ON THE UNIVORM CONTINUITY OF OPERATOR FUNCTIONS AND GENERALIZED POWERS-STORMER INEQUALITIES

By

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DM-442-IR MAY 1987

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Introduction. A few years ago, L.G. Brown used K-Theory together with some clever spectral theory to show that an extension of an AF algebra by an AF algebra was also an AF algebra. At one point in the proof, he used the uniform continuity of the square root function on the positive part of the unit ball in a C*-algebra. At that time, a colleague was working through Brown's theorem and wondered about a proof of this fact. A number of experts in operator theory were consulted and one came up with a rather unpleasant computational proof that was far from being sharp. Eventually, of course, everyone saw the light - it is a trivial consequence of spectral theory (see proposition 1.1).

However, in considering this problem (i.e., before I saw the light) it occurred to me to use the operator monotonicity of the square-root function. This, indeed, provides the very sharp estimate, $\|A^{\frac{1}{2}}-B^{\frac{1}{2}}\| \leq \|A-B\|^{\frac{1}{2}}$ for A and B positive operators of arbitrary norm. This estimate is the uniform-norm analogue of the Powers-Stormer inequality [5, lemma 4.1]. In fact, this same idea leads to a whole family of inequalities, one for every operator-monotone function, f. To wit:

$$\|f(A)-f(B)\| \leq f(\|A-B\|) - f(0)$$

if f is operator-monotone on $[0, \infty)$ and A, B are positive operators. In case $f(x) = x^{\frac{1}{n}}$ for $n \geq 1$ (not necessarily an integer) we have

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\| \leq \|A-B\|^{\frac{1}{n}}.$$

This sharp inequality has interested a number of people, most recently Terry Loring in his difficult matrix estimates.

In talking to B.M. Baker about this inequality (for $n = 2$) he felt that it should be directly related to the Powers-Stormer inequality in the sense that a Powers-Stormer type proof might be found. In the search for such a connection, we discovered two things. First, we discovered a whole continuum of inequalities which contains the Powers-Stormer inequality and the inequality $\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\| \leq \|A-B\|^{\frac{1}{n}}$ as special cases. Second, we found out why Barry Simon calls this sort of thing "the hard analysis of compact operators on Hilbert space" [6, preface].

To state what we call the <u>generalized Powers-Stormer inequality</u>, let $\|\cdot\|_p$ denote the Schatten p-norm for $1 \leq p < \infty$ and let $\|\cdot\|_\infty$ denote the usual operator norm [6]. Let $1 \leq n \leq p \leq \infty$ where $n \neq \infty$, then the aforementioned inequality states that

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p \leq \|A-B\|_{\frac{p}{n}}^{\frac{1}{n}}$$

for A and B positive operators on Hilbert space. In the case that $1 < n < 2$ we must confess that this inequality is still a conjecture unless we also have $n = p$ or $p = \infty$. Perhaps a diagram would help.

The shaded area and solid lines indicate that the inequality has been proved for those values of n and p. The blank area with the question marks indicates values of n and p for which we believe the inequality is true but for which we have no proof. In this blank area we have some numerical evidence in dimensions 2 and 3 which suggest that the conjecture is true. Thanks are due to Dale Olesky at the University of Victoria and Jon Borwein at Dalhousie University for their willingness to perform computer experiments for us. As further evidence that the inequality is true in the blank area, we are able to prove the weaker but still homogeneous inequality:

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p \leq \left[\|A\|_{\frac{p}{n}}^{\frac{1}{2n}} + \|B\|_{\frac{p}{n}}^{\frac{1}{2n}}\right]\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}}$$

for $1 < n < 2$ and $p \geq n$. Of course, this inequality does show that the map $A \mapsto A^{\frac{1}{n}}$ is continuous from the positive operators in $C_{\frac{p}{n}}$ to the positive operations in C_p, and is in fact a Lipschitz functions of exponent $\frac{1}{2n}$ on any bounded set in $C_{\frac{p}{n}}$.

After reducing to the case $A \geq B \geq 0$ via a clever lemma of Heydar Radjavi's, the main tools we use are the operator monotonicity of the functions $f(x) = x^{\frac{1}{n}}$ for $n \geq 1$, a related integral formula for these functions, and the Spectral theorem.

Acknowledgements. I would like to thank my many colleagues whose interest in this problem kept me going even when my enthusiasm began to lag. Special thanks are due to Heydar Radjavi and Peter Fillmore. Besides providing lemma 2.3, Heydar Radjavi was responsible for many stimulating discussions and much encouragement. These latter comments also apply to Peter Fillmore. Thanks are also due to Bob Miers at the University of Victoria for his hospitality and many interesting discussions.

§1. Uniform norm results and operator monotonicity.

