2-Limited Broadcast Domination on Grid Graphs

by

Aaron Slobodin
B.A.Sc., Quest University Canada, 2018

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We acknowledge with respect the Lekwungen peoples on whose traditional territory
the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose
historical relationships with the land continue to this day.

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Supervisory Committee

Dr. Gary MacGillivray, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. Frank Ruskey, Co-Supervisor
(Department of Computer Science)

Dr. Wendy Myrvold, Outside Member
(Department of Computer Science)

ABSTRACT

Suppose there is a transmitter located at each vertex of a graph G. A *k-limited broadcast* on G is an assignment of the integers $0, 1, \ldots, k$ to the vertices of G. The integer assigned to the vertex x represents the strength of the broadcast from x, where strength 0 means the transmitter at x is not broadcasting. A broadcast of positive strength s from x is heard by all vertices at distance at most s from x. A k-limited broadcast is called *dominating* if every vertex assigned 0 is within distance d of a vertex whose transmitter is broadcasting with strength at least d. The *k-limited broadcast domination number* of G is the minimum possible value of the sum of the strengths of the broadcasts in a k-limited dominating broadcast of G.

We establish upper and lower bounds for the 2-limited broadcast domination number of various grid graphs, in particular the Cartesian products of two paths, a path and a cycle, and two cycles. The upper bounds are derived by explicit broadcast constructions for these graphs. The lower bounds are obtained via linear programming duality by finding lower bounds for the fractional 2-limited multipacking number of these graphs. Finally, we present an algorithm to improve the lower bound for the 2-limited broadcast domination number of special sub-families of grids. We conclude this thesis with suggested open problems in broadcast domination and multipackings.

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Chapter 1
Introduction

Consider a city partitioned into neighbourhoods, each of which has a communication tower capable of transmitting a radio broadcast of some non-negative strength. Given this premise, a natural question is: *how can we design a broadcast scheme such that broadcasts are heard by each neighbourhood while reducing the number of communication towers transmitting?* The discrete version of this question leads to broadcast domination, which was introduced by Erwin in 2001 [15].

Given a graph G, each vertex v in G is assigned a broadcast strength $f(v) \in \{0, 1, \ldots, rad(G)\}$. We say that a vertex u *hears* the broadcast from v if $d(u, v) \leq f(v)$. A broadcast f is *dominating* if each vertex of G hears some broadcast. The *cost* of a broadcast f is $\sum_{v \in V(G)} f(v)$. The *broadcast domination number* of G is the minimum cost of a dominating broadcast, denoted $\gamma_b(G)$. Observe that if $f(v) \in \{0, 1\}$ for all $v \in V(G)$, then f is the characteristic function of a dominating set. Given this, it follows that $\gamma_b(G) \leq \gamma(G)$ for all graphs G, where $\gamma(G)$ denotes the domination number of G.

The broadcast domination number $\gamma_b(G)$ of a graph G can also be formulated as follows. Let G be a graph. For each vertex $i \in V(G)$ and $k \in \{1, \ldots, rad(G)\}$ let

$$x_{i,k} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } k \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The broadcast domination number $\gamma_b\,(G)$ of a graph G is the cost of an optimal solution to the following integer linear program (ILP).

$$\text{Minimize:} \quad \sum_{k=1}^{rad(G)} \sum_{i \in V(G)} k \cdot x_{i,k}$$

$$\text{Subject to:} \quad (1) \quad \sum_{k=1}^{rad(G)} \sum_{\substack{i \in V(G) \text{ s.t.} \\ d(i,j) \leq k}} x_{i,k} \geq 1, \quad \text{for each vertex } j \in V(G),$$

$$(2) \quad x_{i,k} \in \{0,1\} \quad \text{for each vertex } i \in V(G) \text{ and } k \in \{1, \ldots, rad(G)\}.$$

The broadcast domination number of a graph was first introduced in Erwin's Ph.D. thesis [15] in 2001 (also see [16]). Erwin proved that, for every non-trivial connected graph,

$$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma_b(G) \leq \min\{rad(G), \gamma(G)\}.$$

A nice corollary of this result is $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$. Broadcast domination in trees was first explored in Herke's thesis [27] and later in [28] and [9]. These works introduce the concepts of *shadow trees*, *split-sets*, and *split-edges* and use these concepts to determine known values and general bounds for the broadcast domination number of trees. In general, their work establishes $\gamma_b(T) \leq \lceil \frac{n}{3} \rceil$, where T is a tree of order n. The broadcast domination number is known for the Cartesian products of two paths [13], two cycles [13], and strong grids [4]. Note that, as $\gamma_b(C_m \square C_n) \leq \gamma_b(P_m \square C_n) \leq \gamma_b(P_m \square P_n)$, the previously stated results provide bounds for $\gamma_b(P_m \square C_n)$.

A broadcast is said to be *efficient* if every vertex hears exactly one broadcast. In general, every graph G has an optimal efficient dominating broadcast [13]. The broadcast domination number of a graph with n vertices can be found in $O(n^6)$ time [25]. There do however exist $O(n), O(n^2)$, and $O(n^3)$ time algorithms for trees [11, 12], interval graphs [8], and strongly chordal graphs [33, 35], respectively.

Broadcast domination has recently been extended to digraphs [18]. A vertex x hears a broadcast of strength s from a vertex y if the directed distance from y to x is at most s. The problem of deciding whether there exists a dominating broadcast of cost at most a given integer B is NP-complete [18].

1.1 k-Limited Broadcast Domination

A restriction of broadcast domination is k-limited broadcast domination. In this restriction, vertices of a graph G may only broadcast with strength $\{1, \ldots, k\}$ for some $1 \leq k \leq rad(G)$ as opposed $\{1, \ldots, rad(G)\}$. The *k-limited broadcast domination number* $\gamma_{b,k}(G)$ of a graph G is the minimum cost of a dominating k-limited broadcast on G. The k-limited broadcast domination number $\gamma_{b,k}(G)$ can also be formulated as follows. Let G be a graph and fix $1 \leq k \leq rad(G)$. For each vertex $i \in V(G)$ and $\ell \in \{1, \ldots, k\}$ let

$$x_{i,\ell} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } \ell \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The broadcast domination number $\gamma_{b,k}(G)$ of a graph G is the cost of an optimal solution to the following ILP.

$$\begin{aligned} \text{Minimize:} \quad & \sum_{\ell=1}^{k} \sum_{i \in V(G)} \ell \cdot x_{i,\ell} \\ \text{Subject to:} \quad & (1) \quad \sum_{\ell=1}^{k} \sum_{\substack{i \in V(G) \text{ s.t.} \\ d(i,j) \leq k}} x_{i,\ell} \geq 1, \quad \text{for each vertex } j \in V(G), \\ & (2) \quad x_{j,\ell} \in \{0,1\} \quad \text{for each vertex } j \in V(G) \text{ and } \ell \in \{1, \ldots, k\}. \end{aligned} \tag{1.1}$$

The k-limited broadcast domination number of a graph was first mentioned, although not explored, in Erwin's Ph.D. thesis [15]. The first major results for k-limited broadcast domination can be found in [6] and are specific to 2-limited broadcasts. These results were quickly generalized for all $k \geq 1$ in [7]. These results determined a best possible upper bound of $\gamma_{b,k}(T) \leq \left\lceil \frac{k+2}{k+1} \cdot \frac{n}{3} \right\rceil$, where T is a tree on n vertices, and showed that, for each fixed integer k, the problem of deciding whether there exists a k-limited dominating broadcast of cost at most a given integer B is NP-complete [7]. Additional complexity results for the k-limited broadcast domination problem can be found in [34]. Notably, the results of [34] establish $O(n^3), O(n^2)$, and $O(n^3)$ time algorithms for strongly chordal graphs, interval graphs, and proper interval bigraphs. It is worth noting that the $O(n^3)$ time algorithm for k-limited broadcast on strongly chordal graphs in [34] is a specialization of the $O(n^3)$ time algorithm for general broadcasts on strongly chordal graphs in [33, 35]. Specific to 2-limited dominating broadcasts, if G is a connected graph of order n, then $\gamma_{b,2}(G) \leq \left\lceil \frac{4n}{9} \right\rceil$ and if G is a graph of order n that contains a dominating path, then $\gamma_{b,2}(G) \leq \left\lceil \frac{2n}{5} \right\rceil$ [6]. Recently it was also shown that, if G is a cubic $(C_4, C_6)-$free graph of order n, then $\gamma_{b,2}(G) \leq \frac{n}{3}$ [34, 26].

1.2 Multipackings

The *multipacking number* of a graph was first introduced in Teshima's Master's thesis [30] in 2012 (also see [32]). It arises from interpreting the linear programming (LP) dual of the LP relaxation of the broadcast domination ILP as a 0-1 ILP. Equality between the multipacking number and the broadcast domination number is known to hold for strongly chordal graphs [33, 35], 2-dimensional square grids [1], and strong grids [29]. General differences between the broadcast and multipacking numbers of graphs are explored in [24]. The multipacking numbers of trees and strongly chordal graphs can be found in $O(n)$ [30, 31, 35] and $O(n^3)$ time [33, 35], respectively. In general, the complexity of deciding whether a given graph G has a multipacking number of at least a given integer B is unknown. There does however exist a $(2 + o(1))$ approximation algorithm for this decision problem on undirected graphs [2] and the problem was recently shown to be NP-complete for digraphs [18].

The dual of the LP relaxation of the k-limited broadcast domination ILP is *fractional k-limited multipacking.* Define $mp_k(G)$ as the maximum cost of a fractional k-limited multipacking on a graph G. For each vertex $i \in V(G)$, define a variable y_i. The fractional k-limited multipacking packing number $mp_k(G)$ of a graph G is the cost of an optimal solution to the following LP.

Maximize: $\displaystyle\sum_{i \in V(G)} y_i$

Subject to: (1) $\displaystyle\sum_{\substack{i \in V(G) \text{ s.t.} \\ d(i,j) \leq \ell}} y_i \leq \ell,$ for each vertex $j \in V(G)$ and $\ell \in \{1, \ldots, k\}$, and

(2) $y_i \geq 0,$ for each vertex $i \in V(G)$.

If the above LP is interpreted as a 0-1 ILP, a formulation of the *k-limited multipacking number* of a graph is obtained. If $k = rad(G)$, the *multipacking number* of a graph is obtained.

Just as the k-limited broadcast domination problem is a restriction of the broadcast domination problem, k-limited multipackings are a restriction of multipackings. Relatively little is known about k-limited multipackings on graphs. For each fixed integer k, the problem of deciding whether a given graph G has a k-limited multipacking number at least a given integer B is NP-complete [34].

1.3 Outline

This thesis focuses on the 2-limited broadcast domination number of the Cartesian product of two paths, a path and a cycle, and two cycles. Note that, as the 1-limited broadcast domination number (i.e. the domination number) of the Cartesian product of two paths is known [22], this is a natural place to start on these graphs. Chapter 2 introduces the concept of constructing 2-limited dominating broadcasts by "tiling" the graph with "broadcast tiles." This method of "tiling" is used to construct 2-limited dominating broadcasts on $P_2 \square P_n$ through $P_{12} \square P_n$ for all $n \geq 2$. For $m, n \geq 13$

a generalized construction is derived from a 2-limited dominating broadcast on the Cartesian plane. Chapters 3 and 4 apply similar methods of "tiling" to construct 2-limited dominating broadcasts on the Cartesian products of path and a cycle and two cycles, respectively. Chapter 5 utilizes the dual of the LP relaxation of 2-limited broadcast domination, *fractional* 2*-limited multipacking* to establish lower bounds for the 2-limited broadcast domination number of the Cartesian product of paths, a path and a cycles, and cycles. Chapter 6 presents a computational technique to improve the lower bounds presented in Chapter 5. Chapter 7 includes suggested questions for future research.

Unless specified otherwise, all computations in this thesis were run on a PowerEdge R7425 server equipped with two AMD EPYC processors and totalling 64 threaded CPU cores. This machine was purchased using NSERC funding by Wendy Myrvold. Unless specified otherwise, all ILPs and LPs were solved in Python 3.9 with the Coin-or branch and cut solver [10]. The figures in this thesis were constructed using a Python script (written by the author) which outputted the TikZ code of a given 2-limited broadcast on a graph (represented as a matrix of 0s, 1s, and 2s).

Chapter 2
Cartesian Products of Two Paths

2.1 Introduction

Given the current difficulty of determining an optimal 2-limited dominating broadcast on $P_m \Box P_n$ for a given $m, n \geq 2$ by computation, it is worth establishing upper bounds. Providing explicit constructions of 2-limited dominating broadcasts is a natural place to start. This chapter describes such constructions. That is, given any $m, n \geq 2$, this chapter provides a construction and associated cost of a 2-limited dominating broadcast on $P_m \Box P_n$. These results establish upper bounds for $\gamma_{b,2}(P_m \Box P_n)$ for all $m, n \geq 2$.

For $2 \leq m \leq 12$ and all $n \geq 2$, the constructed broadcasts are made by using broadcast "tiles" which, when arranged on the graph, determine the broadcast scheme. These broadcast "tiles" act as a certificate that the broadcast scheme dominates the graph and provide easily replicable broadcasts. The broadcast tiles for these cases were identified by hand by examining known optimal solutions, as found by computation, and looking for recurring patterns. This manual process was greatly facilitated by automatically creating PDF visuals of known optimal solutions using a Python script (written by the author). These visuals are not present in this document but are similar to the figures throughout. For $m, n \geq 13$, the constructed broadcasts are modifications of 2-limited dominating broadcasts on the Cartesian plane.

Before proceeding to our constructions, we provide an overview of the "tiling" concept used for $2 \leq m \leq 12$ and all $n \geq 2$. Figure 2.1.1 is an example of a broadcast tile B which can be used to construct broadcast schemes on $P_4 \Box P_n$. The thick blue

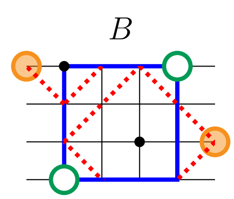


Figure 2.1.1: Toy tile used to dominate $P_4 \Box P_n$.

border indicates the border of B. In this example, B is a 4×4 sized tile. The black circles with a black inner fill indicate vertices on B broadcasting at a non-zero strength. In this example, B has one vertex broadcasting at strength one in the upper left corner and one vertex broadcasting at strength two in the middle lower right. The thick red dotted lines indicate the broadcast ranges of the broadcasting vertices at their centers. Thick green circles with a white inner fill indicate vertices on B which do not hear a broadcast within B. Thick orange circles with an opaque orange inner fill indicate vertices exterior to B which can hear broadcasts from vertices on B.

Given a tile B, $\overline{B}$ denotes flipping B about the horizontal axis, $B^{|}$ indicates flipping B about the vertical axis, and $\overline{B}^{|}$ indicates flipping B about the horizontal and the vertical axes. The four possible states of B are shown in Figure 2.1.2. Given two tiles

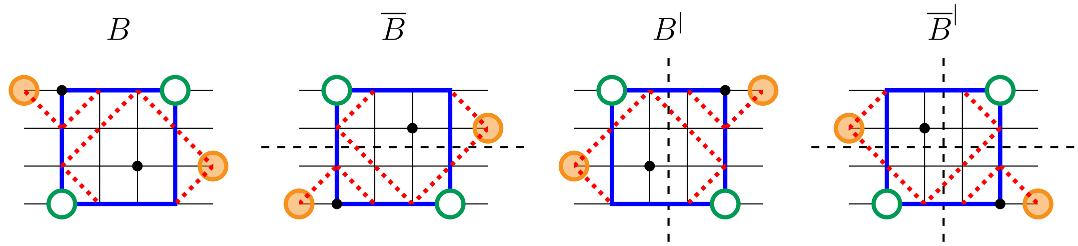


Figure 2.1.2: Possible states of toy tile B.

B_1 and B_2, the sequence B_1B_2 denotes placing the right side of B_1 beside the left side of B_2 such that the vertices along the right border of B_1 and the left border of B_2 are at distance one from one another. Figure 2.1.3 depicts $\overline{B}^{|}B$.

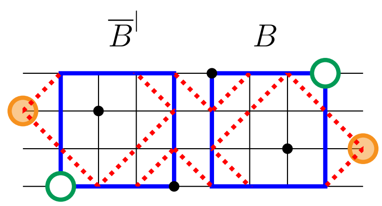


Figure 2.1.3: $\overline{B}^{|}B$ given toy tile B.

Observe that, in Figure 2.1.3, the undominated vertex in the upper right corner of $\overline{B}^{|}$ hears the broadcast from the vertex in the upper left corner of B. Similarly, in Figure 2.1.3, the undominated vertex in the bottom left corner of B hears the broadcast from the vertex in the bottom right corner of $\overline{B}^{|}$.

To see how these tiles can be used to construct dominating broadcast schemes, consider the following example:

Example 2.1.1. Given the tiles B and E in Figure 2.1.4, $E\overline{B}^{|}B\overline{E}^{|}$ provides a dominating 2-limited broadcast scheme on $P_4 \square P_{12}$ of cost 12, as shown in Figure 2.1.5.

Note that the broadcast scheme in Figure 2.1.5 is not optimal as $\gamma_{b,2}(P_4 \square P_{12}) = 11$ (as shown in Section 2.4). This example is simply meant to provide practice with the

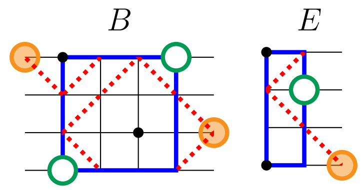


Figure 2.1.4: Toy tiles B and E.

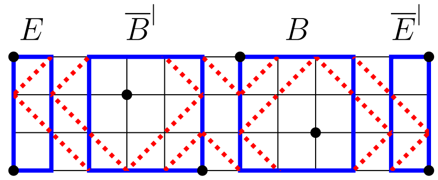


Figure 2.1.5: $E\overline{B}^{|}B\overline{E}^{|}$ given toy tiles B and E.

method of tiling prior to proceeding to our constructions on $P_2 \square P_n$ through $P_{12} \square P_n$, for all $n \geq 2$, in Sections 2.2 through 2.12, respectively.

By constructing 2-limited dominating broadcasts on $P_m \square P_n$ based on broadcast tiles, we hope to sometimes achieve optimal upper bounds for $\gamma_{b,2}(P_m \square P_n)$ as there exists locally optimal 2-limited dominating broadcasts on the infinite Cartesian strip that use broadcast tiles. By "locally optimal," we mean that no sub-region of the infinite Cartesian strip can be dominated with less cost. To see this, fix an integer m and consider a locally optimal 2-limited dominating broadcast f on the infinite Cartesian strip of width m. In a given column, each vertex can broadcast at three possible broadcast strengths (0, 1, or 2). There are therefore 3^m possible broadcasts within a given column. Consider a set of five consecutive columns C. The centre column of C is necessarily dominated by the vertices on C. There are 3^{5m} possible broadcasts on C. Consider a section of infinite Cartesian strip of width m with $5 \cdot 3^{5m} + 5$ columns. By the pigeonhole principle, there exists two distinct sets of five consecutive columns, C_1 and C_2, which have the same broadcast. Without loss of generality, assume that C_1 is left of C_2. Let B be the tile formed by the broadcast in the columns of C_1 and the columns to the right of C_1 up to and not including C_2. Figure 2.1.6 depicts C_1 and C_2, as indicated by the diagonally lined regions, and the broadcast tile B, as indicated by the thick black rectangle with an inner grey fill, on the infinite Cartesian strip of width m.

We can construct another locally optimal 2-limited dominating broadcast on the infinite Cartesian strip of width m with an infinite sequence of B's. This construction is necessarily locally optimal else this would contradict the optimality of f. There therefore exists a locally optimal 2-limited dominating broadcast on an infinite Cartesian strip of width m based on tiling. Given this, it follows that for large enough m, n there are optimal 2-limited dominating broadcasts on $P_m \square P_n$ which use broadcast

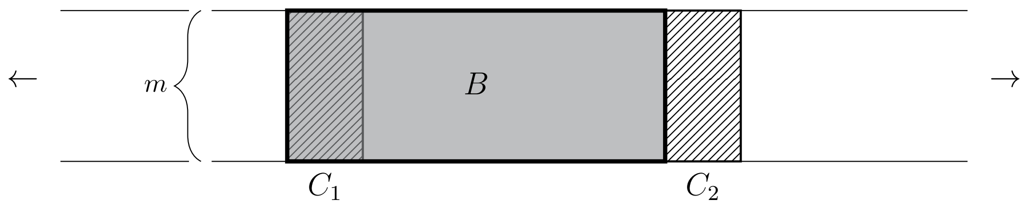


Figure 2.1.6: Existence of a broadcast tile B on infinite Cartesian strip of width m.

tiles. With this justification in hand, we proceed to our constructions in Sections 2.2 through 2.13.

2.2 $P_2 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_2 \square P_n$ for all $n \geq 2$. Figure 2.2.1 contains the tiles used in the broadcast constructions described in Table 2.2.1.

Proposition 2.2.1. *For $n \geq 2$, $\gamma_{b,2}(P_2 \square P_n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

	Construction	Cost
$n \equiv 0 \pmod 4$:	$B_1 \left[B\overline{B}\right]^{\frac{n-4}{4}} BB_1$	$2\left(\frac{n-4}{4}\right) + 3 = \frac{n}{2} + 1$
$n \equiv 1 \pmod 4$:	$B_1 \left[B\overline{B}\right]^{\frac{n-1}{4}}$	$2\left(\frac{n-1}{4}\right) + 1 = \frac{n-1}{2} + 1$
$n \equiv 2 \pmod 4$:	$B_1 \left[B\overline{B}\right]^{\frac{n-2}{4}} B_1$	$2\left(\frac{n-2}{4}\right) + 2 = \frac{n}{2} + 1$
$n \equiv 3 \pmod 4$:	$B_1 \left[B\overline{B}\right]^{\frac{n-3}{4}} B$	$2\left(\frac{n-3}{4}\right) + 2 = \frac{n-1}{2} + 1$

Table 2.2.1: Constructions of 2-limited dominating broadcasts on $P_2 \square P_n$.

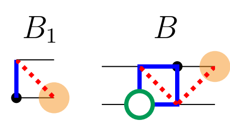


Figure 2.2.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_2 \square P_n$.

The bound given by Proposition 2.2.1 is shown to be tight by Proposition 5.4.2.

2.3 $P_3 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_3 \square P_n$ for all $n \geq 3$. Figure 2.3.1 contains the tiles used in the broadcast constructions described in Table 2.3.1.

Proposition 2.3.1. *For $n \geq 3$, $\gamma_{b,2}(P_3 \square P_n) \leq \left\lceil \frac{2n}{3} \right\rceil$.*

	Construction	Cost
$n \equiv 0 \pmod 3$:	$[B]^{\frac{n}{3}}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$:	$[B]^{\frac{n-1}{3}} B_1$	$\frac{2(n-1)}{3} + 1 = \frac{2n+1}{3}$
$n \equiv 2 \pmod 3$:	$[B]^{\frac{n-2}{3}} B_2$	$\frac{2(n-2)}{3} + 2 = \frac{2n+2}{3}$

Table 2.3.1: Constructions of 2-limited dominating broadcasts on $P_3 \square P_n$.

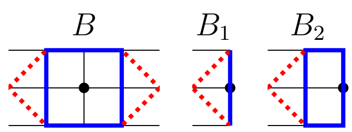


Figure 2.3.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_3 \square P_n$.

The bound given by Proposition 2.3.1 is shown to be tight by Theorem 5.3.1.

2.4 $P_4 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_4 \square P_n$ for all $n \geq 4$. Figure 2.4.1 contains the tiles used in the broadcast constructions described in Table 2.4.2.

Proposition 2.4.1. *For $n \geq 4$,*

$$\gamma_{b,2}(P_4 \square P_n) \leq 8 \left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 2.4.1.

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	1	2	3	4	4	5	6	7	8	8

Table 2.4.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}(P_4 \square P_n)$ stated in Proposition 2.4.1.

	Construction	Cost
$n \equiv 0 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-10}{10}} BB_4$	$8\left(\frac{n-10}{10}\right) + 9 = 8\left(\frac{n}{10}\right) + 1$
$n \equiv 1 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-11}{10}} BB_3$	$8\left(\frac{n-11}{10}\right) + 10 = 8\left(\frac{n-1}{10}\right) + 2$
$n \equiv 2 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-12}{10}} BB_5$	$8\left(\frac{n-12}{10}\right) + 11 = 8\left(\frac{n-2}{10}\right) + 3$
$n \equiv 3 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-3}{10}} \overline{B_2}$	$8\left(\frac{n-3}{10}\right) + 4$
$n \equiv 4 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-4}{10}} \overline{B_1}^{\mid}$	$8\left(\frac{n-4}{10}\right) + 4$
$n \equiv 5 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-5}{10}} \overline{B_4}$	$8\left(\frac{n-5}{10}\right) + 5$
$n \equiv 6 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-6}{10}} \overline{B_3}$	$8\left(\frac{n-6}{10}\right) + 6$
$n \equiv 7 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-7}{10}} \overline{B_5}$	$8\left(\frac{n-7}{10}\right) + 7$
$n \equiv 8 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-8}{10}} BB_2$	$8\left(\frac{n-8}{10}\right) + 8$
$n \equiv 9 \pmod{10}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-9}{10}} BB_1^{\mid}$	$8\left(\frac{n-9}{10}\right) + 8$

Table 2.4.2: Constructions of 2-limited dominating broadcasts on $P_4 \square P_n$.

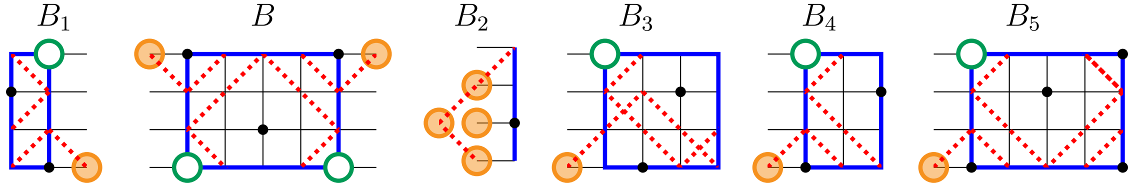


Figure 2.4.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_4 \square P_n$.

The values of Proposition 2.4.1 are optimal for all $n \leq 99$ (as found by computation). A lower bound of $\left\lceil \frac{4n}{5} \right\rceil$ is proven by Theorem 5.3.1. Therefore, the upper bounds given by Proposition 2.4.1 for all $n_{10} \in \{4, 9\}$ are tight.

Conjecture 2.4.1. *The bounds in Proposition 2.4.1 are tight.*

2.5 $P_5 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_5 \square P_n$ for all $n \geq 5$. Figure 2.5.1 contains the tiles used in the broadcast constructions described in Table 2.5.1.

Proposition 2.5.1. *For $n \geq 5$, $\gamma_{b,2}(P_5 \square P_n) \leq n + 1$.*

	Construction	Cost
$n \equiv 0 \pmod 4$:	$B_1[B]^{\frac{n-4}{4}}B_3$	$4\left(\frac{n-4}{4}\right) + 5 = n + 1$
$n \equiv 1 \pmod 4$:	$[B]^{\frac{n-1}{4}}B_2$	$4\left(\frac{n-1}{4}\right) + 2 = n + 1$
$n \equiv 2 \pmod 4$:	$B_1[B]^{\frac{n-2}{4}}B_2$	$4\left(\frac{n-2}{4}\right) + 3 = n + 1$
$n \equiv 3 \pmod 4$:	$[B]^{\frac{n-3}{4}}B_3$	$4\left(\frac{n-3}{4}\right) + 4 = n + 1$

Table 2.5.1: Constructions of 2-limited dominating broadcasts on $P_5 \square P_n$.

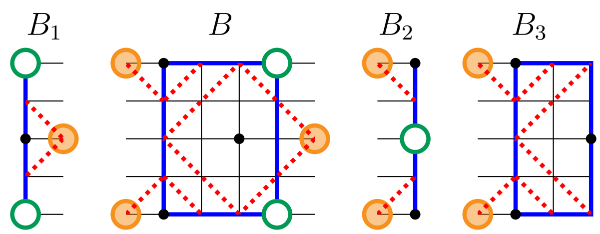


Figure 2.5.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_5 \square P_n$.

The values of Proposition 2.5.1 are optimal for all $n \leq 96$ (as found by computation). A lower bound of n is given for all $n \geq 5$ by Theorem 5.3.1.

Conjecture 2.5.1. *The bound in Proposition 2.5.1 is tight.*

2.6 $P_6 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_6 \square P_n$ for all $n \geq 6$. Figure 2.6.1 contains the tiles used in the broadcast constructions described in Table 2.6.2.

Proposition 2.6.1. *For $n \geq 6$,*

$$\gamma_{b,2}\left(P_6 \square P_n\right) \leq 18 \left\lfloor \frac{n}{16} \right\rfloor + c(n_{16})$$

where $c(n_{16})$ is dependant upon the least residue n_{16} of n modulo 16 and given in Table 2.6.1 with the exception of $n = 6$ which gives a value of $c(n_{16})$ equal to 8.

	Least residue n_{16} of n modulo 16															
n_{16}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Cost:	2	2	4	5	6	7	9	9	11	11	13	14	15	16	18	18

Table 2.6.1: Values of $c(n_{16})$ for the upper bound for $\gamma_{b,2}\left(P_6 \square P_n\right)$ stated in Proposition 2.6.1 where $n \neq 6$.

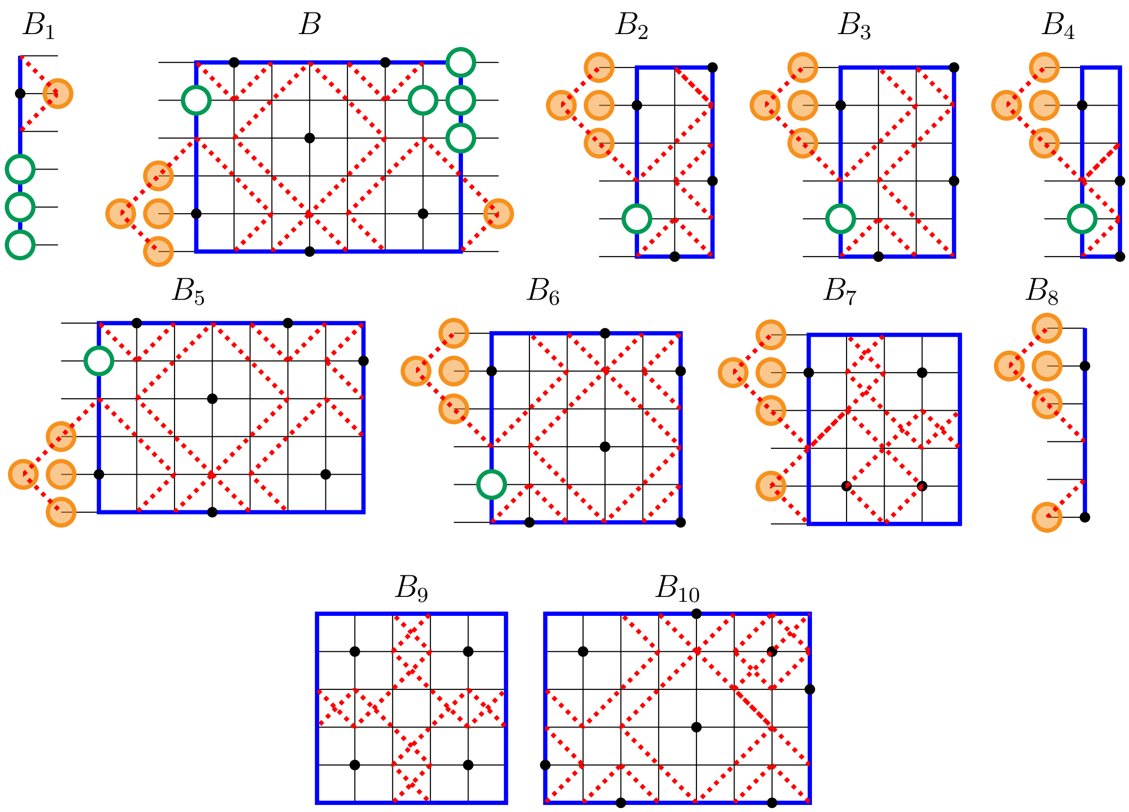


Figure 2.6.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_6 \square P_n$.

The values of Proposition 2.6.1 are optimal for all $n \leq 99$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{17.8}{16}n$.

Conjecture 2.6.1. *The bounds in Proposition 2.6.1 are tight.*

	Construction	Cost
$n \equiv 0 \pmod{16}$:	$\overline{B_5}^{\mid}\left[B\overline{B}\right]^{\frac{n-16}{16}} B_5$	$18\left(\frac{n-16}{16}\right)+20 = 18\left(\frac{n}{16}\right)+2$
$n \equiv 1 \pmod{16}$:	$B_1[B\overline{B}]^{\frac{n-17}{16}} B\overline{B_5}$	$18\left(\frac{n-17}{16}\right)+20 = 18\left(\frac{n-1}{16}\right)+2$
$n \equiv 2 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-2}{16}} \overline{B}_8$	$18\left(\frac{n-2}{16}\right)+4$
$n \equiv 3 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-3}{16}} \overline{B_4}$	$18\left(\frac{n-3}{16}\right)+5$
$n \equiv 4 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-4}{16}} \overline{B_2}$	$18\left(\frac{n-4}{16}\right)+6$
$n \equiv 5 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-5}{16}} \overline{B_3}$	$18\left(\frac{n-5}{16}\right)+7$
$n \equiv 6 \pmod{16}, (n=6)$:	B_9	8
$(n>6)$:	$B_1\left[B\overline{B}\right]^{\frac{n-6}{16}} \overline{B_7}$	$18\left(\frac{n-6}{16}\right)+9$
$n \equiv 7 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-7}{16}} \overline{B_6}$	$18\left(\frac{n-7}{16}\right)+9$
$n \equiv 8 \pmod{16}, (n=8)$:	B_{10}	$11 = 18\left(\frac{n-8}{16}\right)+11$
$(n>8)$:	$\overline{B_5}^{\mid}\left[B\overline{B}\right]^{\frac{n-24}{16}} B\overline{B_5}$	$18\left(\frac{n-24}{16}\right)+29 = 18\left(\frac{n-8}{16}\right)+11$
$n \equiv 9 \pmod{16}$:	$B_1[B\overline{B}]^{\frac{n-9}{16}} B_5$	$18\left(\frac{n-9}{16}\right)+11$
$n \equiv 10 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-10}{16}} BB_8$	$18\left(\frac{n-10}{16}\right)+13$
$n \equiv 11 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-11}{16}} BB_4$	$18\left(\frac{n-11}{16}\right)+14$
$n \equiv 12 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-12}{16}} BB_2$	$18\left(\frac{n-12}{16}\right)+15$
$n \equiv 13 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-13}{16}} BB_3$	$18\left(\frac{n-13}{16}\right)+16$
$n \equiv 14 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-14}{16}} BB_7$	$18\left(\frac{n-14}{16}\right)+18$
$n \equiv 15 \pmod{16}$:	$B_1\left[B\overline{B}\right]^{\frac{n-15}{16}} BB_6$	$18\left(\frac{n-15}{16}\right)+18$

Table 2.6.2: Constructions of 2-limited dominating broadcasts on $P_6 \square P_n$.

2.7 $P_7 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_7 \square P_n$ for all $n \geq 7$. Figure 2.7.1 contains the tiles used in the broadcast constructions described in Table 2.7.2.

Proposition 2.7.1. *For $n \geq 7$,*

$$\gamma_{b,2}(P_7 \square P_n) \leq 18 \left\lfloor \frac{n}{14} \right\rfloor + c(n_{14})$$

where $c(n_{14})$ is dependant upon the least residue n_{14} of n modulo 14 and given in Table 2.7.1.

	Least residue n_{14} of n modulo 14													
n_{14}	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Cost:	2	3	4	5	7	8	9	11	12	13	14	16	17	18

Table 2.7.1: Values of $c(n_{14})$ for the upper bound for $\gamma_{b,2}\left(P_7 \square P_n\right)$ stated in Proposition 2.7.1.

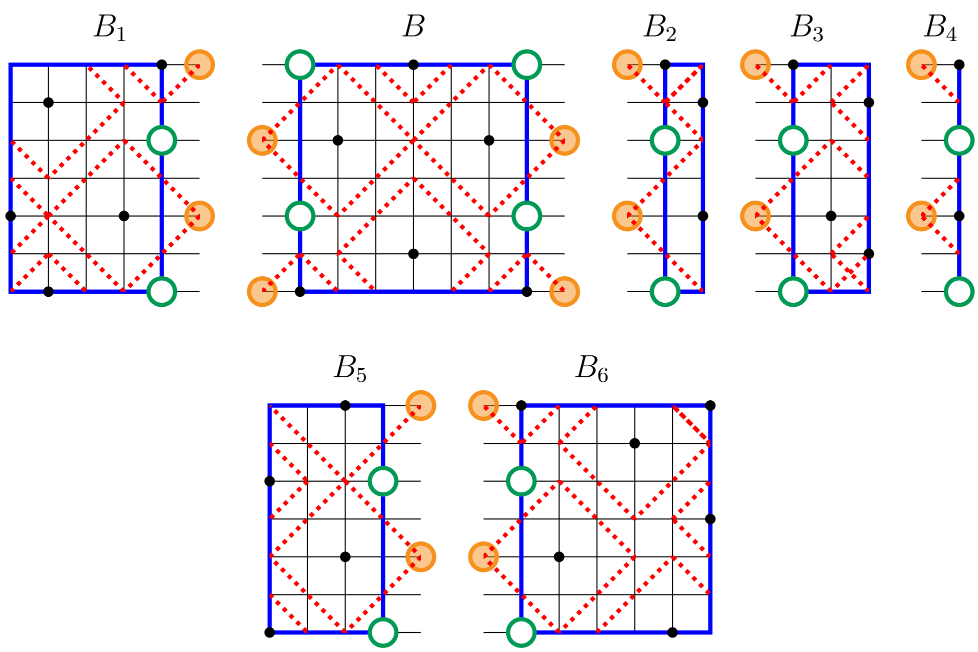


Figure 2.7.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_7 \square P_n$.

	Construction	Cost
$n \equiv 0 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-14}{14}} BB_2$	$18\left(\frac{n-14}{14}\right)+20 = 18\left(\frac{n}{14}\right)+2$
$n \equiv 1 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-15}{14}} BB_3$	$18\left(\frac{n-15}{14}\right)+21 = 18\left(\frac{n-1}{14}\right)+3$
$n \equiv 2 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-16}{14}} BB_5^{|}$	$18\left(\frac{n-16}{14}\right)+22 = 18\left(\frac{n-2}{14}\right)+4$
$n \equiv 3 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-17}{14}} BB_1^{|}$	$18\left(\frac{n-17}{14}\right)+23 = 18\left(\frac{n-3}{14}\right)+5$
$n \equiv 4 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-18}{14}} BB_6$	$18\left(\frac{n-18}{14}\right)+25 = 18\left(\frac{n-4}{14}\right)+7$
$n \equiv 5 \pmod{14}$:	$B_5 \left[B\overline{B}\right]^{\frac{n-5}{14}} \overline{B_4}$	$18\left(\frac{n-5}{14}\right)+8$
$n \equiv 6 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-6}{14}} \overline{B_4}$	$18\left(\frac{n-6}{14}\right)+9$
$n \equiv 7 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-7}{14}} \overline{B_2}$	$18\left(\frac{n-7}{14}\right)+11$
$n \equiv 8 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-8}{14}} \overline{B_3}$	$18\left(\frac{n-8}{14}\right)+12$
$n \equiv 9 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-9}{14}} \overline{B_5^{|}}$	$18\left(\frac{n-9}{14}\right)+13$
$n \equiv 10 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-10}{14}} \overline{B_1^{|}}$	$18\left(\frac{n-10}{14}\right)+14$
$n \equiv 11 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-11}{14}} \overline{B_6}$	$18\left(\frac{n-11}{14}\right)+16$
$n \equiv 12 \pmod{14}$:	$B_5 \left[B\overline{B}\right]^{\frac{n-12}{14}} BB_4$	$18\left(\frac{n-12}{14}\right)+17$
$n \equiv 13 \pmod{14}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-13}{14}} BB_4$	$18\left(\frac{n-13}{14}\right)+18$

Table 2.7.2: Constructions of 2-limited dominating broadcasts on $P_7 \square P_n$.

The values of Proposition 2.7.1 are optimal for all $n \leq 99$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{17.7}{14}n$.

Conjecture 2.7.1. *The bounds in Proposition 2.7.1 are tight.*

2.8 $P_8 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_8 \square P_n$ for all $n \geq 8$. Figure 2.8.1 contains the tiles used in the broadcast constructions described in Table 2.8.2.

Proposition 2.8.1. *For $n \geq 8$,*

$$\gamma_{b,2}\left(P_8 \square P_n\right) \leq 32 \left\lfloor \frac{n}{22} \right\rfloor + c(n_{22})$$

where $c(n_{22})$ is dependant upon the least residue n_{22} of n modulo 22 and given in Table 2.8.1.

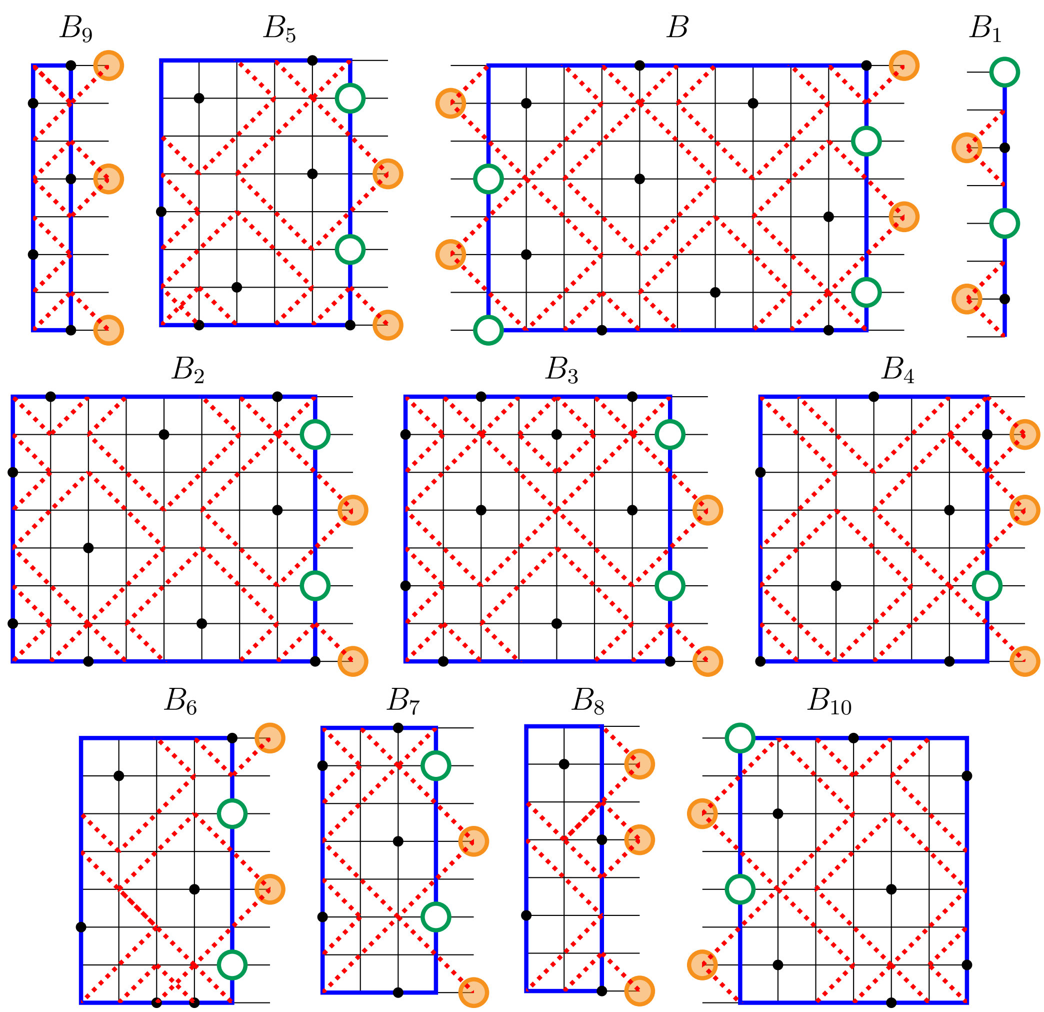
	Least residue n_{22} of n modulo 22										
n_{22}	0	1	2	3	4	5	6	7	8	9	10
Cost:	2	4	5	7	8	9	11	12	14	15	16
n_{22}	11	12	13	14	15	16	17	18	19	20	21
Cost:	18	20	21	23	24	25	27	28	30	31	32

Table 2.8.1: Values of $c(n_{22})$ for the upper bound for $\gamma_{b,2}(P_8 \square P_n)$ stated in Proposition 2.8.1.

	Construction	Cost
$n \equiv 0 \pmod{22}$:	$B_7 \left[B\overline{B}\right]^{\frac{n-22}{22}} BB_{10}$	$32\left(\frac{n-22}{22}\right)+34 = 32\left(\frac{n}{22}\right)+2$
$n \equiv 1 \pmod{22}$:	$\overline{B_{11}} \left[B\overline{B}\right]^{\frac{n-23}{22}} BB_1$	$32\left(\frac{n-23}{22}\right)+36 = 32\left(\frac{n-1}{22}\right)+4$
$n \equiv 2 \pmod{22}$:	$B_5 \left[B\overline{B}\right]^{\frac{n-24}{22}} BB_{10}$	$32\left(\frac{n-24}{22}\right)+37 = 32\left(\frac{n-2}{22}\right)+5$
$n \equiv 3 \pmod{22}$:	$B_9 \left[B\overline{B}\right]^{\frac{n-3}{22}} \overline{B_1}$	$32\left(\frac{n-3}{22}\right)+7$
$n \equiv 4 \pmod{22}$:	$B_8 \left[B\overline{B}\right]^{\frac{n-4}{22}} \overline{B_1}$	$32\left(\frac{n-4}{22}\right)+8$
$n \equiv 5 \pmod{22}$:	$B_7 \left[B\overline{B}\right]^{\frac{n-5}{22}} \overline{B_1}$	$32\left(\frac{n-5}{22}\right)+9$
$n \equiv 6 \pmod{22}$:	$\overline{B_6} \left[B\overline{B}\right]^{\frac{n-6}{22}} \overline{B_1}$	$32\left(\frac{n-6}{22}\right)+11$
$n \equiv 7 \pmod{22}$:	$B_5 \left[B\overline{B}\right]^{\frac{n-7}{22}} \overline{B_1}$	$32\left(\frac{n-7}{22}\right)+12$
$n \equiv 8 \pmod{22}$:	$B_4 \left[B\overline{B}\right]^{\frac{n-8}{22}} \overline{B_1}$	$32\left(\frac{n-8}{22}\right)+14$
$n \equiv 9 \pmod{22}$:	$B_3 \left[B\overline{B}\right]^{\frac{n-9}{22}} \overline{B_1}$	$32\left(\frac{n-9}{22}\right)+15$
$n \equiv 10 \pmod{22}$:	$B_2 \left[B\overline{B}\right]^{\frac{n-10}{22}} \overline{B_1}$	$32\left(\frac{n-10}{22}\right)+16$
$n \equiv 11 \pmod{22}$:	$B_7 \left[B\overline{B}\right]^{\frac{n-11}{22}} \overline{B_{10}}$	$32\left(\frac{n-11}{22}\right)18$
$n \equiv 12 \pmod{22}$:	$\overline{B_{11}} \left[B\overline{B}\right]^{\frac{n-12}{22}} \overline{B_1}$	$32\left(\frac{n-12}{22}\right)+20$
$n \equiv 13 \pmod{22}$:	$B_5 \left[B\overline{B}\right]^{\frac{n-13}{22}} \overline{B_{10}}$	$32\left(\frac{n-13}{22}\right)+21$
$n \equiv 14 \pmod{22}$:	$B_9 \left[B\overline{B}\right]^{\frac{n-14}{22}} BB_1$	$32\left(\frac{n-14}{22}\right)+23$
$n \equiv 15 \pmod{22}$:	$B_8 \left[B\overline{B}\right]^{\frac{n-15}{22}} BB_1$	$32\left(\frac{n-4}{22}\right)+24$
$n \equiv 16 \pmod{22}$:	$B_7 \left[B\overline{B}\right]^{\frac{n-16}{22}} BB_1$	$32\left(\frac{n-5}{22}\right)+25$

$n \equiv 17 \pmod{22}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-17}{22}} BB_1$	$32\left(\frac{n-6}{22}\right)+27$
$n \equiv 18 \pmod{22}$:	$B_5\left[B\overline{B}\right]^{\frac{n-18}{22}} BB_1$	$32\left(\frac{n-7}{22}\right)+28$
$n \equiv 19 \pmod{22}$:	$B_4\left[B\overline{B}\right]^{\frac{n-19}{22}} BB_1$	$32\left(\frac{n-8}{22}\right)+30$
$n \equiv 20 \pmod{22}$:	$B_3\left[B\overline{B}\right]^{\frac{n-20}{22}} BB_1$	$32\left(\frac{n-9}{22}\right)+31$
$n \equiv 21 \pmod{22}$:	$B_2\left[B\overline{B}\right]^{\frac{n-21}{22}} BB_1$	$32\left(\frac{n-10}{22}\right)+32$

Table 2.8.2: Constructions of 2-limited dominating broadcasts on $P_8 \square P_n$.



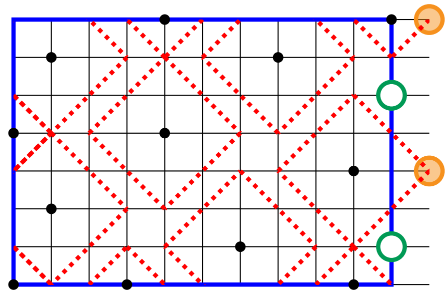


Figure 2.8.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_8 \square P_n$.

The values of Proposition 2.8.1 are optimal for all $n \leq 93$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{31.3}{22}n$.

Conjecture 2.8.1. *The bounds in Proposition 2.8.1 are tight.*

2.9 $P_9 \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_9 \square P_n$ for all $n \geq 9$. Figure 2.9.1 contains the tiles used in the broadcast constructions described in Table 2.9.2.

Proposition 2.9.1. *For $n \geq 9$,*

$$\gamma_{b,2}\left(P_9 \square P_n\right) \leq 16 \left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

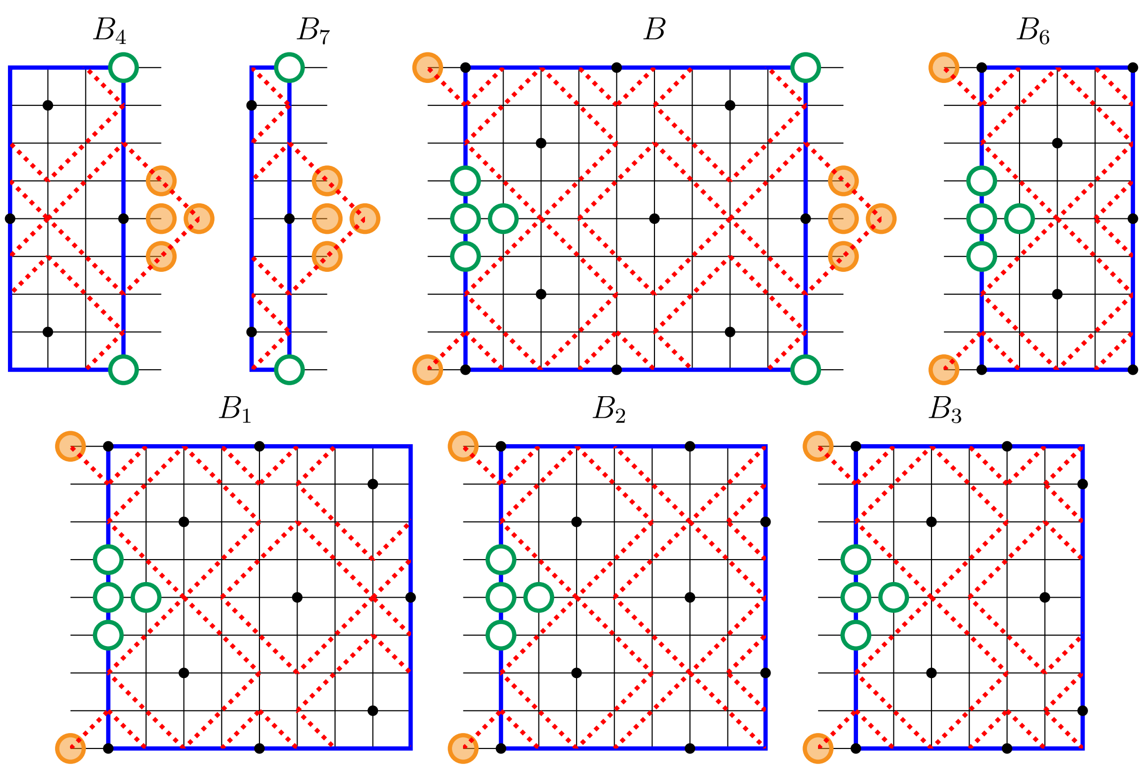
where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 2.9.1.

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	2	3	5	6	8	10	11	13	15	16

Table 2.9.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}\left(P_9 \square P_n\right)$ stated in Proposition 2.9.1.

	Construction	Cost
$n \equiv 0 \pmod{10}$:	$B_7 \left[B\right]^{\frac{n-10}{10}} B_2$	$16\left(\frac{n-10}{10}\right) + 18 = 16\left(\frac{n}{10}\right) + 2$
$n \equiv 1 \pmod{10}$:	$B_4 \left[B\right]^{\frac{n-11}{10}} B_3$	$16\left(\frac{n-11}{10}\right) + 19 = 16\left(\frac{n-1}{10}\right) + 3$
$n \equiv 2 \pmod{10}$:	$B_8 \left[B\right]^{\frac{n-12}{10}} B_5$	$16\left(\frac{n-12}{10}\right) + 21 = 16\left(\frac{n-2}{10}\right) + 5$
$n \equiv 3 \pmod{10}$:	$B_4 \left[B\right]^{\frac{n-13}{10}} B_1$	$16\left(\frac{n-13}{10}\right) + 22 = 16\left(\frac{n-6}{10}\right) + 6$
$n \equiv 4 \pmod{10}$:	$B_7 \left[B\right]^{\frac{n-4}{10}} B_5$	$16\left(\frac{n-4}{10}\right) + 8$
$n \equiv 5 \pmod{10}$:	$B_8 \left[B\right]^{\frac{n-15}{10}} B_6$	$16\left(\frac{n-15}{10}\right) + 26 = 16\left(\frac{n-5}{10}\right) + 10$
$n \equiv 6 \pmod{10}$:	$B_4 \left[B\right]^{\frac{n-6}{10}} B_5$	$16\left(\frac{n-6}{10}\right) + 11$
$n \equiv 7 \pmod{10}$:	$B_8 \left[B\right]^{\frac{n-17}{10}} B_3$	$16\left(\frac{n-17}{10}\right) + 29 = 16\left(\frac{n-7}{10}\right) + 13$
$n \equiv 8 \pmod{10}$:	$B_8 \left[B\right]^{\frac{n-18}{10}} B_2$	$16\left(\frac{n-18}{10}\right) + 31 = 16\left(\frac{n-8}{10}\right) + 15$
$n \equiv 9 \pmod{10}, (n = 9)$:	$B_4 B_6$	$16 = 16\left(\frac{n-9}{10}\right) + 16$
$(n > 9)$:	$B_8 \left[B\right]^{\frac{n-19}{10}} B_1$	$16\left(\frac{n-19}{10}\right) + 32 = 16\left(\frac{n-9}{10}\right) + 16$

Table 2.9.2: Constructions of 2-limited dominating broadcasts on $P_9 \square P_n$.



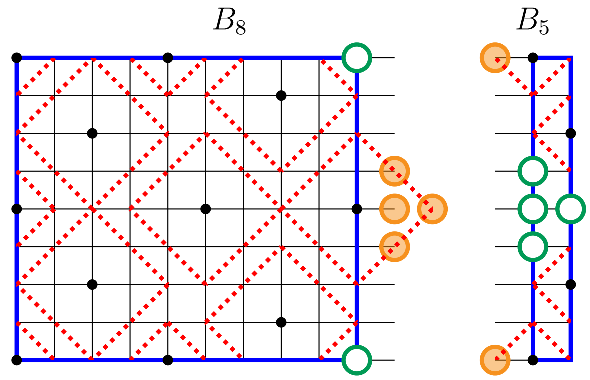


Figure 2.9.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_9 \square P_n$.

The values of Proposition 2.9.1 are optimal for all $n \leq 99$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{15.7}{10}n$.

Conjecture 2.9.1. *The bounds in Proposition 2.9.1 are tight.*

2.10 $P_{10} \square P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{10} \square P_n$ for all $n \geq 10$. Figure 2.10.1 contains the tiles used in the broadcast constructions described in Table 2.10.2. Proposition 2.10.1 provides the bounds established by these constructions.

Proposition 2.10.1. *For $n \geq 10$,*

$$\gamma_{b,2}\left(P_{10} \square P_n\right) \leq 32 \left\lfloor \frac{n}{18} \right\rfloor + c(n_{18})$$

where $c(n_{18})$ is dependant upon the least residue n_{18} of n modulo 18 and given in Table 2.10.1 with the exception of $n = 21$ which gives a value of $c(n_{18})$ equal to 7.

	Least residue n_{18} of n modulo 18																	
n_{18}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Cost:	2	4	6	8	9	11	13	15	16	18	20	22	24	25	27	29	31	32

Table 2.10.1: Values of $c(n_{18})$ for the upper bound for $\gamma_{b,2}\left(P_{10} \square P_n\right)$ stated in Proposition 2.10.1 where $n \neq 21$.

	Construction	Cost
$n \equiv 0 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-18}{18}} B_6^{\mid}$	$32\left(\frac{n-18}{18}\right)+34 = 32\left(\frac{n}{18}\right)+2$
$n \equiv 1 \pmod{18}$:	$B_5^{\mid}\left[B\overline{B}\right]^{\frac{n-19}{18}} BB_1$	$32\left(\frac{n-19}{18}\right)+36 = 32\left(\frac{n-1}{18}\right)+4$
$n \equiv 2 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-20}{18}} BB_3$	$32\left(\frac{n-20}{18}\right)+38 = 32\left(\frac{n-2}{18}\right)+6$
$n \equiv 3 \pmod{18}, (n=21)$:	B_7	$39 = 32\left(\frac{n-3}{18}\right)+7$
$(n > 21)$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-21}{18}} BB_2$	$32\left(\frac{n-21}{18}\right)+40 = 32\left(\frac{n-3}{18}\right)+8$
$n \equiv 4 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-22}{18}} BB_1$	$32\left(\frac{n-22}{18}\right)+41 = 32\left(\frac{n-4}{18}\right)+9$
$n \equiv 5 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-23}{18}} BB_4$	$32\left(\frac{n-23}{18}\right)+43 = 32\left(\frac{n-5}{18}\right)+11$
$n \equiv 6 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-6}{18}} \overline{B_3}$	$32\left(\frac{n-6}{18}\right)13$
$n \equiv 7 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-7}{18}} \overline{B_2}$	$32\left(\frac{n-7}{18}\right)+15$
$n \equiv 8 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-8}{18}} \overline{B_1}$	$32\left(\frac{n-8}{18}\right)+16$
$n \equiv 9 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-27}{18}} B\overline{B_6}^{\mid}$	$32\left(\frac{n-27}{18}\right)+50 = 32\left(\frac{n-9}{18}\right)+18$
$n \equiv 10 \pmod{18}$:	$B_5^{\mid}\left[B\overline{B}\right]^{\frac{n-10}{18}} \overline{B_1}$	$32\left(\frac{n-10}{18}\right)+20$
$n \equiv 11 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-11}{18}} \overline{B_3}$	$32\left(\frac{n-11}{18}\right)+22$
$n \equiv 12 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-12}{18}} \overline{B_2}$	$32\left(\frac{n-12}{18}\right)+24$
$n \equiv 13 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-13}{18}} \overline{B_1}$	$32\left(\frac{n-13}{18}\right)+25$
$n \equiv 14 \pmod{18}$:	$\overline{B_6}\left[B\overline{B}\right]^{\frac{n-14}{18}} \overline{B_4}$	$32\left(\frac{n-14}{18}\right)+27$
$n \equiv 15 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-15}{18}} BB_3$	$32\left(\frac{n-15}{18}\right)+29$
$n \equiv 16 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-16}{18}} BB_2$	$32\left(\frac{n-16}{18}\right)+31$
$n \equiv 17 \pmod{18}$:	$B_1^{\mid}\left[B\overline{B}\right]^{\frac{n-18}{18}} BB_1$	$32\left(\frac{n-18}{18}\right)+32$

Table 2.10.2: Constructions of 2-limited dominating broadcasts on $P_{10}\Box P_n$.

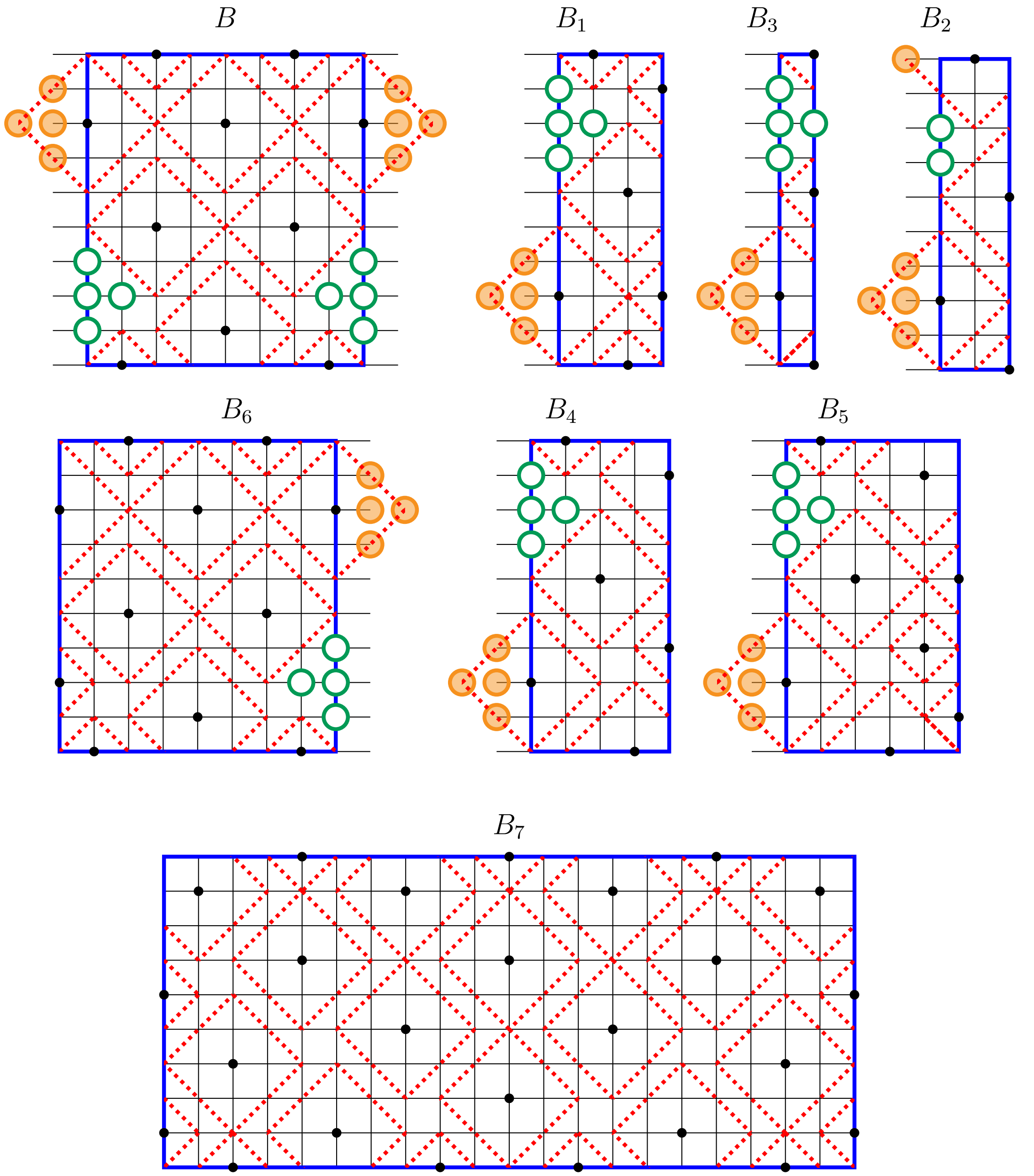


Figure 2.10.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{10} \square P_n$.

The values of Proposition 2.10.1 are optimal for all $n \leq 69$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{31.1}{18}n$.

Conjecture 2.10.1. *The bounds in Proposition 2.10.1 are tight.*

2.11 $P_{11}\Box P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{11}\Box P_n$ for all $n \geq 11$. Figure 2.11.1 contains the tiles used in the broadcast constructions described in Table 2.11.2.

Proposition 2.11.1. *For $n \geq 11$,*

$$\gamma_{b,2}\left(P_{11}\Box P_n\right) \leq 50\left\lfloor\frac{n}{26}\right\rfloor + c(n_{26})$$

where $c(n_{26})$ is dependant upon the least residue n_{26} of n modulo 26 and given in Table 2.11.1.

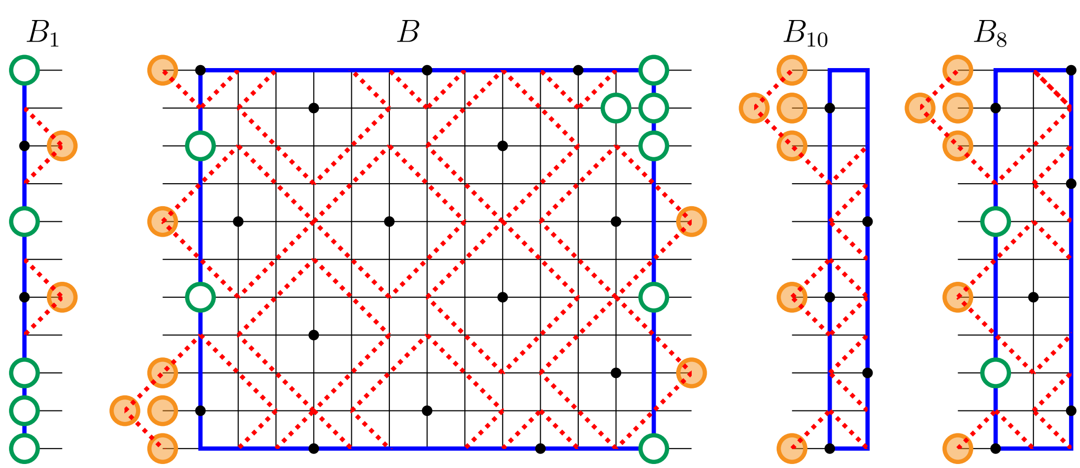
	Least residue n_{26} of n modulo 26												
n_{26}	0	1	2	3	4	5	6	7	8	9	10	11	12
Cost:	3	5	6	8	10	12	14	16	18	19	22	24	25
n_{26}	13	14	15	16	17	18	19	20	21	22	23	24	25
Cost:	28	30	31	33	35	37	39	41	43	44	47	49	50

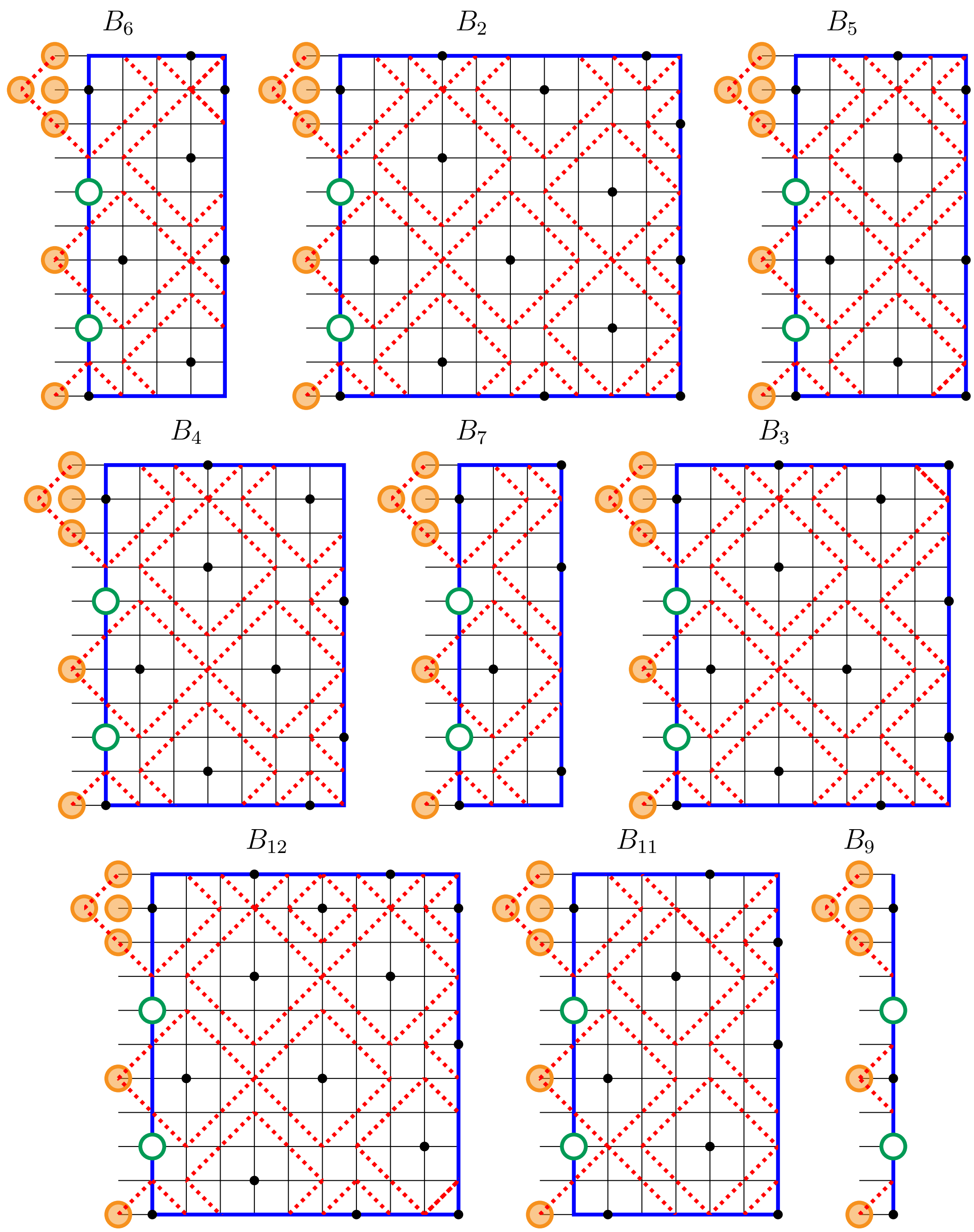
Table 2.11.1: Values of $c(n_{26})$ for the upper bound for $\gamma_{b,2}\left(P_{11}\Box P_n\right)$ stated in Proposition 2.11.1.

	Construction	Cost
$n \equiv 0 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-26}{26}} BB_{13}$	$50\left(\frac{n-26}{26}\right) + 53 = 50\left(\frac{n}{26}\right) + 3$
$n \equiv 1 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-27}{26}} B\overline{B_{14}}$	$50\left(\frac{n-27}{26}\right) + 55 = 50\left(\frac{n-1}{26}\right) + 5$
$n \equiv 2 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-2}{26}} \overline{B_9}$	$50\left(\frac{n-2}{26}\right) + 6$
$n \equiv 3 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-3}{26}} \overline{B_{10}}$	$50\left(\frac{n-3}{26}\right) + 8$
$n \equiv 4 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-4}{26}} \overline{B_8}$	$50\left(\frac{n-4}{26}\right) + 10$
$n \equiv 5 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-5}{26}} \overline{B_7}$	$50\left(\frac{n-7}{26}\right) + 12$
$n \equiv 6 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-6}{26}} \overline{B_6}$	$50\left(\frac{n-6}{26}\right) + 14$
$n \equiv 7 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-7}{26}} \overline{B_5}$	$50\left(\frac{n-7}{26}\right) + 16$
$n \equiv 8 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-8}{26}} \overline{B_{11}}$	$50\left(\frac{n-8}{26}\right) + 18$
$n \equiv 9 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-9}{26}} \overline{B_4}$	$50\left(\frac{n-9}{26}\right) + 19$
$n \equiv 10 \pmod{26}$:	$B_1\left[B\overline{B}\right]^{\frac{n-10}{26}} \overline{B_3}$	$50\left(\frac{n-10}{26}\right) + 22$

$n \equiv 11 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-11}{26}} \overline{B_{12}}$	$50\left(\frac{n-11}{26}\right) + 24$
$n \equiv 12 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-12}{26}} \overline{B_2}$	$50\left(\frac{n-12}{26}\right) + 25$
$n \equiv 13 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-13}{26}} \overline{B_{13}}$	$50\left(\frac{n-13}{26}\right) + 28$
$n \equiv 14 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-14}{26}} B_{14}$	$50\left(\frac{n-14}{26}\right) + 30$
$n \equiv 15 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-15}{26}} BB_9$	$50\left(\frac{n-15}{26}\right) + 31$
$n \equiv 16 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-16}{26}} BB_{10}$	$50\left(\frac{n-16}{26}\right) + 33$
$n \equiv 17 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-17}{26}} BB_8$	$50\left(\frac{n-17}{26}\right) + 35$
$n \equiv 18 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-18}{26}} BB_7$	$50\left(\frac{n-18}{26}\right) + 37$
$n \equiv 19 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-19}{26}} BB_6$	$50\left(\frac{n-19}{26}\right) 39$
$n \equiv 20 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-20}{26}} BB_5$	$50\left(\frac{n-20}{26}\right) + 41$
$n \equiv 21 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-21}{26}} BB_{11}$	$50\left(\frac{n-8}{26}\right) + 43$
$n \equiv 22 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-22}{26}} BB_4$	$50\left(\frac{n-22}{26}\right) + 44$
$n \equiv 23 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-23}{26}} BB_3$	$50\left(\frac{n-23}{26}\right) + 47$
$n \equiv 24 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-24}{26}} BB_{12}$	$50\left(\frac{n-24}{26}\right) + 49$
$n \equiv 25 \pmod{26}$:	$B_1 \left[B\overline{B}\right]^{\frac{n-25}{26}} BB_2$	$50\left(\frac{n-25}{26}\right) + 50$

Table 2.11.2: Constructions of 2-limited dominating broadcasts on $P_{11} \square P_n$.




