus B_r|$.

Suppose $p = 1$. Then $n - (r-1) \leq g(k-1) \leq n + (r-1)$. Observe that when $\beta \geq 6$ and (similarly as above) $k - 1 > 2|A \setminus B_r|$, we have

$$0 > |A \setminus B_r| - |A \cap B_r| \geq f(6, g, r) - \left\lfloor \frac{r-1}{g} \right\rfloor = \left\lfloor \frac{4r-1}{g} \right\rfloor - \left\lceil \frac{2r-1}{g} \right\rceil - \left\lfloor \frac{r-1}{g} \right\rfloor.$$

Here we suppose that $g(k-1)$ is near $n+(r-1)$, so p is indeed 1 and not 2; also, $|A \cap B_r|$ is maximized. Suppose that $g \geq 4r$. Then the following statements hold: we have (1) $|P_0 \cap B_r| = 0$, (2) $|P_1 \cap B_r| \leq 1$, and (3) since $g \leq \lfloor n/2 \rfloor$ and $\beta \geq 6$, $|P_1 \setminus B_r| \geq 1$. Thus in fact $|A \setminus B_r| - |A \cap B_r| \geq 0 \Leftrightarrow 2|A \setminus B_r| - (k-1) \geq 0$ as desired. Now suppose $g < 4r$. Then $\lfloor \frac{4r-1}{g} \rfloor > 0$ and so by similar reasoning as when $p \geq 2$, we can divide both sides of the following inequality by $\lfloor \frac{4r-1}{g} \rfloor$:

$$0 > \left\lfloor \frac{4r-1}{g} \right\rfloor - \left\lceil \frac{2r-1}{g} \right\rceil - \left\lfloor \frac{r-1}{g} \right\rfloor,$$

and similarly get the desired contradiction for the two cases when $g \geq r$ and $g < r$. □

3.2.2 Erdős-Deep Pairs Require $\beta < 6$

Lemmas 2.11 and 3.5 show that an Erdős-deep pair $\mathcal{F}$ must satisfy the conditions of Proposition 3.1 when $\frac{n}{k_1} \geq 6$ where r, g, and k, are substituted for k_1, g_2, and k_2, respectively. That is, these lemmas show that $\frac{n}{6(k_2-1)} \leq g_2 < n - 2k_1 + 1$.

Lemma 3.5 (Upper Bound on g_2)**.** *If $\mathcal{F} = \{A_1, A_2\}$ is an Erdős-deep pair, then $g_2 < 2k_1$ unless $k_1 = k_2 = 3$.*

Proof. First suppose that $\frac{n}{k_1} < 4$, which is equivalent to $\frac{n}{2} < 2k_1$. Then since $g_2 \leq \lfloor n/2 \rfloor \leq n/2$, this implies that $g_2 < 2k_1$. So we assume that $\beta \geq 4$. Let B_{k_1} be the ball with radius k_1 centred at 0. Then since $\mathcal{F}$ is Erdős-deep, $|A_2 \setminus B_{k_1}| = t$ and $|A_2 \cap B_{k_1}| = k_2 - 1 - t$.

Suppose for a contradiction that $g_2 \geq 2k_1$. As in the proof of Lemma 3.2, partition A_2 into passes $P_0, \ldots, P_p$ where we set $r := k_1$ and $g := g_2$. Since $g_2 \geq 2k_1$, $|P_0 \cap B_{k_1}| = 0$ and $|P_j \cap B_{k_1}| \leq 1$ for all $j \in [1, p]$. But since $\beta \geq 4$ and $g_2 \leq \lfloor n/2 \rfloor$, $|P_j \setminus B_{k_1}| \geq 1$; to see this, note that even when there is some $x \in [k_2 - 2]$ such that $g_2 x \equiv k_1 - 1 \pmod{n}$, we have $g_2(x+1) \equiv k_1 - 1 + g_2 \notin B_{k_1}$ because when g_2 is maximized at $g_2 = \lfloor n/2 \rfloor$, $n - (k_1 - 1) - (k_1 - 1 + \lfloor n/2 \rfloor) \geq n/2 - 2(k_1 - 1) \geq 2k_1 - 2(k_1 - 1) > 0$.

Thus $|A_2 \cap B_{k_1}| \leq p \leq |A_2 \setminus B_{k_1}| \Rightarrow k_2 - 1 - t \leq t$, which by Lemma 2.7, can only be the case when $k_1 = k_2 = 3$. □

Proposition 3.6 is the main result of this section, and it shows that when $k_1 \geq k_2 > 3$, if $\mathcal{F} = \{AP(1, k_1, n), AP(g_2, k_2, n)\}$ is Erdős-deep, then $n < 6k_1$.

Proposition 3.6. *There does not exist an Erdős-deep pair when $\frac{n}{k_1} \geq 6$ unless $k_1 = k_2 = 3$.*

Proof. Suppose $\mathcal{F} = \{A_1, A_2\}$ is Erdős-deep, and note that $k_1 \leq \frac{n}{6}$ by assumption. Then by Lemmas 2.11 and 3.5 , we have that $\frac{n}{6(k_2-1)} \leq n - k_1 < g_2 < 2k_1 < n - 2k_1 + 1$. Let B_{k_1} be the ball of radius k_1 centred at 0, then by Lemma 3.4, $k_2 - 1 \leq 2|A \setminus B_{k_1}|$, and since $t = |A \setminus B_{k_1}|$, it follows that there cannot exist an Erdős-deep pair when $k_2 - 1 > 2t$.

Observe that we require $k_2 - 1 - t \geq t$ because by Lemma 2.7, $k_2 - 1 - t \geq \frac{4}{7}(k_2 - 1)$ and $t \leq \frac{3}{7}(k_2 - 1)$. Thus the only case when an Erdős-pair can exist is when $k_2 - 1 = 2t$.

Suppose $k_2 - 1 = 2(k - k_1)$. Then by Observation 3, we have that $(2(k - k_1) + 1)2(k - k_1) + k_1(k_1 - 1) = k(k - 1)$. Expanding and simplifying gives the quadratic $5k_1^2 - (8k + 3)k_1 + (3k^2 + 3k) = 0$, which has solutions $k_1 \in \{k, \frac{3(k+1)}{5}\}$. If $k_1 = k$, then since $k_2 > 0$, $\binom{k_1}{2} + \binom{k_2}{2} = \binom{k}{2}$ has no solution, a contradiction. So we have $5k_1 = 3k + 3 \Leftrightarrow \frac{2k_1 - 3}{3} = k - k_1$, but recall that $\frac{k_2 - 1}{2} = k - k_1$, so we have $4k_1 - 6 = 3k_2 - 3 \Leftrightarrow 4k_1 = 3k_2 + 3$. Since $k_2 \leq k_1$, the only solution to this equation is $k_1 = k_2 = 3$. □

3.3 Condition 2: Maximum Multiplicity

Recall the parameters $t := k - k_1$ and $\beta := \frac{n}{k_1}$. We showed in Section 3.2 that an Erdős-deep pair can only exist if $\beta \in (2, 6)$ unless $k_1 = k_2 = 3$. The main result in this section is that when t is large, there must exist some shared distance with multiplicity in $\Delta(\mathcal{F})$ at least k. This contradicts Condition 2.

3.3.1 Erdős-Deep Pairs Require $n \geq \frac{(k_2 - t)^2}{4}$

Proposition 3.7 applies the pigeonhole principle to show that a shared distance has multiplicity at least k when n is small relative to the number of shared distinct distances.

Proposition 3.7. *If $n < \frac{(k_2 - t)^2}{4}$, then there does not exist an Erdős-deep pair.*

Proof. Set $w = \frac{k_2 - t}{2}$. Thus our assumed inequality is equivalent to $w^2 > n$. We claim that one of the w most frequent shared distinct distances, call it d, is at most w, implying that its total multiplicity is at least $(k_1 - w) + (k_2 - w) = k$, which contradicts the Erdős-deep family condition that the total multiplicity of each distance is at most $k - 1$.

We apply the pigeonhole principle. Partition $\mathbb{Z}_n$ into $\lfloor w \rfloor$ intervals of size at most $\lceil w \rceil$. The pigeonholes are the collection of half-open intervals

$$H = \{I_i \subset \mathbb{Z}_n : I_i = [i\lceil w \rceil, (i + 1)\lceil w \rceil), i \in [0, \lfloor w \rfloor - 1]\}$$

and the pigeons are $P = \{p_j : j \in [0, \lfloor w \rfloor]\}$ where p_j is the least residue of $jg_2 \pmod{n}$.

Note P is well-defined because $\lfloor w \rfloor + 1 < k_2 - 1$, and A_2 is an AP satisfying $k_2 \leq \frac{n}{2\gcd(g_2,n)} < \frac{n}{\gcd(g_2,n)}$, all pigeons are distinct. Since $|P| \geq |H| + 1$, at least two pigeons are in the same pigeonhole; that is, there are $\lfloor w \rfloor$ intervals that partition $\mathbb{Z}_n$, each of size at most $\lceil w \rceil$, and there are $\lfloor w \rfloor + 1$ values of the form $jg_2 \pmod{n}$ with multiple j at most w in A_2, implying that at least two of these values, say p_x and p_y, are in the same interval $I \in H$.

Suppose without loss of generality that $x > y$. Then there exists a distance

$$d = |p_x - p_y|_n \leq w$$

since $p_x, p_y \in I$. Note that even though $|I| \leq \lceil w \rceil$, the maximum distance between a pair of elements in I is at most $\lceil w \rceil - 1$.

Now we need to show that d is a shared distance. Recall that the distinct distances in $\Delta(A_1)$ are exactly $\{1, 2, \ldots, k_1 - 1\}$. Note that since $k_1 + k_2 - k \leq k_2 \leq k_1$, it follows that

$$d \leq w \leq (k_1 + k_2 - k)/2 < k_1 - 1,$$

so d is a shared distance. Recall also that both $x, y \leq w$, thus $x - y \leq w$. Altogether this implies that d has total multiplicity

$$\begin{aligned}(k_1 - d) + (k_2 - (x - y)) &\geq (k_1 - w) + (k_2 - w) \\ &\geq \left(k_1 - \frac{k_1 + k_2 - k}{2}\right) + \left(k_2 - \frac{k_1 + k_2 - k}{2}\right) = k,\end{aligned}$$

which contradicts the requirement that each distance in an Erdős-deep pair has total multiplicity at most $k - 1$. □

3.3.2 Erdős-Deep Pairs Require $t \leq 5\beta - \frac{3}{4}$

Proposition 3.8 shows that when n is large, t is bounded above by a function of β only. Recall from Section 3.2 that we showed $\beta < 6$, so this means that t is bounded above by a constant. In the next section, we show that $\frac{(k_2 - t)^2}{4}$ is bounded above linearly by k_2. So, the upper bound on t from the following Proposition 3.8 allows us to upper bound k_2 later on.

Proposition 3.8. *If $n \geq (k_2 - t)^2/4$, then $t \leq 5\beta - \frac{3}{4}$.*

Proof. Suppose for contradiction that $t > 5\beta - \frac{3}{4}$. Using Lemma 2.2, we have

$$n = \beta k_1 = \frac{\beta k_2(k_2 - 1) - \beta t(t - 1)}{2t}.$$

So, from the assumed bound on n, we obtain the inequality

$$2\beta k_2(k_2 - 1) - 2\beta t(t - 1) - t(k_2 - t)^2 \geq 0. \tag{3.3}$$

The right hand side is a quadratic in k_2 with leading coefficient $2\beta - t < 0$ and zeros at $k_2 = t$ and $k_2 = u$, where $u = \frac{t^2+2\beta(t-1)}{t-2\beta} > t$.

By Lemma 2.7, we have $t \leq \frac{3}{7}(k_2 - 1)$. Also, (3.3) implies $k_2 \leq u$, so $t \leq \frac{3}{7}(u-1)$. Simplifying this, we get $t - 2\beta \leq \frac{3}{7}(t + 2\beta - 1)$, or equivalently $t \leq 5\beta - \frac{3}{4}$. $\square$

3.4 Condition 3: Distinct Multiplicity

In this section, we show that when $n > 18k_2 + 36$, there must be at least two distinct distances with the same multiplicity, which contradicts Condition 3. We begin by proving Lemma 3.9, which applies to general modular APs. Then we apply this lemma in the proof of Proposition 3.12 to show that for an Erdős-deep pair to exist, we require $n \leq 18k_2 + 36$.

3.4.1 Distinct Gaps in Modular APs

Lemma 3.9 states that the number of distinct gaps between the distinct distances in an AP (with sufficiently large modulus) is at most two thirds the size of the AP. This idea of upper bounding the number of gaps is related to the Steinhaus conjecture, originally proved by Sós in 1957 [13], which states that the number of distinct consecutive gaps in an arithmetic progression in $\mathbb{Z}_n$ is at most 3. What is different here is that Lemma 3.9 provides an upper bound on the total number of distinct gaps (not just the consecutive ones) between all distinct distances in the AP.

Lemma 3.9. *Define the set* $D(g,k,n) = \{|xg|_n - |yg|_n : x, y \in \{1, \ldots, k-1\}\}$. *If* $n > 18k$, *then* $|D(g,k,n) \cap \{1, \ldots, k-1\}| \leq \frac{2k}{3}$.

Proof. There are two possibilities for a positive distance in $D(g,k,n)$. Either $|gx|_n - |gy|_n = |g(x-y)|_n$ or $|g(x+y)|_n$. Observe that both distances are contained within the set $\{|g|_n, |2g|_n, \ldots, |(2k-3)g|_n\}$.

Define a $\{0,1\}$-sequence $\mathbf{a} = (a_1, a_2, \ldots, a_{2k-3})$ by

$$a_i = \begin{cases} 1 & \text{if } |gi|_n < k; \\ 0 & \text{otherwise.} \end{cases}$$

The size of the desired intersection is certainly at most the number of ones in $\mathbf{a}$. If the conclusion were not true, there would necessarily exist a subword '11' or '101' in $\mathbf{a}$. But this implies either $g < 2k$ or $g \geq \lfloor n/2 \rfloor - (k-1)$, respectively. This can be seen as follows. Let B_k be the ball with radius k centred at 0 in $\mathbb{Z}_n$. Then the subword 11 can only occur when two consecutive multiples of g intersect B_k. The subword 101 occurs only when there are three consecutive multiples of g, namely $ig, (i+1)g$, and $(i+2)g$ such that $ig, (i+2)g \in B_k$ and $(i+1)g \notin B_k$. If $2k \leq g \leq \lfloor n/2 \rfloor - k$, then every consecutive triple of bits in **a** has at most one 1 in it, which implies that the number of 1s is at most $\frac{2k}{3}$.

We consider the following three cases: (1) $g < 2k$; and (2) $\lfloor n/2 \rfloor - (k-1) \le g < \lfloor n/2 \rfloor$; and (3) $g = \lfloor n/2 \rfloor$. We show that, in each case, $|D(g,k,n) \cap \{1,\dots,k-1\}| \le \frac{2k}{3}$.

Case 1: Suppose $g < 2k$. Let $A = \{0, g, \dots, 2(k-1)g\}$.