We first dispense with an easy but worthwhile observation that takes care of most continuity questions for the operator norm, $\|\cdot\|_\infty$. Of course, this result does not provide sharp estimates.

1.1 Proposition: Let f be a continuous complex-valued function on the compact set $X \subseteq \mathbb{C}$. Let H be a Hilbert space and let $\mathcal{N}_X$ be the set of normal operators on H with spectrum contained in X. Then $f\colon \mathcal{N}_X \to \mathcal{B}(H)$ is uniformly continuous in the operator norm.

proof: Let $\epsilon > 0$ and let p be a polynomial in two variables so that $|f(z) - p(z,\bar{z})| < \frac{\epsilon}{3}$ for all $z \in X$. It is easy to see that there exists $\delta > 0$ so that if $S, T \in \mathcal{N}_X$ and $\|S-T\|_\infty < \delta$ then $\|p(S,S^*)-p(T,T^*)\|_\infty < \frac{\epsilon}{3}$. Thus, $\|f(S)-f(T)\|_\infty < \epsilon$ as required. ■

We now recall that a continuous function $f\colon [0, \infty) \to \mathbb{R}$ is called operator monotone if $S \leq T$ implies $f(S) \leq f(T)$ for all positive operators S and T. As observed in [4, 1.3.7] the functions f_α defined by

$f_\alpha(t) = t(1+\alpha t)^{-1} = \frac{1}{\alpha}\left[1-(1+\alpha t)^{-1}\right]$ are operator monotone for $\alpha > 0$. Moreover, it is also observed in [4, 1.3.12] that any operator monotone function f on $[0, \infty)$ has a unique representation $f(t) = \int_0^\infty f_\alpha(t)d\mu(\alpha) + f(0)$ where μ is a positive measure on $[0, \infty)$.

1.2 Lemma: Let f be operator monotone on $[0, \infty)$ with $f(0) = 0$ and let $s, t \in [0, \infty)$. Then $f(s+t) \le f(s) + f(t)$.

proof: By the preceeding integral formula, it suffices to verify this inequality for the functions f_α. However, this is equivalent to

$$(1+\alpha s)^{-1} + (1+\alpha t)^{-1} \le (1+\alpha s+\alpha t)^{-1} + 1$$

or

$$\frac{1 + \alpha t + 1 + \alpha s}{(1+\alpha s)(1+\alpha t)} \le \frac{1 + 1 + \alpha s + \alpha t}{1 + \alpha s + \alpha t}$$

or

$$1 + \alpha s + \alpha t \le (1+\alpha s)(1+\alpha t) = 1 + \alpha s + \alpha t + \alpha^2 st$$

which is trivially true. ∎

1.3 Lemma: If A and B are commuting positive operators and f is an operator monotone function with $f(0) = 0$ then $f(A+B) \le f(A) + f(B)$.

proof: There is a bounded self-adjoint operator, H so that A and B are both functions of H. That is, if $H = \int_m^M \lambda \, dE_\lambda$ is the spectral decomposition of H then there are positive Borel functions g_1 and g_2 with $A = g_1(H) = \int_m^M g_1(\lambda)dE_\lambda$ and $B = g_2(H) = \int_m^M g_2(\lambda)dE_\lambda$. Thus,

$$f(A+B) = f\left[\int_m^M (g_1(\lambda)+g_2(\lambda)dE_\lambda\right]$$

$$= \int_m^M f(g_1(\lambda)+g_2(\lambda))dE_\lambda$$

$$\le \int_m^M (f(g_1(\lambda)) + f(g_2(\lambda)))dE_\lambda$$

$$= f\left[\int_m^M g_1(\lambda)dE_\lambda\right] + f\left[\int_m^M g_2(\lambda)dE_\lambda\right] = f(A) + f(B). \quad \blacksquare$$

1.4 <u>Remark</u>: If A and B do not commute, this inequality can fail. In particular, if $f(t) = t^{\frac{1}{2}}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Then $f(A) + f(B) = A + B = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ while $f(A+B) = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{\frac{1}{2}}$. Since $A + B$ has eigenvalues $1 \pm \frac{1}{\sqrt{2}}$ it cannot dominate it's square root.

