# FAULT DETECTION IN COMBINATIONAL NETWORKS USING REED-MULLER AND MODIFIED REED-MULLER SPECTRAL TECHNIQUES

by

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## ABSTRACT

The testing of digital circuits has become more difficult and more expensive with the increasing complexity and density of circuits. Many approaches to speeding up testing have been proposed, such as signature analysis and a number of different spectral techniques. In this research, we concentrate on spectral techniques for fault detection.

Reed-Muller (RM) coefficients have been used for canonical expressions for switching functions for a number of years. Recently two authors have proposed their use for fault detection. In this thesis, a further study of their effectiveness is undertaken and a new spectral technique for fault detection based on modified Reed-Muller (MRM) coefficients is developed. An analysis and derivation of the properities of MRM coefficients is given, and it is shown that the many of the Reed-Muller properties follow as special cases of the modified Reed-Muller results. A new matrix method to analyse the properities of Reed-Muller (RM) and modified Reed-Muller spectra is used which simplifies the analysis.

The detection of stuck-at and bridging faults is considered. The testability for stuck-at faults and input bridging faults by using Reed-Muller and modified Reed-Muller coefficients is examined. It is shown that for these faults, a test that is both simple and efficient can be found. A new testing method, constrained RM and MRM spectral testing, is proposed. Constrained Reed-Muller or modified Reed-Muller testing can make the detection of some internal stuck-at faults much easier.

A fault simulation program is developed and used to examine a set of small circuits in detail and compare the various possible signatures.

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# LIST OF SYMBOLS AND NOTATION

| | |
|---|---|
| **n** | number of input variables |
| **N** | $N = 2^n$ |
| $x_i$, i=1,...,n | independent Boolean variables, $x_i$ in {0,1} |
| **X** | $\{x_1, ..., x_n\}$ |
| $f(x_1 \cdots x_n)$ or f(X) | Boolean function of the input variables, f(X) in {0,1} |
| **f,g,h** | Boolean functions |
| **Z** | truth table column vector for f(X), ordered with $x_1$ as least significant variable |
| **u,v** | sub-function identifier |
| $J^n$ | $2^n$ column vector whose top entry is one, all remaining entries are zero |
| **I** | $2^n$ column vector whose entries are all one |
| $\lfloor j \rfloor$ | the order of the integer j |
| **B** | Arithmetic spectrum |
| $b_\alpha$ | entries in B |
| **R** | Rademacher-Walsh spectrum |
| $r_\alpha$ | entries in R |
| **D** | Reed-Muller spectrum |
| $d_\alpha$ | entries in D |
| $D^k$ | The kth polarity Reed-Muller spectrum |
| $d_\alpha^k$ | entries in $D^k$ |
| $\underset{\sim}{D}$ | Modified Reed-Muller spectrum |
| $\underset{\sim}{d}_\alpha$ | entries in $\underset{\sim}{D}$ |
| $\underset{\sim}{D}^k$ | The kth polarity Modified Reed-Muller spectrum |

| | |
|---|---|
| $\underset{\sim}{d}_{\alpha}^{k}$ | entries in $\underset{\sim}{D}^{k}$ |
| $\tilde{Z}$ | $2^n \times 2^n$ polarity output matrix |
| $\tilde{D}$ | $2^n \times 2^n$ polarity Reed-Muller coefficient matrix |
| $\underset{\sim}{\tilde{D}}$ | $2^n \times 2^n$ polarity Modified Reed-Muller coefficient matrix |
| $T^n$ | general $2^n \times 2^n$ transform matrix |
| $RM^n$ | $2^n \times 2^n$ Reed-Muller transform matrix |
| **g/0** | stuck-at 0 fault on line g |
| **g/1** | stuck-at 1 fault on line g |
| **AND-BF**$(x_i,x_j)$ | AND bridging fault between lines $x_i$ and $x_j$ |
| **OR-BF**$(x_i,x_j)$ | OR bridging fault between lines $x_i$ and $x_j$ |

### Sub-function of a Binary Function f(X)

$f_u(x_1, \ldots, x_m) = f(x_1, \ldots, x_m, u_1, \ldots, u_{n-m})$, where $(u_1, \ldots, u_{n-m})$ is the binary expansion of u, i.e., $u = \sum_{i=1}^{n-m} u_i 2^{i-1}$

| | |
|---|---|
| $Z_u$ | minterm column vector for $f_u$, $Z_u \subseteq \{0,1\}$ |
| $R_u$, $D_u$, $\underset{\sim}{D}_u$ | different spectra of $Z_u$ |

### Additional Subscript Notations

| | |
|---|---|
| **i** | lower case letters denote integers |
| **I** | upper case letters denote the binary representation of the corresponding i |
| $\alpha$ | Greek letters denote sets |
| $\lvert i \rvert$ | number of 1's in I |
| $\lvert \alpha \rvert$ | cardinality of $\alpha$ |

### Additional Superscript Notation

| | |
|---|---|
| * | the function or spectrum in the presence of a fault |

**Mathematical Symbols**

| | |
|---|---|
| + | arithmetic addition or logical OR, the context making clear which is used |
| $\times$ | arithmetic multiplication |
| . | multiplication or logical AND (logical AND is also indicated by just concatenating the variables if no ambiguity arises), the context making clear which is used |
| $\oplus$ | exclusive-or, or mod 2 sum |
| $\underline{\oplus}$ | dyadic sum, or componentwise mod 2 sum |
| $\circledast$ | Kronecker product |
| $\otimes$ | matrix multiplication over GF(2) |
| $\odot$ | Hadamard product |

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# CHAPTER 1

# INTRODUCTION

Very Large Scale Integration (VLSI) technology has been developed so that hundreds of thousands or even millions of transistors can be integrated on a single silicon chip. This has enabled the development of very complex systems using a relatively small number of chips and boards. It is evident that the systems must perform more and more reliably. The confidence in the reliability of a system in normal use depends on testing - that is, the determination as to whether circuits have been manufactured properly and behave correctly. For the past twenty years chip complexity and density have increased rapidly. The testing of these circuits has become more difficult, more expensive, and much more time consuming.

The testing process for circuits has to solve two problems:

(1) Finding an appropriate test.

(2) Determining the fault coverage for the test.

As logic networks get larger, conventional test methods based on test sets have been found to often be too costly, complicated or time-consuming for VLSI testing. Moreover, the generation of tests is much more difficult for sequential circuits than that for combinational ones. New test methods with high fault coverage have to be found to reduce the complexity of generating and processing the test data.

One approach to alleviating the testing problem is embodied in a collection of techniques known as *"design for testability"* (DFT) [1]. The main idea of design for testability is to reduce the complexity of testing by adding extra circuitry to a circuit or a chip. There are two concepts in design for testability: *controllability* and *observability*. Controllability is defined as a measure of how easily the internal logic of the circuit can be controlled from its primary inputs. Observability is defined as a measure of how easily the internal logic of the circuit can be observed from its primary outputs.

Various approaches have been suggested to improve the controllability and observability of internal circuits.

A direct approach to enhancing the controllability and observability of a circuit is to use *test points*, that is, additional inputs and outputs are added to the original design. Several techniques to identify how and where to add such test points have been introduced by Hayes and Friedman [2], and Saluja and Reddy [3]. These techniques can achieve a high level of testability, but they have an unreasonably high overhead in extra circuitry and pins on the chip.

In order to increase the controllability and observability in general sequential circuits, several approaches have been introduced. In these, a sequential circuit is designed in such a way that a circuit can easily be set to any desired internal state and the internal states of the circuit can easily be observed. Most of the approaches are based on *scan-path techniques* [4]. A well known scan technique is *Level-Sensitive Scan Design* (LSSD) which has been used in many IBM systems [5].

As digital circuits grow more complex and difficult to test, it becomes increasingly attractive to build some self-testing ability into the chips. This technique is called *Built-in Test* (BIT). The built-in test can be used to enhance testability of VLSI. It is based on the following principles:

(1) Test patterns are generated on-chip.

(2) Responses to the test patterns are also evaluated on-chip.

(3) External operations are required only to initialize the built-in tests and to check the go/no-go test results from a chip.

Built-in test methods can be grouped into two categories: either concurrent or explicit testing. These are described in Chapter 2.

Built-in test has many advantages:

(1) Test patterns are generated automatically inside chips.

(2) Responses to test patterns need not be stored.

(3) Use of expensive test equipment is not necessary.

In recent years, the idea of built-in test has been pursued more aggressively.

Because it requires some test circuitry inside the chip, the test methods and hardware requirement used by built-in test must be as simple as possible, - that is, the generating of the test patterns, the processing of the test data, and the implementation mechanisms must be kept as simple as possible. Many approaches to built-in test have been proposed, such as signature analysis and a variety of different spectral techniques. An overview of these approaches is given in Chapter 2.

The majority of existing methods for designing and analysing a switching circuit are concerned with the properties of Boolean functions in the Boolean domain. Part of the problem with the definition in the Boolean domain is that each of the entries in the output vector tell us precisely the behaviour of the function at a single point but nothing of its behaviour for any other points. This prevents us from recognizing certain characteristics of interest for the entire function by looking at just a few numbers. It is possible to give an alternate representation of a function where the information about the function is much more global in nature. This alternate representation is in the spectral domain. In this research, we concentrate on some specific spectral techniques. The basic idea is to find a simple test generation method, compact the output responses to a reasonable size, and use a small hardware checker to verify the output responses to give a go/no-go result.

Reed-Muller (RM) expressions have been used since their introduction by Muller [48] for both the representation and analysis of switching functions. Until recently, no-one has suggested the use of the values from those expressions, the Reed-Muller coefficients, for fault detection. Damarla and Karpovsky [67] present a new approach using Reed-Muller spectral coefficients. They show that fault detection can be carried out by verification of a few RM coefficients. The hardware requirement for RM testing is very small.

The research in this thesis introduces a new set of coefficients, modified Reed-Muller (MRM) coefficients, and develops all the required properties to enable them to be used for fault detection. It is shown that these approaches give some of the existing Reed-Muller results as special cases, as well as giving a number of new Reed-Muller properties. A new matrix method, which simplifies the analysis, is used to analyse the properties of RM and MRM spectra.

Chapter 2 provides background information about some fault models and a variety of testing methods. First of all, some classical fault models are introduced. Then an overview of various testing methods proposed in the literature to lower the expense of the traditional test set technique is given with some simple examples. It gives a general idea about some new approaches based on data compaction techniques, and enables comparisons to be made. We also give some mathematical background about the spectral techniques used for the development of the later results in Chapter 2.