B_6
B_2
B_5
B_4
B_7
B_3
B_{12}
B_{11}
B_9

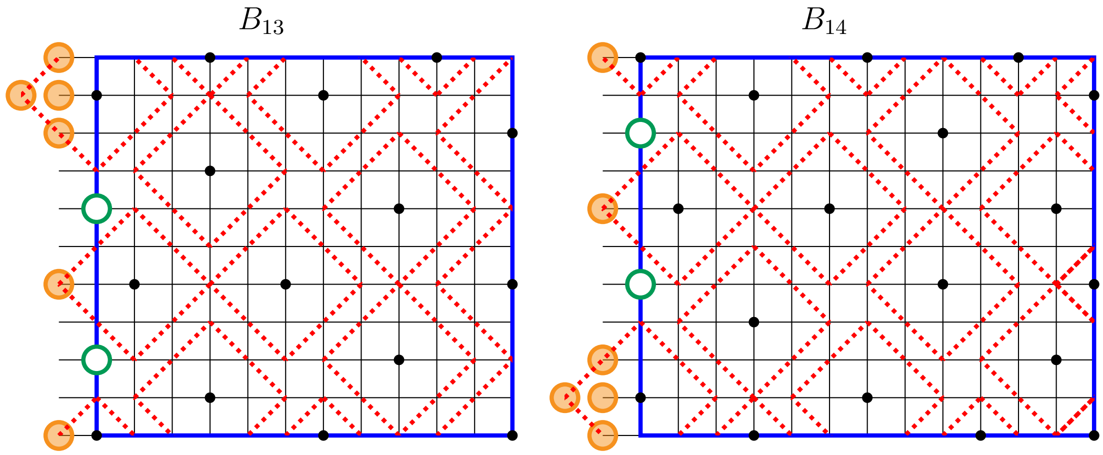


Figure 2.11.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{11}\Box P_n$.

The values of Proposition 2.11.1 are optimal for all $n \leq 88$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{48.9}{26}n$.

Conjecture 2.11.1. *The bounds in Proposition 2.11.1 are tight.*

2.12 $P_{12}\Box P_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{12}\Box P_n$ for all $n \geq 12$. Figure 2.12.1 contains the tiles used in the broadcast constructions described in Table 2.12.2.

Proposition 2.12.1. *For $n \geq 12$,*

$$\gamma_{b,2}\left(P_{12}\Box P_n\right) \leq 50\left\lfloor\frac{n}{24}\right\rfloor + c(n_{24})$$

where $c(n_{24})$ is dependant upon the least residue n_{24} of n modulo 24 and given in Table 2.12.1.

	Least residue n_{24} of n modulo 24											
n_{24}	0	1	2	3	4	5	6	7	8	9	10	11
Cost:	3	4	7	9	11	13	15	17	20	21	24	25

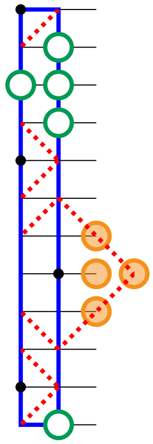
n_{24}	12	13	14	15	16	17	18	19	20	21	22	23
Cost:	28	29	32	34	36	38	40	42	45	46	49	50

Table 2.12.1: Values of $c(n_{24})$ for the upper bound for $\gamma_{b,2}\left(P_{12}\Box P_n\right)$ stated in Proposition 2.12.1.

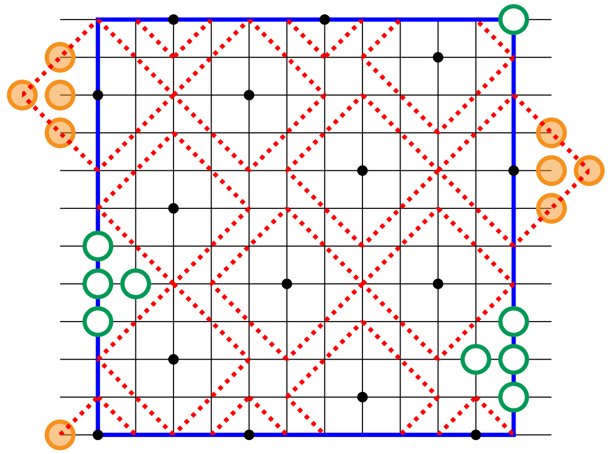
	Construction	Cost
$n \equiv 0 \pmod{24}$:	$\overline{B_{12}}\left[B\overline{B}\right]^{\frac{n-24}{24}} B_{11}$	$50\left(\frac{n-24}{24}\right)+53=50\left(\frac{n}{24}\right)+3$
$n \equiv 1 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-25}{24}} BB_2$	$50\left(\frac{n-25}{24}\right)+54=50\left(\frac{n-1}{24}\right)+4$
$n \equiv 2 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-26}{24}} B\overline{B_{11}}$	$50\left(\frac{n-26}{24}\right)+57=50\left(\frac{n-2}{24}\right)+7$
$n \equiv 3 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-27}{24}} B\overline{B_9}$	$50\left(\frac{n-27}{24}\right)+59=50\left(\frac{n-3}{24}\right)+9$
$n \equiv 4 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-4}{24}} \overline{B_8}$	$50\left(\frac{n-4}{24}\right)+11$
$n \equiv 5 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-29}{24}} B\overline{B_{10}}$	$50\left(\frac{n-29}{24}\right)+63=50\left(\frac{n-5}{24}\right)+13$
$n \equiv 6 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-6}{24}} \overline{B_8}$	$50\left(\frac{n-6}{24}\right)15$
$n \equiv 7 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-7}{24}} \overline{B_7}$	$50\left(\frac{n-7}{24}\right)+17$
$n \equiv 8 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-8}{24}} \overline{B_5}$	$50\left(\frac{n-8}{24}\right)+20$
$n \equiv 9 \pmod{24}$:	$\overline{B_1}\left[B\overline{B}\right]^{\frac{n-9}{24}} \overline{B_4}$	$50\left(\frac{n-9}{24}\right)+21$
$n \equiv 10 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-10}{24}} \overline{B_3}$	$50\left(\frac{n-10}{24}\right)+24$
$n \equiv 11 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-11}{24}} \overline{B_2}$	$50\left(\frac{n-11}{24}\right)+25$
$n \equiv 12 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-12}{24}} \overline{B_3}$	$50\left(\frac{n-12}{24}\right)+28$
$n \equiv 13 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-13}{24}} \overline{B_2}$	$50\left(\frac{n-13}{24}\right)+29$
$n \equiv 14 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-14}{24}} B_{11}$	$50\left(\frac{n-14}{24}\right)+32$
$n \equiv 15 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-15}{24}} B_9$	$50\left(\frac{n-15}{24}\right)+34$
$n \equiv 16 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-16}{24}} BB_8$	$50\left(\frac{n-16}{24}\right)+36$
$n \equiv 17 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-17}{24}} B_{10}$	$50\left(\frac{n-17}{24}\right)+38$
$n \equiv 18 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-18}{24}} BB_8$	$50\left(\frac{n-18}{24}\right)40$
$n \equiv 19 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-19}{24}} BB_7$	$50\left(\frac{n-19}{24}\right)+42$
$n \equiv 20 \pmod{24}$:	$B_6\left[B\overline{B}\right]^{\frac{n-20}{24}} BB_5$	$50\left(\frac{n-20}{24}\right)+45$
$n \equiv 21 \pmod{24}$:	$\overline{B_1}\left[B\overline{B}\right]^{\frac{n-21}{24}} B\overline{B_4}$	$50\left(\frac{n-21}{24}\right)+46$
$n \equiv 22 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-22}{24}} BB_3$	$50\left(\frac{n-22}{24}\right)+49$
$n \equiv 23 \pmod{24}$:	$B_1\left[B\overline{B}\right]^{\frac{n-23}{24}} B\overline{B_2}$	$50\left(\frac{n-23}{24}\right)+50$

Table 2.12.2: Constructions of 2-limited dominating broadcasts on $P_{12}\Box P_n$.

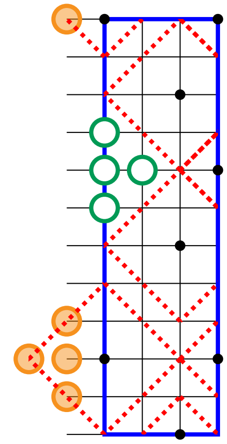
B_1



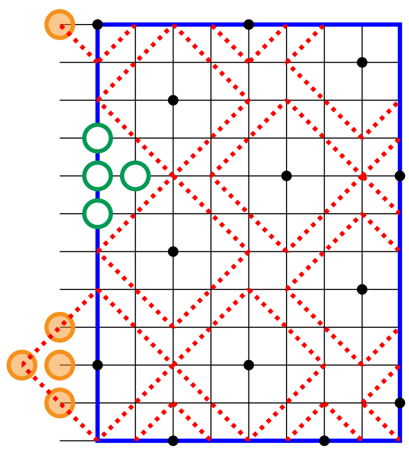
B



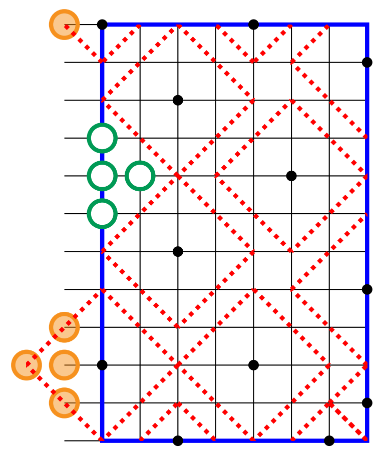
B_5



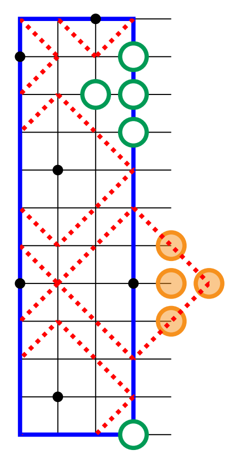
B_2



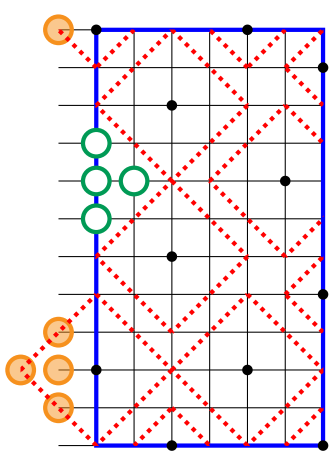
B_3



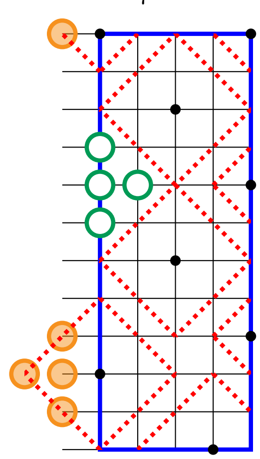
B_6



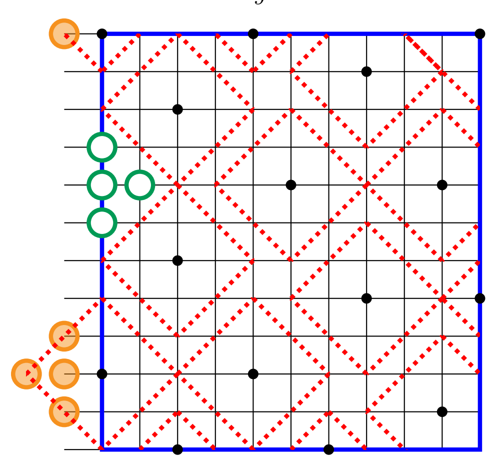
B_4



B_7



B_9



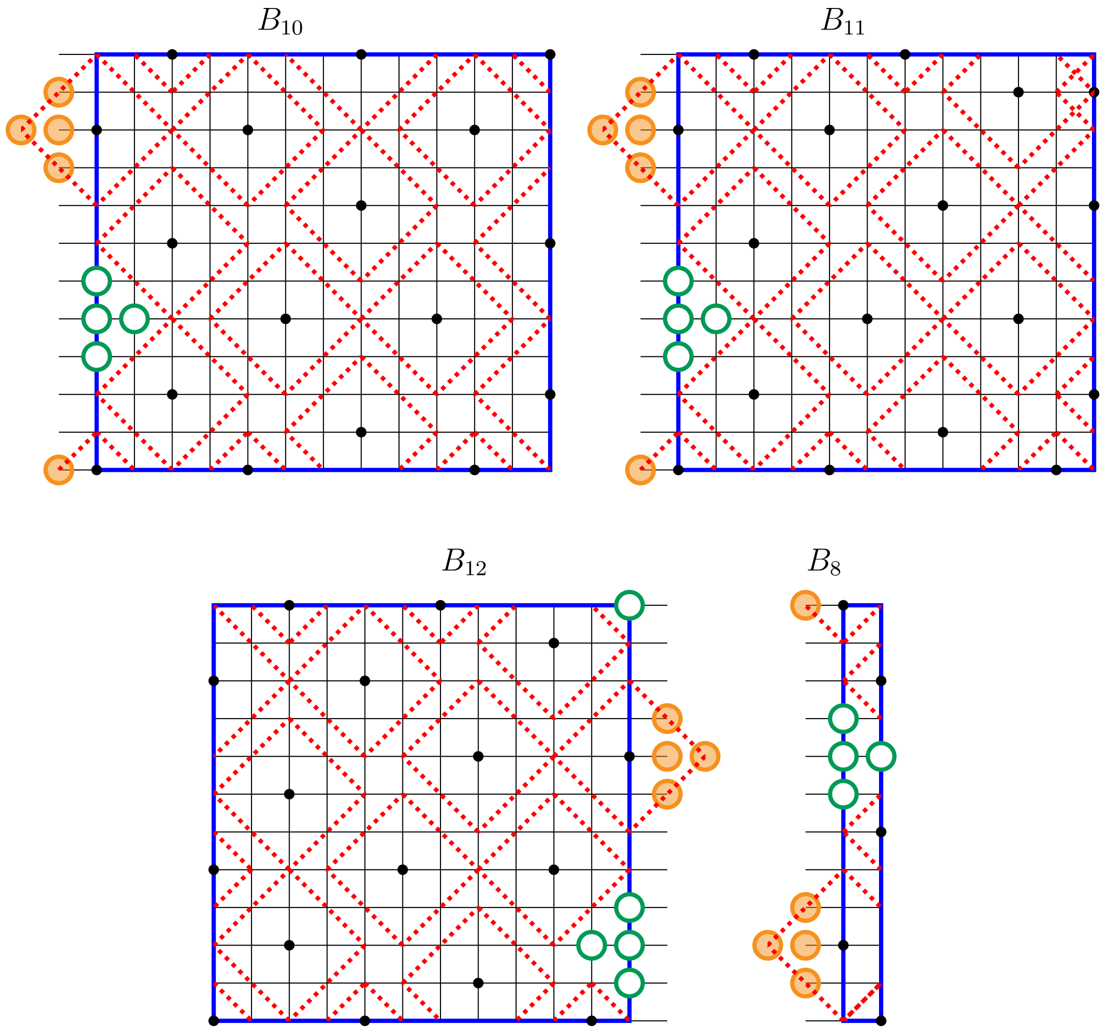


Figure 2.12.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{12}\Box P_n$.

The values of Proposition 2.12.1 are optimal for all $n \leq 76$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{48.8}{24}n$.

Conjecture 2.12.1. *The bounds in Proposition 2.12.1 are tight.*

2.13 $P_{m\geq 13}\Box P_{n\geq 13}$

Sections 2.2 though 2.12 give upper bounds for $\gamma_{b,2}\left(P_m\Box P_n\right)$ for $2 \leq m \leq 12$ and all $n \geq 2$. These upper bounds are made by using sequences of broadcast tiles which, for a fixed m and n, described a 2-limited dominating broadcast on $P_m\Box P_n$. Transposing a 2-limited dominating broadcast on $P_m\Box P_n$ gives a 2-limited dominating broadcast on

$P_n \Box P_m$. Hence, the only cases not yet considered have both m and n at least 13. This section provides upper bounds for $\gamma_{b,2}(P_m \Box P_n)$ for $m, n \geq 13$ by modifying 2-limited dominating broadcasts on the Cartesian plane. We hope to achieve good upper bounds with this approach as, for very large m and n, there likely exists a 2-limited dominating broadcast on $P_m \Box P_n$ which, for much of the graph, resembles this 2-limited dominating broadcast on the Cartesian plane. This methodology was inspired by Michael Farina and Armando Grez and their work on the distance domination number of grids in [17]. We adopt the notation used [17] as well as make use of several of their results.

2.13.1 2-limited dominating broadcasts on $\mathbb{Z}^2$

Let $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ denote the integer lattice. Let $G_{m,n}$ and $Y_{m+4,n+4}$ denote $P_m \Box P_n$ and $P_{m+4} \Box P_{n+4}$, respectively, embedded in $\mathbb{Z}^2$ with their bottom left corners at $(0,0)$ and $(-2,-2)$, respectively. That is, let

$$V(G_{m,n}) = \left\{(i,j) \in \mathbb{Z}^2 : 0 \leq i \leq n-1 \text{ and } 0 \leq j \leq m-1\right\} \text{ and}$$
$$V(Y_{m+4,n+4}) = \left\{(i,j) \in \mathbb{Z}^2 : -2 \leq i \leq n+1 \text{ and } -2 \leq j \leq m+1\right\}.$$

Let $\mathbb{Z}_{13} = \{0, 1, \ldots, 12\}$. Define the function $\phi : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}_{13}$ by

$$(i,j) \mapsto 3i + 2j \pmod{13}.$$

Fix $\ell \in \mathbb{Z}_{13}$. The set given by

$$\phi^{-1}(\ell) = \left\{(i,j) \in \mathbb{Z}^2 : 3i + 2j \equiv \ell \pmod{13}\right\}$$

defines the vertices of $\mathbb{Z}^2$ which, if broadcasting at strength two, give a 2-limited dominating broadcast on $\mathbb{Z}^2$ [14, Lemma V.7]. Figure 2.13.1 depicts $G_{3,4}$, $Y_{7,8}$, and the broadcast vertices and their neighbourhoods given by $\phi^{-1}(4)$. The thick blue and green lines indicate the borders of $G_{3,4}$ and $Y_{7,8}$, respectively. The light blue fill and green inner fill indicate the interiors of $G_{3,4}$ and $Y_{7,8}$, respectively. The thick black lines indicate the axes of $\mathbb{Z}^2$. To see that the broadcast set in Figure 2.13.1 is given by $\phi^{-1}(4)$ observe, for instance, that $(5,2)$ is one such broadcaster and

$$\phi(5,1) = 3(5) + 2(1) = 17 \equiv 4 \pmod{13} \Rightarrow (5,1) \in \phi^{-1}(4).$$

Prior to constructing 2-limited dominating broadcasts on $P_m \Box P_n$ for all $m, n \geq 13$ from 2-limited dominating broadcasts on $\mathbb{Z}^2$, we include Example 2.13.1 to provide an introduction to our methodology.

Example 2.13.1. Suppose we wish to find a 2-limited dominating broadcast on $G_{13,13}$. Overlay one of the 13 possible 2-limited dominating broadcasts of $\mathbb{Z}^2$ given by $\phi^{-1}(\ell)$

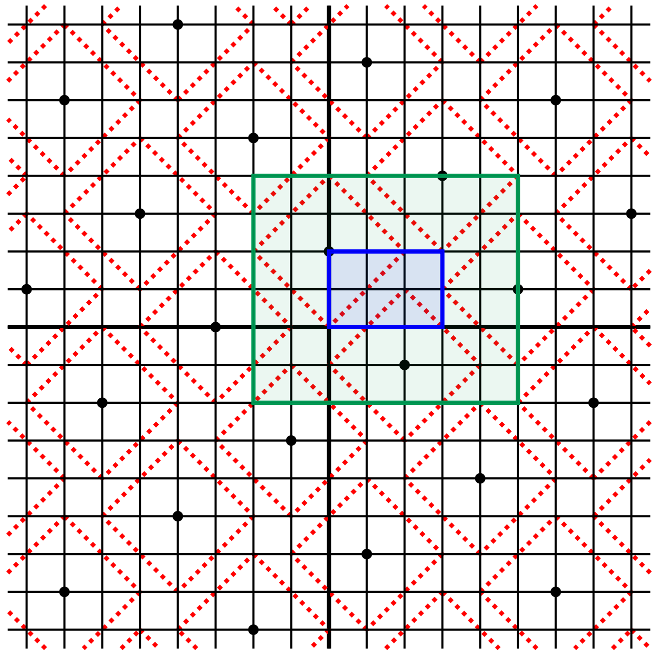


Figure 2.13.1: $G_{3,4}$, $Y_{7,8}$, and the 2-limited dominating broadcast on $\mathbb{Z}^2$ defined by $\phi^{-1}(4)$.

for some $\ell \in \mathbb{Z}_{13}$. The broadcast vertices on $G_{13,13}$ and within distance 2 of the border of $G_{13,13}$ dominate the vertices of $G_{13,13}$. That is, the broadcast vertices on $Y_{17,17}$ dominate the vertices of $G_{13,13}$. Figure 2.13.2 depicts $G_{13,13}$, $Y_{17,17}$, and the overlay of $\phi^{-1}(8)$ where the thick purple circles indicate the broadcast vertices on $Y_{17,17}$ but not $G_{13,13}$.

Moving each of the broadcast vertices exterior to $G_{13,13}$ and on $Y_{17,17}$ to the nearest point on the border of $G_{13,13}$ and reducing their broadcast strength to one produces a 2-limited dominating broadcast on $G_{13,13}$. Figure 2.13.3 depicts the resulting 2-limited dominating broadcast on $G_{13,13}$ given $\phi^{-1}(8)$.

The resulting 2-limited dominating broadcast on $G_{13,13}$ in Figure 2.13.3 establishes $\gamma_{b,2}(P_{13} \square P_{13}) \leq 35$. Note that, although this bound is not tight as $\gamma_{b,2}(P_{13} \square P_{13}) = 32$ (as determined by computation), it is also not too far from the truth. As m and n get large, we show that the ratio of the this upper bound and the lower bound of $\gamma_{b,2}(P_m \square P_n)$ given by Theorem 5.4.1 approaches one.

Example 2.13.1 illustrates that, given some fixed $\ell \in \mathbb{Z}_{13}$ and $m, n \geq 13$, if there are x and y vertices broadcasting at non-zero strength under $\phi^{-1}(\ell)$ on $G_{m,n}$ and $Y_{m+4,n+4}$, respectively, then there is a 2-limited dominating broadcast on $P_m \square P_n$ of cost $2x + (y - x)$. Our general construction, as described in the subsequent section, uses the same approach as Example 2.13.1 but uses a choice of $\ell \in \mathbb{Z}_{13}$ which gives a best possible construction (in terms of cost).

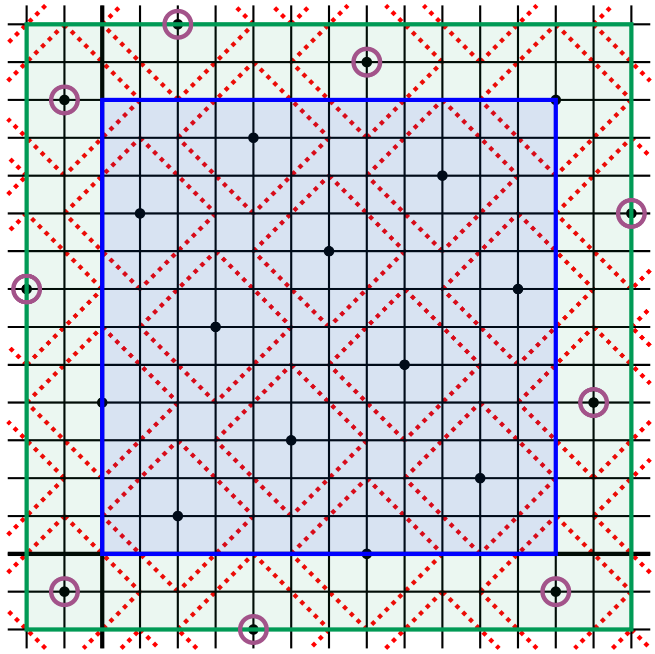


Figure 2.13.2: $G_{13,13}, Y_{17,17}$ and 2-limited dominating broadcast on $\mathbb{Z}^2$ defined by $\phi^{-1}(8)$.

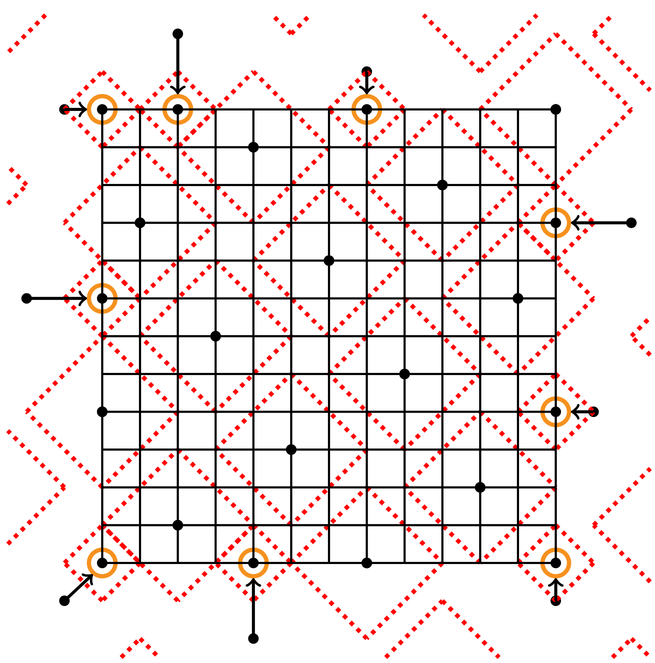


Figure 2.13.3: Resulting 2-limited dominating broadcast on $G_{13,13}$ given $\phi^{-1}(8)$.

2.13.2 Generalized Constructions

This section proves Theorem 2.13.3 which, with Proposition 2.2.1 through 2.12.1, establishes upper bounds for $\gamma_{b,2}(P_m \Box P_n)$ for all $m, n \geq 2$. The proof of Theorem 2.13.3 uses Lemma 2.13.2 from [17], stated as follows.

Lemma 2.13.2. [17, Lemma 3.2] Let $\ell \in \mathbb{Z}_{13}$. If either m or n is a multiple of 13, then for any $\ell \in \mathbb{Z}_{13}$ then

$$\left|\phi^{-1}(\ell) \cap V(G_{m,n})\right| = \frac{mn}{13}.$$

Theorem 2.13.3. *If $m, n \geq 13$ then*

$$\gamma_{b,2}(P_m \Box P_n) \leq 2\left(\frac{mn}{13}\right) + 4\left(\frac{m+n}{13}\right) + \frac{c(m_{13}, n_{13})}{13}$$

where $c(m_{13}, n_{13})$ corresponds with the values given in Table 2.13.1 where m_{13} and n_{13} are the least residues of m and n modulo 13, respectively.

Least residue m_{13} of m modulo 13	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	9	5	1	10	6	2	11	7	16	12	8	4
1	9	16	10	4	11	5	12	6	0	7	1	8	2
2	5	10	2	-6	-1	-9	-4	-12	6	11	3	8	0
3	1	4	-6	10	0	3	6	-4	12	2	5	8	-2
4	10	11	-1	0	-12	2	3	-9	5	6	-6	8	-4
5	6	5	-9	3	2	-12	0	-1	11	10	-4	8	-6
6	2	12	-4	6	3	0	10	-6	4	1	-2	8	5
7	11	6	-12	-4	-9	-1	-6	2	10	5	0	8	3
8	7	0	6	12	5	11	4	10	16	9	2	8	1
9	16	7	11	2	6	10	1	5	9	0	4	8	12
10	12	1	3	5	-6	-4	-2	0	2	4	6	8	10
11	8	8	8	8	8	8	8	8	8	8	8	8	8
12	4	2	0	-2	-4	-6	5	3	1	12	10	8	6

Table 2.13.1: Value of $c(m_{13}, n_{13})$ for the upper bound for $\gamma_{b,2}(P_m \Box P_n)$ stated in Theorem 2.13.3.

Proof. Let $G_{m,n}$, $Y_{m+4,n+4}$, $\mathbb{Z}^2$, $\mathbb{Z}_{13}$, and ϕ be defined as in Section 2.13.1. Fix some $\ell \in \mathbb{Z}^2$. As $Y_{m+4,n+4}$ is the distance two neighbourhood of $G_{m,n}$, the set of vertices

$$V(Y_{m+4,n+4}) \cap \phi^{-1}(\ell),$$

if broadcasting at strength 2, dominate the vertices of $G_{m,n}$. Let

$$B_2 = V(G_{m,n}) \cap \phi^{-1}(\ell) \quad \text{and} \quad B_1 = \phi^{-1}(\ell) \cap (V(Y_{m+4,n+4}) \setminus V(G_{m,n})).$$

For each vertex $v \in B_1$ whose broadcast is heard by a vertex in $G_{m,n}$, let v' be the vertex of $G_{m,n}$ with minimum distance to v. Note that this choice of v' is unique and that v' necessarily hears a broadcast from v. The vertices dominated in $G_{m,n}$ by a broadcast of strength two at v are a subset of the vertices dominated in $G_{m,n}$ by a broadcast of strength one at v'. Define the set

$$B_1' = \left\{ v' \in V(G_{m,n}) \;\middle|\; \begin{array}{c} v' \text{ is the vertex of } G_{m,n} \text{ which hears a broadcast from} \\ \text{and is at minimum distance to } v \text{ for some } v \in B_1 \end{array} \right\}.$$

Informally, B_1' is the resulting collection of vertices formed by moving each vertex in B_1 (whose broadcast is heard by a vertex in $G_{m,n}$) to the nearest vertex in $G_{m,n}$. Each vertex in the plane hears only one broadcast under $\phi^{-1}(\ell)$ [14, Lemma V.7], hence each $v' \in B_1'$ hears only the broadcast from some $v \in B_1$. Therefore $v' \notin B_2$ and $B_2 \cap B_1' = \emptyset$. The broadcast $f : V(G_{m,n}) \mapsto \{0, 1, 2\}$ defined by

$$f(v) = \begin{cases} 2 & \text{if } v \in B_2, \\ 1 & \text{if } v \in B_1', \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

is a 2-limited dominating broadcast of $G_{m,n}$. The cost of f is

$$2|B_2| + |B_1'| \leq \left|V(G_{m,n}) \cap \phi^{-1}(\ell)\right| + \left|V(Y_{m+4,n+4}) \cap \phi^{-1}(\ell)\right|.$$

Note that equality holds in the above expression if and only if the broadcast from each vertex $v \in B_1$ is heard by some vertex $v' \in V(G_{m,n})$. Given this construction, it follows that

$$\gamma_{b,2}(P_m \Box P_n) \leq \min\left\{\left|\phi^{-1}(\ell) \cap V(G_{m,n})\right| + \left|\phi^{-1}(\ell) \cap V(Y_{m+4,n+4})\right| : \ell \in \mathbb{Z}_{13}\right\}.$$

For a fixed $\ell \in \mathbb{Z}^2$, $|\phi^{-1}(\ell) \cap V(G_{m,n})|$ is computed as follows. Let m_{13} and n_{13} be the least residue of m and n modulo 13, respectively. Partition $V(G_{m,n})$ into the following subsets:

$G_1 = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i \leq n - 1 - n_{13} \text{ and } 0 \leq j \leq m - 1\}$,
$G_2 = \{(i, j) \in \mathbb{Z}^2 : n - n_{13} \leq i \leq n - 1 \text{ and } 0 \leq j \leq m - 1 - m_{13}\}$, and
$G_3 = \{(i, j) \in \mathbb{Z}^2 : n - n_{13} \leq i \leq n - 1 \text{ and } m - m_{13} \leq j \leq m - 1\}$.

Figure 2.13.4 depicts $G_{16,18}$ partitioned into G_1, G_2, and G_3.

For each $\ell \in \mathbb{Z}_{13}$,

$$\left|V(G_{m,n}) \cap \phi^{-1}(\ell)\right| = \left|G_1 \cap \phi^{-1}(\ell)\right| + \left|G_2 \cap \phi^{-1}(\ell)\right| + \left|G_3 \cap \phi^{-1}(\ell)\right|.$$

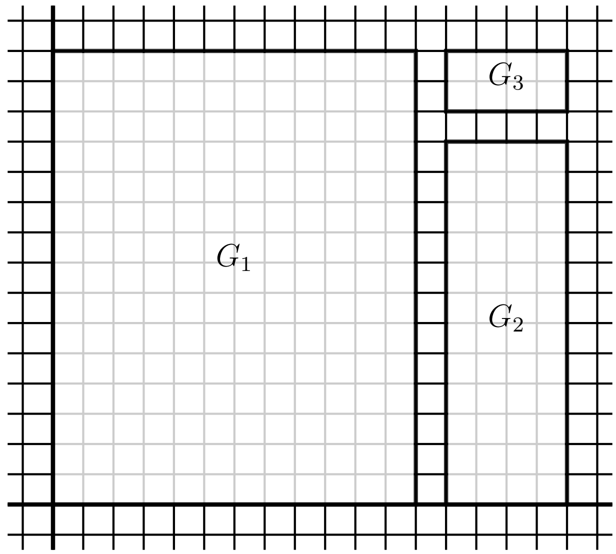


Figure 2.13.4: $G_{16,18}$ partitioned into G_1, G_2, and G_3.

By Lemma 2.13.2,

$$\left|G_1 \cap \phi^{-1}(\ell)\right| = \frac{(m)(n - n_{13})}{13} \quad \text{and} \quad \left|G_2 \cap \phi^{-1}(\ell)\right| = \frac{(n_{13})(m - m_{13})}{13} \tag{2.1}$$

regardless of $\ell \in \mathbb{Z}_{13}$. Hence, to determine $|V(G_{m,n}) \cap \phi^{-1}(\ell)|$, what remains is to find $|G_3 \cap \phi^{-1}(\ell)|$. The same methodology can be applied to $Y_{m+4,n+4}$ by defining m_{13}^Y and n_{13}^Y as the least residue of $(m_{13} + 4)$ and $(n_{13} + 4)$ modulo 13, respectively, and partitioning $V(Y_{m+4,n+4})$ into the following subsets:

$$Y_1 = \left\{(i,j) \in \mathbb{Z}^2 : -2 \le i \le n + 1 - n_{13}^Y \text{ and } -2 \le j \le m + 1\right\},$$
$$Y_2 = \left\{(i,j) \in \mathbb{Z}^2 : n + 2 - n_{13}^Y \le i \le n + 1 \text{ and } -2 \le j \le m + 1 - m_{13}^Y\right\}, \text{ and}$$
$$Y_3 = \left\{(i,j) \in \mathbb{Z}^2 : n + 2 - n_{13}^Y \le i \le n + 1 \text{ and } m + 2 - m_{13}^Y \le j \le m + 1\right\}.$$

Figure 2.13.5 depicts $G_{16,18}$ partitioned into G_1, G_2, and G_3, (all in grey) overlayed by $Y_{20,22}$ partitioned into Y_1, Y_2, and Y_3 (all opaque white).

By Lemma 2.13.2,

$$\left|Y_1 \cap \phi^{-1}(\ell)\right| = \frac{(m+4)\left(n + 4 - n_{13}^Y\right)}{13} \quad \text{and} \quad \left|Y_2 \cap \phi^{-1}(\ell)\right| = \frac{\left(n_{13}^Y\right)\left(m + 4 - m_{13}^Y\right)}{13} \tag{2.2}$$

regardless of $\ell \in \mathbb{Z}_{13}$. Finding a best choice of ℓ for the construction therefore reduces to determining, for each least residue of m and n modulo 13, the minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $\ell \in \mathbb{Z}_{13}$. Said least values are given in Table 2.13.2 and the corresponding largest $\ell \in \mathbb{Z}_{13}$ is given in Table 2.13.3.

The values in Table 2.13.2 and Table 2.13.3 can be verified by an exhaustive search. Let $c'(m_{13}, n_{13})$ correspond with the values given in Table 2.13.2. For a general $\ell \in \mathbb{Z}_{13}$,

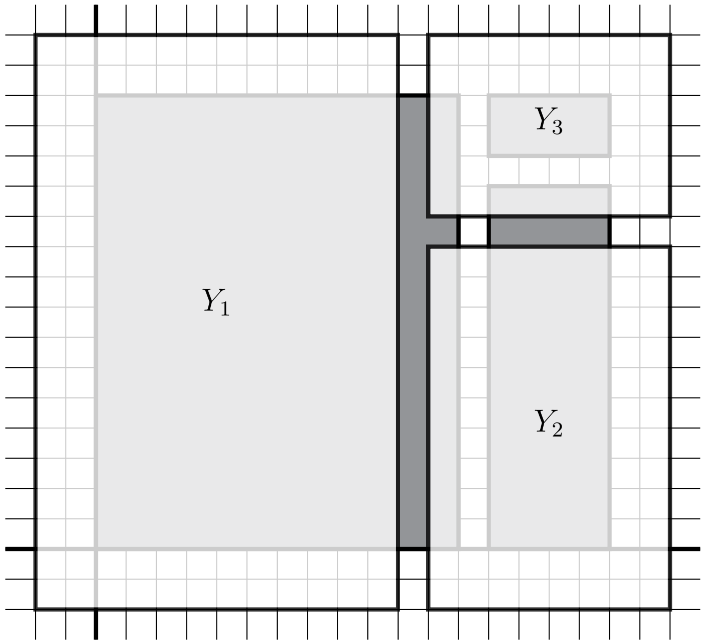


Figure 2.13.5: $G_{16,18}$ partitioned into G_1, G_2, and G_3 overlayed by $Y_{20,22}$ partitioned into Y_1, Y_2, and Y_3.

Least residue m_{13} of m modulo 13	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	1	2	2	2	3	3	0	0	0	0
1	1	2	2	2	3	3	4	4	4	0	0	1	1
2	1	2	2	2	3	3	4	4	6	1	1	2	2
3	1	2	2	4	4	5	6	6	8	1	2	3	3
4	2	3	3	4	4	6	7	7	9	2	2	4	4
5	2	3	3	5	6	6	8	9	11	3	3	5	5
6	2	4	4	6	7	8	10	10	12	3	4	6	7
7	3	4	4	6	7	9	10	12	14	4	5	7	8
8	3	4	6	8	9	11	12	14	16	5	6	8	9
9	0	0	1	1	2	3	3	4	5	5	6	7	8
10	0	0	1	2	2	3	4	5	6	6	7	8	9
11	0	1	2	3	4	5	6	7	8	7	8	9	10
12	0	1	2	3	4	5	7	8	9	8	9	10	11

Table 2.13.2: Minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $\ell \in \mathbb{Z}_{13}$ for $m, n \geq 13$.

Least residue m_{13} of m modulo 13	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	4	10	10	10	12	10	10	12	10	12	11	10	4
1	11	12	11	10	12	11	12	11	7	11	11	11	11
2	10	12	10	10	12	10	12	1	10	12	11	12	11
3	4	4	4	12	11	10	12	4	11	11	11	12	11
4	10	12	10	12	1	10	12	7	9	11	7	12	4
5	10	10	10	12	10	10	12	10	12	11	10	12	4
6	0	11	10	12	11	10	12	7	9	11	10	12	11
7	12	10	10	12	10	12	1	10	12	11	10	12	10
8	6	4	10	12	10	12	4	10	12	11	0	11	4
9	12	10	12	1	10	12	7	9	11	7	9	11	11
10	12	10	12	12	10	12	12	12	1	10	12	12	12
11	11	10	12	12	12	12	12	12	11	11	12	12	12
12	4	10	12	12	12	1	12	12	7	12	12	12	12

Table 2.13.3: Largest $\ell \in \mathbb{Z}_{13}$ corresponding with the minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $m, n \geq 13$.

by Equations 2.1 and 2.2, $|\phi^{-1}(\ell) \cap V(G_{m,n})| + |\phi^{-1}(\ell) \cap V(Y_{m+4,n+4})|$ is simply

$$\frac{m(n-n_{13})}{13} + \frac{(m-m_{13})\,n_{13}}{13} + \frac{(m+4)\left(n+4-n_{13}^{Y}\right)}{13} + \frac{\left(m-m_{13}^{Y}\right)n_{13}^{Y}}{13} + c'(m', n_{13}).$$

As n_{13}^{Y} and m_{13}^{Y} are the least residues of $(m_{13}+4)$ and $(n_{13}+4)$ modulo 13, respectively, the previous equation can be solved for each least residue of m and n modulo 13 by computation. The result gives

$$2\left(\frac{mn}{13}\right) + 4\left(\frac{m+n}{13}\right) + \frac{c(m_{13}, n_{13})}{13}$$

where the values $c(m_{13}, n_{13})$ are given in Table 2.13.1. This proves the theorem.

□

Chapter 3
Cartesian Products of a Path and a Cycle

3.1 Introduction

Having established upper bounds for $\gamma_{b,2}(P_m \Box P_n)$ for all $n, m \geq 2$ in Chapter 2, the natural next question is what are upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 2$ and $n \geq 3$? As $\gamma_{b,2}(P_m \Box C_n) \leq \gamma_{b,2}(P_m \Box P_n)$ for all $m \geq 2$ and $n \geq 3$, upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ follow from Chapter 2. This chapter improves upon these bounds in the hopes of determining exact values for $\gamma_{b,2}(P_m \Box C_n)$, which would then act as lower bounds for $\gamma_{b,2}(P_m \Box P_n)$.

This chapter is structured similarly to Chapter 2. Sections 3.2 through 3.12 provide upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ for $2 \leq m \leq 12$ and all $n \geq 3$ based on tilings. The broadcast tiles for these cases were identified by hand by examining known optimal solutions, as found by computation, and looking for recurring patterns. This manual process was greatly facilitated by automatically creating PDF visuals of known optimal solutions using a Python script (written by the author). These visuals are not present in this document but are similar to the figures throughout. Section 3.13 provides upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ for $13 \leq m \leq 22$ and all $n \geq 3$. These upper bounds also follow from explicit tilings however, unlike Sections 3.2 through 3.12, these tiles are solved for by computation. Section 3.14 provides upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ for all $m \geq 23$ and $n \geq 13$ by a tiling method inspired by the construction in Section 2.13. Sections 3.16 through 3.25 complete the remaining cases by establishing upper bounds for $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 13$ and $3 \leq n \leq 12$ based on tilings.

3.2 $P_2 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_2 \Box C_n$ for all $n \geq 3$. Figure 3.2.1 contains the tiles used in the broadcast constructions described in Table 3.2.1.

Proposition 3.2.1. *For $n \geq 3$, $\gamma_{b,2}(P_2 \Box C_n) \leq \lceil \frac{n}{2} \rceil$.*

	Construction	Cost
$n \equiv 0 \pmod 4$:	$[B\overline{B}]^{\frac{n}{4}}$	$2\left(\frac{n}{4}\right) = \frac{n}{2}$
$n \equiv 1 \pmod 4$:	$[B\overline{B}]^{\frac{n-1}{4}} B_1$	$2\left(\frac{n-1}{4}\right) + 1 = \frac{n+1}{2}$
$n \equiv 2 \pmod 4$:	$[B\overline{B}]^{\frac{n-6}{4}} BB_2$	$2\left(\frac{n-6}{4}\right) + 3 = \frac{n}{2}$
$n \equiv 3 \pmod 4$:	$[B\overline{B}]^{\frac{n-3}{4}} BB_1$	$2\left(\frac{n-3}{4}\right) + 2 = \frac{n+1}{2}$

Table 3.2.1: Constructions of 2-limited dominating broadcasts on $P_2 \Box C_n$.

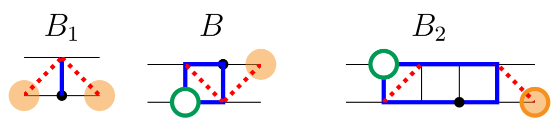


Figure 3.2.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_2 \Box C_n$.

Proposition 3.2.1 improves the bound given by Proposition 2.2.1 by one for all odd n, and otherwise the bounds are the same. The bounds given by Proposition 3.2.1 are shown to be tight by Theorem 5.3.1.

3.3 $P_3 \Box C_n$

Any 2-limited dominating broadcast on $P_3 \Box P_n$ is a 2-limited dominating broadcast on $P_3 \Box C_n$. Therefore $\gamma_{b,2}(C_3 \Box P_n) \leq \gamma_{b,2}(P_3 \Box P_n)$. Proposition 2.3.1 therefore implies Proposition 3.3.1.

Proposition 3.3.1. *For $n \geq 3$, $\gamma_{b,2}(P_3 \Box C_n) \leq \lceil \frac{2n}{3} \rceil$.*

The bounds given by Proposition 3.3.1 are shown to be tight by Theorem 5.3.1.

3.4 $P_4 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_4 \Box C_n$ for all $n \geq 3$. Figure 3.4.1 contains the tiles used in the broadcast constructions described in Table 3.4.2.

Proposition 3.4.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_4 \square C_n\right) \leq 8\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

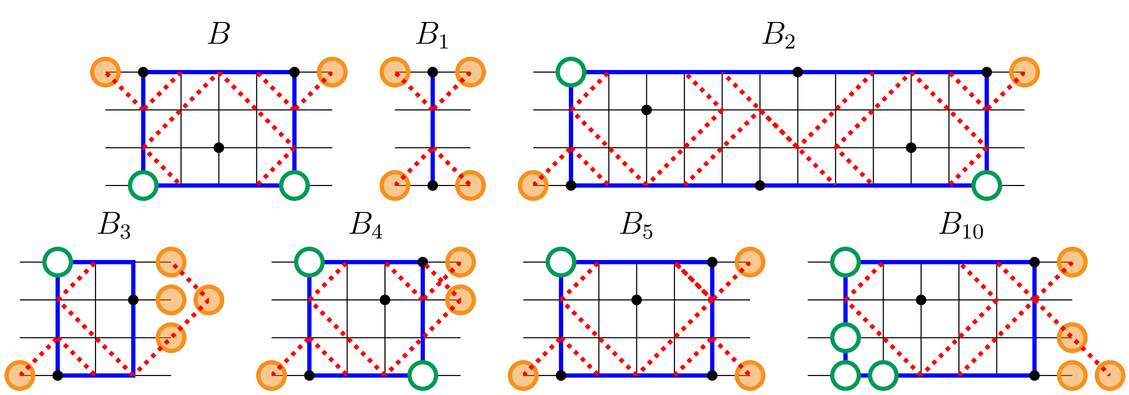
where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 3.4.1.

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	0	2	2	3	4	5	5	6	7	8

Table 3.4.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}\left(P_4 \square C_n\right)$ stated in Proposition 3.4.1.

	Construction	Cost
$n \equiv 0 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n}{10}}$	$8\left(\frac{n}{10}\right)$
$n \equiv 1 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-1}{10}} B_1$	$8\left(\frac{n-1}{10}\right) + 2$
$n \equiv 2 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-12}{10}} B_2$	$8\left(\frac{n-12}{10}\right) + 10 = 8\left(\frac{n-2}{10}\right) + 2$
$n \equiv 3 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-3}{10}} B_3$	$8\left(\frac{n-3}{10}\right) + 3$
$n \equiv 4 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-4}{10}} B_4$	$8\left(\frac{n-4}{10}\right) + 4$
$n \equiv 5 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-5}{10}} B_5$	$8\left(\frac{n-5}{10}\right) + 5$
$n \equiv 6 \pmod{10}, (n = 6)$:	B_{10}	$5 = 8\left(\frac{n-6}{10}\right) + 5$
$(n > 6)$:	$\left[\overline{B}B\right]^{\frac{n-16}{10}} B_6$	$8\left(\frac{n-16}{10}\right) + 13 = 8\left(\frac{n-6}{10}\right) + 5$
$n \equiv 7 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-7}{10}} B_7$	$8\left(\frac{n-7}{10}\right) + 6$
$n \equiv 8 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-8}{10}} B_8$	$8\left(\frac{n-8}{10}\right) + 7$
$n \equiv 9 \pmod{10}$:	$\left[\overline{B}B\right]^{\frac{n-9}{10}} B_9$	$8\left(\frac{n-9}{10}\right) + 8$

Table 3.4.2: Constructions of 2-limited dominating broadcasts on $P_4 \square C_n$.



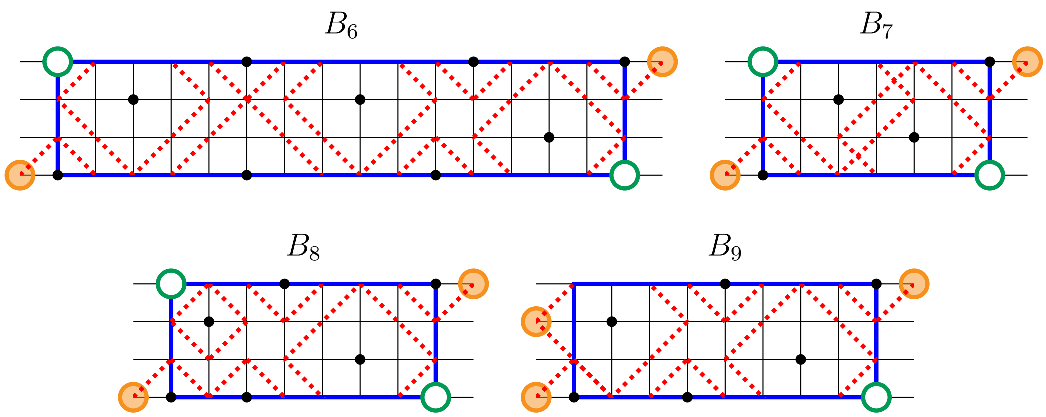


Figure 3.4.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_4 \square C_n$.

Proposition 3.4.1 improves the bounds given by Proposition 2.4.1 by one when $n_{10} \in \{0, 2, 3, 6, 7, 8\}$, and otherwise the bounds are the same. The values of Proposition 3.4.1 are optimal for all $n \leq 99$ (as found by computation). A lower bound of $\left\lceil \frac{4n}{5} \right\rceil$ is proven by Theorem 5.3.1 and this shows that the upper bounds given by Proposition 3.4.1 for all $n_{10} \in \{0, 2, 3, 4, 6, 7, 8, 9\}$ are tight.

Conjecture 3.4.1. *The bounds given by Proposition 3.4.1 are tight.*

3.5 $P_5 \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_5 \square C_n$ for all $n \geq 3$. Figure 3.5.1 contains the tiles used in the broadcast constructions described in Table 3.5.1.

Proposition 3.5.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_5 \square C_n\right) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 2 \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod 2. \end{cases}$$

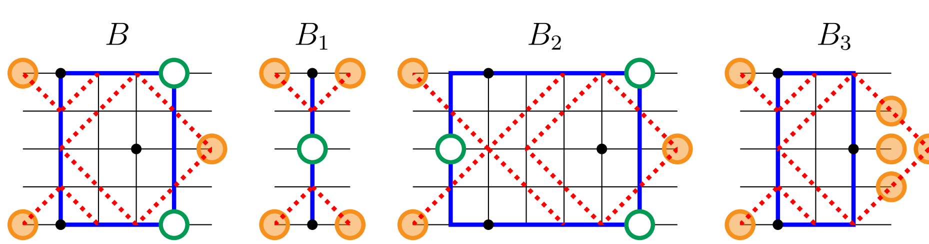


Figure 3.5.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_5 \square C_n$.

	Construction	Cost
$n \equiv 0 \pmod 4$:	$[B]^{\frac{n}{4}}$	$4\left(\frac{n}{4}\right) = n$
$n \equiv 1 \pmod 4$:	$[B]^{\frac{n-1}{4}} B_1$	$4\left(\frac{n-1}{4}\right) + 2 = n + 1$
$n \equiv 2 \pmod 4$:	$[B]^{\frac{n-6}{4}} B_2$	$4\left(\frac{n-6}{4}\right) + 6 = n$
$n \equiv 3 \pmod 4$:	$[B]^{\frac{n-3}{4}} B_3$	$4\left(\frac{n-3}{4}\right) + 4 = n + 1$

Table 3.5.1: Constructions of 2-limited dominating broadcasts on $P_5 \square C_n$.

Proposition 3.5.1 improves the bound given by Proposition 2.5.1 by one for all even n, and otherwise the bounds are the same. The values of Proposition 3.5.1 are optimal for all $n \leq 74$ (as found by computation). A lower bound of n is proven by Theorem 6.4.5 and this shows that the upper bounds given by Proposition 3.5.1 for all even n are tight.

Conjecture 3.5.1. *The bounds in Proposition 3.5.1 are tight.*

3.6 $P_6 \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_6 \square C_n$ for all $n \geq 3$. Figure 3.6.1 contains the tiles used in the broadcast constructions described in Table 3.6.2.

Proposition 3.6.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_6 \square P_n\right) \leq 18 \left\lfloor \frac{n}{16} \right\rfloor + c(n_{16})$$

where $c(n_{16})$ is dependant upon the least residue n_{16} of n modulo 16 and given in Table 3.6.1.

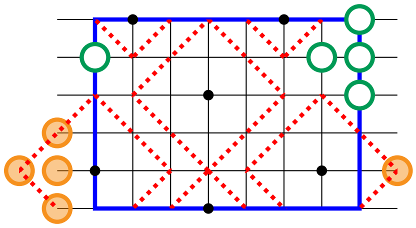
	Least residue n_{16} of n modulo 16															
n_{16}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Cost:	0	2	3	4	5	7	7	9	10	11	12	14	14	16	17	18

Table 3.6.1: Values of $c(n_{16})$ for the upper bound for $\gamma_{b,2}\left(P_6 \square C_n\right)$ stated in Proposition 3.6.1.

	Construction	Cost
$n \equiv 0 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n}{16}}$	$18\left(\frac{n}{16}\right)$
$n \equiv 1 \pmod{16}$:	$[B\overline{B}]^{\frac{n-17}{16}} B_1$	$18\left(\frac{n-17}{16}\right) + 20 = 18\left(\frac{n-1}{16}\right) + 2$
$n \equiv 2 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-18}{16}} B_2$	$18\left(\frac{n-18}{16}\right) + 21 = 18\left(\frac{n-2}{16}\right) + 3$
$n \equiv 3 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-3}{16}} B_3$	$18\left(\frac{n-3}{16}\right) + 4$
$n \equiv 4 \pmod{16}, (n = 4)$:	B_{16}	$5 = 18\left(\frac{n-4}{16}\right) + 5$
$(n > 4)$:	$\left[B\overline{B}\right]^{\frac{n-18}{16}} BB_4$	$18\left(\frac{n-18}{16}\right) + 23 = 18\left(\frac{n-4}{16}\right) + 5$
$n \equiv 5 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-5}{16}} B_5$	$18\left(\frac{n-5}{16}\right) + 7$
$n \equiv 6 \pmod{16}, (n = 6)$:	B_{17}	$7 = 18\left(\frac{n-6}{16}\right) + 7$
$(n > 6)$:	$\left[B\overline{B}\right]^{\frac{n-22}{16}} BB_6$	$18\left(\frac{n-22}{16}\right) + 25 = 18\left(\frac{n-6}{16}\right) + 7$
$n \equiv 7 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-7}{16}} B_7$	$18\left(\frac{n-7}{16}\right) + 9$
$n \equiv 8 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-8}{16}} B_8$	$18\left(\frac{n-8}{16}\right) + 10$
$n \equiv 9 \pmod{16}$:	$[B\overline{B}]^{\frac{n-9}{16}} B_9$	$18\left(\frac{n-9}{16}\right) + 11$
$n \equiv 10 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-10}{16}} B_{10}$	$18\left(\frac{n-10}{16}\right) + 12$
$n \equiv 11 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-11}{16}} B_{11}$	$18\left(\frac{n-11}{16}\right) + 14$
$n \equiv 12 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-12}{16}} B_{12}$	$18\left(\frac{n-12}{16}\right) + 14$
$n \equiv 13 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-13}{16}} B_{13}$	$18\left(\frac{n-13}{16}\right) + 16$
$n \equiv 14 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-14}{16}} B_{14}$	$18\left(\frac{n-14}{16}\right) + 17$
$n \equiv 15 \pmod{16}$:	$\left[B\overline{B}\right]^{\frac{n-15}{16}} B_{15}$	$18\left(\frac{n-15}{16}\right) + 18$

Table 3.6.2: Constructions of 2-limited dominating broadcasts on $P_6 \square C_n$.

B



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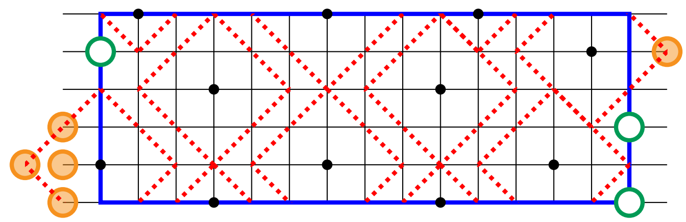

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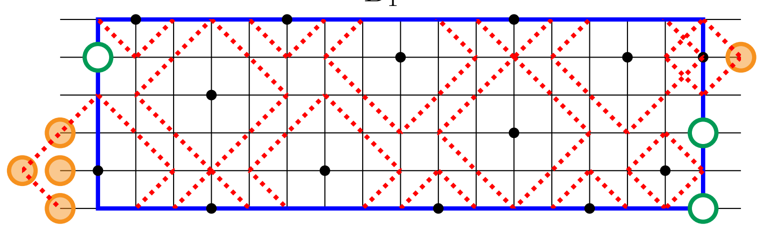

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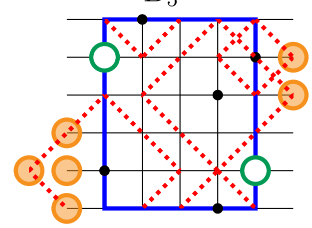

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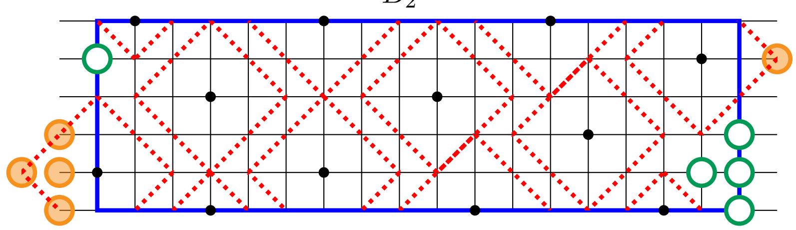
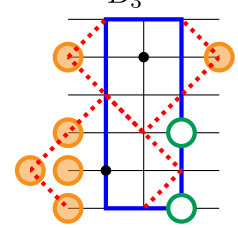


B_3



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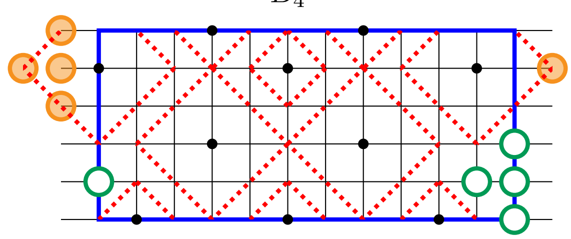
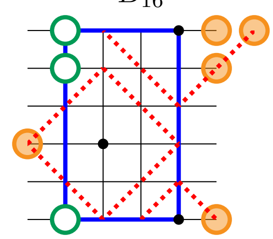
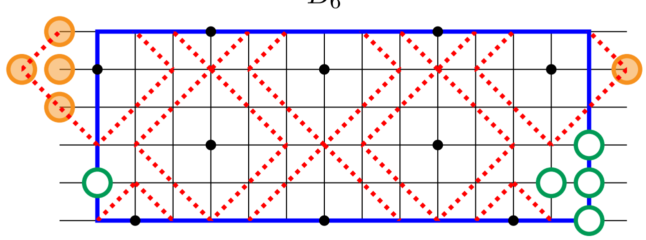


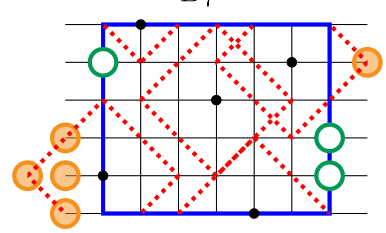
B_{16}



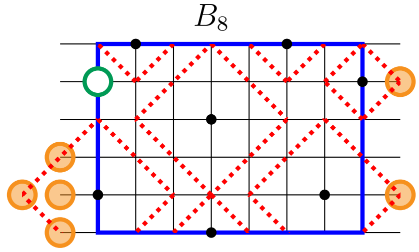
B_6



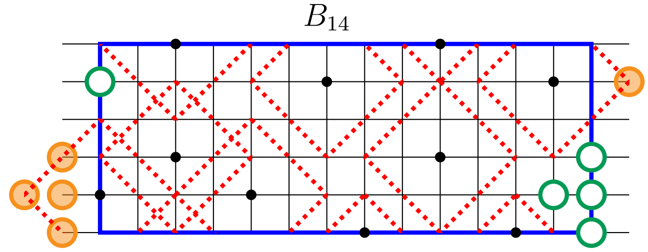
B_7



B_8



B_{14}



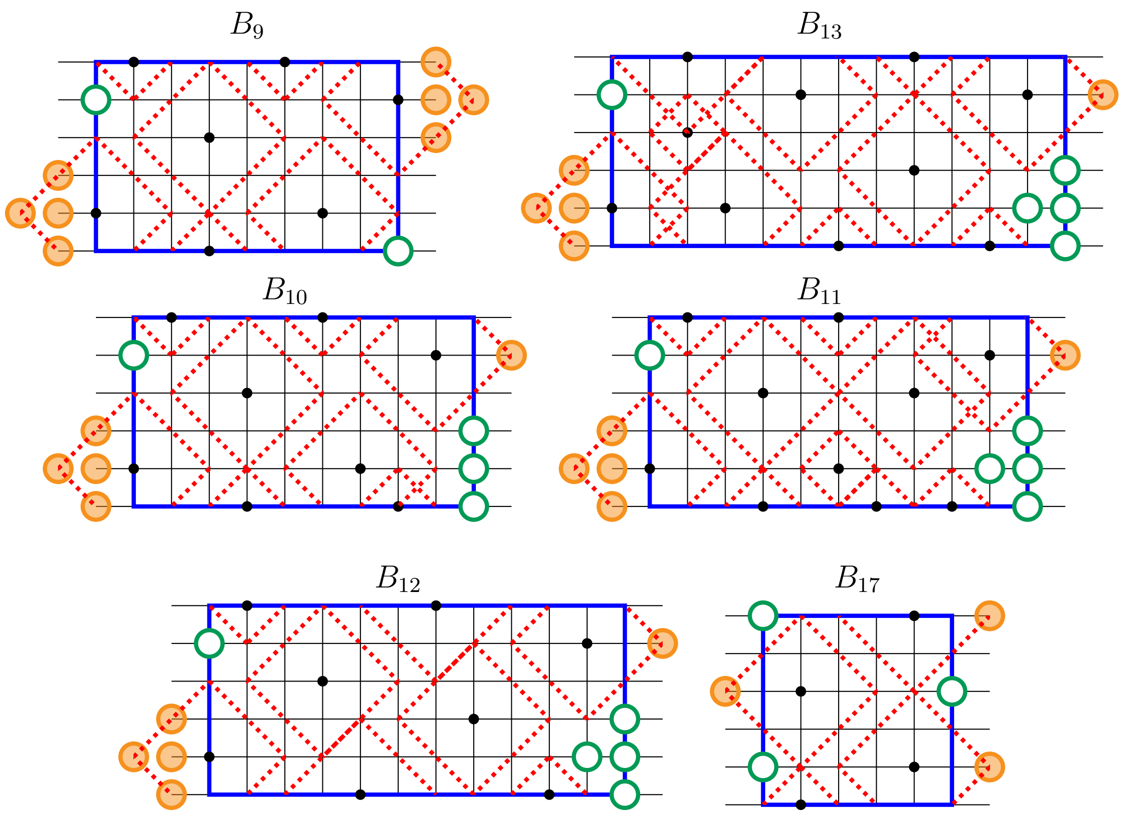


Figure 3.6.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_6 \Box C_n$.

Proposition 3.6.1 improves the bounds given by Proposition 2.6.1 by two when $n_{16} \in \{0, 6\}$, by one when $n_{16} \in \{2, 3, 4, 8, 10, 12, 14\}$, and otherwise the bounds are the same. The values of Proposition 3.6.1 are optimal for all $n \leq 99$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{17.8}{16}n$.

Conjecture 3.6.1. *The bounds given by Proposition 3.6.1 are tight.*

3.7 $P_7 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_7 \Box C_n$ for all $n \geq 3$. Figure 3.7.1 contains the tiles used in the broadcast constructions described in Table 3.7.2.

Proposition 3.7.1. *For $n \geq 3$,*

$$\gamma_{b,2}(P_7 \Box C_n) \leq 18 \left\lfloor \frac{n}{14} \right\rfloor + c(n_{14})$$

where $c(n_{14})$ is dependant upon the least residue n_{14} of n modulo 14 and given in Table 3.7.1.

n_{14}	Least residue n_{14} of n modulo 14													
n_{14}	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Cost:	0	3	3	5	6	8	8	10	11	13	13	15	16	18

Table 3.7.1: Values of $c(n_{14})$ for the upper bound for $\gamma_{b,2}(P_7 \square C_n)$ stated in Proposition 3.7.1.

	Construction	Cost
$n \equiv 0 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n}{14}}$	$18\left(\frac{n}{14}\right)$
$n \equiv 1 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-15}{14}} BB_1$	$18\left(\frac{n-15}{14}\right)+21 = 18\left(\frac{n-1}{14}\right)+3$
$n \equiv 2 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-16}{14}} BB_2$	$18\left(\frac{n-16}{14}\right)+21 = 18\left(\frac{n-2}{14}\right)+3$
$n \equiv 3 \pmod{14}, (n=3)$:	B_{14}	$5 = 18\left(\frac{n-3}{14}\right)+5$
$(n>3)$:	$\left[B\overline{B}\right]^{\frac{n-17}{14}} BB_3$	$18\left(\frac{n-17}{14}\right)+23 = 18\left(\frac{n-3}{14}\right)+5$
$n \equiv 4 \pmod{14}$:	$B_4\left[B\overline{B}\right]^{\frac{n-4}{14}}$	$18\left(\frac{n-4}{14}\right)+6$
$n \equiv 5 \pmod{14}$:	$B_5\left[B\overline{B}\right]^{\frac{n-5}{14}}$	$18\left(\frac{n-5}{14}\right)+8$
$n \equiv 6 \pmod{14}$:	$B_6\left[B\overline{B}\right]^{\frac{n-6}{14}}$	$18\left(\frac{n-6}{14}\right)+8$
$n \equiv 7 \pmod{14}$:	$B_7\left[B\overline{B}\right]^{\frac{n-7}{14}}$	$18\left(\frac{n-7}{14}\right)+10$
$n \equiv 8 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-8}{14}} B_8\overline{B}$	$18\left(\frac{n-8}{14}\right)+11$
$n \equiv 9 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-9}{14}} B_9$	$18\left(\frac{n-9}{14}\right)+13$
$n \equiv 10 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-10}{14}} BB_{10}$	$18\left(\frac{n-10}{14}\right)+13$
$n \equiv 11 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-11}{14}} B_{11}$	$18\left(\frac{n-11}{14}\right)+15$
$n \equiv 12 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-12}{14}} B_{12}$	$18\left(\frac{n-12}{14}\right)+16$
$n \equiv 13 \pmod{14}$:	$\left[B\overline{B}\right]^{\frac{n-13}{14}} B_{13}$	$18\left(\frac{n-13}{14}\right)+18$

Table 3.7.2: Constructions of 2-limited dominating broadcasts on $P_7 \square C_n$.

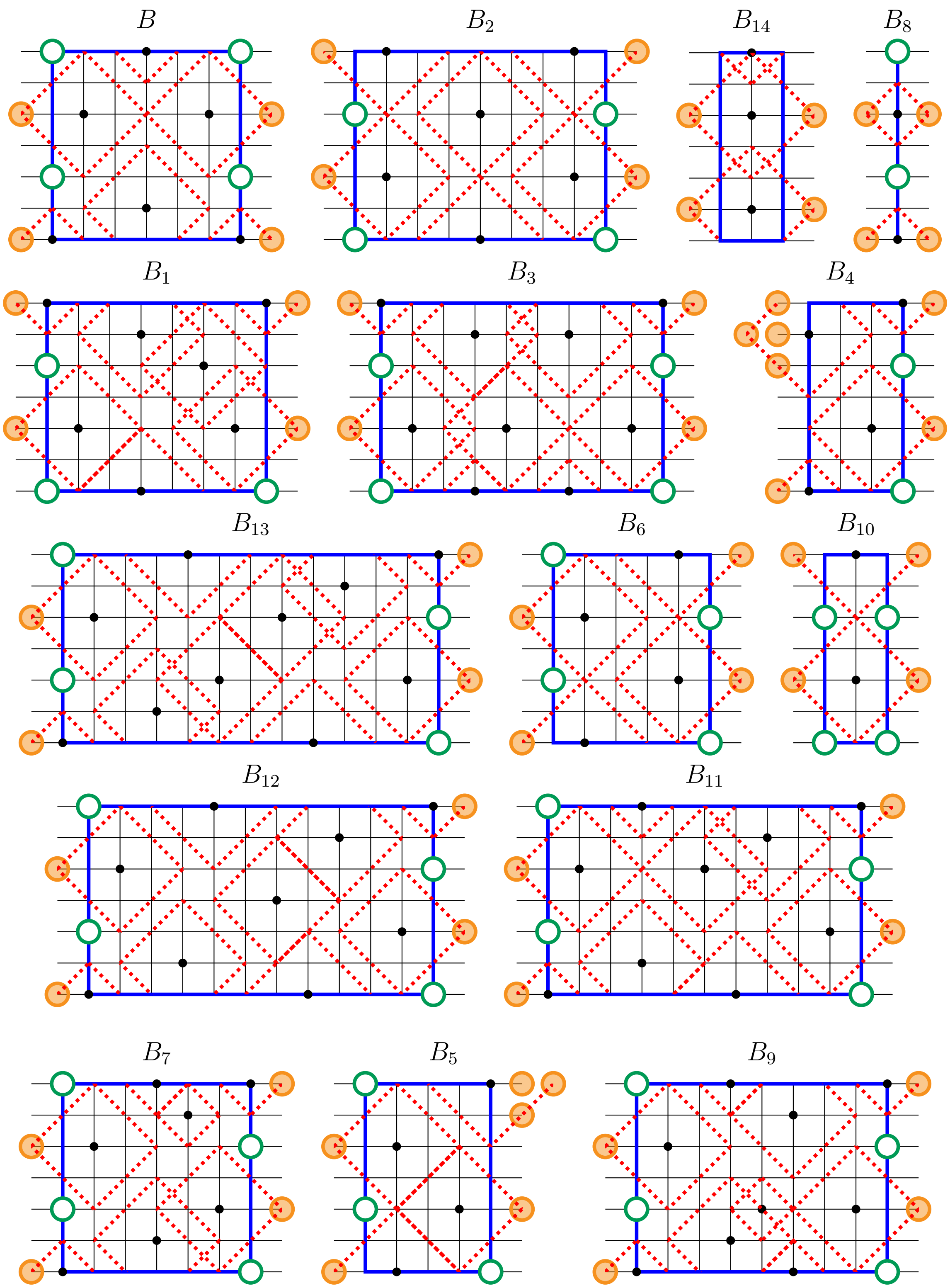


Figure 3.7.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_7 \square C_n$.