Following the proof of Lemma 3.3, partition $A \backslash \{0\}$ into passes around $\mathbb{Z}_n$ as follows:

$$P_0 = \{ig : 0 < i \le \lfloor (k-1)/g \rfloor\},$$
$$P_j = \left\{ ig : \left\lfloor \frac{(j-1)n + k - 1}{g} \right\rfloor < i \le \left\lfloor \frac{jn + k - 1}{g} \right\rfloor \right\} \ \forall j \in [1, p-1],$$
$$P_p = \left\{ ig : \left\lfloor \frac{(p-1)n + k - 1}{g} \right\rfloor < i \le 2(k-1) - 1 \right\},$$

where p is the number of times A crosses k. Let B_k be the ball of radius k about 0. If $p = 0$, then $|A \cap B_k| = |P_0 \cap B_k| \le \lfloor (k-1)/g \rfloor \le k/2 < \frac{2k}{3}$. Suppose $p \ge 1$. Then for every $j \in [1, p-1]$, by Lemma 3.2, $|P_j \cap B_k| \le \lceil \frac{2k-1}{g} \rceil \le \lceil \frac{2k}{g} \rceil$ and $|P_j \setminus B_k| \ge \lfloor \frac{(\beta-2)k+1}{g} \rfloor \ge \lfloor \frac{16k+1}{g} \rfloor \ge \lfloor \frac{16k}{g} \rfloor$. Since $\lfloor 16x \rfloor \ge 4\lceil 2x \rceil$ for all $x \ge 1/2$, it follows by the hypothesis $\frac{k}{g} > \frac{1}{2}$ that $|P_j \cap B_k| \le \frac{|P_j|}{5}$. When $j = p$, observe that if $g \cdot 2(k-1) \pmod{n} \in B_k$, then as before, $|P_p \cap B_k| \le \frac{|P_p|}{5}$; but if $g \cdot 2(k-1) \pmod{n} \notin B_k$, then $|P_p \cap B_k| = 0$ and $|P_p \backslash B_k| \ge 0$. Since $|P_0 \cap B_k| \le \lfloor \frac{k-1}{g} \rfloor \le \max_{j \in [1,p]} |P_j \cap B_k| \le \frac{|P_j|}{5}$, it follows that

$$|A \cap B_k| = \sum_{j=0}^{p} |P_j \cap B_k| \le \sum_{j=0}^{p} \frac{|P_j|}{5} = \frac{2(k-1)}{5} < \frac{2k}{3}.$$

Case 2: Suppose $g \in \{\lfloor n/2 \rfloor - (k-1), \dots, \lfloor n/2 \rfloor - 1\}$. Let $g = \lfloor n/2 \rfloor - \ell$ where $\ell \in \{1, \dots, k-1\}$. Note that the gap between $|ig|_n$ and $|(i+2)g|_n$ is at most $2\ell + 1 < 2k$. Let $g' = n - 2g$, observe that $g' < 2k$, and consider the AP $A' = \{0, g', \dots, (k-1)g'\}$. As in Case 1, partition A' into passes around $\mathbb{Z}_n$ as follows:

$$P'_0 = \{ig' : 0 < i \le \lfloor (k-1)/g' \rfloor\},$$
$$P'_j = \left\{ ig' : \left\lfloor \frac{(j-1)n + k - 1}{g'} \right\rfloor < i \le \left\lfloor \frac{jn + k - 1}{g'} \right\rfloor \right\} \ \forall j \in [1, p'-1],$$
$$P'_{p'} = \left\{ ig' : \left\lfloor \frac{(p'-1)n + k - 1}{g'} \right\rfloor < i \le k - 1 \right\}.$$

Then $|A' \cap B_k| = \sum_{j=0}^{p} |P'_j \cap B_k|$. By identical reasoning to Case 1, $|P'_j \cap B_k| \le \frac{|P'_j|}{5}$, and since $\sum_{j=0}^{p} |P'_j| = k - 1$, we have that $|A' \cap B_k| \le \frac{k-1}{5}$. Let $A^* = \{-g, -g + g', \dots, -g + (k-1)g'\}$. Notice that $-A = A' \cup A^*$ and since $B_k = -B_k$, $|A \cap B_k| \le$

$\frac{2k}{3} \Leftrightarrow |-A \cap B_k| \leq \frac{2k}{3}$. Since A^* is a translate of a proper subset of A', we can partition A^* into sets $P_0^*, \ldots, P_{p^*}^*$ where $p^* \leq p'$ and for every $j \in [0, p^*]$, $P_j^* = P_j - g$; so the same bounds on $|P_j' \cap B_k|$ apply to $|P_j^* \cap B_k|$. Thus $|A^* \cap B_k| \leq \frac{k-1}{5}$ as well. Altogether, we have that $|A \cap B_k| \leq |A' \cap B_k| + |A^* \cap B_k| \leq \frac{2(k-1)}{5} < \frac{2k}{3}$, as desired.

Case 3: Suppose $g = \lfloor n/2 \rfloor$. Let $g' = 2g$. Then $g' = -1$ and A can be partitioned into $A' = \{0, -1, \ldots, -(k-1)\}$ and $A^* = \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, \ldots, \lfloor n/2 \rfloor - (k-1)\}$. Since $n > 18k$, $(A' \cup A^*) \cap \{1, \ldots, k-1\} = \emptyset$. This means that distances of the form $|g(x+y)|_n$ cannot occur and so all distance gaps $|xg|_n - |yg|_n$ intersecting $\{1, \ldots, k-1\}$ have the form $|g(x-y)|_n$. Thus $|D(g,k,n) \cap \{1, \ldots, k-1\}| \leq |\{g, 2g, \ldots, (k-1)g\} \cap B_k|$. Since $|\{g, 2g, \ldots, (k-1)g\} \cap B_k| = |\{g, 2g, \ldots, (k-1)g\} \cap A'|$ and $|\{g, 2g, \ldots, (k-1)g\} \cap A'| \leq \lfloor \frac{k-1}{2} \rfloor$, it follows that $|D(g,k,n) \cap \{1, \ldots, k-1\}| \leq \lfloor \frac{k-1}{2} \rfloor < \frac{2k}{3}$. □

Remark 3.10. The upper bound of $\frac{2k}{3}$ cannot be lowered because when g is near $n/3$, essentially every third bit in **a** is a 1. So, the number of 1s in **a** is very close to, and sometimes exactly $\frac{2k}{3}$.

Remark 3.11. The hypothesis of $n > 18k$ is not the best possible. We suspect that "18" can be reduced to "10", and possibly lower. For the purposes of Proving Proposition 3.12 below, the hypothesis of $n > 18k$ is sufficient.

3.4.2 Erdős-Deep Pairs Require $n \leq 18k_2 + 36$

We now apply Lemma 3.9 to show $n \leq 18k_2 + 36$ must hold for an Erdős-deep pair to exist.

Proposition 3.12. *If $n > 18k_2 + 36$, then there does not exist an Erdős-deep pair.*

Proof. First we show that $k_2 > 6t + 2$. Suppose for a contradiction that $k_2 \leq 6t + 2$. Since $n = \beta k_1$, we have $\beta k_1 > 18k_2 + 36$. Using Lemma 2.2, we know that $k_1 = \frac{k_2(k_2-1)}{2t} - \frac{t-1}{2}$ and so

$$\begin{aligned} \beta k_1 = \beta \frac{k_2(k_2-1)}{2t} - \beta \frac{t-1}{2} > 18k_2 + 36 \\ \Leftrightarrow k_2(k_2-1) > t\Big(\frac{36k_2+72}{\beta}\Big) + t(t-1). \end{aligned}$$

Then applying $\beta < 6$ from Proposition 3.6 and our assumption $t \geq \frac{k_2-2}{6}$ gives:

$$\begin{aligned} k_2(k_2-1) &> \Big(\frac{k_2-2}{6}\Big)\frac{36k_2+72}{\beta} + t(t-1) \\ &= (k_2-2)(k_2+2) + t(t-1), \end{aligned}$$

which is a contradiction when $k_2 \geq 4$ since $t \geq 1$. Note that $k_2 = 3$ implies $k_1 = 3$ by Observation 3 and recall we have classified Erdős-deep pairs for this case in Lemma 2.10.

Since $n > 18k_2 + 36$, we can apply Lemma 3.9 to get that $|D(g_2, k_2, n) \cap \{1, \ldots, k_2 - 1\}| \leq \frac{2k_2}{3}$.

We show by a combinatorial argument that in fact $k_2 - 1 - 2t = |D(g_2, k_2, n) \cap \{1, \ldots, k_2 - 1\}|$. This implies that $k_2 - 1 \leq \frac{2k_2}{3} + 2t \Leftrightarrow \frac{k_2}{3} \leq 2t + 1 \Leftrightarrow k_2 \leq 6t + 2$.

Consider the set of shared distinct distances in $\Delta(\mathcal{F})$. Let $|xg_2|_n$ be a shared distinct distance, then its total multiplicity is given by $(k_1 - |xg_2|_n) + (k_2 - x)$. Exactly t of these shared distinct distances can have total multiplicity at least k_1, and since there are t unshared distinct distances, exactly $k_2 - 1 - 2t = k_2 - (2t+1)$ of distinct distances are shared with total multiplicity $k_1 - z$ where $z \geq 1$.

Recall that the multiplicities in $\Delta(\mathcal{F})$ are distinct. Since z already has multiplicity $k_1 - z$ in $\Delta(A_1)$, if there is some shared distance $|xg_2|_n$ with total multiplicity $k_1 - z$, then there must exist some distance $|yg_2|_n = z$ in $\Delta(A_2)$ so that the multiplicity of z in $\Delta(\mathcal{F})$ exceeds $k_1 - z$, where $1 \leq y < k_2$. Therefore we have the following equation:

$$\begin{aligned} k_1 - |yg_2|_n &= (k_1 - |xg_2|_n) + (k_2 - x) \\ \Leftrightarrow |xg_2|_n - |yg_2|_n &= k_2 - x. \end{aligned}$$

So, there are $k_2 - (2t+1)$ shared distances with total multiplicity at most $k_1 - 1$ of the form $k_2 - x$. That is, there are $k_2 - (2t + 1)$ distinct values of the form $|xg_2|_n - |yg_2|_n$, which are at most $k_2 - 1$. Thus $k_2 - 1 - 2t = |D(g_2, k_2, n) \cap \{1, \ldots, k_2 - 1\}|$, which leads to the contradiction $6t + 2 < k_2 \leq 6t + 2$. □

3.5 Main Result and Computer Search

We present and discuss the computational component of our classification of Erdős-deep pairs and prove Theorem 3.14. We begin with our proof of Theorem 3.14, and for the remainder of the section, we discuss the details of the computational component of the proof. We establish Proposition 3.13, which states the finite parameter space to be searched. We present and remark on our search method. We tabulate the tuples from our search that evade the theoretical arguments in the previous sections.

3.5.1 Main Result

We have prove Theorem 3.14 using theoretical arguments and a computer search. In Proposition 2.10, we addressed the case when $k_1 = k_2 = 3$. In Sections 3.2, 3.3, and 3.4, we showed that when $k_1 \geq k_2 > 3$, it is necessary that the tuples (n, k_1, k_2, g_2) satisfy the bounds stated in Proposition 3.13, which defines space of finitely many tuples. Thus to prove Theorem 3.14 in the case $k_1 \geq k_2 > 3$, it is sufficient to search and check each tuple in this finite parameter space implied by Proposition 3.13. We describe the details of this search in the next subsection.

Proposition 3.13 states the finite parameter space to be searched by computer.

Proposition 3.13. *The tuples* (n, k_1, k_2, g_2) *that are not addressed by the arguments in the previous sections are contained within the following region:* $\beta \in [2, 6)$, $\frac{(k_2-t)^2}{4} \leq n \leq 18k_2 + 36$, $t \in [1, 5\beta - \frac{3}{4}]$, $k_2 \in [\frac{7t}{3} + 1, \frac{36}{\beta}(t+1) + \frac{t-1}{2} + 1)$, *and* $g_2 \in (k_2 - 1 - t, 2k_1)$.

Proof. We apply the main lemmas and propositions from this chapter as follows:

- Lemma 2.6 shows that $g_2 > k_2 - 1 - t$;
- Lemma 3.5 shows that $g_2 < 2k_1$;
- Proposition 3.6 shows that $2 \leq \beta < 6$;
- Proposition 3.7 shows that $n \geq \frac{(k_2-t)^2}{4}$;
- Proposition 3.8 shows that $t \leq 5\beta - 3/4$;
- Proposition 3.12 shows that $n \leq 18k_2 + 36$.

Observe that n is bounded above linearly by k_2 and bounded below by a quadratic in k_2. Since t is bounded above by a function of β, which is itself at most 6, there can be only a finite number of viable values of k_2 and t. By Lemma 2.2, this means that there can be only a finite number of solutions to $\binom{k}{2} = \binom{k_1}{2} + \binom{k_2}{2}$, and in particular, k_1 is bounded above. Since k_1 and β are bounded above, n is bounded above. Another way to see that n is bounded above is that since t is bounded above, only some small values of k_2 can satisfy $\frac{(k_2-t)^2}{4} \leq 18k_2 + 36$, and so only finitely many small values of n satisfy $n \leq 18k_2 + 36$. In any case, we have that there are a finite number of tuples (n, k_1, k_2). Note that $k_2 - t \leq g_2 < 2k_1$.

We now calculate the upper bound on k_2. Since $n = \beta k_1$ and by Remark 2.3, $\beta k_1 = \beta(\frac{k_2(k_2-1)}{2t} - \frac{t-1}{2})$, we have:

$$\beta\left(\frac{k_2(k_2-1)}{2t} - \frac{t-1}{2}\right) \leq 18k_2 + 36$$
$$\Leftrightarrow \beta(k_2(k_2-1) - t(t-1)) \leq 36tk_2 + 72t$$
$$\Leftrightarrow k_2 - 1 \leq \frac{36t}{\beta} + \frac{72t}{\beta k_2} + \frac{t(t-1)}{k_2}.$$

Observe that since $k_2 \geq 2t$, we have that $k_2 - 1 \leq \frac{36}{\beta}(t+1) + \frac{t-1}{2}$. □

We now state and prove Theorem 3.14.

Theorem 3.14. *Let $\mathcal{F} = \{A_1, A_2\}$ be a pair of arithmetic progressions in $\mathbb{Z}_n$ such that $3 \leq k_2 \leq k_1 \leq \lfloor \frac{n}{2\gcd(n,g_1)} \rfloor + 1$, $\gcd(n, g_1, g_2) = 1$, and $g_2 \leq \lfloor n/2 \rfloor$. Then $\mathcal{F}$ is an Erdős-deep pair if and only if*

1. *$k_1 = k_2 = 3$ and $g_2 \in \{2g_1, \frac{n-g_1}{2}\}$ for all $n \geq 7$, or*
2. *$(n, k_1, k_2) \in \{(13, 6, 4), (19, 7, 6), (31, 11, 9)\}$.*

Proof. The case when $k_1 = k_2 = 3$ follows by Lemma 2.10. Let $\beta = n/k_1$. For the case when $k_1 \geq k_2 > 3$, we apply Proposition 3.13 to obtain a necessary finite search space of tuples (n, k_1, k_2, g_2) for Erdős-deep pairs $\{AP(1, k_1, n), AP(g_2, k_2, n)\}$. Recall by Proposition 2.9 that without loss of generality, we assume $g_1 = 1$. Then for each of these tuples and for all $k_2 - t \leq g_2 \leq \min(2k_1 - 1, \lfloor n/2 \rfloor)$, we check whether the family $\{AP(1, k_1, n), AP(g_2, k_2, n)\}$ is Erdős-deep using a computer. Exactly the following Erdős-deep pairs $\mathcal{F} = \{A_1, A_2\}$ can be found from such a search:

- $(n, k_1, k_2, g_2) = (13, 6, 4, 3)$: $\{\{0, 1, 2, 3, 4, 5\}, \{0, 3, 6, 9\}\}$,
- $(n, k_1, k_2, g_2) = (19, 7, 6, 4)$: $\{\{0, 1, 2, 3, 4, 5, 6\}, \{0, 4, 8, 12, 16, 1\}\}$,
- $(n, k_1, k_2, g_2) = (31, 11, 9, 13)$: $\{\{0, 1, \ldots, 10\}, \{0, 13, 26, 8, 21, 3, 16, 29, 11\}\}$.