In this research, we concentrate on Reed-Muller (RM) and modified Reed-Muller (MRM) spectral techniques for fault detection. A new spectral technique based on modified Reed-Muller coefficients is presented. The relevant mathematical background is given in Chapter 3. This background includes the concepts of the generalized Reed-Muller expansion, RM coefficients, and RM transformations. Chapter 3 also includes many of the new theorems for modified Reed-Muller coefficients which have been developed for this thesis.

Some alternative spectral techniques have been proposed by other authors. The conversions which are required for this research are introduced in Chapter 4. Some properties of the RM and MRM spectra are also given in this chapter.

In Chapter 5, the testability using Reed-Muller and modified Reed-Muller spectra is examined. The criteria for RM and MRM techniques to detect stuck-at and bridging faults are derived. In this research, we are only concerned with single or multiple input stuck-at faults, single internal stuck-at faults, and single or multiple input bridging faults. Although there are $2^n$ different polarity RM and MRM spectra, each containing $2^n$ coefficients, only the 0th polarity coefficients are examined in this research. The testability using other polarity coefficients is left to further research. A new approach, constrained RM and MRM testing is presented in this chapter.

The implementation and hardware complexity of computing the RM and MRM spectra is given in Chapter 6, and compared with other spectral techniques. In order to compare testabilities using different spectral techniques, we develop a fault simulation program to simulate the detection for all possible single stuck-at faults in a set of example circuits. The simulation provides the minimum number of test patterns or minimum number of coefficients used by different spectral techniques for each circuit. All of the simulation results are given in Chapter 6.

Finally, conclusions and the directions for further research are given in Chapter 7.

# CHAPTER 2

# BACKGROUND

## 2.1. Introduction

The reliability of a computer depends heavily on testing. However, because of the rapidly increasing circuit density in LSI (large-scale-integration) and VLSI (very-large-scale-integration) technology, testing is getting much more difficult.

A *fault* in a circuit is a physical defect of one or more components. Failures can be classified as *permanent* or *intermittent*. As the name implies a *permanent failure* is one which is always present and does not disappear, or change its nature during testing. Faults that are present in some intervals of time and absent in others are *intermittent faults*. Since such a fault may not be present when a test is applied, we concentrate on permanent faults in this research.

To ensure the proper operation of a system, we must be able to detect a fault when one has occurred and to locate it to a specific component. The former procedure is called *fault detection*, while the latter is called *fault location* or *fault diagnosis*. Fault diagnosis is required in the testing of boards and systems. It is very important in factory testing and in field maintenance. In the testing of VLSI chips it is not necessary to locate faulty gates, since the entire chip must be discarded if any fault was found. Only a "go/no-go" signal is needed after the testing to indicate whether the CUT (circuit under test) is good or not. In this research we consider the problem of fault detection only.

In order to develop a test for a fault effect, the failure must be cast in functional terms. Since an integrated circuit (IC) is too complex to analyse directly, we use some *fault models*. An effective fault model can cover a reasonable set of physical failures.

Most fault models work with a gate level abstraction of the circuit. Physical failures and fault models are introduced in Section 2.2.

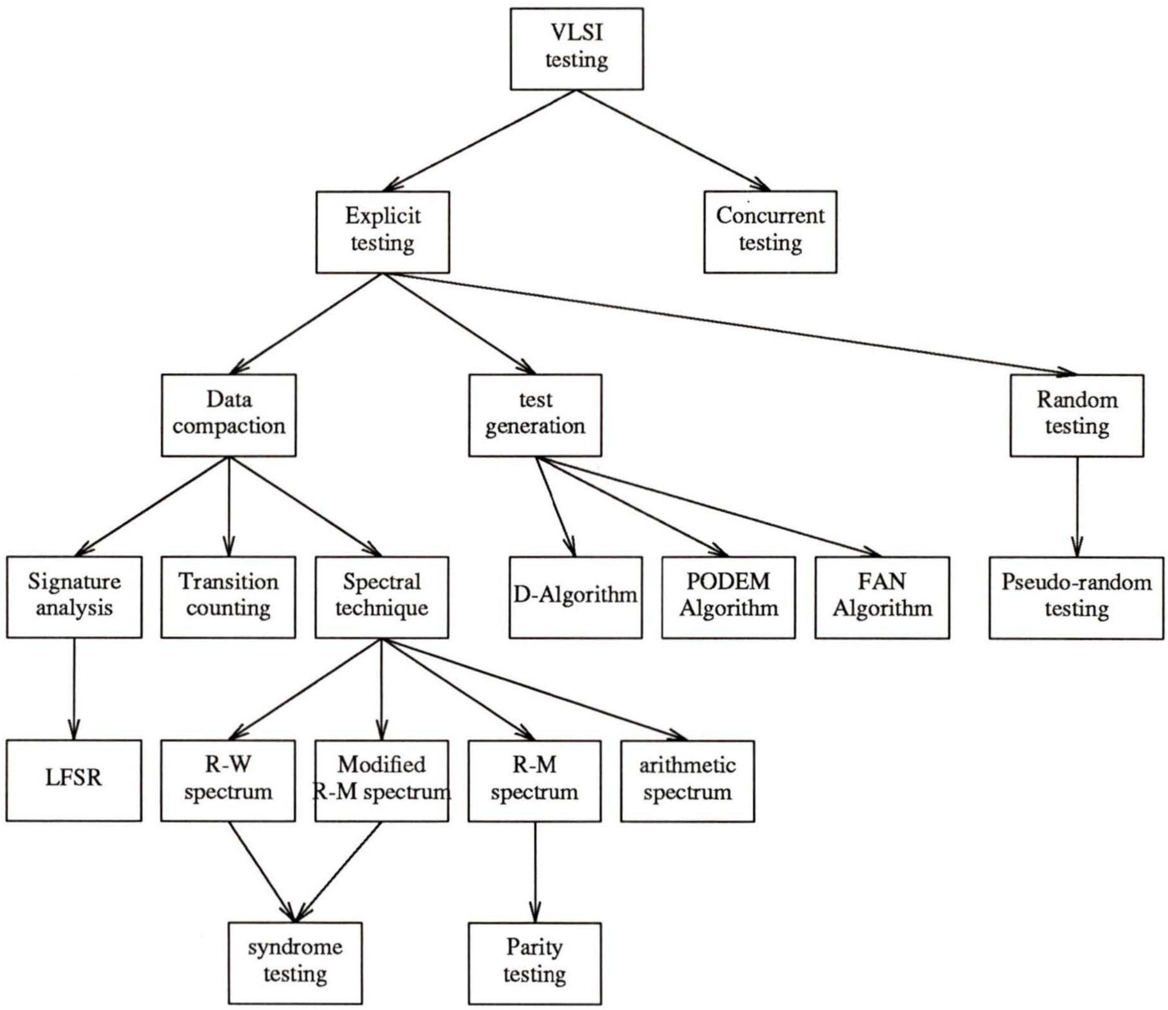



**Figure 2.1** VLSI Testing.

Testing methods for LSI circuits can be divided into two categories - *concurrent* and *explicit* testing. In concurrent approaches, normal user-application input patterns serve as test patterns. Thus testing and normal computation proceed concurrently. In explicit approaches, on the other hand, special input patterns are applied as tests. Hence, normal computation and testing occur at different times. They are discussed in Section 2.2. We only consider explicit testing in this research. Explicit test methods can be distinguished by the techniques used to generate the test patterns and to detect and evaluate the fault responses. A general overview of VLSI testing methods is shown in Fig. 2.1.

The traditional approach is that of a *test set*. Logic circuits are tested by applying a sequence of input patterns that produce erroneous responses when faults are present and then comparing the responses with the correct (expected) ones. Such an input pattern used in testing is called a *test pattern*. In general, a test for a logic circuit consists of some subset of all the possible input patterns. This subset is labeled a *test set*. The complexity of identifying a test set grows rapidly with the size of the network. For large networks this can be prohibitive, and together with large memory requirements, effectively rules out the test set approach as a viable method for on-chip or on-board self-test.

Several alternatives have appeared in the literature. A number use some form of data compaction to reduce the response data to a manageable size. Some examples of such techniques are signature analysis, syndrome testing, and spectral techniques. An overview of data compaction is given in Section 2.4. We concentrate on spectral techniques in this research. Syndrome testing can be considered as a special case of spectral coefficient testing. The background to spectral techniques is given in Section 2.2.4.

## 2.2. Fault Models and Testing Methods

### 2.2.1. Physical Failures and Fault Models

The physical failures in an integrated circuit (IC) differ according to the technology (bipolar, MOS, CMOS etc.), the density of integration, and other factors, such as the operating temperature and operating voltage. A detailed study of physical failure mechanisms is thus beyond the scope of this research.

The large number and complex nature of such physical failures dictates that a practical approach to testing should avoid working directly with the physical failures. In most cases, in fact, one is not concerned with discovering the exact physical failure; what is desired is merely to determine the existence of (or absence of) any physical failure. A physical failure that changes the function of a circuit can be detected by applying an appropriate sequence of input vectors (tests) to the circuit and observing the resulting output behaviour. Any deviation in the response from the known good response of the system under test indicates a failure is present. Many different physical failures may cause the same error under that test.

One approach for solving this problem is to describe the effects of physical failures at some higher level (logic, functional block, etc.). This description is called a fault model. If the fault model is reasonable to describe the effect of physical failures of interest, then one only needs to derive tests to detect all the faults in the fault model.

A fault model can be formulated at any level. At the lowest level the description of the physical failure is the same as the fault model. This is usually at the layout (mask) or transistor level. At a higher level, the fault model is usually formulated in terms of logic gates. It is called a "gate-level fault model". A number of models have been proposed.

### I. Stuck-at Faults.

The most common model used for logical faults is the "single stuck-at fault" [6, 7]. It assumes that a fault in a logic gate results in one of its inputs or the output being fixed to either a logic 0 (stuck-at-0) or a logic 1 (stuck-at-1). Stuck-at-0 and stuck-at-1 faults are often abbreviated to s-a-0 and s-a-1 respectively.

### II. Bridging (Short-circuit) Faults

Bridging faults form an important class of permanent faults which cannot be modeled as stuck-at faults. A bridging fault (BF) occurs when two leads in a logic network are connected accidentally and "wired logic" is performed at the connection. Depending on whether positive or negative logic is being used, and on the technology used, the faults have the effect of ANDing or ORing the signals involved. We limit our discussion to AND bridging faults. Similar results follow for OR bridging faults.