1.5 <u>Theorem</u>: Let f be an operator monotone function on $[0, \infty)$ and let A, B be positive operators. Then,

$$\|f(A)-f(B)\|_\infty \le f(\|A-B\|_\infty) - f(0).$$

<u>proof</u>: Let $g = f - f(0)$ so that g is operator monotone and $g(0) = 0$. Let $\|A-B\|_\infty = \epsilon$ so that $A \le B + \epsilon I$. Then, by lemma 1.3 $g(A) \le g(B+\epsilon I) \le g(B) + g(\epsilon)I$ and so $g(A) - g(B) \le g(\epsilon)I$. By symmetry, we have $g(B) - g(A) \le g(\epsilon)I$ and so $-g(\epsilon)I \le g(A) - g(B) \le g(\epsilon)I$. Since $g(\epsilon) \ge 0$ we have $\|g(A)-g(B)\|_\infty \le g(\epsilon)$. Finally, $\|f(A)-f(B)\|_\infty = \|g(A)-g(B)\|_\infty \le g(\epsilon) = f(\epsilon) - f(0) = f(\|A-B\|_\infty) - f(0)$ as required. ■

1.6 <u>Corollary</u>: Let A and B be positive operators and let $n \ge 1$ be a positive real number. Then,

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_\infty \le \|A-B\|_\infty^{\frac{1}{n}} .$$

<u>proof</u>: $f(t) = t^{\frac{1}{n}}$ is operator monotone by [4, 1.3.8]. ■

1.7 <u>Remark</u>: Another approach to theorem 1.5 is to show that the inequality holds for the functions f_α and then obtain the general case directly from the integral formula used above. However, the easiest way to show that the inequality holds for the functions f_α seems to use monotonicity. As monotonicity is easy to prove for the functions f_α, there is little reason to search for a direct proof of the inequality for these functions.

§2. <u>Inequalities for the Schatten p-norms</u>.

If T is a bounded operator on a Hilbert space H, $|T| = (T^*T)^{\frac{1}{2}}$, and $1 \le p < \infty$ then we define $\|T\|_p = (\text{Trace } |T|^p)^{\frac{1}{p}}$. The Schatten p-class, C_p, is the ideal of all operators, T on H for which $\|T\|_p < \infty$. This, in fact, defines a complete norm on C_p. If $\|T\|_p < \infty$ then $|T|$ (and hence T) is compact and $\|T\|_p^p = \sum \langle |T|_p^p f_i | f_i \rangle = \sum \langle \lambda_i^p f_i | f_i \rangle = \sum \lambda_i^p$ where $\{f_i\}$ is an orthonormal basis consisting of eigenvectors for $|T|$ with corresponding eigenvalues $\{\lambda_i\}$. We use [6] as a convenient reference for the basic properties of $(C_p, \|\cdot\|_p)$.

We record the following useful fact as a lemma. Its proof is a simple application of the Spectral theorem combined with Hölder's inequality: see for example [6, p. 21].

2.1 Lemma: Let A be a positive operator, f a unit vector, and $p \geq 1$ a real number. Then

$$\langle A^p f | f \rangle \geq \langle Af | f \rangle^p.$$

2.2 Corollary: If $A \geq B \geq 0$ are operators and $p > 0$ is a real number, then Trace $A^p \geq$ Trace B^p.

proof: If A is not compact, the left side of the inequality is infinite and we're done. So we may assume A (and hence B) is compact. If $p \geq 1$, let $\{f_i\}$ be an orthonormal basis of eigenvectors for B. Then, Trace $A^p =$ $\sum \langle A^p f_i | f_i \rangle \geq \sum \langle Af_i | f_i \rangle^p \geq \sum \langle Bf_i | f_i \rangle^p = \sum \langle B^p f_i | f_i \rangle =$ Trace B^p. If $p < 1$, we diagonalize A and do a similar computation where we apply the lemma to the operator B^p with $\frac{1}{p} \geq 1$. That is,

$$\langle Bf_i | f_i \rangle^p = \langle (B^p)^{\frac{1}{p}} f_i | f_i \rangle^p \geq \langle B^p f_i | f_i \rangle^{\frac{1}{p} \cdot p} = \langle B^p f_i | f_i \rangle. \blacksquare$$

Of course, this fact has other proofs involving the comparison of the eigenvalues of A and B using the min-max property of the n-th eigenvalues [2, lemma 1.1]. However, our proof is somewhat more elementary and introduces the idea of diagonalizing relative to a specific operator in order to facilitate the computation. We will use this idea again in the proof of the main result.

Now, if T is a self-adjoint operator, then we let T_+ and T_- denote the positive and negative parts of T. That is, $T = T_+ - T_-$ and $T_+ \geq 0$, $T_- \geq 0$ and $T_+T_- = 0 = T_-T_+$. In this case, $|T| = T_+ + T_-$ and $\|T\|_p^p = \|T_+\|_p^p + \|T_-\|_p^p$. Now, if $T = S_1 - S_2$ where $S_i \geq 0$ for each i then $S_1 \geq T$ and so $P_+S_1P_+ \geq P_+TP_+ = T_+$ where P_+ is the range projection of T_+. Thus, by 2.2, $\|S_1\|_p^p \geq \|P_+S_1P_+\|_p^p \geq \|T_+\|_p^p$. Similarly $\|S_2\|_p^p \geq \|T_-\|_p^p$, so that $\|T\|_p^p \leq \|S_1\|_p^p + \|S_2\|_p^p$ in this case. This crucial observation allows us to prove the following lemma which reduces us to the setting of two <u>comparable</u> positive operators. This observation and therefore the following lemma are due to Heydar Radjavi. This lemma, which we call the "Radjavi reduction" replaces a weaker argument which forced us to be content with a factor of 2 in our general inequality. We have rewritten the proof of this lemma so that it more closely resembles our original lemma. This is purely a matter of taste; the idea is still Heydar Radjavi's.