Recall by Lemma 2.5 that for any unit u in $\mathbb{Z}_n$, we may multiply any of these examples of Erdős-deep pairs $\{AP(1, k_1, n), AP(g_2, k_2, n)\}$ by u to obtain another Erdős-deep pair $\{AP(u, k_1, n), AP(g_2 u, k_2, n)\}$. So, when $k_1 \geq k_2 > 3$, $\mathcal{F}$ is Erdős-deep if and only if $(n, k_1, k_2) \in \{(13, 6, 4), (19, 7, 6), (31, 11, 9)\}$.

We provide the details of our search in the following subsection. □

3.5.2 Search

The search method uses a quadruple nested For-loop on the parameters, listed in order of depth, t, k_2, n, and g_2. We need to ensure we have upper bounds on these parameters to guarantee that the search space is finite. Recall k_1 can be determined through Lemma 2.2 using t and k_2, and also that $n \leq 6k_1$ by Proposition 3.6 and $g_2 < 2k_1$ by Lemma 3.5. Since $\beta \leq 6$, $t < 5\beta - \frac{3}{4} < 30$. Observe that since $\beta \geq 2$ we have that $k_2 \leq \frac{36}{\beta}(t+1) + \frac{t-1}{2} + 1 \leq 18(t+1) + t/2 + 1/2 = 18.5(t+1)$. Recall by Lemma 2.10, we may assume $k_2 \geq 4$. We have a lower bound on $k_2 - 1$ of $k_2 - 1 \geq \frac{7t}{3} + 1$ by Lemma 2.7. Recall also by Lemma 2.6 that $g_2 \geq k_2 - t$.

Let $\Delta(X)$ be some distance multiset. Recall that $S(\Delta(X))$ be the set of distinct distances in $\Delta(X)$, and we call this the support of $\Delta(X)$. Recall also that $M(\Delta(X))$ is the set of distinct multiplicities of the distances in $\Delta(X)$.

We now present the search method:

Pseudo-Code

1. For t in $[1, 29]$:

  2. For k_2 in $[\frac{7t}{3}+1, 18.5(t+1)]$:

    3. Calculate integer $k_1 = \frac{k_2(k_2-1)}{2t} - \frac{t-1}{2}$;

    4. Fact: $k_2 \leq k_1$;

    5. Define $A_1 := \{0, 1, \ldots, k_1 - 1\}$;

    6. Define $\Delta(A_1) := \{1^{k_1-1}, 2^{k_1-2}, \ldots, (k_1-1)^1\}$;

    7. For n in $[2k_1, 6k_1)$:

      8. Set $\beta = n/k_1$;

      9. Proposition 3.8: $t < 5\beta - \frac{3}{4}$;

      10. Propositions 3.7 & 3.12: $\frac{(k_2-t)^2}{4} \leq n \leq 18k_2 + 36$;

      11. Proposition 3.13: $k_2 \leq \frac{36}{\beta}(t+1) + \frac{t-1}{2} + 1$;

      12. For g_2 in $[k_2 - t, 2k_1)$:

        13. Check $g_2 \leq \lfloor n/2 \rfloor$;

        14. Define $A_2 := \{0, g_2, \ldots, (k_2-1)g_2\}$;

        15. Define $\Delta(A_2) := \{g_2^{k_2-1}, (2g_2)^{k_2-2}, \ldots, ((k_2-1)g_2)^1\}$;

        16. Define $\mathcal{F} := \{A_1, A_2\}$;

        17. Define $\Delta(\mathcal{F}) := \Delta(A_1) \cup \Delta(A_2)$;

        18. Check max num diffs: $|S(\Delta(\mathcal{F}))| < k$;

        19. Check max mult: $\max(M(\Delta(\mathcal{F}))) < k$;

        20. Check exact num diffs: $|S(\Delta(\mathcal{F}))| = k - 1$;

        21. Check distinct mult: $|M(\Delta(\mathcal{F}))| = k - 1$;

        22. Print($\mathcal{F}$).

Remarks

A key idea for the search method is to leverage the parameterization of $t = k - k_1$ to calculate $k_1 = \frac{k_2(k_2-1)}{2t} - \frac{t(t-1)}{2}$ given t and k_2. Then throughout the nested For-loop over the variables t, k_2, n, g_2, we check the various necessary conditions established throughout this chapter. Finally, after defining $\Delta(\mathcal{F})$, we check the conditions in Steps 18 and 19, which we call the 'box conditions', and they are whether or not $\Delta(\mathcal{F})$ satisfies maximum number of distinct distances and maximum multiplicity. Given these two box conditions, the last two conditions of $|S(\Delta(\mathcal{F}))| = k - 1$ and $|M(\Delta(\mathcal{F}))| = k - 1$ are then sufficient to ensure that $\mathcal{F}$ is an Erdős-deep pair. These final two conditions are checked in Steps 20 and 21, respectively.

If a tuple passes all of the above conditions and is inside the box, then our implementation of the algorithm identifies whether it fails not having enough distinct distances, and/or if it fails having distinct multiplicities.

We list all tuples $(t, k_1, k_2, g_2, n, |S(\Delta(\mathcal{F}))|, |M(\Delta(\mathcal{F}))|)$ that fail the Erdős-deep property but are "inside the box" in the supplemental materials found on the GitHub repository associated with this thesis. Note that the dash '–' for either $|S(\Delta(\mathcal{F}))|$ or $|M(\Delta(\mathcal{F}))|$ indicates that Step 20 or 21 is satisfied, respectively. A tuple permitting an Erdős-deep pair is indicated by the presence of two dashes.

3.5.3 Results of Search

The results of our search for Erdős-deep pairs are located at the GitHub repository that contains the supplementary materials for this thesis. The link to this repository is: `www.github.com/taogaede/MScThesis`.

Refer to the"EDPairSearchResults.txt" file, which reports all tuples (t, k_1, k_2, g_2, n, $|S(\Delta(\mathcal{F}))|$, $|M(\Delta(\mathcal{F}))|$) that pass Step 19 of the search algorithm. Note that if either Steps 20 or 21 are passed, then the $|S(\Delta(\mathcal{F}))|$ or $|M(\Delta(\mathcal{F}))|$ values in the tuple are replaced with '–', respectively. Observe that the only tuples that have two dashes in the final two positions satisfy $(n, k_1, k_2) \in \{(13, 6, 4), (19, 7, 6), (31, 11, 9)\}$.

Figure 3.1 shows the multiplicity histograms of the Erdős-deep pairs found in the search. The tuples (t, k_1, k_2, g_2, n) for the Erdős-deep pairs are, in order from left to right, $(1, 6, 4, 3, 13)$, $(2, 7, 6, 4, 19)$, and $(3, 11, 9, 13, 31)$.

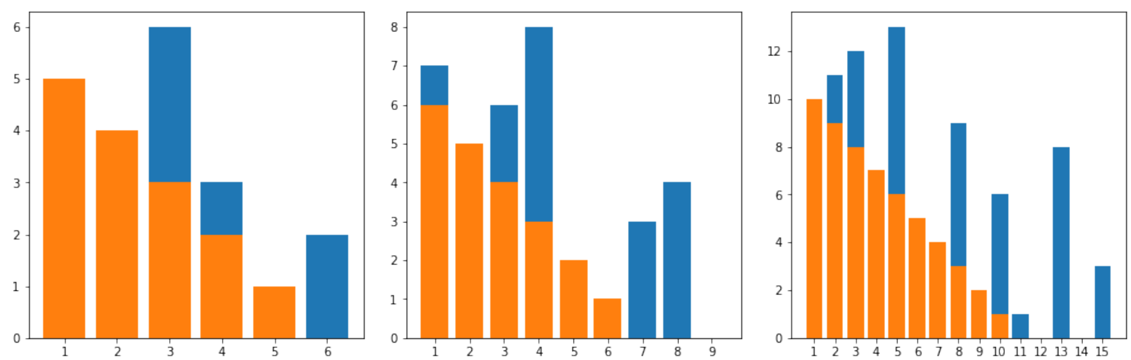


Figure 3.1: This figure shows the multiplicity histograms for notable tuples (t, k_1, k_2, g_2, n) found in the computer search that permit existence of an Erdős-deep pair.

Chapter 4
Larger Erdős-Deep Families of Arithmetic Progressions

Now we consider an Erdős-deep family $\mathcal{F} = \{A_1, \dots, A_s\}$ where $s > 2$ and $A_i = AP(g_i, k_i, n)$ such that $k_i \le \lfloor \frac{n}{2\gcd(g_i,n)} \rfloor$ and $k_1 \ge k_2 \ge \cdots \ge k_s \ge 3$. Our techniques from Chapter 3 do not seem to generalize for larger families so easily. However we present a general construction for square integers s, a conjectured classification for $s = 3$, and a basic upper bound on the size of $\mathcal{F}$.

A key reason why our results from Chapter 3 do not appear to generalize easily for larger s is that we cannot always assume that all generators are coprime with n. It was very convenient that we could assume $A_1 = \{0, 1, \dots, k_1 - 1\}$ when $s = 2$. However, when $s > 2$, there can be a generator that is not a unit while $\gcd(n, g_1, \dots, g_s) = 1$. Another obstruction when $s > 2$ is that since there are many subfamilies of sets in $\mathcal{F}$, it becomes difficult to account for the various combinations of shared distances in $\Delta(\mathcal{F})$. For instance, when $s = 2$, we could easily partition the distinct distances of $\Delta(A_2)$ into two parts: (1) the unshared part with size $t = k - k_1$, and (2) the shared part with size $k_2 - 1 - t$. But when s is larger, one must partition the distinct distances of each constituent distance multiset $\Delta(A_i)$ into many shared parts as well as an unshared part.

4.1 Erdős-Deep Families When s is a Square

In this section, we present a general construction for Erdős-deep families with size the square of an integer. Note that we allow repeated sets in Erdős-deep families. For a positive integer i and set A, we use the notation $i \cdot A$ to denote i repetitions of A.

Let a and h be positive integers. Then the following identity holds:

$$\binom{ah}{2} = \binom{h+1}{2}\binom{a}{2} + \binom{h}{2}\binom{a+1}{2}. \tag{4.1}$$

The next construction shows that the identity (4.1) can be used to construct an

Erdős-deep family.

Proposition 4.1. *Let $h, a \in \mathbb{Z}$ satisfying $a \geq 3$ and $h \geq 2$. Let $D = \{g_1, \ldots, g_h\}$ where for each $i \in \{1, \ldots, h\}$ and $j \in \{1, \ldots, a\}$, all $|jg_i|_n$ are distinct. For $i \in \{1, \ldots, h\}$, let $\mathcal{F}_i$ be the family consisting of i copies of $AP(g_i, a, n)$ together with $h - i$ copies of $AP(g_i, a+1, n)$. Then $\mathcal{F} = \cup_{i=1}^{h} \mathcal{F}_i$ is an Erdős-deep family of $s = h^2$ sets and $\binom{k}{2}$ distances, where $k = ha$.*

Proof. Note that $\binom{h+1}{2} + \binom{h}{2} = \frac{h((h+1)+(h-1))}{2} = \frac{2h^2}{2} = h^2 = s$, so $\mathcal{F}$ has the correct number of sets. Identity 4.1 can be shown by an easy calculation, which we show presently:

$$\begin{aligned}
\binom{h+1}{2}\binom{a}{2} + \binom{h}{2}\binom{a+1}{2} &= \frac{a(a-1)(h+1)(h) + (a+1)a(h)(h-1)}{4} \\
&= \frac{ah((a-1)(h+1) + (a+1)(h-1))}{4} \\
&= \frac{ah(2(a-1)h + 2(h-1))}{4} \\
&= \frac{ah((a-1)h + h - 1)}{2} \\
&= \frac{ah(ah-1)}{2} \\
&= \binom{ah}{2}.
\end{aligned}$$

Now, we show that $\mathcal{F}$ is an Erdős-deep family of APs by considering each $\mathcal{F}_i$ separately.

Note that the distances $|jg_h|_n$, only occur in the h copies of $AP(g_h, a, n)$ found in $\mathcal{F}_h$. So $m(|jg_h|_n, \mathcal{F}) = h(a-j)$. Now we calculate the multiplicities of the distances $|jg_i|_n$ when $i \in \{1, \ldots, h-1\}$ and $j \in \{1, \ldots, a\}$. Observe that $m(|jg_i|_n, \mathcal{F}_i) = i(a-j) + (h-i)(a+1-j)$, where $|jg_i|_n$ occurs $i(a-j)$ times in copies of $AP(g_i, a, n)$ and $(h-i)(a+1-j)$ times in the copies of $AP(g_i, a+1, n)$. That is, we have that

$$\begin{aligned}
m(|jg_i|_n, \mathcal{F}_i) &= i(a-j) + (h-i)(a+1-j) \\
&= ai - ij + (a+1)h - hj - (a+1)i + ij \\
&= (a+1)h - hj - i \\
&= h(a+1-j) - i.
\end{aligned}$$

Let $M_1 = \{h(a+1-j) - i : i \in \{1, \ldots, h-1\}, j \in \{1, \ldots, a\}\}$ and $M_2 = \{h(a-j) : j \in \{1, \ldots, a\}$. Then altogether, we have that there exists a $k \in \mathbb{Z}^+$ such that

$$M(\Delta(\mathcal{F})) = M_1 \cup M_2 = \{1, \ldots, k-1\}.$$

Therefore, $\Delta(\mathcal{F})$ is Erdős-deep. Note that there are $\binom{k}{2}$ distances, and the distance with highest multiplicity is g_1 with multiplicity $ah - 1$, so $k = ah - 1 + 1 = ah$. □

Example 4.2. Let $s = 4$, $a = 3$, and $n \geq 13$. Then by Proposition 4.1, we can set $D = \{g_1, g_2\} = \{1, 4\}$ to get the family $\mathcal{F} = \{\{0, 1, 2, 3\}, \{0, 1, 2\}, \{0, 4, 8\}, \{0, 4, 8\}\}$, where $\Delta(\mathcal{F}) = \{3^1, 8^2, 2^3, 4^4, 1^5\}$.

We include examples of Erdős-deep families resulting from Proposition 4.1 for $s = 9$ and $s = 16$.