With stuck-at faults, if there are n lines in the circuit, there are 2n possible single stuck-at faults, and $(3^n - 1)$ possible multiple stuck-at faults. With bridging faults, if bridging between any s lines in a circuit are considered, the number of single bridging faults alone will be $\binom{q}{s}$, where q is the number of lines to be considered in the circuit. Consider a fan-out node in the circuit as shown in Fig.2.2. For single stuck-at faults, the number n counts all lines, including separate lines for stem 1, and branches 2, 3 in a fan-out situation. A bridging fault does not need this separation since the signal performed by the wired logic on any part of the fan-out is carried both forward and backward to stems and branches. Thus $q < n$, except for a fan-out free network where $q = n$. The number of multiple bridging faults will be very much larger. A more accurate analysis is given in [8].

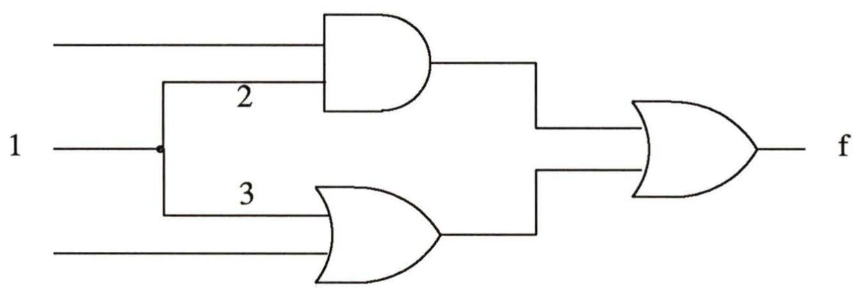



**Figure 2.2** A Circuit with Fan-out nodes.

There are two main types of bridging faults[14]:

**(a). Feedback Bridging Faults (FBF):**

If there exists at least one path between the input line x and output line y, then the BF(x,y) creates a feedback loop and changes the combinational circuit into a sequential one. This kind of bridging fault is called a feedback bridging faults. More about FBF and testing of FBF is proposed by P.K.Lui and J.C.Muzio [9, 10].

**(b). Non-Feedback Bridging Faults (NFBF):**

There exists no path between the lines x and y and thus no feedback loop is created. We will consider the NFBF only in this research.

III. **Stuck-open Faults**

For most practical purposes, logical faults are successfully modeled by stuck-at faults or bridging faults. However, not all faults can be modeled by these classical faults. *Stuck-open* faults are a peculiarity of CMOS integrated circuits They are not equivalent to classical stuck-at and bridging faults. A stuck-open fault assumes the physical failure results in a transistor which can never conduct current and is thus in switching level terms permanently open. More detail about stuck-open faults is given in [11, 12].

A stuck-open fault requires a sequence of tests for its detection. There is considerable discussion in the industry today as to the relative importance of stuck-open faults. In this research, we limit the fault models to stuck-at faults and non-feedback bridging faults.

### 2.2.2. Testing Methods

There are many test methods for LSI circuits, each with its own way of generating and processing test data. These approaches can be divided into two broad categories - concurrent and explicit [15].

**Concurrent testing**

*Concurrent testing* is executed at the same time as the normal computer operation. Data patterns from the normal operation of the system serve as test patterns, and a failure caused by some faults can be detected by built-in monitoring circuits. Concurrent testing is also called *on-line testing*. The emphasis is on ensuring the correctness of specific outputs, rather than the detection of hardware failures.

Concurrent testing is usually implemented by *redundancy techniques: information redundancy* and *hardware redundancy*. Information redundancy approaches include some coding schemes such as parity and Hamming codes. The extra bits, called check-bits, are appended to the data bits to detect faults. The simplest form of hardware redundancy is duplication, where two identical versions of a circuit are run concurrently with the same input signals, the outputs are subsequently compared to each other and any mismatch is flagged as an error. The reader who is interested in concurrent testing is referred to Sayers et al [13]. The basic idea of concurrent testing is shown in Fig. 2.3.

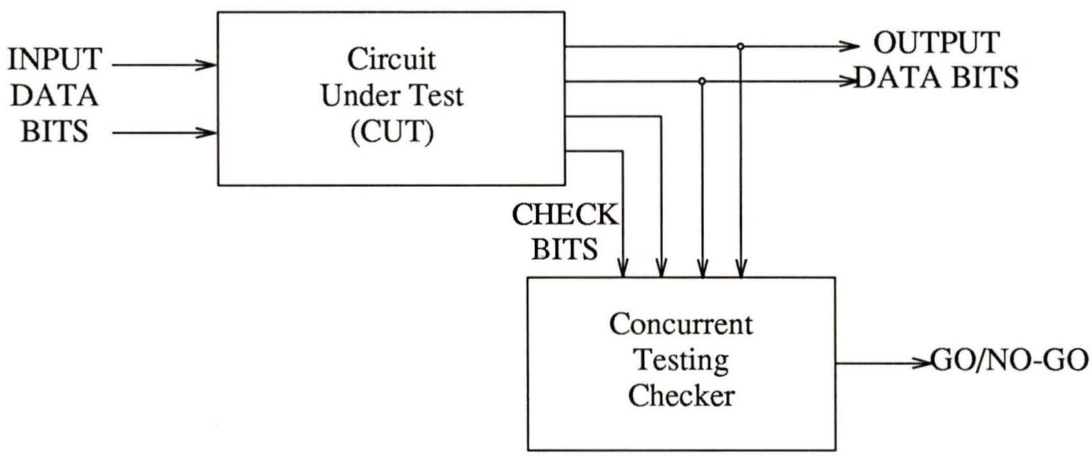



**Figure 2.3** Concurrent Testing.

Since the data patterns used in normal operation serve as test patterns, explicit testing expenses are eliminated during the life time of the system. The faults are detected during the normal use of the LSI chip.

Unfortunately, the concurrent testing approach suffers from several problems. For example, the application patterns may not exercise all the storage elements or all the internal connection lines. Defects may exist in places that are not exercised, hence some faults may go undetected for a long period of time. Using error detecting codes to code the information signals used in an LSI chip requires at least two, sometimes three or four, extra pins as error signal indicators. Additional hardware circuitry is required to implement the checkers and increase the width of the data paths. At the same time, testing of the checker itself is another problem. The problems mentioned above have limited the use of concurrent testing in LSI. However, as digital systems grow more complex and difficult to test, it becomes increasingly attractive to make use of concurrent testing.

**Explicit testing.**

All explicit testing methods separate the testing process from the normal operation of the circuit as shown in Fig 2.4.

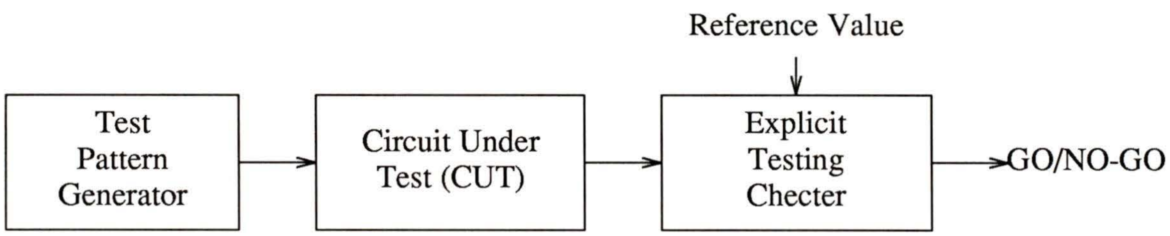



**Figure 2.4** Explicit Testing.

**In general, an explicit testing process involves three steps:**

(1) Generating the test patterns.

The goal of this step is to produce a set of test patterns to exercise the CUT (circuit under test) under different modes of operation and to cover some predefined set of faults. The test patterns may be generated exhaustively or non-exhaustively depending on the test methods used.

(2) Applying the test to the CUT.

(3) Evaluating the responses obtained from the CUT.

This step is designed with one of two goals in mind. The first is the detection of an erroneous response, which indicates the existence of one or more faults (go/no-go testing or fault detection). The other is the isolation of the fault, if one exists, in an easily replaceable module (fault location testing or fault diagnosis).

Many explicit test methods have evolved in the last decade. They can be distinguished by the techniques used to generate the test patterns and to detect and evaluate the faulty responses. In what follows, we concentrate on go/no-go explicit testing.

## 2.3. Test Generation

Each output of a combinational logic circuit realizes a logic (Boolean) function. Let $f(x_1,x_2, \cdots ,x_n)$ be a logic function of the n Boolean input variables $x_1, x_2, \cdots ,x_n$. There are $2^n$ possible row vectors which generate all input combinations to the circuit, producing an output vector of $2^n$ bits. A test set is a subset of the possible input combinations to the circuit. Each input combination in the test set is a test vector, the set being selected so that any fault within the target fault class, e.g. single stuck-at faults, gives different responses for the faulty and the fault-free circuits for at least one vector.

As an example of a test set, consider the simple circuit shown in Fig. 2.5. There are six labeled lines, including three input lines and one output line. The test set 011, 101, 110, and 111 can detect all of the twelve possible single stuck-at faults.

Several different faults may cause a circuit to malfunction in precisely the same way. These faults are called the *equivalent faults*. We can group these *equivalent faults* into equivalence classes. This procedure is called *fault collapsing*. For example, for AND (NAND) gates, all the inputs stuck-at 0 and the output stuck-at 0 (1) form an equivalence class. Similarly, for OR (NOR) gates, the inputs stuck-at 1 faults and the output stuck-at 1 (0) fault form an equivalence class. In the circuit shown in Fig. 2.5, the malfunctions caused by 6/1, 4/1, 5/1, 1/0, 2/0, and 3/0 are all the same. All of these faults can be collapsed into an equivalence class. It is sufficient to consider only one fault from the equivalence class. The fault coverage table is shown in Table 2.1. Applying the test set on the input lines, any fault will change one or more of the output value corresponding to the test patterns.