Proposition 3.7.1 improves the bounds given by Proposition 2.7.1 by two when $n_{16} = 0$, by one when $n_{16} \in \{2, 4, 6, 7, 8, 10, 11, 12\}$, and otherwise the bounds are the same. The values of Proposition 3.7.1 are optimal for all $n \leq 99$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{17.7}{14}n$.

Conjecture 3.7.1. *The bounds given by Proposition 3.7.1 are tight.*

3.8 $P_8 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_8 \Box C_n$ for all $n \geq 3$. Figure 3.8.1 contains the tiles used in the broadcast constructions described in Table 3.8.2.

Proposition 3.8.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_8 \Box C_n\right) \leq 32\left\lfloor \frac{n}{22} \right\rfloor + c(n_{22})$$

where $c(n_{22})$ is dependant upon the least residue n_{22} of n modulo 22 and given in Table 3.8.1 with the exception of $n = 4, 8,$ and 14 which give values of $c(n_{22})$ equal to $6, 12,$ and 21, respectively.

	Least residue n_{22} of n modulo 22										
n_{22}	0	1	2	3	4	5	6	7	8	9	10
Cost:	0	3	4	6	7	9	9	11	13	14	15
n_{22}	11	12	13	14	15	16	17	18	19	20	21
Cost:	18	18	20	22	23	24	26	27	29	30	32

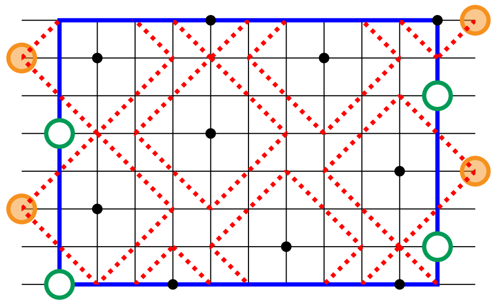
Table 3.8.1: Values of $c(n_{22})$ for the upper bound for $\gamma_{b,2}\left(P_8 \Box C_n\right)$ stated in Proposition 3.8.1 where $n \notin \{4, 8, 14\}$.

	Construction	Cost
$n \equiv 0 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n}{22}} B$	$32\left(\frac{n}{22}\right)$
$n \equiv 1 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-23}{22}} BB_1$	$32\left(\frac{n-23}{22}\right) + 35 = 32\left(\frac{n-1}{22}\right) + 3$
$n \equiv 2 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-24}{22}} BB_2$	$32\left(\frac{n-24}{22}\right) + 36 = 32\left(\frac{n-2}{22}\right) + 4$
$n \equiv 3 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-3}{22}} B_3$	$32\left(\frac{n-3}{22}\right) + 6$
$n \equiv 4 \pmod{22}, (n = 4)$:	B_{22}	6
$(n > 4)$:	$\left[B\overline{B}\right]^{\frac{n-4}{22}} B_4$	$32\left(\frac{n-4}{22}\right) + 7$

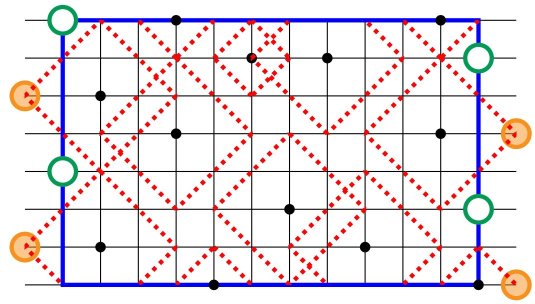
$n \equiv 5 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-5}{22}} B_5$	$32\left(\frac{n-5}{22}\right)+9$
$n \equiv 6 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-6}{22}} B_6$	$32\left(\frac{n-6}{22}\right)+9$
$n \equiv 7 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-7}{22}} B_7$	$32\left(\frac{n-7}{22}\right)+11$
$n \equiv 8 \pmod{22}, (n=8)$:	B_{23}	12
$(n>8)$:	$\left[B\overline{B}\right]^{\frac{n-8}{22}} B_8$	$32\left(\frac{n-8}{22}\right)+13$
$n \equiv 9 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-31}{22}} BB_9$	$32\left(\frac{n-31}{22}\right)+46 = 32\left(\frac{n-9}{22}\right)+14$
$n \equiv 10 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-10}{22}} B_{10}$	$32\left(\frac{n-10}{22}\right)+15$
$n \equiv 11 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-11}{22}} B_{11}$	$32\left(\frac{n-11}{22}\right)+18$
$n \equiv 12 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-12}{22}} B_{12}$	$32\left(\frac{n-12}{22}\right)+18$
$n \equiv 13 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-13}{22}} B_{13}$	$32\left(\frac{n-13}{22}\right)+20$
$n \equiv 14 \pmod{22}, (n=14)$:	B_{25}	21
$(n>14)$:	$\left[B\overline{B}\right]^{\frac{n-14}{22}} B_{14}$	$32\left(\frac{n-14}{22}\right)+22$
$n \equiv 15 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-15}{22}} B_{15}$	$32\left(\frac{n-15}{22}\right)+23$
$n \equiv 16 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-16}{22}} B_{16}$	$32\left(\frac{n-16}{22}\right)+24$
$n \equiv 17 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-17}{22}} B_{17}$	$32\left(\frac{n-17}{22}\right)+26$
$n \equiv 18 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-18}{22}} B_{18}$	$32\left(\frac{n-18}{22}\right)+28$
$n \equiv 19 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-19}{22}} B_{19}$	$32\left(\frac{n-19}{22}\right)+29$
$n \equiv 20 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-20}{22}} BB_{20}$	$32\left(\frac{n-20}{22}\right)+30$
$n \equiv 21 \pmod{22}$:	$\left[B\overline{B}\right]^{\frac{n-21}{22}} B_{21}\overline{B}$	$32\left(\frac{n-21}{22}\right)+32$

Table 3.8.2: Constructions of 2-limited dominating broadcasts on $P_8 \square C_n$.

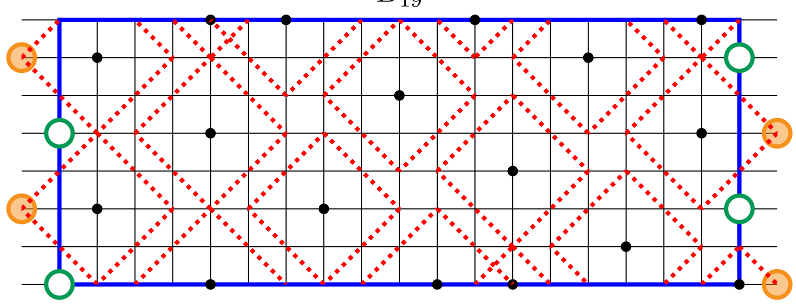
B



B_1



B_{19}



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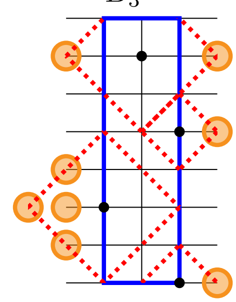

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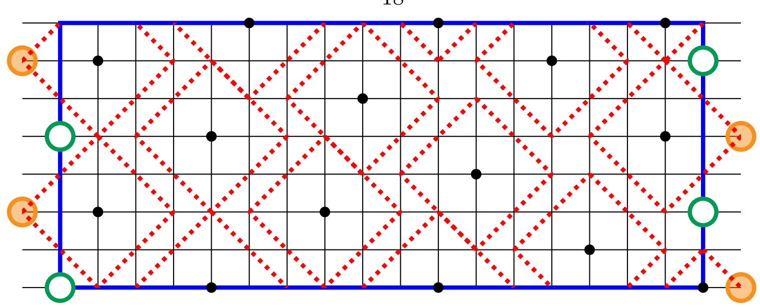

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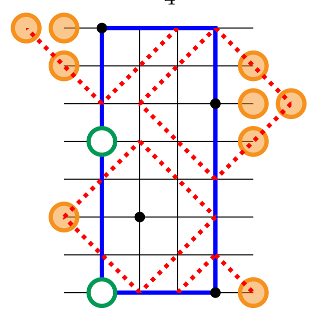
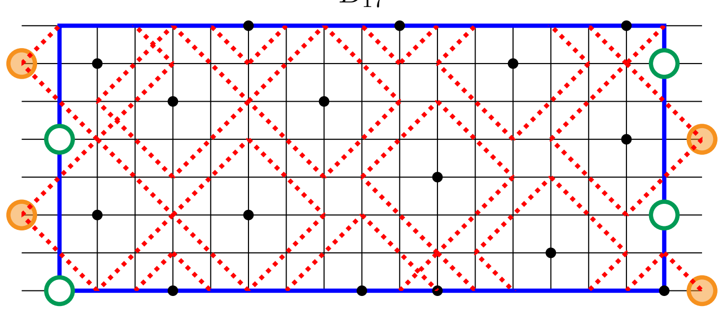
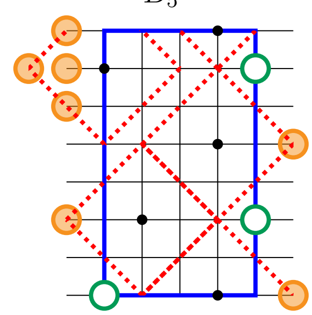


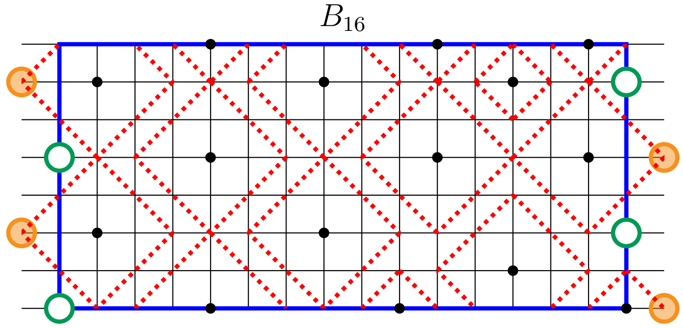
B_{17}



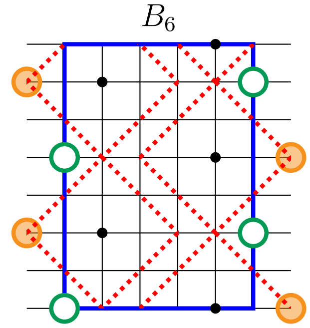
B_5



B_{16}



B_6



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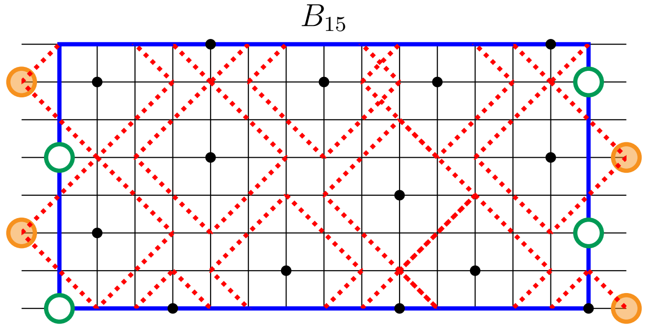
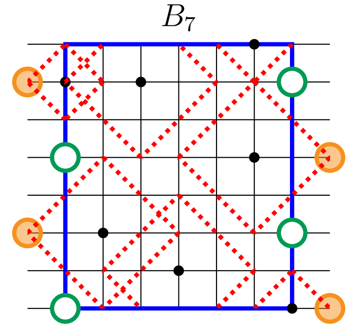
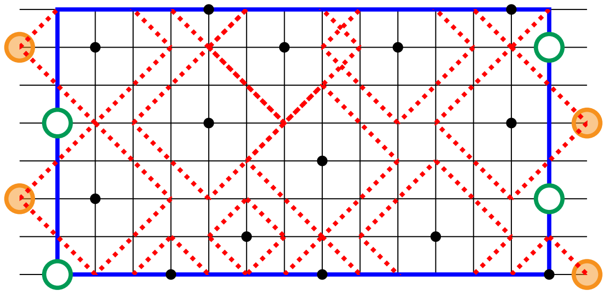


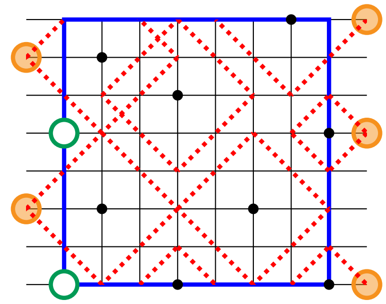
B_7



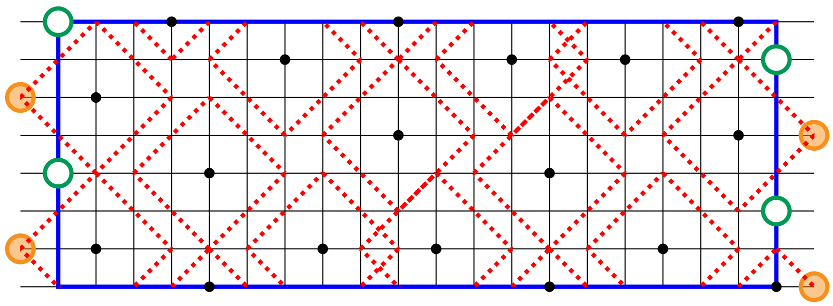
B_{14}



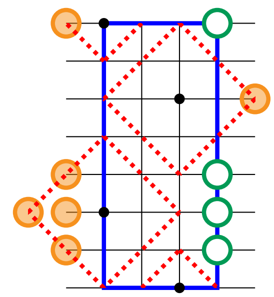
B_8



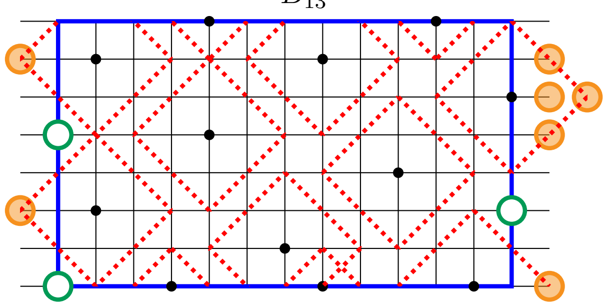
B_9



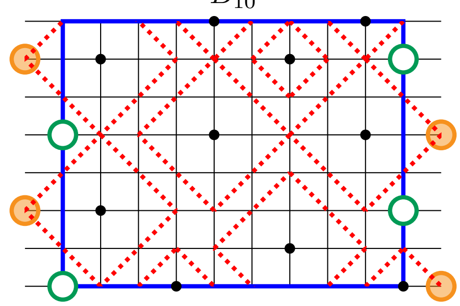
B_{22}



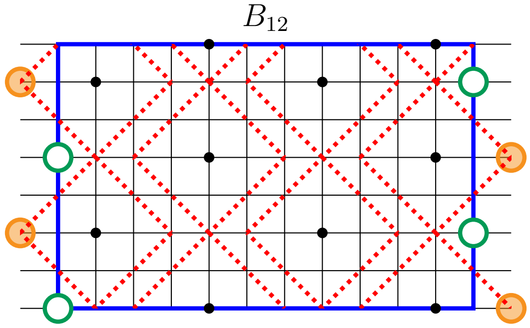
B_{13}



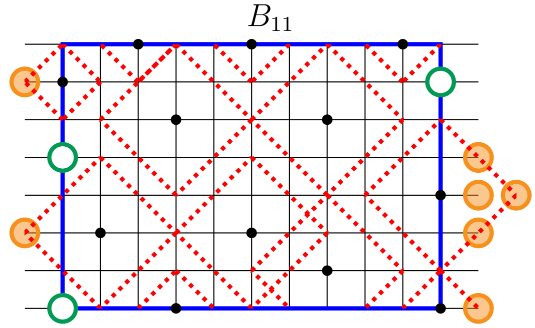
B_{10}



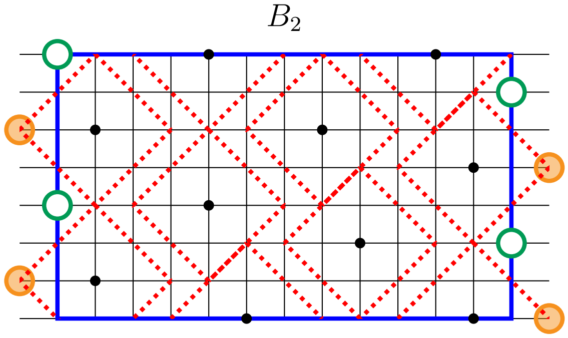
B_{12}



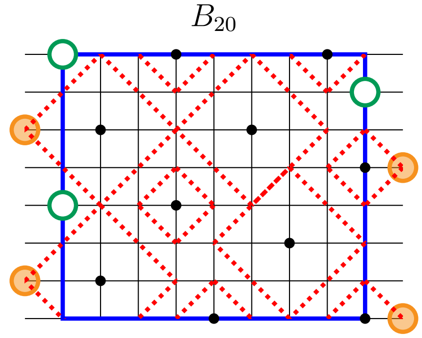
B_{11}



B_2



B_{20}



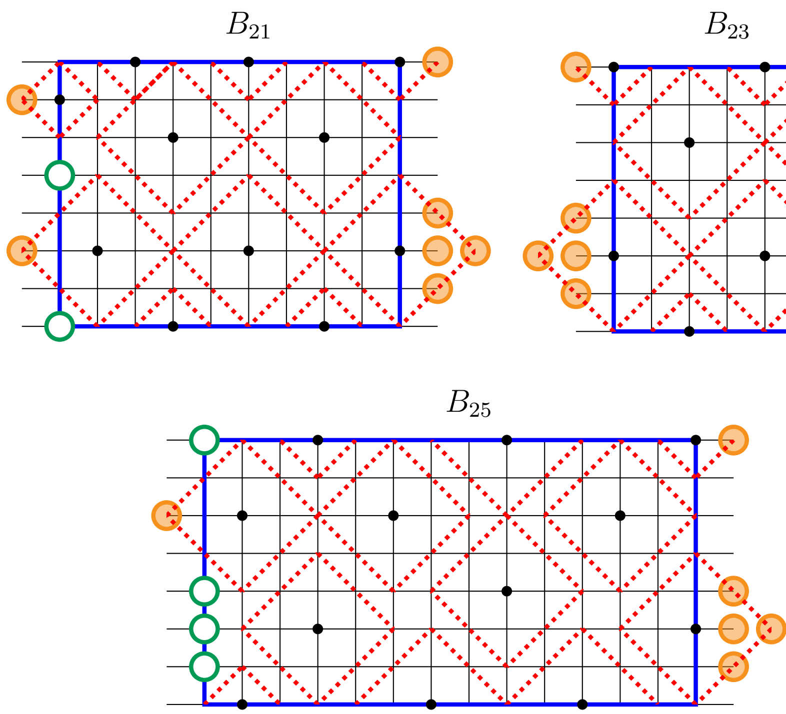


Figure 3.8.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_8 \square C_n$.

Proposition 3.8.1 improves the bounds given by Proposition 2.8.1 by two when $n_{22} \in \{0, 6, 12\}$, by one when $n_{22} \in \{1, 2, 3, 4, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20\}$, and otherwise the bounds are the same. The values of Proposition 3.8.1 are optimal for all $n \leq 82$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{31.3}{22}n$.

Conjecture 3.8.1. *The bounds given by Proposition 3.8.1 are tight.*

3.9 $P_9 \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_9 \square C_n$ for all $n \geq 3$. Figure 3.9.1 contains the tiles used in the broadcast constructions described in Table 3.9.2.

Proposition 3.9.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_9 \square C_n\right) \leq 16 \left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

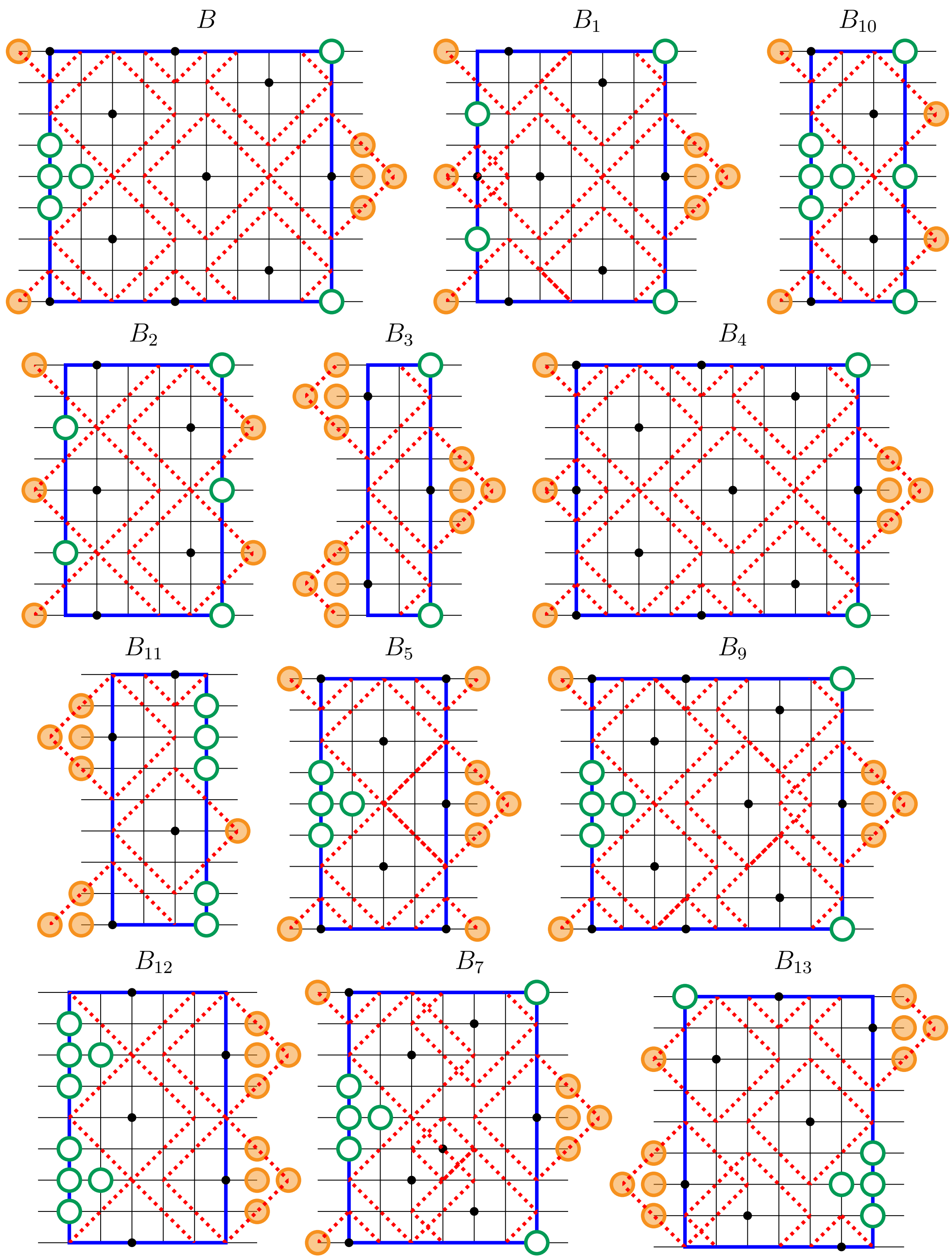
where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 3.9.1 with the exception of $n = 7$ which gives a value of $c(n_{10})$ equal to 12.

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	0	3	4	6	7	10	10	13	14	16

Table 3.9.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}\left(P_9 \square C_n\right)$ stated in Proposition 3.9.1 where $n \neq 7$.

	Construction	Cost
$n \equiv 0 \pmod{10}$:	$[B]^{\frac{n}{10}}$	$16\left(\frac{n}{10}\right)$
$n \equiv 1 \pmod{10}$:	$[B]^{\frac{n-11}{10}} B_{10}B_1$	$16\left(\frac{n-11}{10}\right) + 19 = 16\left(\frac{n-1}{10}\right) + 3$
$n \equiv 2 \pmod{10}, (n = 12)$:	B_{12}^2	$20 = 16\left(\frac{n-2}{10}\right) + 4$
$(n > 12)$:	$B_8\,[B]^{\frac{n-22}{10}} B_{10}B_2B_6$	$16\left(\frac{n-22}{10}\right) + 36 = 16\left(\frac{n-2}{10}\right) + 4$
$n \equiv 3 \pmod{10}$:	$[B]^{\frac{n-3}{10}} B_3$	$16\left(\frac{n-3}{10}\right) + 6$
$n \equiv 4 \pmod{10}, (n = 4)$:	B_{11}	$7 = 16\left(\frac{n-4}{10}\right) + 7$
$(n > 4)$:	$[B]^{\frac{n-14}{10}} B_{10}B_4$	$16\left(\frac{n-14}{10}\right) + 23 = 16\left(\frac{n-4}{10}\right) + 7$
$n \equiv 5 \pmod{10}$:	$[B]^{\frac{n-5}{10}} B_5$	$16\left(\frac{n-5}{10}\right) + 10$
$n \equiv 6 \pmod{10}, (n = 6)$:	B_{12}	$10 = 16\left(\frac{n-6}{10}\right) + 10$
$(n > 6)$:	$[B]^{\frac{n-16}{10}} B_{10}B_6$	$16\left(\frac{n-16}{10}\right) + 26 = 16\left(\frac{n-6}{10}\right) + 10$
$n \equiv 7 \pmod{10}, (n = 7)$:	B_{13}	12
$(n > 7)$:	$[B]^{\frac{n-7}{10}} B_7$	$16\left(\frac{n-7}{10}\right) + 13$
$n \equiv 8 \pmod{10}$:	$[B]^{\frac{n-8}{10}} B_8$	$16\left(\frac{n-8}{10}\right) + 14$
$n \equiv 9 \pmod{10}$:	$[B]^{\frac{n-9}{10}} B_9$	$16\left(\frac{n-9}{10}\right) + 16$

Table 3.9.2: Constructions of 2-limited dominating broadcasts on $P_9 \square C_n$.


B
B_1
B_{10}
B_2
B_3
B_4
B_{11}
B_5
B_9
B_{12}
B_7
B_{13}

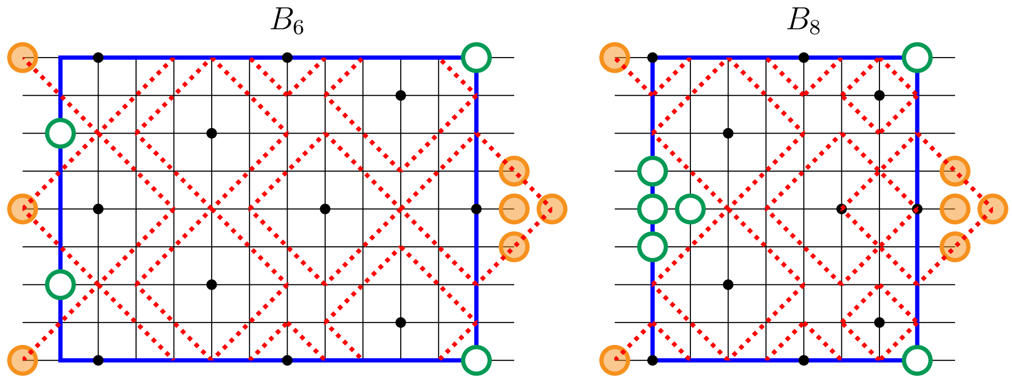


Figure 3.9.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_9 \square C_n$.

Proposition 3.9.1 improves the bounds given by Proposition 2.9.1 by two when $n_{10} = 0$, by one when $n_{10} \in \{2, 4, 6, 8\}$, and otherwise the bounds are the same. The values of Proposition 3.9.1 are optimal for all $n \le 94$ (as found by computation) except for cases given in Table 3.9.3. Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{15.7}{10}n$.

Conjecture 3.9.1. *The bounds given by Proposition 3.9.1 are tight except when* $n \equiv 7 \pmod{20}$.

n	Optimal	Bound
27	44	45
47	76	77
67	108	109
87	140	141

Table 3.9.3: Known values of n for which the bound in Proposition 3.9.1 is not optimal.

3.10 $P_{10} \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{10} \square C_n$ for all $n \ge 3$. Figure 3.10.1 contains the tiles used in the broadcast constructions described in Table 3.10.2.

Proposition 3.10.1. *For* $n \ge 3$,

$$\gamma_{b,2}\left(P_{10} \square C_n\right) \le 32 \left\lfloor \frac{n}{18} \right\rfloor + c(n_{18})$$

where $c(n_{18})$ *is dependant upon the least residue* n_{18} *of* n *modulo* 18 *and given in Table 3.10.1 with the exception of* $n = 5$ *and* 9 *which gives values of* $c(n_{18})$ *equal to* 11 *and* 18, *respectively.*

	Least residue n_{18} of n modulo 18																	
n_{18}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Cost:	0	3	4	7	8	10	11	14	15	17	18	21	22	24	25	28	29	32

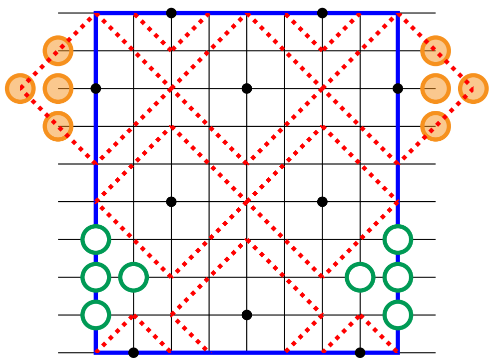
Table 3.10.1: Values of $c(n_{18})$ for the upper bound for $\gamma_{b,2}(P_{10}\Box C_n)$ stated in Proposition 3.10.1 where $n \notin \{5, 9\}$.

	Construction	Cost
$n \equiv 0 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n}{18}}$	$32\left(\frac{n}{18}\right)$
$n \equiv 1 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-19}{18}} B_1$	$32\left(\frac{n-19}{18}\right) + 35 = 32\left(\frac{n-1}{18}\right) + 3$
$n \equiv 2 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-20}{18}} BB_2$	$32\left(\frac{n-20}{18}\right) + 36 = 32\left(\frac{n-2}{18}\right) + 4$
$n \equiv 3 \pmod{18}, (n = 3)$:	B_{18}	$7 = 32\left(\frac{n-3}{18}\right) + 7$
$(n > 3)$:	$\left[B\overline{B}\right]^{\frac{n-21}{18}} BB_3$	$32\left(\frac{n-21}{18}\right) + 39 = 32\left(\frac{n-21}{18}\right) + 7$
$n \equiv 4 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-4}{18}} B_4$	$32\left(\frac{n-4}{18}\right) + 8$
$n \equiv 5 \pmod{18}, (n = 5)$:	B_{19}	11
$(n > 5)$:	$\left[B\overline{B}\right]^{\frac{n-23}{18}} B_5$	$32\left(\frac{n-23}{18}\right) + 42 = 32\left(\frac{n-5}{18}\right) + 10$
$n \equiv 6 \pmod{18}, (n = 6)$:	B_{20}	$11 = 32\left(\frac{n-6}{18}\right) + 11$
$(n > 6)$:	$\left[B\overline{B}\right]^{\frac{n-24}{18}} BB_6$	$32\left(\frac{n-24}{18}\right) + 43 = 32\left(\frac{n-6}{18}\right) + 11$
$n \equiv 7 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-7}{18}} B_7$	$32\left(\frac{n-7}{18}\right) + 14$
$n \equiv 8 \pmod{18}, (n = 8)$:	B_{21}	$15 = 32\left(\frac{n-8}{18}\right) + 15$
$(n > 8)$:	$\left[B\overline{B}\right]^{\frac{n-26}{18}} BB_8$	$32\left(\frac{n-17}{18}\right) + 31 = 32\left(\frac{n-8}{18}\right) + 15$
$n \equiv 9 \pmod{18}, (n = 9)$:	B_{22}	18
$(n > 9)$:	$\left[B\overline{B}\right]^{\frac{n-27}{18}} BB_9$	$32\left(\frac{n-18}{18}\right) + 33 = 32\left(\frac{n-9}{18}\right) + 17$
$n \equiv 10 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-10}{18}} B_{10}$	$32\left(\frac{n-10}{18}\right) + 18$
$n \equiv 11 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-11}{18}} B_{11}$	$32\left(\frac{n-11}{18}\right) + 21$
$n \equiv 12 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-12}{18}} B_{12}$	$32\left(\frac{n-12}{18}\right) + 22$
$n \equiv 13 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-13}{18}} B_{13}$	$32\left(\frac{n-13}{18}\right) + 24$
$n \equiv 14 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-14}{18}} B_{14}$	$32\left(\frac{n-14}{18}\right) + 25$
$n \equiv 15 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-15}{18}} B_{15}$	$32\left(\frac{n-15}{18}\right) + 28$

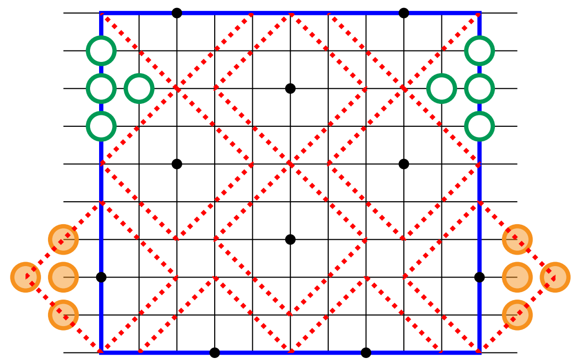
$n \equiv 16 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-16}{18}} B_{16}$	$32\left(\frac{n-16}{18}\right) + 29$
$n \equiv 17 \pmod{18}$:	$\left[B\overline{B}\right]^{\frac{n-18}{18}} B_{17}$	$32\left(\frac{n-18}{18}\right) + 32$

Table 3.10.2: Constructions of 2-limited dominating broadcasts on $P_{10}\Box C_n$.

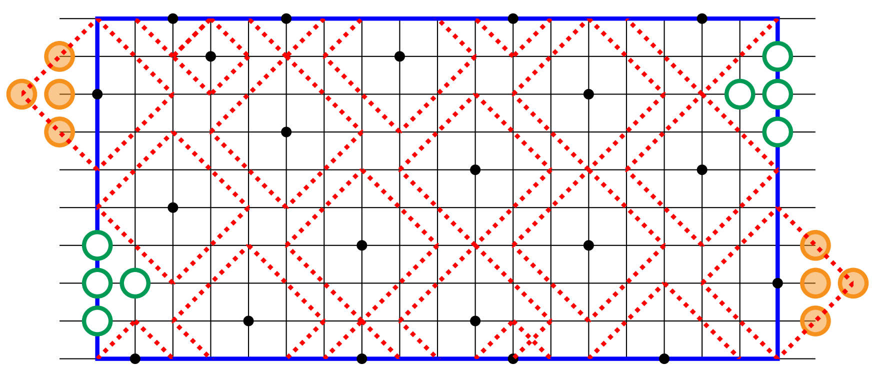
B



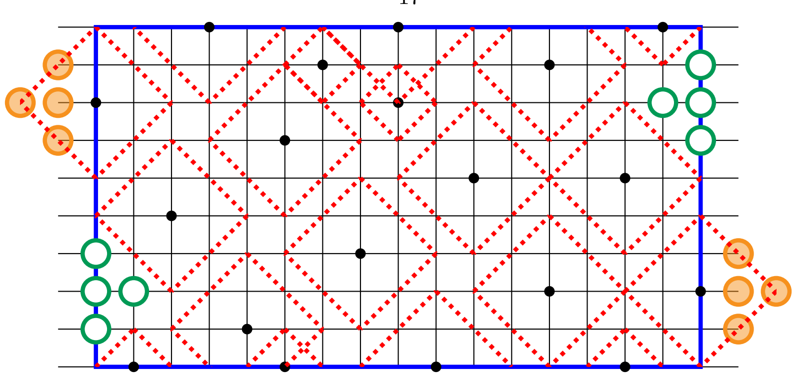
B_2



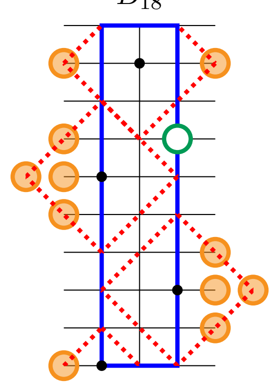
B_1



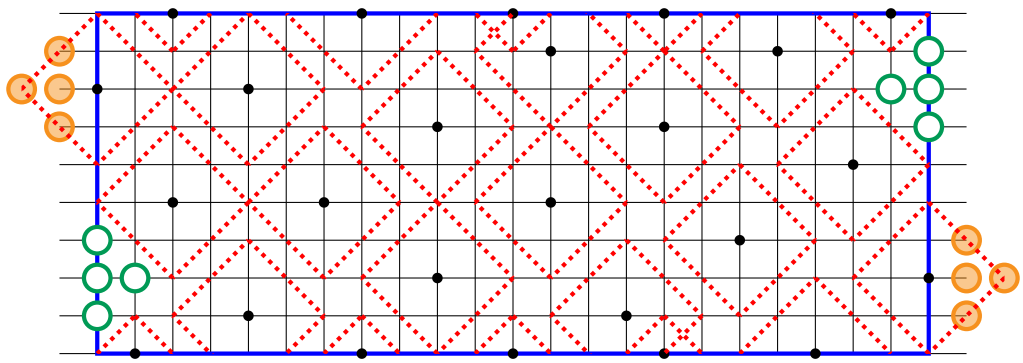
B_{17}



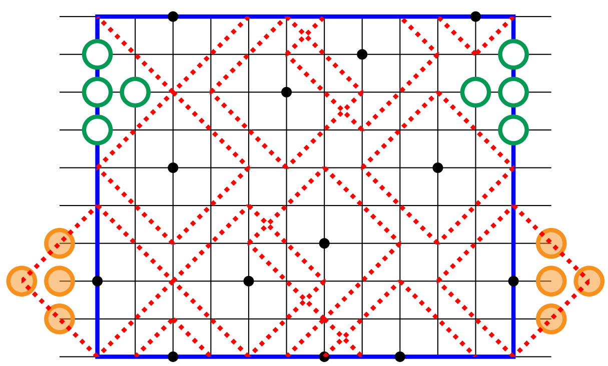
B_{18}



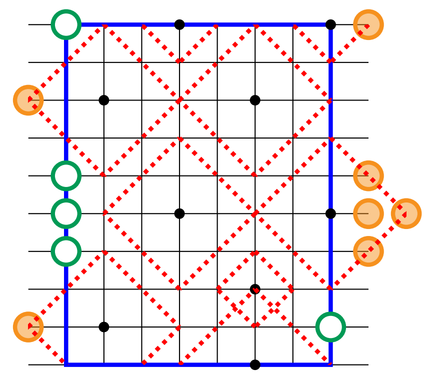
B_5



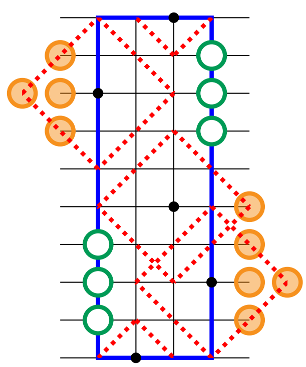
B_3



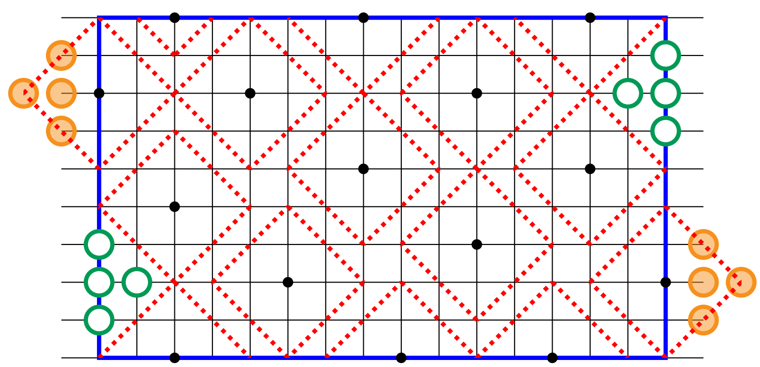
B_{21}



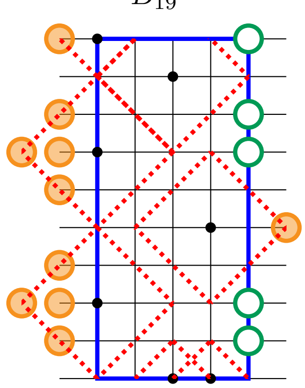
B_4



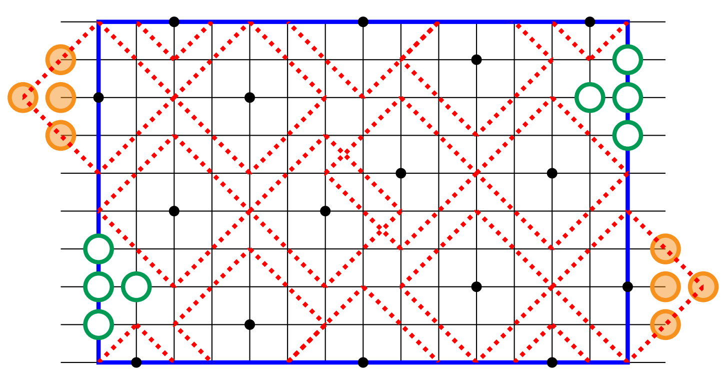
B_{16}



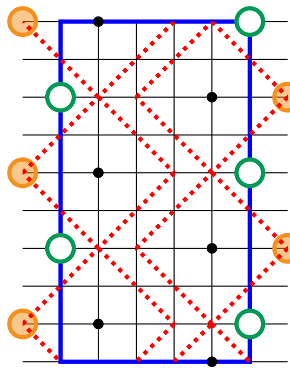
B_{19}



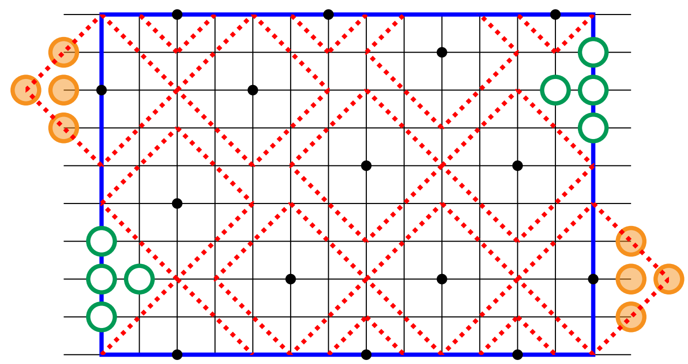
B_{15}



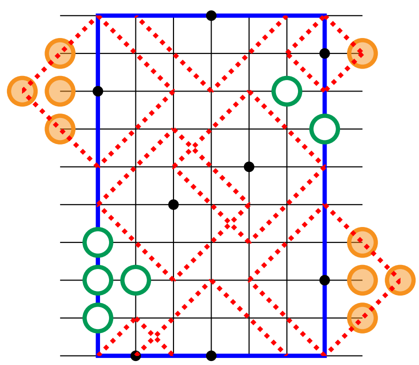
B_{20}



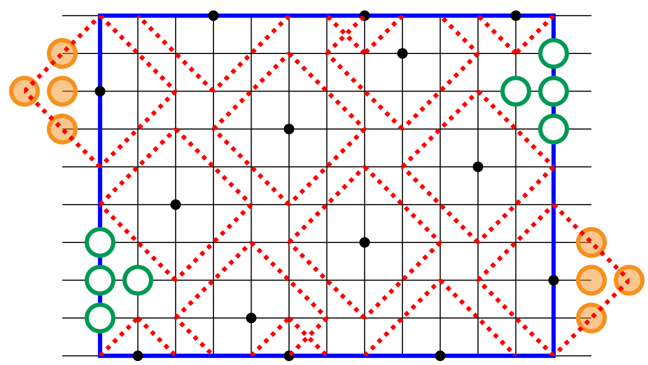
B_{14}



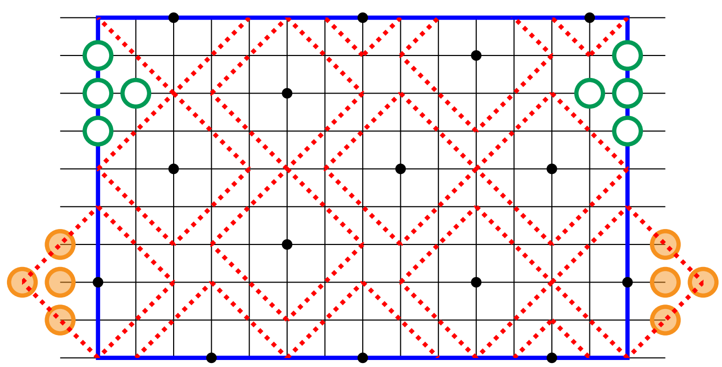
B_7



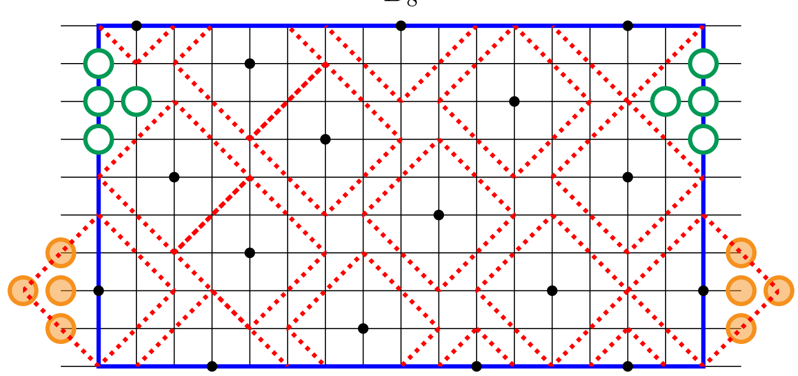
B_{13}



B_6



B_8



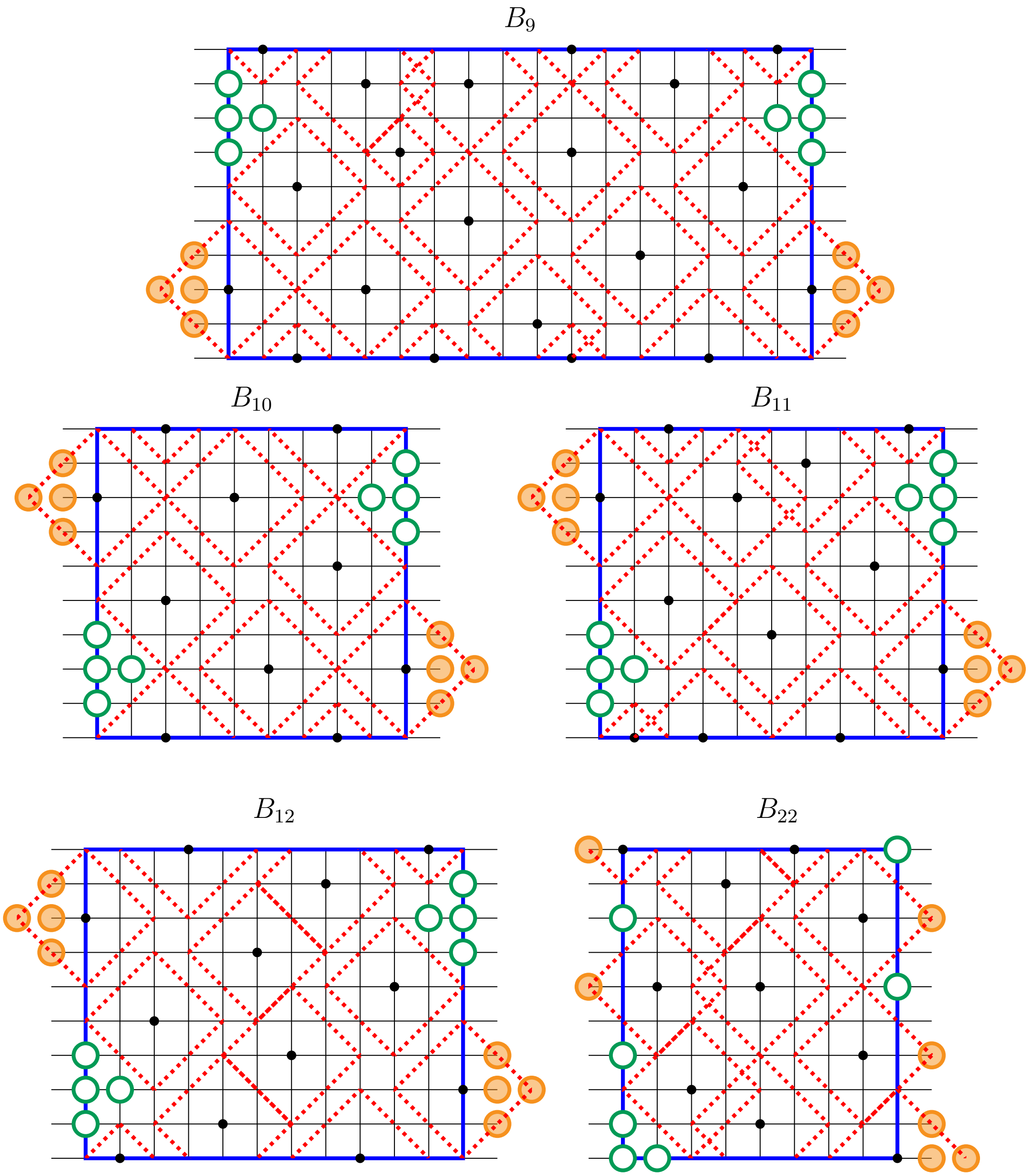


Figure 3.10.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{10}\Box C_n$.

Proposition 3.10.1 improves the bounds given by Proposition 2.10.1 by two when $n_{18} \in \{0, 2, 6, 10, 12, 14, 16\}$, by one when $n_{18} \in \{1, 3, 4, 5, 7, 8, 9, 11, 13, 15\}$, and otherwise the bounds are the same. The values of Proposition 3.10.1 are optimal for all $n \leq 60$ (as found by computation). Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{31.1}{18}n$.

Conjecture 3.10.1. *The bounds given by Proposition 3.10.1 are tight.*

3.11 $P_{11}\Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{11}\Box C_n$ for all $n \geq 3$. Figure 3.11.1 contains the tiles used in the broadcast constructions described in Table 3.11.2.

Proposition 3.11.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_{11}\Box C_n\right) \leq 50\left\lfloor \frac{n}{26} \right\rfloor + c(n_{26})$$

where $c(n_{26})$ is dependant upon the least residue n_{26} of n modulo 26 and given in Table 3.11.1 with the exception of $n = 4$ and 8 which give values of $c(n_{26})$ equal to 8 and 16, respectively.

	Least residue n_{26} of n modulo 26												
n_{26}	0	1	2	3	4	5	6	7	8	9	10	11	12
Cost:	0	4	5	8	9	12	12	15	17	19	20	23	24
n_{26}	13	14	15	16	17	18	19	20	21	22	23	24	25
Cost:	27	28	31	31	34	36	38	39	42	43	46	48	50

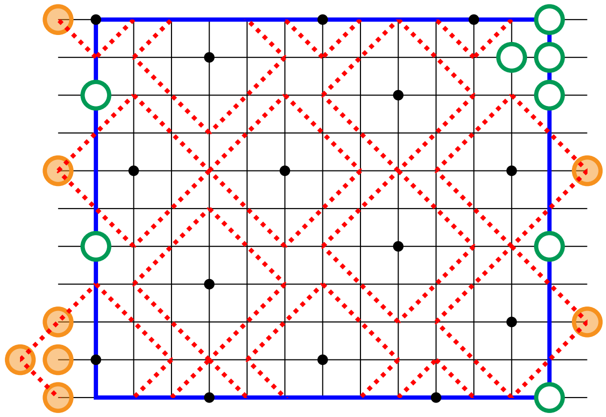
Table 3.11.1: Values of $c(n_{26})$ for the upper bound for $\gamma_{b,2}\left(P_{11}\Box C_n\right)$ stated in Proposition 3.11.1 where $n \notin \{4, 8\}$.

	Construction	Cost
$n \equiv 0 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n}{26}}$	$50\left(\frac{n}{26}\right)$
$n \equiv 1 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-25}{26}} BB_1$	$50\left(\frac{n-14}{26}\right) + 29 = 50\left(\frac{n-1}{26}\right) + 4$
$n \equiv 2 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-28}{26}} B_2$	$50\left(\frac{n-28}{26}\right) + 55 = 50\left(\frac{n-2}{26}\right) + 5$
$n \equiv 3 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-3}{26}} B_3$	$50\left(\frac{n-3}{26}\right) + 8$
$n \equiv 4 \pmod{26}, (n = 4)$:	B_{26}	8
$(n > 4)$:	$\left[B\overline{B}\right]^{\frac{n-4}{26}} B_4$	$50\left(\frac{n-4}{26}\right) + 9$
$n \equiv 5 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-5}{26}} B_5$	$50\left(\frac{n-5}{26}\right) + 12$
$n \equiv 6 \pmod{26}, (n = 6)$:	B_{29}	$12 = 50\left(\frac{n-6}{26}\right) 12$
$(n > 6)$:	$\left[B\overline{B}\right]^{\frac{n-32}{26}} BB_6$	$50\left(\frac{n-19}{26}\right) + 37 = 50\left(\frac{n-6}{26}\right) + 12$
$n \equiv 7 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-7}{26}} B_7$	$50\left(\frac{n-7}{26}\right) + 15$

$n \equiv 8 \pmod{26}, (n=8)$:	B_{27}	16
$(n > 8)$:	$\left[B\overline{B}\right]^{\frac{n-8}{26}} B_8$	$50\left(\frac{n-8}{26}\right) + 17$
$n \equiv 9 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-9}{26}} B_9$	$50\left(\frac{n-9}{26}\right) + 19$
$n \equiv 10 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-10}{26}} B_{10}$	$50\left(\frac{n-10}{26}\right) + 20$
$n \equiv 11 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-11}{26}} B_{11}$	$50\left(\frac{n-11}{26}\right) + 23$
$n \equiv 12 \pmod{26}, (n=12)$:	B_{28}	$50\left(\frac{n-12}{26}\right) + 24$
$(n > 12)$:	$\left[B\overline{B}\right]^{\frac{n-38}{26}} BB_{12}$	$50\left(\frac{n-25}{26}\right) + 49 = 50\left(\frac{n-12}{26}\right) + 24$
$n \equiv 13 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-13}{26}} B_{13}$	$50\left(\frac{n-13}{26}\right) + 27$
$n \equiv 14 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-14}{26}} B_{14}$	$50\left(\frac{n-14}{26}\right) + 28$
$n \equiv 15 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-15}{26}} B_{15}$	$50\left(\frac{n-15}{26}\right) + 31$
$n \equiv 16 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-16}{26}} B_{16}$	$50\left(\frac{n-16}{26}\right) + 31$
$n \equiv 17 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-17}{26}} B_{17}$	$50\left(\frac{n-17}{26}\right) + 34$
$n \equiv 18 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-18}{26}} B_{18}$	$50\left(\frac{n-18}{26}\right) 36$
$n \equiv 19 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-19}{26}} B_{19}$	$50\left(\frac{n-19}{26}\right) + 38$
$n \equiv 20 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-20}{26}} B_{20}$	$50\left(\frac{n-20}{26}\right) + 39$
$n \equiv 21 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-21}{26}} B_{21}$	$50\left(\frac{n-21}{26}\right) + 42$
$n \equiv 22 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-22}{26}} B_{22}$	$50\left(\frac{n-22}{26}\right) + 43$
$n \equiv 23 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-23}{26}} B_{23}$	$50\left(\frac{n-23}{26}\right) + 46$
$n \equiv 24 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-24}{26}} B_{22}$	$50\left(\frac{n-22}{26}\right) + 48$
$n \equiv 25 \pmod{26}$:	$\left[B\overline{B}\right]^{\frac{n-25}{26}} B_{23}$	$50\left(\frac{n-23}{26}\right) + 50$

Table 3.11.2: Constructions of 2-limited dominating broadcasts on $P_{11} \Box C_n$.

B



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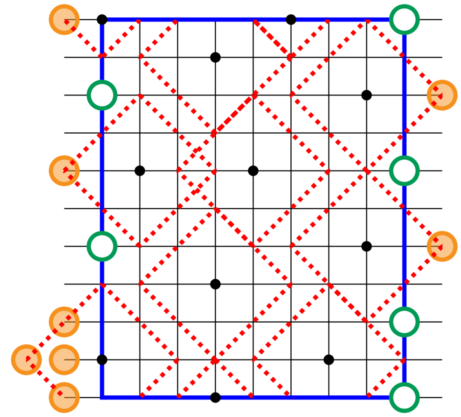

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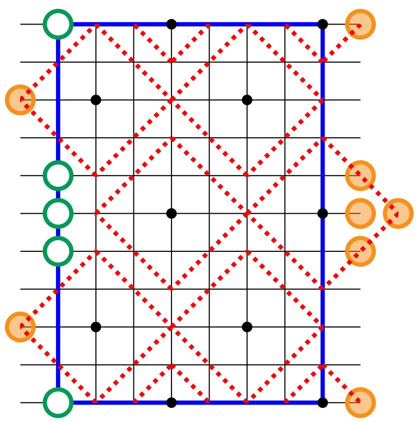

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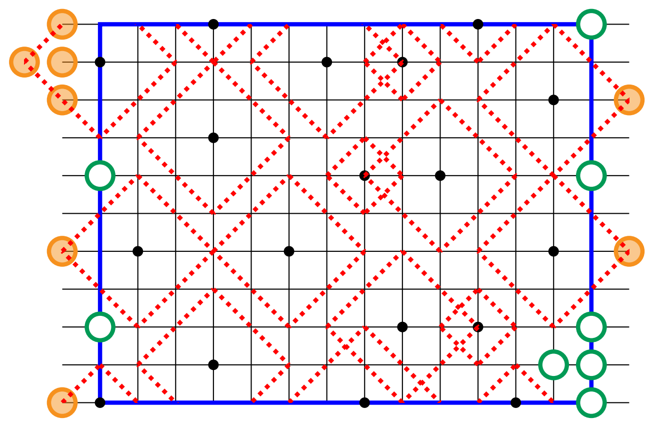

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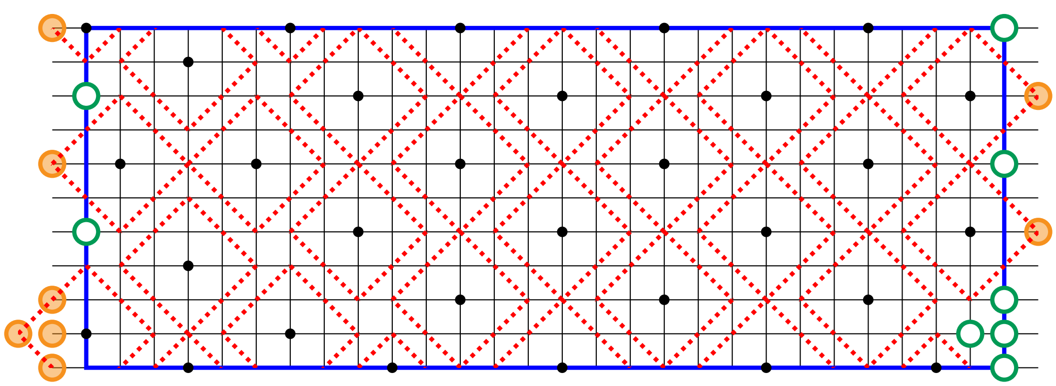

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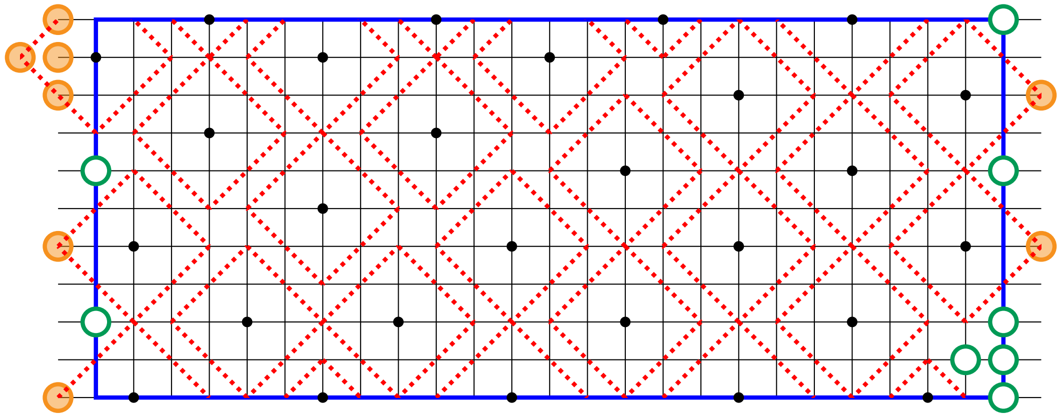

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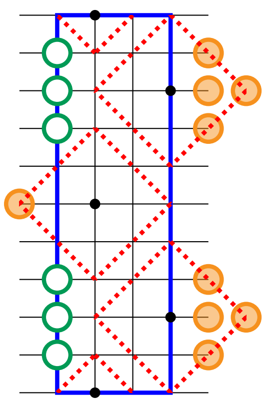

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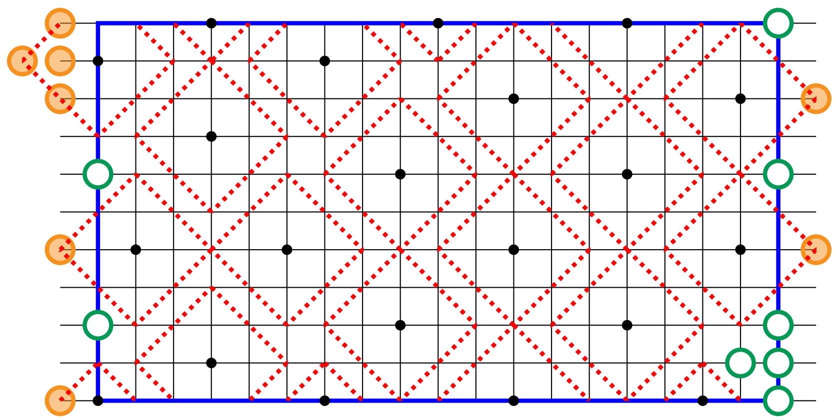

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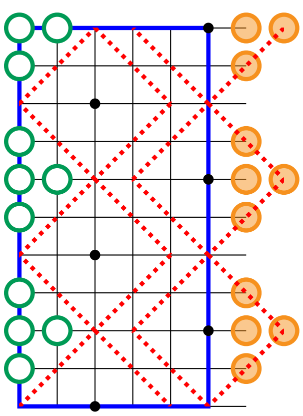

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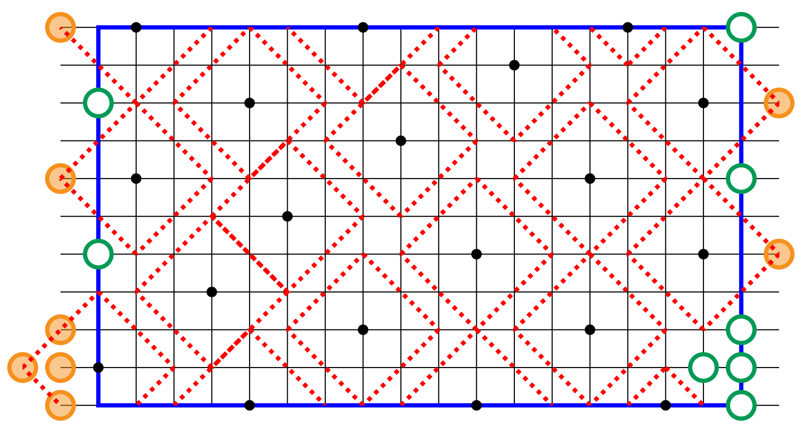

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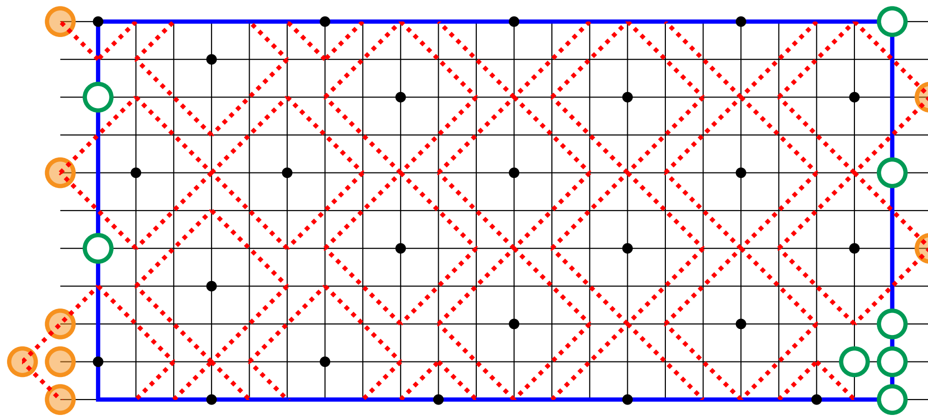

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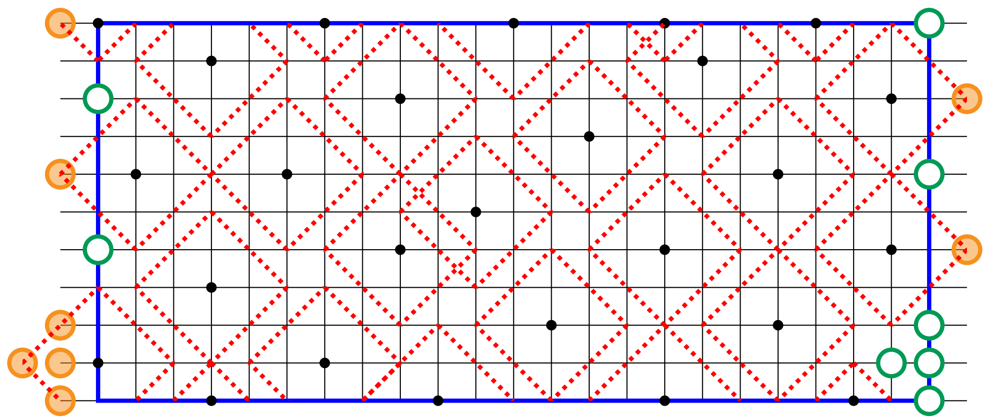

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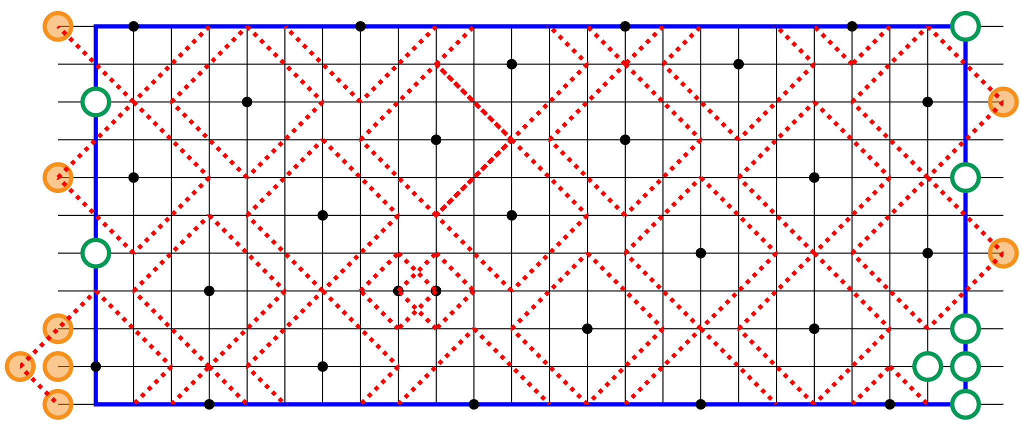

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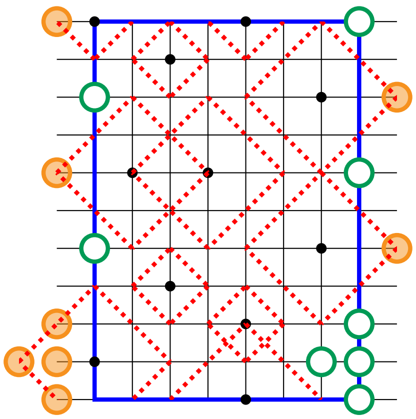

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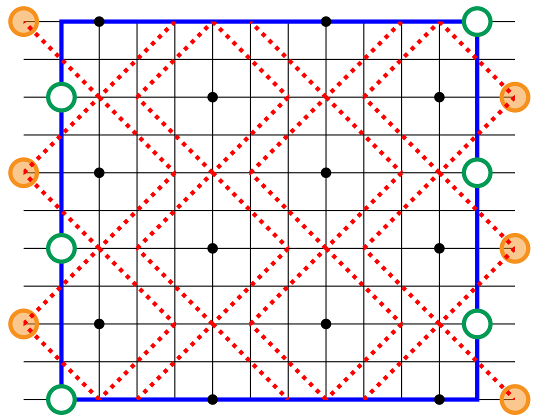

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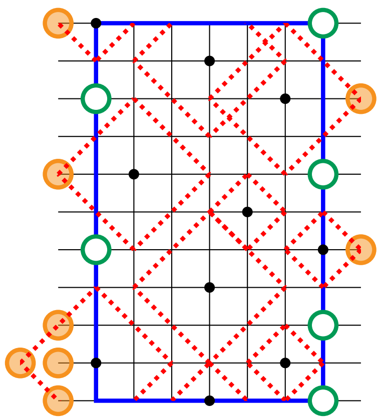

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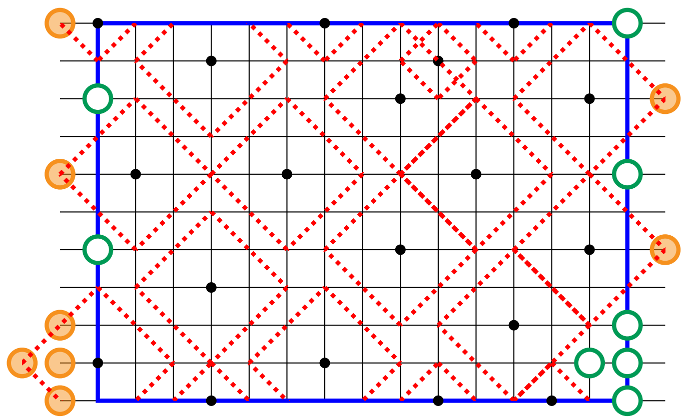

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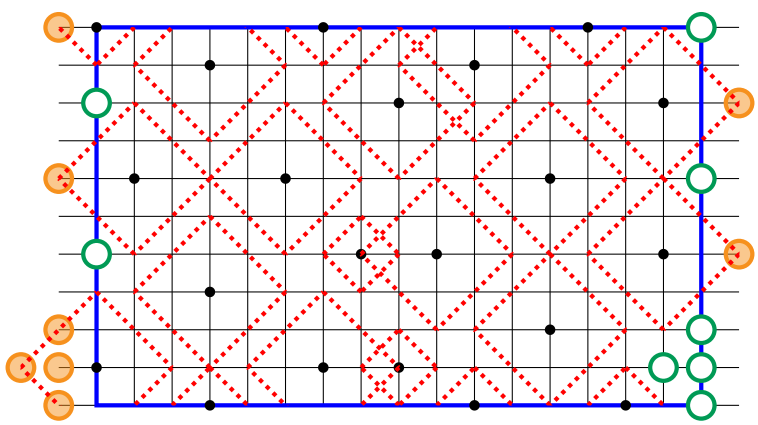

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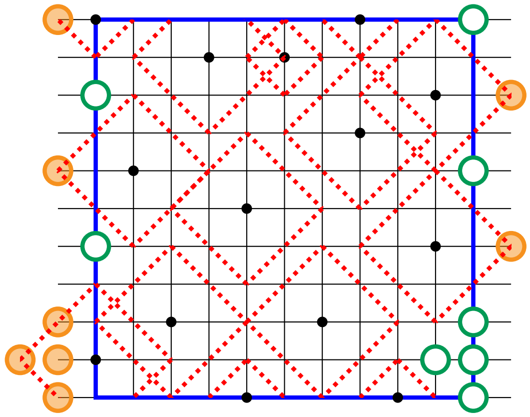

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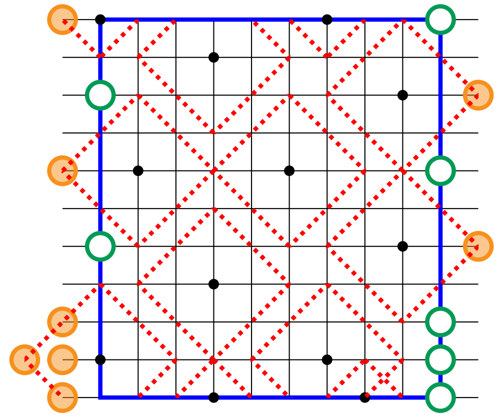

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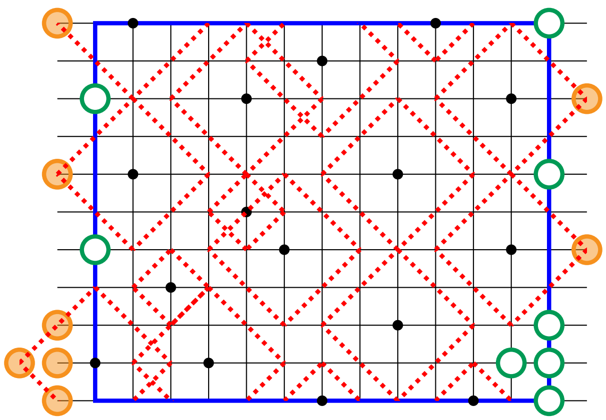

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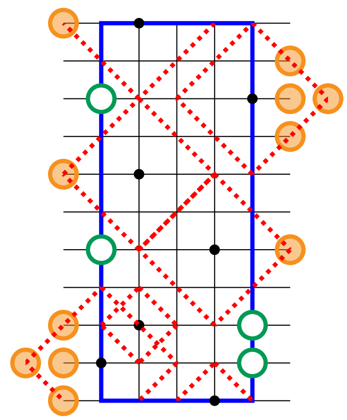

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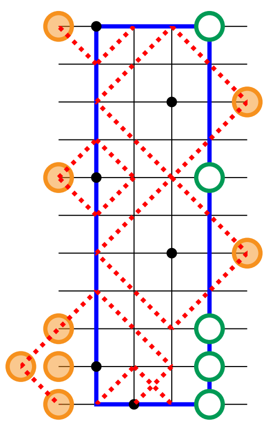
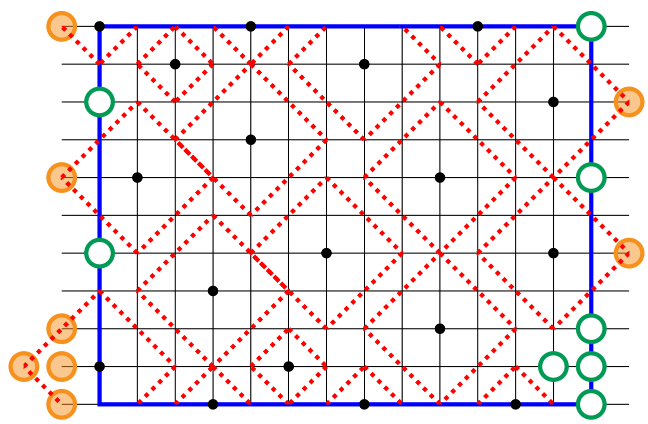


B_{14}



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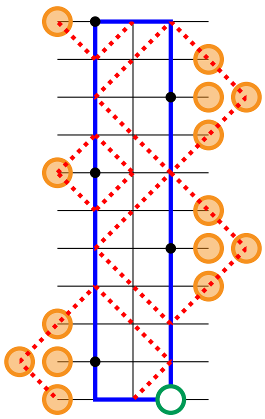
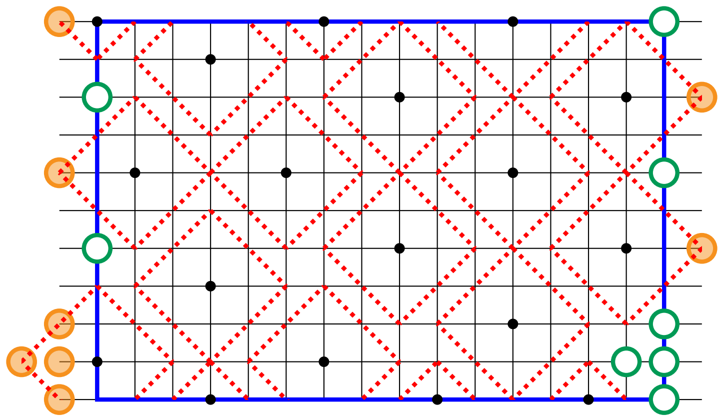
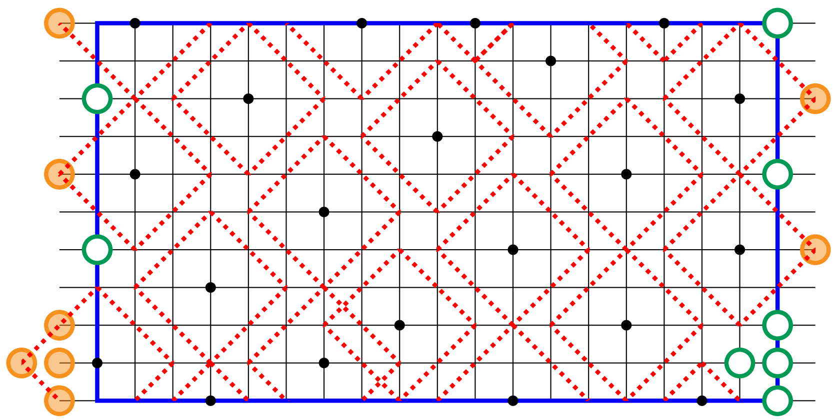


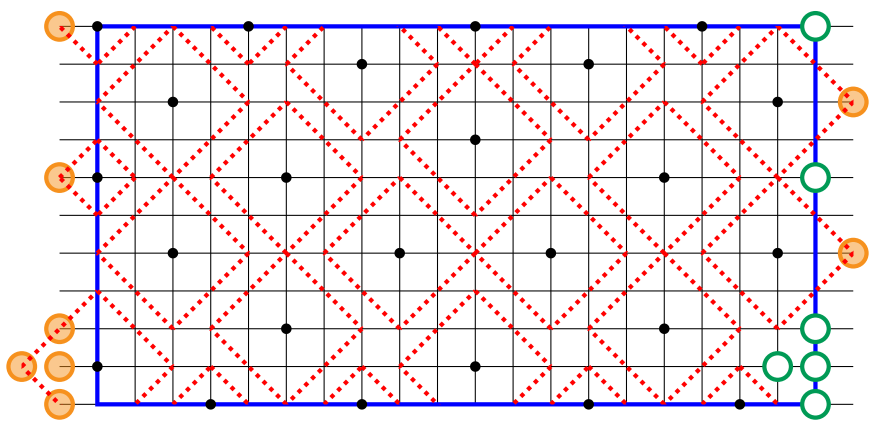
B_{16}



B_{19}



B_{20}



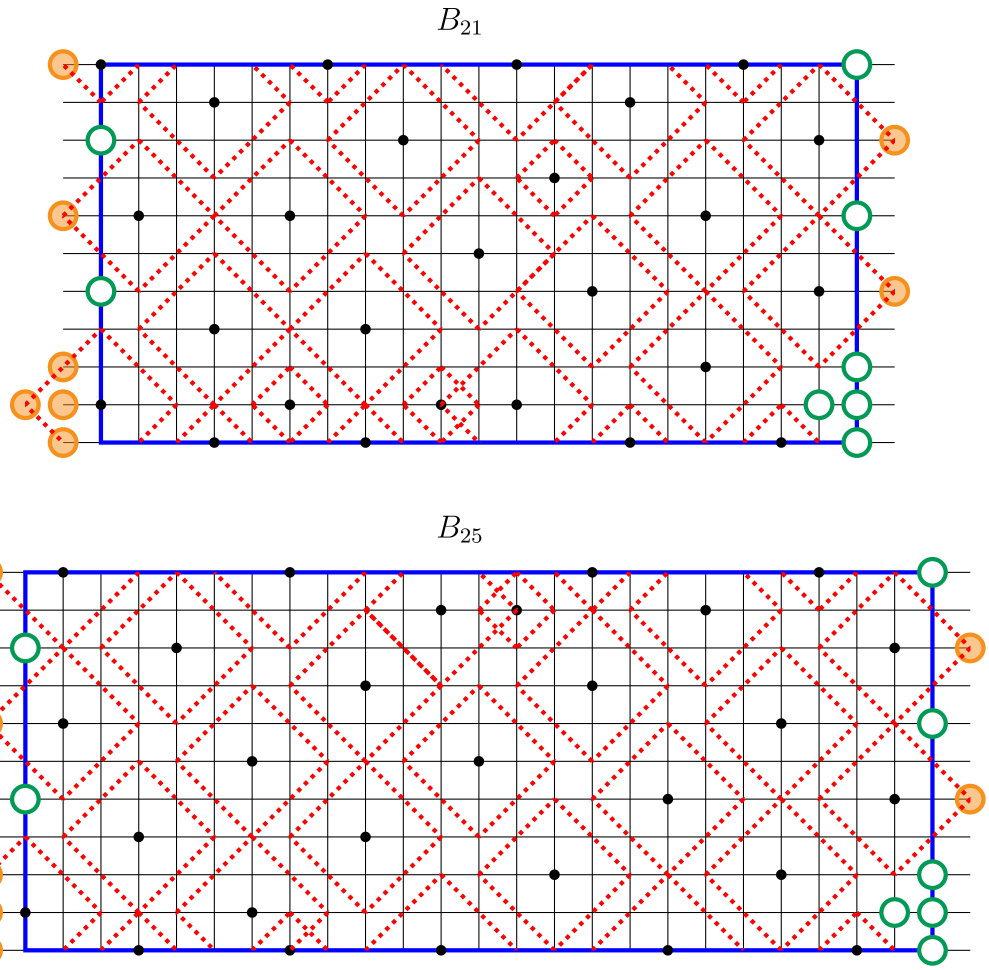


Figure 3.11.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{11}\square C_n$.