2.3 <u>Lemma</u>: Suppose $p \geq n \geq 1$ are positive real numbers and that $\|A_1^{\frac{1}{n}} - B_1^{\frac{1}{n}}\|_p \leq \|A_1 - B_1\|_{\frac{p}{n}}^{\frac{1}{n}}$ whenever A_1 and B_1 are positive operators with

$A_1 \leq B_1$ or $A_1 \geq B_1$. Then, this same inequality holds for arbitrary pairs of positive operators, A, B.

<u>proof</u>: Let $C = \frac{1}{2}(A+B) + \frac{1}{2}|A-B|$. Then, $C - A = \frac{1}{2}(B-A) + \frac{1}{2}|B-A| = (B-A)_+ = (A-B)_-$. In particular $C \geq A$. Similarly, $C - B = (A-B)_+$ so that $C \geq B$. Thus, $C^{\frac{1}{n}} \geq A^{\frac{1}{n}}$ and $C^{\frac{1}{n}} \geq B^{\frac{1}{n}}$, so that $A^{\frac{1}{n}} - B^{\frac{1}{n}} = (C^{\frac{1}{n}}-B^{\frac{1}{n}}) - (C^{\frac{1}{n}}-A^{\frac{1}{n}})$ and therefore $\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p^p \leq \|C^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p^p + \|C^{\frac{1}{n}}-A^{\frac{1}{n}}\|_p^p$

$$\leq \|C-B\|_{\frac{p}{n}}^{\frac{p}{n}} + \|C-A\|_{\frac{p}{n}}^{\frac{p}{n}}$$

$$= \|(A-B)_+\|_{\frac{p}{n}}^{\frac{p}{n}} + \|(A-B)_-\|_{\frac{p}{n}}^{\frac{p}{n}}$$

$$= \|A-B\|_{\frac{p}{n}}^{\frac{p}{n}} .$$

Taking p-th roots completes the proof. ∎

We now proceed to the proof of the main theorem. Although the form of this proof resembles the Powers-Stormer argument our proof depends on some lemmas which are proven later. We do things in this order to make these lemmas seem more natural: of course, these lemmas do not logically depend on the theorem.

2.4 Theorem: Suppose $2 \leq n \leq p < \infty$ are real numbers and A, B are positive operators. Then

$$\|A^{\frac{1}{n}} - B^{\frac{1}{n}}\|_p \leq \|A-B\|_{\frac{p}{n}}^{\frac{1}{n}} .$$

proof: By the Radjavi reduction, we can assume $A \geq B$. We also assume $\|A-B\|_{\frac{p}{n}}^{\frac{1}{n}} < \infty$ otherwise there is nothing to prove. In particular, $A - B$ is compact and so $\pi(A) = \pi(B)$ where π is the Calkin map. Thus $\pi(A^{\frac{1}{n}}) = \pi(A)^{\frac{1}{n}} = \pi(B)^{\frac{1}{n}} = \pi(B^{\frac{1}{n}})$ and so $A^{\frac{1}{n}} - B^{\frac{1}{n}}$ is compact. Let $z = A^{\frac{1}{n}} - B^{\frac{1}{n}} \geq 0$ and let $y = B^{\frac{1}{n}}$. Then $(y+z)^n = A \geq B = y^n$. We let $\{f_i\}$ be an orthonormal basis of eigenvectors for z and calculate:

$$\|A-B\|_{\frac{p}{n}}^{\frac{p}{n}} = \text{Trace}[(y+z)^n - y^n]^{\frac{p}{n}}$$

$$= \sum \langle ((y+z)^n - y^n)^{\frac{p}{n}} f_i | f_i \rangle$$

$$\geq \sum \langle ((y+z)^n - y^n) f_i | f_i \rangle^{\frac{p}{n}} \quad \text{by lemma 2.1,}$$

$$\geq \sum \langle z^n f_i | f_i \rangle^{\frac{p}{n}} \quad \text{by lemma 2.8,}$$

$$= \sum \langle z^p f_i | f_i \rangle$$

$$= \text{Trace}(A^{\frac{1}{n}} - B^{\frac{1}{n}})^p = \|A^{\frac{1}{n}} - B^{\frac{1}{n}}\|_p^p . \quad \blacksquare$$