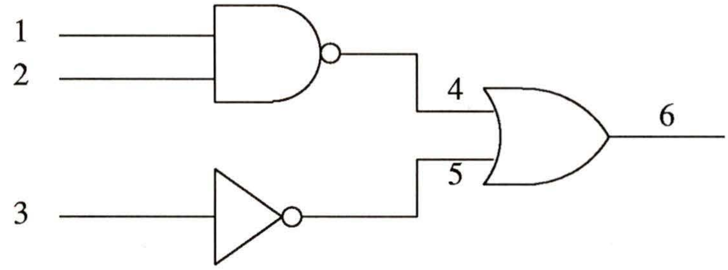



**Figure2.5** An Example Circuit.

| $f(x_1,x_2,x_3) = \overline{x}_1+\overline{x}_2+\overline{x}_3$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Test patterns | | | Fault coverage | | | | | | | | | | | |
| $x_3$ | $x_2$ | $x_1$ | 1/0 | 1/1 | 2/0 | 2/1 | 3/0 | 3/1 | 4/0 | 4/1 | 5/0 | 5/1 | 6/0 | 6/1 |
| 0 | 1 | 1 | | | | | | × | | | × | | × | |
| 1 | 0 | 1 | | | | × | | | × | | | | × | |
| 1 | 1 | 0 | | × | | | | | × | | | | × | |
| 1 | 1 | 1 | × | | × | | × | | | × | | × | | × |

**Table 2.1.** Test Set Fault Coverage Table.

Many algorithms for test set generation have been proposed over the years. Most of those approaches were of more theoretical than practical significance. Only a few approaches are of practical use. The most widely used algorithm is the D-algorithm [16]. This is a complete algorithm in the sense that it will generate a test for any logical fault, if such a test exists. However, it has been pointed out that the D-algorithm is inefficient in generating tests for circuits with many XOR gates. To improve this defect of the D-algorithm, another test-generation algorithm called PODEM (Path-Oriented Decision Marking) was proposed [17]. A fanout-oriented test-generation algorithm (called FAN) has been developed [18] that generates tests much more efficiently than the D-algorithm or PODEM.

The determination of test vectors is computationally complex and very time-consuming. However, it is a one-time design cost. The necessity to store the test set and the associated expected responses and the time required to apply the tests and to verify the responses are the serious drawbacks to the test set approach. However, it remains by far the most frequently used method in the industry today.

### 2.3.1. Random Testing

In order to avoid the expense of test set generation, two basic approaches have been proposed. The first is *random test generation*, the second is *data compaction*. In random test generation, the test vectors are generated pseudo-randomly, typically using a feedback shift register or, possibly, a CALBO generator [19, 20], and it is hoped that the test patterns generated will cover the vast majority of all possible faults. Several random test methods have been proposed. More information on these methods can be found in [21].

Random test generation alone is unsuitable for build-in self test (BIST). It still needs to store correct output values. The application of a random pattern test may require a costly quality verification especially when high fault coverage is required.

## 2.4. Data Compaction

In actually testing a physical chip it is necessary to analyse the output response to the test inputs in order to determine whether a fault has been detected. The complexity of identifying a test set grows rapidly with the size of the network. The size of the test set and the corresponding response data is much more serious. Both must be stored and manipulated at test time. For complex networks, this can be prohibitive.

Data compaction techniques have been proposed to reduce the response data to a manageable size. Ones count[22], edge count[23] and transition count[24] testing are examples. Fujiwara and Kinoshita[25] have presented a scheme where the response data for any network is compacted to two bits.

These techniques do not eliminate the need to compute, store and apply a test set. In fact, to maintain single stuck-at fault detection, the number of tests applied often far exceeds the number of tests in a minimal single stuck-at fault detection test set.

Other techniques, eliminate the test set entirely. For example the functional attribute testing of Tzidon et al [26] exhaustively applies all input assignments. Simple data compactors, selected according to the network under test, are used to reduce the response data to a manageable size. This work is restricted to certain network topologies.

McCluskey and Bozorgui-Nesbat [27] proposed a technique for designing autonomously testable circuits. The proposed method is based on built-in testing in which test patterns are applied internally by built-in test equipment. A linear-feedback shift register (LFSR) is used to generate the test patterns. All $2^n$ input patterns except the (0,0,...,0) pattern for an n-input circuit can be generated from an LFSR. Further, an LFSR can compact the output pattern of CUT by generating a "signature," which is compared with the precomputed signature of the fault-free circuit. In this way, LFSR's can be used as both the test-pattern source and the response evaluator.

Savir presented a method of designing combinational circuits such that the storage requirement is restricted to only one number, no matter how large the circuit is [28, 29, 30, 31]. This number, which is called the syndrome of the circuit, is based on the number of minterms realized by the switching function.

Syndrome analysis is a special case of more general spectral techniques, presented by Hurst, Miller and Muzio [34]. The research presented here is based on the general theory of such methods.

### 2.4.1. Signature Analysis

#### 2.4.1.1. Linear-feedback Shift Register (LFSR)

The signature analysis approach was first introduced by Hewlett-Packard Corporation[35]. This data compaction technique is based on polynomial division implemented by a linear feedback shift register. In this method, each output response is passed through a n-bit linear-feedback register (LFSR). The contents of the LFSR after all the test patterns have been applied, is called the test *signature*. This signature is compared with the precomputed signature of the fault-free circuit. Fig. 2.6 showns an example of a LFSR used in signature analysis.

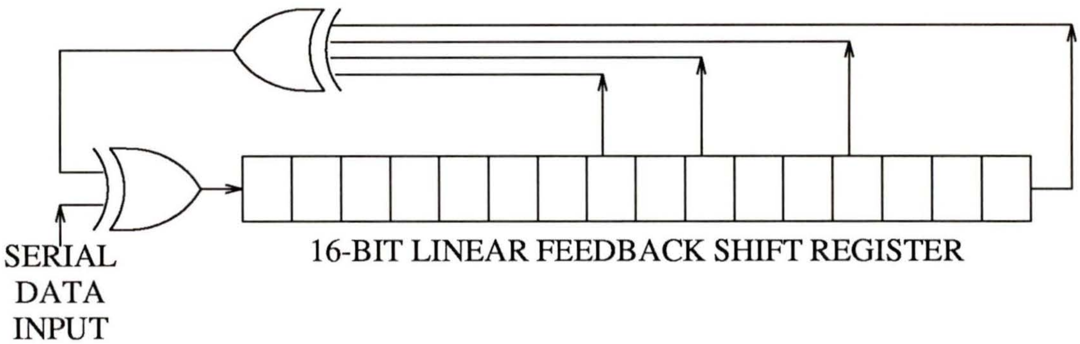



**Figure 2.6** LFSR Testing.

It is possible that an input stream to the LFSR which represents faulty circuit behaviour may give the same signature as for a fault-free circuit. This is called aliasing. Nadig [36] has shown that the probability of aliasing when using 16-bit signature analyzer is

$$\frac{2^{k-16}-1}{2^k-1} \doteq 2^{-16}, \quad \text{for } k \gg 16.$$

where k is the length of the data stream. The aliasing probabilities of a number of common data compaction techniques are investigated by J.C.Muzio et al [37]. It is clear that the aliasing probability can be reduced by using a longer shift register.

The LFSR technique is of considerable interest due to the comparatively low chip area overhead required compared with other build-in self test (BIST) techniques. LFSR's based on data compaction are in common use in the IC industry today.

### 2.4.2. Syndrome Testing

The syndrome S(f) of a logic function is defined as

$$S(f) = \frac{W(f)}{2^n}$$

where W(f) is the number of minterms realized by the function and n the number of binary input lines.

The syndrome is a functional property. Thus, various realization of the same function will have the same syndrome. Clearly, $0 \le S \le 1$, where the boundaries are attained by the constant functions.

The test procedure for the syndrome-testable circuits is shown in Fig.2.7. Every possible input combination is applied to the CUT exactly once. The syndrome-register is a counter which counts the number of ones appearing on the output of the CUT. The equality checker checks the register's contents with the expected syndrome. If the syndromes are equal, the CUT is reported to be fault-free; otherwise a fault is detected and the CUT is declared faulty. Savir [30] has presented a functional approach to determining the syndrome-testability of individual single-stuck-at faults in combinational circuits.

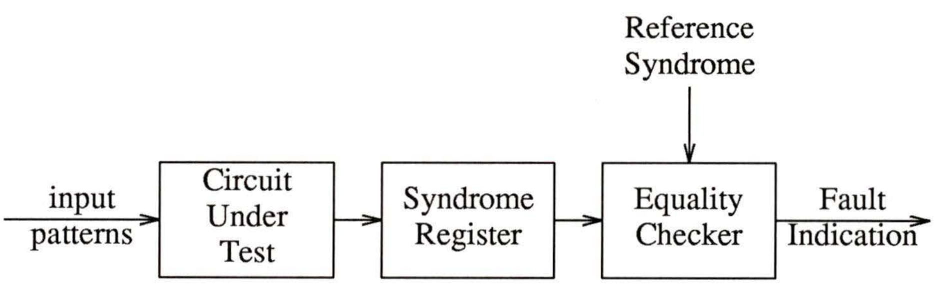



**Figure 2.7.** Testing Scheme For Syndrome-testable Circuits

Any line g in a network can be expressed in the form $f(X) = A(X)g(X) + B(X)\overline{g}(X) + C(X)$, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of boolean variables, A(X), B(X) and C(X) are independent of g(X), the function realized by the network line labeled g. In this context, g may be either an input or an internal line. Also, let S(f) denote the syndrome of the function f. We have the following results [30]:

**Theorem 2.1:**[30] The fault g stuck-at-0 (g/0) is syndrome-untestable if and only if

$$S(A\overline{C}g) = S(B\overline{C}g).$$

**Theorem 2.2:** [30] The fault g stuck-at-1 (g/1) is syndrome-untestable if and only if

$$S(A\overline{C}\overline{g}) = S(B\overline{C}\overline{g}).$$

**Definition 2.1:** The function $f(x_1, \cdots, x_i, \cdots x_n)$ is said to be *positive (negative)* unate in the variable $x_i$, if there exists a disjunctive or conjunctive expression for it in which $x_i$ appears only in uncomplemented (complemented) form.

**Definition 2.2:** The function $f(x_1, \cdots, x_i, \cdots x_n)$ is said to be *unate* in the variable $x_i$, if it is either positive or negative unate in $x_i$.

Consider the functions $f(x_1,x_2,x_3) = \overline{x}_1x_3 + x_2\overline{x}_3$, f is positive unate in $x_2$, negative unate in $x_1$, unate in $x_1$ or $x_2$ but not unate in $x_3$ (of course, this single expression does not necessarily prove that the function is not unate in $x_3$, which requires further verification).

**Theorem 2.3:**[30] If the function f is unate in line g, the stuck-at faults on g are syndrome-testable.

The full conditions for syndrome testability for a single-output circuit were developed and proved in the spectral domain by Miller and Muzio [38]. They result in much more straightforward calculations based on the Rademacher-Walsh transform to determine the syndrome testability. In this research, we show that syndrome testing is also a special case of modified Reed-Muller spectrum technique.

Markowsky [28] has shown that any network can be modified by the addition of control lines and gates so that single stuck-at faults, including those affecting the additional circuitry, are syndrome-testable. Miller and Eris [39] have shown that a two-level network can be made syndrome-testable for single stuck-at faults by the addition of one control line and at most one gate. This is of practical interest since programmable logic arrays (PLA's) are functionally two-level networks. This method requires the addition of one input thereby doubling the length of the syndrome test.