Proposition 3.11.1 improves the bounds given by Proposition 2.11.1 by three when $n_{26} = 0$, by two when $n_{26} \in \{6, 10, 14, 16, 20\}$, by one when $n_{26} \in \{1, 2, 4, 7, 8, 11, 12, 13, 17, 18, 19, 21, 22, 23, 24\}$, and otherwise the bounds are the same. The values of Proposition 3.11.1 are optimal for all $n \leq 66$ (as found by computation) except for cases given in Table 3.11.3. Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{48.9}{26}n$.

Conjecture 3.11.1. *The bounds given by Proposition 3.11.1 are tight except when* $n \equiv 13 \pmod{26}$.

3.12 $P_{12}\square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_{12}\square C_n$ for all $n \geq 3$. Figure 3.12.1 contains the tiles used in the

n	Optimal	Bound
13	26	27
39	76	77
65	126	127

Table 3.11.3: Known values of n for which the bound in Proposition 3.11.1 is not optimal.

broadcast constructions described in Table 3.12.2.

Proposition 3.12.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(P_{12}\Box C_n\right) \leq 50\left\lfloor\frac{n}{24}\right\rfloor + c(n_{24})$$

where $c(n_{24})$ is dependant upon the least residue n_{24} of n modulo 24 and given in Table 3.12.1 with the exception of $n = 5$ and 9 which give values of $c(n_{24})$ equal to 13 and 21, respectively.

	Least residue n_{24} of n modulo 24											
n_{24}	0	1	2	3	4	5	6	7	8	9	10	11
Cost:	0	4	5	8	9	12	13	16	18	20	21	25
n_{24}	12	13	14	15	16	17	18	19	20	21	22	23
Cost:	26	29	30	33	34	37	39	41	42	46	47	50

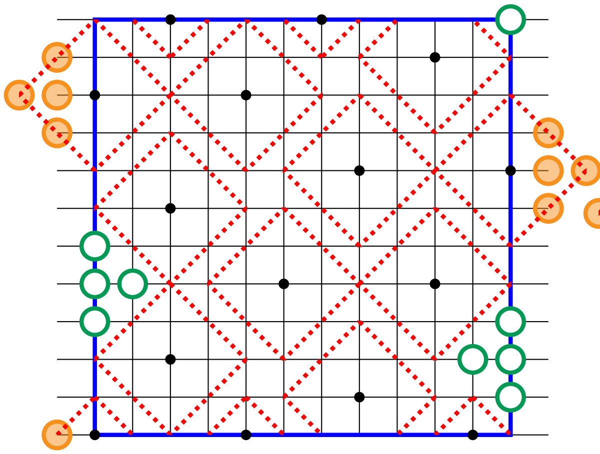
Table 3.12.1: Values of $c(n_{24})$ for the upper bound for $\gamma_{b,2}\left(P_{12}\Box C_n\right)$ stated in Proposition 3.12.1 where $n \notin \{5, 9\}$.

	Construction	Cost
$n \equiv 0 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n}{24}}$	$50\left(\frac{n}{24}\right)$
$n \equiv 1 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-25}{24}} BB_1$	$50\left(\frac{n-25}{24}\right) + 29 = 50\left(\frac{n-1}{24}\right) + 4$
$n \equiv 2 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-26}{24}} B_2$	$50\left(\frac{n-26}{24}\right) + 55 = 50\left(\frac{n-2}{24}\right) + 5$
$n \equiv 3 \pmod{24}, (n = 3)$:	B_{24}	$8 = 50\left(\frac{n-3}{24}\right) + 8$
$(n > 3)$:	$\left[B\overline{B}\right]^{\frac{n-27}{24}} BB_3$	$50\left(\frac{n-27}{24}\right) + 58 = 50\left(\frac{n-3}{24}\right) + 8$
$n \equiv 4 \pmod{24}, (n = 4)$:	B_{28}	$9 = 50\left(\frac{n-4}{24}\right) + 9$
$(n > 4)$:	$\left[B\overline{B}\right]^{\frac{n-28}{24}} BB_4$	$50\left(\frac{n-28}{24}\right) + 59 = 50\left(\frac{n-4}{24}\right) + 9$
$n \equiv 5 \pmod{24}, (n = 5)$:	B_{26}	13

$(n > 5)$:	$\left[B\overline{B}\right]^{\frac{n-29}{24}} B_5$	$50\left(\frac{n-29}{24}\right) + 62 = 50\left(\frac{n-5}{24}\right) + 12$
$n \equiv 6 \pmod{24}, (n = 6)$:	B_{27}	$13 = 50\left(\frac{n-6}{24}\right) 13$
$(n > 6)$:	$\left[B\overline{B}\right]^{\frac{n-30}{24}} BB_6$	$50\left(\frac{n-30}{24}\right) + 63 = 50\left(\frac{n-6}{24}\right) 13$
$n \equiv 7 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-7}{24}} B_7$	$50\left(\frac{n-7}{24}\right) + 16$
$n \equiv 8 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-8}{24}} B_8$	$50\left(\frac{n-8}{24}\right) + 18$
$n \equiv 9 \pmod{24}, (n = 9)$:	B_{28}	21
$(n > 9)$:	$\left[B\overline{B}\right]^{\frac{n-33}{24}} BB_9$	$50\left(\frac{n-33}{24}\right) + 70 = 50\left(\frac{n-9}{24}\right) + 20$
$n \equiv 10 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-10}{24}} B_{10}$	$50\left(\frac{n-10}{24}\right) + 21$
$n \equiv 11 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-11}{24}} B_{11}$	$50\left(\frac{n-11}{24}\right) + 25$
$n \equiv 12 \pmod{24}, (n = 12)$:	B_{30}	$26 = 50\left(\frac{n-12}{24}\right) + 26$
$(n > 12)$:	$\left[B\overline{B}\right]^{\frac{n-36}{24}} BB_{12}$	$50\left(\frac{n-36}{24}\right) + 76 = 50\left(\frac{n-12}{24}\right) + 26$
$n \equiv 13 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-13}{24}} B_{13}$	$50\left(\frac{n-13}{24}\right) + 29$
$n \equiv 14 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-14}{24}} B_{14}$	$50\left(\frac{n-14}{24}\right) + 30$
$n \equiv 15 \pmod{24}, (n = 15)$:	B_{32}	$33 = 50\left(\frac{n-15}{24}\right) + 33$
$(n > 15)$:	$\left[B\overline{B}\right]^{\frac{n-39}{24}} BB_{15}$	$50\left(\frac{n-39}{24}\right) + 83 = 50\left(\frac{n-15}{24}\right) + 33$
$n \equiv 16 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-16}{24}} B_{16}$	$50\left(\frac{n-16}{24}\right) + 34$
$n \equiv 17 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-17}{24}} B_{17}$	$50\left(\frac{n-17}{24}\right) + 37$
$n \equiv 18 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-18}{24}} B_{18}$	$50\left(\frac{n-18}{24}\right) 39$
$n \equiv 19 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-19}{24}} B_{19}$	$50\left(\frac{n-19}{24}\right) + 41$
$n \equiv 20 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-20}{24}} B_{20}$	$50\left(\frac{n-20}{24}\right) + 42$
$n \equiv 21 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-21}{24}} B_{21}$	$50\left(\frac{n-21}{24}\right) + 46$
$n \equiv 22 \pmod{24}, (n = 22)$:	B_{33}	$47 = 50\left(\frac{n-22}{24}\right) + 47$
$(n > 22)$:	$\left[B\overline{B}\right]^{\frac{n-46}{24}} BB_{22}$	$50\left(\frac{n-46}{24}\right) + 97 = 50\left(\frac{n-22}{24}\right) + 47$
$n \equiv 23 \pmod{24}$:	$\left[B\overline{B}\right]^{\frac{n-23}{24}} B_{23}$	$50\left(\frac{n-23}{24}\right) + 50$

Table 3.12.2: Constructions of 2-limited dominating broadcasts on $P_{12}\Box C_n$.

B



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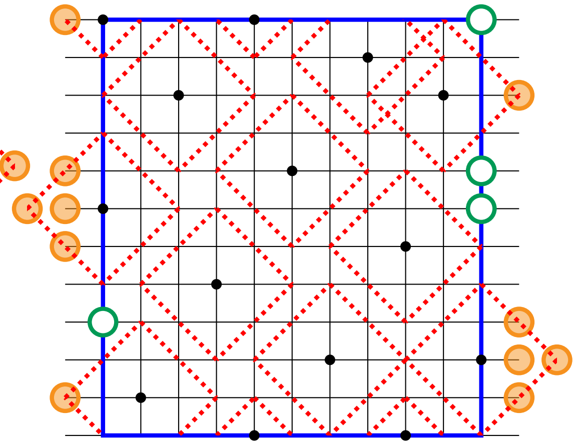

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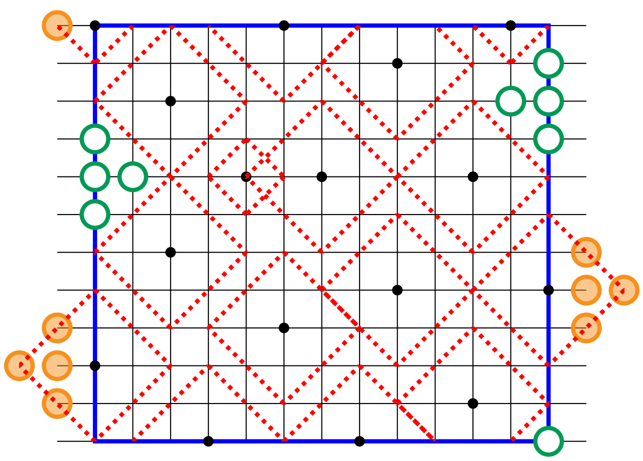

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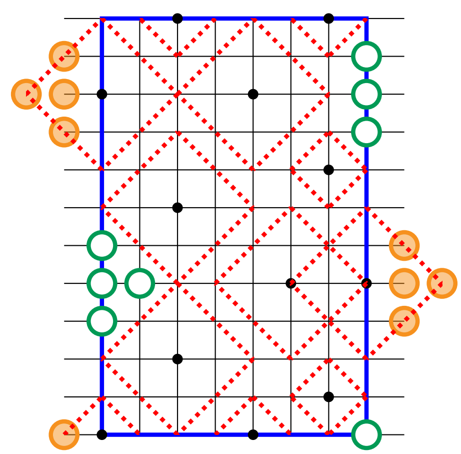

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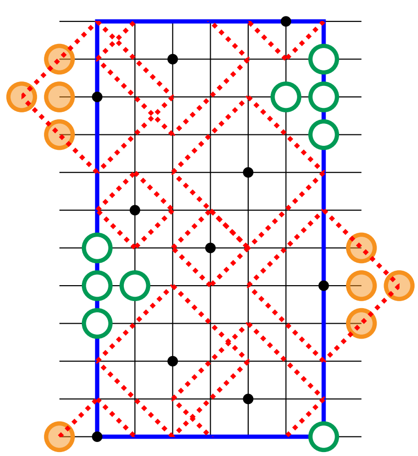

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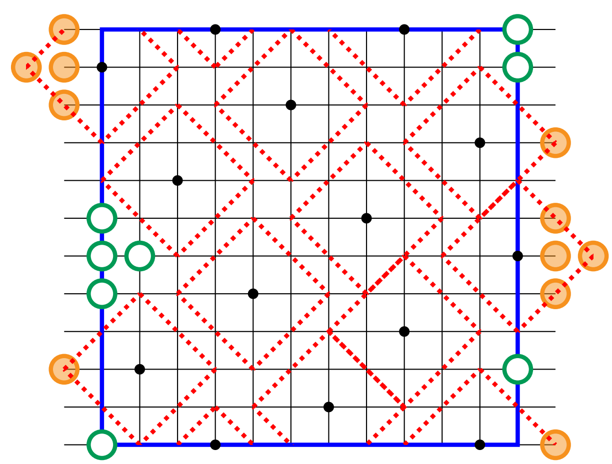

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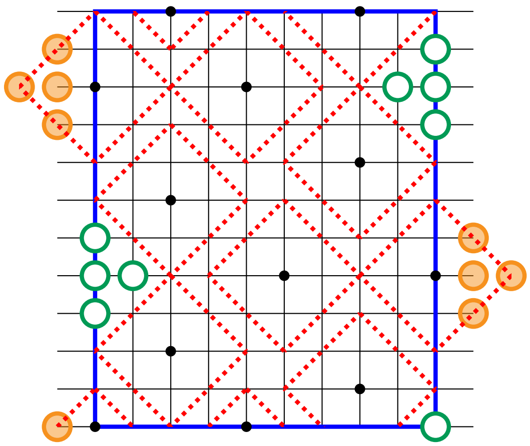

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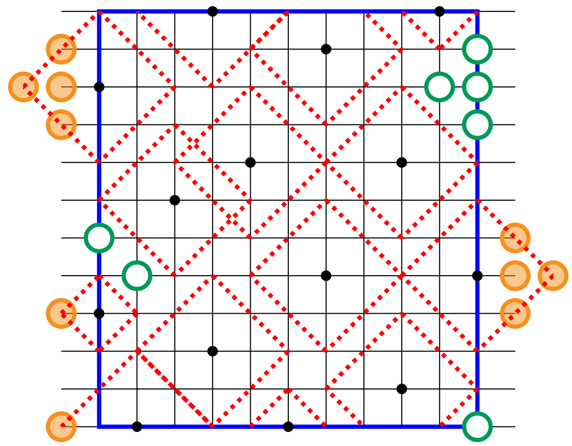

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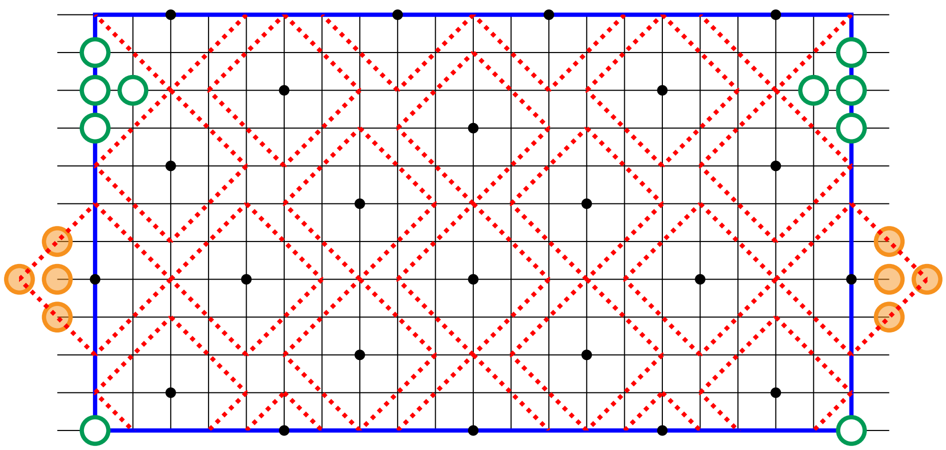

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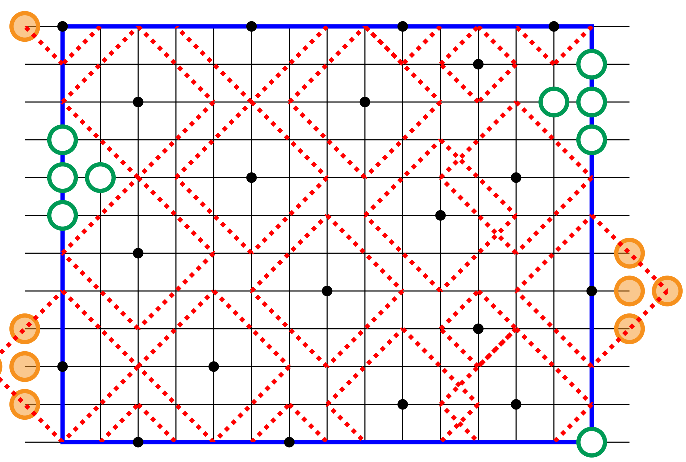

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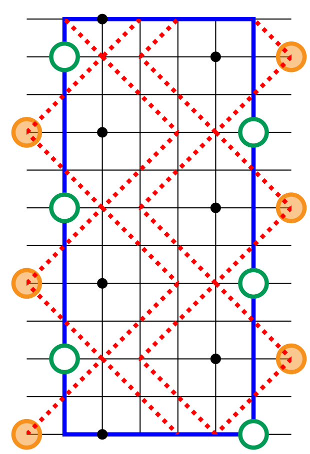

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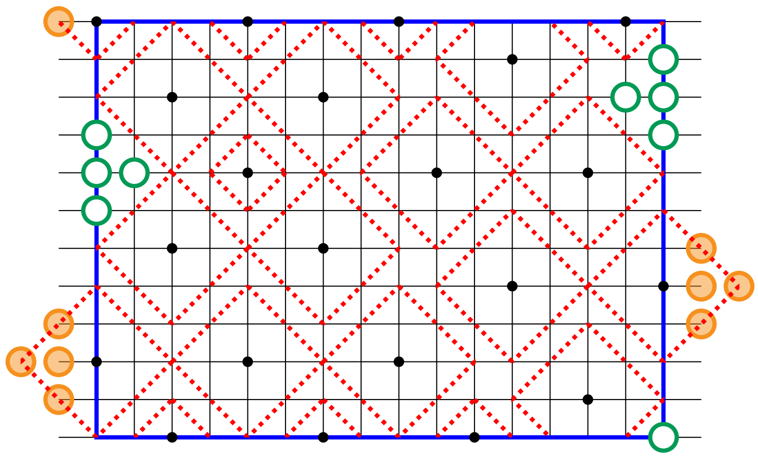

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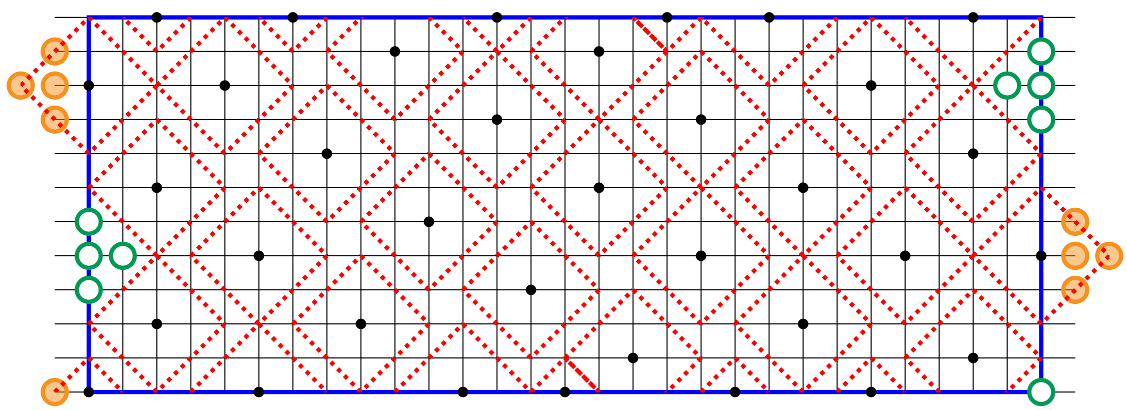

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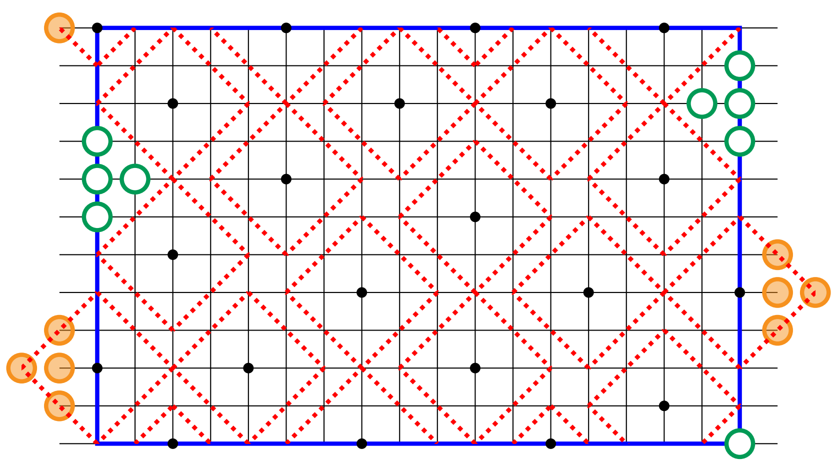

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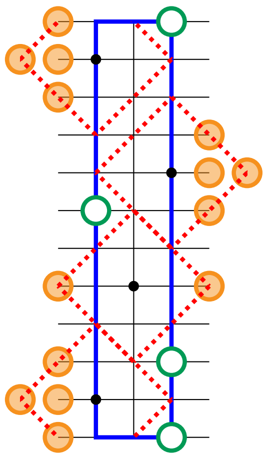
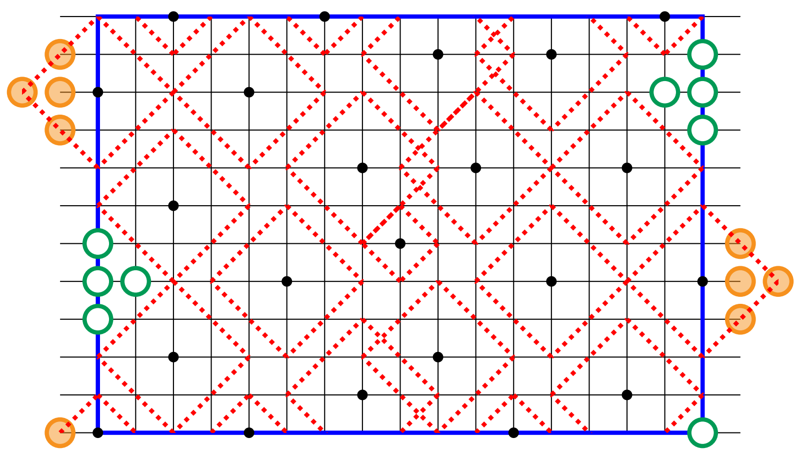
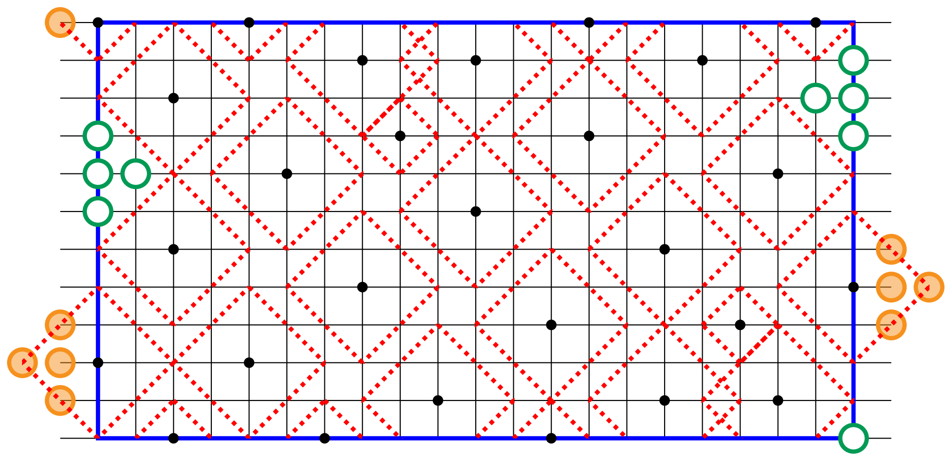


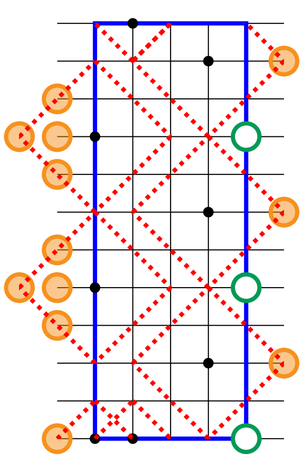
B_{17}



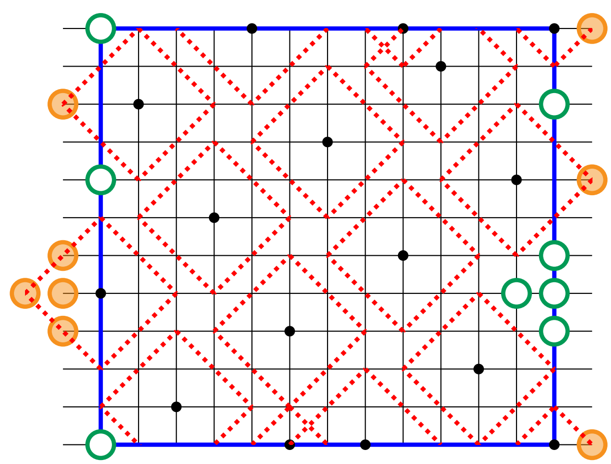
B_9



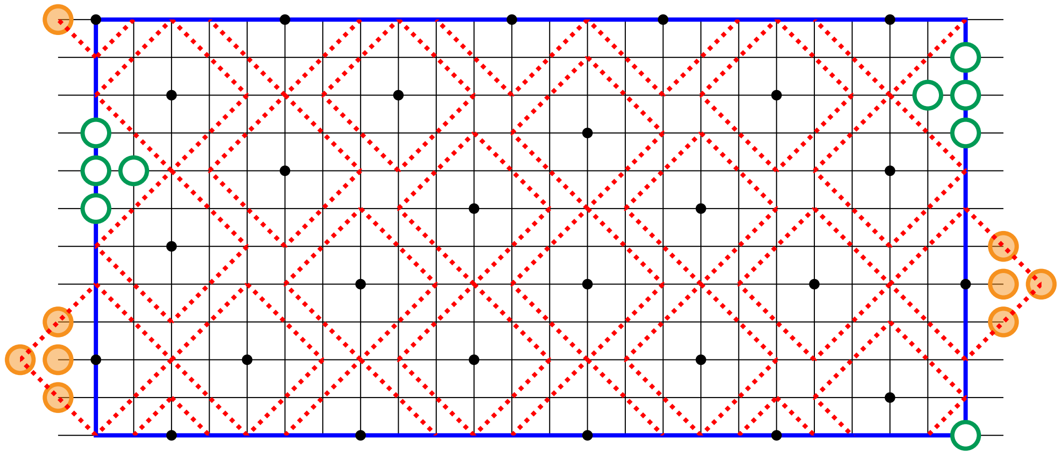
B_{26}



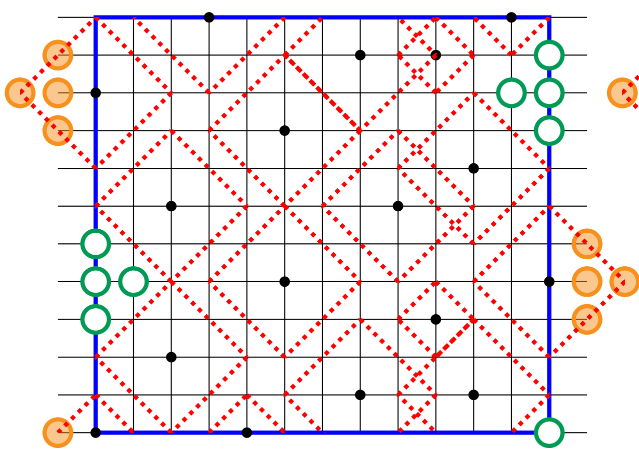
B_{31}



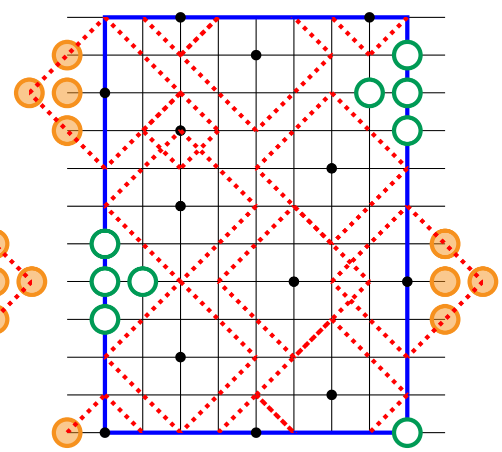
B_{12}



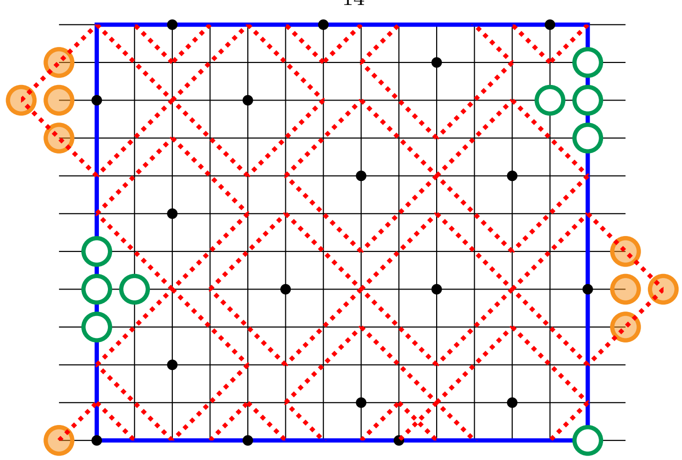
B_{13}



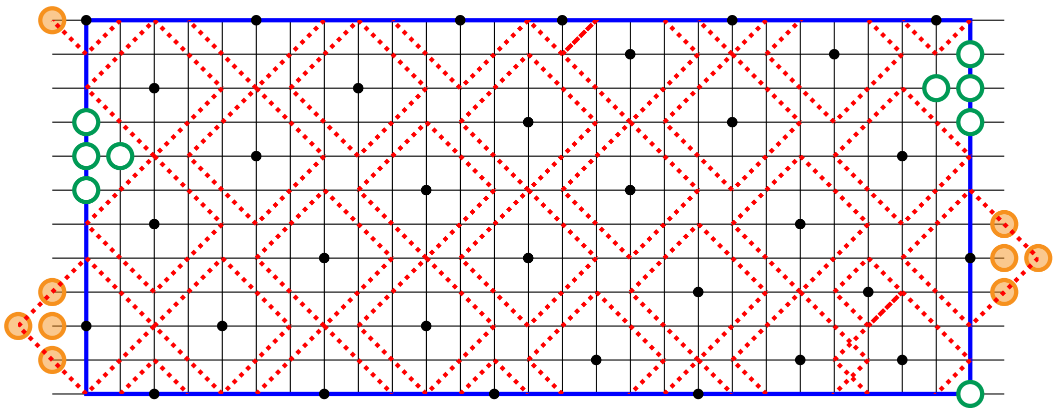
B_{28}



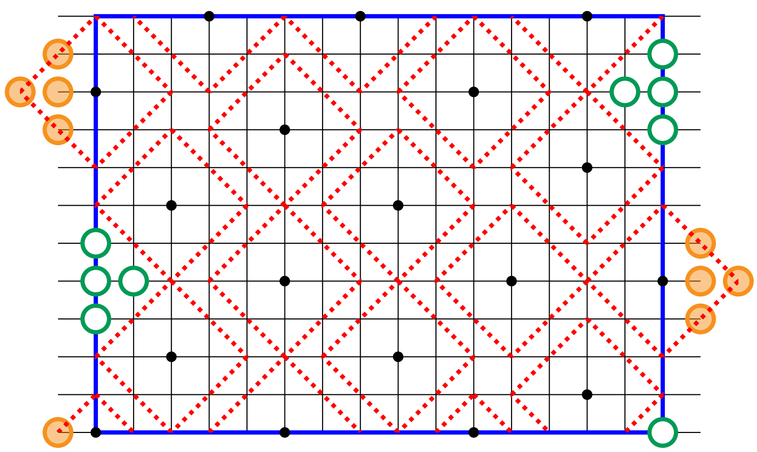
B_{14}



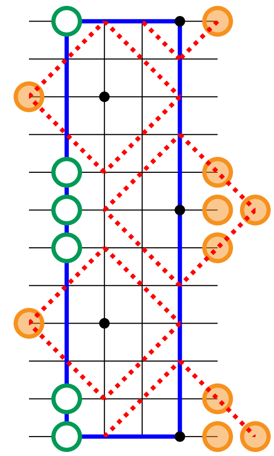
B_{15}



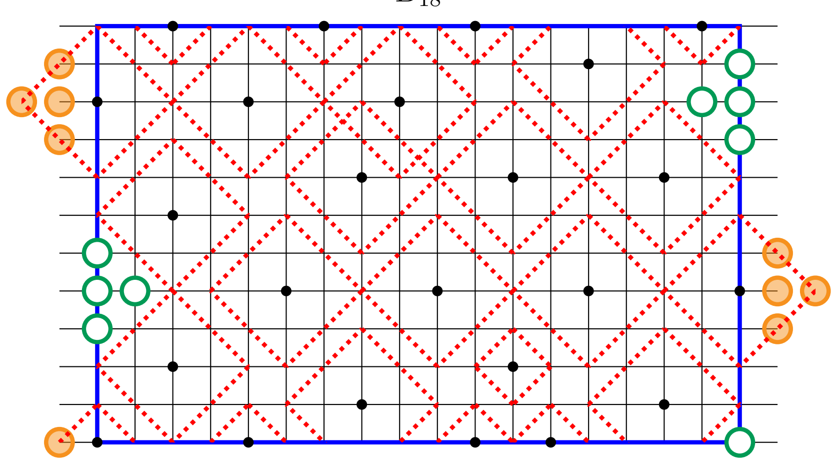
B_{16}



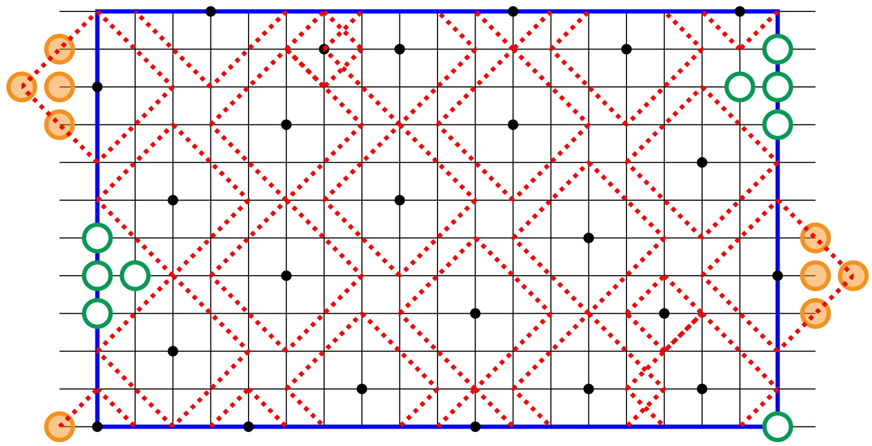
B_{25}



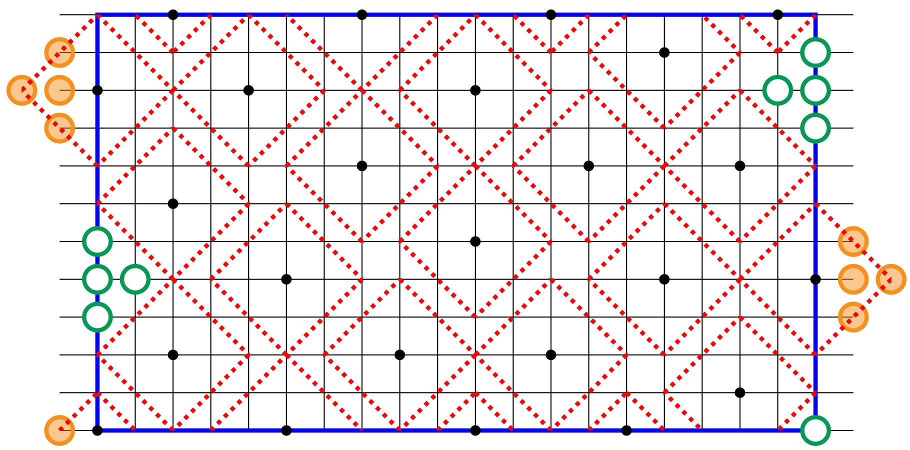
B_{18}



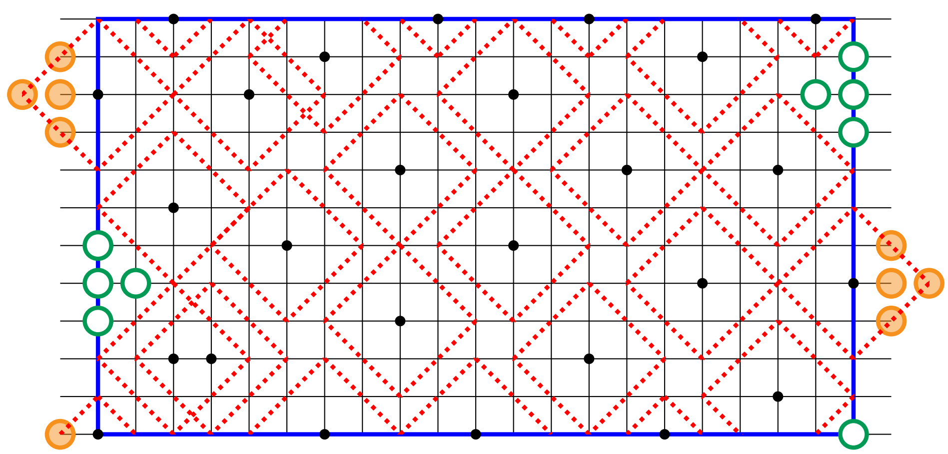
B_{19}



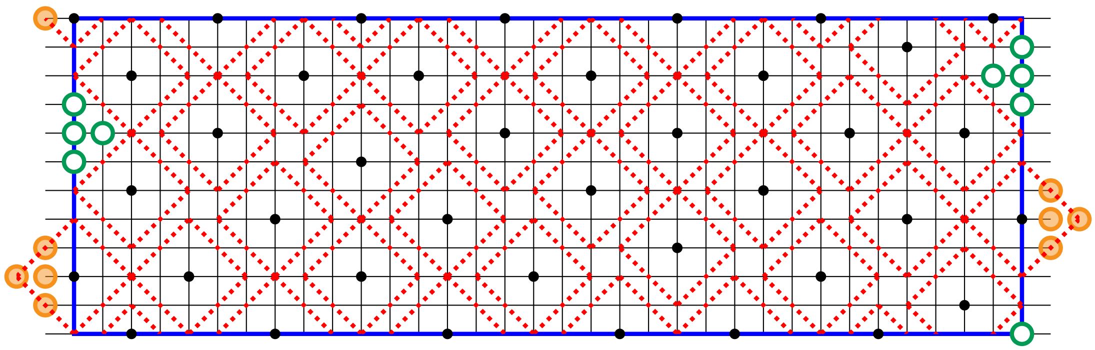
B_{20}



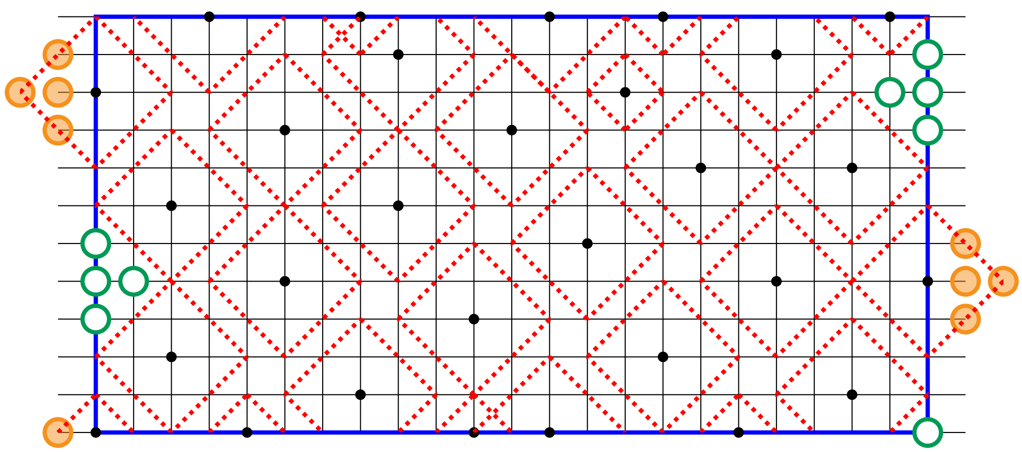
B_{21}



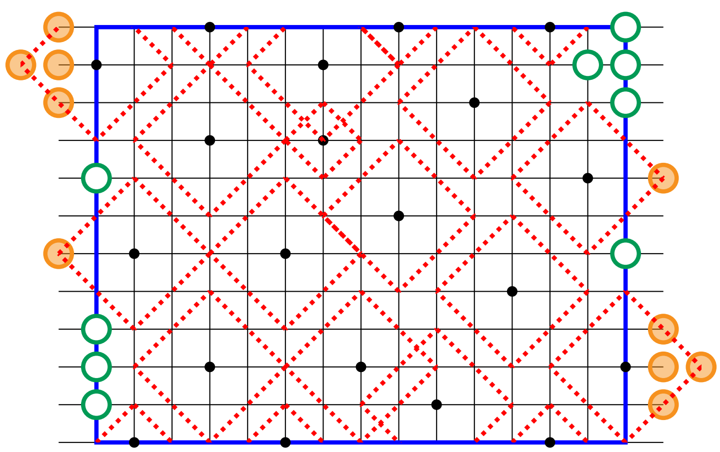
B_{22}



B_{23}



B_{32}



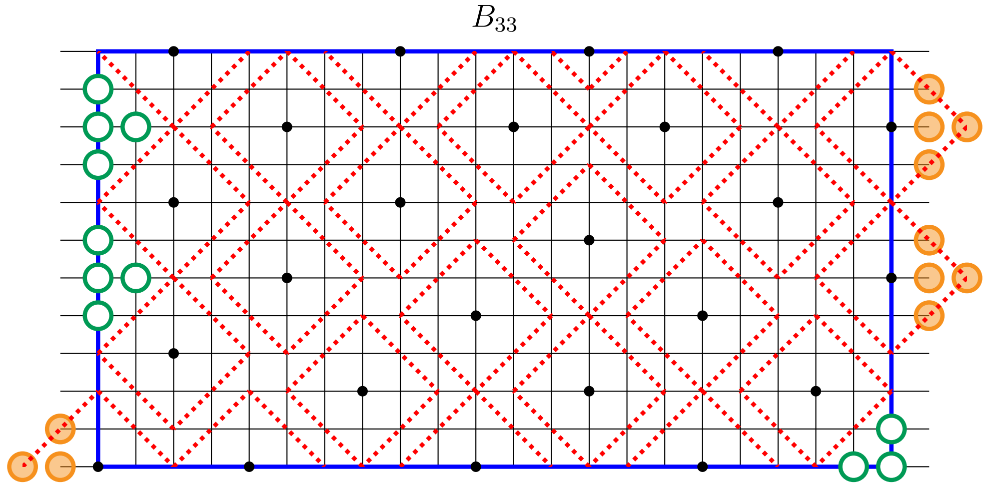


Figure 3.12.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{12}\square C_n$.

Proposition 3.12.1 improves the bounds given by Proposition 2.12.1 by three when $n_{24} \in \{0, 10, 20\}$, by two when $n_{24} \in \{2, 4, 6, 8, 12, 14, 16, 22\}$, by one when $n_{24} \in \{3, 5, 7, 9, 15, 17, 18, 19\}$, and otherwise the bounds are the same. The values of Proposition 3.12.1 are optimal for all $n \leq 54$ (as found by computation) except for cases given in Table 3.12.3. Theorem 5.3.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{48.8}{24}n$.

Conjecture 3.12.1. *The bounds given by Proposition 3.12.1 are tight except when* $n \equiv 11, 13 \pmod{24}$.

n	Optimal	Bound
11	24	25
13	28	29
35	74	75
37	78	79

Table 3.12.3: Known values of n for which the bound in Proposition 3.12.1 is not optimal.

3.13 $P_{13\leq m\leq 22}\square C_n$

Sections 3.2 through 3.12 give upper bounds for $\gamma_{b,2}(P_m\square C_n)$ for $2 \leq m \leq 12$ and all $n \geq 3$. These bounds were made by analyzing known optimal broadcasts of $P_m\square C_n$ for $2 \leq m \leq 12$ and varied values of n to find recurring broadcast tiles which were then used to create general constructions. This section provides upper bounds for $\gamma_{b,2}(P_m\square C_n)$

for $13 \leq m \leq 22$ and all $n \geq 3$. These bounds also follow from explicit broadcast constructions. However, unlike the previous sections, the constructions are more naïve in nature. For each fixed $13 \leq m \leq 22$, we find an optimal 2-limited dominating broadcast on $P_m \square C_{13}$ by computation. Given this optimal broadcast on $P_m \square C_{13}$, 2-limited dominating broadcasts on $P_m \square C_{n \geq 13}$ are then constructed by repeating the broadcast on $P_m \square C_{13}$ as many times as possible and then filling in the remainder of the broadcast based on the least residue of n modulo 13. The remaining cases of $P_m \square C_{3 \leq n \leq 12}$ are then solved via computation. This methodology establishes the following theorem which, with Propositions 3.2.1 through 3.12.1, establishes upper bounds for $\gamma_{b,2}(P_m \square C_n)$ for all $2 \leq m \leq 22$ and $n \geq 3$. Theorem 3.13.1 periodically improves the bounds given in Theorem 2.13.3 by a positive constant.

Theorem 3.13.1. *If $13 \leq m \leq 22$ and $n \geq 3$ then*

$$\gamma_{b,2}(P_m \square C_n) \leq (30 + 2(m-13)) \left\lfloor \frac{n}{13} \right\rfloor + c(m, n_{13}) - c'(m,n)$$

where $c(m, n_{13})$ corresponds with the value of Table 3.13.1 dependant upon m and the least residue n_{13} of n modulo 13 and $c'(m,n)$ corresponds with the value in Table 3.13.2 dependent upon m and n if $n \leq 12$ and 0 otherwise.

	Least residue n_{13} of n modulo 13												
m	0	1	2	3	4	5	6	7	8	9	10	11	12
13	0	6	8	9	11	14	15	17	20	22	24	27	29
14	0	6	8	11	11	15	16	18	21	24	26	28	30
15	0	7	9	11	12	16	18	20	23	26	27	30	32
16	0	7	9	12	12	17	19	22	24	28	29	32	34
17	0	7	9	13	14	18	20	22	25	29	31	33	36
18	0	8	10	14	14	19	21	24	27	31	32	36	38
19	0	8	11	15	15	20	22	25	28	33	35	37	40
20	0	8	11	15	16	21	23	26	30	34	36	39	42
21	0	9	12	17	17	22	24	27	31	36	38	41	44
22	0	9	12	16	18	23	26	28	33	37	40	42	46

Table 3.13.1: Values of $c(m, n_{13})$ for the upper bound for $\gamma_{b,2}(P_m \square C_n)$ for $13 \leq m \leq 22$ and $n \geq 3$.

Proof. For each $13 \leq m \leq 22$, $\gamma_{b,2}(P_m \square C_{13})$ is given in Table 3.13.3. These values are easily solved via computation. Observe that, for all $13 \leq m \leq 22$,

$$\gamma_{b,2}(P_m \square C_{13}) = 30 + 2(m-13).$$

Fix $13 \leq m \leq 22$, $n \geq 13$, and let B correspond with a 2-limited dominating broadcast

		n									
		3	4	5	6	7	8	9	10	11	12
m	13	0	1	0	1	0	1	0	1	0	1
	14	1	1	0	1	0	1	0	2	0	0
	15	1	2	0	2	0	1	0	1	0	0
	16	1	0	0	2	1	1	1	1	0	0
	17	1	2	0	2	0	1	0	2	0	0
	18	2	1	0	2	1	1	1	1	0	0
	19	2	1	0	2	1	1	1	3	0	0
	20	1	2	0	2	0	2	0	2	0	0
	21	3	2	0	2	0	1	1	2	0	0
	22	1	2	0	3	0	2	0	3	0	0

Table 3.13.2: Values of $c'(m, n)$ for the upper bound for $\gamma_{b,2}(P_m \square C_n)$ for $13 \leq m \leq 22$ and $n \leq 12$.

m	13	14	15	16	17	18	19	20	21	22
$\gamma_{b,2}(P_m \square C_{13})$	30	32	34	36	38	40	42	44	46	48

Table 3.13.3: Cost of an optimal 2-limited dominating broadcast on $P_m \square C_{13}$ for $13 \leq m \leq 22$.

of $P_m \square C_{13}$ of cost equal to that given in Table 3.13.3. For instance, if $m = 16$ then a possible form of B corresponds with the tile in Figure 3.13.1.

Let n_{13} be the least residue of n modulo 13. Form a 2-limited broadcast on $P_m \square C_n$ by the sequence of broadcast tiles $[B]^{(n-n_{13})/13}$. This broadcast is dominating if and only if $n_{13} = 0$. Assuming $n_{13} \neq 0$, to make the broadcast dominating, it suffices to determine the additional broadcast vertices required to extend the broadcast given by $[B]^{(n-n_{13})/13}$ to dominate the remaining vertices of $P_m \square C_n$. This reduces to determining the additional broadcast vertices required to extend the broadcast given by B to dominate $P_m \square C_{13+n_{13}}$. For each $13 \leq m \leq 22$ and each least residue of n modulo 13, the costs of these additional broadcast vertices required to extend the broadcast given by B to dominate $P_m \square C_{13+n_{13}}$ are given in Table 3.13.1. It therefore follows that for all $13 \leq m \leq 22$ and $n \geq 13$,

$$\gamma_{b,2}(P_m \square C_n) \leq (30 + 2(m - 13)) \left\lfloor \frac{n}{13} \right\rfloor + c(m, n) - c'(m, n).$$

Recall that $c'(m, n)$ is 0 when $n \geq 13$. See Example 3.13.2 for an example of this construction for $m = 16$.

To complete the proof it remains to consider the cases that have $3 \leq n \leq 12$. Table 3.13.4 gives $\gamma_{b,2}(P_m \square C_n)$ for $13 \leq m \leq 22$ and all $3 \leq n \leq 12$ (determined via computation). As the term

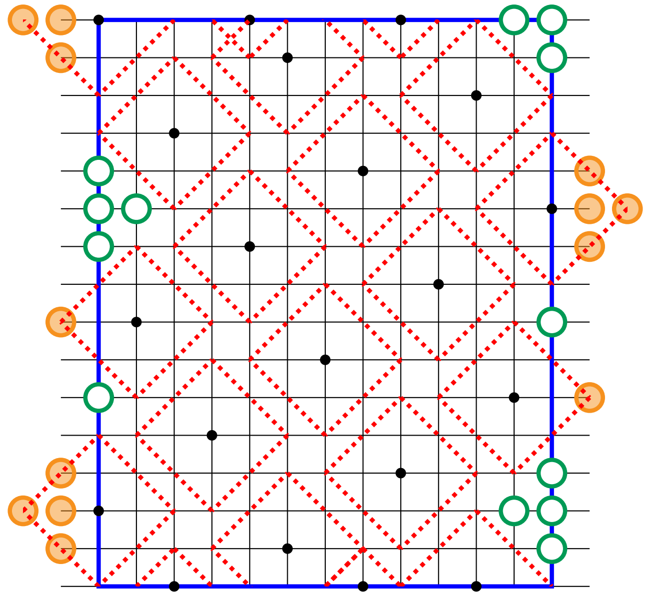


Figure 3.13.1: An optimal 2-limited dominating broadcast of $P_{16} \square C_{13}$.

m \ n	3	4	5	6	7	8	9	10	11	12
13	9	10	14	14	17	19	22	23	27	28
14	10	10	15	15	18	20	24	24	28	30
15	10	10	16	16	20	22	26	26	30	32
16	11	12	17	17	21	23	27	28	32	34
17	12	12	18	18	22	24	29	29	33	36
18	12	13	19	19	23	26	30	31	36	38
19	13	14	20	20	24	27	32	32	37	40
20	14	14	21	21	26	28	34	34	39	42
21	14	15	22	22	27	30	35	36	41	44
22	15	16	23	23	28	31	37	37	42	46

Table 3.13.4: $\gamma_{b,2}(P_m \square C_n)$ for $13 \leq m \leq 22$ and all $3 \leq n \leq 12$.

$$\left(30+2\left(m-13\right)\right)\left\lfloor\frac{n}{13}\right\rfloor$$

is equal to 0 when $n \leq 12$, the upper bound given in Theorem 3.13.1 is tight if $c'(m,n)$ is the difference of the corresponding values of Tables 3.13.1 and 3.13.4. Table 3.13.2 contains such values for $c'(m,n)$. $\square$

Example 3.13.2. If $m=16$, B corresponds with the tile in Figure 3.13.1, and $n_{13}=5$. The additional broadcast vertices required to extend the broadcast given by B to dominate $P_{16}\square C_{18}$ correspond with the vertices at the centres of the shaded broadcast regions given in Figure 3.13.2. The total cost of this broadcast is $36+17=53$. This construction easily generalizes. Given $P_{16}\square C_{n\geq 13}$ such that $n \equiv 5 \pmod{13}$, there exists a 2-limited dominating broadcast on $P_{16}\square C_{n\geq 13}$ formed by starting with $[B]^{\left\lfloor \frac{n}{13}\right\rfloor}$ and then adding additional broadcast vertices in the same was as in Figure 3.13.2. The cost of such a broadcast is

$$36\left\lfloor\frac{n}{13}\right\rfloor+17=\left(30+2\left(m-13\right)\right)\left\lfloor\frac{n}{13}\right\rfloor+c(m,n)+c'(m,n).$$

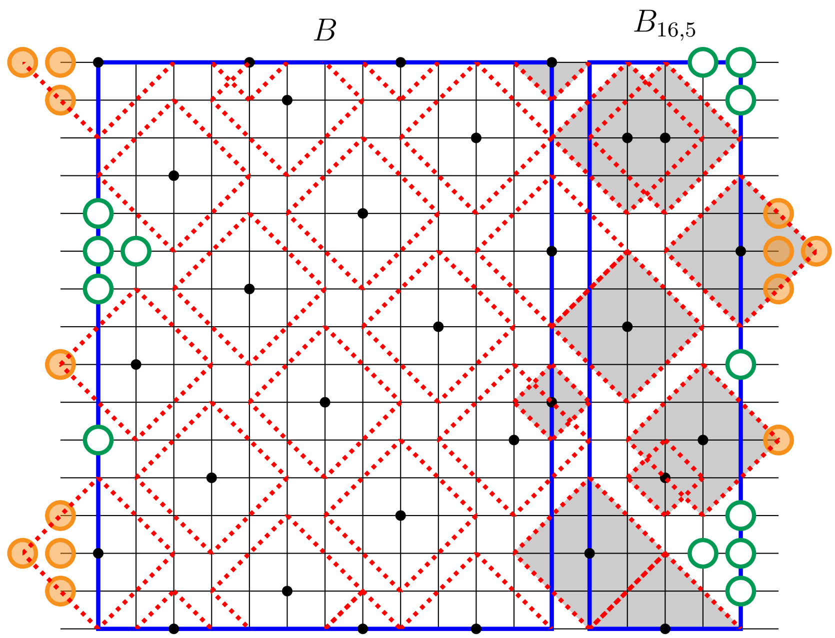


Figure 3.13.2: A minimum cost 2-limited dominating broadcast of $P_{16}\square C_{18}$ given fixed broadcast tile B.

3.14 $P_{m\geq 23}\square C_{n\geq 13}$

This section considers upper bounds for the 2-limited broadcast domination number of $P_m\square C_n$ where $m \geq 23$ and $n \geq 13$. The upper bounds are established by a generalized construction of a 2-limited dominating broadcast on $P_m \Box C_n$ for all $m \geq 23$ and $n \geq 13$. We provide an intuition of our methodology and then state the resulting theorem.

Fix $m \geq 23$ and $n \geq 13$, let m'_{13} and n_{13} be the least residue of $(m-10)$ and n modulo 13, respectively. Partition $P_m \Box C_n$ into $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 as depicted in Figure 3.14.1. A concrete example of such a partition is given in Figure 3.14.10 for $m = 26$ and $n = 16$. Observe the following facts about this partition.

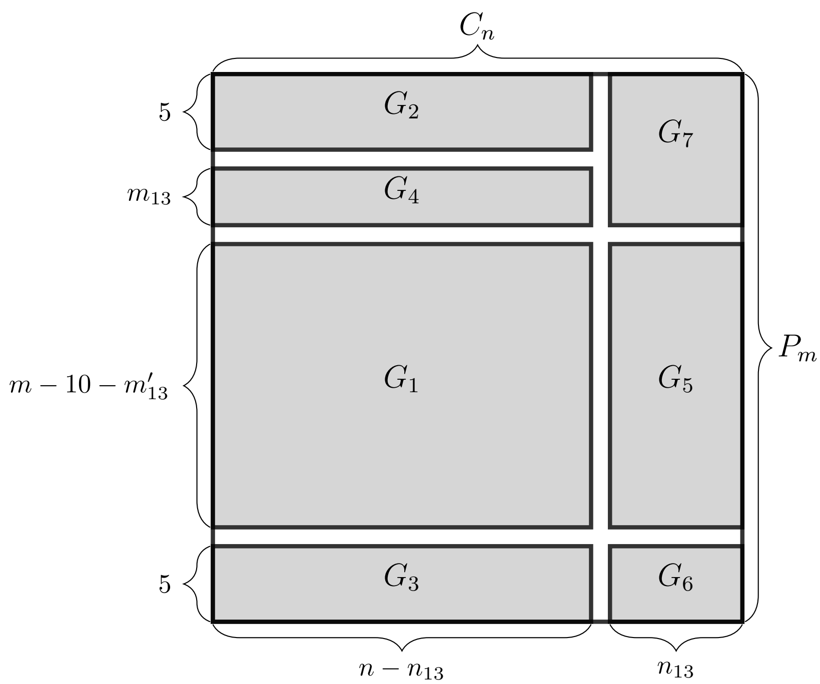


Figure 3.14.1: $P_m \Box C_n$ partitioned into G_1, G_2, G_3, G_4, G_5 and G_6.

1. The region G_1 occupies the majority of the "center" of $P_m \Box C_n$ and can be partitioned into
$$\frac{(m-10-m'_{13})(n-n_{13})}{13^2}$$
13×13 blocks.

2. The regions G_2 and G_3 occupy the majority of the top and bottom "borders" of $P_m \Box C_n$ and, with a fixed height of five, can each be partitioned into
$$\frac{n-n_{13}}{13}$$
5×13 blocks. The choice of a fixed height of five for G_2 and G_3 is in some way arbitrary. That said, when testing fixed heights of two through thirteen, we found that a fixed height of five was 'good enough' in the sense that the improvements in the resulting bounds when using larger fixed heights were negligible.

3. The region G_4 occupies the bounded regions between G_1 and G_2 and, with a fixed height of m_{13}, can be partitioned into

$$\frac{n - n_{13}}{13}$$

$m'_{13} \times 13$ blocks.

4. The region G_5 is on the middle of the right hand side "border" of $P_m \square C_n$ and, with a fixed width of n_{13}, can be partitioned into

$$\frac{(m - 10 - m'_{13})\, n_{13}}{13}$$

$13 \times n_{13}$ blocks.

5. The regions G_6 and G_7 occupy the top right and bottom right "corners" of $P_m \square C_n$ with sizes of $5 \times n_{13}$ and $(5 + m'_{10}) \times n_{13}$, respectively.

The ability to partition each of $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 into blocks is the key to our construction. Specifically, these methods determine a general 13×13 tile to tile G_1 and, for each of the least residues of $(m - 10)$ and n modulo 13,

1. 5×13 tiles to tile G_2 and G_3,
2. an $m'_{13} \times 13$ tile to tile G_4,
3. a $13 \times n_{13}$ tile to tile G_5,
4. $5 \times n_{13}$ and $(5 + m'_{10}) \times n_{13}$ tiles to tile G_6 and G_7, respectively, and
5. a collection of additional broadcast vertices to be added to $G_1, G_2, G_3, G_4, G_5, G_6$, and G_7 to complete the broadcast.

Once combined, the tiles and additional broadcast vertices create a 2-limited dominating broadcast on $P_m \square C_n$ for all $m \geq 23$ and $n \geq 13$. Figure 3.14.2 depicts how these tiles are "glued" together to tiles $P_m \square C_n$.

These construction establish Theorem 3.14.1. For all least residues of n modulo 13, with the exception of a least residue of three, Theorem 3.14.1 improves upon the bounds given by Theorem 2.13.3 by a linear factor of n. For the case of $n \equiv 3 \pmod{13}$, Theorem 3.14.1 improves upon the bounds given by Theorem 2.13.3 by up to a constant value.

Theorem 3.14.1. *If $m \geq 23$ and $n \geq 13$ then*

$$\gamma_{b,2}\,(P_m \square C_n) \leq 2\left(\frac{mn}{13}\right) + \frac{4m}{13} + \frac{b(n)}{13} + \frac{c(m'_{13}, n_{13})}{13}$$

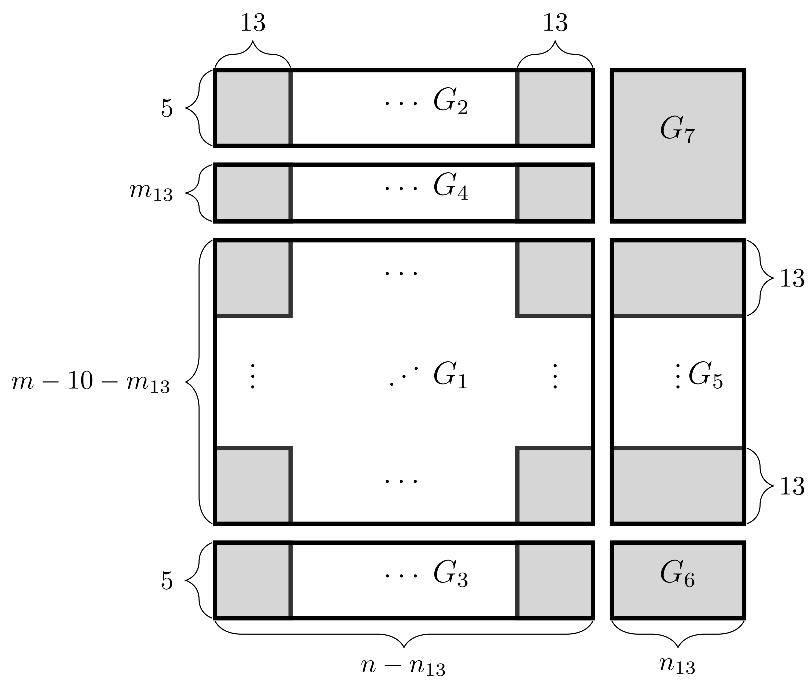


Figure 3.14.2: Tiling of $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 to construct a 2-limited dominating broadcast for $P_m \square C_n$

where

$$b(n) = \begin{cases} 0 & \text{for } n \equiv 0 \pmod{13}, \\ 2n & \text{for } n \equiv 4, 7, 11, \text{or } 12 \pmod{13}, \\ 3n & \text{for } n \equiv 1, 2, 5, 6, 8, 9, \text{or } 10 \pmod{13}, \text{ and} \\ 4n & \text{for } n \equiv 3 \pmod{13} \end{cases}$$

and $c(m'_{13}, n_{13})$ *corresponds with the values given in Table 3.14.1 where* m'_{13} *and* n_{13} *are the least residues of* $(m - 10)$ *and* n *modulo 13, respectively.*

Proof. Proof of the general case will follow from the cases that have $23 \leq m \leq 35$ and $13 \leq n \leq 25$. Fix $23 \leq m \leq 35$ and $13 \leq n \leq 25$. Let m'_{13} and n_{13} be the least residues of $(m - 10)$ and n modulo 13, respectively. Define $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 to be the blocks depicted in Figure 3.14.1. G_1 has dimensions 13×13. Start by tiling G_1 by the tile in Figure 3.14.3. An astute reader may notice that said tile is one of the 13 possible 13×13 cross-sections of the 2-limited dominating broadcast of the Cartesian plane discussed in Section 2.13 (defined by $\ell = 0$).

The block defining G_4 has dimensions $m'_{13} \times 13$. Let G_4 be tiled by the first m'_{13} bottom rows of the tile given in Figure 3.14.3. This choice of tile for G_4 ensures that the tiling of G_1 and G_4 "fit" together. An example of this tiling of G_1 and G_4 is given in Figure 3.14.4 for $m = 26$.