Example 4.3. When $(h, a) = (3, 3)$ and $D = \{1, 4, 5\}$, we have the following Erdős-deep family for $(n, s) = (19, 9)$ by Proposition 4.1:

$$\begin{cases} \{0,1,2,3\}, \{0,4,8,12\}, \{0,5,10\}, \\ \{0,1,2,3\}, \{0,4,8\}, \{0,5,10\}, \\ \{0,1,2\}, \{0,4,8\}, \{0,5,10\}. \end{cases}$$

Example 4.4. When $(h, a) = (4, 4)$ and $D = \{1, 5, 9, 14\}$, we similarly have the following Erdős-deep family when $(n, s) = (48, 16)$:

$$\begin{cases} \{0,1,2,3,4\}, \{0,9,18,27,36\}, \{0,14,28,42,8\}, \{0,5,10,15\}, \\ \{0,1,2,3,4\}, \{0,9,18,27,36\}, \{0,14,28,42\}, \{0,5,10,15\}, \\ \{0,1,2,3,4\}, \{0,9,18,27\}, \{0,14,28,42\}, \{0,5,10,15\}, \\ \{0,1,2,3\}, \{0,9,18,27\}, \{0,14,28,42\}, \{0,5,10,15\}. \end{cases}$$

4.2 Remark on the Case $s = 3$ and an Upper Bound on s

In this section we restate our conjectured classification for Erdős-deep triples that we stated originally as Conjecture 4.5 and present an upper bound on the size of an Erdős-deep family.

4.2.1 Conjectured Classification of Erdős-Deep Triples

At the end of Chapter 1, we presented a conjectured classification of Erdős-deep AP triples, which we restate as follows:

Conjecture 4.5. *The only tuples (k, k_1, k_2, k_3) that permit an Erdős-deep family of three arithmetic progressions in $\mathbb{Z}_n$ with sizes $3 \leq k_3 \leq k_2 \leq k_1 \leq \lfloor \frac{n}{2\gcd(n,g_1)} \rfloor + 1$ are of the form*

1. $(k, k_1, k_2, k_3) = (6, 4, 4, 3)$ *for infinitely many* $n \geq 13$*, and* $(k, k_1, k_2, k_3) = (7, 6, 3, 3)$ *for infinitely many* $n \geq 15$*;*

2. the remaining tuples exist for only finitely many n*, and they are:*

$$\{(8,6,5,3),(9,6,6,4),(10,6,6,6),(10,7,7,3),(10,9,4,3),(11,8,7,4),(12,8,8,5),\\ (12,10,6,4),(13,12,4,4),(14,13,5,3),(15,13,7,4),(17,16,5,4),(22,21,6,4)\}.$$

The tuples from this conjecture were the Erdős-deep triple results of a search in a finite parameter space, which we describe using the following simplified pseudo-code:

Pseudo-code

1. For $k \in [4, 28]$;
  2. For $g_1, g_2, g_3 \leq \lfloor n/2 \rfloor$;
    3. For $3 \leq k_3 \leq k_2 \leq k_1 \leq \lfloor \frac{n}{2\gcd(n,g_1)} \rfloor$;
      4. For $n \in [2k_1, 60]$;
        5. If $\gcd(g_1, n) = 1$, then without loss of generality, check $g_1 = 1$;
        6. If $\gcd(g_1, n) > 1$, then check $\gcd(n, g_1, g_2, g_3) = 1$.
        7. Check exact num diffs: $|S(\Delta(\mathcal{F}))| = k - 1$;
        8. Check distinct mult: $|M(\Delta(\mathcal{F}))| = k - 1$;
        9. Print($\mathcal{F}$).

4.2.2 Bound on the Number of Sets in an Erdős-Deep Family

Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be an Erdős-deep family in some metric space $\mathcal{M}$. We show a basic result using the average value of the number of distances that some $\Delta(A_i)$ contains for $i \in \{2, \ldots, s\}$. The technique is that if one knows the average value of a set of quantities, then there are quantities both below and above the average value.

Lemma 4.6. *Let* $k_1, \ldots, k_s$ *be positive integers such that* $k_1 \geq \cdots \geq k_s$*. Suppose* $k(k-1) = \sum_{i=1}^{s} k_i(k_i - 1)$ *for some integer* k*, and let* $t = k - k_1$*. Then there exists* k_i *and* k_j *where* $i, j \in [2, s]$*, such that* $k_i - 1 > \sqrt{\frac{2kt - t(t+1)}{s-1}} - 1$ *and* $k_j - 1 < \sqrt{\frac{2kt - t(t+1)}{s-1}}$*.*

Proof. First we rearrange the hypothesis equation:

$$\begin{aligned} \sum_{i=2}^{s} k_i(k_i - 1) &= k(k-1) - k_1(k_1 - 1) \\ &= k^2 - k - k_1^2 + k_1 \\ &= k^2 - (k-t)^2 - t \\ &= 2kt - t(t+1). \end{aligned}$$

Thus the average value of $k_i(k_i-1)$ is

$$\frac{2kt - t(t+1)}{s-1}.$$

So for some $i \in [2, s]$ there exists a k_i such that $k_i^2 > k_i(k_i-1) \geq \frac{2kt-t(t+1)}{s-1}$. Taking the square root of both sides gives

$$k_i - 1 > \sqrt{\frac{2kt - t(t+1)}{s-1}} - 1.$$

Similarly for some $j \in [2, s]$ we have

$$k_j - 1 < \sqrt{\frac{2kt - t(t+1)}{s-1}}.$$

□

We use Lemma 4.6 to prove Proposition 4.7 below, which directly implies an upper bound on the number of sets in an Erdős-deep family.

Proposition 4.7. *Let $k_1, \ldots, k_s$ be positive integers such that $k_1 \geq \cdots \geq k_s \geq 3$. Suppose $k(k-1) = \sum_{i=1}^{s} k_i(k_i-1)$ for some k and $t = k - k_1$. Then $s - 1 < \frac{kt - \binom{t+1}{2}}{2}$ where $t = k - k_1$.*

Proof. Suppose for a contradiction that $s - 1 \geq \frac{kt-\binom{t+1}{2}}{2}$. By Lemma 4.6, we know there exists a k_i such that

$$k_i - 1 < \sqrt{\frac{2kt - t(t+1)}{s-1}}.$$

Since $k_i \geq 3$, applying $s - 1 \geq \frac{kt-\binom{t+1}{2}}{2} = \frac{2kt-t(t+1)}{4}$ gives

$$2 \leq k_i - 1 < \sqrt{4},$$

a contradiction. □

Remark 4.8. We remark that if $|A_i| = k_i$ for each $i \in \{1, \ldots, s\}$, then since $\mathcal{F}$ is Erdős-deep, $|A_1|, \ldots, |A_s|$ satisfy $\binom{k}{2} = \sum_{i=1}^{s} \binom{|A_i|}{2}$ for some $k \in \mathbb{Z}^+$. Thus by Proposition 4.7, $|\mathcal{F}| < \frac{kt-\binom{t+1}{2}}{2} + 1$.

Chapter 5
Winograd Families of Arithmetic Progressions

In this chapter, we study Winograd families of Erdős-deep arithmetic progressions in $(\mathbb{Z}_n, |\cdot|_n)$ and $(\mathbb{Z}, |\cdot|)$. We restate our definitions of $|\cdot|_n$ and $|\cdot|$ as follows: for any $x, y \in \mathbb{Z}_n$, we say that the distance between x and y is $|x-y|_n = \min(x-y, n-(x-y))$; similarly, for any $x, y \in \mathbb{Z}$, the distance is $|x-y|$. A Winograd family $\mathcal{F} = \{A_1, \ldots, A_s\}$ is a type of Erdős-deep family that satisfies the additional condition that $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$.

Recall by Theorem 1.18 that Erdős-deep sets in $\mathbb{Z}_n$ are APs of the form $\{0, g, \ldots, g(k-1)\}$, and their translates, where $k \leq \lfloor \frac{n}{2\gcd(g,n)} \rfloor + 1$. Also, it is easy to see that all APs in $\mathbb{Z}$ are Erdős-deep. Throughout this chapter, we discuss only Winograd families of Erdős-deep APs in $(\mathbb{Z}_n, |\cdot|_n)$ and $(\mathbb{Z}, |\cdot|)$, so in the former we assume these APs are sufficiently small as in Theorem 1.18 and we make no assumption for the latter. Though we focus on Winograd families of Erdős-deep APs in $(\mathbb{Z}_n, |\cdot|_n)$ and $(\mathbb{Z}_n, |\cdot|)$ in this chapter, for completeness, we introduce the notion of a Winograd family in a general metric space $\mathcal{M}$, which we define presently.

Definition 5.1 (Winograd-Deep Family)**.** Let $\mathcal{M}$ be a metric space. Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be an Erdős-deep family in $\mathcal{M}$ where $M(\mathcal{F}) = \{1, \ldots, k-1\}$ for some $k \in \mathbb{Z}^+$. Then we call $\mathcal{F}$ *Winograd-deep* if $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$ as well.

We present computational and theoretical results on Winograd families of APs in both $(\mathbb{Z}_n, |\cdot|_n)$ and $(\mathbb{Z}, |\cdot|)$; in particular, we conjecture that no Winograd families of APs exist in $(\mathbb{Z}, |\cdot|)$. We discuss how proving this conjecture would show that the possibility of satisfying the Winograd property is a distinguishing property between families of APs in $\mathbb{Z}_n$ and families of APs in $\mathbb{Z}$.

In an unpublished term paper for a music theory course in 1966 [17], Terry Winograd originally defined the notion of a *Winograd set*, which is a set X in $\mathbb{Z}_n$ satisfying the Erdős-deep property but also where the distance values are precisely $1, \ldots, \lfloor n/2 \rfloor$ as well. That is, $\Delta(X) = \{1^{m_1}, \ldots, (\lfloor n/2 \rfloor)^{m_{\lfloor n/2 \rfloor}}\}$, where $m_i \in \{1, \ldots, \lfloor n/2 \rfloor\}$ and

$m_i \neq m_j$. So if a set has the property of being Erdős-deep with every distance in $\{1, \ldots, \lfloor n/2 \rfloor\}$ having distinct multiplicity, then the set is Winograd. A more detailed discussion and a variety of references on Winograd sets can be found in [4].

There are many examples of Winograd families of APs in $\mathbb{Z}_n$ (See Figure 5.1 below), however Winograd families appear to not exist when the constituent sets are APs in $\mathbb{Z}$. The main goal of this chapter is to establish, through partial results, an investigation into proving our conjecture that Winograd families do not exist in $(\mathbb{Z}, |\cdot|)$.

5.1 Winograd Families of Arithmetic Progressions in $(\mathbb{Z}_n, |\cdot|_n)$

We begin with a basic bound for Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$, then we conclude with several known examples of such Winograd families involving up to four constituent APs. Lemma 5.2 bounds n in terms of k for Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$.

Lemma 5.2. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}_n, |\cdot|_n)$ where k is an integer such that $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$, $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$, and $k_1 \geq \cdots \geq k_s \geq 3$. If for some $j \in \{1, \ldots, s\}$ and $\ell \in \{2, \ldots, k_i - 1\}$, $g_j \ell > \lfloor n/2 \rfloor$, then $n \leq 3(k-1)$.*

Proof. Following the notation from Chapter 3, let B_k be the ball of radius k about 0. Observe that since $\mathcal{F}$ is Winograd, for every $A_i \in \mathcal{F}$, $A_i \cap (\mathbb{Z}_n \setminus B) = \emptyset$. Note that $g_i < k$. Suppose for a contradiction that $n > 3(k-1)$ and let $j \in \{1, \ldots, s\}$ such that $\ell g_j > \lfloor n/2 \rfloor$, and suppose ℓ is smallest with this property. Then $n - k < \ell g_j < n$ and $(\ell - 1) g_j < k$ since $\ell g_j, (\ell-1) g_j \in A_j \cap B$. Recall that $|A_i| \geq 3$, $|2g_i|_n \in \Delta(\mathcal{F})$, which means that such an ℓ exists. Since $g_j < k$ and $n > 3(k-1)$, ℓg_j cannot be greater than $n - k$, a contradiction. □

Note that $k - 1 \leq \lfloor n/2 \rfloor$ because otherwise $S(\Delta(\mathcal{F}))$ would contain a distance outside $\{1, \ldots, \lfloor n/2 \rfloor\}$, which is impossible by the definition of $|\cdot|_n$ (see Definition 1.17). So this fact along with Lemma 5.2 together imply the following Corollary 5.3.

Corollary 5.3. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}_n, |\cdot|_n)$ where k is an integer such that $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$, $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$, and $k_1 \geq \cdots \geq k_s \geq 3$. If for some $j \in \{1, \ldots, s\}$ and $\ell \in \{2, \ldots, k_i - 1\}$, $g_j \ell > \lfloor n/2 \rfloor$, then $2(k-1) \leq n \leq 3(k-1)$.*

Figure 5.1 shows some known examples of Winograd Families of APs in $(\mathbb{Z}_n, |\cdot|_n)$ determined by computer search.

While Figure 5.1 is not a complete classification of Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$ for all $s \in [2, 4]$, it does contain enough examples to speculate about their

s	n	k	$(k_1, \ldots, k_s)$	$(g_1, \ldots, g_s)$
2	7	4	$(3,3)$	$(1,2)$
2	13	7	$(6,4)$	$(1,3)$
2	19	9	$(7,6)$	$(1,4)$
3	11	6	$(4,4,3)$	$(1,2,4)$
3	13	7	$(6,3,3)$	$(1,3,5)$
3	17	9	$(6,6,4)$	$(1,3,8)$
3	17	8	$(6,5,3)$	$(1,6,6)$
3	22	10	$(9,4,3)$	$(1,3,9)$
3	22	12	$(10,6,4)$	$(1,7,5)$
3	29	14	$(13,5,3)$	$(1,4,12)$

s	n	k	$(k_1, \ldots, k_s)$	$(g_1, \ldots, g_s)$
4	11	6	$(4,3,3,3)$	$(1,1,3,4)$
4	13	7	$(4,4,4,3)$	$(1,2,5,5)$
4	17	7	$(4,4,4,3)$	$(1,2,6,6)$
4	17	9	$(6,6,3,3)$	$(1,3,1,5)$
4	19	9	$(6,6,3,3)$	$(6,8,3,4)$
4	19	9	$(9,3,3,3)$	$(1,3,5,8)$
4	25	12	$(12,4,3,3)$	$(1,4,6,12)$
4	26	12	$(12,4,3,3)$	$(1,5,5,7)$
4	35	17	$(16,5,3,3)$	$(1,8,13,2)$
4	37	17	$(16,5,3,3)$	$(1,4,12,16)$
4	23	12	$(10,6,3,3)$	$(1,7,5,11)$
4	22	11	$(8,7,3,3)$	$(1,9,7,9)$
4	23	11	$(8,7,3,3)$	$(1,9,5,7)$
4	17	8	$(7,4,4,3)$	$(1,3,5,2)$
4	19	8	$(7,4,4,3)$	$(6,1,7,4)$
4	31	16	$(15,4,4,3)$	$(1,5,13,7)$
4	21	11	$(9,5,4,3)$	$(1,3,5,4)$

s	n	k	$(k_1, \ldots, k_s)$	$(g_1, \ldots, g_s)$
4	23	11	$(9,5,4,3)$	$(8,1,2,3)$
4	19	10	$(7,6,4,3)$	$(1,3,6,4)$
4	22	10	$(7,6,4,3)$	$(1,9,9,7)$
4	23	10	$(7,6,4,3)$	$(8,1,7,4)$
4	23	12	$(9,7,4,3)$	$(1,6,9,4)$
4	31	16	$(12,10,4,3)$	$(1,7,6,8)$
4	23	11	$(7,7,5,3)$	$(1,5,10,7)$
4	37	18	$(16,6,6,3)$	$(15,4,12,1)$
4	17	8	$(5,4,4,4)$	$(6,1,6,7)$
4	19	9	$(5,5,5,4)$	$(2,4,6,7)$
4	17	9	$(5,5,5,4)$	$(1,2,6,3)$
4	29	14	$(9,8,7,4)$	$(10,4,2,3)$
4	19	10	$(6,5,5,5)$	$(1,4,8,9)$
4	22	10	$(6,5,5,5)$	$(9,1,7,9)$
4	23	10	$(6,5,5,5)$	$(8,1,2,7)$
4	37	19	$(17,6,5,5)$	$(1,4,14,17)$
4	22	11	$(6,6,6,5)$	$(5,8,9,10)$

Figure 5.1: Examples of Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$ involving two, three, and four APs.

structure. Based on these examples, we speculate that Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$ only exist when k is very close to $n/2$, namely $k-1 \in \{\lfloor n/2\rfloor - 2, \lfloor n/2\rfloor - 1, \lfloor n/2\rfloor\}$. We state this conjecture formally as follows:

Conjecture 5.4. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}_n, |\cdot|_n)$ where k is an integer such that $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$, $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$, and $k_1 \geq \cdots \geq k_s \geq 3$. If for some $j \in \{1, \ldots, s\}$ and $\ell \in \{2, \ldots, k_i - 1\}$, $g_j \ell > \lfloor n/2\rfloor$, then $k \in \{\lfloor n/2\rfloor - 2, \lfloor n/2\rfloor - 1, \lfloor n/2\rfloor\}$.*

It would be interesting to know why Winograd families of APs in $(\mathbb{Z}_n, |\cdot|_n)$ do not seem to exist when $n/3 \leq k-1 \leq \lfloor n/2\rfloor - 3$. For some reason, the support of the distance multiset of a Winograd family of APs in $(\mathbb{Z}_n, |\cdot|_n)$ must be nearly as large as possible.