Constrained syndrome testing is an alternative to hardware modification for networks with syndrome-untestable faults. In a constrained syndrome test, certain of the inputs are held at constant values, the constraint, while the remaining inputs are exhaustively exercised. Savir [31] has shown that a set of constrained syndrome tests can always be found to protect a single-output combinational network against single stuck-at faults.

Savir's constrained syndrome method, developed in the Boolean domain, relies on choosing constraints to make lines with syndrome -untestable stuck-at faults unate, the result following from Theorem 2.3. The spectral coefficient conditions for constrained syndrome-testing were proposed by Miller and Muzio [40]. They do not require lines with syndrome-untestable faults to be made unate. The spectral conditions are thus less stringent than Savir's Boolean conditions and allow more flexibility in the choice of constraints.

### 2.4.2.1. Parity Testing

Parity testing (parity-bit checking) [32, 33] calculates the (0 or 1) parity, i.e., the exclusive-OR sum of the set of outputs, of a given function. Given an n-variable function f, the $2^n$ output values are compacted to a *parity-bit signature.* The signature, s(f), consists of n+1 bits, $p_0, p_1, \cdots, p_n$, where

$$p_0 = p(f), \quad \text{and} \quad p_i = p(f_0^{x_i}), \quad i = 1, \cdots, n$$

Here $p_0$ is the *parity of the function*, i.e., the exclusive-OR sum of the entire sequence of outputs. $(f_0^{x_i})$ is defined as the function obtained by setting the ith variable in f equal to 0.

Consider a function f $(x_1,x_2,x_3,x_4)$ defined by the truth table shown in Table 2.2, the parity-bit signature is given in the last line of Table 2.2.

Parity testing is very easy to implement. But parity testing alone is not very useful for fault detection because of the aliasing problem. The fault coverage of single faults is very low. Carter [72] has shown that many single faults are not detected by this simple parity test.

In this research, we found out that each parity-bit in the parity-bit signature is one of the Reed-Muller spectral coefficients. Parity testing is a special case of Reed-Muller spectral coefficients testing. The relationship between parity-bit signatures and the Reed-Muller spectrum is introduced in later chapters.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | f | $f_0^{x_4}$ | $f_0^{x_3}$ | $f_0^{x_2}$ | $f_0^{x_1}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | | |
| 0 | 1 | 0 | 0 | 0 | 0 | | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | | 0 | |
| 0 | 1 | 1 | 0 | 0 | 0 | | | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | | | |
| 1 | 0 | 0 | 0 | 1 | | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | | 1 | 1 | |
| 1 | 0 | 1 | 0 | 1 | | 1 | | 1 |
| 1 | 0 | 1 | 1 | 0 | | 0 | | |
| 1 | 1 | 0 | 0 | 1 | | | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | | | 0 | |
| 1 | 1 | 1 | 0 | 1 | | | | 1 |
| 1 | 1 | 1 | 1 | 0 | | | | |
| Signature | | | | 0 | 1 | 0 | 0 | 1 |

**Table 2.2** Parity-bit Signature.

### 2.4.3. Spectral Techniques

#### 2.4.3.1. The Spectral Domain

A Boolean variable is one which can take any of the two distinct values 0 and 1. A Boolean function (switching function) $f(x_1,x_2, \cdots ,x_n)$ is a mapping which associates any one of 0 and 1 with each of the $2^n$ combinations of the Boolean variables $x_1,x_2, \cdots ,x_n$. An assignment of $x_1,x_2, \cdots ,x_n$ is said to be an input combination to the function, and the value after the mapping is termed the output value of the function corresponding to that input combination. A Boolean function may be represented by a switching expression based on some switching algebra, or by a truth table which tabulates its outputs for all possible input combinations [41]. Consider as an example, the truth table for the Boolean function $f(x_1,x_2,x_3) = x_1\bar{x}_2\bar{x}_3+\bar{x}_1x_2\bar{x}_3+x_1x_2x_3$ is shown in Table 2.3.

The binary input combinations are usually ordered in ascending order with respect to the binary number system with $x_n$ the high order variable. If this order is observed, the column vector of function outputs is termed the characteristic sequence(CHS) of the function.

The majority of existing methods for designing and analysing a switching circuit are concerned with the properties of Boolean functions in the Boolean domain. Part of the problem with the definition in the Boolean domain is that each of the entries in the output vector tell us precisely the behaviour of the function at a single point but nothing of its behaviour for any other points. This prohibits us from recognizing certain characteristics of interest for the entire function by looking at just a few numbers.

| $f(x_1,x_2,x_3) = x_1\bar{x}_2\bar{x}_3+\bar{x}_1x_2\bar{x}_3+x_1x_2x_3$ | | | |
|---|---|---|---|
| $x_3$ | $x_2$ | $x_1$ | $f(x_1,x_2,x_3)$(CHS) |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

**Table 2.3** Truth Table of $f(x_1,x_2,x_3) = x_1\bar{x}_2\bar{x}_3+\bar{x}_1x_2\bar{x}_3+x_1x_2x_3$

It is possible to give an alternate representation of a function where the information about the function is much more global in nature. This alternate representation is in the spectral domain.

The basic idea of the spectral domain is illustrated in Fig. 2.8. If we are to avoid losing information, we have to ensure that the transform can be reversed, that is, we can move to and from the spectral domain without any loss of information.

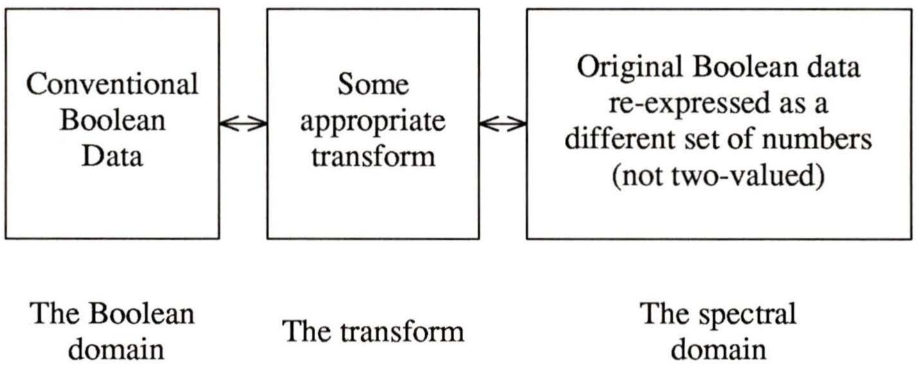



**Figure 2.8.** The Spectral Transform

To transform the Boolean representation into the spectral domain, let the characteristic sequence (CHS) for a Boolean function be labelled as the vector **Z** with a particular square matrix **T** of the correct size to give the spectrum **R** = **TZ** for the function. This vector **R** (of the same size as **Z**) is an alternative representation for the function as long as **T** has an inverse.

### 2.4.3.2. The Spectra of Discrete Functions

We present only the background necessary for the development below. Readers interested in a more detailed discussion of the spectral domain should consult Hurst [42] or Karpovsky [43].

Consider a combinational network which realizes f(X), $X = \{x_1, x_2, \cdots, x_n\}$. **Z** will denote the truth column vector of f(X), i.e., $Z_v = f(v)$ where f(v) is the value of f(X) when $x_i = v_i$, $1 \le i \le n$, and $v = \sum_{i=1}^{n} 2^{i-1} v_i$.

I. **The Rademacher-Walsh Spectrum**

The Rademacher-Walsh spectrum of f(X) is defined as

$$R = T^n Z \tag{2.1}$$

where $T^n$ is defined recursively as

$$T^0 = [1]$$

$$T^n = \begin{bmatrix} T^{n-1} & T^{n-1} \\ T^{n-1} & -T^{n-1} \end{bmatrix} \quad , \quad n = 1,2,... \tag{2.2}$$

For example, the spectrum of $f(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3$ is

$$R = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \\ -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{2.3}$$

$$\qquad R \qquad\qquad\qquad T^3 \qquad\qquad\qquad Z$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} r_0 & 0 \\ r_1 & x_1 \\ r_2 & x_2 \\ r_{12} & x_1 \oplus x_2 \\ r_3 & x_3 \\ r_{13} & x_1 \oplus x_3 \\ r_{23} & x_2 \oplus x_3 \\ r_{123} & x_1 \oplus x_2 \oplus x_3 \end{matrix} \tag{2.4}$$

The spectrum can be computed more efficiently using the fast Hadamard transform procedure [34, 43] which is a direct consequence of the recursive structure of the Rademacher-Walsh transform matrix (2.1). For an n-input function, this procedure requires $n \times 2^n$ additions and subtractions and $2^n$ memory cells.

Examining (2.2), we find $(T^n)^{-1} = 2^{-n}T^n$. Hence,

$$Z = 2^{-n}T^nR \tag{2.5}$$

The identical fast transform procedure with a final scaling factor $2^{-n}$ can thus be used to compute the inverse transform.

There is a one-to-one relationship between the Rademacher-Walsh functions and the exclusive-OR functions. Each row of $T^n$ is a Rademacher-Walsh function. The set of Rademacher-Walsh functions is homomorphic to the set of exclusive-OR functions. For each row, the variables involved in the corresponding to 1's in the binary expansion of the row index, with $x_1$ associated with the least significant bit. The spectral coefficients are identified by subscripts denoting the variables involved in the corresponding exclusive-OR function. $r_0$ denotes the first coefficient which corresponds to the constant 0 function, i.e., the exclusive-OR of no variables. The case of n = 3 is depicted in (2.3).

The ordering of the coefficients, illustrated in (2.4), is important. In particular, if R is divided in half, i.e.,

$$R = \begin{bmatrix} R^0 \\ R^1 \end{bmatrix}$$

where $R^0$ and $R^1$ each have $2^{n-1}$ coefficients, no coefficient in the top half involve $x_n$, while every coefficient in the bottom half does. The same holds for $x_{n-1}$. Further, partitioning of R, e.g.,

$$R = \begin{bmatrix} R^0 \\ R^1 \\ R^2 \\ R^3 \end{bmatrix}$$

divides the coefficients according to the involvement of $x_n$, $x_{n-1}$. The same holds for $x_{n-2}, x_{n-3}, \cdots, x_1$.

Permuting the variables of f(X) merely permutes the spectral coefficients. For example, if $x_i$ and $x_j$ are interchanged, all i's become j's and j's become i's in the coefficient subscripts [34, 38].