2.5 Lemma: Let k be a nonnegative real number, y and z positive operators and f a unit vector for which $zf = \lambda f$ for some nonnegative number λ. Then, $\langle (y+z)^k yf | f\rangle \geq 0$ and $\langle y(y+z)^k f|f\rangle \geq 0$.

proof: We first assume $0 \leq k \leq 1$. By [4, p. 8] we have the formula:

$$x^k = C_k \int_0^\infty \alpha^{-k}\ x(1+\alpha x)^{-1}\ d\alpha \quad \text{for} \quad x \in \mathbb{R}^+$$

where the Riemann sums converge uniformly to x^k for x in a bounded interval. Hence

$$(y+z)^k = C_k \int_0^\infty \alpha^{-k}(y+z)(1+\alpha(y+z))^{-1}\ d\alpha$$

and the integral converges in the norm. Thus

$$\langle (y+z)^k yf|f\rangle = C_k \int_0^\infty \alpha^{-k}\langle (y+z)(1+\alpha(y+z))^{-1}yf|f\rangle d\alpha.$$

So, it suffices to see that $\langle (y+z)(1+\alpha(y+z))^{-1}yf|f\rangle$ is nonnegative for all $\alpha \geq 0$. Since this quantity equals $\langle (y(1+\alpha(y+z))^{-1}yf|f\rangle +$ $\lambda\langle (1+\alpha(y+z))^{-1}yf|f\rangle$, it suffices to see that $\langle (1+\alpha(y+z))^{-1}yf|f\rangle \geq 0$. Now, replacing αy with y and αz with z we see that it suffices to see that

$$\langle (1+y+z)^{-1}yf|f\rangle \geq 0.$$

To see this, let $\tilde{z} = (1+z)^{-\frac{1}{2}}$ so that $\tilde{z}f = \tilde{\lambda}f$ where $\tilde{\lambda} = (1+\lambda)^{-\frac{1}{2}}$. Then, $\langle (1+y+z)^{-1}yf|f\rangle = \langle [(1+z)^{\frac{1}{2}}(1+\tilde{z}y\tilde{z})(1+z)^{\frac{1}{2}}]^{-1}yf|f\rangle = \langle \tilde{z}(1+\tilde{z}y\tilde{z})^{-1}\tilde{z}yf|f\rangle =$ $\langle (1+\tilde{z}y\tilde{z})^{-1}\tilde{z}yf|\tilde{z}f\rangle = \langle (1+\tilde{z}y\tilde{z})^{-1}\tilde{z}yf|\tilde{\lambda}f\rangle = \langle (1+\tilde{z}y\tilde{z})^{-1}\tilde{z}y\tilde{z}f|f\rangle \geq 0$ as required.

Now, suppose we know $\langle (y+z)^k yf|f\rangle \geq 0$ for all k with $0 \leq k \leq n$ where n is a positive integer. In particular, we have just shown that this is true for $n = 1$. Now, let $0 \leq k \leq n+1$. If $0 \leq k \leq n$ we're done, so suppose $n \leq k \leq n+1$. Then,

$$\langle (y+z)^k yf|f\rangle = \langle (y+z)(y+z)^{k-1}yf|f\rangle = \langle y(y+z)^{k-1}yf|f\rangle + \lambda\langle (y+z)^{k-1}yf|f\rangle$$

which is nonnegative since $0 \le k-1 \le n$. Therefore, by induction, $\langle (y+z)^k yf|f\rangle \ge 0$ for all $k \ge 0$. Since, $\langle (y+z)^k yf|f\rangle$ is real, we also have $\langle y(y+z)^k f|f\rangle = \langle (y+z)^k yf|f\rangle \ge 0$ for all $k \ge 0$. ■

2.6 <u>Lemma</u>: Let $2 \le n \le 3$, let y and z be positive operators and let f be a unit eigenvector for z. Then, $\langle (y+z)^n f|f\rangle \ge \langle y^n f|f\rangle + \langle z^n f|f\rangle$.

<u>proof</u>: Let $n = 2 + k$ with $0 \le k \le 1$. Then

$$\begin{aligned}
\langle (y+z)^n f|f\rangle &= \langle (y+z)(y+z)^k (y+z)f|f\rangle \\
&= \langle y(y+z)^k yf|f\rangle + \langle z(y+z)^k zf|f\rangle + \langle y(y+z)^k zf|f\rangle + \langle z(y+z)^k yf|f\rangle \\
&\ge \langle yy^k yf|f\rangle + \langle zz^k zf|f\rangle + \lambda[\langle y(y+z)^k f|f\rangle + \langle (y+z)^k yf|f\rangle] \\
&\ge \langle y^n f|f\rangle + \langle z^n f|f\rangle \text{ by lemma 2.5.} \quad ■
\end{aligned}$$