Least residue n_{13} of n modulo 13

m'_{13}	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	-2	13	-21	1	-7	-18	-6	-14	-12	-10	2	-9
1	0	6	6	-18	-9	-7	-20	-9	-7	-7	-7	4	4
2	0	1	12	-15	-6	-7	-22	1	0	-15	-17	6	-9
3	0	9	5	-12	-3	-7	-24	-2	-6	-10	-14	-5	-9
4	0	4	-2	-22	-13	-20	-26	-5	-12	-18	-11	-3	4
5	0	-1	4	-19	3	-7	-15	-8	-5	-13	-21	-1	-9
6	0	7	10	-16	-7	-7	-17	-11	-11	-8	-18	1	-9
7	0	2	16	-13	-4	-7	-19	-1	-4	-16	-15	3	-9
8	0	10	9	-23	-1	-7	-21	-4	-10	-11	-12	-8	-9
9	0	5	2	-20	-11	-7	-23	-7	-16	-6	-9	7	-9
10	0	0	8	-17	-8	-7	-25	-10	-9	-14	-19	-4	-9
11	0	8	1	-14	-5	-7	-27	-13	-15	-9	-16	-2	-9
12	0	3	7	-24	-2	-7	-29	-3	-8	-17	-13	0	4

Table 3.14.1: Constant term $c(m'_{13}, n_{13})$ for the upper bound for $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 23$ and $n \geq 13$.

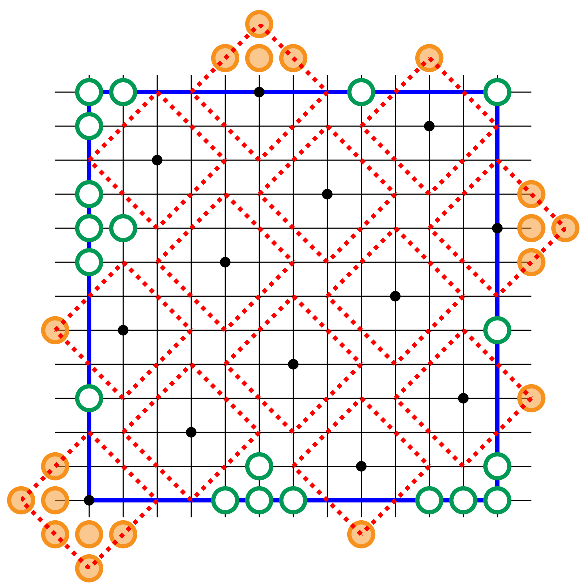


Figure 3.14.3: 13×13 tile used repeatedly in G_1

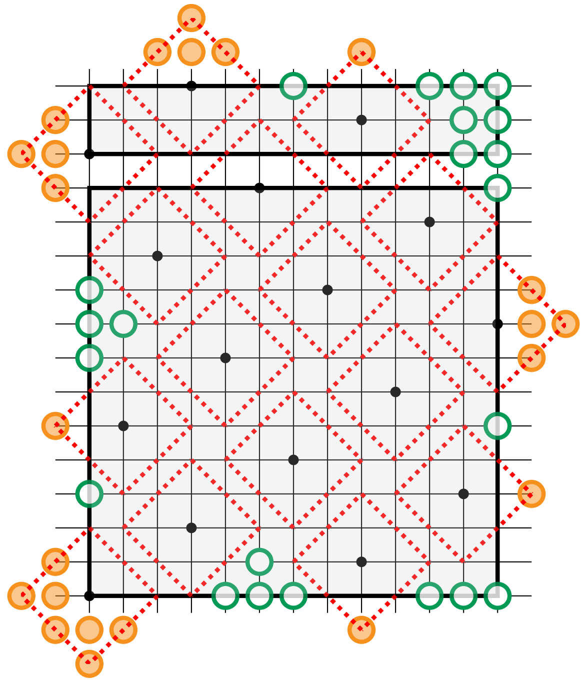


Figure 3.14.4: Tiling of G_1 and G_4 for $m = 26$.

The second step is to determine the appropriate tiles for G_2 and G_3 given the tiling of G_1 and G_4. Consider the graph $P_m \square C_{13}$ formed by G_1, G_2, G_3, and G_4. For example, Figure 3.14.5 depicts $P_{26} \square C_{13}$ partitioned into G_1, G_2, G_3, and G_4.

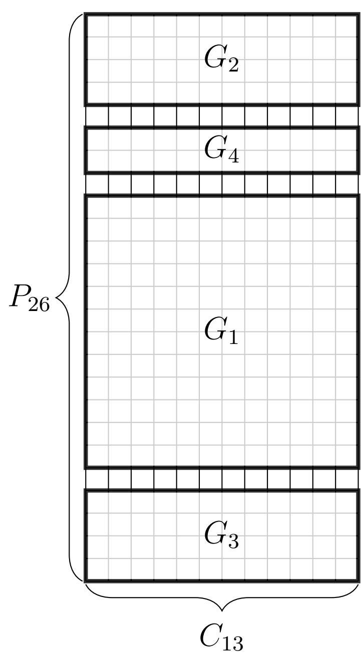


Figure 3.14.5: $P_{26} \square C_{13}$ partitioned into G_1, G_2, G_3 and G_4.

To complete the broadcast on $P_m \square C_{13}$, like in Section 3.13, it reduces to determine the additional broadcast vertices required to extend the broadcast given by the tilings of G_1 and G_4 to dominate $P_m \square C_{13}$. This is easily solved by computation. For every least residue of $(m-10)$ modulo 13, the minimum cost of the additional broadcast vertices required to extend the broadcast of the tiling of G_1 and G_4 to dominate $P_m \square C_{13}$ is 12. Moreover, for each least residue of $(m-10)$, there exist a broadcast tile of G_2 and G_3, with a cumulative cost of 12, which when "glued" appropriately to G_1 and G_4 dominates $P_m \square C_{13}$. Figure 3.14.6 depicts a 2-limited dominating broadcast of $P_{26} \square C_{13}$ given by this tiling of G_1, G_2, G_3, and G_4.

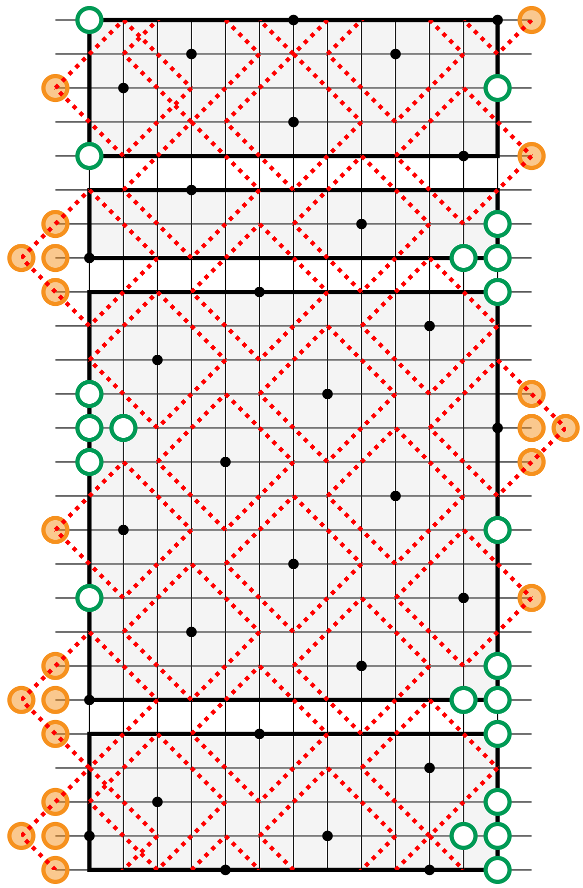


Figure 3.14.6: Resulting 2-limited dominating broadcast of $P_{26} \square C_{13}$ given by the tilings of G_1, G_2, G_3, and G_4.

Observe that, as G_2 and G_3 have been solved for on $P_m \square C_{13}$, the tiles just solved for can be used to construct 2-limited dominating broadcast on $P_m \square C_n$ where $n \equiv 0 \pmod{13}$. For instance, repeating copies of the tiling of $P_{26} \square C_{13}$ given in Figure 3.14.6 gives a 2-limited dominating broadcast on $P_{26} \square C_{26}$. The ability to repeat the tiles solved for in these cases is key to our general construction.

Having established the tiles for G_1, G_2, G_3, and G_4, the next step determines an appropriate tile for G_5 given G_1. Recall that n_{13} is the least residue of n modulo 13. Consider the graph $C_{13} \square C_n$ formed by G_1 and G_5. Figure 3.14.7 depicts $C_{13} \square C_{16}$

partitioned into G_1 and G_5.

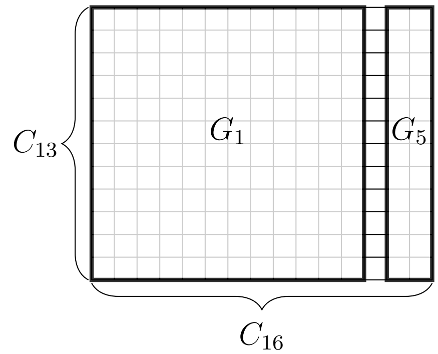


Figure 3.14.7: $C_{13}\square C_{16}$ partitioned into G_1 and G_5.

To complete the broadcast on $C_{13}\square C_n$, like in the previous step, it suffices to determine the additional broadcast vertices required to extend the broadcast given by the tiling of G_1 to dominate $C_{13}\square C_n$. This is easily solved by computation. For each least residue n_{13} of n modulo 13 the minimum cost $b(n_{13})$ of additional broadcast vertices to dominate $C_{13}\square C_n$ is given in Table 3.14.2. Figure 3.14.8 depicts a minimum 2-limited dominating broadcast of $C_{13}\square C_{16}$ as given by Table 3.14.2 for $n_{13} = 3$.

	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
$b(n_{13})$	0	5	7	10	10	13	15	16	19	21	23	24	26

Table 3.14.2: Minimum cost of the additional broadcast vertices required to extend the broadcast of G_1 to dominate $C_{13}\square C_n$.

Observe that, as the additional broadcast vertices have been solved for on $C_{13}\square C_n$, this construction can be generalized. For instance, $C_{26}\square C_{29}$ can be dominated by repeating four of the 13×13 tiles in Figure 3.14.3 and adding two sets of the additional broadcast vertices as shown in Figure 3.14.8. Figure 3.14.9 illustrates a partitioning of $C_{26}\square C_{29}$ into four 13×13 blocks and two 13×3 blocks from which such a broadcast can be constructed.

Let G_1, G_2, G_3, G_4, and G_5 contain the broadcasts as described previously. To complete the broadcast it suffices to determine the additional broadcast vertices required to extend the broadcast given by G_1, G_2, G_3, G_4, and G_5 to dominate $P_m\square C_n$. For each of the least residues of $(m-10)$ modulo 13 and n modulo 13 the minimum cost of the additional broadcast vertices required to extend the broadcast of G_1, G_2, G_3, G_4 and G_5 is given in Table 3.14.3.

Figure 3.14.10 gives a 2-limited dominating broadcast of $P_{26}\square C_{16}$ given by this methodology of cost 76. In general, the cost of a 2-limited dominating broadcast given

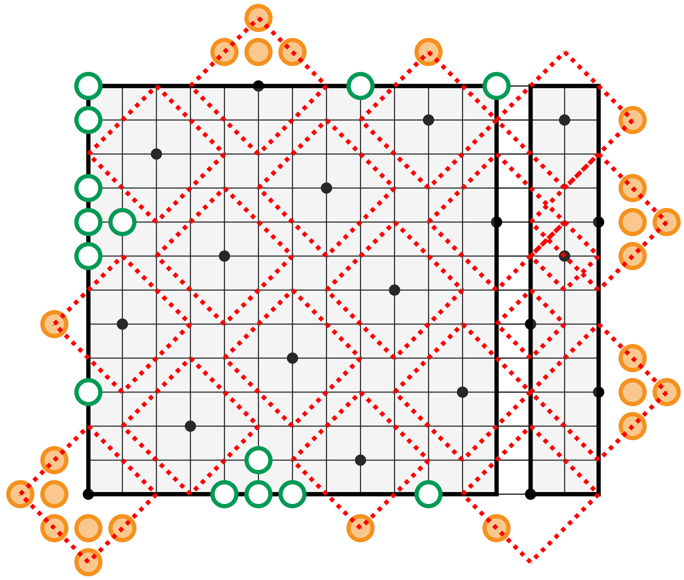


Figure 3.14.8: Least cost dominating broadcast of $C_{13} \square C_{16}$ given the broadcast of G_1.

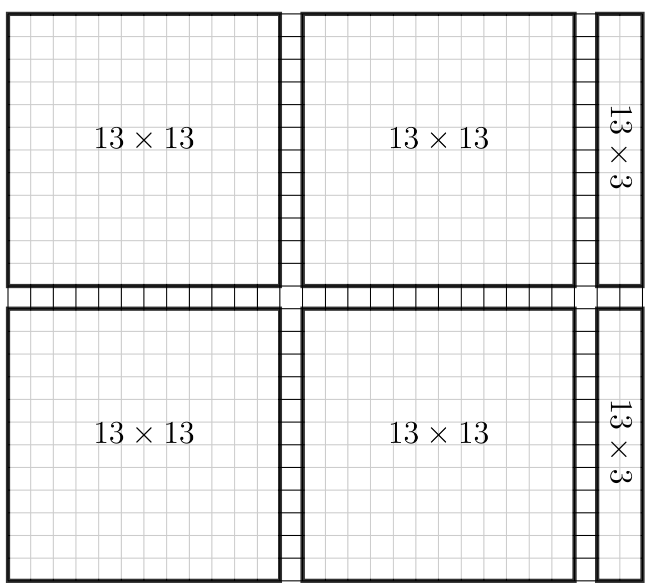


Figure 3.14.9: $C_{26} \square C_{29}$ partitioned into four 13×13 blocks and two 13×3 blocks.

m'_{13}	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	4	7	7	9	11	12	14	16	18	20	22	23
1	0	5	7	8	9	12	13	15	18	20	22	24	26
2	0	5	8	9	10	13	14	17	20	21	23	26	27
3	0	6	8	10	11	14	15	18	21	23	25	27	29
4	0	6	8	10	11	14	16	19	22	24	27	29	32
5	0	6	9	11	13	16	18	20	24	26	28	31	33
6	0	7	10	12	13	17	19	21	25	28	30	33	35
7	0	7	11	13	14	18	20	23	27	29	32	35	37
8	0	8	11	13	15	19	21	24	28	31	34	36	39
9	0	8	11	14	15	20	22	25	29	33	36	39	41
10	0	8	12	15	16	21	23	26	31	34	37	40	43
11	0	9	12	16	17	22	24	27	32	36	39	42	45
12	0	9	13	16	18	23	25	29	34	37	41	44	48

Table 3.14.3: Minimum cost of the additional broadcast vertices required to extend the broadcast of G_1, G_2, G_3, G_4, and G_5 to dominate $P_m \square C_n$.

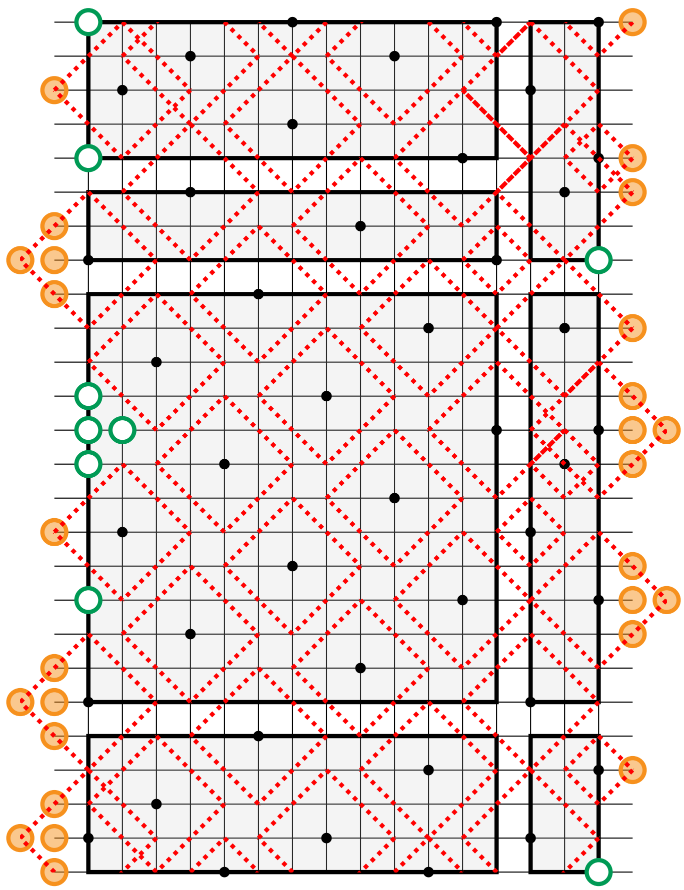


Figure 3.14.10: $P_{26} \square C_{16}$ partitioned into $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 and the 2-limited dominating broadcast solved for via our construction.

by this construction is

$$26 + 24 + 2m_{13} + b(n_{13}) + c(m_{13}, n_{13}) \tag{3.1}$$

where $b(n_{13})$ corresponds with the border cost given in Table 3.14.2 and $c(m_{13}, n_{13})$ corresponds with the corner cost associated with m'_{13} and n_{13} in Table 3.14.3.

As alluded to earlier, this construction is easily generalized for all $m \geq 23$ and $n \geq 13$. Fix $m \geq 23$ and $n \geq 13$ and let m'_{13} and n_{13} be the least residue of $(m-10)$ and n modulo 13. Define $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 as described in this section. Construct a 2-limited dominating broadcast on $P_m \square C_n$ via the following procedure.

1. Tile G_1 with
$$\frac{(m - 10 - m'_{13})(n - n_{13})}{13^2}$$
copes of the 13×13 tiles given in Figure 3.14.3.

2. Tile G_2 and G_3 with
$$\frac{n - n_{13}}{13}$$
5×13 tiles of cost 12 as computed previously.

3. Tile G_4 with
$$\frac{n - n_{13}}{13}$$
$m'_{13} \times 13$ tiles of the first m'_{13} bottom rows of the tile in Figure 3.14.3.

4. Add
$$\frac{m - 10 - m'_{13}}{13}$$
additional broadcasts given by the broadcast vertices corresponding with G_5 as found in Table 3.14.2.

5. Add the additional broadcast vertices to $G_1, G_2, G_3, G_4, G_5, G_6$ and G_7 as associated with the value of Table 3.14.3 given m'_{13} and n_{13} to complete the broadcast.

The resulting broadcast is a 2-limited dominating broadcast on $P_m \square C_n$. Given Expression 3.1, the cost of such a broadcast is

$$26 \left[\frac{(m - 10 - m_{13})\,(n - n_{13})}{13^2} \right] + 24 \left(\frac{n - n_{13}}{13} \right) + 2m_{13} \left(\frac{n - n_{13}}{13} \right)$$
$$+ b(n_{13}) \left(\frac{m - 10 - m_{13}}{13} \right) + c(m_{13}, n_{13}).$$

The previous expression can be evaluated for each of the least residues of $(m-10)$ and n modulo 13. The results give Theorem 3.14.1.

□

3.15 $P_{m\geq 23}\square C_{3\leq n\leq 12}$

Section 3.14 established upper bounds for $\gamma_{b,2}(P_m\square C_n)$ for $m\geq 23$ and $n\geq 13$. The cases which remain to be considered are $m\geq 23$ and $3\leq n\leq 12$. These cases are given in Sections 3.16 through 3.25. As the number of columns (i.e. the value of n) of $P_m\square C_n$ is fixed in Sections 3.16 through 3.25, we note a subtle difference in the figures in the follow sections. The difference is best explained via an example. Figure 3.15.1 depicts three broadcast tiles used in the construction of 2-limited dominating broadcasts on $P_m\square C_4$. Note that here thick green squares with a white inner fill indicate vertices

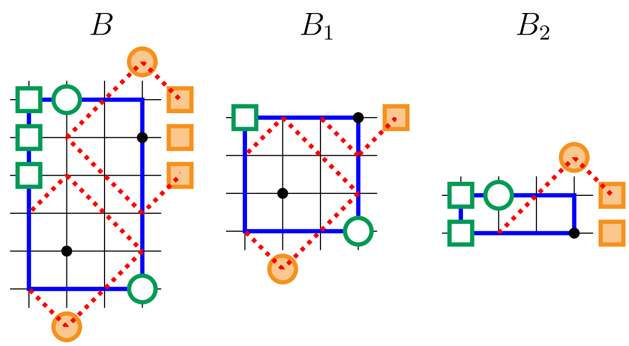


Figure 3.15.1: Toy tiles used to depict change in diagrams for $P_m\square C_4$.

which hear a broadcast from some vertex via the "wrap around" of the cycle in $P_m\square C_n$. Similarly, thick orange tiles with an opaque inner fill indicate vertices whose broadcast is heard via the "wrap around" of the cycle in $P_m\square C_n$. For instance, the three upper left most vertices in the broadcast tiles B of Figure 3.15.1 are dominated by the vertex broadcasting at strength two in the right most column of B.

Given the tiles in Figure 3.15.1, for all $m\geq 10$ where $m\equiv 0 \pmod 6$ the sequence

$$B_1\,[B]^{\frac{m-6}{6}}\,B_2$$

constructs a 2-limited dominating broadcast on $P_m\square C_4$ where the tiles are tiled from top to bottom as opposed to left to right. Figure 3.15.2 depicts the sequence B_1BB_2.

Sections 3.16 through 3.25 are the only sections where a sequence of tiles are tiled from top to bottom as opposed to left to right. We now proceed to our constructions in Sections 3.16 through 3.25.

3.16 $P_{m\geq 23}\square C_3$

Any 2-limited dominating broadcast on $P_3\square P_n$ is a 2-limited dominating broadcast on $P_m\square C_3$. Therefore $\gamma_{b,2}(P_m\square C_3)\leq\gamma_{b,2}(P_3\square P_n)$. Proposition 2.3.1 therefore implies Proposition 3.16.1.

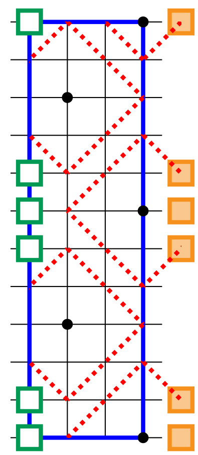


Figure 3.15.2: Toy tiling of $P_{12}\square C_4$.

Proposition 3.16.1. *For all $m \geq 23$,*

$$\gamma_{b,2}\left(P_m \square C_3\right) \leq \left\lceil \frac{2m}{3} \right\rceil .$$

The bounds given by Proposition 3.16.1 are shown to be tight by Corollary 6.4.3.

3.17 $P_{m\geq 23}\square C_4$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\square C_4$ for all $m \geq 13$. Figure 3.17.1 contains the tiles used in the broadcast constructions described in Table 3.17.1.

Proposition 3.17.1. *For all $m \geq 23$,*

$$\gamma_{b,2}\left(P_m \square C_4\right) \leq 4\left\lfloor \frac{m}{6} \right\rfloor + \begin{cases} 1 & \text{for } m \equiv 0 \pmod 6, \\ 2 & \text{for } m \equiv 1 \text{ or } 2 \pmod 6, \\ 3 & \text{for } m \equiv 3 \pmod 6, \text{ and} \\ 4 & \text{for } m \equiv 4 \text{ or } 5 \pmod 6. \end{cases}$$

	Construction	Cost
$m \equiv 0 \pmod 6$:	$B_1\,[B]^{\frac{m-6}{6}}\,B_2$	$4\left(\frac{m-6}{6}\right)+5=4\left(\frac{m}{6}\right)+1$
$m \equiv 1 \pmod 6$:	$B_3\,[B]^{\frac{m-7}{6}}\,B_4$	$4\left(\frac{m-7}{6}\right)+6=4\left(\frac{m-1}{6}\right)+2$
$m \equiv 2 \pmod 6$:	$B_1\,[B]^{\frac{m-8}{6}}\,B_4$	$4\left(\frac{m-8}{6}\right)+6=4\left(\frac{m-2}{6}\right)+2$
$m \equiv 3 \pmod 6$:	$B_5\,[B]^{\frac{m-9}{6}}\,B_4$	$4\left(\frac{m-9}{6}\right)+7=4\left(\frac{m-3}{6}\right)+3$
$m \equiv 4 \pmod 6$:	$B_6\,[B]^{\frac{m-10}{6}}\,B_4$	$4\left(\frac{m-10}{6}\right)+8=4\left(\frac{m-4}{6}\right)+4$
$m \equiv 5 \pmod 6$:	$B_7\,[B]^{\frac{m-11}{6}}\,B_4$	$4\left(\frac{m-11}{6}\right)+8=4\left(\frac{m-5}{6}\right)+4$

Table 3.17.1: Constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\square C_4$.

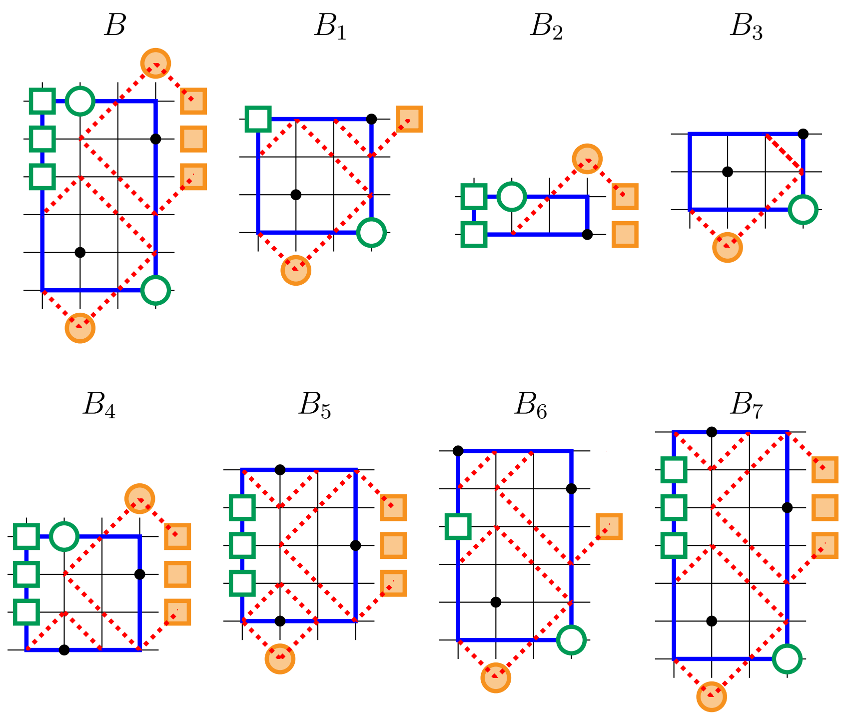


Figure 3.17.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\square C_4$.

Proposition 3.17.1 improves the leading term in the bounds given by Proposition 2.4.1 from $8\lfloor m/10\rfloor$ to $4\lfloor m/6\rfloor$. The values of Proposition 3.17.1 are optimal for all $m \leq 99$ (as found by computation). A lower bound of $\frac{4}{6}m$ is proven by Corollary 6.4.4 and this shows that the upper bounds given by Proposition 3.17.1 for all $n \equiv 2$ and 4 (mod 6) are tight.

Conjecture 3.17.1. *The bounds given by Proposition 3.17.1 are tight.*

3.18 $P_{m\geq 23}\Box C_5$

Any 2-limited dominating broadcast on $P_5\Box P_n$ is a 2-limited dominating broadcast on $P_m\Box C_5$. Therefore $\gamma_{b,2}\left(P_m\Box C_5\right) \leq \gamma_{b,2}\left(P_5\Box P_n\right)$. Proposition 2.5.1 therefore implies Proposition 3.18.1.

Proposition 3.18.1. *For all $m \geq 23$, $\gamma_{b,2}\left(P_m\Box C_5\right) \leq m+1$.*

The values of Proposition 3.18.1 are optimal for all $m \leq 33$ (as found by computation). A lower bound of m is proven by Corollary 6.4.8.

Conjecture 3.18.1. *The bound given by Proposition 3.18.1 is tight.*

3.19 $P_{m\geq 23}\Box C_6$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\Box C_6$ for all $m \geq 13$. Figure 3.19.1 contains the tiles used in the broadcast constructions described in Table 3.19.1.

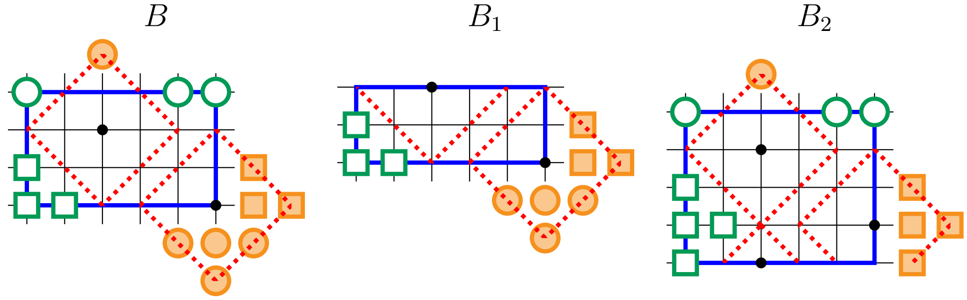
Proposition 3.19.1. *For $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\Box C_6\right) \leq m+1.$$

	Construction	Cost
$m \equiv 0 \pmod 4$:	$B_1[B]^{\frac{m-8}{4}}B_2$	$4\left(\frac{m-8}{4}\right)+9 = m+1$
$m \equiv 1 \pmod 4$:	$B_3\,[B]^{\frac{m-1}{4}}$	$4\left(\frac{m-1}{4}\right)+2 = m+1$
$m \equiv 2 \pmod 4$:	$B_4\,[B]^{\frac{m-2}{4}}$	$4\left(\frac{m-2}{4}\right)+3 = m+1$
$m \equiv 3 \pmod 4$:	$B_1\,[B]^{\frac{m-3}{4}}$	$4\left(\frac{m-3}{4}\right)+4 = m+1.$

Table 3.19.1: Constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_6$.

B B_1 B_2



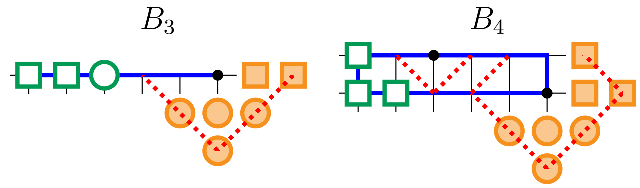


Figure 3.19.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_6$.

Proposition 3.19.1 improves the leading term in the bounds given by Proposition 2.6.1 from $18\lfloor m/16 \rfloor$ to m. The values of Proposition 3.19.1 are optimal for all $m \leq 82$ (as found by computation). A lower bound of m is proven by Corollary 6.4.9.

Conjecture 3.19.1. *The bound given by Proposition 3.19.1 is tight.*

3.20 $P_{m\geq 23}\Box C_7$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\Box C_7$ for all $m \geq 13$. The tile B in Figure 3.20.1 is the main tile used in this construction. Given $P_m\Box C_7$, the tile B is used $\lfloor \frac{n}{35} \rfloor$ times and, for each least residue of m modulo 35, an integer program was so solved in order to find a minimum extension of $B^{\lfloor \frac{n}{35} \rfloor}$ to dominate $P_m\Box C_7$.

Proposition 3.20.1. *For all $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\Box C_7\right) \leq 42\left\lfloor \frac{m}{35} \right\rfloor + c(m_{35}) - c'(m_{35})$$

where $c(m_{35})$ is given in Table 3.20.1, and $c'(m_{35})$ is 0 if $m \geq 35$ and given in Table 3.20.2 otherwise.

	Least residue m_{35} of m modulo 35											
m_{35}	0	1	2	3	4	5	6	7	8	9	10	11
$c(m_{35})$:	2	4	5	6	7	8	10	11	12	13	14	16
m_{35}	12	13	14	15	16	17	18	19	20	21	22	23
$c(m_{35})$:	17	18	19	20	22	23	24	25	26	28	29	30
m_{35}	24	25	26	27	28	29	30	31	32	33	34	
$c(m_{35})$:	31	32	34	35	36	37	38	40	41	42	43	

Table 3.20.1: Values of $c(m_{35})$ for the upper bound for $\gamma_{b,2}\left(P_m\Box C_7\right)$ for $m \geq 23$.

	Least residue m_{35} of m modulo 35										
m_{35}	13	14	15	16	17	18	19	20	21	22	23
$c'(m_{35})$:	1	1	0	1	1	1	1	0	1	1	1

m_{35}	24	25	26	27	28	29	30	31	32	33	34
$c'(m_{35})$:	1	0	1	1	1	1	0	1	1	1	1

Table 3.20.2: Values of $c'(m_{35})$ for the upper bound for $\gamma_{b,2}(P_m \square C_7)$ for $23 \leq m \leq 34$.

Proof. The cases of $23 \leq m \leq 34$ are solved by computation. The 2- limited broadcast domination numbers of $P_m \square C_7$ for $23 \leq m \leq 34$ are given in Table 3.20.3. For $23 \leq m \leq 34$, the term $26 \left\lfloor \frac{m}{35} \right\rfloor$ in Proposition 3.20.1 is 0. As Table 3.20.2 corresponds with the difference of Table 3.20.1 and 3.20.3, the bound is exact for $23 \leq m \leq 34$.

m	13	14	15	16	17	18	19	20	21	22	23
$\gamma_{b,2}(P_m \square C_7)$:	17	18	20	21	22	23	24	26	27	28	29

m	24	25	26	27	28	29	30	31	32	33	34
$\gamma_{b,2}(P_m \square C_7)$:	30	32	33	34	35	36	38	39	40	41	42

Table 3.20.3: $\gamma_{b,2}(P_m \square C_7)$ for $23 \leq m \leq 35$.

Fix $m \geq 35$ and let m_{35} be the least residue of m modulo 35. Consider $P_{35+m_{35}} \square C_7$. Tile the top 35 rows of $P_{35+m_{35}} \square C_7$ with B as shown in Figure 3.20.1. The minimum cost of additional broadcast vertices required to extend the broadcast given by B to dominate $P_{35+m_{35}} \square C_7$ corresponds with the value associated with m_{35} in 3.20.1. These values can be checked by computation. Tile $P_m \square C_7$ by $[B]^{\left\lfloor \frac{n}{35} \right\rfloor}$. Adding the additional broadcast vertices, as solved previously, to extend the broadcast given by $[B]^{\left\lfloor \frac{n}{35} \right\rfloor}$ based on the least residue of m modulo 35 forms a 2-limited dominating broadcast on $P_m \square C_7$. The cost of said broadcast corresponds with the bounds given in Proposition 3.20.1. $\square$

Proposition 3.20.1 improves the leading term in the bounds given by Proposition 2.7.1 from $18 \lfloor m/14 \rfloor$ to $42 \lfloor m/35 \rfloor$. The values of Proposition 3.20.1 are optimal for all $m \leq 49$ (as found by computation) except for cases given in Table 3.20.4. Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{37.6}{35}m$.

n	36	37	38	39	41	42	43	44	46	47	48	49
Bound	46	47	48	49	52	53	54	55	58	59	60	61
Optimal	45	46	47	48	51	52	53	54	57	58	59	60

Table 3.20.4: Known values of n for which the bound in Proposition 3.20.1 is not optimal.

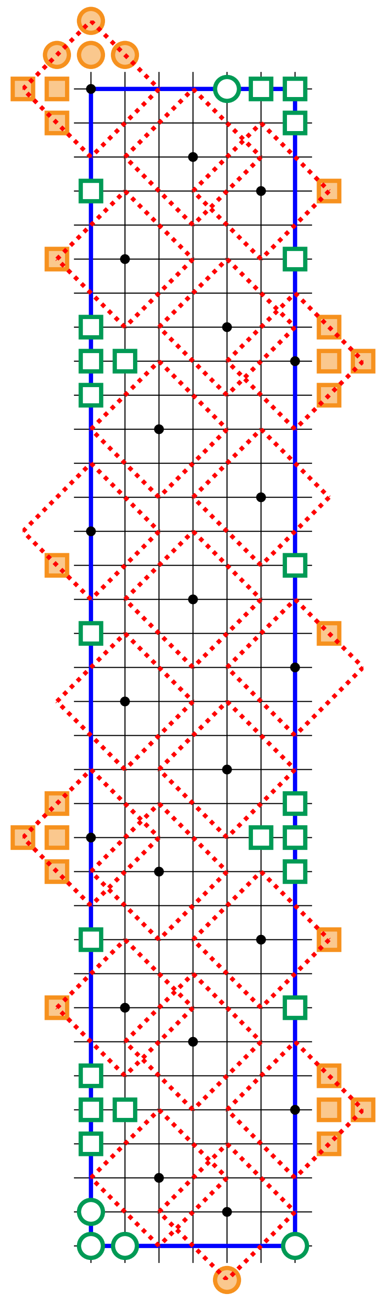


Figure 3.20.1: Tile used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\square C_7$.

3.21 $P_{m\geq 23}\square C_8$

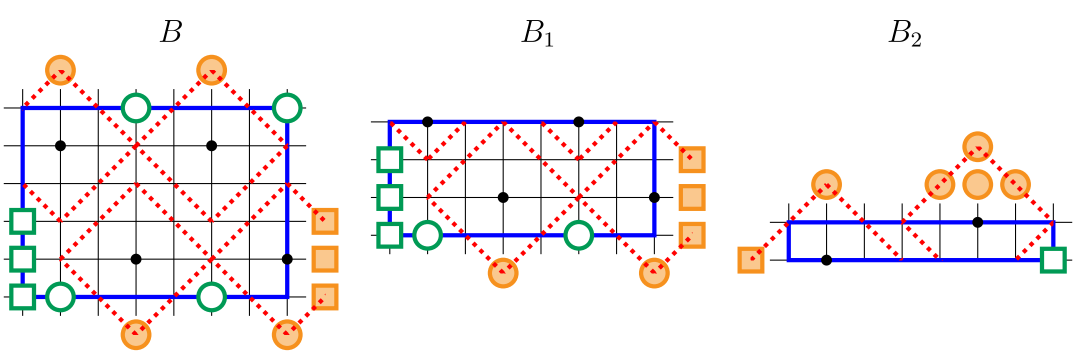
This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\square C_8$ for all $m \geq 13$. Figure 3.21.1 contains the tiles used in the broadcast constructions described in Table 3.21.1.

Proposition 3.21.1. *For $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\square C_8\right) \leq 8\left\lfloor\frac{m}{6}\right\rfloor + \begin{cases} 2 & \text{for } m \equiv 0 \pmod 6, \\ 3 & \text{for } m \equiv 1 \pmod 6, \\ 4 & \text{for } m \equiv 2 \pmod 6, \\ 6 & \text{for } m \equiv 3 \pmod 6, \\ 7 & \text{for } m \equiv 4 \pmod 6, \text{ and} \\ 8 & \text{for } m \equiv 5 \pmod 6. \end{cases}$$

	Construction	Cost
$m \equiv 0 \pmod{10}$:	$B_1\,[B]^{\frac{m-6}{6}}\,B_2$	$8\left(\frac{m-6}{6}\right)+10 = 8\left(\frac{m}{6}\right)+2$
$m \equiv 1 \pmod{10}$:	$B_1\,[B]^{\frac{m-7}{6}}\,B_3$	$8\left(\frac{m-7}{6}\right)+11 = 8\left(\frac{m-1}{6}\right)+3$
$m \equiv 2 \pmod{10}$:	$B_1\,[B]^{\frac{m-8}{6}}\,B_4$	$8\left(\frac{m-8}{6}\right)+12 = 8\left(\frac{m-2}{6}\right)+4$
$m \equiv 3 \pmod{10}$:	$B_5\,[B]^{\frac{m-3}{6}}\,B_2$	$8\left(\frac{m-3}{6}\right)+6$
$m \equiv 4 \pmod{10}$:	$B_5\,[B]^{\frac{m-4}{6}}\,B_3$	$8\left(\frac{m-4}{6}\right)+7$
$m \equiv 5 \pmod{10}$:	$B_5\,[B]^{\frac{m-5}{6}}\,B_4$	$8\left(\frac{m-5}{6}\right)+8$

Table 3.21.1: Constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\square C_8$.



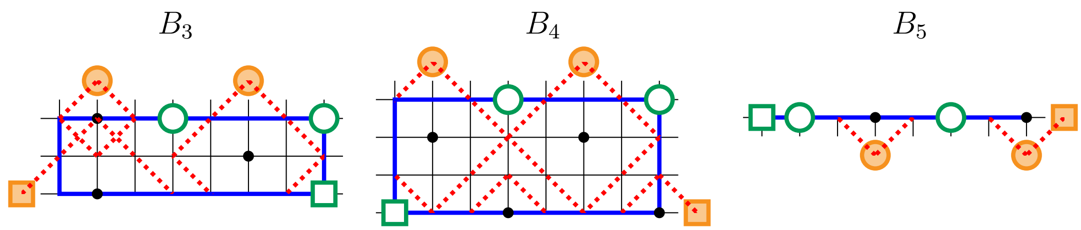


Figure 3.21.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_8$.

Proposition 3.21.1 improves the leading term in the bounds given by Proposition 2.8.1 from $32\lfloor m/22\rfloor$ to $8\lfloor m/6\rfloor$. The values of Proposition 3.21.1 are optimal for all $m \leq 60$ (as found by computation). Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{7.3}{6}m$.

Conjecture 3.21.1. *The bounds in Proposition 3.21.1 are tight.*

3.22 $P_{m\geq 23}\Box C_9$

Any 2-limited dominating broadcast on $P_9\Box P_n$ is a 2-limited dominating broadcast on $P_m\Box C_9$. Therefore $\gamma_{b,2}(P_m\Box C_9) \leq \gamma_{b,2}(P_9\Box P_n)$. Proposition 2.9.1 therefore implies Proposition 3.22.1.

Proposition 3.22.1. *For $m \geq 23$,*

$$\gamma_{b,2}(P_m\Box C_9) \leq 16\left\lfloor \frac{m}{10}\right\rfloor + c(m_{10})$$

where $c(m_{10})$ is dependant upon the least residue m_{10} of m modulo 10 and given in Table 3.22.1.

	Least residue m_{10} of m modulo 10									
m_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	2	3	5	6	8	10	11	13	15	16

Table 3.22.1: Values of $c(m_{10})$ for the upper bound for $\gamma_{b,2}(P_m\Box C_9)$ stated in Proposition 3.22.1.

The values of Proposition 3.22.1 are optimal for all $m \leq 29$ (as found by computation). Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{13.8}{10}m$.

Conjecture 3.22.1. *The bounds in Proposition 3.22.1 are tight.*

3.23 $P_{m\geq 23}\Box C_{10}$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\Box C_{10}$ for all $m \geq 13$. Figure 3.23.1 contains the tiles used in the broadcast constructions described in Table 3.23.2.

Proposition 3.23.1. *For $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\Box C_{10}\right) \leq 16\left\lfloor \frac{m}{10} \right\rfloor + c(m_{10})$$

where $c(m_{10})$ is dependant upon the least residue n_{10} of m modulo 10 and given in Table 3.23.1.

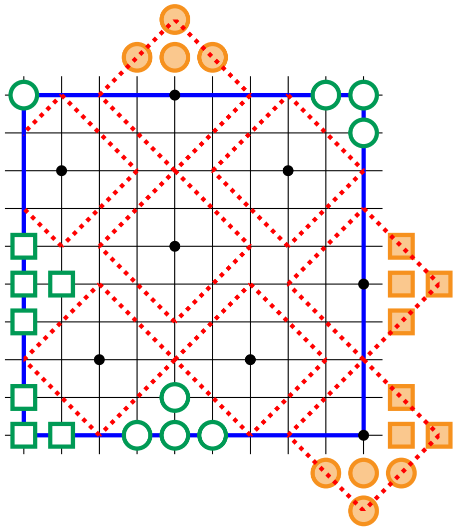
	Least residue m_{10} of m modulo 10									
m_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	2	4	5	7	8	10	12	13	15	16

Table 3.23.1: Values of $c(m_{10})$ for the upper bound for $\gamma_{b,2}\left(P_m\Box C_{10}\right)$ stated in Proposition 3.23.1.

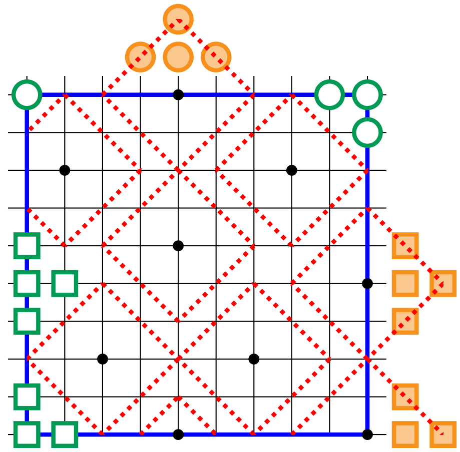
	Construction	Cost
$m \equiv 0 \pmod{10}$:	$B_1\,[B]^{\frac{m-10}{10}}\,B_2$	$16\left(\frac{m-10}{10}\right)+18 = 16\left(\frac{m-1}{10}\right)+2$
$m \equiv 1 \pmod{10}$:	$B_3\,[B]^{\frac{m-11}{10}}\,B_2$	$16\left(\frac{m-11}{10}\right)+20 = 16\left(\frac{m-1}{10}\right)+4$
$m \equiv 2 \pmod{10}$:	$B_1\,[B]^{\frac{m-12}{10}}\,B_4$	$16\left(\frac{m-12}{10}\right)+21 = 16\left(\frac{m-2}{10}\right)+5$
$m \equiv 3 \pmod{10}$:	$B_3\,[B]^{\frac{m-13}{10}}\,B_4$	$16\left(\frac{m-13}{10}\right)+23 = 16\left(\frac{m-3}{10}\right)+7$
$m \equiv 4 \pmod{10}$:	$B_1\,[B]^{\frac{m-4}{10}}\,B_5$	$16\left(\frac{m-4}{10}\right)+8$
$m \equiv 5 \pmod{10}$:	$B_3\,[B]^{\frac{m-5}{10}}\,B_5$	$16\left(\frac{m-5}{10}\right)+10$
$m \equiv 6 \pmod{10}$:	$B_3\,[B]^{\frac{m-6}{10}}\,B_6$	$16\left(\frac{m-6}{10}\right)+12$
$m \equiv 7 \pmod{10}$:	$B_7\,[B]^{\frac{m-7}{10}}\,B_5$	$16\left(\frac{m-7}{10}\right)+13$
$m \equiv 8 \pmod{10}$:	$B_7\,[B]^{\frac{m-8}{10}}\,B_6$	$16\left(\frac{m-8}{10}\right)+15$
$m \equiv 9 \pmod{10}$:	$B_8\,[B]^{\frac{m-9}{10}}\,B_5$	$16\left(\frac{m-9}{10}\right)+16$

Table 3.23.2: Constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_{10}$.

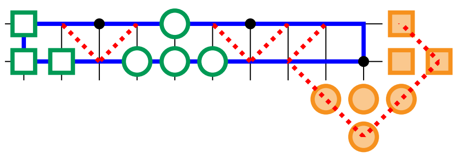
B



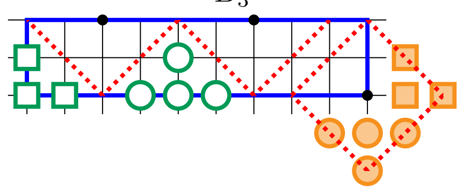
B_4



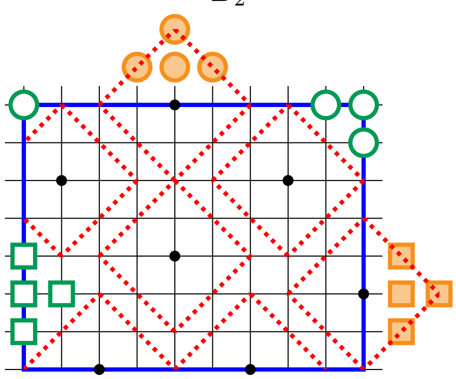
B_1



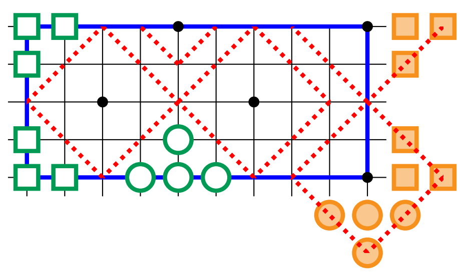
B_3



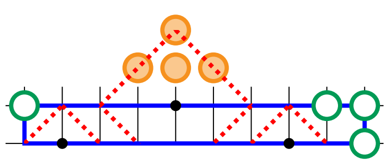
B_2



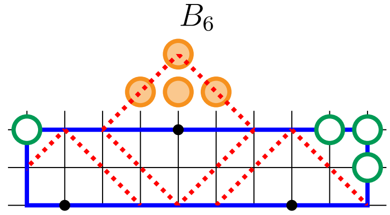
B_7



B_5



B_6



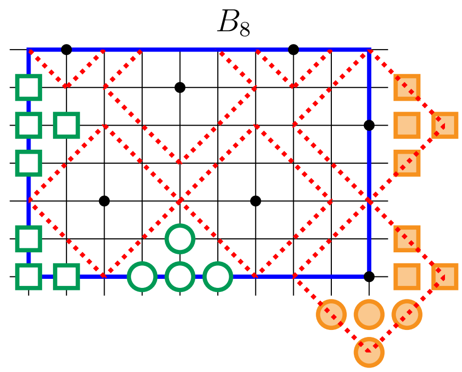


Figure 3.23.1: Tiles used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_{10}$.

Proposition 3.23.1 improves the leading term in the bounds given by Proposition 2.10.1 from $32\lfloor m/18\rfloor$ to $16\lfloor m/10\rfloor$. The values of Proposition 3.23.1 are optimal for all $m \leq 77$ (as found by computation). Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{15.3}{10}m$.

Conjecture 3.23.1. *The bounds in Proposition 3.23.1 are tight.*

3.24 $P_{m\geq 23}\Box C_{11}$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\Box C_{11}$ for all $m \geq 23$. These constructions are found by the same method as discussed in Section 3.20.

Proposition 3.24.1. *For $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\Box C_{11}\right) \leq 24\left\lfloor\frac{m}{13}\right\rfloor + c(m_{13})$$

where $c(m_{13})$ is dependant upon the least residue n_{13} of m modulo 13 and given in Table 3.24.1.

	Least residue m_{10} of m modulo 13												
m_{13}	0	1	2	3	4	5	6	7	8	9	10	11	12
Cost:	3	5	6	9	10	12	14	15	18	19	21	23	24

Table 3.24.1: Values of $c(m_{13})$ for the upper bound for $\gamma_{b,2}\left(P_m\Box C_{11}\right)$ stated in Proposition 3.24.1.

Proof. Use the same method in the proof of Proposition 3.7.1 given the tile in Figure 3.24.1. $\square$

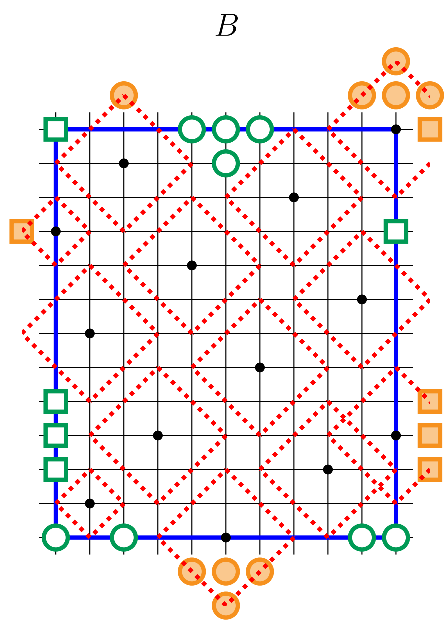


Figure 3.24.1: Tile used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23}\Box C_{11}$.

Proposition 3.24.1 improves the leading term in the bounds given by Proposition 2.11.1 from $50\lfloor m/26\rfloor$ to $24\lfloor m/13\rfloor$. Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{22}{13}m$.

3.25 $P_{m\geq 23}\Box C_{12}$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $P_m\Box C_{12}$ for all $m \geq 23$. These constructions are found by the same method as discussed in Section 3.20. Proposition 3.25.1 provides the bounds established by these constructions.

Proposition 3.25.1. *For $m \geq 23$,*

$$\gamma_{b,2}\left(P_m\Box C_{12}\right) \leq 26\left\lfloor \frac{m}{13}\right\rfloor + c(m_{13})$$

where $c(m_{13})$ is dependant upon the least residue n_{13} of m modulo 13 and given in Table 3.25.1.

Proof. Use the same method in the proof of Proposition 3.7.1 given the tile in Figure 3.25.1. □

Proposition 3.25.1 improves the leading term in the bounds given by Proposition 2.12.1 from $50\lfloor m/24\rfloor$ to $26\lfloor m/13\rfloor$. Theorem 5.2.1 gives a lower bound for this case that is bounded below by the function $\frac{24}{13}m$.

	Least residue m_{10} of m modulo 13												
m_{13}	0	1	2	3	4	5	6	7	8	9	10	11	12
Cost:	2	5	7	9	11	13	15	17	19	21	23	25	27

Table 3.25.1: Values of $c(m_{13})$ for the upper bound for $\gamma_{b,2}(P_m \square C_{12})$ stated in Proposition 3.25.1.

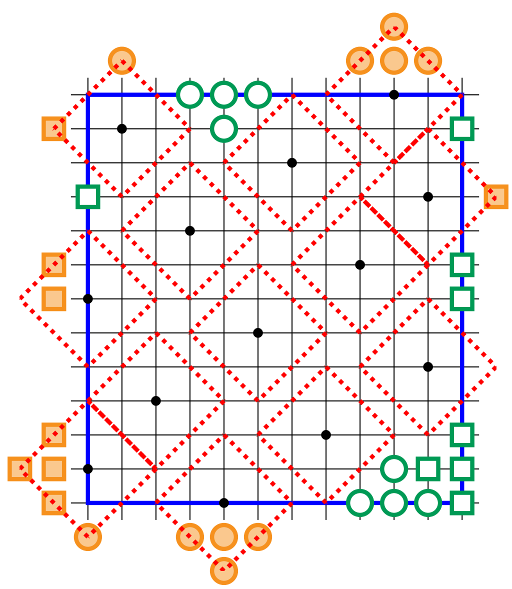


Figure 3.25.1: Tile used in constructions of 2-limited dominating broadcasts on $P_{m\geq 23} \square C_{12}$.

Chapter 4
Cartesian Products of Two Cycles

4.1 Introduction

Having considered the Cartesian products of two paths and a path and a cycle in Chapters 2 and 3, respectively, the natural next step is the Cartesian products of two cycles. As

$$\gamma_{b,2}\left(C_m \square C_n\right) \leq \gamma_{b,2}\left(P_m \square C_n\right) \leq \gamma_{b,2}\left(P_m \square P_n\right),$$

exact values for $\gamma_{b,2}\left(C_m \square C_n\right)$ provide lower bounds for $\gamma_{b,2}\left(P_m \square C_n\right)$ and $\gamma_{b,2}\left(P_m \square P_n\right)$. This chapter is structured similarly to that of Chapters 2 and 3. Sections 4.2 through 4.9 establishes upper bounds for $\gamma_{b,2}\left(C_m \square C_n\right)$ for $3 \leq m \leq 10$ and $n \geq 3$ by using broadcast tiles. The broadcast tiles for these cases were identified by hand by examining known optimal solutions, as found by computation, and looking for recurring patterns. This manual process was greatly facilitated by automatically creating PDF visuals of known optimal solutions using a Python script (written by the author). These visuals are not present in this document but are similar to the figures throughout. Section 4.10 establishes upper bounds for $\gamma_{b,2}\left(C_m \square C_n\right)$ for $11 \leq m \leq 25$ and $m \geq 3$ by using broadcast tiles found by computation in a similar fashion to that of Section 3.11. Finally, Section 4.11 establishes upper bounds of $\gamma_{b,2}\left(C_m \square C_n\right)$ for $m, n \geq 26$ by describing constructions using tilings inspired by the construction in Section 2.13.

Before proceeding to our constructions, we note a subtle difference in the figures in this chapter. Due to the ability of broadcast vertices of $C_m \square C_n$ to "wrap around" the toroidal grid, figures in this chapter will resemble that of Sections 3.16 through 3.25. For example, Figure 4.1.1 depicts a broadcast tiles which is entirely dominated as the vertex in the lower left corner is dominated by the vertex of strength one in the upper left, as indicated by the squares in the figure. We now proceed to our constructions in Sections 4.2 through 4.11.

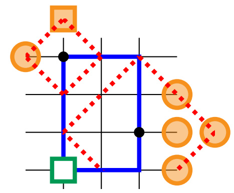


Figure 4.1.1: Toy tile used to depict the "wrap around" on toroidal tiles.

4.2 $C_3 \square C_n$

Any 2-limited dominating broadcast on $P_3 \square P_n$ is a 2-limited dominating broadcast on $C_3 \square C_n$. Therefore $\gamma_{b,2}(C_3 \square C_n) \leq \gamma_{b,2}(P_3 \square P_n)$. Proposition 2.3.1 therefore implies Proposition 4.2.1.

Proposition 4.2.1. *For $n \geq 3$, $\gamma_{b,2}(C_3 \square C_n) \leq \left\lceil \frac{2n}{3} \right\rceil$.*

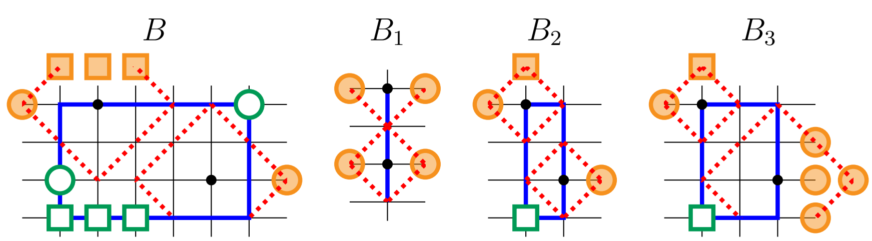
The bound given by Proposition 4.2.1 is shown to be tight by Theorem 6.4.1.

4.3 $C_4 \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_4 \square C_n$ for all $n \geq 4$. Figure 4.3.1 contains the tiles used in the broadcast constructions described in Table 4.3.1.

Proposition 4.3.1. *For $n \geq 4$,*

$$\gamma_{b,2}(C_4 \square C_n) \leq 4\left\lfloor \frac{n}{6} \right\rfloor + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 6, \\ 2 & \text{for } n \equiv 1, \text{or } 2 \pmod 6, \\ 3 & \text{for } n \equiv 3, \text{or } 4 \pmod 6, \text{ and} \\ 4 & \text{for } n \equiv 5 \pmod 6. \end{cases}$$



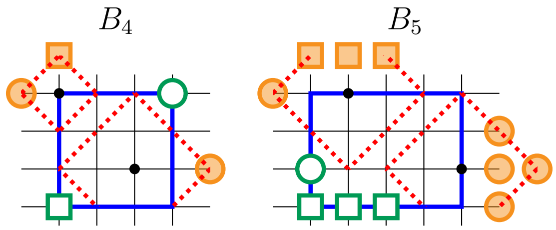


Figure 4.3.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_4 \square C_n$.

	Construction	Cost
$n \equiv 0 \pmod 6$:	$[B]^{\frac{n}{6}}$	$4\left(\frac{n}{6}\right)$
$n \equiv 1 \pmod 6$:	$[B]^{\frac{n-1}{6}} B_1$	$4\left(\frac{n-1}{6}\right) + 2$
$n \equiv 2 \pmod 6$:	$[B]^{\frac{n-2}{6}} B_2$	$4\left(\frac{n-2}{6}\right) + 2$
$n \equiv 3 \pmod 6$:	$[B]^{\frac{n-3}{6}} B_3$	$4\left(\frac{n-3}{6}\right) + 3$
$n \equiv 4 \pmod 6$:	$[B]^{\frac{n-4}{6}} B_4$	$4\left(\frac{n-4}{6}\right) + 3$
$n \equiv 5 \pmod 6$:	$[B]^{\frac{n-5}{6}} B_5$	$4\left(\frac{n-5}{6}\right) + 4$

Table 4.3.1: Constructions of 2-limited dominating broadcasts on $C_4 \square C_n$.

Proposition 4.3.1 improves the leading term given by the bound in Proposition 3.4.1 from $8\lfloor n/10 \rfloor$ to $4\lfloor n/6 \rfloor$ and improves the bound given by Proposition 3.17.1 by one when $n \equiv 0$ and $4 \pmod 6$. The values of Proposition 4.3.1 are optimal for all $n \leq 99$ (as found by computation). A lower bound of $\frac{4}{6}n$ is proven by Corollary 6.4.2 and this shows that the upper bound given for all $n \equiv 0, 2, 3$ and $4 \pmod 6$ are tight.

Conjecture 4.3.1. *The bounds given by Proposition 4.3.1 are tight.*

4.4 $C_5 \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_5 \square C_n$ for all $n \geq 5$. Figure 4.4.1 contains the tiles used in the broadcast constructions described in Table 4.4.1.

Proposition 4.4.1. *For $n \geq 5$, $\gamma_{b,2}(C_5 \square C_n) \leq n$.*

	Construction	Cost
$n \equiv 0 \pmod 4$:	$[B]^{\frac{n}{4}}$	$4\left(\frac{n}{4}\right) = n$
$n \equiv 1 \pmod 4$:	$[B]^{\frac{n-5}{4}}\ B_1$	$4\left(\frac{n-5}{4}\right) + 5 = n$
$n \equiv 2 \pmod 4$:	$[B]^{\frac{n-6}{4}}\ B_2$	$4\left(\frac{n-6}{4}\right) + 6 = n$
$n \equiv 3 \pmod 4), (n = 7)$:	B_5	$7 = n$
$(n > 7)$:	$[B]^{\frac{n-7}{4}}\ B_3$	$4\left(\frac{n-7}{4}\right) + 7 = n$

Table 4.4.1: Constructions of 2-limited dominating broadcasts on $C_5 \Box C_n$.

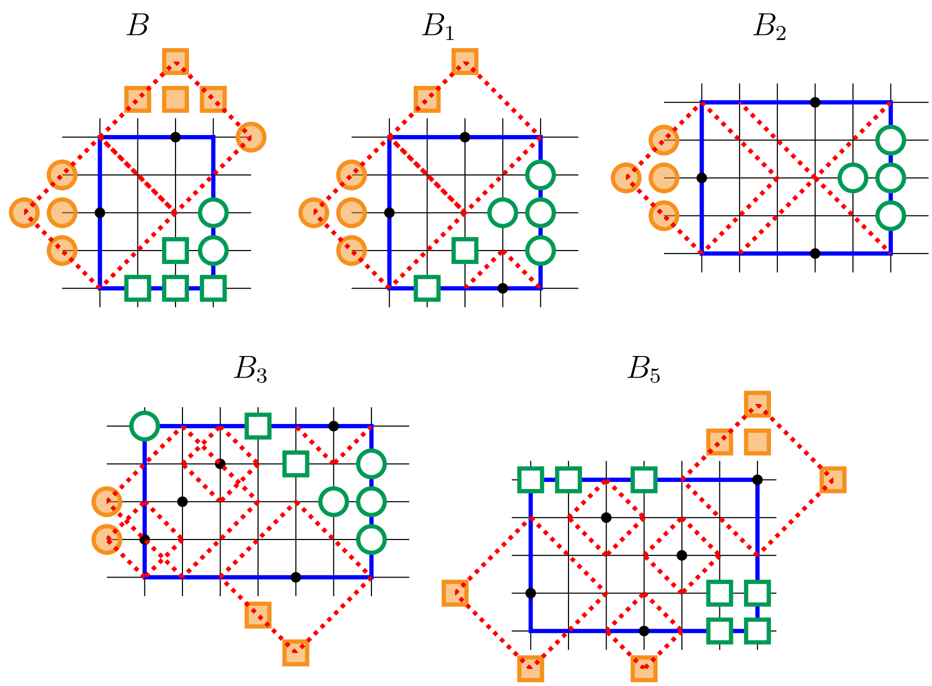


Figure 4.4.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_5 \Box C_n$.

Proposition 4.4.1 improves the bound given by Proposition 3.5.1 by one for odd n and improves the bound given by Proposition 3.18.1 by one. The bound given by Proposition 4.4.1 is shown to be tight by Theorem 6.4.5.

4.5 $C_6 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_6 \Box C_n$ for all $n \geq 6$. Figure 4.5.1 contains the tiles used in the broadcast constructions described in Table 4.5.1.

Proposition 4.5.1. *For $n \geq 6$,*

$$\gamma_{b,2}(C_6 \square C_n) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 4 \text{ and} \\ 1 & \text{for } n \equiv 1, 2, \text{or } 3 \pmod 4. \end{cases}$$

	Construction	Cost
$n \equiv 0 \pmod 4$:	$[B]^{\frac{n}{4}}$	$4\left(\frac{n}{4}\right) = n$
$n \equiv 1 \pmod 4$:	$[B]^{\frac{n-5}{4}} B_1$	$4\left(\frac{n-5}{4}\right) + 6 = n+1$
$n \equiv 2 \pmod 4, (n = 6)$:	B_4	$7 = n+1$
$(n > 6)$:	$[B]^{\frac{n-6}{4}} B_2$	$4\left(\frac{n-6}{4}\right) + 7 = n+1$
$n \equiv 3 \pmod 4$:	$[B]^{\frac{n-3}{4}} B_3$	$4\left(\frac{n-3}{4}\right) + 4 = n+1$

Table 4.5.1: Constructions of 2-limited dominating broadcasts on $C_6 \square C_n$.

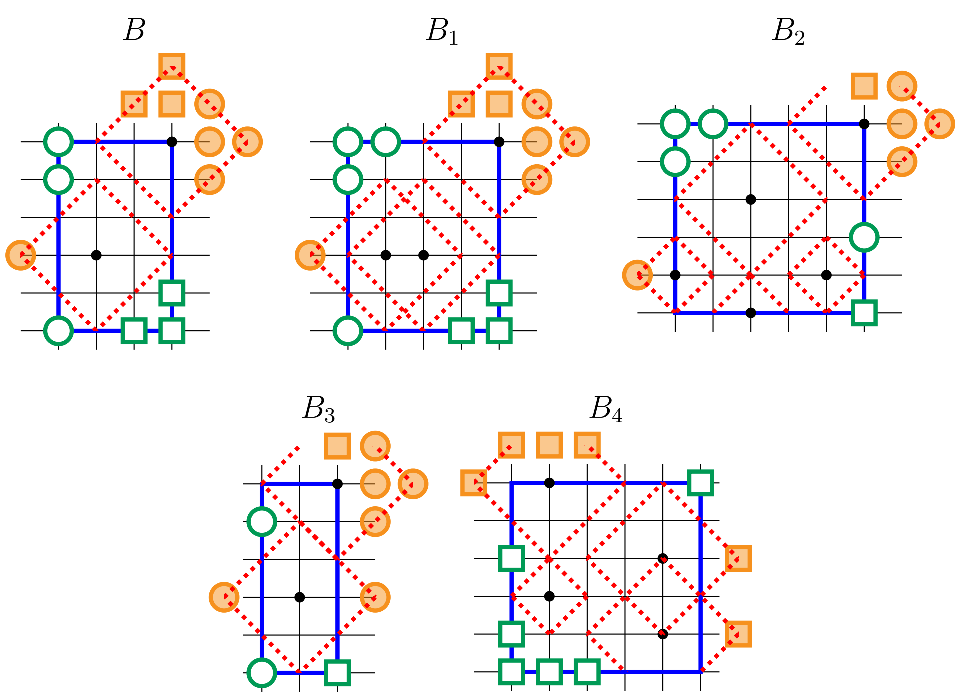


Figure 4.5.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_6 \square C_n$.

Proposition 4.5.1 improves the leading term given by the bound in Proposition 3.6.1 from $18\lfloor n/16 \rfloor$ to n and improves the bound given by Proposition 3.19.1 by one when $n \equiv 0 \pmod 4$. The values of Proposition 4.5.1 are optimal for all $n \leq 72$ (as found by computation). The bound given by Proposition 4.5.1 is shown to be tight when $n \equiv 0 \pmod 4$ by Theorem 6.4.6.

Conjecture 4.5.1. *The bound given by Proposition 4.5.1 is tight.*

4.6 $C_7 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_7 \Box C_n$ for all $n \geq 7$. Figure 4.6.1 contains the tiles used in the broadcast constructions described in Table 4.6.2.

Proposition 4.6.1. *For* $n \geq 7$

$$\gamma_{b,2}(C_7 \Box C_n) \leq 42 \left\lfloor \frac{n}{35} \right\rfloor + c(n_{35})$$

where $c(n_{35})$ *is dependant upon the least residue* n_{35} *of* n *modulo 35 and given in Table 4.6.1.*

	Least residue n_{35} of n modulo 35											
n_{35}	0	1	2	3	4	5	6	7	8	9	10	11
Cost:	0	3	4	5	6	7	8	9	10	12	13	14
n_{35}	12	13	14	15	16	17	18	19	20	21	22	23
Cost:	16	16	18	19	20	21	22	24	25	26	27	29
n_{35}	24	25	26	27	28	29	30	31	32	33	34	
Cost:	30	31	32	33	34	36	37	38	39	40	42	

Table 4.6.1: Values of $c(n_{35})$ in the upper bound for $\gamma_{b,2}(C_7 \Box C_n)$ for $n \geq 7$.