We make use of the subfunctions

$$f_u(x_1,x_2,\cdots,x_k) = f(x_1,x_2,....,x_k,u_1,u_2,\cdots,u_{n-k}) \tag{2.6}$$

where $0 \le u \le 2^{n-k}-1$, and $u = \sum_{i=1}^{n} 2^{i-1} u_i$. $Z_u$ and $R_u$ denote the truth column vector and spectrum of $f_u(x_1,x_2,\cdots,x_k)$, respectively. Coefficients of $R_u$ are identified as $r_{u,0}$, $r_{u,1}$, $r_{u,2}$, $r_{u,12}$, through $r_{u,12...k}$.

## II. The arithmetic Spectrum

Any Boolean function $f(x_1,x_2,\cdots,x_n)$ has a *canonic arithmetic expansion* [D&D78]. For any three-variable function, $f(x_1,x_2,x_3)$, we have:

$$f(X) = b_0 + b_1x_1 + b_2x_2 + b_3x_1x_2 + b_4x_3$$

$$+ b_5x_1x_3 + b_6x_2x_3 + b_7x_1x_2x_3, \tag{2.7}$$

where the addition is arithmetic, not Boolean disjunction. The general expression for any f(X) is given by:

$$f(X) = \sum_{j=0}^{2^n-1} b_j X_j, \tag{2.8}$$

where $X_j \in \{ 1, x_1, x_2, x_1x_2, \cdots, x_1x_2\cdots x_n \}$. Each $b_j$ is an arithmetic coefficient, and is not confined to positive values. The vector B of the $b_j$ coefficients is the arithmetic spectrum of f(X).

For Example, $f(X) = x_1 + x_2 + x_3 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3x_4$ is the arthmetic representation for $f(X) = x_1 \oplus x_2 \oplus x_3$.

The arithmetic spectrum B of a Boolean function f(X) is a $2^n \times 1$ column vector defined by $B = T^n.Z$, where $T^n$ is a $2^n \times 2^n$ arithmetic transform matrix defined by

$$T^0 = [1]$$

$$T^n = \begin{bmatrix} T^{n-1} & 0 \\ -T^{n-1} & T^{n-1} \end{bmatrix} \quad , \quad n = 1,2,... \tag{2.9}$$

### 2.4.3.3. Shannon Decompositions

In order to explore the application of spectral testing, some results about decompositions of spectra are very useful [16]. For synthesis purposes we normally seek the decomposition of a given function f(X). Algebraic techniques for the decomposition of binary functions have been extensively investigated [44, 45, 46]. Here we consider the Shannon decomposition of a given function f(X):

$$\begin{aligned} f(X) &= \bar{x}_n f(x_1, \cdots, x_{n-1}, 0) + x_n f(x_1, \cdots, x_{n-1}, 1) \\ &= \bar{x}_n f_0(x_1, \cdots, x_{n-1}) + x_n f_1(x_1, \cdots, x_{n-1}) \end{aligned} \tag{2.10}$$

(2.10) can also be written in an exclusive-OR form:

$$\begin{aligned} f(X) &= \bar{x}_n f(x_1, \cdots, x_{n-1}, 0) \oplus x_n f(x_1, \cdots, x_{n-1}, 1) \\ &= \bar{x}_n f_0(x_1, \cdots, x_{n-1}) \oplus x_n f_1(x_1, \cdots, x_{n-1}) \end{aligned} \tag{2.11}$$

Further decompositions with respect to $x_{n-1}$ and $x_n$ gives

$$\begin{aligned} f(X) = {} & \bar{x}_{n-1}\bar{x}_n f_0(x_1, \cdots, x_{n-2}) + x_{n-1}\bar{x}_n f_1(x_1, \cdots, x_{n-2}) \\ & + \bar{x}_{n-1} x_n f_2(x_1, \cdots, x_{n-2}) + x_{n-1} x_n f_3(x_1, \cdots, x_{n-2}) \end{aligned} \tag{2.12}$$

where

$$f_0(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 0, 0),$$

$$f_1(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 1, 0),$$

$$f_2(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 0, 1),$$

$$f_3(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 1, 1).$$

Consider the decomposition of (2.10). Let R, $R_0$, and $R_1$ be the Rademacher-Walsh spectra for the functions f(X), $f_0(x_1, \cdots, x_{n-1})$, $f_1(x_1, \cdots, x_{n-1})$, respectively. The full spectrum R is given by

$$R = \begin{bmatrix} T^{n-1} & T^{n-1} \\ T^{n-1} & -T^{n-1} \end{bmatrix} \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} \tag{2.13}$$

where $Z_0$ and $Z_1$ represent the column truth table vectors for $f_0(x_1, \cdots, x_{n-1})$ and $f_1(x_1, \cdots, x_{n-1})$, respectively. Hence,

$$R = \begin{bmatrix} T^{n-1}Z_0 + T^{n-1}Z_1 \\ T^{n-1}Z_0 - T^{n-1}Z_1 \end{bmatrix} \tag{2.14}$$

However, $T^{n-1}Z_0 = R_0$ and $T^{n-1}Z_1 = R_1$, whence it follows that if we partition R into two proper halves, such that

$$R = \begin{bmatrix} R^0 \\ R^1 \end{bmatrix}$$

then from (2.14),

$$R^0 = R_0 + R_1 \qquad \text{and} \qquad R^1 = R_0 - R_1$$

Rearranging this gives

$$R_0 = \frac{1}{2}(R^0 + R^1) \qquad \text{and} \qquad R_1 = \frac{1}{2}(R^0 - R^1) \tag{2.15}$$

This may readily be extended to higher orders; for example, consider the decomposition of (2.12). This divides the truth vector into four parts, namely $Z_0,Z_1,Z_2,Z_3$, and the respective spectra are $R_0,R_1,R_2,R_3$, obtained by $R_k = T^{n-2}Z_k$, $0 \le k \le 3$. If the spectrum R of Z is partitioned as

$$R = T^n \cdot \begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} R^0 \\ R^1 \\ R^2 \\ R^3 \end{bmatrix}$$

the following relationships exist,

$$R^0 = R_0 + R_1 + R_2 + R_3, \quad R^1 = R_0 - R_1 + R_2 - R_3,$$

$$R^2 = R_0 + R_1 - R_2 - R_3, \quad R^3 = R_0 - R_1 - R_2 + R_3, \qquad (2.16a)$$

and conversely

$$R_0 = \frac{1}{4}(R^0 + R^1 + R^2 + R^3), \quad R_1 = \frac{1}{4}(R^0 - R^1 + R^2 - R^3),$$

$$R_2 = \frac{1}{4}(R^0 + R^1 - R^2 - R^3), \quad R_3 = \frac{1}{4}(R^0 - R^1 - R^2 + R^3). \qquad (2.16b)$$

For the general case, R and Z can be partitioned into $2^{n-m}$ subvectors with $m < n-1$, such as

$$R = \begin{bmatrix} R^0 \\ R^1 \\ \cdot \\ \cdot \\ R^\beta \end{bmatrix}, \quad Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \cdot \\ \cdot \\ Z_\beta \end{bmatrix} \quad \text{where } \beta = 2^{n-m} - 1. \qquad (2.17)$$

It can be shown that [47]

$$[R^0 R^1 \cdots R^\beta] = [R_0 R_1 \cdots R_\beta] T^{n-m}$$

$$\text{and} \quad [R_0 R_1 \cdots R_\beta] = \frac{1}{2^{n-m}} [R^0 R^1 \cdots R^\beta] T^{n-m} \qquad (2.18)$$

Shannon decompositions are frequently used in this research.

### 2.4.3.4. Spectral Coefficient Testability

Miller and Muzio have developed the concept of spectral coefficient (Rademacher-Walsh) testability. A few of the results for spectral coefficient testability are given in this section. Readers interested in a detailed spectral coefficient testability should consult [38, 39, 40].

First consider single input stuck-at faults.

**Theorem 2.4.** [40] $x_i/0$ and $x_i/1$ are $r_\alpha$ -testable and $r_{\alpha,i}$ -testable, where $\alpha \subseteq \{1,2,...,i\text{-}1,i\text{+}1,...,n\}$ or $\alpha = \varnothing$, if and only if $r_{\alpha,i} \neq 0$. For all the following results, $\alpha$ is allowed to be empty in which case $r_\alpha$ is denoted by $r_0$.

A multiple stuck-at input fault is one where two or more primary input lines are each stuck-at 0 or stuck-at 1. We denote such a fault $(x_{i_1}, x_{i_2}, \cdots, x_{i_p})/u$, $1 \le i_j \le n$, $1 \le p \le j$. This fault represents a set of single stuck input lines $x_{i_j}/u_j$, $1 \le j \le p$, where $u = \sum_{j=1}^{p} u_j 2^{j-1}$.

**Theorem 2.5.** [40] A multiple-input stuck-at fault involving p or more inputs is $r_\alpha$ -testable if $r_\alpha$ is not an integer multiple of $2^p$.

**Theorem 2.6.** [40] $(x_{i_1}, x_{i_2}, \cdots, x_{i_p})/u$ is $r_\alpha$ -testable, $\alpha \cap \{i_1, i_2, \cdots, i_p\} = \varnothing$, if and only if

$$r_\alpha \neq [\, r_{\gamma_0}, r_{\gamma_1}, \cdots, r_{\gamma_\beta} \,]\ T_u^p \qquad (2.19)$$

where $\gamma_0 = \alpha$, $\gamma_1 = \alpha \cup \{i_1\}$, $\gamma_2 = \alpha \cup \{i_2\}$, $\gamma_3 = \alpha \cup \{i_1, i_2\}$, $\gamma_\beta = \alpha \cup \{i_1, i_2, \cdots, i_p\}$, $\beta = 2^p - 1$, and $T_u^p$ is the $u^{th}$ column of $T^p$.

Theorem 2.4 is the special case of Theorem 2.6 with p = 1, i.e., single input stuck-at faults. The above results include syndrome testability as a special case, i.e., $r_0$-testability. The results depend only on f(X) and its spectrum. They are independent of the details of the network which realizes f(X).

Consider an internal line g in a single-output combinational network as shown in Fig. 2.9.