2.7 <u>Remarks</u>: The conclusion of lemma 2.6 can fail for $1 < n < 2$. For example, let $y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $z = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, $f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $n = \frac{3}{2}$. Then, $(y+z)^{\frac{3}{2}} \doteq \begin{bmatrix} 1.2828 & 2.3140 \\ 2.3140 & 8.2247 \end{bmatrix}$ so that $\langle (y+z)^{\frac{3}{2}} f|f\rangle \doteq 1.2828$ while $\langle y^{\frac{3}{2}} f|f\rangle = \sqrt{2}$ and $\langle z^{\frac{3}{2}} f|f\rangle = 0$. However, in order to extend Theorem 2.4 to the case $1 \le n \le 2$ we need only prove lemma 2.6 for these values of n under the extra hypothesis: $(y+z)^n \ge y^n$. We firmly believe that this is true, but we have been unable to prove it.

We now extend lemma 2.6 to include all $n \ge 2$. However, we need the extra hypothesis $(y+z)^{n-2} \ge y^{n-2}$ which is automatically satisfied if $n \le 3$ or if $(y+z)^n \ge y^n$ as is the case in theorem 2.4.

2.8 Lemma: Let $n \geq 2$ be a positive real number, let y and z be positive operators with $(y+z)^{n-2} \geq y^{n-2}$, and let f be a unit eigenvector for z. Then

$$\langle (y+z)^n f|f\rangle \geq \langle y^n f|f\rangle + \langle z^n f|f\rangle.$$

proof: We first assume $3 \leq n \leq 4$. The case $2 \leq n \leq 3$ follows from 2.6 where the extra hypothesis is not needed. Then, $n = 3 + k$ with $0 \leq k \leq 1$. We calculate:

$$\begin{aligned}
\langle (y+z)^n f|f\rangle &= \langle (y+z)(y+z)^{1+k}(y+z)f|f\rangle \\
&= \langle y(y+z)^{1+k}yf|f\rangle + \langle z(y+z)^{1+k}zf|f\rangle \\
&\quad + \lambda[\langle y(y+z)^{1+k}f|f\rangle + \langle (y+z)^{1+k}yf|f\rangle] \\
&\geq \langle yy^{1+k}yf|f\rangle + \langle z(y+z)^k(y+z)zf|f\rangle \\
&= \langle y^n f|f\rangle + \lambda^2\langle (y+z)^k yf|f\rangle + \lambda^3\langle (y+z)^k f|f\rangle \\
&\geq \langle y^n f|f\rangle + \lambda^3\langle z^k f|f\rangle = \langle y^n f|f\rangle + \langle z^n f|f\rangle.
\end{aligned}$$

To prove the lemma for $n \geq 4$ we use induction. Let $k > 2$ be a positive integer and let P_k be the statement of this lemma with the added hypothesis $n \leq k$. We have already shown that P_3 and P_4 are true. Thus, let $k \geq 5$ and assume P_j is true for all integers j with $2 \leq j \leq k-1$.

Now, suppose $2 \leq n \leq k$. If, in fact, $n \leq k-1$, then since P_{k-1} is true, we are done. So, suppose $k-1 \leq n \leq k$. Then $(n-2) \geq (k-3) \geq 2$ and by monotonicity, $(y+z)^{n-2} \geq y^{n-2}$ implies $(y+z)^m \geq y^m$ for all m with $0 \leq m \leq n-2$. In particular, the hypothesis of P_{k-2} hold and so $\langle (y+z)^{n-2}f|f\rangle \geq \langle y^{n-2}f|f\rangle + \langle z^{n-2}f|f\rangle$. Finally, we calculate:

$$\begin{aligned}
\langle (y+z)^n f|f\rangle &= \langle (y+z)(y+z)^{n-2}(y+z)f|f\rangle \\
&= \langle y(y+z)^{n-2}yf|f\rangle + \langle z(y+z)^{n-2}zf|f\rangle
\end{aligned}$$

$$+ \lambda[\langle y(y+z)^{n-2}f|f\rangle + \langle (y+z)^{n-2}yf|f\rangle]$$

$$\geq \langle yy^{n-2}yf|f\rangle + \lambda^2\langle (y+z)^{n-2}f|f\rangle$$

$$\geq \langle y^n f|f\rangle + \lambda^2[\langle y^{n-2}f|f\rangle + \langle z^{n-2}f|f\rangle]$$

$$\geq \langle y^n f|f\rangle + \lambda^2\langle z^{n-2}f|f\rangle = \langle y^n f|f\rangle + \langle z^n f|f\rangle,$$

as required. hence, P_k is true. Thus, P_k is true for all positive integers $k \geq 3$ and so the lemma is proved. ∎

In case $p = n$ we can prove the theorem even for n in the interval $1 \leq n \leq 2$.