5.2 Winograd Families of Arithmetic Progressions in $(\mathbb{Z}, |\cdot|)$

We now direct our attention to Winograd families of APs in $(\mathbb{Z}, |\cdot|)$. Let $\mathcal{F}$ be a Winograd family consisting of APs $A_1, \ldots, A_s$ where for every $i \in \{1, \ldots, s\}$, $A_i = \{0, g_i, \ldots, (k_i - 1)g_i\}$ for some positive integers k_i and g_i satisfying $k_i \geq 3$. As before, we assume without loss of generality that $k_1 \geq \cdots \geq k_s$. Firstly, since the ground set in our metric space is $\mathbb{Z}$ rather than $\mathbb{Z}_n$, we have the following condition:

Winograd Condition 1 (Area Condition)**.** Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family where $A_i = \{0, g_i, \ldots, (k_i - 1)g_i\}$. Then for every $i \in \{1, \ldots, s\}$, it holds that $(k_i - 1)g_i < k$.

Winograd Condition 1 holds because each element of every set in $\mathcal{F}$ must be a distance value in $\Delta(\mathcal{F})$. Since $\mathcal{F}$ is Winograd, its highest distance value is $k-1$. Since $g_i(k_i - 1)$ is a bounded product of two parameters, we can think of Winograd Condition 1 as an "area condition".

Another condition on Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ is that $A_1 = \{0, \ldots, k_1 - 1\}$. We state the second Winograd condition formally as follows:

Winograd Condition 2 (A_1 is a Large Interval)**.** Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family where $A_i = \{0, g_i, \ldots, (k_i - 1)g_i\}$. Then $A_1 = \{0, 1, \ldots, k_1 - 1\}$.

Our proof of Winograd Condition 2 is presented after Proposition 5.9 at the end of this section.

Another Winograd condition follows from Winograd Condition 1 and the fact that we assume each constituent set has size at least 3. We state this third condition as follows:

Winograd Condition 3 (Generator Upper Bound)**.** Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family where $A_i = \{0, g_i, \ldots, (k_i - 1)g_i\}$. Then $g_i < k/2$.

Observe that Winograd Condition 3 follows immediately from the requirements $g_i(k_i - 1) < k$ and $k_i - 1 \geq 2$.

Since Winograd families are Erdős-deep as well, we recall Conditions 2 and 3 mentioned in Chapter 3 and restate them in terms of Erdős-deep families of APs in $(\mathbb{Z}, |\cdot|)$.

Erdős-Deep Condition 1 (Maximum Multiplicity)**.** Let $\mathcal{F}$ be an Erdős-deep family of APs in $(\mathbb{Z}, |\cdot|)$. Then for every $d \in \Delta(\mathcal{F})$, $m(d, \Delta(\mathcal{F})) \leq k - 1$.

Erdős-Deep Condition 2 (Distinct Multiplicity)**.** Let $\mathcal{F}$ be an Erdős-deep family of APs in $(\mathbb{Z}, |\cdot|)$. Then for every $d, d' \in \Delta(\mathcal{F})$ such that $d \neq d'$, $m(d, \Delta(\mathcal{F})) \neq m(d, \Delta(\mathcal{F}))$.

5.2.1 Main Conjecture

We continue to use the parameter t where $t := k - k_1$, which was used in previous chapters, since it is significant for Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ as well. We show in Proposition 5.9 that $t < k/2$, which implies that $k_1 > k/2$ and so by Winograd Condition 1, $g_1 = 1$. Winograd Condition 2 implies that the interval $E = [k - t, k - 1]$ is significant; the reason why E is significant is that no distance values in E occur in $\Delta(A_1)$, and so we require the remaining APs $A_2, \ldots, A_s$ to ensure that the distances in E all have distinct positive multiplicities. Since E contains the largest distances in the support of a Winograd family, which is at the end of the interval $\{1, \ldots, k - 1\}$, we call E the *end segment*.

Since $k_1 > k/2$, small distances have large multiplicity in $\Delta(\mathcal{F})$ from $\Delta(A_1)$. Since the multiplicity of every distance value in $\{1, \ldots, k-1\}$ must be at most $k-1$ (Erdős-Deep Condition 1), the APs $A_2, \ldots, A_s$ cannot be too large. Otherwise, small multiples of $g_2, \ldots, g_s$ would have large multiplicity from $\Delta(A_2), \ldots, \Delta(A_s)$, as well as the already large multiplicity from $\Delta(A_1)$, which could cause a small distance in $\{1, \ldots, k - 1\}$ to have multiplicity in $\Delta(\mathcal{F})$ greater than $k - 1$.

Another factor to note is that since each of the t distances in $[k - t, k - 1]$ needs positive multiplicity m from some subset of $\{\Delta(A_2), \ldots, \Delta(A_s)\}$, if $m < k_1$, then there is a distance in $[1, k - t)$ that also needs more multiplicity from these sets. Otherwise, there would be at least two distances with equal multiplicity, which contradicts Erdős-Deep Condition 2.

So, we require $A_2, \ldots, A_s$ not to be too long; however, we also require that a sufficient number of $A_2, \ldots, A_s$ intersect the end segment. These two opposing requirements together suggest that Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ are unlikely to exist. We expect that if Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ do exist, then the average sizes of $A_2, \ldots, A_s$ would be small enough to ensure small distances don't have too high multiplicity in $\Delta(\mathcal{F})$, and s is sufficiently large to ensure that the large distances found in E occur frequently enough. Based on the above, some computer experimentation, and the arguments to follow, we think it is likely that no Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ exist. We formalize this speculation in the following conjecture.

Conjecture 5.5 (Main Conjecture)**.** *There does not exist a Winograd family of APs in* $(\mathbb{Z}, |\cdot|)$*.*

5.2.2 Preliminary Non-Existence Results

What follows are partial non-existence results on Winograd families of APs in $(\mathbb{Z}, |\cdot|)$. Our broad approach is to bound t as a function of k. First we show that the end segment cannot contain a prime, which implies that $t < k/2$ and for sufficiently large k, $t < k^{0.525}$. Then we present partial results that seem to suggest that $t > \sqrt{\frac{k}{2}}$.

We use Bertrand's postulate, proven originally by Pafnuty Chebyshev in 1852 (see [15]), which states the following:

Lemma 5.6 (Bertrand's Postulate)**.** *Let n be a positive integer. Then there exists a prime p such that $p \in (n, 2n)$.*

Recall that $E = [k-t, k-1]$, and we call E the *end segment*.

Lemma 5.7 (Prime-Free End Segment)**.** *If $\mathcal{F}$ is a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$, then there is no prime in E.*

Proof. Suppose for a contradiction that p is a prime such that $p \in E$. Suppose p is the largest such prime. Then since p has multiplicity at least 1, there exists an AP $A \in \mathcal{F}$ containing p. Since p is prime, it cannot be a multiple of any integer except itself and 1, so p must be the generator of A. Recall that every set in $\mathcal{F}$ has size at least 3, so $2p \in A$, which implies that $2p \leq k-1$. But by Lemma 5.6, there is a prime $q \in (p, 2p)$, and so $k - t \leq p < q \leq k-1$. But this contradicts the maximality of p, so there is no prime in E. □

We use a result of R. C. Baker, G. Harman, and J. Pintz from 2001 [1], which is similar to Bertrand's postulate, but valid for sufficiently large n, stated as follows:

Lemma 5.8 (R. C. Baker, G. Harman, and J. Pintz 2001 [1])**.** *Let n be a sufficiently large positive integer. Then there exists a prime p such that $p \in [n - n^{0.525}, n]$.*

Proposition 5.9. *If $\mathcal{F}$ is a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$, then $t < \frac{k}{2}$ for all k, and if k is sufficiently large, $t < k^{0.525}$.*

Proof. By Lemma 5.6, there exists a prime between $\frac{k}{2}$ and k, so $\mathcal{F}$ cannot be Winograd by Lemma 5.7. Similarly, by Lemma 5.8, for sufficiently large k, there is a prime in the end segment if $t \geq k^{0.525}$. □

The truth of Winograd Condition 2 follows directly from Proposition 5.9 and Winograd Condition 1.

Proof of Winograd Condition 2. By Proposition 5.9, $t < k/2$. Since $t = k - k_1$, we have that $k_1 > k - k/2 = k/2$. Since $A_1 = \{0, g_1, \ldots, g_1(k_1 - 1)\}$ and $g_1(k_1 - 1) < k$ by Condition 1, it must be the case that $g_1 = 1$. □

5.2.3 Necessary Condition When $k \geq 2t^2$

In this section, we show that Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ must contain an AP with size at least $t - 1$ that intersects the end segment when $k \geq 2t^2$. In the next section, we conjecture that such an AP cannot exist due to a combination of the following: (1) a restriction on its generator value (see Lemma 5.14), and (2) a possible issue with long APs reaching the end segment contradicting Erdős-Deep Condition 2.

We now prove Proposition 5.12, which is a necessary condition on Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ when $k \geq 2t^2$.

Lemma 5.10. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$ where $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$. If $k \geq 2t^2$, then for every $i \in \{2, \ldots, s\}$, $A_i \cap E \in \{\emptyset, \{g_i(k_i - 1)\}\}$.*

Proof. Suppose otherwise, then we have for some A_i, $k-t \leq g_i(k_i-2) < g_i(k_i-1) < k$, implying that $g_i < t$ and so we have that $k_i - 2 \geq \frac{k-t}{g_i} > \frac{k-t}{t} = \frac{k}{t} - 1$. Thus $k_i - 1 > k/t$, which means by Winograd Condition 2 and Lemma 2.6 that g_i has multiplicity in $\Delta(\mathcal{F})$ at least $(k-t) - g_i + k/t \geq k - 2t + k/t$. That is, since $g_i < t$, $k_i - 1 < 2t$; otherwise g_i will have multiplicity in $\Delta(\mathcal{F})$ at least k. Thus $k/t < 2t$ and so $k < 2t^2$, a contradiction. □

Since every distance in E needs positive multiplicity in $\Delta(\mathcal{F})$, Lemma 5.10 implies that for every distance $d \in E$ there exists some A_i such that $d = g_i(k_i - 1)$. Our goal is to show that there must be some distance in E with multiplicity 0 unless there is some AP intersecting E with size at least $t - 1$. We show this in Proposition 5.12.

We define the prime counting function $\pi(n)$, which is the number of primes at most n, where $n \in \mathbb{Z}^+$. The prime counting function is of major interest in a variety of areas of mathematics, and many mathematicians have dedicated much effort to estimate it. The prime number theorem states that $\pi(n)$ is asymptotically equivalent to $\frac{n}{\ln(n)}$, but explicit bounds on $\pi(n)$, which are close to $\frac{n}{\ln(n)}$ are difficult to attain, and they usually only hold for sufficiently large n. Lemma 5.11 states the estimate of $\pi(n)$ that we use in the proof of Proposition 5.12.

Lemma 5.11 (Prime Counting Function Estimate of Dusart 2018 [6])**.** *The following upper bound on $\pi(n)$ holds: If $n \geq 2$, then $\pi(n) \leq \frac{n}{\ln(n)}\left(1 + \frac{1}{\ln(n)} + \frac{2}{\ln(n)^2} + \frac{7.59}{\ln(n)^3}\right)$.*

Proposition 5.12. *Let $\mathcal{F} = \{A_1, \dots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$ where $A_i = \{0, g_i, \dots, g_i(k_i-1)\}$ and $k_1 \geq \dots \geq k_s \geq 3$. If $k \geq 2t^2$ then there exists $i \in \{2, \dots, s\}$ such that $A_i \cap E \neq \emptyset$ and $k_i \geq t-1$.*

Proof. Observe that if $t \leq 4$, then there must be APs in $\mathcal{F}$ of size at least 3 intersecting E by the assumption that $k_i \geq 3$ and the distances in E need positive multiplicity in $\Delta(\mathcal{F})$. We therefore suppose that $t \geq 5$.

Suppose otherwise that every AP intersecting E has size at most $t-2$. By Lemma 5.10, for every $d \in E$, there exists an A_i such that $d = g_i(k_i-1)$. Since $k_i - 1 \in [2, t-3]$, we show that there cannot exist enough generator values cumulatively across the range of possible values of $k_i - 1 \in [2, t-3]$ to permit every distance in E to have positive multiplicity. This contradicts the requirement that $S(\Delta(\mathcal{F})) = \{1, \dots, k-1\}$ from Definition 5.1.

Note that for every $k_i - 1 \in [2, t-3]$, $g_i \in [\frac{k-t}{k_i-1}, \frac{k-1}{k_i-1}]$, implying that there are at most $\lfloor \frac{t-1}{k_i-1} \rfloor + 1$ possible values for g_i. Suppose we are in the extremal case in which there is an AP with final element in E for every possible g_i value.