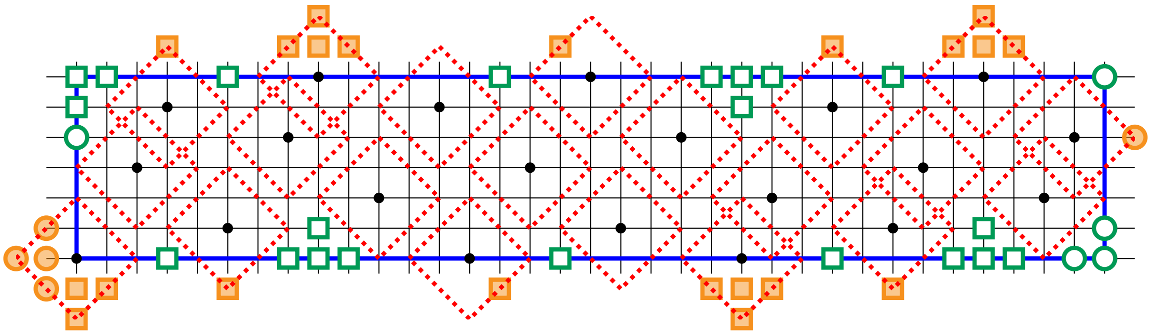
	Construction	Cost
$n \equiv 0 \pmod{35}$:	$[B]^{\frac{n}{35}}$	$42\left(\frac{n}{35}\right)$
$n \equiv 1 \pmod{35}$:	$[B]^{\frac{n-1}{35}} B_1$	$42\left(\frac{n-1}{35}\right) + 3$
$n \equiv 2 \pmod{35}$:	$[B]^{\frac{n-2}{35}} B_2$	$42\left(\frac{n-2}{35}\right) + 4$
$n \equiv 3 \pmod{35}$:	$[B]^{\frac{n-3}{35}} B_3$	$42\left(\frac{n-3}{35}\right) + 5$
$n \equiv 4 \pmod{35}$:	$[B]^{\frac{n-4}{35}} B_4$	$42\left(\frac{n-4}{35}\right) + 6$
$n \equiv 5 \pmod{35}$:	$[B]^{\frac{n-5}{35}} B_5$	$42\left(\frac{n-5}{35}\right) + 7$
$n \equiv 6 \pmod{35}$:	$[B]^{\frac{n-6}{35}} B_6$	$42\left(\frac{n-6}{35}\right) + 8$
$n \equiv 7 \pmod{35}$:	$[B]^{\frac{n-7}{35}} B_7$	$42\left(\frac{n-7}{35}\right) + 9$

$n \equiv 8 \pmod{35}$:	$[B]^{\frac{n-8}{35}} B_8$	$42\left(\frac{n-8}{35}\right)+10$
$n \equiv 9 \pmod{35}$:	$[B]^{\frac{n-9}{35}} B_9$	$42\left(\frac{n-9}{35}\right)+12$
$n \equiv 10 \pmod{35}$:	$[B]^{\frac{n-10}{35}} B_{10}$	$42\left(\frac{n-10}{35}\right)+13$
$n \equiv 11 \pmod{35}$:	$[B]^{\frac{n-11}{35}} B_{11}$	$42\left(\frac{n-11}{35}\right)+14$
$n \equiv 12 \pmod{35}$:	$[B]^{\frac{n-12}{35}} B_{12}$	$42\left(\frac{n-12}{35}\right)+16$
$n \equiv 13 \pmod{35}$:	$[B]^{\frac{n-13}{35}} B_{13}$	$42\left(\frac{n-13}{35}\right)+16$
$n \equiv 14 \pmod{35}$:	$[B]^{\frac{n-14}{35}} B_{14}$	$42\left(\frac{n-14}{35}\right)+18$
$n \equiv 15 \pmod{35}$:	$[B]^{\frac{n-15}{35}} B_{15}$	$42\left(\frac{n-15}{35}\right)+19$
$n \equiv 16 \pmod{35}$:	$[B]^{\frac{n-16}{35}} B_{16}$	$42\left(\frac{n-16}{35}\right)+20$
$n \equiv 17 \pmod{35}$:	$[B]^{\frac{n-17}{35}} B_{17}$	$42\left(\frac{n-17}{35}\right)+21$
$n \equiv 18 \pmod{35}$:	$[B]^{\frac{n-18}{35}} B_{18}$	$42\left(\frac{n-18}{35}\right)+22$
$n \equiv 19 \pmod{35}$:	$[B]^{\frac{n-19}{35}} B_{19}$	$42\left(\frac{n-19}{35}\right)+24$
$n \equiv 20 \pmod{35}$:	$[B]^{\frac{n-20}{35}} B_{20}$	$42\left(\frac{n-20}{35}\right)+25$
$n \equiv 21 \pmod{35}$:	$[B]^{\frac{n-21}{35}} B_{21}$	$42\left(\frac{n-21}{35}\right)+26$
$n \equiv 22 \pmod{35}$:	$[B]^{\frac{n-22}{35}} B_{22}$	$42\left(\frac{n-22}{35}\right)+27$
$n \equiv 23 \pmod{35}$:	$[B]^{\frac{n-23}{35}} B_{23}$	$42\left(\frac{n-23}{35}\right)+29$
$n \equiv 24 \pmod{35}$:	$[B]^{\frac{n-24}{35}} B_{24}$	$42\left(\frac{n-24}{35}\right)+30$
$n \equiv 25 \pmod{35}$:	$[B]^{\frac{n-25}{35}} B_{25}$	$42\left(\frac{n-25}{35}\right)+31$
$n \equiv 26 \pmod{35}$:	$[B]^{\frac{n-26}{35}} B_{26}$	$42\left(\frac{n-26}{35}\right)+32$
$n \equiv 27 \pmod{35}$:	$[B]^{\frac{n-27}{35}} B_{27}$	$42\left(\frac{n-27}{35}\right)+33$
$n \equiv 28 \pmod{35}$:	$[B]^{\frac{n-28}{35}} B_{28}$	$42\left(\frac{n-28}{35}\right)+34$
$n \equiv 29 \pmod{35}$:	$[B]^{\frac{n-29}{35}} B_{29}$	$42\left(\frac{n-29}{35}\right)+36$
$n \equiv 30 \pmod{35}$:	$[B]^{\frac{n-30}{35}} B_{30}$	$42\left(\frac{n-30}{35}\right)+37$
$n \equiv 31 \pmod{35}$:	$[B]^{\frac{n-31}{35}} B_{31}$	$42\left(\frac{n-31}{35}\right)+38$
$n \equiv 32 \pmod{35}$:	$[B]^{\frac{n-32}{35}} B_{32}$	$42\left(\frac{n-32}{35}\right)+39$
$n \equiv 33 \pmod{35}$:	$[B]^{\frac{n-33}{35}} B_{33}$	$42\left(\frac{n-33}{35}\right)+40$

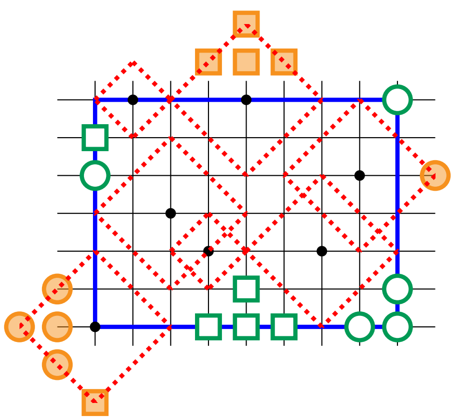
$n \equiv 34 \pmod{35}$:	$[B]^{\frac{n-34}{35}}\, B_{34}$	$42\left(\frac{n-34}{35}\right) + 42$

Table 4.6.2: Constructions of 2-limited dominating broadcasts on $C_7 \square C_n$.

B



B_9



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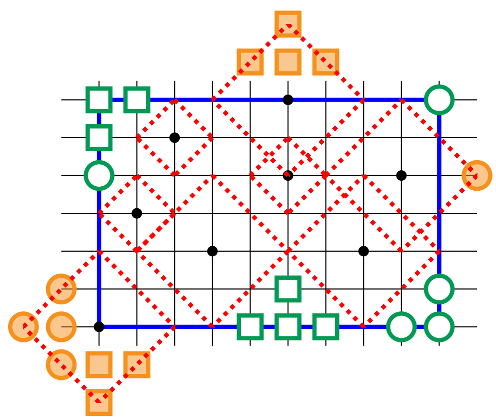

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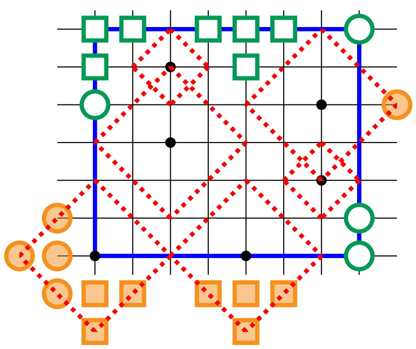

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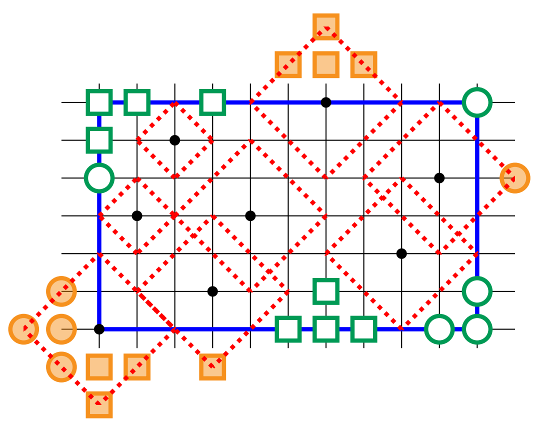
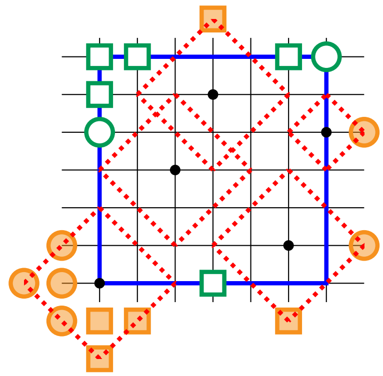


B_7



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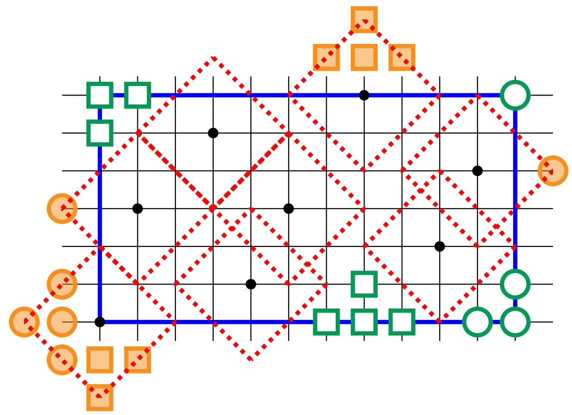

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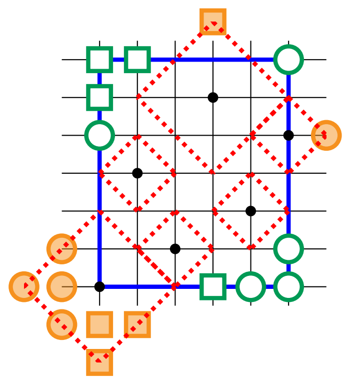

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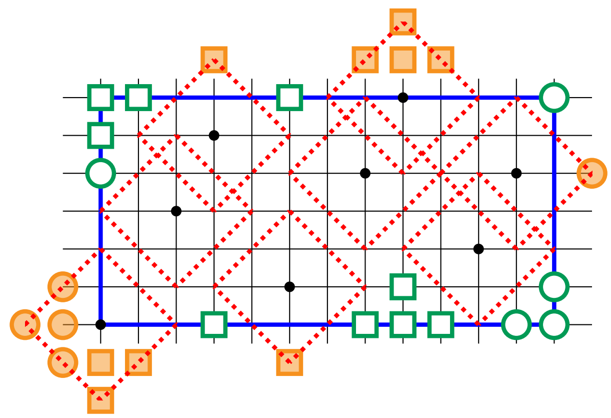

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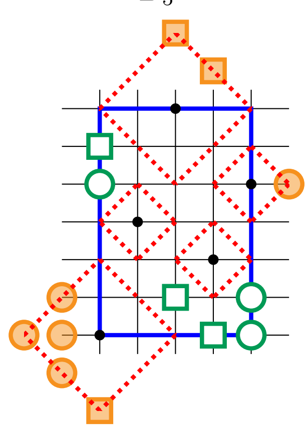

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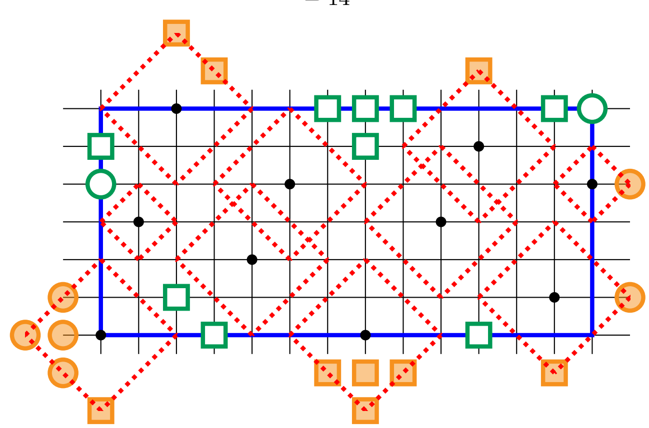

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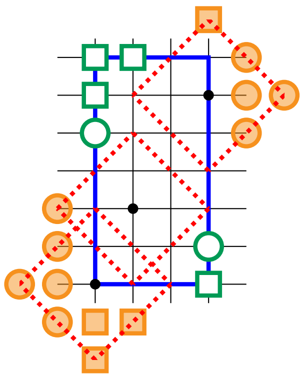

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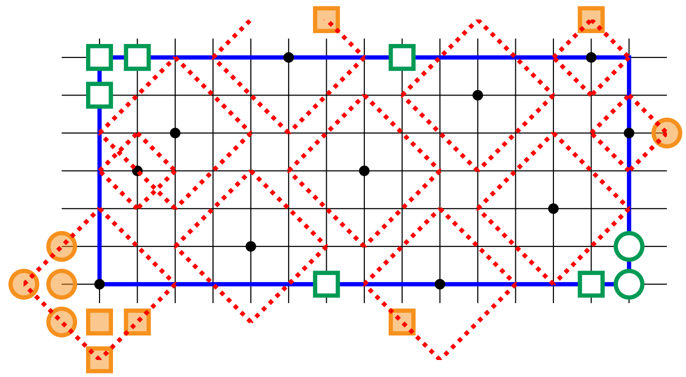

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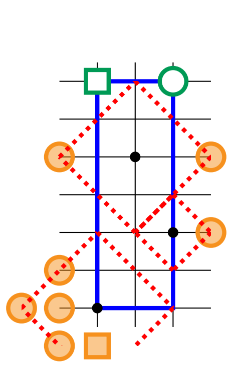

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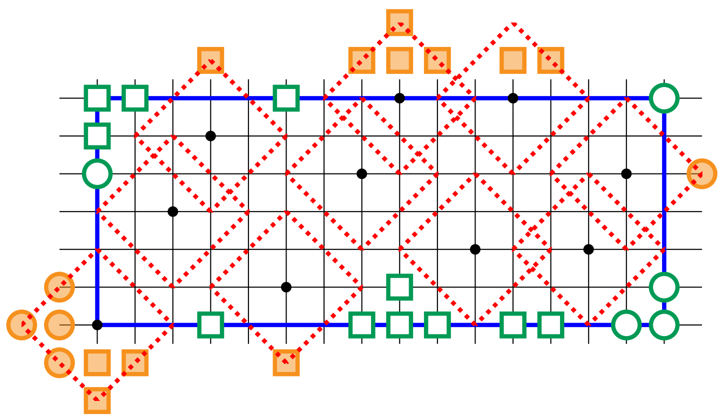
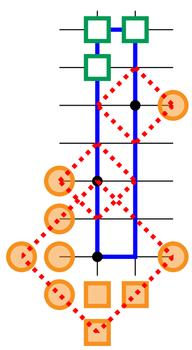


B_2



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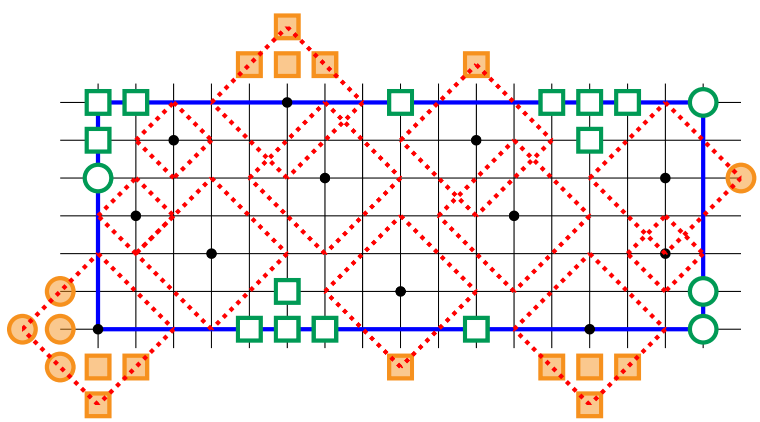
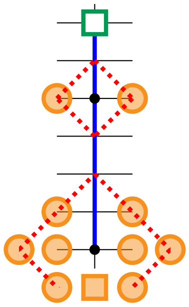


B_1



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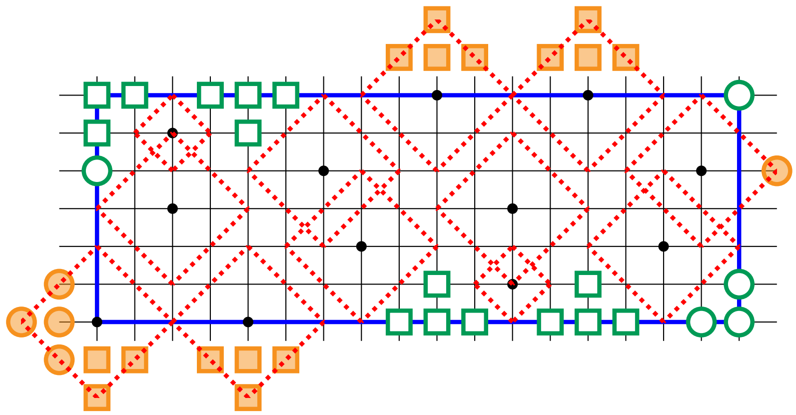

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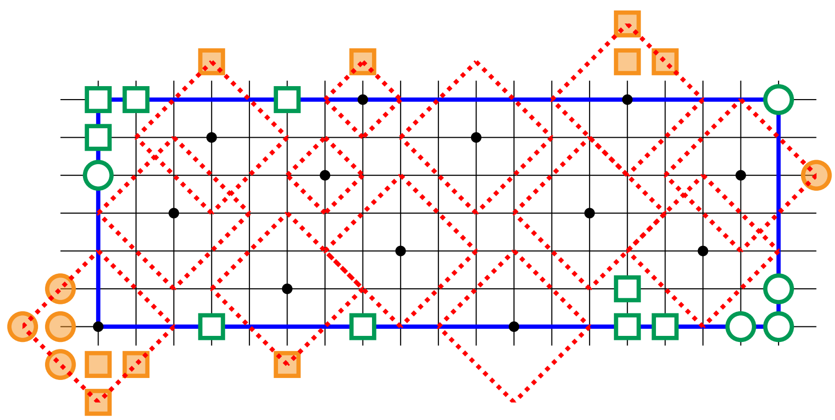
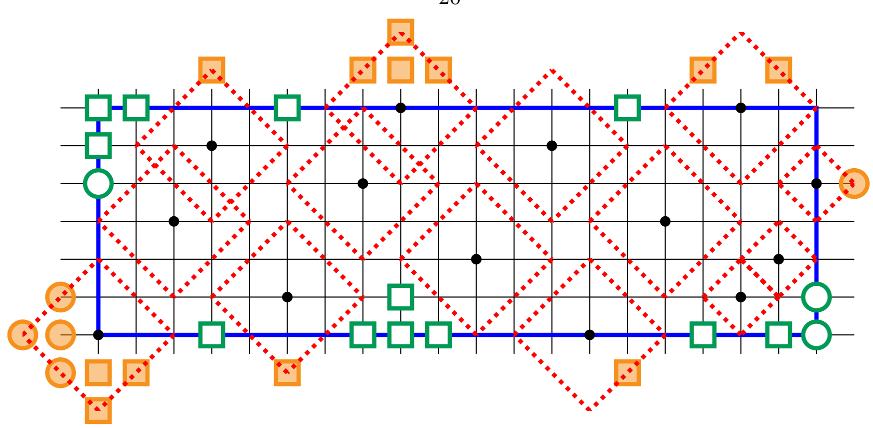


B_{20}



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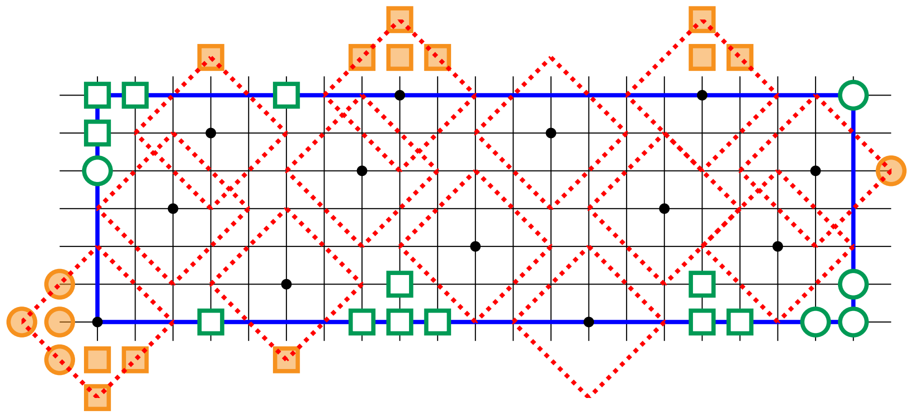

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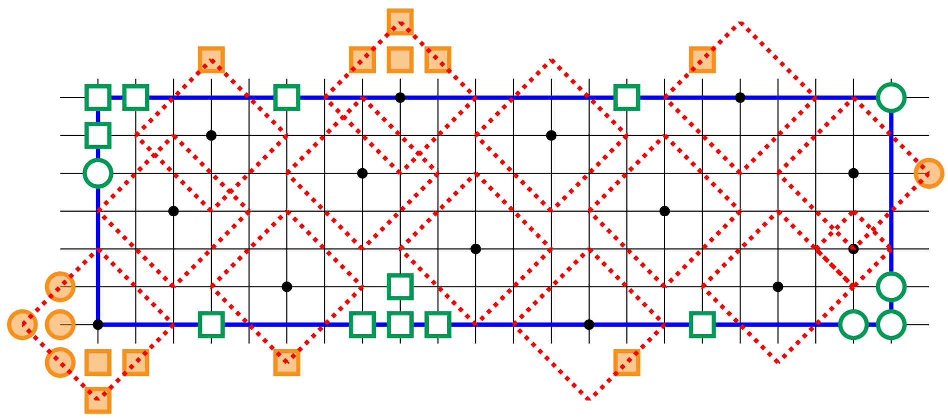

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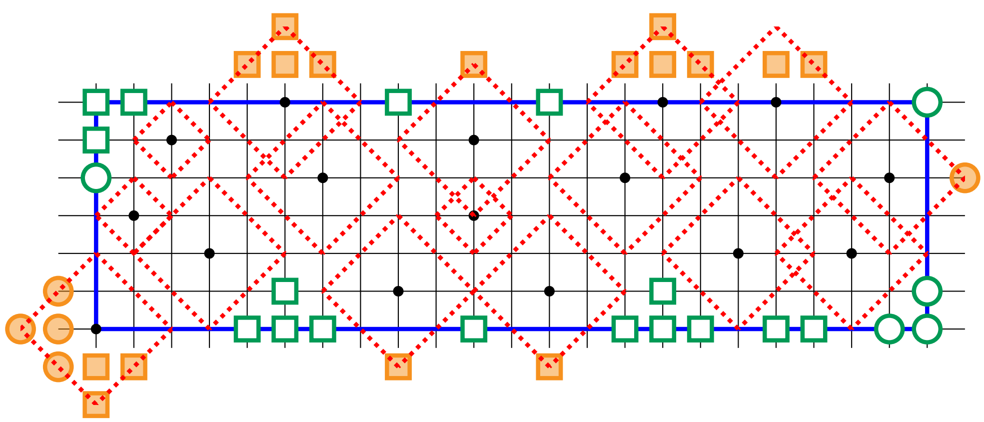

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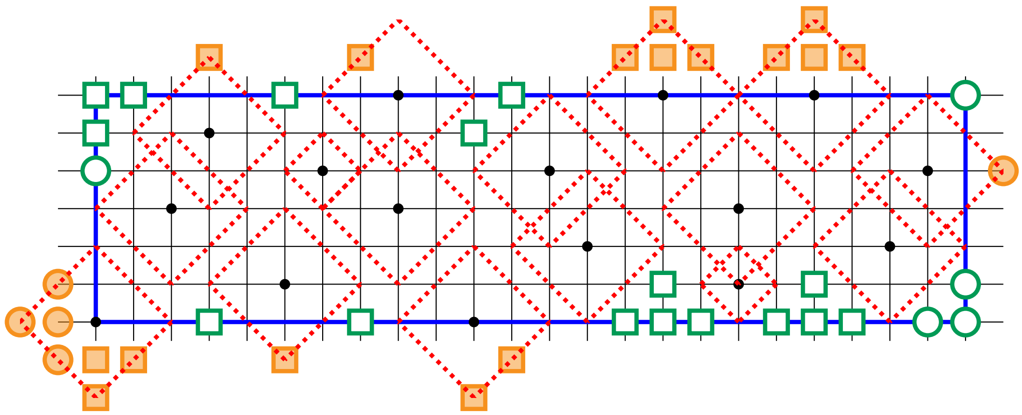

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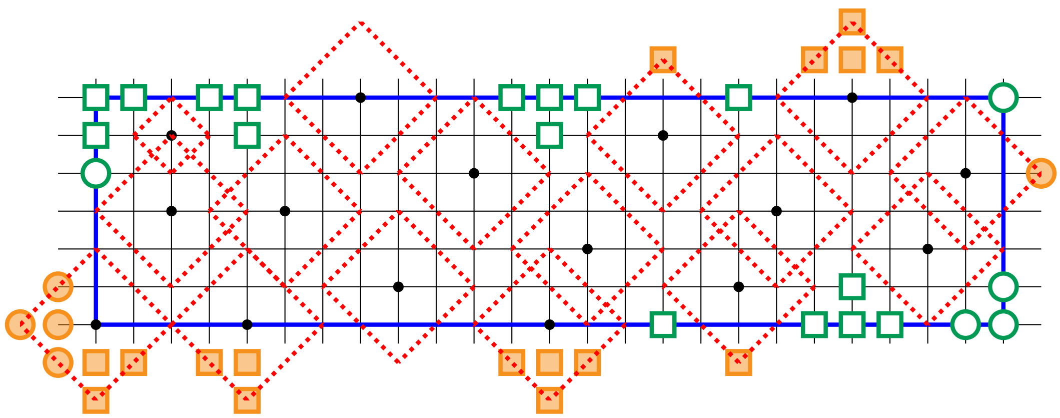
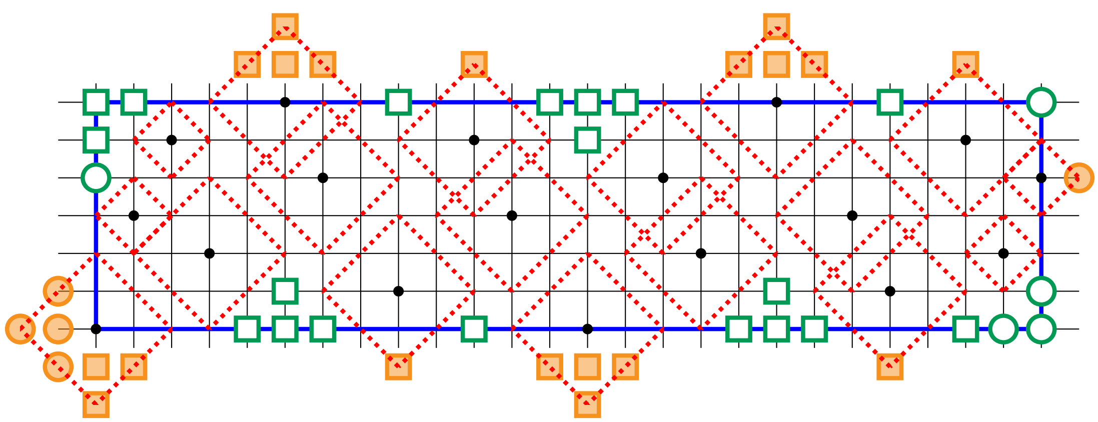


B_{26}



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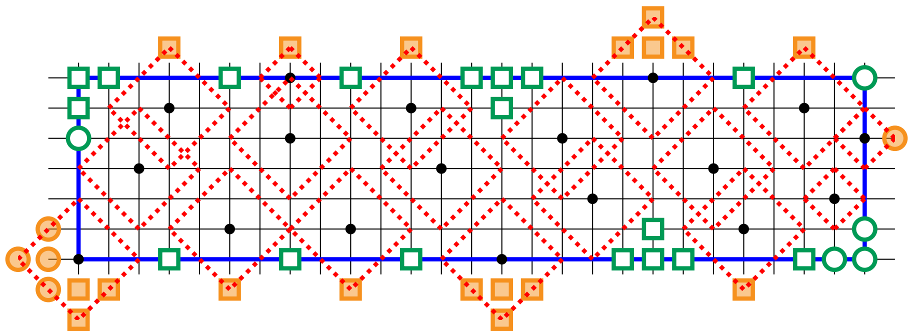

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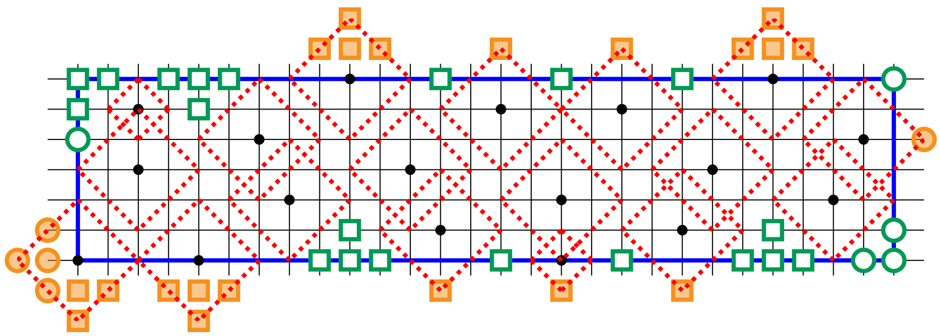
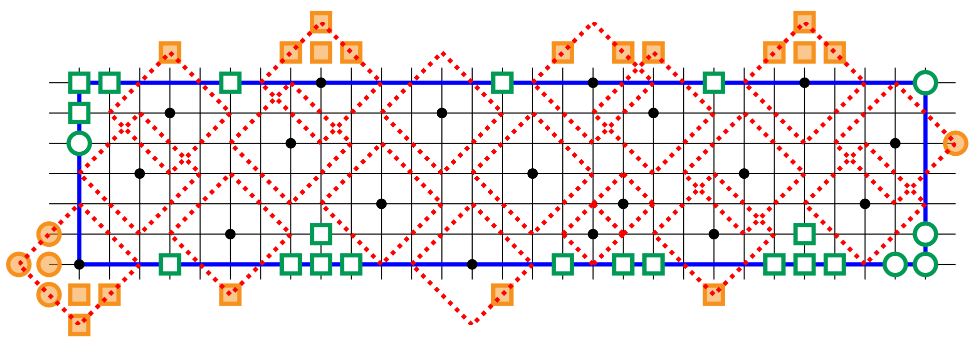
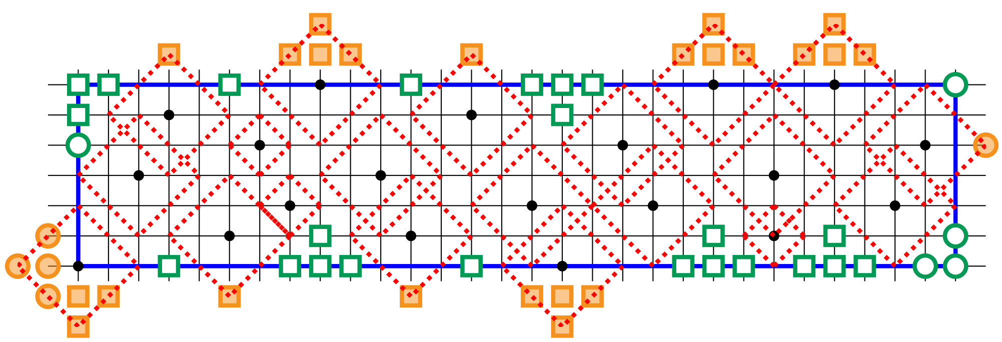


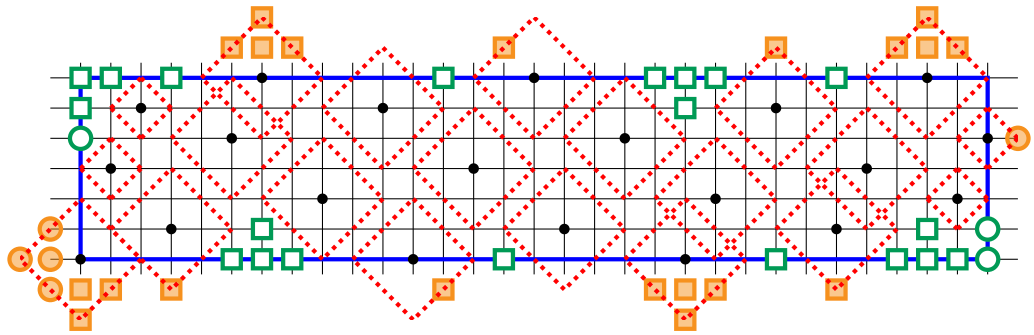
B_{29}



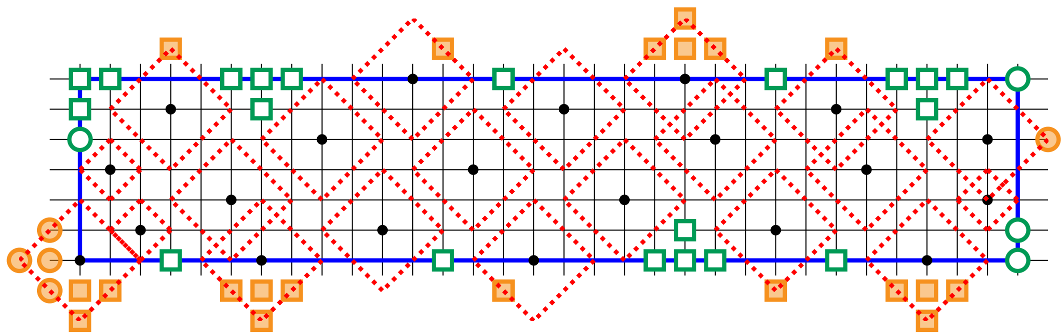
B_{30}



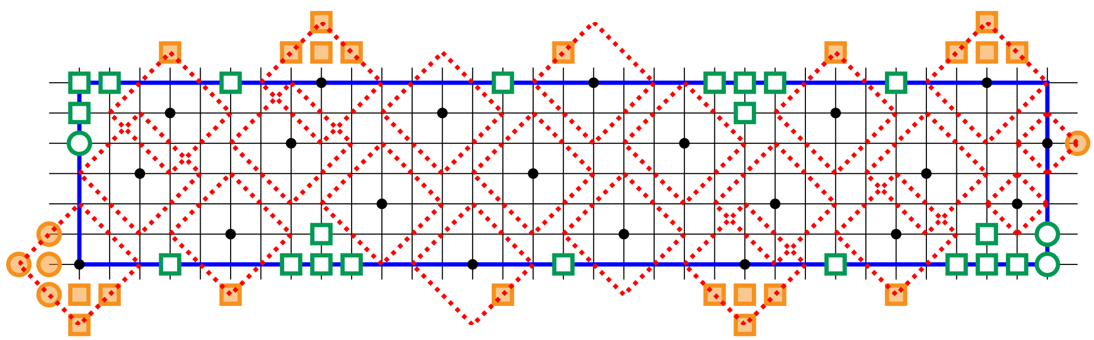
B_{31}



B_{32}



B_{33}



B_{34}

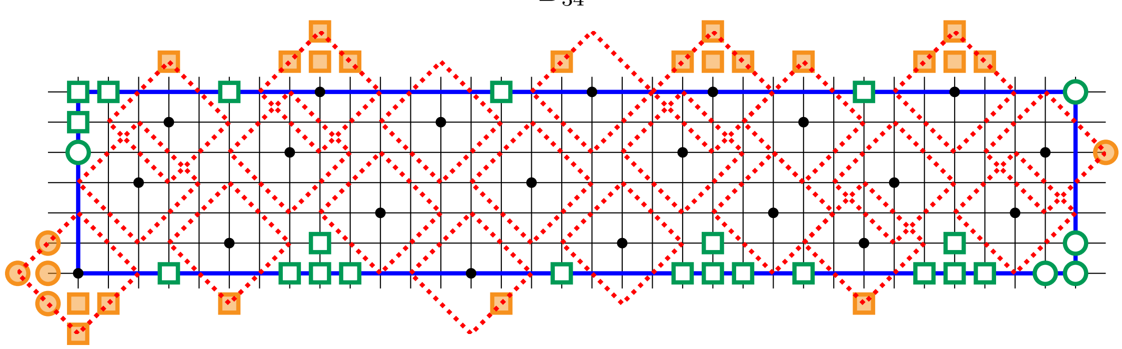


Figure 4.6.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_7 \square C_n$.

Proposition 4.6.1 improves the leading term given by the bound in Proposition 3.7.1 from $18\lfloor n/14 \rfloor$ to $42\lfloor n/35 \rfloor$. Proposition 4.6.1 improves the bound given by Proposition 3.20.1 by two when $n_{35} \in \{0,6,7,8,11,13,16,17,18,21,22,26,27,28,31,32,33\}$, and by one otherwise. The values of Proposition 4.6.1 are optimal for all $n \le 48$ (as found by computation) except for cases given in Table 4.6.3. Theorem 5.2.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{36.7}{35}n$.

Conjecture 4.6.1. *For $n \ge 7$, $\left\lceil \frac{42n}{35} \right\rceil \le \gamma_{b,2}(C_7 \Box C_n)$.*

n	Optimal	Bound
36	44	45
37	45	46
38	46	47
47	57	58

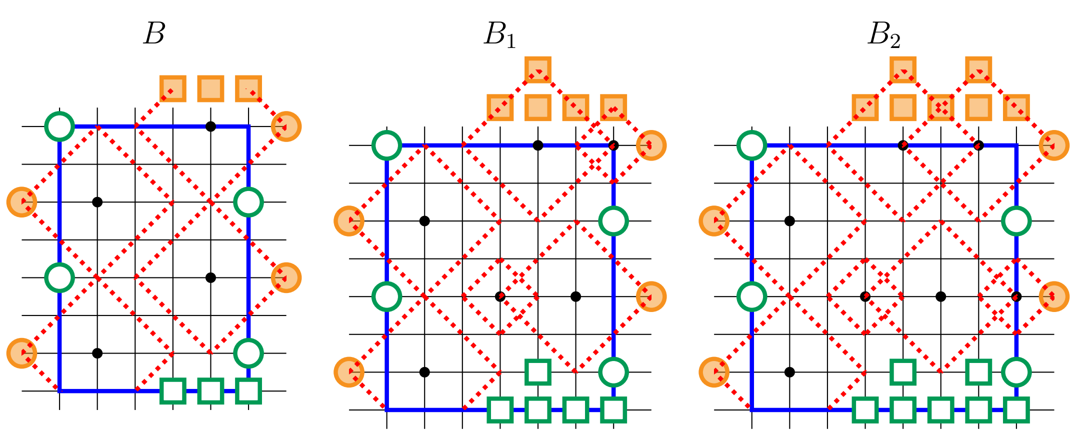
Table 4.6.3: Known values of n for which the bound in Proposition 4.6.1 is not optimal.

4.7 $C_8 \Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_8 \Box C_n$ for all $n \ge 8$. Figure 4.7.1 contains the tiles used in the broadcast constructions described in Table 4.7.1.

Proposition 4.7.1. *For $n \ge 8$,*

$$\gamma_{b,2}(C_8 \Box C_n) \le 8\left\lfloor \frac{n}{6} \right\rfloor + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 6, \\ 2 & \text{for } n \equiv 1 \pmod 6, \\ 4 & \text{for } n \equiv 2 \pmod 6, \\ 6 & \text{for } n \equiv 3 \text{ or } 4 \pmod 6, \text{ and} \\ 8 & \text{for } n \equiv 5 \pmod 6. \end{cases}$$



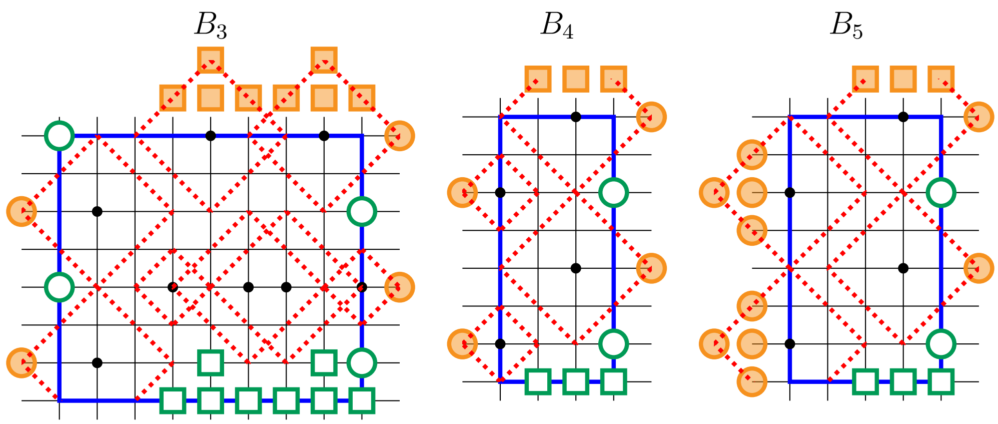


Figure 4.7.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_8\Box C_n$.

	Construction	Cost
$n \equiv 0 \pmod 6$:	$[B]^{\frac{n}{6}}$	$8\left(\frac{n}{6}\right)$
$n \equiv 1 \pmod 6$:	$[B]^{\frac{n-7}{6}} B_1$	$8\left(\frac{n-7}{6}\right) + 10 = 8\left(\frac{n-1}{6}\right) + 2$
$n \equiv 2 \pmod 6$:	$[B]^{\frac{n-8}{6}} B_2$	$8\left(\frac{n-8}{6}\right) + 12 = 8\left(\frac{n-2}{6}\right) + 4$
$n \equiv 3 \pmod 6$:	$[B]^{\frac{n-9}{6}} B_3$	$8\left(\frac{n-9}{6}\right) + 14 = 8\left(\frac{n-3}{6}\right) + 6$
$n \equiv 4 \pmod 6$:	$[B]^{\frac{n-4}{6}} B_4$	$8\left(\frac{n-4}{6}\right) + 6$
$n \equiv 5 \pmod 6$:	$[B]^{\frac{n-5}{6}} B_5$	$8\left(\frac{n-5}{6}\right) + 8$

Table 4.7.1: Constructions of 2-limited dominating broadcasts on $C_8\Box C_n$.

Proposition 4.7.1 improves the leading term given by the bound in Proposition 3.8.1 from $32\lfloor n/22\rfloor$ to $8\lfloor n/6\rfloor$. Proposition 4.7.1 improves the bound given by Proposition 3.21.1 by two when $n \equiv 0 \pmod 6$, by one when $n \equiv 1$ and $4 \pmod 6$, and the otherwise the bounds are the same. The values of Proposition 4.7.1 are optimal for all $n \leq 50$ (as found by computation). Theorem 5.2.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{7.3}{6}n$.

Conjecture 4.7.1. *The bounds given by Proposition 4.7.1 are tight.*

4.8 $C_9\Box C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_9\Box C_n$ for all $n \geq 9$. Figure 4.8.1 contains the tiles used in the broadcast constructions described in Table 4.8.2.

Proposition 4.8.1. *For $n \geq 9$,*

$$\gamma_{b,2}\left(C_9 \Box C_n\right) \leq 16\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 4.8.1.

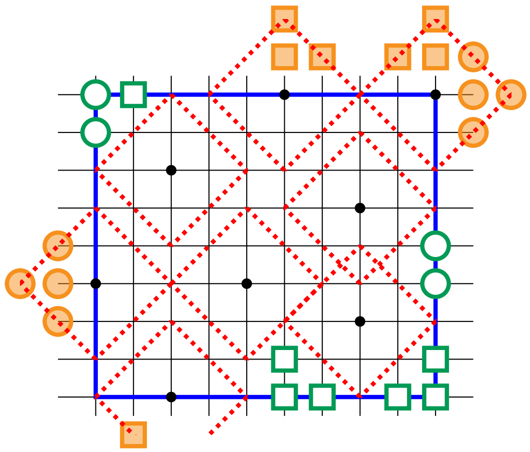
	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	0	2	4	5	7	9	10	12	13	15

Table 4.8.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}\left(C_9 \Box C_n\right)$ stated in Proposition 4.8.1.

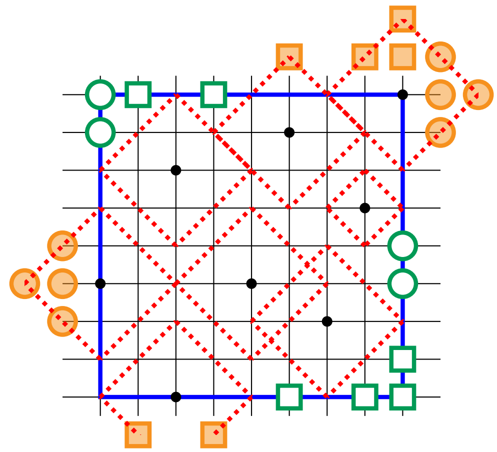
	Construction	Cost
$n \equiv 0 \pmod{10}$:	$[B]^{\frac{n}{10}}$	$16\left(\frac{n}{10}\right)$
$n \equiv 1 \pmod{10}$:	$[B]^{\frac{n-11}{10}}\ B_1$	$16\left(\frac{n-11}{10}\right) + 18 = 16\left(\frac{n-1}{10}\right) + 2$
$n \equiv 2 \pmod{10}$:	See $P_9 \Box C_n$	$16\left(\frac{n-2}{10}\right) + 4$
$n \equiv 3 \pmod{10}$:	$[B]^{\frac{n-13}{10}}\ B_3$	$16\left(\frac{n-13}{10}\right) + 21 = 16\left(\frac{n-3}{10}\right) + 5$
$n \equiv 4 \pmod{10}$:	See $P_9 \Box C_n$	$16\left(\frac{n-4}{10}\right) + 7$
$n \equiv 5 \pmod{10}$:	$[B]^{\frac{n-5}{10}}\ B_5$	$16\left(\frac{n-5}{10}\right) + 9$
$n \equiv 6 \pmod{10}$:	See $P_9 \Box C_n$	$16\left(\frac{n-6}{10}\right) + 10$
$n \equiv 7 \pmod{10}$:	$[B]^{\frac{n-7}{10}}\ B_7$	$16\left(\frac{n-7}{10}\right) + 12$
$n \equiv 8 \pmod{10}$:	$[B]^{\frac{n-8}{10}}\ B_8$	$16\left(\frac{n-19}{10}\right) 29 = 16\left(\frac{n-8}{10}\right) + 13$
$n \equiv 9 \pmod{10}$:	$[B]^{\frac{n-9}{10}}\ B_9$	$16\left(\frac{n-9}{10}\right) + 15$

Table 4.8.2: Constructions of 2-limited dominating broadcasts on $C_9 \Box C_n$.

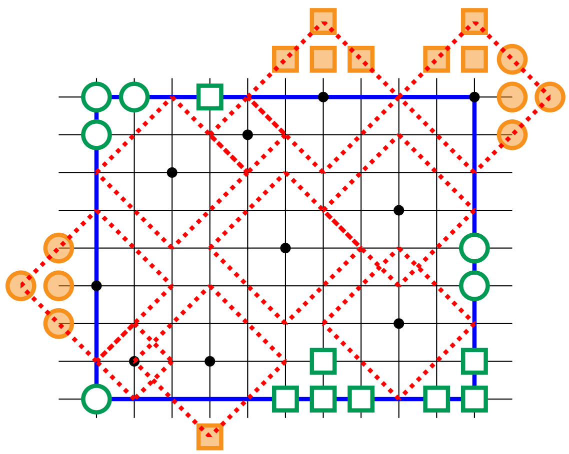
B



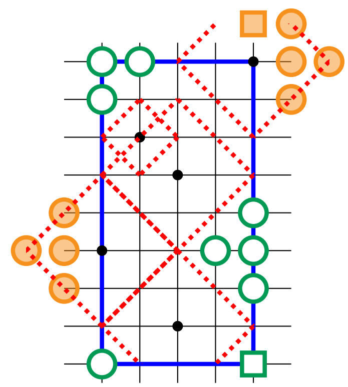
B_9



B_1



B_5



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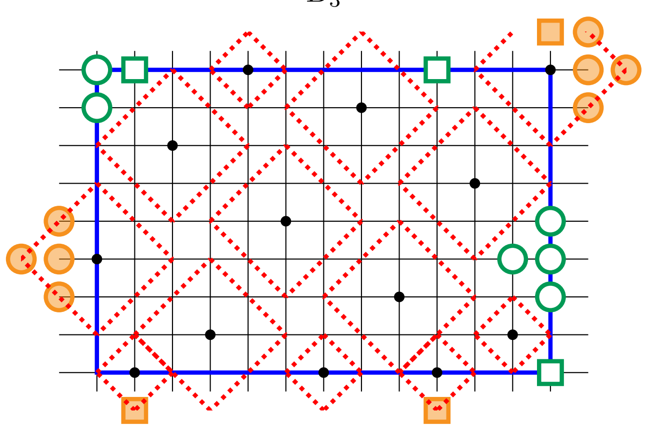
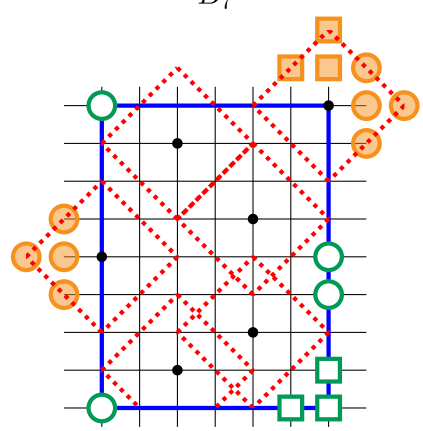


B_7



B_8

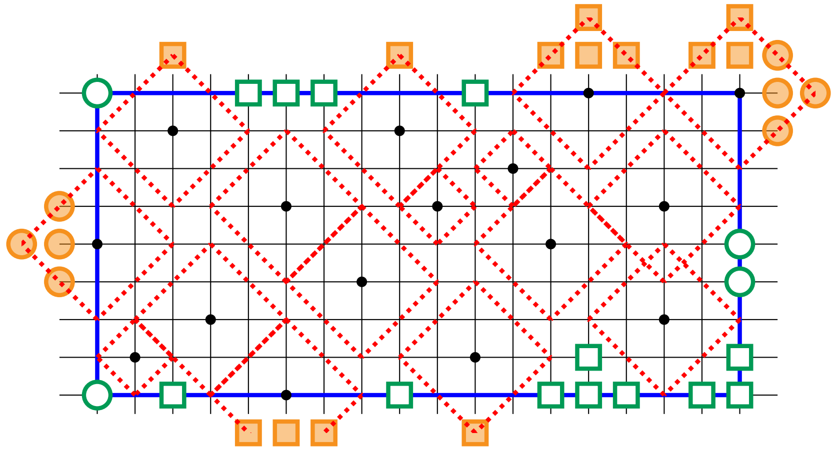


Figure 4.8.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_9 \square C_n$.

Proposition 4.8.1 improves the bound given by Proposition 3.9.1 by one when $n_{10} \in \{1, 3, 5, 7, 8, 9\}$, and otherwise the bounds are the same. Proposition 4.8.1 improves the bound given by Proposition 3.22.1 by two when $n_{10} \in \{0, 8\}$, and by one otherwise. The values of Proposition 4.8.1 are optimal for all $n \leq 30$ (as found by computation) except for cases given in Table 4.8.3. Theorem 5.2.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{13.8}{10}n$.

n	Optimal	Bound
15	24	25
25	40	41

Table 4.8.3: Known values of n for which the bound in Proposition 4.8.1 is not optimal.

4.9 $C_{10} \square C_n$

This section describes the constructions and associated costs of 2-limited dominating broadcasts on $C_{10} \square C_n$ for all $n \geq 10$. Figure 4.9.1 contains the tiles used in the broadcast constructions described in Table 4.9.2.

Proposition 4.9.1. *For $n \geq 10$,*

$$\gamma_{b,2}\left(C_{10} \square C_n\right) \leq 16 \left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$$

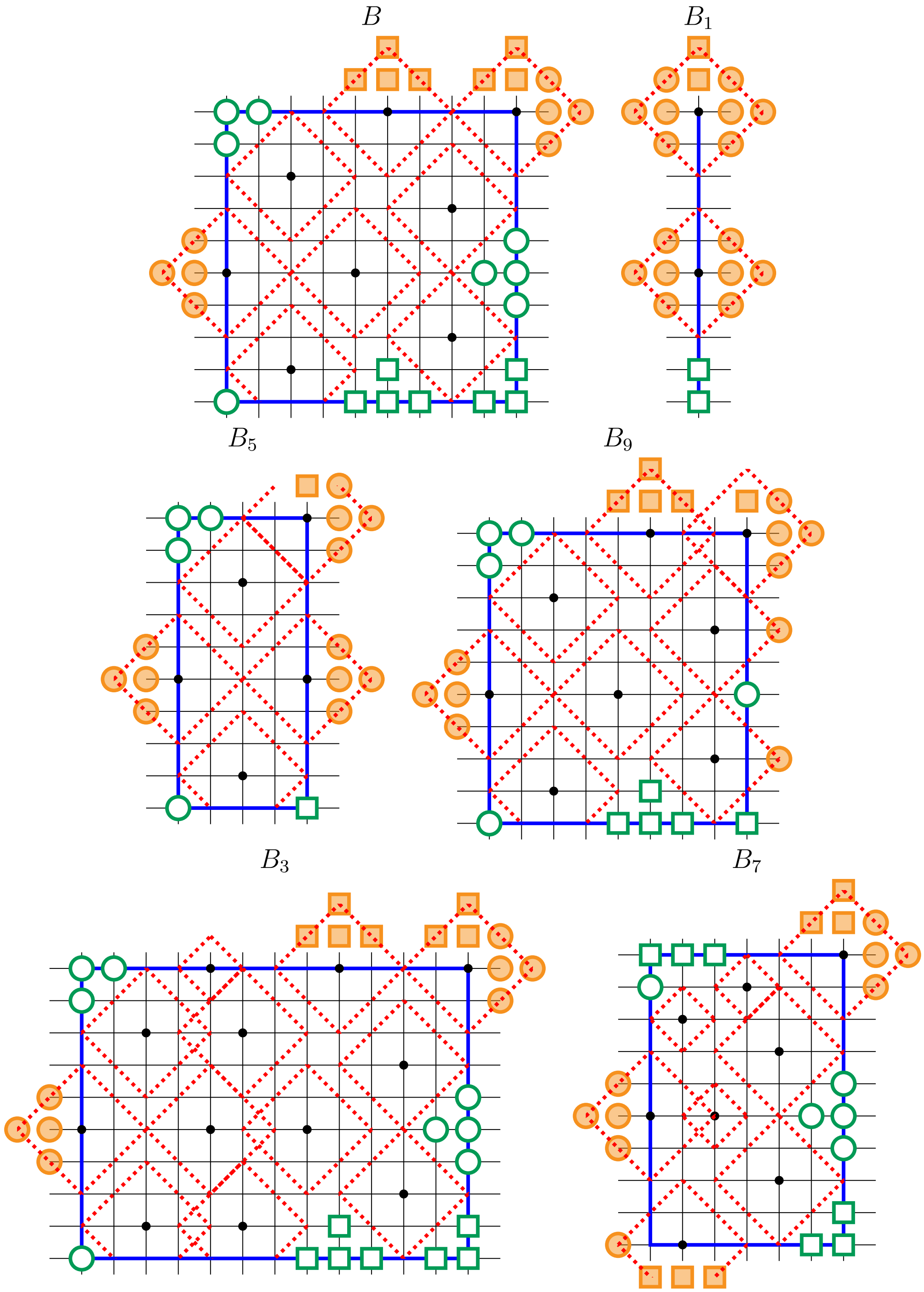
where $c(n_{10})$ is dependant upon the least residue n_{10} of n modulo 10 and given in Table 4.9.1.

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
Cost:	0	4	5	7	7	10	11	13	14	16

Table 4.9.1: Values of $c(n_{10})$ for the upper bound for $\gamma_{b,2}(C_{10}\Box C_m)$ stated in Proposition 4.9.1.

	Construction	Cost
$n \equiv 0 \pmod{10}$:	$[B]^{\frac{n}{10}}$	$16\left(\frac{n}{10}\right)$
$n \equiv 1 \pmod{10}$:	$[B]^{\frac{n-1}{10}}\ B_1$	$16\left(\frac{n-1}{10}\right)+4$
$n \equiv 2 \pmod{10}$:	$[B]^{\frac{n-12}{10}}\ B_2$	$16\left(\frac{n-12}{10}\right)+21=16\left(\frac{n-2}{10}\right)+5$
$n \equiv 3 \pmod{10}$:	$[B]^{\frac{n-13}{10}}\ B_3$	$16\left(\frac{n-13}{10}\right)+23=16\left(\frac{n-3}{10}\right)+7$
$n \equiv 4 \pmod{10}$:	$[B]^{\frac{n-14}{10}}\ B_4$	$16\left(\frac{n-14}{10}\right)+23=16\left(\frac{n-4}{10}\right)+7$
$n \equiv 5 \pmod{10}$:	$[B]^{\frac{n-5}{10}}\ B_5$	$16\left(\frac{n-5}{10}\right)+10$
$n \equiv 6 \pmod{10}$	$[B]^{\frac{n-6}{10}}\ B_6$	$16\left(\frac{n-6}{10}\right)+11$
$n \equiv 7 \pmod{10}$:	$[B]^{\frac{n-7}{10}}\ B_7$	$16\left(\frac{n-7}{10}\right)+13$
$n \equiv 8 \pmod{10}$:	$[B]^{\frac{n-8}{10}}\ B_8$	$16\left(\frac{n-8}{10}\right)+14$
$n \equiv 9 \pmod{10}$:	$[B]^{\frac{n-9}{10}}\ B_9$	$16\left(\frac{n-9}{10}\right)+16$

Table 4.9.2: Constructions of 2-limited dominating broadcasts on $C_{10}\Box C_n$.


B
B_1
B_5
B_9
B_3
B_7

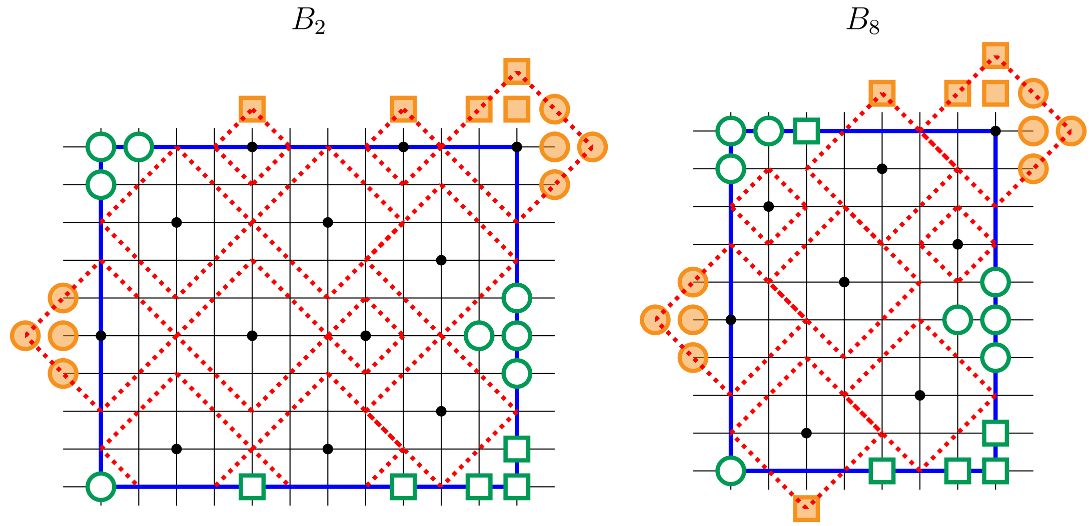


Figure 4.9.1: Tiles used in constructions of 2-limited dominating broadcasts on $C_{10}\square C_n$.

Proposition 4.9.1 improves the leading term given by the bound in Proposition 3.10.1 from $32\lfloor n/18\rfloor$ to $16\lfloor n/10\rfloor$. Proposition 4.9.1 improves the bound given by Proposition 3.23.1 by two when $n_{10}=0$, by one when $n_{10}\in\{4,5,8\}$, and otherwise the bounds are the same. The values of Proposition 4.9.1 are optimal for all $n\le 61$ (as found by computation). Theorem 5.2.1 gives a lower bound for this case that is bounded below by the simpler function $\frac{15.3}{10}n$.

Conjecture 4.9.1. *The bounds given by Proposition 4.9.1 are tight.*

4.10 $C_{11\le m\le 25}\square C_{n\ge m}$

Sections 4.2 through 4.9 gave upper bounds for $\gamma_{b,2}(C_m\square C_n)$ for $3\le m\le 10$ and all $n\ge 3$. This section provides upper bounds for $\gamma_{b,2}(C_m\square C_n)$ for $11\le m\le 25$ and all $n\ge m$. The bounds are derived in the same fashion as that of Section 3.13. That is, for each fixed $11\le m\le 25$, we find an optimal 2-limited dominating broadcast on $C_m\square C_{13}$ by computation then use this broadcast to tile $C_m\square C_{n\ge 13}$ and complete the remainder of the broadcast based on the least residue of n modulo 13. This methodology establishes the following Theorem which, with Sections 4.2 through 4.9, establishes upper bounds for $\gamma_{b,2}(C_m\square C_n)$ for all $3\le m\le 25$ and $n\ge 3$.

Theorem 4.10.1. *If $11\le m\le 25$ and $n\ge m$ then*

$$\gamma_{b,2}(C_m\square C_n)\le b(m)\left\lfloor\frac{n}{13}\right\rfloor+c(m,n_{13})$$

where $b(m)$ corresponds with the values in Table 4.10.1 and $c(m,n_{13})$ corresponds with the value of Table 4.10.2 where n_{13} is the least residue of n modulo 13.

m	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$b(m)$	24	26	26	31	33	36	36	39	41	42	45	48	49	50	52

Table 4.10.1: Value of $b(m)$ in the upper bound for $\gamma_{b,2}(P_m \Box C_n)$ for $11 \leq m \leq 25$ and, equivalently, the cost of an optimal 2-limited dominating broadcast on $C_m \Box C_{13}$ for $11 \leq m \leq 25$.

		Least residue n_{13} of n modulo 13												
		0	1	2	3	4	5	6	7	8	9	10	11	12
m	11	0	5	6	8	9	11	13	14	16	18	20	21	24
	12	0	4	6	8	10	11	14	16	16	20	22	24	26
	13	0	5	7	10	10	13	15	16	19	21	23	24	26
	14	0	5	8	10	11	14	16	18	21	23	25	27	30
	15	0	6	8	11	12	15	17	19	22	25	27	29	32
	16	0	6	8	12	12	16	18	20	22	26	28	30	33
	17	0	7	9	13	13	17	19	21	25	28	30	32	35
	18	0	7	9	13	13	18	21	23	26	29	32	35	36
	19	0	7	10	14	14	19	22	24	28	31	34	36	40
	20	0	8	11	15	16	20	23	25	29	33	35	37	42
	21	0	8	11	15	15	21	23	25	30	34	36	39	42
	22	0	9	12	16	18	22	25	27	32	36	39	41	46
	23	0	9	12	17	17	23	26	29	33	37	39	43	48
	24	0	9	13	18	19	24	27	30	34	39	43	45	50
	25	0	9	12	18	19	25	28	32	34	40	45	47	51

Table 4.10.2: Values of $c(m, n_{13})$ for the upper bound for $\gamma_{b,2}(C_m \Box C_n)$ for $11 \leq m \leq 25$ and $n \geq m$.

Proof. For each $11 \leq m \leq 25$, $\gamma_{b,2}(C_m \Box C_{13})$ is equal to $b(m)$ as found in Table 4.10.1. These values are easily solved via computation. Fix $11 \leq m \leq 25$, $n \geq 13$, and let B be an optimal 2-limited dominating broadcast of $C_m \Box C_{13}$ of cost equal to that given in Table 4.10.1. For instance, if $m = 16$ then a possible form of B is the tile in Figure 4.10.1. Let n_{13} be the least residue of n modulo 13. Form a 2-limited broadcast on $C_m \Box C_n$ by the sequence of broadcast tiles $[B]^{(n-n_{13})/13}$. This broadcast is dominating if and only if $n_{13} = 0$. Assuming $n_{13} \neq 0$, to make the broadcast dominating, it suffices to determine the additional broadcast vertices required to extend the broadcast given by $[B]^{(n-n_{13})/13}$ to dominate the remaining vertices of $C_m \Box C_n$. This reduces to determining the additional broadcast vertices required to extend the broadcast given by B to dominate $C_m \Box C_{13+n_{13}}$. For each $11 \leq m \leq 22$ and each least residue of n modulo 13, the costs of these additional broadcast vertices used to extend the broadcast given by B to dominate $C_m \Box C_{13+n_{13}}$ are given in Table 4.10.2. It therefore follows that for

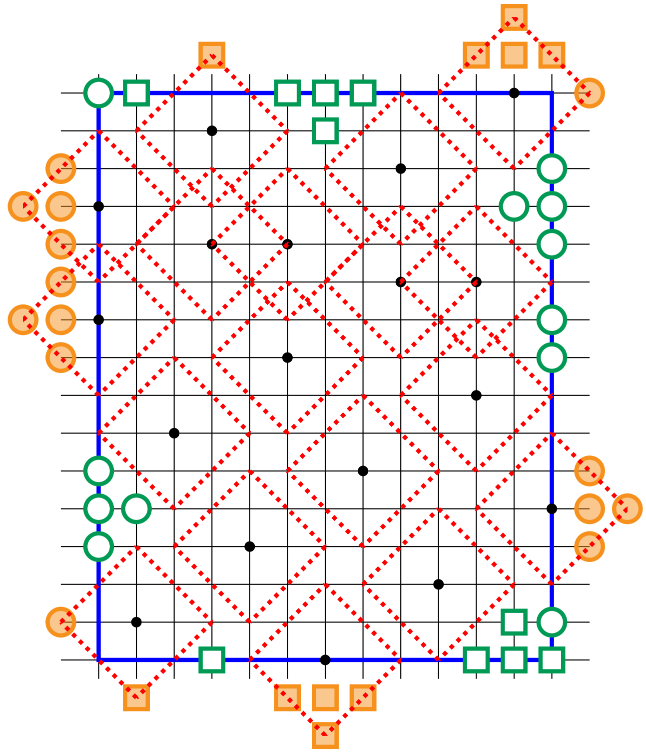


Figure 4.10.1: An optimal 2-limited dominating broadcast of $C_{16}\Box C_{13}$.

all $11 \leq m \leq 25$ and $n \geq 13$,

$$\gamma_{b,2}\left(C_m \Box C_n\right) \leq b(m)\left\lfloor \frac{n}{13} \right\rfloor + c(m, n_{13})$$

To complete the claim it suffices to consider $\gamma_{b,2}\left(C_{11}\Box C_{11}\right)$, $\gamma_{b,2}\left(C_{11}\Box C_{12}\right)$, and $\gamma_{b,2}\left(C_{12}\Box C_{12}\right)$. These are easily solved via computation to be 21, 24, and 24, respectively. These values for $\gamma_{b,2}\left(C_{11}\Box C_{11}\right)$ and $\gamma_{b,2}\left(C_{11}\Box C_{12}\right)$ are tight with that of Table 4.10.1 and the bound of Table 4.10.1 holds for $\gamma_{b,2}\left(C_{12}\Box C_{12}\right)$ as $24 \leq 26$. $\Box$

We conclude with an example of our construction for $C_{16}\Box C_n$.

Example 4.10.2. If $m = 16$, B is the tile in Figure 4.10.1, and $n_{13} = 5$. The additional broadcast vertices required to extend the broadcast given by B to dominate $C_{16}\Box C_{18}$ correspond with the vertices at the centre of the shaded broadcast region given in Figure 4.10.2. The total cost of this broadcast is $36 + 16 = 52$. Note that this construction corresponds with a broadcast cost one better than that of $P_{16}\Box C_{18}$ as found in Example 3.13.2. This construction easily generalizes. Given $C_{16}\Box C_{n\geq m}$ such that $n \equiv 5 \pmod{13}$, there exists a 2-limited dominating broadcast on $C_{16}\Box C_{n\geq m}$ formed by starting with $[B]^{\left\lfloor \frac{n}{13} \right\rfloor}$ and then adding the additional broadcast vertices given in Figure 4.10.2. The cost of such a broadcast is $36\left\lfloor \frac{n}{13} \right\rfloor + 16$.

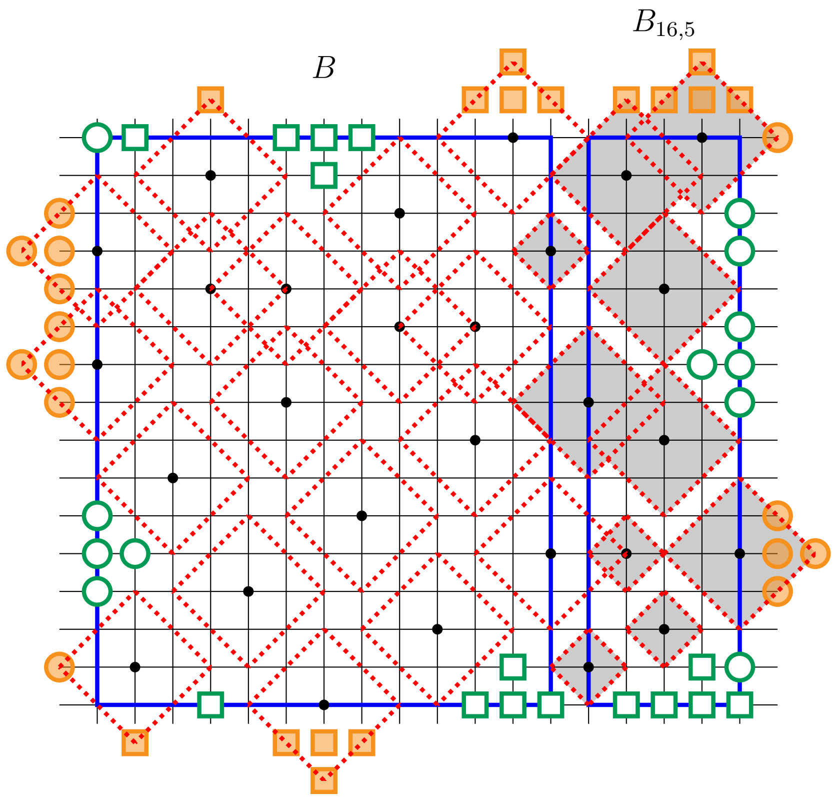


Figure 4.10.2: A minimum cost 2-limited dominating broadcast of $C_{16} \square C_{18}$ given fixed broadcast tile B.

4.11 $C_{m \geq 26} \square C_{n \geq 26}$

To finish establishing upper bounds for the 2-limited dominating broadcast number for the Cartesian products of two cycles, it suffices to consider $m, n \geq 26$. This section completes these upper bounds by presenting generalized tilings of $C_m \square C_n$ for all $m, n \geq 26$. We provide an intuition of our methodology and then state the resulting theorem.

Fix $m, n \geq 26$ and let m_{13} and n_{13} be the least residue of m and n modulo 13, respectively. Partition $C_m \square C_n$ into G_1, G_2, G_3, G_4, G_5, and G_6 as depicted in Figure 4.11.1. A concrete example of such a partition is given in Figure 4.11.7 for $m = 30$ and $n = 29$. Observe the following facts about this partition.

1. The region G_1, for large m and n, occupies the majority of the "center" of $C_m \square C_n$ and can be partitioned into

$$\frac{(m - 13 - m_{13})(n - 13 - n_{13})}{13^2}$$

13×13 blocks.

2. The regions G_4 and G_5 occupy the majority of the "border" of $C_m \square C_n$ and, with

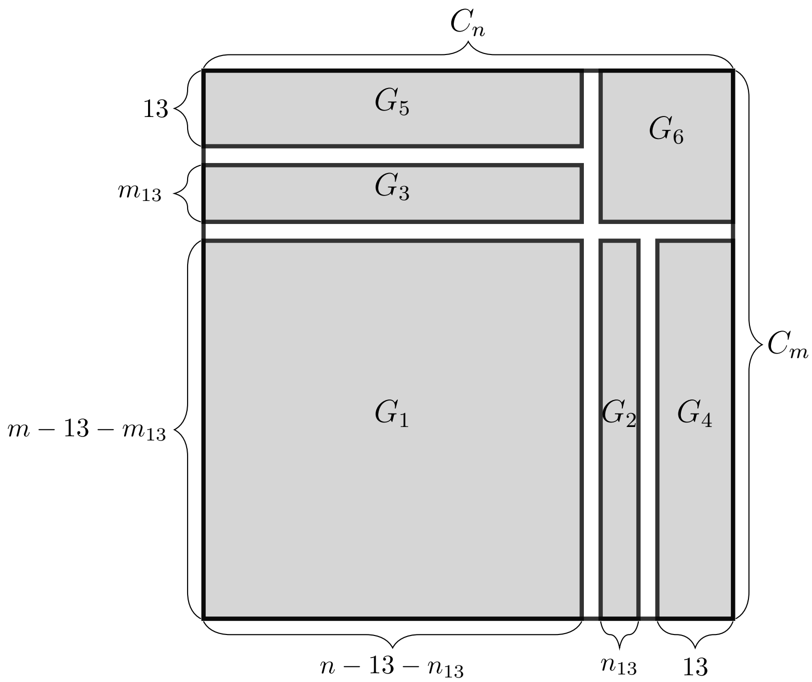


Figure 4.11.1: $C_m \square C_n$ partitioned into G_1, G_2, G_3, G_4, G_5 and G_6.

a fixed width and height of 13, respectively, can be partitioned into

$$\frac{m-13-m_{13}}{13} \quad \text{and} \quad \frac{n-13-n_{13}}{13}$$

13×13 blocks, respectively.

3. The regions G_2 and G_3 occupy bounded regions between the "center" and "border" of $C_m \square C_n$ and, with a fixed width and height of n_{13} and m_{13}, respectively, can be partitioned into

$$\frac{m-13-m_{13}}{13} \quad \text{and} \quad \frac{n-13-n_{13}}{13}$$

$13 \times n_{13}$ and $m_{13} \times 13$ blocks, respectively.

4. The region G_6 occupies the "corner" of $C_m \square C_n$ with a size of $(m_{13}+13)(n_{13}+13)$.

The ability to partition each of G_1, G_2, G_3, G_4, G_5, and G_6 into tiles is the key to our construction. Specifically, these methods determine a general 13×13 tile to tile G_1 and, for each least residue of m and n modulo 13,

1. a $13 \times n_{13}$ tile to tile G_2,
2. 13×13 tiles to tile G_4 and G_5,

3. a $m_{13} \times 13$ tile to tile G_3,

4. a $(m_{13} + 13)\,(n_{13} + 13)$ tile for G_6, and

5. a collection of additional broadcast vertices to be added to G_1, G_2, G_3, G_4, G_5, and G_6 to complete the broadcast.