2.9 <u>Theorem</u>: Suppose $1 \leq n \leq 2$ is a real number and A, B are positive operators. Then

$$\|A^{\frac{1}{n}} - B^{\frac{1}{n}}\|_n \leq \|A-B\|_1^{\frac{1}{n}}.$$

<u>proof</u>: As in the proof of 2.4, we assume $A \geq B \geq 0$. Let $z = A^{\frac{1}{n}} - B^{\frac{1}{n}} \geq 0$ and $y = B^{\frac{1}{n}} \geq 0$. Then, $(y+z)^n - y^n \geq 0$ and we want to show that Trace $z^n \leq$ Trace$[(y+z)^n - y^n]$. Let $n = 1 + k$ and assume that all the operators are finite rank. Then

$$\text{Trace}(y+z)^n = \text{Trace}(y+z)(y+z)^k$$

$$= \text{Trace } y(y+z)^k + \text{Trace } z(y+z)^k$$

$$= \text{Trace } y^{\frac{1}{2}}(y+z)^k\, y^{\frac{1}{2}} + \text{Trace } z^{\frac{1}{2}}(y+z)^k\, z^{\frac{1}{2}}$$

$$\geq \text{Trace } y^{\frac{1}{2}}y^k y^{\frac{1}{2}} + \text{Trace } z^{\frac{1}{2}}z^k z^{\frac{1}{2}}$$

$$= \text{Trace } y^n + \text{Trace } z^n.$$

Since all quantities are finite, this does it.

Now, in the general case $A \geq B \geq 0$, we let P be a finite rank projection, so that $PAP \geq PBP \geq 0$. Thus, by the result just proved $\|(PAP)^{\frac{1}{n}}-(PBP)^{\frac{1}{n}}\|_n \leq \|PAP-PBP\|_1^{\frac{1}{n}} \leq \|A-B\|_1^{\frac{1}{n}}$. Now, by an argument similar to that of proposition 1.1 we see that $(PAP)^{\frac{1}{n}} \to A^{\frac{1}{n}}$ in the strong operator topology as P increases to I. Thus, $[(PAP)^{\frac{1}{n}}-(PBP)^{\frac{1}{n}}]^n \to [A^{\frac{1}{n}}-B^{\frac{1}{n}}]^n$ in the strong operator topology and one easily concludes that $\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_n \leq \sup_P\|(PAP)^{\frac{1}{n}}-(PBP)^{\frac{1}{n}}\|_n$. This concludes the proof. ∎

However, if $1 \leq n \leq 2$ and $p \geq n$, we are only able to prove the much weaker inequality below.

2.10 <u>Proposition</u>: Suppose $1 \leq n \leq 2$ and $p \geq n$ are real numbers and A, B are positive operators. Then, $\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p \leq (\|A\|_{\frac{p}{n}}^{\frac{1}{2n}}+\|B\|_{\frac{p}{n}}^{\frac{1}{2n}})\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}}$.

<u>proof</u>: We calculate:

$$\begin{aligned}\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p &= \|(A^{\frac{1}{2n}})^2-(B^{\frac{1}{2n}})^2\|_p \\ &\leq \|A^{\frac{1}{2n}}(A^{\frac{1}{2n}}-B^{\frac{1}{2n}})\|_p + \|(A^{\frac{1}{2n}}-B^{\frac{1}{2n}})B^{\frac{1}{2n}}\|_p \\ &\leq \|A^{\frac{1}{2n}}\|_{2p}\|A^{\frac{1}{2n}}-B^{\frac{1}{2n}}\|_{2p} + \|A^{\frac{1}{2n}}-B^{\frac{1}{2n}}\|_{2p}\|B^{\frac{1}{2n}}\|_{2p}\end{aligned}$$

by Hölder's inequality since $\frac{1}{2p}+\frac{1}{2p}=\frac{1}{p}$ [6, theorem 2.8]. Now, $2p \geq 2n \geq 2$ and so Theorem 2.4 applies. Thus,

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_p \le (\|A^{\frac{1}{2n}}\|_{2p}+\|B^{\frac{1}{2n}}\|_{2p})\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}}$$

$$= (\|A\|_{\frac{p}{n}}^{\frac{1}{2n}}+\|B\|_{\frac{p}{n}}^{\frac{1}{2n}})\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}}$$

as claimed. ■

2.11 Remark: The inequality in 2.10 is always weaker than the desired inequality, 2.4 as the following calculation shows.