For each $\ell \in [2, t-3]$, define

$$E_\ell = \{g \in \mathbb{Z}^+ : g\ell \in E, \text{ and } \forall j \in [2, \ell-1], g \notin E_j\}.$$

Note that 'ℓ' is taking the place of '$k_i - 1$' in the product $g_i(k_i-1)$, and so E_ℓ contains all possible values of g_i such that $g_i\ell \in E$ subject to the condition that $g_i j \notin E$ for all $j \in [2, \ell-1]$. Let P_{t-3} be the set of all primes less than or equal to $t-3$ and observe that $|P_{t-3}| = \pi(t-3)$. Then we have for example $|E_2| \leq \lfloor \frac{t-1}{2} \rfloor + 1, |E_3| \leq \lfloor \frac{t-1}{2\cdot 3} \rfloor + 1$, and $|E_4| = 0$; and in general we have

$$|E_5| \leq \left\lfloor \frac{t-1}{2\cdot 3\cdot 5} \right\rfloor + 1, \dots, |E_{t-3}| \leq \left\lfloor \frac{t-1}{\prod_{p\in P_{t-3}} p} \right\rfloor + 1.$$

Observe $E_\ell = \emptyset$ unless ℓ is prime. Thus the distances in E that intersect an AP of size at most $t-2$ must all come from the set $\tau = \bigcup_{\ell\in[2,t-3]} E_\ell$. Since all t distances in E intersect some AP of size at most $t-2$, we require that $|\tau| = t$.

We show that in fact $|\tau| \leq t-1$. Observe that we have

$$|\tau| = \sum_{q\in P_{t-3}} |E_q| \leq \sum_{q\in P_{t-3}} \left\lfloor \frac{t-1}{\prod_{p\in P_q} p} \right\rfloor + 1.$$

Let $\mathbb{P}$ denote the set of all primes. Observe that $\sum_{q\in P_{t-3}} \frac{1}{\prod_{p\in P_q} p} < \sum_{q\in\mathbb{P}}^{\infty} \frac{1}{\prod_{p\in P_q} p}$, which is the infinite sum of reciprocals of the primorials. According to the online encyclopedia of integer sequences, this infinite sum converges to a number less than $0.706 < \frac{1}{\sqrt{2}}$ (see sequence A064648 on OEIS [12]). Note that $\lfloor x \rfloor \leq x$ and by Lemma 5.11, $\pi(t-3) \leq \frac{t-3}{\ln(t-3)}\left(1 + \frac{1}{\ln(t-3)} + \frac{2}{\ln(t-3)^2} + \frac{7.59}{\ln(t-3)^3}\right) =: U$ for all $t \geq 5$. Thus we have

$$|\tau| \leq \sum_{q \in P_{t-3}} \left\lfloor \frac{t-1}{\prod_{p \in P_q} p} \right\rfloor + 1 \leq \sum_{q \in P_{t-3}} \frac{t-1}{\prod_{p \in P_q} p} + \pi(t-3)$$
$$< \frac{t-1}{\sqrt{2}} + \pi(t-3)$$
$$< \frac{t-1}{\sqrt{2}} + U,$$

which is at most $t-1$ when $t \geq 109$, a contradiction. Note that this argument fundamentally relies on the fact that the minimum constituent set size in $\mathcal{F}$ is 3. That is, if $\mathcal{F}$ contained a set of size 2, then if an AP of size 2 intersected E, we would require $E_1 \subseteq \tau$, in which case $|\tau| \geq t$ because $|E_1| = t$.

The case when $t \in [5, 108]$ is addressed by a computer search. For brevity let $\omega = \sum_{q \in P_{t-3}} \left\lfloor \frac{t-1}{\prod_{p \in P_q} p} \right\rfloor + 1$. The tuples (t, ω) are tabulated below; observe that for every $t \in [5, 108]$, $\omega \leq t-1$, which implies that $|\tau| \leq t-1$, a contradiction.

$(5,3)$	$(6,4)$	$(7,6)$	$(8,7)$	$(9,8)$	$(10,9)$	$(11,10)$	$(12,10)$	$(13,12)$
$(14,13)$	$(15,14)$	$(16,15)$	$(17,16)$	$(18,16)$	$(19,18)$	$(20,19)$	$(21,20)$	$(22,21)$
$(23,22)$	$(24,22)$	$(25,24)$	$(26,25)$	$(27,26)$	$(28,26)$	$(29,27)$	$(30,27)$	$(31,30)$
$(32,31)$	$(33,32)$	$(34,33)$	$(35,34)$	$(36,34)$	$(37,36)$	$(38,36)$	$(39,37)$	$(40,38)$
$(41,39)$	$(42,39)$	$(43,41)$	$(44,42)$	$(45,43)$	$(46,44)$	$(47,45)$	$(48,45)$	$(49,47)$
$(50,48)$	$(51,49)$	$(52,49)$	$(53,50)$	$(54,50)$	$(55,52)$	$(56,53)$	$(57,54)$	$(58,54)$
$(59,55)$	$(60,55)$	$(61,58)$	$(62,59)$	$(63,60)$	$(64,61)$	$(65,62)$	$(66,62)$	$(67,64)$
$(68,64)$	$(69,65)$	$(70,66)$	$(71,67)$	$(72,67)$	$(73,69)$	$(74,70)$	$(75,71)$	$(76,72)$
$(77,73)$	$(78,73)$	$(79,75)$	$(80,75)$	$(81,76)$	$(82,77)$	$(83,78)$	$(84,78)$	$(85,80)$
$(86,81)$	$(87,82)$	$(88,82)$	$(89,83)$	$(90,83)$	$(91,86)$	$(92,87)$	$(93,88)$	$(94,88)$
$(95,89)$	$(96,89)$	$(97,91)$	$(98,91)$	$(99,92)$	$(100,93)$	$(101,94)$	$(102,94)$	$(103,96)$
$(104,97)$	$(105,98)$	$(106,99)$	$(107,100)$	$(108,100)$				

Table 5.1: This table shows (t, ω) pairs for $t \in [5, 108]$, where $\omega = \sum_{q \in P_{t-3}} \left\lfloor \frac{t-1}{\prod_{p \in P_q} p} \right\rfloor + 1$.

Thus there exists at least one AP in $\mathcal{F}$ with size at least t that intersects E. □

5.2.4 Conjectured Necessary Condition When $k \geq 2t^2$

We conjecture that when $k \geq 2t^2$, the maximum size of an AP intersecting E is less than t. If this statement is true, then Proposition 5.12 is contradicted, which would mean that Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ cannot exist when $k \geq 2t^2$.

Formally, we conjecture the following:

Conjecture 5.13. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$ where $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$. Then for every $i \in \{1, \ldots, s\}$ such that $g_i(k_i - 1) \in E$, it holds that $k_i < t - 1$ for all $t \geq 1$.*

Observe that we only want to show an **upper** bound on the $k_i - 1$ values of the APs that intersect E. At the end of this section, we show in Lemma 5.15 that for all APs in a Winograd family, $\max(k_i - 1) < \sqrt{k} + t$. Having only to prove an upper bound could be accessible since we expect there to be a lot of small APs intersecting E.

The following Lemma 5.14 might be helpful to prove Conjecture 5.13. In what follows, we use the notation (mod 1) as an operator to refer to the fractional part of a number; that is, for any rational number x, we say that $x \pmod 1$ is the fractional part of x. For example $10.73 \pmod 1 = 0.73$.

Lemma 5.14. *Let $\mathcal{F}$ be a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$ and suppose $A \in \mathcal{F}$ where $A = \{0, g, \ldots, g\ell\}$. If $A \cap E \neq \emptyset$ and $\ell \geq t$, then $\frac{t-1}{\ell} \geq \frac{k-1}{\ell} \pmod 1$ and $g = \lfloor \frac{k-1}{\ell} \rfloor = \lceil \frac{k-t}{\ell} \rceil$.*

Proof. Suppose $\ell \geq t$. Since $A \cap E \neq \emptyset$, we have $k - t \leq g\ell \leq k - 1 \Leftrightarrow \frac{k-t}{\ell} \leq g \leq \frac{k-1}{\ell}$. Because g is an integer, $g \in [\lceil \frac{k-t}{\ell} \rceil, \lfloor \frac{k-1}{\ell} \rfloor]$, and this interval contains at most one integer value. In particular, g can be an integer only when $\frac{t-1}{\ell}$ is larger than the fractional part of $\frac{k-1}{\ell}$. This is because for any positive rationals x and y where $y \in (0, 1)$, $\lfloor x \rfloor - \lceil x - y \rceil \leq 0$ and $y \geq x \pmod 1$ if and only if $\lfloor x \rfloor - \lceil x - y \rceil = 0$. Thus for $x = \frac{k-1}{\ell}$ and $y = \frac{t-1}{\ell}$, we have $g_i = \lfloor \frac{k-1}{\ell} \rfloor = \lceil \frac{k-t}{\ell} \rceil$ and $\frac{t-1}{\ell} \geq \frac{k-1}{\ell} \pmod 1$. □

Multiplicity Paths

We believe a proof of Conjecture 5.13 likely must account for the following structure. Note that each distance d_1 in E has a nearest distance outside E such that the multiplicity of d_1 is equal to the multiplicity of some distance $d_2 \in \Delta(A_1)$. This implies that there are other APs besides A_1 that contain d_2 so that d_2 has distinct multiplicity from d_1. Then the same pattern occurs for some distance d_3 in $\Delta(A_1)$ relative to d_2, and so on, until the final distance in the path has multiplicity at least k_1. Perhaps it is the case that if an AP with size at least $t-1$ intersects E at some distance d, somehow the multiplicities of its at least $t-1$ distinct distances interfere with the multiplicities in the multiplicity path beginning at d by forcing some of them to not be distinct in $\Delta(\mathcal{F})$. This would contradict Erdős-Deep Condition 2. In any case, contradicting this multiplicity path structure could be the missing piece required to resolve Conjecture 5.5.

Upper Bound on Sizes of Constituent Sets in a Winograd Family of APs in $(\mathbb{Z}, |\cdot|)$

In Lemma 5.15, we show an elementary upper bound on the sizes of constituent sets in a Winograd Family.

Lemma 5.15. *Let $\mathcal{F} = \{A_1, \ldots, A_s\}$ be a Winograd family of APs in $(\mathbb{Z}, |\cdot|)$ where $A_i = \{0, g_i, \ldots, g_i(k_i - 1)\}$. Then $\max_{i \in [2,s]}(k_i - 1) < \frac{1}{2}(\sqrt{t^2 + 4k} + t)$.*

Proof. By Winograd Condition 2, $A_1 = \{0, 1, \ldots, k_1 - 1\}$. Then by Lemma 2.6, for each $i \in \{1, \ldots, s\}$, $k_1 - g_i + k_i - 1 < k$. Additionally, by Winograd Condition 1 it holds that $g_i < \frac{k}{k_i - 1}$. So we have:

$$\begin{aligned} k_1 - g_i + k_i - 1 &< k \\ \Rightarrow k_i - 1 - t - \frac{k}{k_i - 1} &< 0 \\ \Leftrightarrow (k_i - 1)^2 - t(k_i - 1) - k &< 0. \end{aligned}$$

Solving the quadratic gives $k_i - 1 < \frac{1}{2}(\sqrt{t^2 + 4k} + t)$. □

Remark 5.16. Note that $\frac{1}{2}(\sqrt{t^2 + 4k} + t) < \frac{1}{2}(\sqrt{t^2} + \sqrt{4k} + t) = \sqrt{k} + t$ since $k, t > 0$.

5.2.5 When $k < 2t^2$

We have not addressed the case when $k < 2t^2$. However, since Lemma 5.7 implies that E cannot contain a prime, it is unlikely for Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ to exist when $t > \sqrt{\frac{k}{2}}$. So, we have focused our attention on what we think is the harder case when $k \geq 2t^2$.

5.3 Conclusion

If Winograd families of APs in $(\mathbb{Z}, |\cdot|)$ do not exist, then this means that the only way for families of integer APs to be Winograd, is if the ground set is $\mathbb{Z}_n$. This implies that the possibility of a family of integer APs satisfying the Winograd property of $S(\Delta(\mathcal{F})) = \{1, \ldots, k-1\}$ distinguishes between families of APs in $\mathbb{Z}_n$ and families of APs in $\mathbb{Z}$.

Chapter 6
Music

In this chapter, we explore possible applications of Erdős-deep families of arithmetic progressions to families of musical rhythms. The main aspect of rhythms that we focus our attention on is the cumulative set of distances between all pairs of onsets in each rhythm. We begin with some terminology and examples, as well as motivation for the musical significance of Erdős-deep families. Then we provide audio demonstrations of various features of Erdős-deep rhythms, followed by simple example compositions by the author. The audio for these demonstrations and compositions can be accessed on the GitHub repository that contains the supplementary material for this thesis found at `www.github.com/taogaede/MScThesis`.

6.1 Rhythms with Specified Inter-Onset Distances

We introduce terms associated with a rhythm R represented as a set in $\mathbb{Z}_n$. We say that n is the *timespan* of R, which means that there are n indivisible and evenly spread out units of time. Some of these time units correspond to note-hits, called *onsets*, while the other time units correspond to *rests*. We imagine R to be played repeatedly and indefinitely, so we avoid favouring any particular time unit as primary.

Example 6.1. Figure 6.1 shows an example of an arithmetic progression in $\mathbb{Z}_{16}$ with common difference, or generator, 5, realized as a rhythm. The set and music notation are as follows:

$$\{0, 5, 10, 15, 4, 9, 14, 3\},$$

Figure 6.1: This is a rhythm over time span 16. The elements of the set correspond to the quarter notes in the rhythm.

Given this cyclic and periodic context, we wish to study the cumulative collection of shortest distances between every pair of onsets in R; we call these the *inter-onset distances* of R, and we call the collection of these distances the *distance multiset* of R, denoted by $\Delta(R)$. We call the number of times that an inter-onset distance d in $\Delta(R)$ occurs the *multiplicity* of d.

It may be the case that larger inter-onset distances occur with higher multiplicity in $\Delta(R)$ than short ones do, or vice-versa. It could be interesting to survey rhythms in each of these two cases and compare whether they sound significantly different.

We are interested in a related but different type of question. Instead of inquiring about audibly discernible differences that depend on the *values* of inter-onset distances of rhythms, we are interested in whether there are noticeable differences that depend solely on the *multiplicities* of the distances. For instance, we do not distinguish between long and short distances, we only care about how the multiplicities of the distances are distributed. For example, given a rhythm R, we may ask whether there are few distances with large multiplicity in $\Delta(R)$, or perhaps whether the multiplicities are all relatively uniform.

In Example 6.2, we show the distance multiset for the rhythm in Example 6.1.

Example 6.2. Let R be the rhythm from Example 6.1 where $R = \{0, 5, 10, 15, 4, 9, 14, 3\}$. Note that the distance '5' occurs 7 times and the distance '10' occurs 6 times, but since the time span is 16 and the rhythm is treated as a cycle, the distance '10' is not the shortest distance, rather, 6 is. So, the distance multiset for R is $\Delta(R) = \{5^7, 6^6, 1^5, 4^4, 7^3, 2^2, 3^1\}$.

Notice in the example above that the distance multiplicities are consecutive beginning at 1, namely the multiplicity set is $\{1, 2, 3, 4, 5, 6, 7\}$. Observe that each distance value in $\Delta(R)$ has distinct multiplicity. Rhythms whose distance values have distinct multiplicity are called *deep rhythms* and they are discussed at great length in Godfried Toussaint's book "The Geometry of Musical Rhythm" [16]. Toussaint provides many examples and excellent visualizations of deep rhythms found in a variety of different types of music, as well as a measure for how deep a general rhythm is. Many of the examples of deep rhythms that Toussaint discusses satisfy the additional property that the multiplicities are an interval beginning at 1; such deep rhythms are called *Erdős-deep rhythms*. Note that the rhythm from Example 6.1 is Erdős-deep.