Once combined, the tiles and the additional broadcast vertices create 2-limited dominating broadcast on $C_m \Box C_n$ for all $m, n \geq 13$. Figure 4.11.2 depicts how these tiles are "glued" together to tile $C_m \Box C_n$. These constructions establish Theorem 4.11.1

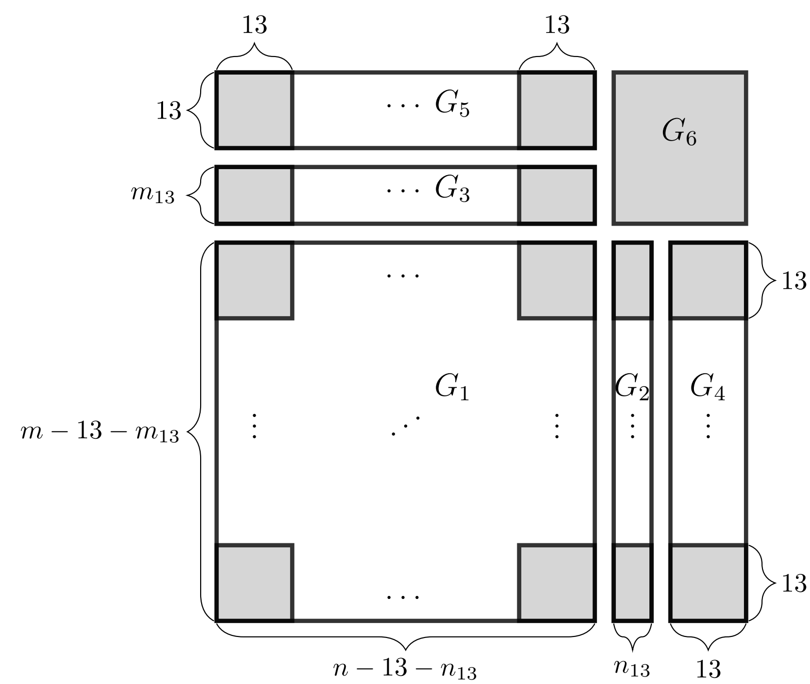


Figure 4.11.2: Tiling of G_1, G_2, G_3, G_4, G_5 and G_6 to construct a 2-limited dominating broadcast for $C_m \Box C_n$

Theorem 4.11.1. *If $m, n \geq 26$ then*

$$\gamma_{b,2}\left(C_m \Box C_n\right) \leq 2\left(\frac{mn}{13}\right) + \frac{b(m) + b(n)}{13} - c(m_{13}, n_{13})$$

where

$$b(x) = \begin{cases} 0 & \text{if } 0 \equiv x \pmod{13}, \\ 4x & \text{if } 3 \equiv x \pmod{13}, \\ 2x & \text{if } 4, 7, 11, \text{or } 12 \equiv x \pmod{13}, \text{ and} \\ 3x & \text{if } 1, 2, 5, 6, 8, 9, \text{or } 10 \equiv x \pmod{13} \end{cases}$$

and $c(m_{13}, n_{13})$ corresponds with the values given in Table 4.11.1 where m_{13} and n_{13} are the least residues of m and n modulo 13, respectively.

Least residue m_{13} of m modulo 13	Least residue n_{13} of n modulo 13												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	21	26	58	22	15	33	24	30	35	40	18	23
2	0	26	20	55	19	15	35	14	23	43	37	16	10
3	0	71	55	81	33	46	82	37	63	47	70	51	35
4	0	22	19	46	35	36	59	13	27	24	47	14	11
5	0	28	28	59	23	54	28	10	28	54	41	10	23
6	0	33	48	82	46	28	30	26	47	49	64	34	36
7	0	11	14	37	13	10	26	9	32	22	25	21	-2
8	0	17	23	50	27	28	34	32	33	26	45	30	23
9	0	35	43	60	37	41	49	22	52	34	68	28	36
10	0	27	37	70	47	41	51	25	58	42	65	26	10
11	0	18	16	51	14	10	34	21	43	28	26	26	11
12	0	23	23	35	24	36	23	-2	75	23	23	11	-2

Table 4.11.1: Constant term $c(m_{13}, n_{13})$ in the upper bound for $\gamma_{b,2}(C_m \square C_n)$ for $m, n \geq 26$.

Proof. Proof of the general case will follow from the cases that have $26 \leq m, n \leq 38$. Fix $26 \leq m, n \leq 38$ and let m_{13} and n_{13} be the least residues of m and n modulo 13, respectively. Define G_1, G_2, G_3, G_4, G_5, and G_6 to be the blocks depicted in Figure 4.11.1. The block G_1 has dimensions 13×13. Let G_1 be tiled by the tile in Figure 4.11.3. An astute reader may notice that said tile is one of the 13 possible 13×13 cross-sections of the 2-limited dominating broadcast of the Cartesian plane discussed in Section 2.13 (for $\ell = 0$). The block G_2 has dimensions $13 \times n_{13}$. Let G_2 be tiled by the first n_{13} left most columns of the tile given in Figure 4.11.3. This choice of tile for G_2 ensures that the tiling of G_1 and G_2 "fit" together. Similarly, as G_3 has dimensions $m_{13} \times 13$, let G_3 be tiled by the first m_{13} bottom rows of the tile given in Figure 4.11.3. An example of this tiling of G_1, G_2, and G_3 is given in Figure 4.11.4 for $m = 29$ and $n = 28$.

The second step is to determine the appropriate tiles for G_4 and G_5 given the tiling of G_1, G_2, and G_3. We describe the approach for G_4 as the methodology is the same for G_5. Consider $C_{13} \square C_n$ tiled by G_1, G_2, and G_4. For example, Figure 4.11.5 depicts $C_{13} \square C_{29}$ partitioned by G_1, G_2, and G_4. To complete the broadcast on $C_{13} \square C_n$, like in Section 4.10, it reduces to determine the additional broadcast vertices required to extend the broadcast given by the tilings of G_1 and G_2 to dominate $C_{13} \square C_n$. This is easily solved by computation. For each least residue of n modulo 13, the minimum costs of the additional broadcast vertices used to extend the broadcast of the tiling of G_1 and G_2 to dominate $C_{13} \square C_n$ are given in Table 4.11.2 for n_{13}. For example, Figure 4.11.6 depicts a 2-limited dominating broadcast of $C_{13} \square C_{29}$ given by this tiling of G_1,

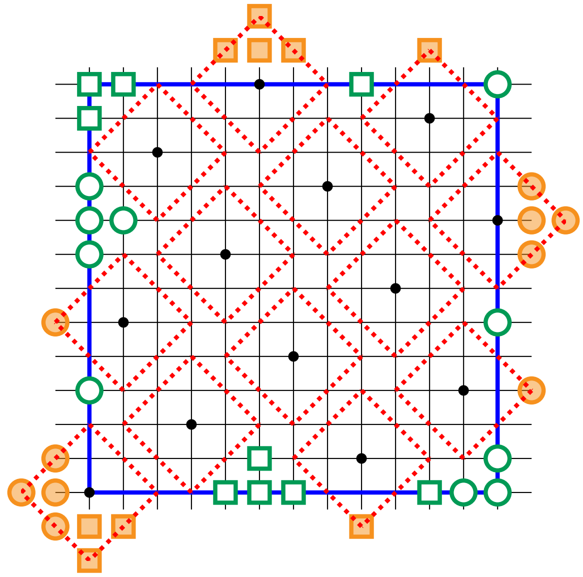


Figure 4.11.3: 13×13 tile used repeatedly in G_1.

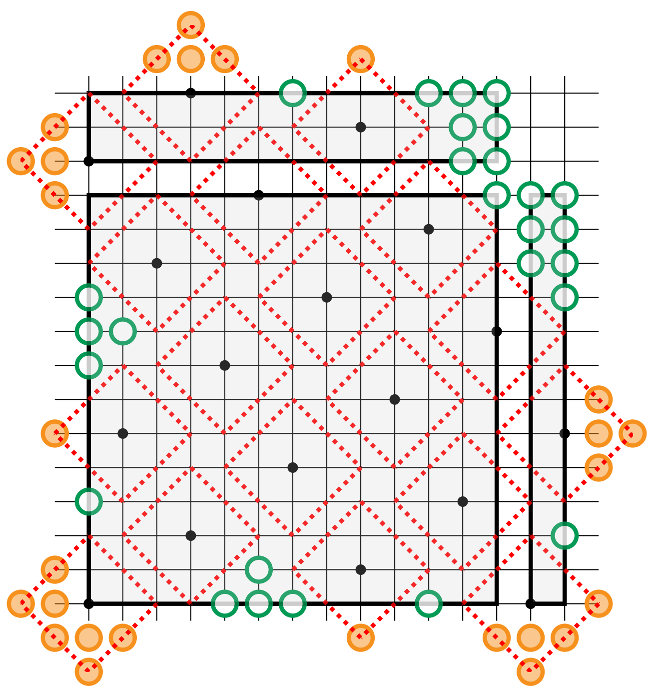


Figure 4.11.4: Tiling of G_1, G_2, and G_3 for $m = 29$ and $n = 28$.

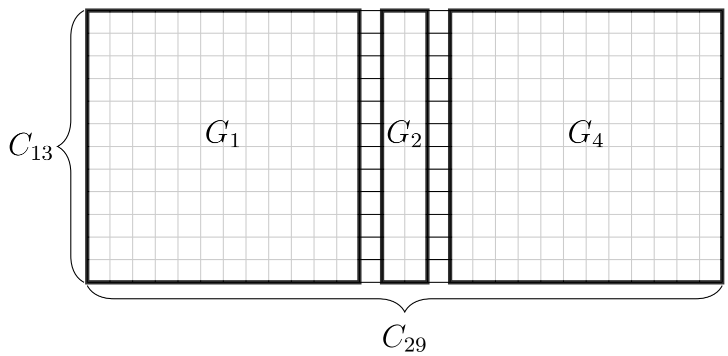


Figure 4.11.5: $C_{13}\square C_{29}$ partitioned by G_1, G_2, and G_4.

G_2, and G_5.

m_{13} or n_{13}	0	1	2	3	4	5	6	7	8	9	10	11	12
Cost:	26	29	29	30	28	29	29	28	29	29	29	28	28

Table 4.11.2: The values of the minimum costs of the additional broadcasts required to dominate $C_{13}\square C_n$ given the tilings of G_1 and G_2 and the values of the minimum costs of the additional broadcasts required to dominate $C_m\square C_{13}$ given the tilings of G_1 and G_3.

The same methodology when applied to G_5 yields the same associated cost as given in Table 4.11.2 for m_{13}. Add the additional broadcast vertices to G_1, G_2, G_3, G_4, and G_5 as determined by Table 4.11.2. The broadcast formed by this tiling of G_1, G_2, G_3, G_4, and G_5 forms a partial broadcast on $C_m\square C_n$. To complete the broadcast it suffices to determine the additional broadcast vertices required to extend the broadcast given by this tiling of G_1, G_2, G_3, G_4, and G_5 to dominate $C_m\square C_n$. For each least residue of m modulo 13 and n modulo 13 the cost of the additional broadcast vertices required to extend the broadcast of G_1, G_2, G_3, G_4 and G_5 is given in Table 4.11.3.

Let G_1, G_2, G_3, G_4, G_5 and G_6 contain the additional broadcast vertices associated with the value of Table 4.11.3 given n_{13} and m_{13}. For example, Figure 4.11.7 gives a 2-limited dominating broadcast of $C_{30}\square C_{29}$ given by this methodology of cost 144. In general, the cost of a 2-limited dominating broadcast given by this construction is

$$2\left(\frac{mn}{13}\right) - 2\,(m+n) - 2\left(\frac{m_{13}n_{13}}{13}\right) + 26 + b(m_{13}) + b(n_{13}) + c(m_{13}, n_{13}) \qquad (4.1)$$

where $b(x)$ corresponds with the border cost associated with x in Table 4.11.2 and $c(m_{13}, n_{13})$ corresponds with the corner cost associated with m_{13} and n_{13} in Table 4.11.3.

This construction is easily generalized for all $m, n \geq 26$. Fix $m, n \geq 26$ and let m_{13} and n_{13} be the least residue of m and n modulo 13. Define G_1, G_2, G_3, G_4, G_5, and G_6 as described at the beginning of this section. Construct a 2-limited dominating

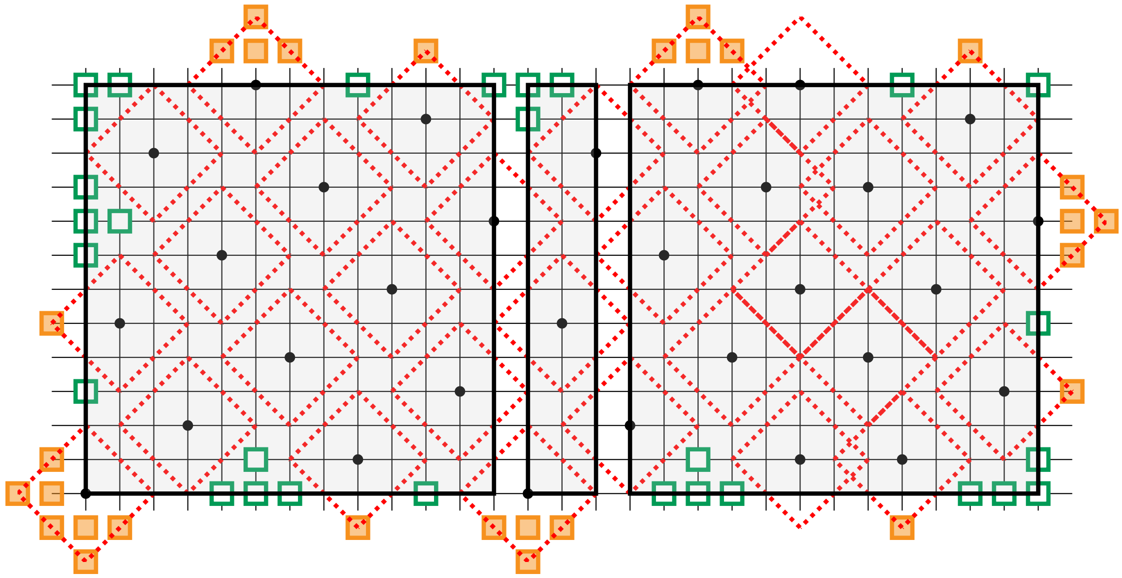


Figure 4.11.6: Least cost dominating broadcast of $C_m \square C_{13}$ given the vertices broadcasting at non-zero strength in G_1 and G_2.

Least residue m_{13} of m modulo 13	Least residue n_{13} of n modulo 13: 0	1	2	3	4	5	6	7	8	9	10	11	12
0	26	31	33	36	36	39	41	42	45	47	49	50	52
1	31	35	37	38	41	45	46	48	51	53	55	58	60
2	33	37	40	41	44	48	49	52	55	56	59	62	65
3	36	37	41	43	47	50	50	55	57	61	62	65	69
4	36	41	44	46	47	51	52	57	60	63	64	68	71
5	39	44	47	49	52	54	59	62	65	66	70	74	76
6	41	46	48	50	53	59	62	64	67	70	72	76	79
7	42	49	52	55	57	62	64	67	70	74	77	79	84
8	45	52	55	58	60	65	68	70	75	79	81	84	88
9	47	53	56	60	62	67	70	74	77	82	83	88	91
10	49	56	59	62	64	70	73	77	80	85	87	92	97
11	50	58	62	65	68	74	76	79	83	88	92	94	99
12	52	60	64	69	70	75	80	84	84	92	96	99	104

Table 4.11.3: Associated cost of the additional broadcast vertices required to extend the broadcast of G_1, G_2, G_3, G_4, and G_5 to dominate $C_m \square C_n$.

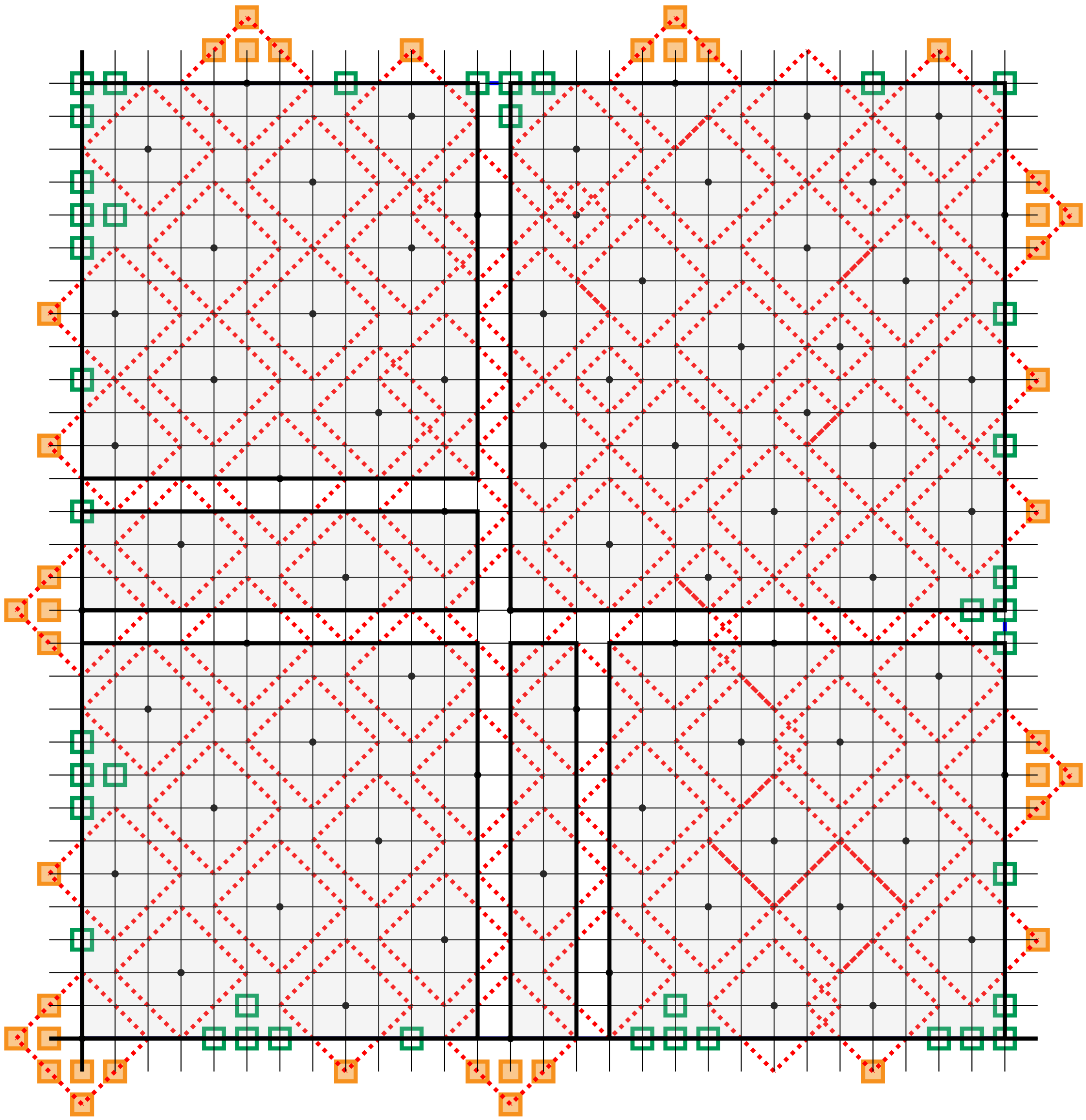


Figure 4.11.7: $C_{30} \square C_{29}$ partitioned into G_1, G_2, G_3, G_4, G_5 and G_6 and the 2-limited dominating broadcast solved for via our construction.

broadcast on $C_m \Box C_n$ via the follow procedure.

1. Tile G_1 with
$$\frac{(m-13-m_{13})(n-13-n_{13})}{13^2}$$
copies of the 13×13 tiles given in Figure 4.11.3.

2. Tile G_2 with
$$\frac{m-13-m_{13}}{13}$$
$13 \times n_{13}$ tiles of the first n_{13} left most columns of the tile in Figure 4.11.3.

3. Tile G_3 with
$$\frac{n-13-n_{13}}{13}$$
$m_{13}\times$ tiles of the first m_{13} bottom rows of the tile in Figure 4.11.3.

4. Add
$$\frac{m-13-m_{13}}{13} \quad \text{and} \quad \frac{n-13-n_{13}}{13}$$
additional sets of broadcast vertices corresponding with G_4 and G_5 as computed in Table 4.11.2 for n_{13} and m_{13}, respectively.

5. Add the remaining additional vertices G_1, G_2, G_3, G_4, G_5, and G_6 associated with the values of Table 4.11.3 given n_{13} and m_{13} to complete the broadcast.

The resulting broadcast is a 2-limited dominating broadcast on $C_m \Box C_n$. Applying equation 4.1, the cost of such a broadcast is

$$2\left(\frac{mn}{13}\right) - 2\left(m+n\right) - 2\left(\frac{m_{13}n_{13}}{13}\right) + 26 +$$
$$b(m_{13})\left(\frac{n-13-n_{13}}{13}\right) + b(n_{13})\left(\frac{m-13-m_{13}}{13}\right) + c(m,n).$$

The previous equation can be solved for each least residue of m and n modulo 13. This completes the proof of Theorem 4.11.1.

$\square$

Chapter 5
Multipacking

5.1 Introduction

Chapters 2, 3, and 4 established upper bounds for the 2-limited broadcast domination number of the Cartesian products of two paths, a path and a cycle, and two cycles. This chapter is devoted to determining corresponding lower bounds for the 2-limited broadcast domination number of these graphs. These bounds are obtained via linear programming duality by finding the lower bounds for the *fractional 2-limited multipacking number* of these graphs. All results, with the exception of those stated in Section 5.2, were found by using an exact LP solver [21, 20] as a part of the SoPlex distribution [19]. All computations in this chapter were run on the author's 2012 4GB MacBook Pro with a 2.5 GHz Intel Dual-Core i5 processor.

This chapter focuses on *fractional 2-limited multipackings*, the definition of which is recalled from Chapter 1. Define $mp_2(G)$ as the maximum cost of a fractional 2-limited multipacking on a graph G. For each vertex $i \in V(G)$, define the variable y_i. The fractional 2-limited multipacking packing number $mp_2(G)$ of a graph G is the cost of an optimal solution to the following LP.

$$
\begin{aligned}
\text{Maximize:} \quad & \sum_{i \in V(G)} y_i \\
\text{Subject to:} \quad & (1) \quad \sum_{i \in V(G) \text{ s.t. } d(i,j) \le 1} y_i \le 1, \quad \text{for each vertex } j \in V(G), \\
& (2) \quad \sum_{i \in V(G) \text{ s.t. } d(i,j) \le 2} y_i \le 2, \quad \text{for each vertex } j \in V(G), \text{ and} \\
& (3) \quad y_i \ge 0 \quad \text{for each vertex } j \in V(G).
\end{aligned}
\tag{5.1}
$$

If the above LP is interpreted as a 0-1 ILP, a formulation of the *2-limited multipacking number* of a graph is obtained.

As fractional 2-limited multipackings on a graph G do not lend themselves to visuals in the same way that 2-limited dominating broadcast on a graph do, this chapter will have far less figures that that of the previous chapters. Example 5.1.1 presents a fractional 2-limited multipacking on $C_4 \square C_{10}$ to aid the reader in their familiarity with multipackings.

Example 5.1.1. Consider $P_4 \square C_{10}$ with rows labelled from 1 to 4 and columns labelled from 1 to 10. Let $\mathbf{M}$ be the 4×10 matrix as show in Figure 5.1.1. For each $1 \leq i \leq 4$ and $1 \leq j \leq 10$, define the fractional 2-limited multipacking on $P_4 \square C_{10}$ by letting the variable $x_{i,j}$, corresponding with the vertex in row i and column j, be the value of the element in the ith row and jth column of $\mathbf{M}$. To check that this is a valid 2-limited multipacking, it suffices to check that, for each vertex, the sum of the weights within distances one and two are less than or equal to one and two, respectively. From this result we have that $8 \leq mp_2(P_4 \square C_{10})$.

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Figure 5.1.1: Matrix $\mathbf{M}$ used to establish an 2-limited multipacking on $P_4 \square C_{10}$.

As a prelude to the results in this chapter, observe that $\mathbf{M}$ can act as a "multipacking tile," similar to the broadcast tiles used in the previous chapters, on $P_4 \square C_{10}$. Let B be the "multipacking tile" as given by $\mathbf{M}$. The tiling $[B]^{\frac{n}{10}}$ gives a 2-limited multipacking on $P_4 \square C_n$ for all $n \equiv 0 \pmod{10}$. This construction establishes the following lower bound

$$8 \left\lfloor \frac{n}{10} \right\rfloor \leq mp_2(P_4 \square C_n)$$

for $n \geq 10$. As $mp_2(P_4 \square C_n) \leq \gamma_{b,2}(P_4 \square C_n)$, coupled with the bound given in Proposition 3.4.1, the above construction establishes

$$8 \left\lfloor \frac{n}{10} \right\rfloor \leq \gamma_{b,2}(P_4 \square C_n) \leq 8 \left\lfloor \frac{n}{10} \right\rfloor + \begin{cases} 0 & n \equiv 0 \pmod{10}, \\ 2 & n \equiv 1 \text{ or } 2 \pmod{10}, \\ 3 & n \equiv 3 \pmod{10}, \\ 4 & n \equiv 4 \pmod{10}, \\ 5 & n \equiv 5 \text{ or } 6 \pmod{10}, \\ 6 & n \equiv 7 \pmod{10}, \\ 7 & n \equiv 8 \pmod{10}, \text{ and} \\ 8 & n \equiv 9 \pmod{10}. \end{cases}$$

Observe that these bounds are tight for $n \equiv 0 \pmod{10}$. Given this, we conclude that

$$mp_2(P_4 \square C_n) = \gamma_{b,2}(P_4 \square C_n) = \frac{4n}{5}$$

for $n \equiv 0 \pmod{10}$. These results are given formally in Section 5.3.

Having provided motivation for using the fractional 2-limited multipacking number as a method to establish lower bounds for 2-limited broadcast domination number in Example 5.1.1, we now provide an outline of the structure for the remainder of the chapter. Section 5.2 establishes optimal values for $mp_2(C_m \square C_n)$ for all $m, n \geq 3$ by describing general fractional 2-limited multipackings on $C_m \square C_n$ and general fractional 2-limited dominating broadcasts on $C_m \square C_n$. Sections 5.3 and 5.4 establish lower bounds for $mp_2(P_m \square C_n)$ and $mp_2(P_m \square P_n)$, respectively, via general multipacking constructions.

5.2 Cartesian Products of Two Cycles

This section describes a general fractional 2-limited multipacking on $C_m \square C_n$ for $m, n \geq 3$. These results establish Theorem 5.2.1. The proof of Theorem 5.2.1 utilizes the LP relaxation of the 2-limited broadcast domination number of a graph. Denote $\gamma_{f,b,2}(G)$ as the *fractional* 2*-limited broadcast domination number* of a graph G. The relaxation of the ILP 1.1 defines $\gamma_{f,b,2}(G)$.

Theorem 5.2.1. *For $m, n \geq 3$,*

$$\frac{2mn}{13} = mp_2(C_m \square C_n).$$

Proof. Fix $m, n \geq 3$. Define y_i for $i \in V(C_m \square C_n)$ as in LP 5.1. Define the fractional 2-limited multipacking on $C_m \square C_n$ by $y_i = \frac{2}{13}$ for all $i \in V(C_m \square C_n)$. As $C_m \square C_n$ is vertex transitive, for each vertex $i \in V(C_m \square C_n)$, there are at most 5 and 13 vertices within distance 1 and 2, respectively, of i. To see that this is a valid fractional 2-limited multipacking observe that

$$(1) \quad \sum_{\substack{i \in V(G) \text{ s.t} \\ d(i,j) \leq 1}} y_i = \sum_{\substack{i \in V(G) \text{ s.t} \\ d(i,j) \leq 1}} \frac{2}{13} \leq \frac{10}{13} \leq 1, \quad \text{for each vertex } j \in V(C_m \square C_n),$$

$$(2) \quad \sum_{\substack{i \in V(G) \text{ s.t} \\ d(i,j) \leq 2}} y_i = \sum_{\substack{i \in V(G) \text{ s.t} \\ d(i,j) \leq 2}} \frac{2}{13} \leq 13\left(\frac{2}{13}\right) = 2, \quad \text{for each vertex } j \in V(C_m \square C_n), \ \&$$

$$(3) \quad y_j = \frac{2}{13} \geq 0 \quad \text{for each vertex } j \in V(C_m \square C_n).$$

Thus the constraints of the LP 5.1 are satisfied. As the cost of this multipacking is $\frac{2mn}{13}$, it follows that

$$\frac{2mn}{13} \leq mp_2(C_m \square C_n).$$

To prove the opposite inequality, consider the following fractional 2-limited dominating broadcast. For $i \in V(C_m \square C_n)$ and $k \in \{1, 2\}$ define $x_{i,1}$ and $x_{i,2}$ as the fractional broadcasts of vertex i at strengths 1 and 2, respectively. Define the fractional 2-limited broadcast by $x_{i,1} = 0$ and $x_{i,2} = \frac{1}{13}$ for all $i \in V(C_m \square C_n)$. As $m, n \geq 3$, for each $i \in V(C_m \square C_n)$, there are at least 9 vertices within distance 2 of i. Given this, to see that is a valid fractional 2-limited dominating broadcast observe that

$$(1) \quad \sum_{k=1}^{2} \sum_{\substack{i \in V(G) \text{ s.t} \\ d(i,j) \leq 2}} k \cdot x_{i,k} \geq 9\left(\frac{2}{13}\right) \geq 1, \quad \text{for each vertex } j \in V(C_m \square C_n), \text{ and}$$

$$(2) \quad \frac{1}{13} = x_{j,2} \in [0,1] \text{ and } 0 = x_{j,1} \in [0,1] \quad \text{for each vertex } j \in V(C_m \square C_n).$$

Thus the constraints of the relaxation of the ILP 1.1 are satisfied. The cost of this broadcast is $\frac{2mn}{13}$ which establishes

$$\gamma_{f,b,2}(C_m \square C_n) \leq \frac{2mn}{13}.$$

By the duality theorem of linear programming $mp_2(C_m \square C_n) = \gamma_{f,b,2}(C_m \square C_n)$.

□

Combined, the results found in Chapter 4 and Theorem 5.2.1 establish the upper and lower bounds, respectively, in Table 5.2.1 for the 2-limited broadcast domination number of Cartesian products of two cycles. The values of the constant terms in the upper bounds in Table 5.2.1 can be found in their respective propositions in Chapter 4 and are omitted here for the sake of brevity. To easily compare the upper and lower bounds in Table 5.2.1 we also, for some bounds, include a simpler lower bound.

Lower Bound	$\gamma_{b,2}(C_m \square C_n)$	Upper Bound
$1.3\left(\frac{n}{3}\right) \leq \left\lceil 6\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_3 \square C_n)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$3.6\left(\frac{n}{6}\right) \leq \left\lceil 8\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_4 \square C_n)$	$\leq 4\left\lfloor \frac{n}{6} \right\rfloor + O(1)$
$\left\lceil 10\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_5 \square C_n)$	$\leq n$
$\left\lceil 12\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_6 \square C_n)$	$\leq n + O(1)$
$37.6\left(\frac{n}{35}\right) \leq \left\lceil 14\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_7 \square C_n)$	$\leq 42\left\lfloor \frac{n}{35} \right\rfloor + O(1)$
$7.3\left(\frac{n}{6}\right) \leq \left\lceil 16\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_8 \square C_n)$	$\leq 8\left\lfloor \frac{n}{6} \right\rfloor + O(1)$
$13.8\left(\frac{n}{10}\right) \leq \left\lceil 18\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}(C_9 \square C_n)$	$\leq 16\left\lfloor \frac{n}{10} \right\rfloor + O(1)$

$15.3\left(\frac{n}{10}\right) \leq \left\lceil 20\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{10}\Box C_n\right)$	$\leq 16\left\lfloor\frac{n}{10}\right\rfloor + O(1)$
$\left\lceil 22\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{11}\Box C_n\right)$	$\leq 24\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 24\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{12}\Box C_n\right)$	$\leq 26\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 26\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{13}\Box C_n\right)$	$\leq 26\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 28\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{14}\Box C_n\right)$	$\leq 31\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 30\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{15}\Box C_n\right)$	$\leq 33\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 32\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{16}\Box C_n\right)$	$\leq 36\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 34\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{17}\Box C_n\right)$	$\leq 36\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 36\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{18}\Box C_n\right)$	$\leq 39\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 38\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{19}\Box C_n\right)$	$\leq 41\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 40\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{20}\Box C_n\right)$	$\leq 42\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 42\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{21}\Box C_n\right)$	$\leq 45\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 44\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{22}\Box C_n\right)$	$\leq 48\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 46\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{23}\Box C_n\right)$	$\leq 49\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 48\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{24}\Box C_n\right)$	$\leq 50\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 50\left(\frac{n}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{25}\Box C_n\right)$	$\leq 52\left\lfloor\frac{n}{13}\right\rfloor + O(1)$
$\left\lceil 2\left(\frac{mn}{13}\right)\right\rceil \leq$	$\gamma_{b,2}\left(C_{m\geq 26}\Box C_{n\geq 26}\right)$	$\leq 2\left(\frac{mn}{13}\right) + O(m+n)$

Table 5.2.1: Upper and lower bounds for $\gamma_{b,2}\left(C_m\Box C_n\right)$ for $m, n \geq 3$.

Observe that the bounds in Table 5.2.1 give optimal values of $\gamma_{b,2}\left(C_m\Box C_n\right) = 2\left(\frac{mn}{13}\right)$ for all $m, n \geq 3$ such that $m, n \equiv 0 \pmod{13}$.

5.3 Cartesian Products of a Path and a Cycle

This section describes the construction and associated costs of fractional 2-limited multipackings on $P_m\Box C_n$ for $m \geq 2$ and $n \geq 3$. For each $2 \leq m \leq 22$, an explicit fractional 2-limited multipacking is given on $P_m\Box C_n$ for all $n \geq 3$. These multipackings were determined by examining fractional 2-limited multipackings on $P_m\Box C_5$ for $2 \leq m \leq 22$. These constructions are used to prove Theorem 5.3.1. For $m \geq 23$ and all $n \geq 3$ a general fractional 2-limited multipacking is given on $P_n\Box P_m$. This general multipacking was determined by examining fractional 2-limited multipackings on $P_{23}\Box C_5$. This general construction is used to prove Theorem 5.3.2.

Theorem 5.3.1. *For $2 \leq m \leq 22$ and all $n \geq 3$*

$$f(n) \leq mp_2(P_m\Box C_n)$$

where $f(n)$ is given in Table 5.3.1.

m	2	3	4	5	6	7	8
$f(n)$	$\frac{n}{2}$	$\frac{2n}{3}$	$\frac{4n}{5}$	$\frac{26n}{27}$	$\frac{29n}{26}$	$\frac{19n}{15}$	$\frac{212n}{149}$

m	9	10	11	12	13	14	15
$f(n)$	$\frac{52n}{33}$	$\frac{780n}{451}$	$\frac{81n}{43}$	$\frac{273n}{134}$	$\frac{5042n}{2301}$	$\frac{2324n}{991}$	$\frac{5690n}{2277}$

m	16	17	18	19	20	21	22
$f(n)$	$\frac{15593n}{5878}$	$\frac{28417n}{10125}$	$\frac{103240n}{34873}$	$\frac{3896n}{1251}$	$\frac{337976n}{103415}$	$\frac{304705n}{89043}$	$\frac{548313n}{153338}$

Table 5.3.1: Values of $f(n)$ for lower bounds for $mp_2(P_m \square C_n)$ where $2 \leq m \leq 22$ and $n \geq 3$.

Proof. Fix $2 \leq m \leq 22$. Let $\mathbf{v}$ be the $m \times 1$ vector as given in either Table 5.3.2 or Table 5.3.3 (dependant upon m).

Let $\mathbf{J}_{m,n}$ be the all ones matrix of size $m \times n$. The product $\mathbf{v}^T\mathbf{J}_{m,n}$ defines a $m \times n$ matrix $\mathbf{M}$ where the elements within each row, respectively, have identical entries. For example if $m = 5$ and $n = 6$ then

$$\mathbf{M} = \begin{bmatrix} \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} \end{bmatrix}.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, define the fractional 2-limited multipacking f on $P_m \square C_n$ by letting the variable $x_{i,j}$ (corresponding to the vertex in row i and column j of $P_m \square C_n$) be the value in the ith row and jth column of $\mathbf{M}$. As the elements within each row are identical, to check that f is a valid fractional 2-limited multipacking it suffices to check that the constraints of the LP 5.1 are satisfied for a given column of

						m					
2	3	4	5	6	7	8	9	10	11	12	13
$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{3}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}$	$\begin{bmatrix} \frac{8}{27} \\ \frac{1}{9} \\ \frac{4}{27} \\ \frac{1}{9} \\ \frac{8}{27} \end{bmatrix}$	$\begin{bmatrix} \frac{4}{13} \\ \frac{1}{13} \\ \frac{9}{52} \\ \frac{9}{52} \\ \frac{1}{13} \\ \frac{4}{13} \end{bmatrix}$	$\begin{bmatrix} \frac{3}{10} \\ \frac{1}{10} \\ \frac{2}{15} \\ \frac{1}{5} \\ \frac{2}{15} \\ \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}$	$\begin{bmatrix} \frac{45}{149} \\ \frac{14}{149} \\ \frac{23}{149} \\ \frac{24}{149} \\ \frac{24}{149} \\ \frac{23}{149} \\ \frac{14}{149} \\ \frac{45}{149} \end{bmatrix}$	$\begin{bmatrix} \frac{10}{33} \\ \frac{1}{11} \\ \frac{5}{33} \\ \frac{2}{11} \\ \frac{4}{33} \\ \frac{2}{11} \\ \frac{5}{33} \\ \frac{1}{11} \\ \frac{10}{33} \end{bmatrix}$	$\begin{bmatrix} \frac{136}{451} \\ \frac{43}{451} \\ \frac{6}{41} \\ \frac{81}{451} \\ \frac{64}{451} \\ \frac{64}{451} \\ \frac{81}{451} \\ \frac{6}{41} \\ \frac{43}{451} \\ \frac{136}{451} \end{bmatrix}$	$\begin{bmatrix} \frac{13}{43} \\ \frac{4}{43} \\ \frac{13}{86} \\ \frac{15}{86} \\ \frac{6}{43} \\ \frac{7}{43} \\ \frac{6}{43} \\ \frac{15}{86} \\ \frac{13}{86} \\ \frac{4}{43} \\ \frac{13}{43} \end{bmatrix}$	$\begin{bmatrix} \frac{81}{268} \\ \frac{25}{268} \\ \frac{10}{67} \\ \frac{12}{67} \\ \frac{9}{67} \\ \frac{43}{268} \\ \frac{43}{268} \\ \frac{9}{67} \\ \frac{12}{67} \\ \frac{10}{67} \\ \frac{25}{268} \\ \frac{81}{268} \end{bmatrix}$	$\begin{bmatrix} \frac{695}{2301} \\ \frac{72}{767} \\ \frac{343}{2301} \\ \frac{136}{767} \\ \frac{320}{2301} \\ \frac{119}{767} \\ \frac{28}{177} \\ \frac{119}{767} \\ \frac{320}{2301} \\ \frac{136}{767} \\ \frac{343}{2301} \\ \frac{72}{767} \\ \frac{695}{2301} \end{bmatrix}$

Table 5.3.2: Vectors used in the proof of the lower bound for $mp_2(P_m \Box C_n)$ for $2 \leq m \leq 13$

				m				
14	15	16	17	18	19	20	21	22
$\frac{599}{1982}$	$\frac{688}{2277}$	$\frac{888}{2939}$	$\frac{6119}{20250}$	$\frac{10537}{34873}$	$\frac{42}{139}$	$\frac{31248}{103415}$	$\frac{26905}{89043}$	$\frac{2155}{7132}$
$\frac{185}{1982}$	$\frac{71}{759}$	$\frac{275}{2939}$	$\frac{631}{6750}$	$\frac{3262}{34873}$	$\frac{13}{139}$	$\frac{9671}{103415}$	$\frac{2776}{29681}$	$\frac{667}{7132}$
$\frac{297}{1982}$	$\frac{340}{2277}$	$\frac{1757}{11756}$	$\frac{1514}{10125}$	$\frac{5211}{34873}$	$\frac{187}{1251}$	$\frac{15458}{103415}$	$\frac{26617}{178086}$	$\frac{11460}{76669}$
$\frac{351}{1982}$	$\frac{45}{253}$	$\frac{2085}{11756}$	$\frac{599}{3375}$	$\frac{144}{811}$	$\frac{74}{417}$	$\frac{18357}{103415}$	$\frac{10537}{59362}$	$\frac{13608}{76669}$
$\frac{136}{991}$	$\frac{104}{759}$	$\frac{405}{2939}$	$\frac{278}{2025}$	$\frac{4792}{34873}$	$\frac{172}{1251}$	$\frac{384}{2795}$	$\frac{12238}{89043}$	$\frac{10537}{76669}$
$\frac{317}{1982}$	$\frac{40}{253}$	$\frac{464}{2939}$	$\frac{119}{750}$	$\frac{5515}{34873}$	$\frac{22}{139}$	$\frac{16376}{103415}$	$\frac{4697}{29681}$	$\frac{1129}{7132}$
$\frac{303}{1982}$	$\frac{359}{2277}$	$\frac{458}{2939}$	$\frac{3151}{20250}$	$\frac{5454}{34873}$	$\frac{65}{417}$	$\frac{1241}{7955}$	$\frac{13898}{89043}$	$\frac{47835}{306676}$
$\frac{303}{1982}$	$\frac{112}{759}$	$\frac{1791}{11756}$	$\frac{508}{3375}$	$\frac{5241}{34873}$	$\frac{21}{139}$	$\frac{3114}{20683}$	$\frac{8945}{59362}$	$\frac{11559}{76669}$
$\frac{317}{1982}$	$\frac{359}{2277}$	$\frac{1791}{11756}$	$\frac{1591}{10125}$	$\frac{5416}{34873}$	$\frac{194}{1251}$	$\frac{16119}{103415}$	$\frac{27665}{178086}$	$\frac{11920}{76669}$
$\frac{136}{991}$	$\frac{40}{253}$	$\frac{458}{2939}$	$\frac{508}{3375}$	$\frac{5416}{34873}$	$\frac{64}{417}$	$\frac{15848}{103415}$	$\frac{4572}{29681}$	$\frac{11770}{76669}$
$\frac{351}{1982}$	$\frac{104}{759}$	$\frac{464}{2939}$	$\frac{3151}{20250}$	$\frac{5241}{34873}$	$\frac{194}{1251}$	$\frac{15848}{103415}$	$\frac{13625}{89043}$	$\frac{47169}{306676}$
$\frac{297}{1982}$	$\frac{45}{253}$	$\frac{405}{2939}$	$\frac{119}{750}$	$\frac{5454}{34873}$	$\frac{21}{139}$	$\frac{16119}{103415}$	$\frac{4572}{29681}$	$\frac{47169}{306676}$
$\frac{185}{1982}$	$\frac{340}{2277}$	$\frac{2085}{11756}$	$\frac{278}{2025}$	$\frac{5515}{34873}$	$\frac{65}{417}$	$\frac{3114}{20683}$	$\frac{27665}{178086}$	$\frac{11770}{76669}$
$\frac{599}{1982}$	$\frac{71}{759}$	$\frac{1757}{11756}$	$\frac{599}{3375}$	$\frac{4792}{34873}$	$\frac{22}{139}$	$\frac{1241}{7955}$	$\frac{8945}{59362}$	$\frac{11920}{76669}$
	$\frac{688}{2277}$	$\frac{275}{2939}$	$\frac{1514}{10125}$	$\frac{144}{811}$	$\frac{172}{1251}$	$\frac{16376}{103415}$	$\frac{13898}{89043}$	$\frac{11559}{76669}$
		$\frac{888}{2939}$	$\frac{631}{6750}$	$\frac{5211}{34873}$	$\frac{74}{417}$	$\frac{384}{2795}$	$\frac{4697}{29681}$	$\frac{47835}{306676}$
			$\frac{6119}{20250}$	$\frac{3262}{34873}$	$\frac{187}{1251}$	$\frac{18357}{103415}$	$\frac{12238}{89043}$	$\frac{1129}{7132}$
				$\frac{10537}{34873}$	$\frac{13}{139}$	$\frac{15458}{103415}$	$\frac{10537}{59362}$	$\frac{10537}{76669}$
					$\frac{42}{139}$	$\frac{9671}{103415}$	$\frac{26617}{178086}$	$\frac{13608}{76669}$
						$\frac{31248}{103415}$	$\frac{2776}{29681}$	$\frac{11460}{76669}$
							$\frac{26905}{89043}$	$\frac{667}{7132}$
								$\frac{2155}{7132}$

Table 5.3.3: Vectors used in the proof of the lower bound for $mp_2(P_m \square C_n)$ for $14 \leq m \leq 22$

$P_m \Box C_n$. Thus it suffices to check that

$$(1) \qquad \sum_{i \in V(G) \text{ s.t. } d(i,1) \leq 1} x_{i,1} \leq x_{i-1,1} + 3x_{i,1} + x_{i+1,1} \leq 1, \text{ and}$$

$$(2) \qquad \sum_{i \in V(G) \text{ s.t. } d(i,1) \leq 2} x_{i,1} \leq x_{i-2,1} + 3x_{i-1,1} + 5x_{i,1} + 3x_{i+1,1} + x_{i+2,1} \leq 2$$

where $x_{i,1} = 0$ if $i \notin [1, m]$. The check can be done by computation. Let $v_1, \ldots v_m$ denote the elements of $\mathbf{v}$. The cost of f is $n \sum_{i=1}^{m} v_i$. For each $2 \leq m \leq 22$ the cost of f corresponds with the values in Table 5.3.1. This completes the proof.

$\square$

Theorem 5.3.2. *For $m \geq 23$ and all $n \geq 3$*

$$\frac{2mn}{13} + \frac{2620n}{13767} \leq mp_2(P_m \Box C_n).$$

Proof. The general case follows from the case $m = 23$. Letting $\mathbf{v}$ be the vector in Figure 5.3.1 for $m = 23$ and proceeding with as in the proof of Theorem 5.3.1 yields a fractional 2-limited multipacking on $P_{23} \Box C_n$ for all $n \geq 3$. The cost of such a multipacking is

$$(m - 16)\left(\frac{2n}{13}\right) + \frac{36508n}{13767} = 7\left(\frac{2n}{13}\right) + \frac{36508n}{13767}.$$

Observe that the centre most row of this multipacking on $P_{23} \Box C_n$ is composed solely of vertices with weight $2/13$. Moreover, all vertices within distance 2 of the centre row also have weight $2/13$. As this row does not violate the fractional 2-limited multipacking on $P_{23} \Box C_n$, adding additional centre rows, all of which contain vertices with weight $2/13$, will yield valid fractional 2-limited multipacking on $P_{m \geq 23} \Box C_n$. The cost of such a multipacking will be

$$(m - 16)\left(\frac{2n}{13}\right) + \frac{36508n}{13767},$$

which when simplified proves the theorem.

$\square$

Combined, the results found in Chapter 3 and Theorems 5.3.1 and 5.3.2 establish the upper and lower bounds in Table 5.3.4 for the 2-limited broadcast domination number of the Cartesian products of a path and a cycle. The value of the constant terms in the upper bounds in Table 5.3.4 can be found in the respective propositions in Chapter 3 and are omitted here for the sake of brevity. To easily compare the upper and lower bounds in Table 5.3.4 we also, for some bounds, include a simpler lower bound.

$$
\left[\begin{array}{c}
\frac{16639}{55068}\\
\frac{1717}{18356}\\
\frac{686}{4589}\\
\frac{814}{4589}\\
\frac{1895}{13767}\\
\frac{2903}{18356}\\
\frac{8573}{55068}\\
\frac{697}{4589}\\
\hdashline
\frac{2}{13}\\
\vdots\\
\frac{2}{13}\\
\hdashline
\frac{697}{4589}\\
\frac{8573}{55068}\\
\frac{2903}{18356}\\
\frac{1895}{13767}\\
\frac{814}{4589}\\
\frac{686}{4589}\\
\frac{1717}{18356}\\
\frac{16639}{55068}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \\ \\
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} m-16 \text{ entries of } \tfrac{2}{13}\\
\\ \\ \\ \\ \\ \\ \\ \\
\end{array}
$$

Figure 5.3.1: Vector used in the proof of the lower bound for $mp_2(P_m \square C_n)$ for $m \geq 23$

Lower Bound	$\gamma_{b,2}(P_m \Box C_n)$	Upper Bound
$\left\lceil \frac{n}{2} \right\rceil \leq$	$\gamma_{b,2}(P_2 \Box C_n)$	$\leq \left\lceil \frac{n}{2} \right\rceil$
$\left\lceil \frac{2n}{3} \right\rceil \leq$	$\gamma_{b,2}(P_3 \Box C_n)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{5} \right\rceil \leq$	$\gamma_{b,2}(P_4 \Box C_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$\left\lceil \frac{26n}{27} \right\rceil \leq$	$\gamma_{b,2}(P_5 \Box C_n)$	$\leq n + O(1)$
$17.8\left(\frac{n}{16}\right) \leq \left\lceil \frac{29n}{26} \right\rceil \leq$	$\gamma_{b,2}(P_6 \Box C_n)$	$\leq 18 \left\lfloor \frac{n}{16} \right\rfloor + O(1)$
$17.7\left(\frac{n}{14}\right) \leq \left\lceil \frac{19n}{15} \right\rceil \leq$	$\gamma_{b,2}(P_7 \Box C_n)$	$\leq 18 \left\lfloor \frac{n}{14} \right\rfloor + O(1)$
$31.3\left(\frac{n}{22}\right) \leq \left\lceil \frac{212n}{149} \right\rceil \leq$	$\gamma_{b,2}(P_8 \Box C_n)$	$\leq 32 \left\lfloor \frac{n}{22} \right\rfloor + O(1)$
$15.7\left(\frac{n}{10}\right) \leq \left\lceil \frac{52n}{33} \right\rceil \leq$	$\gamma_{b,2}(P_9 \Box C_n)$	$\leq 16 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$31.1\left(\frac{n}{18}\right) \leq \left\lceil \frac{780n}{451} \right\rceil \leq$	$\gamma_{b,2}(P_{10} \Box C_n)$	$\leq 32 \left\lfloor \frac{n}{18} \right\rfloor + O(1)$
$48.9\left(\frac{n}{26}\right) \leq \left\lceil \frac{81n}{43} \right\rceil \leq$	$\gamma_{b,2}(P_{11} \Box C_n)$	$\leq 50 \left\lfloor \frac{n}{26} \right\rfloor + O(1)$
$48.8\left(\frac{n}{24}\right) \leq \left\lceil \frac{273n}{134} \right\rceil \leq$	$\gamma_{b,2}(P_{12} \Box C_n)$	$\leq 50 \left\lfloor \frac{n}{24} \right\rfloor + O(1)$
$28.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5042n}{2301} \right\rceil \leq$	$\gamma_{b,2}(P_{13} \Box C_n)$	$\leq 30 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$30.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{2324n}{991} \right\rceil \leq$	$\gamma_{b,2}(P_{14} \Box C_n)$	$\leq 32 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$32.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5690n}{2277} \right\rceil \leq$	$\gamma_{b,2}(P_{15} \Box C_n)$	$\leq 34 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$34.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{15593n}{5878} \right\rceil \leq$	$\gamma_{b,2}(P_{16} \Box C_n)$	$\leq 36 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$36.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{28417n}{10125} \right\rceil \leq$	$\gamma_{b,2}(P_{17} \Box C_n)$	$\leq 38 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$38.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{103240n}{34873} \right\rceil \leq$	$\gamma_{b,2}(P_{18} \Box C_n)$	$\leq 40 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{3896n}{1251} \right\rceil \leq$	$\gamma_{b,2}(P_{19} \Box C_n)$	$\leq 42 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$42.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{337976n}{103415} \right\rceil \leq$	$\gamma_{b,2}(P_{20} \Box C_n)$	$\leq 44 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{304705n}{89043} \right\rceil \leq$	$\gamma_{b,2}(P_{21} \Box C_n)$	$\leq 46 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$46.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{548313n}{153338} \right\rceil \leq$	$\gamma_{b,2}(P_{22} \Box C_n)$	$\leq 48 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 2\left(\frac{mn}{13}\right) + \frac{2620n}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_{n\geq 13})$	$\leq 2\left(\frac{mn}{13}\right) + \frac{4m}{13} + O(n)$
$1.3\left(\frac{m}{3}\right) \leq \left\lceil \frac{6m}{13} + \frac{2620}{4586} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_3)$	$\leq \left\lceil \frac{2m}{3} \right\rceil$
$3.6\left(\frac{m}{3}\right) \leq \left\lceil \frac{8m}{13} + \frac{10480}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_4)$	$\leq 4 \left\lfloor \frac{m}{6} \right\rfloor + O(1)$
$\left\lceil \frac{10m}{13} + \frac{13100}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_5)$	$\leq m + 1$
$\left\lceil \frac{12m}{13} + \frac{5240}{4589} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_6)$	$\leq m + 1$
$37.6\left(\frac{m}{35}\right) \leq \left\lceil \frac{14m}{13} + \frac{18340}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_7)$	$\leq 42 \left\lfloor \frac{m}{35} \right\rfloor + O(1)$
$7.3\left(\frac{n}{6}\right) \leq \left\lceil \frac{16m}{13} + \frac{20960}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_8)$	$\leq 8 \left\lfloor \frac{m}{6} \right\rfloor + O(1)$
$13.8\left(\frac{n}{10}\right) \leq \left\lceil \frac{18m}{13} + \frac{7860}{4589} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_9)$	$\leq 16 \left\lfloor \frac{m}{10} \right\rfloor + O(1)$
$15.3\left(\frac{n}{10}\right) \leq \left\lceil \frac{20m}{13} + \frac{26200}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_{10})$	$\leq 16 \left\lfloor \frac{m}{10} \right\rfloor + O(1)$
$\left\lceil \frac{22m}{13} + \frac{28820}{13767} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_{11})$	$\leq 24 \left\lfloor \frac{m}{13} \right\rfloor + O(1)$
$\left\lceil \frac{24m}{13} + \frac{10480}{4589} \right\rceil \leq$	$\gamma_{b,2}(P_{m\geq 23} \Box C_{12})$	$\leq 26 \left\lfloor \frac{m}{13} \right\rfloor + O(1)$

Table 5.3.4: Upper and lower bounds for $\gamma_{b,2}(P_m \Box C_n)$ for $m, n \geq 3$.

Observe that the bounds in Table 5.3.4 give exact values for $\gamma_{b,2}\left(P_2 \square C_n\right)$ and $\gamma_{b,2}\left(P_3 \square C_n\right)$. These bounds also give optimal values of $\gamma_{b,2}\left(P_4 \square C_n\right)$ for all $n \geq 3$ such that $n \not\equiv 1$ and $5 \pmod{10}$.

5.4 Cartesian Products of Two Paths

This section describes the general construction and associated costs of fractional 2-limited multipackings on $P_m \square P_n$ for $m, n \geq 23$. For $2 \leq m \leq 22$, Section 5.3 provides lower bounds for $\gamma_{b,2}\left(P_m \square P_n\right)$ as $mp_2\left(P_m \square C_n\right) \leq mp_2\left(P_m \square P_n\right)$. This general construction was determined by examining fractional 2-limited multipackings on $P_{23} \square P_{23}$. This construction establishes Theorem 5.4.1. The proof of Theorem 5.4.1 utilizes tools from Theorem 5.3.2 while taking the general approach of tiling as described in Example 5.1.1.

Theorem 5.4.1. *For $m, n \geq 23$*

$$\frac{2mn}{13} + \frac{14287568}{75254411}(m+n) - \frac{177612468}{978307343} \leq mp_2(P_m \square P_n).$$

Proof. The general case follows from the case of $m = n = 23$. Partition $V\left(P_{23} \square P_{23}\right)$ into $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$, and G_9 as depicted in Figure 5.4.1. Let C be the

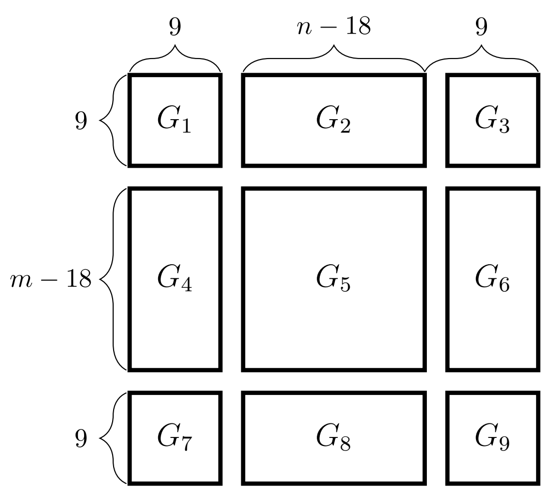


Figure 5.4.1: Tiling of $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$ and G_9 to construct a fractional 2-limited multipacking on $P_m \square P_n$

9×9 multipacking tile in Figure 5.4.2 and B be the 1×9 multipacking tile in Figure 5.4.3.

Let B_{90} and B_{270} denote rotating B by 90 and 270 degrees, respectively, clockwise to make 9×1 multipacking tiles. As in Chapters 2, 3, and 4, let $\overline{C}$ denote flipping C about the horizontal axis, $C^{|}$ denote flipping C about the vertical axis, and $\overline{C}^{|}$ denote flipping C about the horizontal and vertical axes. The same notation applies to B. Construct a fractional 2-limited multipacking on $P_{23} \square P_{23}$ via the following procedure.

$$
\begin{bmatrix}
\frac{12281674}{26440739} & \frac{14159065}{52881478} & \frac{14159065}{52881478} & \frac{716876929}{1956614686} & \frac{285882828}{978307343} & \frac{605062525}{1956614686} & \frac{290931945}{978307343} & \frac{45829467}{150508822} & \frac{295571672}{978307343} \\
\frac{14159065}{52881478} & 0 & \frac{74722744}{978307343} & \frac{5541796}{75254411} & \frac{31454788}{978307343} & \frac{197922615}{1956614686} & \frac{86952600}{978307343} & \frac{187824381}{1956614686} & \frac{178544927}{1956614686} \\
\frac{14159065}{52881478} & \frac{74722744}{978307343} & \frac{160889754}{978307343} & \frac{199257368}{978307343} & \frac{138934177}{978307343} & \frac{305011283}{1956614686} & \frac{305038839}{1956614686} & \frac{296920287}{1956614686} & \frac{11241560}{75254411} \\
\frac{716876929}{1956614686} & \frac{5541796}{75254411} & \frac{199257368}{978307343} & \frac{184527975}{978307343} & \frac{154930157}{978307343} & \frac{365582793}{1956614686} & \frac{171891256}{978307343} & \frac{164408802}{978307343} & \frac{173517195}{978307343} \\
\frac{285882828}{978307343} & \frac{31454788}{978307343} & \frac{138934177}{978307343} & \frac{154930157}{978307343} & \frac{115284064}{978307343} & \frac{135425169}{978307343} & \frac{128205527}{978307343} & \frac{133002879}{978307343} & \frac{135422438}{978307343} \\
\frac{605062525}{1956614686} & \frac{197922615}{1956614686} & \frac{305011283}{1956614686} & \frac{365582793}{1956614686} & \frac{135425169}{978307343} & \frac{13556138}{75254411} & \frac{157996738}{978307343} & \frac{152825964}{978307343} & \frac{155986569}{978307343} \\
\frac{290931945}{978307343} & \frac{86952600}{978307343} & \frac{305038839}{1956614686} & \frac{171891256}{978307343} & \frac{128205527}{978307343} & \frac{157996738}{978307343} & \frac{150611239}{978307343} & \frac{2}{13} & \frac{153260037}{978307343} \\
\frac{45829467}{150508822} & \frac{187824381}{1956614686} & \frac{296920287}{1956614686} & \frac{164408802}{978307343} & \frac{133002879}{978307343} & \frac{152825964}{978307343} & \frac{2}{13} & \frac{11090729}{75254411} & \frac{4016465}{26440739} \\
\frac{295571672}{978307343} & \frac{178544927}{1956614686} & \frac{11241560}{75254411} & \frac{173517195}{978307343} & \frac{135422438}{978307343} & \frac{155986569}{978307343} & \frac{153260037}{978307343} & \frac{4016465}{26440739} & \frac{154197504}{978307343}
\end{bmatrix}
$$

Figure 5.4.2: Multipacking tile used for the corner partitions of $V(P_m \square P_n)$ in the proof of the lower bound for $mp_2(P_m \square P_n)$ for $m, n \geq 23$.

$$
\begin{bmatrix}
\frac{295571672}{978307343} & \frac{91592327}{978307343} & \frac{11241560}{75254411} & \frac{173517195}{978307343} & \frac{135013048}{978307343} & \frac{153976400}{978307343} & \frac{152518991}{978307343} & \frac{4016465}{26440739} & \frac{2}{13}
\end{bmatrix}
$$

Figure 5.4.3: Multipacking tile used for the border partitions of $V(P_m \square P_n)$ in the proof of the lower bound for $mp_2(P_m \square P_n)$ for $m, n \geq 23$.

1. Tile G_1 with C,
2. tile G_2 with $[B_{90}]^{n-18}$,
3. tile G_3 with $C^{|}$,
4. tile G_4 with $[B]^{m-18}$,
5. tile G_6 with $\left[B^{|}\right]^{m-18}$,
6. tile G_7 with $\overline{C}$,
7. tile G_8 with $[B_{270}]^{n-18}$,
8. tile G_9 with $\overline{C}^{|}$, and
9. let every vertex of G_5 broadcast at strength $2/13$.

The resulting fractional 2-limited multipacking satisfies the constraints of Equation 5.1. This can be checked by computation. The cost of such a multipacking on $P_{23} \square P_{23}$ is

$$\frac{87985460234}{978307343}.$$

Put in terms of a general m and n, the cost of this multipacking on $P_{23} \square P_{23}$ is

$$\frac{2mn}{13} + \frac{14287468}{75254411}(m+n) - \frac{177612468}{978307343}$$

As $m - 18 = n - 18 = 5$, by the same logic as the proof of Theorem 5.3.2, the procedure described previously holds for all $P_m \square P_n$ such that $m, n \geq 23$. The cost of such a multipacking is given by Theorem 5.4.1.

□

Combined, the results found in Chapter 2 and Theorems 5.3.1 and 5.4.1 establish the upper and lower bounds, respectively, in Table 5.4.1 for the 2-limited broadcast domination numbers of the Cartesian products of two paths. The value of the constant terms in the upper bounds in Table 5.4.1 can be found in the respective propositions in Chapter 2 and are omitted here for the sake of brevity. To easily compare the upper and lower bounds in Table 5.4.1 we also, for some bounds, include a simpler lower bound.

Lower Bound	$\gamma_{b,2}(P_m \square P_n)$	Upper Bound
$\left\lceil \frac{n}{2} \right\rceil \leq$	$\gamma_{b,2}(P_2 \square P_n)$	$\leq \left\lfloor \frac{n}{2} \right\rfloor + 1$
$\left\lceil \frac{2n}{3} \right\rceil \leq$	$\gamma_{b,2}(P_3 \square P_n)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{5} \right\rceil \leq$	$\gamma_{b,2}(P_4 \square P_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$\left\lceil \frac{26n}{27} \right\rceil \leq$	$\gamma_{b,2}(P_5 \square P_n)$	$\leq n + 1$
$17.8\left(\frac{n}{16}\right) \leq \left\lceil \frac{29n}{26} \right\rceil \leq$	$\gamma_{b,2}(P_6 \square P_n)$	$\leq 18 \left\lfloor \frac{n}{16} \right\rfloor + O(1)$
$17.7\left(\frac{n}{14}\right) \leq \left\lceil \frac{19n}{15} \right\rceil \leq$	$\gamma_{b,2}(P_7 \square P_n)$	$\leq 18 \left\lfloor \frac{n}{14} \right\rfloor + O(1)$
$31.3\left(\frac{n}{22}\right) \leq \left\lceil \frac{212n}{149} \right\rceil \leq$	$\gamma_{b,2}(P_8 \square P_n)$	$\leq 32 \left\lfloor \frac{n}{22} \right\rfloor + O(1)$
$15.7\left(\frac{n}{10}\right) \leq \left\lceil \frac{52n}{33} \right\rceil \leq$	$\gamma_{b,2}(P_9 \square P_n)$	$\leq 16 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$31.1\left(\frac{n}{18}\right) \leq \left\lceil \frac{780n}{451} \right\rceil \leq$	$\gamma_{b,2}(P_{10} \square P_n)$	$\leq 32 \left\lfloor \frac{n}{18} \right\rfloor + O(1)$
$48.9\left(\frac{n}{26}\right) \leq \left\lceil \frac{81n}{43} \right\rceil \leq$	$\gamma_{b,2}(P_{11} \square P_n)$	$\leq 50 \left\lfloor \frac{n}{26} \right\rfloor + O(1)$
$48.8\left(\frac{n}{24}\right) \leq \left\lceil \frac{273n}{134} \right\rceil \leq$	$\gamma_{b,2}(P_{12} \square P_n)$	$\leq 50 \left\lfloor \frac{n}{24} \right\rfloor + O(1)$
$28.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5042n}{2301} \right\rceil \leq$	$\gamma_{b,2}(P_{13} \square P_n)$	$\leq 30 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$30.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{2324n}{991} \right\rceil \leq$	$\gamma_{b,2}(P_{14} \square P_n)$	$\leq 32 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$32.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5690n}{2277} \right\rceil \leq$	$\gamma_{b,2}(P_{15} \square P_n)$	$\leq 34 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$34.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{15593n}{5878} \right\rceil \leq$	$\gamma_{b,2}(P_{16} \square P_n)$	$\leq 36 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$36.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{28417n}{10125} \right\rceil \leq$	$\gamma_{b,2}(P_{17} \square P_n)$	$\leq 38 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$38.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{103240n}{34873} \right\rceil \leq$	$\gamma_{b,2}(P_{18} \square P_n)$	$\leq 40 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{3896n}{1251} \right\rceil \leq$	$\gamma_{b,2}(P_{19} \square P_n)$	$\leq 42 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$42.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{337976n}{103415} \right\rceil \leq$	$\gamma_{b,2}(P_{20} \square P_n)$	$\leq 44 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{304705n}{89043} \right\rceil \leq$	$\gamma_{b,2}(P_{21} \square P_n)$	$\leq 46 \left\lfloor \frac{n}{13} \right\rfloor + O(1)$

$46.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{548313n}{153338} \right\rceil \leq$	$\gamma_{b,2}\left(P_{22}\Box P_n\right)$	$\leq 48\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil \frac{2mn}{13} + \frac{14287568}{75254411}(m+n) - O(1) \right\rceil \leq$	$\gamma_{b,2}\left(P_{m\geq 23}\Box P_n\right)$	$\leq \frac{2mn}{13} + \frac{4}{13}(m+n) + O(1)$

Table 5.4.1: Upper and lower bounds for $\gamma_{b,2}\left(P_m\Box P_n\right)$ for $m, n \geq 2$.

Observe that the bounds in Table 5.4.1 give optimal values for $\gamma_{b,2}\left(P_2\Box P_n\right)$ for odd n, optimal values for $\gamma_{b,2}\left(P_3\Box P_n\right)$, and optimal values for $\gamma_{b,2}\left(P_4\Box P_n\right)$ when $n \equiv 4$ and $9 \pmod{10}$. The periodically optimal values for $\gamma_{b,2}\left(P_2\Box P_n\right)$ allow for an easy proof of Proposition 5.4.2.

Proposition 5.4.2. *For $n \geq 2$,*

$$\gamma_{b,2}\left(P_2\Box P_n\right) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proof. If $n \in \{2,3,4,5\}$ the claim is easily verified. Fix $n \geq 6$. Suppose n is odd. As

$$\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2} = \frac{n-1}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

by the bounds for $\gamma_{b,2}\left(P_2\Box P_n\right)$ in Theorem 5.3.1 and Proposition 2.2.1, the claim holds.

Suppose now that n is even and that there exists a 2-limited dominating broadcast f of $P_2\Box P_n$ of cost $\leq \left\lfloor \frac{n}{2} \right\rfloor$. Observe that a vertex broadcast at strength one or two on $P_2\Box P_n$ can be heard by at most four or eight vertices, respectively. Therefore, given $v \in V(P_2\Box P_n)$, the broadcast from v can be heard by at most $4f(v)$ vertices. Consider the leftmost column of $P_2\Box P_n$. Any 2-limited broadcast of $P_2\Box P_n$ which dominates this column necessarily "wastes" a portion of a broadcast from a vertex. By "wastes" we mean that, in dominating the leftmost column, there exists a vertex v whose broadcast either overlaps with the broadcast from some other vertex v' or is heard by at least one less than $4f(v)$ vertices. In either case, a broadcast from such a vertex is heard by at least one less than $4f(v)$ vertices. The same logic applies to the rightmost column of $P_2\Box P_n$. This follow since $n \geq 6$. Therefore there cannot exist a vertex which dominates a vertex in the left and rightmost columns. By assumption therefore, f can dominate at most

$$4\left\lfloor \frac{n}{2} \right\rfloor - 2 = 4\left(\frac{n}{2}\right) - 2 = 2n - 2 < 2n = |V(P_2\Box P_n)|$$

vertices. This is a contradiction. $\square$

Chapter 6
Eliminating Induced Sub-broadcasts of Fixed Cost

Chapters 2 through 5 established upper and lower bounds for the 2-limited broadcast domination numbers of the Cartesian products of two paths, a path and a cycle, and two cycles. Some of the bounds produced yield optimal or periodically optimal values for the 2-limited broadcast domination numbers of these graphs. This chapter is devoted to improving the lower bounds found in Chapter 5 by computational methods.

Our approach completes an exhaustive search of all possible *small induced sub-broadcasts* of given costs on a graph. A computational approach is used to eliminate cases which provably cannot be part of an optimal broadcast. For example, cases where the same region can be dominated by a broadcast with less cost. All ILPs in this chapter are solved with a Gurobi solver [23]. Section 6.1 provides an informal example of one of the tactics used in our search. Section 6.2 describes two main subroutines used. Section 6.3 describes our algorithm in full and Section 6.4 presents the results found.

6.1 Intuition

This chapter presents a method to prove lower bounds for $\gamma_{b,2}(P_m \square C_n)$ and $\gamma_{b,2}(C_m \square C_n)$. Throughout the description of our method, we use examples specific to $P_5 \square C_n$ to assist understanding.

For example, from Chapter 3, we have that

$$\gamma_{b,2}(P_5 \square C_n) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \text{ or } 2 \pmod 4 \text{ and} \\ 1 & \text{for } n \equiv 1 \text{ or } 3 \pmod 4. \end{cases}$$

Suppose we wish to establish $n \leq \gamma_{b,2}(P_5 \square C_n)$ to yield periodically optimal values for $\gamma_{b,2}(P_5 \square C_n)$. From the computational results in Section 3.5, $n \leq \gamma_{b,2}(P_5 \square C_n)$ for $3 \leq n \leq 16$. Suppose the bound does not hold and let n be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast f on $P_5 \square C_n$ of cost at most $n-1$. If every sequence of eight consecutive columns of f has cost at least eight, then the total cost of f is at least n. By assumption then, there must exist some sequence of eight consecutive columns with cost at most seven. Let C be the subgraph of $P_5 \square C_n$ induced by the vertices appearing in a minimum cost set of eight consecutive columns with respect to f. For each integer $x \leq 7$, we consider all possible broadcasts of cost x within C and try to reach a contradiction. If this is done for all such broadcasts, then this establishes $n \leq \gamma_{b,2}(P_5 \square C_n)$ as desired.

We require the following natural definitions.

Definition 6.1.1. Let f be a 2-limited broadcast on the graph G and let $X \subseteq V(G)$. Define the *sub-broadcast g induced by X* by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.1.2. Given two 2-limited broadcasts f and g on a graph G, define $h = f \oplus g$ by $h(x) = \max\{f(x), g(x)\}$ for all $x \in V(G)$. Similarly, define $h = f \ominus g$ for all $x \in V(G)$ by

$$h(x) = \begin{cases} 0 & \text{if } g(x) > 0 \text{ and} \\ f(x) & \text{otherwise.} \end{cases}$$

Definition 6.1.3. If f is a broadcast on G, then we say f *dominates* $y \in V(G)$ if there exists a vertex x such that y hears the broadcast from x under f. Further, we say that f *dominates* $X \subseteq V(G)$ if it *dominates* every vertex $x \in X$.

Definition 6.1.4. Let f be a broadcast on G. The *broadcast range* of f is the set of vertices which hear a broadcast under f.