$$\|A-B\|_{\frac{p}{n}}^{\frac{1}{n}} = (\|A-B\|_{\frac{p}{n}}^{\frac{p}{n}})^{\frac{1}{2p}}\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}}$$

$$\le (\|A\|_{\frac{p}{n}}^{\frac{p}{n}}+\|B\|_{\frac{p}{n}}^{\frac{p}{n}})^{\frac{1}{2p}}\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}} \quad \text{by remarks before 2.3}$$

$$\le (\|A\|_{\frac{p}{n}}^{\frac{1}{2n}}+\|B\|_{\frac{p}{n}}^{\frac{1}{2n}})\|A-B\|_{\frac{p}{n}}^{\frac{1}{2n}} \quad \text{by lemma 1.2.}$$

2.12 Remark: If $n \ge 2$ and A, B are positive operators with $A - B$ in $C_{\frac{p}{n}}$ for some $p \ge n$, then $A - B$ is in $C_{\frac{p'}{n}}$ for any $p' \ge p$. Thus, $A^{\frac{1}{n}} - B^{\frac{1}{n}}$ is in $C_{p'}$ for all $p' \ge p$. If we let $p \to \infty$ in theorem 2.4 then we obtain $\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_\infty \le \|A-B\|_\infty^{\frac{1}{n}}$. Now, if A and B are arbitrary positive operators and P is a finite rank projection, then we obtain

$$\|(PAP)^{\frac{1}{n}}-(PBP)^{\frac{1}{n}}\|_\infty \le \|PAP-PBP\|_\infty^{\frac{1}{n}} \le \|A-B\|_\infty^{\frac{1}{n}} .$$

Now, by an argument similar to that of proposition 1.1 we see that $(PAP)^{\frac{1}{n}} \to A^{\frac{1}{n}}$ in the strong operator topology as P increases to I. Thus,

$(PAP)^{\frac{1}{n}} - (PBP)^{\frac{1}{n}} \to A^{\frac{1}{n}} - B^{\frac{1}{n}}$ in the strong operator topology, and so

$$\|A^{\frac{1}{n}}-B^{\frac{1}{n}}\|_\infty \le \overline{\lim}\|(PAP)^{\frac{1}{n}}-(PBP)^{\frac{1}{n}}\|_\infty \le \|A-B\|_\infty^{\frac{1}{n}}$$

which is the conclusion of 1.6 (at least for $n \ge 2$). Thus, we are forced to admit that Mitch Baker was right (although we hate to admit it): one can indeed obtain the uniform norm estimate (1.6) from Powers-Stormer-type estimates (2.4). However, given the difficulty of proving 2.4 and the ease of proving 1.6, it is pretty silly to use 2.4 to prove 1.6.

Finally, as a simple application of 2.4 we show the continuity of the absolute value function on C_p for $p \ge 2$. We note that for $p = 2$, the stronger estimate $\||A|-|B|\|_2 \le \sqrt{2}\|A-B\|_2$ was obtained by H. Araki and S. Yamagami [1]: see also [3] for a much simpler proof. Their arguments do not seem to generalize to the case $p \ne 2$, however. We suspect that a much better estimate than ours holds in general.

2.13 <u>Proposition</u>: If $2 \le p \le \infty$ is a positive number and A, B are operators in C_p, then

$$\||A|-|B|\|_p \le (\|A\|_p+\|B\|_p)^{\frac{1}{2}}\|A-B\|_p^{\frac{1}{2}} .$$

<u>proof</u>: $\||A|-|B|\|_p^2 = \|(A^*A)^{\frac{1}{2}}-(B^*B)^{\frac{1}{2}}\|_p^2$

$\le \|A^*A-B^*B\|_{\frac{p}{2}}$ by theorem 2.4

$\le \|A^*(A-B)\|_{\frac{p}{2}} + \|(A^*-B^*)B\|_{\frac{p}{2}}$

$\leq \|A^*\|_p \|A-B\|_p + \|A^*-B^*\|_p \|B\|_p$ by Hölder's inequality

$= (\|A\|_p + \|B\|_p)\|A-B\|_p$. ∎

REFERENCES

1. H. Araki and S. Yamagami, An inequality for the Hilbert-Schmidt norm, Commun. Math. Phys., 81(1981), 89-96.

2. I.C. Gohberg and M.G. Krein, Introduction to the theory of linear non-selfadjoint operators, Transl. Math. Monog., 18(1969), Amer. Math. Soc., Providence, R.I.

3. F. Kittaneh, Inequalities for the Schatten p-norm, III, Commun. Math. Phys., 104(1986), 307-310.

4. G.K. Pedersen, C*-algebras and their automorphism groups, L.M.S. Monographs, 14(1979), Academic Press, London, New York, San Francisco.

5. R.T. Powers and E. Stormer, Free states of the canonical anticommutation relations, Commun. Math. Phys., 16(1970), 1-33.

6. B. Simon, Trace ideals and their applications, L.M.S. Lecture Note Series, 35(1979), Cambridge University Press, Cambridge.