It is easy to generate an Erdős-deep rhythm. So long as the rhythm does not have too many onsets, any arithmetic progression is Erdős-deep. A feature of Erdős-deep rhythms that distinguishes them from other deep ones is that their multiplicities are consecutive, which means they are as close to one another as possible while still being distinct. Put another way, the set of multiplicities of Erdős-deep rhythms are as close as possible to those of non-deep rhythm in which a pair of multiplicities are equal.

The first five chapters of this thesis concerns the mathematical structure of families of Erdős-deep rhythms whereby the cumulative distances from each rhythm in the family also has distance multiplicities being an interval beginning at 1. We call such families *Erdős-deep families*. Since Erdős-deep rhythms occur in a variety of music, and we have introduced and studied the notion of an Erdős-deep family of Erdős-deep rhythms at length, we propose some possibly musically interesting questions about Erdős-deep rhythm families.

For the remainder of this chapter, we outline some possible applications of Erdős-deep families to musical rhythms. Recall that each set R in $\mathbb{Z}_n$ can represent a rhythm with timespan n where the elements of R correspond to onsets, or note-hits. The main way we interpret a family of rhythms is as follows: A family of sets in $\mathbb{Z}_n$ corresponds to a family of rhythms, whereby each rhythm corresponds to the onsets of a particular voice or instrument in a musical phrase.

6.2 Arithmetic Progressions as Rhythms

One of the more fascinating and convenient aspects of Erdős-deep rhythms is that, except for one case, they are always arithmetic progressions, which have a very regular structure to them. See Theorem 1.18 for the precise characterization of Erdős-deep sets in $\mathbb{Z}_n$, which is restated from [4].

One of the key arguments for why Erdős-deep rhythms are interesting musically is that the complexity of their inter-onset distance arrangement is complemented by the simplicity of being an arithmetic progression. Since Erdős-deep rhythms are arithmetic progressions, the distance between every pair of onsets is always a multiple of some common difference. This common difference property makes arithmetic progressions, as rhythms, easier to predict since their onsets are arranged in a generally regular fashion. However, we allow for the onsets in the arithmetic progressions to wrap around the time span, which, due to mathematical reasons from the Euclidean algorithm, causes onsets to form periodic clusters. Figure 6.2 shows a visualization of this clustering.

As Figure 6.2 illustrates, Erdős-deep rhythms have their onsets distributed around the time span in clusters. While the common distance of an arithmetic progressions is perhaps easier to predict audibly, the clustering pattern is not so obvious - but it is nonetheless regular in the sense that the clusters are distributed uniformly around the time span. So, we can think of modular arithmetic progressions as easy to follow due to their common distance, but less predictable due to the clustering of their onsets.

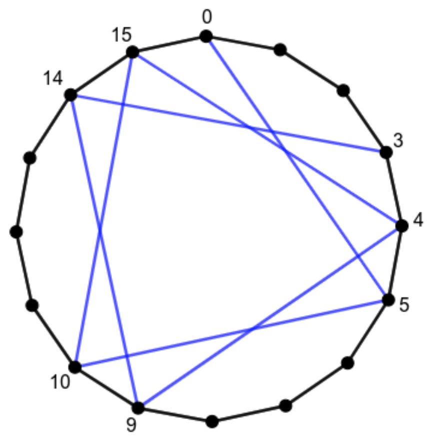


Figure 6.2: This illustration shows a modular arithmetic progression with generator 5, size 8 over $\mathbb{Z}_{16}$. The blue lines represent the progression beginning at 0 and ending at 3. Note that the set begins at the element '0', increments clockwise by multiples of 5, and ends until there are 8 elements.

6.3 Erdős-Deep Rhythm Families

6.3.1 Erdős-Deep Property

The main features of an Erdős-deep family of interest to musical rhythms is both that the constituent sets are arithmetic progressions and the family has the Erdős-deep property. We discussed the significance of arithmetic progression rhythms in the previous section. Recall that the multiset of distances for a family is the union of distance multisets of each rhythm in the family. For a family of rhythms to be Erdős-deep, the multiset of distances for the family has multiplicities $1, \ldots, k-1$ for some k. We discuss the significance of the Erdős-deep property for families presently.

6.3.2 Almost Uniform Distinct Multiplicities

A key aspect of the Erdős-deep property is that the multiplicities are as uniform as possible while still being distinct. This aspect may contribute to the rhythm family sounding more balanced. In particular, the maximum difference between the maximum and minimum multiplicities is $k-1$, which is relatively small.

It is not so easy to find families with distances having consecutive multiplicities. Most families do not satisfy this property. As an example, we could have a family of rhythms like the following:

$$\{\{0,1,2\}^3, \{0,1,2,3\}^2, \{0,1,2,3,4\}\},$$

which has distance multiset $\{1^{16}, 2^{10}, 3^4, 4^1\}$. In this example, the distance 1 occurs with relative multiplicity to 4 on the order of $x^{\log_x(16)} = x^2$ where x is the number of distinct distances. This $x^2 : 1$ ratio contrasts with Erdős-deep families where the most frequent distance occurs relative to the least frequent distance at essentially a $x : 1$ ratio.

We have that the Erdős-deep property ensures this ratio between maximum and minimum multiplicities for deep rhythms is small. Note that the ratio is minimized when the multiplicities form an interval and the minimum multiplicity is large. See [5] for examples of families of APs in $\mathbb{Z}_n$ with interval multiplicities whose minimum is larger than 1. In this sense, we can think of Erdős-deep rhythms and Erdős-deep rhythm families as minimizing the max : min multiplicity ratio given that there are $k-1$ distinct distances and $\binom{k}{2}$ total number of distances. In any case, the max : min multiplicity ratio is quite small for Erdős-deep families, compared other families.

6.3.3 Erdős-Deep Families and Small Timespans

In the previous section, we argued that Erdős-deep families sound more balanced because their distance multiplicities are more uniform. However, if balance due to uniform multiplicity is the feature we wish to optimize for, then our requirement that multiplicities be distinct is unnecessary. There is a class of well studied families called *Difference families* that have this uniform multiplicity property (see Definition 1.24 and [14] for more details). In fact, we suspect that when the timespan is large, Erdős-deep and difference families become less audibly distinguishable based on their distance multiplicities. However, in our demonstrations, we concern ourselves with smaller timespans in which n is usually around $7-24$, and at most 64. Since n is small, the distinctness of the multiplicities are more likely to be noticeable. So, Erdős-deep families are an ideal rhythm structure to use for studying musical significance of rhythm families with distinct and almost uniform distance multiplicities.

6.3.4 Maximal Evenness to Translate Rhythms in Erdős-Deep Families

Recall that the translations of the constituent sets in an Erdős-deep family has no effect on the family maintaining its Erdős-deep property, so if there are s sets in the family, there are n^s possible translation configurations. Since n^s can be a very large number indeed, a musician has many choices on how to realize the Erdős-deep family; however, certain configurations seem more useful than others. The musical examples in the next section use a particular translation configuration using maximally even sets.

Let $\mathcal{F} = \{R_1, \ldots, R_s\}$ be an Erdős-deep family and let t_i be the translation index for the R_i constituent rhythm; this means that $R_i = t_i + \{0, g_i, \ldots, (k_i - 1)g_i\} =$

$\{t_i, t_i+g_i, \ldots, t_i+(k_i-1)g_i\}$ for some generator g_i and size k_i. Now set the translation indices to come from a maximally even set; this means that $\{t_1, \ldots, t_s\}$ is a maximally even set. The reason why we translate the rhythms according to a maximally even set is because we expect the onsets to generally be more evenly distributed throughout the timespan n. This standardized way of translating rhythms makes it easier to compare other variables in our demonstrations.

The following example is of an Erdős-deep family in $\mathbb{Z}_n$ of size 4 with translation indices coming from a maximally even set of size 4 in $\mathbb{Z}_n$.

Example 6.3. The following is an Erdős-deep family of four rhythms with $n = 24$: $\mathcal{F} = \{\{\mathbf{0}, 1, 2, 3, 4\}, \{\mathbf{6}, 11, 16, 21\}, \{\mathbf{12}, 13, 14, 15\}, \{\mathbf{18}, 23, 4, 9\}\}$. Here the translation index set is $\{0, 6, 12, 18\}$, which is maximally even in $\mathbb{Z}_{24}$.

6.4 Demonstrations

We present several demonstrations of Erdős-deep rhythm families. Each demonstration is meant to emphasize a particular musically interesting aspect of Erdős-deep rhythm families. We refer the reader to the GitHub repository that contains the supplementary material for this thesis: `www.github.com/taogaede/MScThesis`. In particular, all audio examples can be found in the 'Chapter6_Supplementary_Material' directory of this repository.

We begin by discussing how varying the number of distinct distances between onsets $k-1$ and time span n affects the complexity of the family. Then we show a way to combine rhythm families when n is a power of 2. We show two examples of Erdős-deep rhythm families that result from Proposition 4.1 that involve a lot of constituent rhythms - namely, 9 and 16. Finally, we conclude the demonstration section with an example that shows, among other things, how to use Erdős-deep rhythm families in common time despite the fact that they often do not exist when the time span is a multiple of 4.

Recall that the Erdős-deep family structure does not require that each constituent rhythm is translated a particular way, so for demonstration purposes, it is desirable to have a standardized way of translating the rhythms. We mentioned in the previous section that translating constituent rhythms using a maximally even set was a standardized way to distribute the onsets across the timespan more evenly. We apply this maximally even translation scheme to all of our musical demonstrations in this section.

6.4.1 Varying the Time Span and Number of Distances

We use our construction from Chapter 4 (See Proposition 4.1) to generate Erdős-deep rhythm families involving four rhythms. The examples are chosen to be as similar as possible to one another except that they each involve varying number of onsets and timespan n. When there are more onsets, this increases the number of distinct distances, which corresponds to the value $k-1$. Our goal with the following demonstrations is to make two observations: (1) Erdős-deep families with smaller n and k are simpler and easier to follow than those with larger n and k; and (2) it is possible to combine Erdős-deep families with specially chosen timespans to create larger rhythm families that sound stable despite their complexity. The second observation suggests that Erdős-deep families can be used to create interesting rhythm families with larger n and number of distinct distances. Unlike the first observation, the second may not be expected; what seems to happen is that since each timespan is a power of 2, downbeats occur at multiples of powers of 2, which creates a stable sounding emergent metre.

The Erdős-deep rhythm family examples discussed below are described by the tuples (n, k, k_1, k_2, k_3, k_4, g_1, g_2, g_3, g_4) and we list them with maximally even translates as follows:

$$(16,6,4,3,3,3,1,5,1,5) \sim \begin{cases} \{\mathbf{0},1,2,3\},\{\mathbf{4},9,14\}, \\ \{\mathbf{8},9,10\},\{\mathbf{12},1,6\} \end{cases}$$

$$(32,10,6,5,5,5,1,6,1,6) \sim \begin{cases} \{\mathbf{0},1,2,3,4,5\},\{\mathbf{8},14,20,26,0\}, \\ \{\mathbf{16},17,18,19,20\},\{\mathbf{24},30,4,10,16\} \end{cases}$$

$$(64,14,8,7,7,7,1,9,1,9) \sim \begin{cases} \{\mathbf{0},1,2,3,4,5,6,7\},\{\mathbf{16},25,34,43,52,61,6\}, \\ \{\mathbf{32},33,34,35,36,37,38\},\{\mathbf{48},57,2,11,20,29,38\} \end{cases}$$

We chose the rhythm sizes to be what they are so that k increases linearly across the three examples; this keeps the difference in number of distinct distances across the three families small. We chose n to be the smallest power of 2 for the given rhythm sizes k_1, k_2, k_3, and k_4 so that (1) each family can be easily presented in the $\frac{4}{4}$ time signature; and (2), the three families can be conveniently layered in a way that ensures frequent onset intersections across the largest of the three timespans.

We first consider the three Erdős-deep rhythm families independently. Each contains four rhythms and has time span 16, 32, and 64, respectively. We interpret each rhythm in each family as providing the onsets for when a particular note is played.

Example 6.4. The family in this example has time span 16 and is shown below in both set and music notation. Observe that each rhythm corresponds to a particular register of C (denoted above the rhythm). Set notation:

$$\overset{\text{C5}}{\{0,1,2,3\}},\overset{\text{C3}}{\{4,9,14\}},\overset{\text{C6}}{\{8,9,10\}},\overset{\text{C4}}{\{12,1,6\}}.$$

Refer to 'EDFamilyDemonstrations16.mp3' in the supplementary material and the following score example:

Figure 6.3: Music notation of an Erdős-deep rhythm family of four rhythms over time span 16. Each rhythm corresponds to one of four registers of C, namely C3, C4, C5, and C6.

In Example 6.4, the treble rhythms $\{0, 1, 2, 3\}$ and $\{8, 9, 10\}$ establish a 4 beat metre, which acts as a steady foundation for the more offset and fluid bass clef rhythms $\{4, 9, 14\}$ and $\{12, 1, 6\}$. It is significant here that the treble rhythms were chosen to have generator 1, as well as for the timespan to be so short at $n = 16$; these two facts together ensure that the two consecutive sixteenth note clusters occur close enough to one another to preserve the metre. The next two demonstrations also have two rhythms with generator 1, and we will see in that when the timespan increases, similar metres are not so easily established solely by the treble rhythms.

Example 6.5. The family in this example has time span 32 and is shown below in both set and music notation. Observe that each rhythm corresponds to a particular register of F (denoted above the rhythm). Set notation:

$$\{0, 1, \overset{\text{F5}}{2}, 3, 4, 5\}, \{8, 14, \overset{\text{F3}}{20}, 26, 0\}, \{16, 17, \overset{\text{F6}}{18}, 19, 20\}, \{24, 30, \overset{\text{F4}}{4}, 10, 16\}.$$

Refer to 'EDFamilyDemonstrations32.mp3' in the supplementary material and the following score example:

Figure 6.4: Music notation of an Erdős-deep rhythm family of four rhythms over time span 32. Each rhythm corresponds to one of four registers of F, namely F3, F4, F5, and F6.

Even though the treble rhythms with generator 1 in Example 6.5 are spread out further across the timespan from one another than in Example 6.4, the bass rhythms happen to be translated in such a way so that their onsets altogether preserve a similar metre as in Example 6.4. So, unlike in Example 6.4, the bass rhythms are less sporadic here and much more inline with the metre.

Example 6.6. The family in this example has time span 64 and is shown below in both set and music notation. Observe that each rhythm corresponds to a particular register of B♭ (denoted above the rhythm). Set notation:

$$\begin{cases} \{0,1,2,\overset{\text{B}\flat 4}{3},4,5,6,7\}, \{16,25,34,\overset{\text{B}\flat 3}{43},52,61,6\}, \\ \{32,33,34,\overset{\text{B}\flat 5}{35},36,37,38\}, \{48,57,2,\overset{\text{B}\flat 2}{11},20,29,38\}. \end{cases}$$

Refer to 'EDFamilyDemonstrations64.mp3' in the supplementary material and the following score example:

Figure 6.5: Score of an Erdős-deep rhythm family of four rhythms over time span 64. Each rhythm corresponds to one of four registers of B♭, namely B♭2, B♭3, B♭4, and B♭5.

Out of the three examples, the music in both Examples 6.4 and 6.5 seem to have a steady metre with four beats per bar, while the metre in Example 6.6 seems driven solely by the bass rhythms and teeters unsteadily. Since the bass rhythms are arithmetic progressions, their onsets are predictable enough to establish the metre, but since the number of onsets in each of the two bass rhythms is 7, which does not divide the timespan of 64, the metre breaks the trend of the previous two examples by having 14 beats, rather than the expected 16.