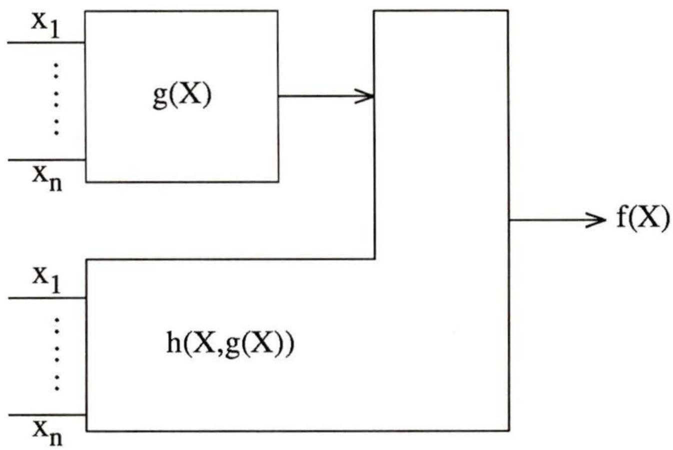



**Figure 2.9** Internal Line Model.

Such a line fits the functional model

$$f(X) = h(X, g(X)) \tag{2.20}$$

h(X,g) is a function of n+1 inputs. We let g correspond to $x_{n+1}$. $\hat{R}$ will denote the spectrum of h(X,g). g/0 and g/1 denote stuck-at 0 and stuck-at 1 faults on g, respectively.

**Theorem 2.7** [40] g/0 is $r_\alpha$ -testable , $\alpha \subseteq \{1,2,...,n\}$ or $\alpha = \varnothing$, if and only if

$$r_\alpha \neq \frac{1}{2}(\hat{r}_\alpha + \hat{r}_{\alpha n+1}) \tag{2.21a}$$

g/0 is $r_\alpha$ -testable , $\alpha \subseteq \{1,2,...,n\}$, if and only if

$$r_\alpha \neq \frac{1}{2}(\hat{r}_\alpha - \hat{r}_{\alpha n+1}) \tag{2.21b}$$

**Theorem 2.8** [40] Any stuck-at fault which results in an $f^1(X)$ independent of some subset of its inputs, $\{x_{i_1}, x_{i_2}, \cdots, x_{i_p}\}$, $1 \leq j \leq p$, is $r_\alpha$ -testable for $\alpha \cap \{i_1, i_2, \cdots, i_p\} \neq \varnothing$, if and only if $r_\alpha \neq 0$.

**Theorem 2.9** [40] Any stuck-at fault which results in an $f^1(X)$ independent of p primary inputs is $r_\alpha$ -testable, $\alpha \subseteq \{1,2,...,n\}$, if $r_\alpha$ is not an integer multiple of $2^p$.

## 2.5. Conclusion

In this chapter, we give an overview of testing problems and some definitions needed in this research. Two kinds of fault models are introduced. The fault models that we adopt are those of the stuck-at fault and the non-feedback bridging fault. It has been shown that this gives a very reasonable coverage of most practical faults that occur. Two broad categories of test methods are discussed. There are still a number of serious problems using concurrent testing in practice so we concentrate on explicit testing in this research. Traditional methods based on test sets have been shown to be inadequate in many situations and various new approaches have been proposed. In recent years, fault detection techniques using data compaction have been developed. Two of these involve syndrome testing and spectral coefficient testing. A spectral coefficient has a higher information content about the function than an entry in the output vector. The well-established mathematics behind the transform makes it easy to formulate or formalize theories and algorithms. Spectral techniques have wide applications in analysing Boolean functions.

# CHAPTER 3

# REED-MULLER AND MODIFIED REED-MULLER EXPANSION

## 3.1. Introduction

As we have seen in chapter 2, with the advent of VLSI technology, the cost of testing computer hardware is increasing and, in some cases, may be higher than the costs of development and manufacturing. Many testing and easily testable network design methods have been proposed. Recent work in fault detection has moved away from the idea of using a test set with all its associated cost in terms of storage requirements and set-up time. Some data compaction techniques have been developed. One of these is the spectral technique based on Rademacher-Walsh (R-W) transforms for fault detection. It was shown [14, 38, 40, 65] that by verifying a few R-W spectral coefficients for a Boolean function it is possible to detect faults in a network implementing the Boolean function. Rademacher-Walsh (R-W) spectral techniques require an exhaustive testing method, where all possible input combinations are applied to the CUT. A small number of spectral coefficients required for fault detection constitute a spectral signature. Some of the testability criteria were given in section 2.4.3.4.

There are a variety of spectral coefficients available by using different transformations. Arithmetic and Reed-Muller spectral coefficients are possible examples. The concept of the Reed-Muller canonic expansion was first introduced by Muller [48]. It was shown that any arbitrary logic function f(X) has an unique Reed-Muller expansion. A realization for an arbitrary logic function, using AND-EXOR (Exclusive-OR) gates, based on Reed-Muller canonic expansion was proposed by Reddy [52].

The existing research on the Reed-Muller spectral technique has two directions - minimization of Reed-Muller canonic expansions and testability for AND-EXOR logic circuits based on Reed-Muller expansion. This work is primarily based on test sets for fault detection. It has been shown that for an n variable arbitrary Boolean function, there are $2^n$ different polarity Reed-Muller expansions. Different polarity RM expansion may have different numbers of non-zero terms. It is obvious that the fewer the terms in the RM expansion the fewer the gates and lines needed to implement the function, and may be, the easier to detect the faults in an AND-EXOR logic circuit. Reddy [52] showed that AND-EXOR logic circuits possess many of the desirable properties of easily testable networks.

### 3.2. Generalized Reed-Muller Expansion

Any arbitrary logic function $f(x_1 \cdots x_n)$ has an unique *Generalized Reed-Muller (GRM) expansion* [50, 52] as given in (3.1):

$$f(x_1x_2 \cdots x_n) = d_0 \oplus d_1\dot{x}_1 \oplus d_2\dot{x}_2 \oplus \cdots \oplus d_n\dot{x}_n$$

$$\oplus\, d_{n+1}\dot{x}_1\dot{x}_n \oplus \cdots \oplus d_{2^n-1}\dot{x}_1\dot{x}_2 \cdots \dot{x}_n \qquad (3.1)$$

where $\dot{x}_i$ is either $x_i$ or the complement of $x_i$ ($\overline{x_i}$) and $d_j$ is a binary constant ( 1 or 0 ) called Reed-Muller coefficient (RM coefficient). GRM expansion is also called *Reed-Muller canonical (RMC) expansion.* In this research, we use the term GRM rather than RMC.

Only fixed polarity variables appear; that is, a given variable appears only in complemented or uncomplemented form but not in both forms. There are $2^n$ possible expansions corresponding to the $2^n$ possible combinations of $\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n$. These expansions are unique for any given function. Let j and k be integers, $0 \le j, k \le 2^n-1$, with binary representations $(j_n \cdots j_1)$ and $(k_n \cdots k_1)$ ($j_1$ and $k_1$ are the least significant bits). Let coefficient $d_j^k$ be associated with the product term consisting of variables corresponding to the 1's among the $j_i$ where k, the polarity of the expression, is the integer whose binary expansion contains 1's in locations corresponding to complemented variables. The $2^n$ RM expressions are given in (3.2):

$$f(x_1x_2 \cdots x_n) = d_0^0 \oplus d_1^0x_1 \oplus \cdots \oplus d_N^0x_1x_2 \cdots x_n$$

$$\cdots\ \cdots\ \cdots\ \cdots$$

$$= d_0^k \oplus d_1^k x_1^{k_1} \oplus \cdots \oplus d_N^k x_1^{k_1} \cdots x_n^{k_n}$$

$$\cdots\ \cdots\ \cdots\ \cdots$$

$$= d_0^N \oplus d_1^N \bar{x}_1 \oplus \cdots \oplus d_N^N \bar{x}_1 \cdots \bar{x}_n \qquad (3.2a)$$

where $N = 2^n - 1$, $x_i^0 = x_i$, $x_i^1 = \bar{x}_i$, i, j, and $k \in \{0, 1, \ldots, N\}$, and $d_j^k \in \{0,1\}$.

(3.2a) can be written in a general form

$$f(x_1 \cdots x_n) = \bigoplus_{j=0}^{2^n-1} d_j^k [\prod_{\text{All } i,\, j_i=1} x_i^{k_i}] \qquad (3.2b)$$

$d_j^k$ is called the kth RM coefficient, $k \in \{ 0, N \}$. The RM expansion associated with the kth RM coefficients is called the kth polarity RM expansion for a given Boolean function f(X).

**Example 3.1**

Consider a three-input function $f(x_1,x_2,x_3) = x_1 + \bar{x}_2 x_3$. There are $2^3 = 8$ different polarity RM expansions. In the 0th polarity RM expansion, all variables appear in uncomplemented form, $f(X) = f(x_1,x_2,x_3) = x_1 \oplus x_1x_2 \oplus x_3 \oplus x_1x_3 \oplus x_1x_2x_3$. The 0th polarity RM expansion is also called *positive canonic RM expansion* (PRM). For $k = 5$ ( $k_3k_2k_1 = 101$ ), the 5th polarity RM expansion is a function of $\bar{x}_3$, $x_2$, and $\bar{x}_1$, $f(X) = 1 \oplus \bar{x}_3\bar{x}_1 \oplus \bar{x}_3 x_2 \oplus \bar{x}_3 x_2 \bar{x}_1$.

Reddy [52] has shown that any switching function can be realized by a *Reed-Muller canonical circuit*, or an *AND-EXOR logic circuit*, which are direct realizations of Reed-Muller expansions using AND and EXOR gates only. For example, consider

$$f(x_1,x_2,x_3,x_4) = 1 \oplus x_1x_2 \oplus x_1x_3 \oplus x_1x_3x_4 \oplus x_2x_3x_4$$

The Reed-Muller canonical circuit realizing f(X) is given in Fig. 3.1, Each AND gate forms a product term in the expression. An extra input $x_0$ is used to supply $d_0 = 1$.

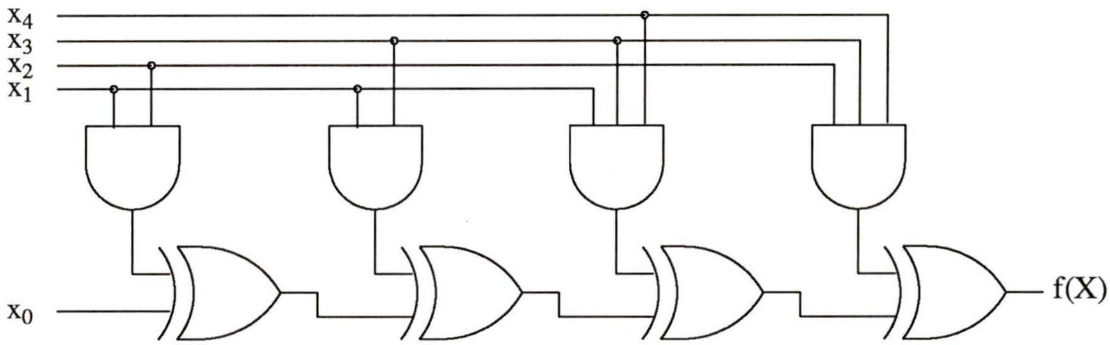



**Figure 3.1** Reed-Muller Canonical Circuit Realizing f(X)

In order to get the transformation between the Boolean domain and RM spectral domain, we define the concept of *subnumber* first.