Continuing with the previously mentioned assumption, suppose we are considering sub-broadcasts of cost at most seven induced by $V(C)$ on a 2-limited dominating broadcast f on $P_5 \square C_n$ of cost at most $n-1$. One such possible broadcast is given by g in Figure 6.1.1. Let $f' = f \ominus g$. Let R be the range of g. The broadcast f' dominates

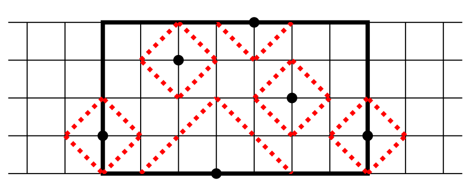


Figure 6.1.1: Assumed sub-broadcast g induced by $V(C)$ of cost seven.

all vertices of $P_5 \square C_n$ with the possible exception of the vertices of R. Suppose we delete four columns from the grid, as indicated in Figure 6.1.2, and "patch" the grid back together in the natural way such that the resulting graph is $P_5 \square C_{n-4}$. Note that, in this way, $V(P_5 \square C_n)$ has a natural correspondence with $V(P_5 \square C_{n-4})$. Let f'' be

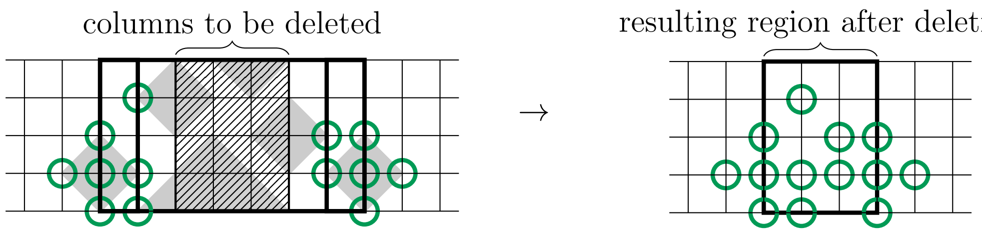


Figure 6.1.2: Procedure which reduces $P_5 \square C_n$ to $P_5 \square C_{n-4}$.

the broadcast formed by restricting f' to this reduced graph in the natural way. Let $R' = \{v \in P_5 \square C_{n-4} : v \in R\}$. The vertices of R' are indicated by the green circles in Figures 6.1.2 and 6.1.3. By the same logic as before, f'' dominates all of $P_5 \square C_{n-4}$ with the possible exception of the vertices of R'. However, R' can be dominated by the broadcast h of cost three on $P_5 \square C_{n-4}$ defined by the broadcast in Figure 6.1.3. Let

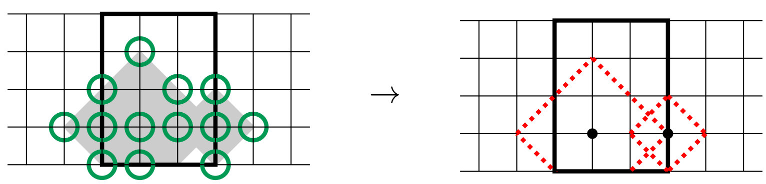


Figure 6.1.3: Broadcast of cost three which dominates R' from Figure 6.1.2.

$f''' = f'' \oplus h$. The broadcast f''' is a 2-limited dominating broadcast on $P_5 \square C_{n-4}$. The cost of f''' is at most

$$cost(f'') + 3 = cost(f \ominus g) + 3 \leq (n-1) - 7 + 3 = n - 5.$$

However, this is a contradiction as, by the assumption of the minimality of n, $n - 4 \leq \gamma_{b,2}(P_5 \square C_{n-4})$. Therefore, if there exists a 2-limited dominating broadcast f on $P_5 \square C_n$ of cost at most $n-1$, it cannot contain, as a sub-broadcast induced by $V(C)$, the broadcast g in Figure 6.1.1. The goal of our method is to complete an exhaustive search of all possible induced sub-broadcasts and, for each broadcast, find a contradiction. If this is done, we prove the bound.

This type of contradiction is made rigorous in Section 6.2 and is used here simply to give an intuition for one of our methods. Additional methods for yielding contradictions are also described in Sections 6.2 and 6.3.

6.2 Applying Induction to Reach Contradictions

Our method repeatedly utilizes two main ways of obtaining contradictions. The first of these tactics, as described in Algorithm 1: *HasBroadcastScheme*, provides a routine which, given a graph G and set of vertices $R \subseteq V(G)$, returns the truth value of the statement "R can be dominated with cost at most x." Example 6.2.1 provides an application of *HasBroadcastScheme* on $P_5 \square C_n$.

Algorithm 1: Routine to determine whether a given set of vertices on a graph can dominated with cost at most x.

---

function HasBroadcastScheme (G, R, x);
Input : A graph G, a set of vertices $R \subseteq V(G)$, and desired cost x to dominate R.
Output: Value of the truth statement: "R can be dominated with cost at most x."
Let f be a minimum cost 2-limited dominating broadcast of R on G;
if *cost(f)* $\leq x$ **then**
  return True;
end
return False;

Example 6.2.1. Consider the broadcast g of cost six on $P_5 \square C_n$, for some fixed $n \geq 8$, defined by the left-hand figure in Figure 6.2.1. Let R be the range of g. Observe that R can be dominated with cost five, as depicted in the right-hand figure of Figure 6.2.1. Therefore, $HasBroadcastScheme(P_5 \square C_n, R, 5)$ is true. Hence, no optimal broadcast

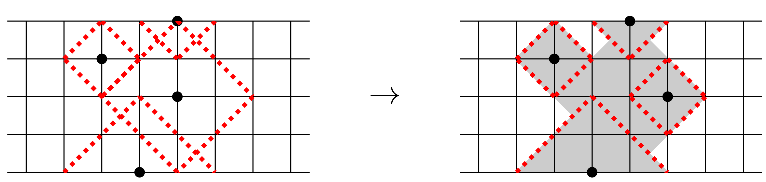


Figure 6.2.1: Broadcast g of cost six and broadcast of cost five which dominates the range of g on $P_5 \square C_{n \geq 8}$.

on $P_5 \square C_{n \geq 8}$ contains g as an induced sub-broadcast.

Our second method, as described in Algorithm 2: *ResultsInMinimalityContradiction*, is a generalization of the example described in Section 6.1. Prior to a formal description of *ResultsInMinimalityContradiction*, we include an example to aid in its explanation.

Example 6.2.2. From the computational results in Section 3.5, $B(n) = n \leq \gamma_{b,2}(P_5 \square C_n)$ for $3 \leq n \leq 16$. Suppose that $B(n)$ does not hold for all n and let n

be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast f on $P_5 \Box C_n$ of cost at most $n-1$. Suppose f contains the induced sub-broadcast g of cost 12 on $P_5 \Box C_n$ as defined by Figure 6.2.2. Let R be the range of g.

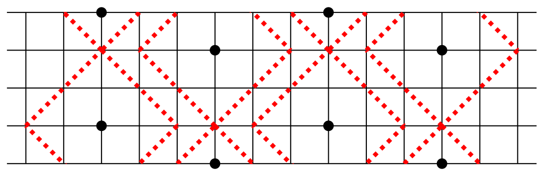


Figure 6.2.2: Assumed induced sub-broadcast g of f of cost 12.

Let $f' = f \ominus g$. The broadcast f' dominates all of $P_5 \Box C_n$ with the possible exception of the vertices of R. Suppose we delete six columns of the grid, as indicated in Figure 6.2.3, and "patch" the grid back together in the natural way such that the resulting graph is $P_5 \Box C_{n-6}$. Note that $V(P_5 \Box C_n)$ has a natural correspondence with $V(P_5 \Box C_{n-6})$. The columns deleted could contain vertices broadcasting with non-zero strength under f'. Our goal is to create a new broadcast f'' such that every vertex of $P_5 \Box C_{n-6}$, which heard a broadcast under f', hears a broadcast under f''. Furthermore, we want the cost of f'' to be at most the cost of f'.

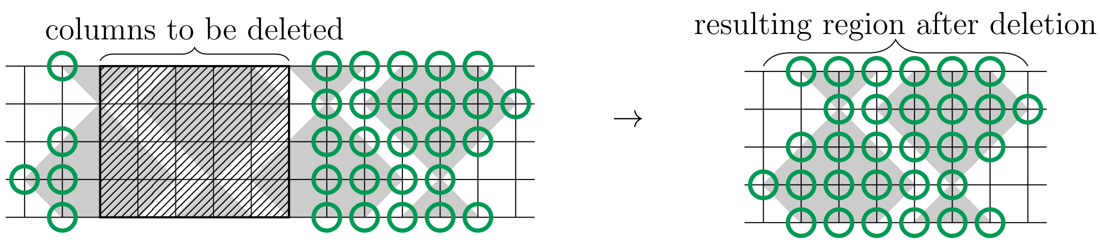


Figure 6.2.3: Procedure which reduces $P_5 \Box C_n$ to $P_5 \Box C_{n-6}$.

For each vertex $v \in V(P_5 \Box C_{n-6})$ that is not in one of the deleted columns, define the broadcast f'' on $P_5 \Box C_{n-6}$ by $f''(v) = f'(v)$. In order to preserve f', for each vertex v in a deleted column, pick a vertex u in the same row as v and in a nearest column undeleted to v and let $f''(u) = \max\{f'(u), f'(v)\}$. For example, suppose f' contains the vertex v broadcasting at strength one in the leftmost deleted column in the left-hand figure of Figure 6.2.4. After deleting these columns, the broadcast of v is re-allocated to one column to the left, as depicted in the right-hand figure in Figure 6.2.4. We make the following observation about this process of redistribution.

Observation 1. *Suppose f is a 2-limited dominating broadcast on $G = (P_m \text{ or } C_m) \Box C_n$. Let g be an induced sub-broadcast of f and let R be the range of g. Define $f' = f \ominus g$. Let G' be the graph formed by deleting columns $c_{\alpha_1}, \ldots, c_{\alpha_z}$*

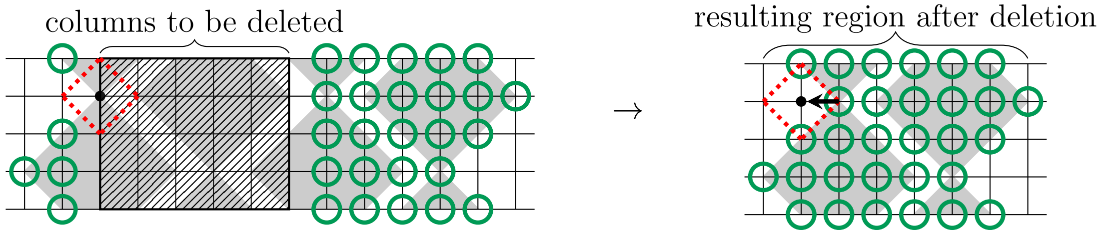


Figure 6.2.4: Procedure which reduces $P_5 \square C_n$ to $P_5 \square C_{n-6}$ and preserves f'.

which contain vertices of R from G and "patching" the graph together in the natural way. Let f'' be the broadcast formed by restricting f' to this reduced graph in the natural way. For each deleted vertex v broadcasting with non-zero strength under f', pick a vertex u in the same row as v and in a nearest column undeleted to v and let $f''(u) = \max\{f'(u), f'(v)\}$. The cost$(f'') \leq$ cost(f') and the vertices of G' that belong to the set $(V(G) \setminus [V(c_{\alpha_1}) \cup \cdots \cup V(c_{\alpha_z})]) \setminus R$ each hear a broadcast under f'' (possibly other vertices do too).

Proof. The broadcast f'' is defined such that, for each vertex v, broadcasting with non-zero strength under f', on a column in $\{c_{\alpha_1}, \ldots, c_{\alpha_z}\}$, there exists a vertex u in the same row as v and in a nearest undeleted column such that $f''(u) = \max\{f'(u), f'(v)\}$. As the strength of the broadcast from u did not decrease, any vertex on this reduced graph that heard a broadcast from u under f' hears a broadcast from u under f''. Redistributing the strength of v's broadcast to u ensures that any vertex which hears a broadcast from v under f' hears a broadcast from u under f''. This follows because, for each vertex w which heard a broadcast from v under f', since $f'(v) \leq 2$, either

1. w is deleted from the graph and therefore does not need to hear a broadcast,

2. u was originally on the same column as w or w is on the column between u and v and, being in the same row as v and broadcasting at strength at least $f'(v)$, is heard by w under f'',

3. v was originally on a column between w and u and, as u is on a nearest column to v undeleted, the column of u has, at worst, taken the place of the column originally containing v from the viewpoint of w. As u is in the same row as v and broadcasting at strength at least $f'(v)$, w hears a broadcast from u under f''.

Therefore, for any vertex $w \in V(G)$ dominated by f', if $w \in V(G')$, w is dominated by f''. Moreover, the cost$(f'') \leq$ cost(f') since, given a broadcast vertex v on a deleted column and an appropriate vertex u, the strength of u is at most $f'(u) + f'(v)$. $\square$

For this example, by Observation 1, f'' dominates all the vertices of $P_5 \square C_{n-6}$ with the possible exception of the vertices indicated by the green circles in Figure 6.2.3. Let

$R' = \{v \in V(P_5 \square C_{n-6}) : v \in R\}$. The vertices of R' might not be dominated by f''. The vertices in R' can be dominated by the broadcast h of cost six on $P_5 \square C_{n-6}$, as defined in Figure 6.2.5. Let $f''' = f'' \oplus h$. The broadcast f''' is a 2-limited dominating

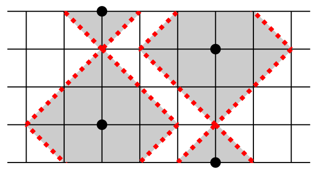


Figure 6.2.5: Broadcast h of cost six which dominates the vertices circled in green in Figure 6.2.3.

broadcast of $P_5 \square C_{n-6}$ of cost at most

$$cost(f'') + 6 \leq cost(f \ominus g) + 6 \leq (n-1) - 12 + 6 = n - 7.$$

However, this is a contradiction since, by the assumption of the minimality of n, $n-6 \leq \gamma_{b,2}(P_5 \square C_{n-6})$.

There are several important things to take away from Example 6.2.2. Given some assumed induced sub-broadcast g of a broadcast f whose broadcast range is contained within k consecutive columns on a larger graph, we contradict the minimality of n by "patching" the graph back together after deleting some number of columns and inferring the existence of a dominating broadcast of a bounded cost on a smaller graph. To derive a contradiction, we need not know what the broadcast f looks like, we simply need to know its cost. The example in Section 6.1 and Example 6.2.2 delete four and six columns, respectively, to yield contradictions. In general, given some induced sub-broadcast g whose broadcast range R is contained within k columns, we attempt to find a contradiction of minimality after deleting one selection of i columns, for each i from 1 to k. If we are trying to prove that $B(n) \leq \gamma_{b,2}(P_m \square C_n)$ or $\gamma_{b,2}(C_m \square C_n)$, if the vertices of R not in the i columns that are deleted can be dominated with cost at most

$$cost(g) - (B(n) - B(n-i))$$

then this implies a contradiction of the minimality of n since it gives a 2-limited dominating broadcast on the reduced graph of cost at most

$$cost(f \ominus g) + cost(g) - (B(n) - B(n-i)) < B(n) - B(n) + B(n-i) = B(n-i).$$

Given some fixed k, our approach first verifies that the lower bound holds for $n \leq k+2$ by computation. We then assume that $n \geq k+3$. This assumption ensures that, after deleting at most k columns, the reduced graph has at least three columns and therefore this process is well defined.

Put more formally, *ResultsInMinimalityContradiction* takes, as an input, a graph $G_{m,k} = (P_m \text{ or } C_m)\,\Box P_k$, some set of vertices $R \subseteq V(G_{m,k})$, and the previous cost x used to dominate R by some induced sub-broadcast g (of a broadcast f) whose broadcast range is contained within $G_{m,k}$. The graph $G_{m,k}$ is some set of k consecutive columns of a larger graph $(P_m \text{ or } C_m)\,\Box C_n$ where $n > k+2$. Moreover, n is assumed to be the smallest integer greater than $k+2$ such that there exists an optimal 2-limited dominating broadcast on this larger graph $(P_m \text{ or } C_m)\,\Box C_n$ of cost at most $B(n)-1$ where $B(n)$ is the bound we are attempting to prove (dependent upon P_m or C_m). To determine whether or not the cost x of this induced sub-broadcast g, which dominates R on $G_{m,k}$, results in a contradiction of minimality, the values $m_1, \ldots, m_k$, defined subsequently, are also passed to *ResultsInMinimalityContradiction*. For each i from 1 to k, define m_i as the maximum value, over all $n > k+2$, of $B(n) - B(n-i)$.

For instance, given $B(n) = 4\left\lfloor \frac{n}{6} \right\rfloor$ and $k = 14$, the values of m_i are given in the bottom row of Table 6.2.1. Table 6.2.1 also includes the calculation of the m_i's, broken down according to the least residue of n modulo 6.

i	1	2	3	4	5	6	
$n \equiv 0 \pmod 6$	4	4	4	4	4	4	Value of $4\left\lfloor \frac{n}{6} \right\rfloor - 4\left\lfloor \frac{n-i}{6} \right\rfloor$
$n \equiv 1 \pmod 6$	0	4	4	4	4	4	
$n \equiv 2 \pmod 6$	0	0	4	4	4	4	
$n \equiv 3 \pmod 6$	0	0	0	4	4	4	
$n \equiv 4 \pmod 6$	0	0	0	0	4	4	
$n \equiv 5 \pmod 6$	0	0	0	0	0	4	
m_i	4	4	4	4	4	4	

Table 6.2.1: Calculated values of m_i given $B(n) = 4\left\lfloor \frac{n}{6} \right\rfloor$ and $k = 14$.

The values $m_1, \ldots, m_k$ are used in *ResultsInMinimalityContradiction* as follows. Suppose we delete $i \leq k$ columns of $G_{m,k}$ and "patch" the grid together in the natural way to obtain $G' = G_{m,k-i}$. Let $R' = \{v \in R : v \in G'\}$. If R' can be dominated with cost $x - m_i$, then there exists a broadcast of $G_{m,n-i}$ that has cost less than $B(n-i)$. This follows as, by our choices of m_i, and the arguments discussed above, this implies there exists a 2-limited dominating broadcast on G' of cost at most

$$cost(f \ominus g) + x - m_i < B(n) - m_i \leq B(n) - B(n) + B(n-i) = B(n-i).$$

This contradicts the minimality of n. To determine whether R' can be dominated with cost at most $x - m_i$, it suffices to determine whether $HasBroadcastScheme(G_{m,k-i}, R', x - m_i)$ is true.

Algorithm 2 provides the pseudo-code for *ResultsInMinimalityContradiction* and Theorem 6.2.3 proves the correctness of this approach.

Algorithm 2: Routine to determine a minimality contradiction.

function ResultsInMinimalityContradiction $(G_{m,k}, R, x, m_1, \ldots, m_k)$;

Input : A graph $G_{m,k} = (P_m \text{ or } C_m)\Box P_k$ with columns labelled from left to right by $c_1, \ldots, c_k$, and rows from 1 to m, a set of vertices $R \subseteq V(G_{m,k})$, the cost x used to dominate R by some broadcast g whose broadcast range lies entirely within $G_{m,k}$, and m_i (for each $i = 1$ to k) defined as the maximum values over all $n > k + 2$ of $B(n) - B(n - i)$ where $B(n)$ is the bound we are attempting to prove.

Output: Value of the truth statement: "given the cost x used to dominate a region R of a larger graph with $n > k + 2$ columns with a broadcast of cost less than $B(n)$, there is a $1 \leq i \leq k$ such that there exists a broadcast on a graph with $n - i$ columns with a broadcast cost less than $B(n - i)$."

Create a sorted list L of the columns that contain at least one vertex of R so that they are first ordered from maximum to minimum according to the number of vertices that are in R. Resolve ties by sorting so that c_i comes before c_j if $i < j$;

Let the vertices of $G_{m,k}$ be labelled according to their row number and column label;

for i **from** 1 **to** *length*(L) **do**

  Let S be the set of columns contained in the first i entries of list L;

  Let G be the graph $G_{m,k-i}$;

  Number the columns of G using the set of column labels $\{c_1, \ldots, c_k\} \setminus S$ going from left to right in order of the column indices;

  Let the vertices of G be labelled according to their row number and column label;

  Set $R' = \{v \in R : v \in G\}$;

  if *OptimalityContradiction*$(G, R', x - m_i)$ **then**

    return True;

  end

end

return False;

Theorem 6.2.3. *Fix $k \geq 1$ and let $B(n)$ be a lower bound for the 2-limited broadcast domination number of $G_{m,n} = (P_m \text{ or } C_m) \Box C_n$ which holds for all $n \leq k+2$. Suppose the bound does not hold and let n be the smallest integer greater than $k+2$ such that there exists an optimal 2-limited dominating broadcast f on $G_{m,n}$ of cost at most $B(n)-1$. Let g be an induced sub-broadcast of f whose broadcast range is contained in some k-column induced subgraph $G_{m,k}$ of $G_{m,n}$ which is isomorphic to $(P_m \text{ or } C_m) \Box P_k$. Let R be the range of g. Define m_i (for each $i = 1$ to k) as the maximum value over all $n > k+2$ of $B(n) - B(n-i)$. If*

$$ResultsInMinimalityContradiction(G_{m,k}, R, cost(g), m_1, \ldots, m_k)$$

is true, there exists a $1 \leq i \leq k$ such that the graph $G_{m,n-i} = (P_m \text{ or } C_m) \Box C_{n-i}$ has a 2-limited dominating broadcast of cost less than $B(n-i)$.

Proof. Assume the conditions of Theorem 6.2.3 and suppose

$$ResultsInMinimalityContradiction(G_{m,k}, R, cost(g), m_1, \ldots, m_k)$$

is true. Let $f' = f \ominus g$. Let S, G, and R' be defined as in Algorithm 2. Let G' be the resulting graph $G_{m,n-i}$ constructed by removing the set of columns S that results in *ResultsInMinimalityContradiction* returning true on line 11 and "patching" the grid back together in the natural way such that the resulting graph is $(P_m \text{ or } C_m) \Box C_{n-i}$. The resulting set of vertices R' of G defined on line 9 can be dominated by an 2-limited broadcast h on G' with cost $\leq x - m_i$ since the function call on line 10 returns true. Let f'' be the broadcast formed by restricting f' on this reduced graph in the natural way and redistributing the broadcasts of vertices deleted as described in Observation 1. By Observation 1, f'' dominates all of G' with the possible exception of the vertices of R'. Let $f''' = f'' \oplus h$. The broadcast f''' is a 2-limited dominating broadcast on G'. The cost of such a broadcast is at most

$$cost(f \ominus g) + x - m_i < B(n) - B(n) + B(n-i) = B(n-i).$$

This establishes $\gamma_{b,2}(G') < B(n-i)$. This contradicts the minimality of n.

□

6.3 Main Methodology

Fix m and suppose we wish to establish $B(n)$ as a lower bound for the 2-limited broadcast domination number of $G_{m,n}$ where $G_{m,n}$ is either $P_m \Box C_n$ or $C_m \Box C_n$. Our computational approach starts by assuming there is some minimum values of n for which there is a broadcast f of cost less than $B(n)$. A positive fixed number $r \geq 5$ of columns is chosen. This constraint on r is not necessary in general but is chosen here as it is sufficient for our methods and makes the explanation of our sub-routines simpler. Let C be the subgraph of $G_{m,n}$ induced by the vertices appearing in a minimum cost set (with respect to f) of r consecutive columns of $G_{m,n}$. Our approach completes an exhaustive search of all possible sub-broadcasts g induced by $V(C)$ of f and subjects each broadcast g to a series of tests to try and reach a contradiction.

Given C, four columns are added to both the left and right of C in order to ensure that the subgraph considered is large enough to include the region dominated by any vertex that could potentially dominate some vertex in C. That is, our algorithm takes as an input, $(P_m \text{ or } C_m) \square P_k$ centred at C where $k = r + 8$.

All cases with $n \leq k + 2$ are first checked by computation to ensure that the bound holds. As we assume $n > k + 2$, deleting up to k columns is well defined.

By our choice of C, we can bound the possible costs of the sub-broadcasts g induced by C between some fixed values s and t. Let $C' \subseteq V(C)$ be the set of the vertices of C which are distance greater than one from the leftmost and rightmost columns in C. Informally, C' can be thought of as the "middle" of C. Vertices in C' can only hear the broadcasts from vertices in C. For example, given $G_{m,k} = P_5 \square P_{16}$, C is formed by the eight consecutive columns centred in $G_{m,k}$ and C' is the vertices in the four consecutive columns centred in C. Figure 6.3.1 depicts $G_{m,k}$, C (indicated by the thick black rectangle), C' (indicated by the shaded grey region), and possible broadcast vertices in and outside of C. The value of s can be set as the minimum cost

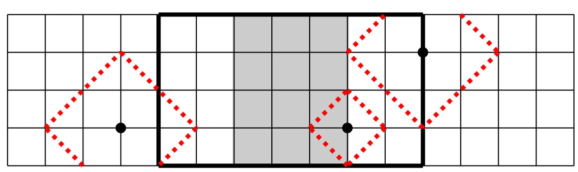


Figure 6.3.1: Depiction of C embedded in $P_5 \square P_{16}$ to illustrate that the middle four columns of C must be dominated by vertices within C.

of a broadcast required to dominate C'. The value of t is set to equal the maximum possible cost of a minimum r consecutive column induced sub-broadcast for a graph $G_{m,n}$ that hypothetically has a broadcast of cost less than $B(n)$. For example, given $B(n) = 4\left\lfloor \frac{n}{6} \right\rfloor$, if every six consecutive column induced subgraph has cost at least four, then the total cost is at least $\left\lceil \frac{4n}{6} \right\rceil$ and hence the bound $B(n)$ holds. But if a broadcast has one or more six consecutive column induced subgraphs with cost at most three, then it is possible that the total cost is less than $B(n)$. So for this example, $t = 3$.

Let $\mathcal{C}$ be the set of all possible sub-broadcasts induced by $V(C)$ of f of costs s to t. Fix some induced sub-broadcast $g \in \mathcal{C}$. The following five functions are designed to try and yield a contradiction for each $g \in \mathcal{C}$.

6.3.1 Dominating the Middle Contradiction

Since the vertices of C' can only hear a broadcast from a vertex in C, they must be dominated by g. The function *DoesNotDominateMiddle* returns true if any vertices of C' are not dominated by g, and false otherwise.

Algorithm 3: Routine to determine if a broadcast dominates the "middle" of a graph.

<u>function DoesNotDominateMiddle</u> $(G_{m,k}, g)$;
Input : A graph $G_{m,k} = (P_m \text{ or } C_m)\,\square P_k$ with columns labelled from left to right by $c_1, \ldots, c_k$, where $k \geq 13$, and a broadcast g.
Output: Value of the truth statement: "g does not dominate C'."
Let C' be formed by the vertices of columns c_7 to c_{k-6};
if *C' is not a subset of the range of g* **then**
  return True;
end
return False;

6.3.2 Forbidden Broadcasts Structure Contradiction

If g contains a vertex broadcasting at strength one within distance one of a vertex broadcasting at strength two, then f is trivially not optimal. If there is an optimal broadcast h of cost $< B(n)$ that has two vertices broadcasting at strength one within distance two of each other, then there is another optimal broadcast h' such that $cost(h) = cost(h')$ where these two vertices are replaced by a vertex broadcasting at strength two. Proving that there is no optimal broadcast like h' of cost $< B(n)$ proves that there is no optimal broadcast like h as h' can be modified to construct h. Therefore, if g contains two strength one vertices within distance two of each other, g need not be considered.

The function *ForbiddenBroadcast* returns true if g contains either such broadcast structure, and false otherwise.

6.3.3 Necessary Broadcasts Contradiction

Recall that C is the subgraph of $G_{m,n}$ induced by the vertices appearing in a minimum cost set (with respect to f) of r consecutive columns of $G_{m,n}$, g is the sub-broadcast induced by $V(C)$, and R is the range of g.

Let D be the set of vertices at distance one from either the leftmost or rightmost column in C which are not in R. For example, given $G_{m,k} = P_5 \square P_{16}$, Figure 6.3.2 depicts such a sub-broadcast g induced by $V(C)$, where the vertices of D are indicated by thick orange circles. As the vertices broadcasting at non-zero strength under g

Algorithm 4: Routine to determine if a broadcast contains a forbidden structure.

```
function ForbiddenBroadcast (G, g);
Input  : A graph G and a broadcast g.
Output: Value of the truth statement: "g contains a forbidden structure."
if g contains a vertex broadcasting at strength one within distance one of a
  vertex broadcasting at strength two then
  | return True;
end
if g contains two vertices broadcasting at strength one within distance two of
  each other then
  | return True;
end
return False;
```

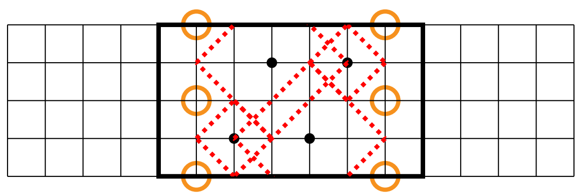


Figure 6.3.2: Sub-broadcast g induced by $V(C)$ and undominated vertices D.

are the only vertices broadcasting at non-zero strength under f when restricted to C, the only vertices whose broadcast can be heard by the vertices of D are vertices broadcasting at strength two in the same row, respectively, at distance one from either the leftmost or rightmost column of C. Figure 6.3.3 depicts the necessary vertices broadcasting at strength two forced by the vertices of D of Figure 6.3.2. Therefore, for each $v \in D$, there exists a vertex u, in the same row as v and at distance one from the leftmost or rightmost column of C, such that $f(u) = 2$. Note that, as $|C| = r \geq 5$, this choice of u is unique. Let $\mathcal{U}$ be the set of such vertices u as dictated by the vertices of D. Let g' be the sub-broadcast induced by $V(C) \cup \mathcal{U}$. The routine $NecessaryBroadcastsContradiction$ returns true if g' leads to a contradiction, and false otherwise.

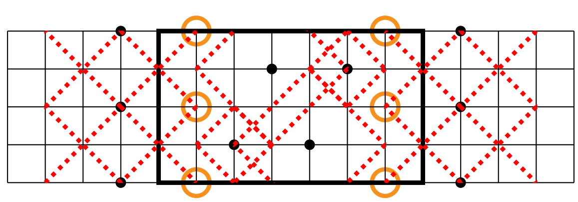


Figure 6.3.3: Sub-broadcast g' induced by $V(C) \cup \mathcal{U}$ as forced by D in Figure 6.3.2.

Algorithm 5: Routine to determine if necessary broadcasts force a contradiction.

function NecessaryBroadcastsContradiction ($G_{m,k}, g, m_1, \ldots, m_k$);

Input : A graph $G_{m,k} = (P_m \text{ or } C_m) \square P_k$ with columns labelled from left to right by $c_1, \ldots, c_k$, where $k \geq 13$, and rows labelled from 1 to m, a broadcast g, and m_i (for each $i = 1$ to k) defined as the maximum values over all $n > k + 2$ of $B(n) - B(n - i)$ where $B(n)$ is the bound we hope to prove.

Output: Value of the truth statement: "necessary broadcasts force a contradiction."

Let R be the range of g;
Set $g' = g$;
foreach $r = 1$ *to* m **do**
    Let v be the vertex in row r and columns c_6;
    if $v \notin R$ **then**
        Set $g'(u) = 2$ where u is the vertex in row r and column c_4;
    end
    Let v be the vertex in row r and columns c_{k-5};
    if $v \notin R$ **then**
        Set $g'(u) = 2$ where u is the vertex in row r and column c_{k-3};
    end
end
Let R' be the range of g';
if $HasBroadcastScheme(G_{m,k}, R', cost(g') - 1)$ **then**
    return True;
end
if $ResultsInMinimalityContradiction(G_{m,k}, R', cost(g'), m_1, \ldots, m_k)$ **then**
    return True;
end
return False;

6.3.4 Maximum Range Contradiction

Recall that C is the subgraph of $G_{m,n}$ induced by the vertices appearing in a minimum cost set (with respect to f) of r consecutive columns of $G_{m,n}$, g is the sub-broadcast induced by $V(C)$, R is the range of g, and $G_{m,k}$ is a k-column induced subgraph of $G_{m,n}$, where $k = r + 8$.

Let g' be the sub-broadcast induced by $V(C) \cup \mathcal{U}$, where $\mathcal{U}$ is the set of vertices at distance one from the leftmost or rightmost column of C that are necessarily broadcasting at strength two under f given g (as defined formally in Section 6.3.3).

Suppose there exists vertices of $V(C)$ which do not hear a broadcast under g'. For example, given $G_{m,k} = P_5 \Box P_{16}$, Figure 6.3.4 depicts a possible sub-broadcast g induced by $V(C)$ which, as shown by the circled vertex in Figure 6.3.5, does not result in an sub-broadcast g' induced by $V(C) \cup \mathcal{U}$ which dominates $V(C)$.

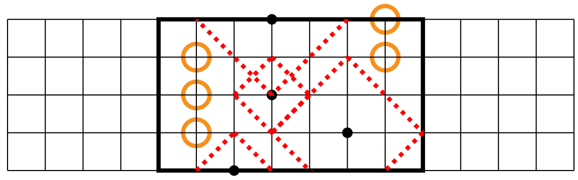


Figure 6.3.4: Sub-broadcast g induced by $V(C)$ and vertex set D.

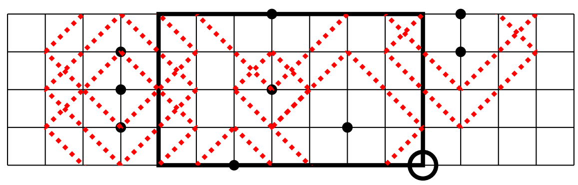


Figure 6.3.5: Sub-broadcast g' induced by $V(C) \cup \mathcal{U}$ forced by g in Figure 6.3.4 which does not dominate the circled vertex.

Let h be a minimum cost broadcast on $G_{m,k}$ which extends the broadcast of g' to dominate $V(C)$. For example, Figure 6.3.6 depicts a possible minimum cost broadcast h given g' in Figure 6.3.5. Let h' be the minimum cost induced sub-broadcast of f which dominates $V(C)$. Note that, though we do not know what h' looks like, we know that it contains g' as a sub-broadcast. Therefore, by our choice of h, $cost(h) \leq cost(h')$. Set $h'' = g'$. For each vertex $u \notin V(C)$ which is within distance two of a vertex $v \in V(C)$ not dominated by g', let $h''(u) = 2$. Let R'' be the range of h''. Observe that the range R' of h' is a subset of R''. Therefore, if R'' can be dominated with cost less than $cost(h)$, then R' can be dominated with cost less than $cost(h')$. This means that h' cannot be an induced sub-broadcast of an optimal broadcast f. This contradicts the choice of h'. For example, Figure 6.3.7 depicts h'' as determined from g' in Figure 6.3.5. The range R'' of h'' can be dominated with cost 14, as seen in Figure 6.3.8. This broadcast costs less than h as found in Figure 6.3.6.

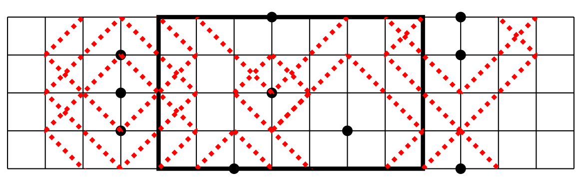


Figure 6.3.6: A minimum cost broadcast h which extends g' to dominate $V(C)$ from Figure 6.3.5.

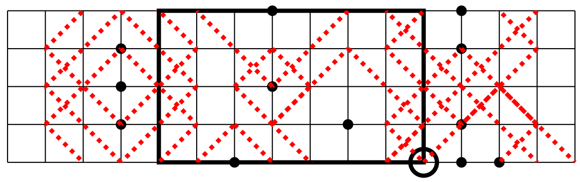


Figure 6.3.7: The broadcast h'' given g' in Figure 6.3.5.

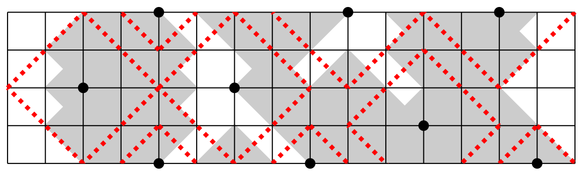


Figure 6.3.8: Broadcast of cost 14 which dominates R'' as seen in Figure 6.3.7.

The routine *MaximumRangeContradiction* returns true if h and h'' lead to a contradiction, and false otherwise.

6.3.5 Contradiction when Considering All Possible Sub-Cases

Recall that C is the subgraph of $G_{m,n}$ induced by the vertices appearing in a minimum cost set (with respect to f) of r consecutive columns of $G_{m,n}$, g is the sub-broadcast induced by $V(C)$, and R is the range of g.

Let g' be the sub-broadcast induced by $V(C) \cup \mathcal{U}$, where $\mathcal{U}$ is the set of vertices at distance one from the leftmost or rightmost column of C that are necessarily broadcasting at strength two under f given g (as defined formally in Section 6.3.3).

Let D' denote the vertices of C which do not hear a broadcast under g'. As the broadcast of f beyond $V(C) \cup \mathcal{U}$ is not defined explicitly, there are many possible ways f could dominate D'. We consider all possible ways f could dominate D'. That is, we construct every possible induced sub-broadcast g'' of f which extends the broadcast of g' to dominate D'. Let $\mathcal{C}'$ be the collection of all possible induced sub-broadcasts of f formed by extending the broadcast of g' until D' is dominated. For example, given $G_{m,k} = P_5 \square C_{16}$, Figure 6.3.11 depicts a possible induced sub-broadcast $h \in \mathcal{C}'$ constructed from g' in Figure 6.3.10 which was forced by g in Figure 6.3.9. The

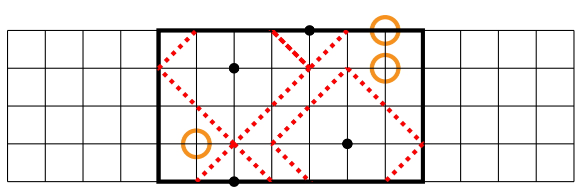


Figure 6.3.9: Sub-broadcast g induced by $V(C)$.

routine *NecessaryBroadcastsContradiction* returns true if, for all $h \in \mathcal{C}'$, h leads to a contradiction, and false otherwise.

Algorithm 6: Routine to determine if a contradiction arises from the maximum range test.

function MaximumRangeContradiction ($G_{m,k}, g, m_1, \ldots, m_k$);

Input : A graph $G_{m,k} = (P_m \text{ or } C_m) \square P_k$ with columns labelled from left to right by $c_1, \ldots, c_k$, where $k \geq 13$, a broadcast g, and m_i (for each $i = 1$ to k) defined as the maximum values over all $n > k + 2$ of $B(n) - B(n-i)$ where $B(n)$ is the bound we are attempting to prove..

Output: Value of the truth statement: "maximum range test force a contradiction."

Construct g' as described in Algorithm 5;

Let D' be the vertices in columns c_5 and c_{k-4} which do not hear a broadcast under g';

Let h be a minimum cost broadcast which extends the broadcast of g' to dominate D' constructed by altering the broadcasts of vertices in $V(G_{m,k}) \setminus [V(c_5) \cup \cdots \cup V(c_{k-4})]$;

Set $h'' = g'$;

foreach $v \in D'$ **do**

  foreach $x \in V(G_{m,k}) \setminus [V(c_5) \cup \cdots \cup V(c_{k-4})]$ **do**

    if $d(x, v) \leq 2$ **then**

      $h''(x) = 2$;

    end

  end

end

Let R'' be the range of h'';

if $HasBroadcastScheme(G_{m,k}, R'', cost(h) - 1)$ **then**

  return True;

end

if $ResultsInMinimalityContradiction(G_{m,k}, R'', cost(h), m_1, \ldots, m_k)$ **then**

  return True;

end

return False;

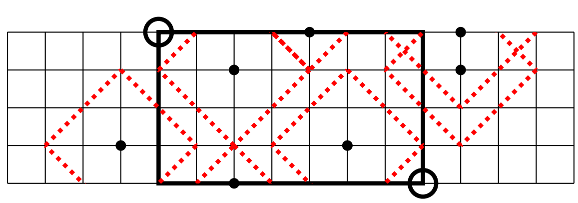


Figure 6.3.10: Sub-broadcast g' induced by $V(C) \cup \mathcal{U}$ forced by g in Figure 6.3.9 which does not dominate $V(C)$.

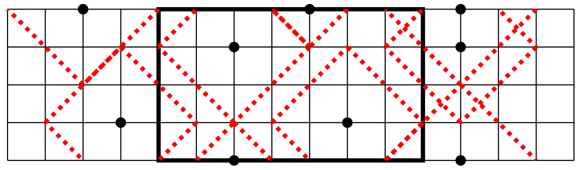


Figure 6.3.11: A possible induced sub-broadcast $h \in \mathcal{C}'$ given g' in Figure 6.3.10.

Algorithm 7: Routine to determine if every possible sub-case yields a contradiction.

function <u>ContradictionForEverySubCase</u> $(G_{m,k}, g, m_1, \ldots, m_k)$;

Input : A graph $G_{m,k} = (P_m \text{ or } C_m)\square P_k$ with columns labelled from left to right by $c_1, \ldots, c_k$, where $k \geq 13$, a broadcast g, and m_i (for each $i = 1$ to k) defined as the maximum values over all $n > k+2$ of $B(n) - B(n-i)$ where $B(n)$ is the bound we are attempting to prove.

Output: Value of the truth statement: "every possible sub-case yields a contradiction."

Construct g' as described in Algorithm 5;

Let D be the set of vertices which do not hear a broadcast under g in columns c_6 and c_{k-5};

Let $\mathcal{C}'$ be the collection of broadcasts formed by extending the broadcast of g' by altering the broadcasts of vertices in $V(G_{m,k}) \setminus [V(c_5) \cup \cdots \cup V(c_{k-4})]$ until every vertex of D' is dominated;

foreach $h \in \mathcal{C}'$ **do**

  Let R be the range of h;

  If $HasBroadcastScheme(G_{m,k}, R, cost(h) - 1)$ **then** break;

  If $ResultsInMinimalityContradiction(G_{m,k}, R, cost(h), m_1, \ldots, m_k)$ **then** break;

  return False;

end

return True;

6.3.6 Algorithm to Prove Lower Bounds

Algorithm 8: *ProvedLowerBound* implements the battery of tests described in Section 6.3 and Sections 6.3.1 through 6.3.5 to try and find contradictions for all possible sub-broadcasts induced on r consecutive columns on $P_m \square P_k$ or $C_m \square P_k$. The routine *ProvedLowerBound* returns true if every such possible induced sub-broadcast results in a contradiction and the bound is proven, and false otherwise.

Algorithm 8: Routine to prove lower bound.

function ProvedLowerBound $(G_{m,k}, s, t, m_1, \ldots, m_k)$;

Input : A graph $G_{m,k} = (P_m \text{ or } C_m)\Box P_k$, where $k \geq 13$, with columns labelled from left to right by $c_1, \ldots, c_k$, the minimum s and maximum t possible costs of a sub-broadcast g of $G_{m,k}$ in the $r = k-8$ consecutive columns centred in $G_{m,k}$, and m_i (for each $i = 1$ to k) defined as the maximum values over all $n > k+2$ of $B(n) - B(n-i)$ where $B(n)$ is the bound we are attempting to prove.

Output: Value of the truth statement: "lower bound $B(n)$ proven."

Let $\mathcal{C}$ be all possible sub-broadcasts g of costs s to t induced by the vertices of the $r = k-8$ consecutive columns centred in $G_{m,k}$;

foreach $g \in \mathcal{C}$ **do**

  if $DoesNotDominateMiddle(G_{m,k}, g)$ **then** break;

  if $ForbiddenBroadcast(G, g)$ **then** break;

  Let R be the range of g;

  if $HasBroadcastScheme(G_{m,k}, R, cost(g) - 1)$ **then** break;

  if $ResultsInMinimalityContradiction(G_{m,k}, R, cost(g), m_1, \ldots, m_k)$ **then** break;

  if $NecessaryBroadcastsContradiction(G_{m,k}, g, m_1, \ldots, m_k)$ **then** break;

  if $MaximumRangeContradiction(G_{m,k}, g, m_1, \ldots, m_k)$ **then** break;

  if $ContradictionForEverySubCase(G_{m,k}, g, m_1, \ldots, m_k)$ **then** break;

  return False;

end

return True;

6.4 Implementation

We have implemented *ProvedLowerBound* in Python 3.9 for $P_m \Box C_n$ and $C_m \Box C_n$. The code is compiled with Cython [3] and all ILP calls are run with a Gurobi solver [23]. All computations in this section were run on the author's 2012 4GB MacBook Pro with a 2.5 GHz Intel Dual-Core i5 processor.

Our implementation of *ProvedLowerBound* was first designed for proving lower bounds on $P_m \Box C_n$. Given this, we included a canonicity test for $P_m \Box P_n$ so as to only consider the set $\mathcal{C}$ of all possible sub-broadcast induced by the vertices of some set of r consecutive columns with costs s to t up to isomorphism. This test was done by checking that each broadcast (when expressed as a sequence) was the least lexicographically when compared to all broadcasts isomorphic to it. When adapting the code to work on $C_m \Box C_n$, we did not update the canonicity test to reduce the number of cases up to isomorphism on $C_m \square C_n$ from $P_m \square C_n$. Fortunately, for the bounds we tested, this redundancy was acceptable in terms of run times.

Our implementation of *ProvedLowerBound* has allowed us to prove Theorems 6.4.1 and 6.4.5, and their respective corollaries. The values of $|\mathcal{C}|$ in Tables 6.4.1 and 6.4.2 have been verified by Pólya's Theorem (see [5, Theorem 14.3.3]) as 2-limited broadcasts can simply be thought of as three-colourings of a graph (note that the colouring need not be proper).

Theorem 6.4.1. *For $n \geq 3$,*

$$\gamma_{b,2}\left(C_3 \square C_n\right) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. Proposition 4.2.1 states that $\gamma_{b,2}\left(C_3 \square C_n\right) \leq \left\lceil \frac{2n}{3} \right\rceil$. Fix $r = 6$. Let $k = 14 = r+8$. From our computational results in Section 4.2, we know that $\gamma_{b,2}\left(C_3 \square C_n\right) = \left\lceil \frac{2n}{3} \right\rceil$ for all $3 \leq n \leq 16$. Suppose the bound does not hold and let n be the smallest integer greater than 16 such that there exists an optimal 2-limited dominating broadcast f on $C_3 \square C_n$ of cost $\leq \left\lceil \frac{2n}{3} \right\rceil - 1$. As $B(n) = \left\lceil \frac{2n}{3} \right\rceil$ is the bound we hope to prove, for $1 \leq i \leq 14 = k$, the m_i's take on the following values:

$$\{m_1, \ldots, m_{14}\} = (1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10).$$

As $r - 4 = 2$ and $\gamma_{b,2}\left(C_3 \square P_2\right) = 2$, set $s = 2$. Set $t = 3$. Observe if $t > 3$ then $\left\lceil \frac{4n}{6} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil \leq \gamma_{b,2}\left(C_3 \square C_n\right)$. As

$$ProvedLowerBound\left(G_{3,14}, 2, 3, m_1, \ldots, m_{14}, \right)$$

is true, this proves the theorem. Running *ProvedLowerBound* for the above values took under six seconds. See Table 6.4.1 for a summary of the number of broadcasts rejected at each step of the algorithm per considered cost. Steps with zero cases are omitted from Table 6.4.1. Note that, as *ContradictionForEverySubCase* considers all possible induced sub-broadcast of a given case, the total number of cases considered will be at least $|\mathcal{C}|$.

	Cost 2	Cost 3
$\lvert\mathcal{C}\rvert$:	54	302
$DoesNotDominateMiddle$:	48	231
$ForbiddenBroadcast$:	4	45
$ResultsInMinimalityContradiction$:	0	12
$NecessaryBroadcastsContradiction$ + $HasBroadcastScheme$:	0	8
$NecessaryBroadcastsContradiction$ + $ResultsInMinimalityContradiction$:	2	3
$ContradictionForEverySubCase$ + $HasBroadcastScheme$:	0	336
$ContradictionForEverySubCase$ + $ResultsInMinimalityContradiction$:	0	96

Table 6.4.1: Cases considered in the proof of Theorem 6.4.1.

□

Observe the following corollaries of Theorem 6.4.1.

Corollary 6.4.2. *For* $n \geq 3$*,*

$$\left\lceil \frac{4n}{6} \right\rceil \leq \gamma_{b,2}\left(C_4 \square C_n\right) \leq 4\left\lfloor \frac{n}{6} \right\rfloor + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 6, \\ 2 & \text{for } n \equiv 1 \text{ or } 2 \pmod 6, \\ 3 & \text{for } n \equiv 3 \text{ or } 4 \pmod 6, \text{ and} \\ 4 & \text{for } n \equiv 5 \pmod 6. \end{cases}$$

Proof. The upper bound of Corollary 6.4.2 is stated in Proposition 4.3.1. Any 2-limited dominating broadcast on $C_4 \square C_n$ can be modified to give a 2-limited dominating broadcast on $C_3 \square C_n$. Therefore $\gamma_{b,2}\left(C_3 \square C_n\right) \leq \gamma_{b,2}\left(C_4 \square C_n\right)$ and the result follows from Theorem 6.4.1.

□

Corollary 6.4.3. *For* $m \geq 23$*,*

$$\gamma_{b,2}\left(P_m \square C_3\right) = \left\lceil \frac{2m}{3} \right\rceil$$

Proof. The upper bound of Corollary 6.4.3 is stated in Proposition 3.16.1. Any 2-limited dominating broadcast on $P_m \square C_3$ is a 2-limited dominating broadcast on $C_m \square C_3$. Therefore $\gamma_{b,2}\left(C_3 \square C_m\right) \leq \gamma_{b,2}\left(P_m \square C_3\right)$ and the result follows from Theorem 6.4.1.

$\square$

Corollary 6.4.4. *For $m \geq 23$,*

$$\left\lceil \frac{4m}{6} \right\rceil \leq \gamma_{b,2}\left(P_m \square C_4\right) \leq 4\left\lfloor \frac{m}{6} \right\rfloor + \begin{cases} 1 & \text{for } m \equiv 0 \pmod 6, \\ 2 & \text{for } m \equiv 1 \text{ or } 2 \pmod 6, \\ 3 & \text{for } m \equiv 3 \pmod 6, \text{ and} \\ 4 & \text{for } m \equiv 4 \text{ or } 5 \pmod 6. \end{cases}$$

Proof. The upper bound of Corollary 6.4.4 is stated in Proposition 3.17.1. Any 2-limited dominating broadcast on $P_m \square C_4$ is a 2-limited dominating broadcast on $C_m \square C_4$. Therefore $\gamma_{b,2}\left(C_3 \square C_m\right) \leq \gamma_{b,2}\left(P_m \square C_4\right)$ and the result follows from Theorem 6.4.1.

$\square$

Theorem 6.4.5. *For $n \geq 5$,*

$$\gamma_{b,2}\left(C_5 \square C_n\right) = n.$$

Proof. Proposition 4.4.1 states that $\gamma_{b,2}\left(C_5 \square C_n\right) \leq n$. Fix $r = 8$. Let $k = 16 = r + 8$. From our computational results in Section 4.4, we know that $\gamma_{b,2}\left(C_5 \square C_n\right) = n$ for all $3 \leq n \leq 18$. Suppose the bound does not hold and let n be the smallest integer greater than 18 such that there exists an optimal 2-limited dominating broadcast f on $C_5 \square C_n$ of cost $\leq n - 1$. As $B(n) = n$ is the bound we hope to prove, for $1 \leq i \leq 16 = k$, the m_i's take on the following values:

$$\{m_1, \ldots, m_{16}\} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16).$$

As $r - 4 = 4$ and $\gamma_{b,2}\left(C_5 \square P_4\right) = 5$, set $s = 4$. Set $t = 7$. Observe if $t > 7$ then $n \leq \gamma_{b,2}\left(C_3 \square C_n\right)$. As

$$ProvedLowerBound\left(G_{5,16}, 5, 7, m_1, \ldots, m_{16}\right)$$

is true, this proves the theorem. Running *ProvedLowerBound* for the above values took under eight minutes. See Table 6.4.2 for a summary of the number of broadcasts rejected at each step of the algorithm per considered cost. Steps with zero cases are omitted from Table 6.4.2. Note that, as $ContradictionForEverySubCase$ considers all possible induced sub-broadcast of a given case, the total number of cases considered will be at least $|\mathcal{C}|$.

	Cost 5	Cost 6	Cost 7
$\lvert\mathcal{C}\rvert$:	264,148	1,925,104	12,162,548
$DoesNotDominateMiddle$:	264,115	1,922,880	12,103,722
$ForbiddenBroadcast$:	8	1423	48,899
$HasBroadcastScheme$:	0	161	5,198
$ResultsInMinimalityContradiction$:	25	632	4,696
$NecessaryBroadcastsContradiction$ + $ResultsInMinimalityContradiction$:	0	5	27
$ContradictionForEverySubCase$ + $HasBroadcastScheme$:	0	27	0
$ContradictionForEverySubCase$ + $ResultsInMinimalityContradiction$:	0	48	30

Table 6.4.2: Cases considered in the proof of Theorem 6.4.5.

□

Observe the following corollaries of Theorem 6.4.5.

Corollary 6.4.6. *For $n \geq 6$,*

$$n \leq \gamma_{b,2}\left(C_6 \square C_n\right) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 4 \text{ and} \\ 1 & \text{for } n \equiv 1, 2, \text{ or } 3 \pmod 4. \end{cases}$$

Proof. The upper bound of Corollary 6.4.6 is stated in Proposition 4.5.1. Any 2-limited dominating broadcast on $C_6 \square C_n$ can be modified to give a 2-limited dominating broadcast on $C_5 \square C_n$. Therefore $\gamma_{b,2}\left(C_5 \square C_n\right) \leq \gamma_{b,2}\left(C_6 \square C_n\right)$ and the result follows from Theorem 6.4.5.

□

Corollary 6.4.7. *For $n \geq 3$,*

$$n \leq \gamma_{b,2}\left(P_5 \square C_n\right) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 2 \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod 2. \end{cases}$$

Proof. For $n = 3$ and 4 the bound follows from the computational results in Section 3.5. For $n \geq 5$, the upper bound of Corollary 6.4.7 is stated in Proposition 3.5.1. Any 2-limited dominating broadcast on $P_5 \square C_n$ is a 2-limited dominating broadcast on $C_5 \square C_n$. Therefore $\gamma_{b,2}\left(C_5 \square C_n\right) \leq \gamma_{b,2}\left(P_5 \square C_n\right)$ and the result follows from Theorem 6.4.5.

□

Corollary 6.4.8. *For $m \geq 23$,*

$$m \leq \gamma_{b,2}\left(P_m \square C_5\right) \leq m+1.$$

Proof. For $n \geq 23$, the upper bound of Corollary 6.4.8 is stated in Proposition 3.18.1. Any 2-limited dominating broadcast on $P_m \square C_5$ is a 2-limited dominating broadcast on $C_5 \square C_m$. Therefore $\gamma_{b,2}\left(C_5 \square C_m\right) \leq \gamma_{b,2}\left(P_m \square C_5\right)$ and the result follows from Theorem 6.4.5.

□

Corollary 6.4.9. *For $m \geq 23$,*

$$m \leq \gamma_{b,2}\left(P_m \square C_6\right) \leq m+1.$$

Proof. For $n \geq 23$, the upper bound of Corollary 6.4.9 is stated in Proposition 3.19.1. Any 2-limited dominating broadcast on $P_m \square C_6$ can be modified to give a 2-limited dominating broadcast on $C_5 \square C_m$. Therefore $\gamma_{b,2}\left(C_5 \square C_m\right) \leq \gamma_{b,2}\left(P_m \square C_6\right)$ and the result follows from Theorem 6.4.5.

□

Corollary 6.4.10. *For $n \geq 2$,*

$$n \leq \gamma_{b,2}\left(P_5 \square P_n\right) \leq n+1.$$

Proof. For $n = 2$, 3, and 4 the bound follows from the computational results in Section 2.5. For $n \geq 5$, the upper bound of Corollary 6.4.10 is stated in Proposition 2.5.1. Any 2-limited dominating broadcast on $P_5 \square P_n$ is a 2-limited dominating broadcast on $C_5 \square C_n$. Therefore $\gamma_{b,2}\left(C_5 \square C_n\right) \leq \gamma_{b,2}\left(P_5 \square P_n\right)$ and the result follows from Theorem 6.4.5.

□

Chapter 7
Concluding Remarks

Chapters 2 through 4 established upper bounds for the 2-limited broadcast domination number of the Cartesian products of two paths, a path and a cycle, and two cycles. Chapter 5 determined lower bounds for the 2-limited broadcast domination number for these graph classes by bounding the fractional 2-limited multipacking number for these graph classes. Chapter 6 presented an algorithm to prove lower bounds for the 2-limited broadcast domination number of these graph classes. This algorithm was then used to improve some of the bounds found in Chapter 5. Section 7.1 presents the cumulative bounds produced by our work. Section 7.2 concludes this thesis with open problems for future research.

7.1 Final Bounds

Tables 7.1.1, 7.1.2, and 7.1.3 present the lower and upper bounds for the 2-limited broadcast domination number of the Cartesian products of two paths, a path and a cycle, and two cycles, respectively, as proved in this thesis. The constant terms in the upper bounds can be found in their respective propositions in Chapters 2, 3, and 4. To easily compare the upper and lower bonds we have, for some bounds, included a simpler lower bounds.

Lower Bound	$\gamma_{b,2}(P_m \Box P_n)$	Upper Bound
$\left\lceil \frac{n}{2} \right\rceil \leq$	$\gamma_{b,2}(P_2 \Box P_n)$	$\leq \left\lfloor \frac{n}{2} \right\rfloor + 1$
$\left\lceil \frac{2n}{3} \right\rceil \leq$	$\gamma_{b,2}(P_3 \Box P_n)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{5} \right\rceil \leq$	$\gamma_{b,2}(P_4 \Box P_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$n \leq$	$\gamma_{b,2}(P_5 \Box P_n)$	$\leq n + 1$
$17.8 \left(\frac{n}{16}\right) \leq \left\lceil \frac{29n}{26} \right\rceil \leq$	$\gamma_{b,2}(P_6 \Box P_n)$	$\leq 18 \left\lfloor \frac{n}{16} \right\rfloor + O(1)$
$17.7 \left(\frac{n}{14}\right) \leq \left\lceil \frac{19n}{15} \right\rceil \leq$	$\gamma_{b,2}(P_7 \Box P_n)$	$\leq 18 \left\lfloor \frac{n}{14} \right\rfloor + O(1)$
$31.3 \left(\frac{n}{22}\right) \leq \left\lceil \frac{212n}{149} \right\rceil \leq$	$\gamma_{b,2}(P_8 \Box P_n)$	$\leq 32 \left\lfloor \frac{n}{22} \right\rfloor + O(1)$
$15.7 \left(\frac{n}{10}\right) \leq \left\lceil \frac{52n}{33} \right\rceil \leq$	$\gamma_{b,2}(P_9 \Box P_n)$	$\leq 16 \left\lfloor \frac{n}{10} \right\rfloor + O(1)$

$31.1\left(\frac{n}{18}\right) \leq \left\lceil \frac{780n}{451} \right\rceil \leq$	$\gamma_{b,2}\left(P_{10}\square P_n\right)$	$\leq 32\left\lfloor \frac{n}{18} \right\rfloor + O(1)$
$48.9\left(\frac{n}{26}\right) \leq \left\lceil \frac{81n}{43} \right\rceil \leq$	$\gamma_{b,2}\left(P_{11}\square P_n\right)$	$\leq 50\left\lfloor \frac{n}{26} \right\rfloor + O(1)$
$48.8\left(\frac{n}{24}\right) \leq \left\lceil \frac{273n}{134} \right\rceil \leq$	$\gamma_{b,2}\left(P_{12}\square P_n\right)$	$\leq 50\left\lfloor \frac{n}{24} \right\rfloor + O(1)$
$28.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5042n}{2301} \right\rceil \leq$	$\gamma_{b,2}\left(P_{13}\square P_n\right)$	$\leq 30\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$30.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{2324n}{991} \right\rceil \leq$	$\gamma_{b,2}\left(P_{14}\square P_n\right)$	$\leq 32\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$32.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5690n}{2277} \right\rceil \leq$	$\gamma_{b,2}\left(P_{15}\square P_n\right)$	$\leq 34\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$34.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{15593n}{5878} \right\rceil \leq$	$\gamma_{b,2}\left(P_{16}\square P_n\right)$	$\leq 36\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$36.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{28417n}{10125} \right\rceil \leq$	$\gamma_{b,2}\left(P_{17}\square P_n\right)$	$\leq 38\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$38.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{103240n}{34873} \right\rceil \leq$	$\gamma_{b,2}\left(P_{18}\square P_n\right)$	$\leq 40\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{3896n}{1251} \right\rceil \leq$	$\gamma_{b,2}\left(P_{19}\square P_n\right)$	$\leq 42\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$42.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{337976n}{103415} \right\rceil \leq$	$\gamma_{b,2}\left(P_{20}\square P_n\right)$	$\leq 44\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{304705n}{89043} \right\rceil \leq$	$\gamma_{b,2}\left(P_{21}\square P_n\right)$	$\leq 46\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$46.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{548313n}{153338} \right\rceil \leq$	$\gamma_{b,2}\left(P_{22}\square P_n\right)$	$\leq 48\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil \frac{2mn}{13} + \frac{14287568}{75254411}(m+n) - O(1) \right\rceil \leq$	$\gamma_{b,2}\left(P_{m\geq 23}\square P_n\right)$	$\leq \frac{2mn}{13} + \frac{4}{13}(m+n) + O(1)$

Table 7.1.1: Upper and lower bounds for $\gamma_{b,2}\left(P_m\square P_n\right)$ for $m, n \geq 2$ where the respective constants can be found in Chapter 2.

Lower Bound	$\gamma_{b,2}\left(P_m\square C_n\right)$	Upper Bound
$\left\lceil \frac{n}{2} \right\rceil \leq$	$\gamma_{b,2}\left(P_2\square C_n\right)$	$\leq \left\lceil \frac{n}{2} \right\rceil$
$\left\lceil \frac{2n}{3} \right\rceil \leq$	$\gamma_{b,2}\left(P_3\square C_n\right)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{5} \right\rceil \leq$	$\gamma_{b,2}\left(P_4\square C_n\right)$	$\leq 8\left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$n \leq$	$\gamma_{b,2}\left(P_5\square C_n\right)$	$\leq n + O(1)$
$17.8\left(\frac{n}{16}\right) \leq \left\lceil \frac{29n}{26} \right\rceil \leq$	$\gamma_{b,2}\left(P_6\square C_n\right)$	$\leq 18\left\lfloor \frac{n}{16} \right\rfloor + O(1)$
$17.7\left(\frac{n}{14}\right) \leq \left\lceil \frac{19n}{15} \right\rceil \leq$	$\gamma_{b,2}\left(P_7\square C_n\right)$	$\leq 18\left\lfloor \frac{n}{14} \right\rfloor + O(1)$
$31.3\left(\frac{n}{22}\right) \leq \left\lceil \frac{212n}{149} \right\rceil \leq$	$\gamma_{b,2}\left(P_8\square C_n\right)$	$\leq 32\left\lfloor \frac{n}{22} \right\rfloor + O(1)$
$15.7\left(\frac{n}{10}\right) \leq \left\lceil \frac{52n}{33} \right\rceil \leq$	$\gamma_{b,2}\left(P_9\square C_n\right)$	$\leq 16\left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$31.1\left(\frac{n}{18}\right) \leq \left\lceil \frac{780n}{451} \right\rceil \leq$	$\gamma_{b,2}\left(P_{10}\square C_n\right)$	$\leq 32\left\lfloor \frac{n}{18} \right\rfloor + O(1)$
$48.9\left(\frac{n}{26}\right) \leq \left\lceil \frac{81n}{43} \right\rceil \leq$	$\gamma_{b,2}\left(P_{11}\square C_n\right)$	$\leq 50\left\lfloor \frac{n}{26} \right\rfloor + O(1)$
$48.8\left(\frac{n}{24}\right) \leq \left\lceil \frac{273n}{134} \right\rceil \leq$	$\gamma_{b,2}\left(P_{12}\square C_n\right)$	$\leq 50\left\lfloor \frac{n}{24} \right\rfloor + O(1)$
$28.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5042n}{2301} \right\rceil \leq$	$\gamma_{b,2}\left(P_{13}\square C_n\right)$	$\leq 30\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$30.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{2324n}{991} \right\rceil \leq$	$\gamma_{b,2}\left(P_{14}\square C_n\right)$	$\leq 32\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$32.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{5690n}{2277} \right\rceil \leq$	$\gamma_{b,2}\left(P_{15}\square C_n\right)$	$\leq 34\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$34.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{15593n}{5878} \right\rceil \leq$	$\gamma_{b,2}\left(P_{16}\square C_n\right)$	$\leq 36\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$36.4\left(\frac{n}{13}\right) \leq \left\lceil \frac{28417n}{10125} \right\rceil \leq$	$\gamma_{b,2}\left(P_{17}\square C_n\right)$	$\leq 38\left\lfloor \frac{n}{13} \right\rfloor + O(1)$

$38.4\left(\frac{n}{13}\right) \le \left\lceil \frac{103240n}{34873} \right\rceil \le$	$\gamma_{b,2}\left(P_{18}\square C_n\right)$	$\le 40\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \le \left\lceil \frac{3896n}{1251} \right\rceil \le$	$\gamma_{b,2}\left(P_{19}\square C_n\right)$	$\le 42\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$42.4\left(\frac{n}{13}\right) \le \left\lceil \frac{337976n}{103415} \right\rceil \le$	$\gamma_{b,2}\left(P_{20}\square C_n\right)$	$\le 44\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$40.4\left(\frac{n}{13}\right) \le \left\lceil \frac{304705n}{89043} \right\rceil \le$	$\gamma_{b,2}\left(P_{21}\square C_n\right)$	$\le 46\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$46.4\left(\frac{n}{13}\right) \le \left\lceil \frac{548313n}{153338} \right\rceil \le$	$\gamma_{b,2}\left(P_{22}\square C_n\right)$	$\le 48\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 2\left(\frac{mn}{13}\right) + \frac{2620n}{13767} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_n\right)$	$\le 2\left(\frac{mn}{13}\right) + \frac{4m}{13} + O(n)$
$\left\lceil \frac{2m}{3} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_3\right)$	$\le \left\lceil \frac{2m}{3} \right\rceil$
$\left\lceil \frac{4m}{6} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_4\right)$	$\le 4\left\lfloor \frac{m}{6} \right\rfloor + O(1)$
$m \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_5\right)$	$\le m+1$
$m \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_6\right)$	$\le m+1$
$37.6\left(\frac{m}{35}\right) \le \left\lceil \frac{14m}{13} + \frac{18340}{13767} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_7\right)$	$\le 42\left\lfloor \frac{m}{35} \right\rfloor + O(1)$
$7.3\left(\frac{n}{6}\right) \le \left\lceil \frac{16m}{13} + \frac{20960}{13767} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_8\right)$	$\le 8\left\lfloor \frac{m}{6} \right\rfloor + O(1)$
$13.8\left(\frac{n}{10}\right) \le \left\lceil \frac{18m}{13} + \frac{7860}{4589} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_9\right)$	$\le 16\left\lfloor \frac{m}{10} \right\rfloor + O(1)$
$15.3\left(\frac{n}{10}\right) \le \left\lceil \frac{20m}{13} + \frac{26200}{13767} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_{10}\right)$	$\le 16\left\lfloor \frac{m}{10} \right\rfloor + O(1)$
$\left\lceil \frac{22m}{13} + \frac{28820}{13767} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_{11}\right)$	$\le 24\left\lfloor \frac{m}{13} \right\rfloor + O(1)$
$\left\lceil \frac{24m}{13} + \frac{10480}{4589} \right\rceil \le$	$\gamma_{b,2}\left(P_{m\ge 23}\square C_{12}\right)$	$\le 26\left\lfloor \frac{m}{13} \right\rfloor + O(1)$

Table 7.1.2: Upper and lower bounds for $\gamma_{b,2}\left(P_m\square C_n\right)$ for $m, n \ge 3$ where the respective constants can be found in Chapter 3.

Lower Bound	$\gamma_{b,2}\left(C_m\square C_n\right)$	Upper Bound
$\left\lceil \frac{2n}{3} \right\rceil \le$	$\gamma_{b,2}\left(C_3\square C_n\right)$	$\le \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{6} \right\rceil \le$	$\gamma_{b,2}\left(C_4\square C_n\right)$	$\le 4\left\lfloor \frac{n}{6} \right\rfloor + O(1)$
$n \le$	$\gamma_{b,2}\left(C_5\square C_n\right)$	$\le n$
$n \le$	$\gamma_{b,2}\left(C_6\square C_n\right)$	$\le n + O(1)$
$37.6\left(\frac{n}{35}\right) \le \left\lceil 14\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_7\square C_n\right)$	$\le 42\left\lfloor \frac{n}{35} \right\rfloor + O(1)$
$7.3\left(\frac{n}{6}\right) \le \left\lceil 16\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_8\square C_n\right)$	$\le 8\left\lfloor \frac{n}{6} \right\rfloor + O(1)$
$13.8\left(\frac{n}{10}\right) \le \left\lceil 18\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_9\square C_n\right)$	$\le 16\left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$15.3\left(\frac{n}{10}\right) \le \left\lceil 20\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{10}\square C_n\right)$	$\le 16\left\lfloor \frac{n}{10} \right\rfloor + O(1)$
$\left\lceil 22\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{11}\square C_n\right)$	$\le 24\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 24\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{12}\square C_n\right)$	$\le 26\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 26\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{13}\square C_n\right)$	$\le 26\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 28\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{14}\square C_n\right)$	$\le 31\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 30\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{15}\square C_n\right)$	$\le 33\left\lfloor \frac{n}{13} \right\rfloor + O(1)$
$\left\lceil 32\left(\frac{n}{13}\right) \right\rceil \le$	$\gamma_{b,2}\left(C_{16}\square C_n\right)$	$\le 36\left\lfloor \frac{n}{13} \right\rfloor + O(1)$

$\lceil 34\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{17}\square C_n\right)$	$\leq 36\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 36\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{18}\square C_n\right)$	$\leq 39\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 38\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{19}\square C_n\right)$	$\leq 41\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 40\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{20}\square C_n\right)$	$\leq 42\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 42\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{21}\square C_n\right)$	$\leq 45\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 44\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{22}\square C_n\right)$	$\leq 48\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 46\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{23}\square C_n\right)$	$\leq 49\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 48\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{24}\square C_n\right)$	$\leq 50\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 50\left(\frac{n}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{25}\square C_n\right)$	$\leq 52\left\lfloor\frac{n}{13}\right\rfloor+O(1)$
$\lceil 2\left(\frac{mn}{13}\right)\rceil \leq$	$\gamma_{b,2}\left(C_{m\geq 26}\square C_{n\geq 26}\right)$	$\leq 2\left(\frac{mn}{13}\right)+O(m+n)$

Table 7.1.3: Upper and lower bounds for $\gamma_{b,2}\left(C_m\square C_n\right)$ for $m,n\geq 3$ where the respective constants can be found in Chapter 4.

7.2 Future Work

Although we have bounded the 2-limited broadcast domination number for the Cartesian products of two paths, a path and a cycle, and two cycles, few of our bounds are tight. Additionally, given a graph G, since $\gamma_{b,k}(G)\leq\gamma_{b,k-1}(G)$, our bounds provide upper bounds for the k-limited broadcast domination number for these graph classes where $k\geq 2$. However, these bounds are likely very far from the truth with respect to large grids. Given this, we present the following problems.

Problem 7.1. *Determine $\gamma_{b,k}\left(P_m\square P_n\right)$ for all $m,n\geq 2$ for $k\geq 2$.*

Problem 7.2. *Determine $\gamma_{b,k}\left(P_m\square C_n\right)$ for all $m\geq 2$ and $n\geq 3$ for $k\geq 2$.*

Problem 7.3. *Determine $\gamma_{b,k}\left(C_m\square C_n\right)$ for all $m,n\geq 3$ for $k\geq 2$.*

Little is known about the k-limited broadcast domination number for many graph classes but even less is known about the k-limited multipacking number. In this thesis, we determined lower bounds for the fractional 2-limited multipacking number for the Cartesian products of two paths, a path and a cycle, and two cycles. However, we did not include (nor find) any results with relation to the 2-limited multipacking number of these graphs. More generally, nothing is known about the k-limited multipacking number mp_k of these graphs. Given this, we present the following problems.

Problem 7.4. *Determine $mp_k\left(P_m\square P_n\right)$ for all $m,n\geq 2$ for $k\geq 2$.*

Problem 7.5. *Determine $mp_k\left(P_m\square C_n\right)$ for all $m\geq 2$ and $n\geq 3$ for $k\geq 2$.*

Problem 7.6. *Determine* $mp_k(C_m \square C_n)$ *for all* $m, n \geq 3$ *for* $k \geq 2$*.*

Many of the bounds for the 2-limited broadcast domination number of the Cartesian products of two paths, a path and a cycle, and two cycles found in this thesis were informed by known results found by computational methods. Future work on these graph classes would benefit from a better understanding of the complexity results relevant to these graph classes.

Problem 7.7. *Although the* k*-limited broadcast domination problem is NP-complete, does there exist polynomial time algorithms for this problem for the Cartesian products of two paths, a path and a cycle, and two cycles?*

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