Listen to 'EDFamilyDemonstrations-powers-of-two-altogether.mp3' to hear the three families playing together. In this demonstration, we allow the onsets in the rhythm families to change notes according to a harmonic progression. Observe that combining

Erdős-deep rhythm families, whose timespans are all powers of two, seems to produce a complex rhythm family over a relatively long timespan that nonetheless has a clear metre. We note that the combined rhythm family is not itself Erdős-deep, the purpose of this demonstration is to show that Erdős-deep families can be combined to create complex rhythm families with a long timespan and clear metre.

6.4.2 Larger Rhythm Families

In this section, we show two rhythm families, one with 9 rhythms, and the other with 16. These families result from our construction in Proposition 4.1. Since there are many rhythms, we may expect their combinations to destabilize the metre such that it is hard to follow. However steady metre emerges in both cases. Along with this steady metre, the rhythms combine to form a variety of complex rhythmic phrases. This rhythmic complexity could be partially explained by the interval multiplicity pattern of the inter-onset distances. In each example, the timespan is relatively short; so, this suggests that a shorter time span, rather than fewer rhythms, may cause Erdős-deep rhythm families to have a steady metre and rhythmic complexity.

Example 6.7. The following is an example of an Erdős-deep rhythm family with 9 rhythms over time span 27 translated according to the maximally even set $\{0, 3, 6, 9, 12, 15, 18, 21, 24\}$. Each rhythm corresponds to the eighth note onsets of a particular pitch and register indicated above the rhythm. Refer to 'EDFamilyDemonstrations-9-voices.mp3' in the supplemental materials.

$$\begin{cases} \{0, \overset{\text{C6}}{1}, 2, 3\}, \{3, 7, \overset{\text{G5}}{11}, 15\}, \{6, \overset{\text{C5}}{11}, 16\}, \\ \{9, 10, \overset{\text{G4}}{11}, 12\}, \{12, \overset{\text{C4}}{16}, 20\}, \{15, \overset{\text{G3}}{20}, 25\}, \\ \{18, \overset{\text{C3}}{19}, 20\}, \{21, \overset{\text{G2}}{25}, 2\}, \{24, \overset{\text{C2}}{2}, 7\} \end{cases}$$

Below is the scored version of this rhythm family.

Figure 6.6: Music notation for an Erdős-deep family involving nine rhythms.

Observe that the metre in Example 6.7 has 6 beats across the span of 27 eighth notes (here subdivided into 9 bars).

Example 6.8. We show an example of an Erdős-deep rhythm family with 16 rhythms over time span 48 translated according to the maximally even set $\{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45\}$. Each rhythm corresponds to the eighth note onsets of a particular pitch and register indicated above the rhythm. The audio file for this example can be found in the supplementary materials and is titled 'EDFamilyDemonstrations-16-voices.mp3'.

$$\begin{cases} \{0,1,\overset{\text{C7}}{2},3,4\}, \{3,12,\overset{\text{B}\flat\text{6}}{21},30,39\}, \{6,20,\overset{\text{G6}}{34},0,14\}, \{9,14,\overset{\text{F6}}{19},24\}, \\ \{12,13,\overset{\text{E}\flat\text{6}}{14},15,16\}, \{15,24,\overset{\text{C6}}{33},42,3\}, \{18,32,\overset{\text{B}\flat\text{5}}{46},12\}, \{21,\overset{\text{G5}}{26},31,36\}, \\ \{24,25,\overset{\text{F5}}{26},27,28\}, \{27,\overset{\text{E}\flat\text{5}}{36},45,6\}, \{30,\overset{\text{C5}}{44},10,24\}, \{33,\overset{\text{B}\flat\text{4}}{38},43,48\}, \\ \{36,37,\overset{\text{G4}}{38},39\}, \{39,\overset{\text{F4}}{0},9,18\}, \{42,\overset{\text{E}\flat\text{4}}{8},22,36\}, \{45,\overset{\text{C4}}{2},7,12\} \end{cases}$$

Below is the scored version of this rhythm family.

Figure 6.7: Music notation for an Erdős-deep rhythm family involving 16 rhythms.

In Example 6.8, the emergent metre appears to have 12 beats with beats $1, 4, 7$, and 10 being especially emphasized. Note the time span is 48, so this is an example where the number of beats in the metre divides the timespan.

6.4.3 Scales Demonstration

Now we use Erdős-deep pairs in which both rhythms have 3 onsets and the time span is 7. We show that while these families have time span 7, it is possible to use them in a way so that the time signature is a more common one, namely $\frac{4}{4}$. Our main mathematical result from Chapter 3 (see Theorem 3.14 for details) showed that Erdős-deep pairs hardly ever exist, and when they do, it is sometimes only for sporadic values of n that correspond to uncommon timespans. So, it is possibly worthwhile to devise a way to use Erdős-deep rhythm families for more commonly used timespans. To fit an Erdős-deep rhythm into $\frac{4}{4}$ time, we can repeat the family a multiple of 4 times, so that the time span is divisible by 4. Then we ensure that each bar has a multiple of 4 beats. In the demonstrations below, we repeat the families 8 times so that each bar contains 8 eighth notes. For instance, in Example 6.9, the rhythm $\{0, 1, 2\}$ in the Erdős-deep pair is repeated in such a way that the onsets within the translates $\{0, 1, 2\}, \{7, 8, 9\}, \ldots, \{49, 50, 51\}$ altogether form the rhythm for one of the two voices in the demonstration. Similarly, $\{2, 3, 6\}, \{9, 10, 13\}, \ldots, \{51, 52, 55\}$ form the rhythm for the other voice.

Despite the fact that the Erdős-deep families used in the following scale examples is the simplest mathematically, in each example, the two rhythms combine to produce relatively complex emergent rhythms from the interacting onsets as well as clear metres.

Example 6.9. In this example, the Erdős-deep rhythm pair is $\{\{0, 1, 2\}, \{2, 3, 6\}\}$. Each voice ascends and then descends the C-Major scale. Note that since each rhythm has the same number of onsets, the two voices can ascend and descend at relatively the same rate. The rhythm of the top voice is characterized by $\{0, 1, 2\}$, while $\{2, 3, 6\}$ characterizes the bottom voice. In the supplemental materials, the audio file for this example is titled 'EDFamilyDemonstrations-CMajor(0,1,2),(3,6,2).mp3'. The score for this example is as follows:

The following are two similar examples, but using different Erdős-deep pairs. Observe that each the interactions between the two voices are distinctive for each example.

Example 6.10. The Erdős-deep pair $\{\{0, 1, 2\},\ \{1, 3, 5\}\}$ realized as larger rhythm pair analogously to the one in Example 6.9. Refer to 'EDFamilyDemonstrationsCMajor(0,1,2),(1,3,5).mp3' in the supplemental materials. Below is the score.

Example 6.11. Analogously to the previous two example, but here the Erdős-deep pair is $\{\{0, 2, 4\},\ \{0, 3, 4\}\}$. The score for this example is below, and the corresponding audio is titled 'EDFamilyDemonstrationsCMajor(0,2,4),(3,4,0).mp3'.


(0,2,4)
(3,4,0)

6.5 Compositions

This section contains remarks on three compositions by the author, which use Erdős-deep rhythm families. Each voice in Compositions 1 and 2 follows a particular rhythm in that they change notes only when there is an onset. The final composition uses an Erdős-deep rhythm family as the main developing theme. Scores of the first two compositions are included below, and audio presentations of all three are included in the supplemental materials.

6.5.1 Compositions Using Onsets to Change Notes

We present two brief compositions that use Erdős-deep families to indicate when instruments change notes. In both compositions, the time span is 13, and so they are written in the $\frac{13}{8}$ time signature. In Theorem 3.14, we have that one of the few existing Erdős-deep pairs satisfies $n = 13$. So unlike the scales demonstrations from the previous section, here we decide to accept the uncommon timespan implied by the Erdős-deep family, and create music subject to this constraint.

Composition 1: Variable Rhythm Translations

In this short piece, each instrument corresponds to a rhythm, and the onsets of each rhythm in the Erdős-deep family indicates when the corresponding instrument changes notes. The Erdős-deep rhythm family is $\mathcal{F} = \{\{0,3,6\}, \{0,2,4,6\}, \{0,3,6,9\}, \{0,4,8,12\}\}$.

Creative liberty was taken to translate the rhythms within various bars to suit the musical development of the piece. This means that each bar is an Erdős-deep rhythm family, however the Erdős-deep property may fail when considering a time span of $n = 13$ across bars. While this decision interferes with the Erdős-deep rhythm family structure, it avails the composer of an interesting creative problem to align the rhythms in a musical way without fully breaking the Erdős-deep structure. Note that the rhythms therefore no longer conform to standardized maximally even translates like in the demonstrations.

Refer to 'EDFamilyComposition1.mp3' in the supplemental materials. The scores of Compositions 1 and 2 are after the following brief discussion on Composition 2.

Composition 2: Fixed Rhythm Translations

In this piece, unlike Composition 1, the translations of the rhythms are left fixed. This seems to contribute to the repetitive rhythmic drive that underlies the piece. The Erdős-deep family is $\{\{0,2,5,10\},\{0,4,5,6,11,12\}\}$. Each of these two rhythms corresponds to the onsets within a bar. Observe that each instrument changes its notes according to the onsets of one of these rhythms.

Refer to 'EDFamilyComposition2.mp3' in the supplemental materials. The score of Composition 2 follows that of Composition 1, which is on the next page.

Composition 1

Variable Rhythm Translations

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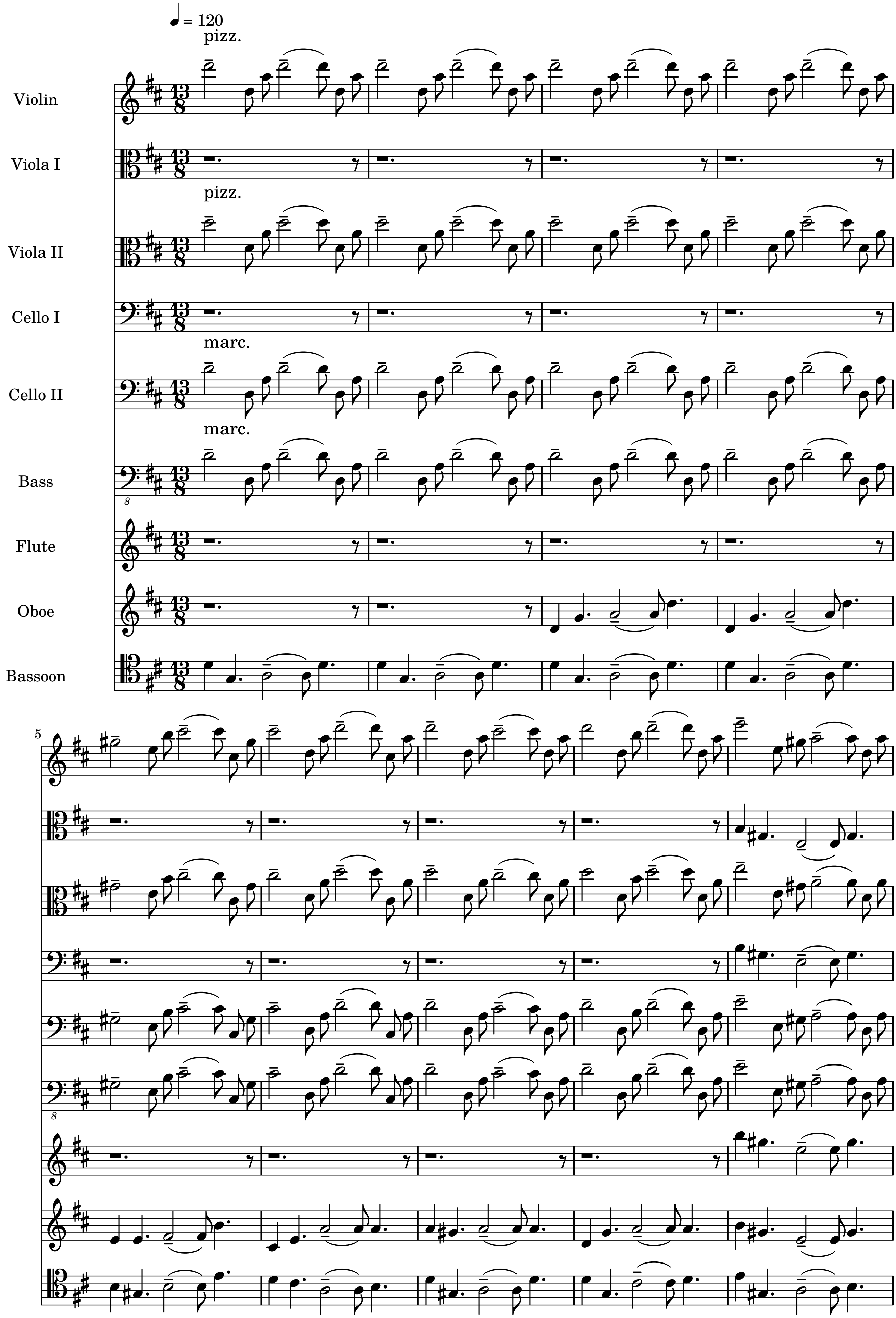
Composition 2

Fixed Rhythm Translations



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6.5.2 Composition With Erdős-Deep Families as Main Theme

We remark on a composition that uses Erdős-deep families as main themes. Refer to 'EDFamilyComposition(Hold).mp3' in the supplementary materials.

The families $\mathcal{A} = \{\{0, 9, 4, 13\}, \{5, 8, 11, 0\}, \{10, 13, 2\}\}$ with time span 14, and $\mathcal{B} = \{\{0, 5, 10, 15, 4\},\ \{4, 7, 10, 13\},\ \{13, 6, 15, 8\},\ \{12, 3, 10, 1\}\}$ with timespan 16 are used in this composition. For $\mathcal{A}$, using the same method described in Example 6.9, we "expand" it so that its timespan is doubled to 28 instead of 14. That is, we actually use the following family:

$\mathcal{A}' = \{\{0, 9, 4, 13, 14, 18, 23, 27\}, \{5, 8, 11, 0, 19, 22, 25, 14\}, \{10, 13, 2, 24, 27, 16\}\}$.

The piece alternates between slow sections containing no Erdős-deep rhythm families and fast sections, in which the Erdős-deep family is the focal point. The piece begins with a theme involving family $\mathcal{A}'$, and is followed by a development section involving a theme based on family $\mathcal{B}$. The piece ends with a variation on the beginning theme based on $\mathcal{A}'$. The theme involving $\mathcal{A}'$ occurs in 7 bar phrases and the theme involving $\mathcal{B}$ is the main developing idea throughout the middle, and it occurs in 4 bar phrases.

A key goal behind this piece is to determine whether the emergent rhythmic complexity and metrical pattern of Erdős-deep families is sufficient to enable them to act as the main theme for music that aims to be complex. We use creative liberty in the form of harmonic progressions and minimal rhythmic accompaniment to help express the Erdős-deep families in a variety of contexts. We note that the main development section in the middle of the piece contains little to no rhythmic accompaniment, so the main transition in the piece is achieved primarily by the harmonic progression and Erdős-deep family structure.

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