**Definition 3.1**

Let j, k, and m be three integers, $0 \le j, k, m \le 2^n - 1$, with binary representations $(j_n \cdots j_1)$, $(k_n \cdots k_1)$, and $(m_n \cdots m_1)$ ($j_1, k_1$, and $m_1$ are the least significant bits), respectively. j is a k subnumber of m, denoted by $j \supseteq_k m$, if and only if $j_i = k_i$ or $m_i = \overline{k_i}$ for each $i = 1, \ldots, n$.

If k = 0, $j \supseteq_0 m$ can be denoted by $j \supseteq m$.

For example, let n = 3, and m = 6. The binary representations of n, m, k, and j are shown in Table 3.1.

| j | $j_3$ | $j_2$ | $j_1$ | m=6 | | | k=0 | | | k=5 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $m_3$ | $m_2$ | $m_1$ | $k_3$ | $k_2$ | $k_1$ | $k_3$ | $k_2$ | $k_1$ |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 2 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 4 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 5 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 6 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |

**Table 3.1** The 0th and 5th subnumbers of m = 6.

From Table 3.1, we find out that j = 0, 2, 4, 6 are the 0th subnumbers of m = 6, and j = 4, 5, 6, 7 are the 5th subnumbers of m = 6.

Fisher [59] has given a general expression to calculate the kth RM coefficient $d_j^k$ of a given Boolean function f(X).

**Theorem 3.1** [59]

Let f be a Boolean function and $f(x_1 \cdots x_n) = d_0^0 \oplus \cdots \oplus d_N^0, x_1 x_2 \cdots x_n$ be the 0th polarity RM expansion for f. Then

$$f(m) = \bigoplus_{j \supseteq_0 m} d_j^0 \tag{3.3a}$$

$$d_j^0 = \bigoplus_{m \supseteq_0 j} f(m) \tag{3.3b}$$

**Theorem 3.2** [59]

Let $0 \le k \le 2^n - 1$ and let the kth polarity RM expansion for a Boolean function f be

$$f(x_1 \cdots x_n) = \bigoplus_{j=0}^{2^n-1} d_j^k [\prod_{\text{All } i,\, j_i=1} x_i^{k_i}]$$

Then

$$f(m) = \bigoplus_{j \supseteq_0 m \overline{\oplus} k} d_j^k \tag{3.4a}$$

$$d_j^k = \bigoplus_{m \overline{\oplus} k \supseteq_0 j} f(m) \tag{3.4b}$$

where $m \overline{\oplus} k$ denotes the *dyadic sum* of m and k, the integer whose binary expansion is the componentwise mod 2 (exclusive-OR) sum of the binary components of m and k.

(3.4a) and (3.4b) can be written in another form

$$f(m \overline{\oplus} k) = \bigoplus_{j \supseteq_0 m} d_j^k \tag{3.5a}$$

$$d_j^k = \bigoplus_{m \supseteq_0 j} f(m \overline{\oplus} k) \tag{3.5b}$$

From (3.2) we can see that there are $2^n$ different polarity RM expansions corresponding to $2^n$ possible combinations of $(\dot{x}_1, \cdots, \dot{x}_n)$. In order to discuss the relationship between the output values of a Boolean function f(X) and the different polarity RM expansions, we define a $2^n \times 2^n$ *polarity output matrix* $\tilde{Z}$, *and a* $2^n \times 2^n$ *polarity coefficient matrix* $\tilde{D}$.

**Definition 3.2**

The *polarity-output (PO) matrix* is a $2^n \times 2^n$ matrix

$$\tilde{Z} = [\, Z^0\, Z^1 \,\cdots\, Z^{2^n-1}\,]. \tag{3.6}$$

where the kth column of $\tilde{Z}$, $Z^k$, is a $2^n \times 1$ column vector. The jth entry of $Z^k$, denoted by $z_j^k$, is the output value corresponding to the input pattern $j\overline{\oplus}k$, f $(j\overline{\oplus}k)$. $Z^k$ is called the kth polarity output column vector, obtained by

$$Z^k = \begin{bmatrix} f(0\overline{\oplus}k) \\ f(1\overline{\oplus}k) \\ \cdots \\ f(N\overline{\oplus}k) \end{bmatrix} \tag{3.7}$$

**Example 3.2** Let n = 3. Consider the function in example 3.1, $f(X) = x_1 + \bar{x}_2 x_3$, the 0th polarity and 5th polarity output column vectors will be

$$Z^0 = \begin{bmatrix} f(0) \\ f(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(7) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{,and } Z^5 = \begin{bmatrix} f(0\overline{\oplus}5) \\ f(1\overline{\oplus}5) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(7\overline{\oplus}5) \end{bmatrix} = \begin{bmatrix} f(5) \\ f(4) \\ f(7) \\ f(6) \\ f(1) \\ f(0) \\ f(3) \\ f(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$Z^k$, $0 \le k \le 2^n-1$, changes the order of output values of f(X) $2^n$ times.

**Definition 3.3**

The *polarity-coefficient (PC) matrix* is a $2^n \times 2^n$ matrix

$$\tilde{D} = [\, D^0\, D^1 \,\cdots\, D^{2^n-1}\,]. \tag{3.8}$$

where the kth column of $\tilde{D}$, $D^k$, is a $2^n \times 1$ column vector corresponding to the kth GRM expansion. The jth entry of $D^k$ is a RM coefficient $d_j^k$ calculated by (3.4b) or (3.5b).

In this research, we pay more attention to the 0th polarity RM spectra. $D^0$ and $Z^0$ are simply denoted by D and Z, and $d_j$ stands for $d_j^0$.

Using the Reed-Muller (RM) transform matrix, we can transform our conventional digital data into the alternative RM spectral domain. The transform matrix and the transformation are introduced in Section 3.4.

### 3.3. The Meaning and Order of RM Spectral Coefficients

Initially consider the meaning and order of RM spectral coefficients. Each RM coefficient has an obvious interpretation, which is explained below. In (3.2) the RM spectral coefficients $d_j^k$ are given in decimal order.

There are 3 alternative labellings for the subscripts of $d_j^k$. These are illustrated in (3.9).

$$D^k = \begin{bmatrix} d_0{}^k \\ d_1{}^k \\ d_2{}^k \\ d_3{}^k \\ d_4{}^k \\ \cdots \\ d_{2^n-1}{}^k \end{bmatrix} = \begin{bmatrix} d^k_{0...000} \\ d^k_{0...001} \\ d^k_{0...010} \\ d^k_{0...011} \\ d^k_{0...100} \\ \cdots \\ d^k_{11...1} \end{bmatrix} = \begin{bmatrix} d_0{}^k \\ d_1{}^k \\ d_2{}^k \\ d_{1,2}{}^k \\ d_3{}^k \\ \cdots \\ d_{1,...,n}{}^k \end{bmatrix} \quad \begin{bmatrix} 0 \\ x_1{}^{k_1} \\ x_2{}^{k_2} \\ x_1{}^{k_1}x_2{}^{k_2} \\ x_3{}^{k_3} \\ \cdots \\ x_1{}^{k_1} \cdots x_n{}^{k_n} \end{bmatrix} \tag{3.9}$$

where $k_n\, k_{n-1} \cdots k_1$ is the binary representation of k, $k_i \in \{ 0,1 \}$, $x_i^0 = x_i$, and $x_i^1 = \overline{x_i}$

The decimal labelling has been used above. We use Latin subscripts, $d_j^k$, stand for the decimal labelling, where j is an integer. The second form uses a binary labelling. A capital Latin subscripts is used for this labelling. The third explicitly includes the variable number if the corresponding bit in the binary expansion is 1. In this labelling the coefficient $d_{1,2}{}^k$, for example, is that associated with the product term $x_1{}^{k_1}x_2{}^{k_2}$ as shown in (3.9). We use Greek subscripts, $d_\alpha^k$, for this kind of labelling, where $\alpha$ is a set.

As an example, consider a three-variable function $f(X) = \overline{x}_1\overline{x}_2x_3 + \overline{x}_1x_2\overline{x}_3 + x_1x_2x_3$.

**Example 3.3**

$f(X) = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1x_2x_3.$

The 0th polarity spectrum D is as follows:

$$D = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix} = \begin{bmatrix} d_{000} \\ d_{001} \\ d_{010} \\ d_{011} \\ d_{100} \\ d_{101} \\ d_{110} \\ d_{111} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_{1,2} \\ d_3 \\ d_{1,3} \\ d_{2,3} \\ d_{1,2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

If we write the RM coefficients in the form of $d_0, d_1, d_2, d_{1,2}, \cdots, d_{1,2,3}$, (3.2a) is written as follows

$$f(X) = d_0 \oplus d_1x_1 \oplus d_2x_2 \oplus d_{1,2}x_1x_2 \oplus d_3x_3 \oplus d_{1,3}x_1x_3 \oplus d_{2,3}x_2x_3 \oplus d_{1,2,3}x_1x_2x_3$$

In this example, the PRM expansion of f(X) is

$$f(X) = x_1 \oplus x_2 \oplus x_1x_3 \oplus x_2x_3 \oplus x_1x_2x_3$$

$d_{1,2}=0$, so that the PRM of function f(X) did not include the term $x_1x_2$, while $d_{1,3}=1$, so the PRM of f(X) includes the term $x_1x_3$.

Because of the functional meaning of the individual coefficients of D, we frequently classify them according to their *order*, this referring to the number of variables associated with the coefficient. The order of the coefficients, in Greek labelling $d_\alpha$, is given by the size of $\alpha$ denoted by $|\alpha|$, that is:

| The coefficients | The order |
|---|---|
| $d_0$: | the zero-order coefficient, $\mid\alpha\mid = 0$ |
| $d_i$: $i = 1$ to n. | the primary or first-order coefficients, $\mid\alpha\mid = 1$ |
| $d_{i,j}$: $i,j = 1,2;\ 1,3;\ \ldots$ . | second-order coefficients, $\mid\alpha\mid = 2$ |
| $d_{i,j,k}$: $i,j,k = 1,2,3;\ 1,2,4,\ 1,3,4;\ \ldots$ . | third-order coefficients, $\mid\alpha\mid = 3$ |
| ... | ... |

For n = 4, we can write the coefficients in the following order, as distinct from the column vector order by which they are generated by the RM transform:

$d_0$; $d_1$ $d_2$ $d_3$ $d_4$; $d_{1,2}$ $d_{1,3}$ $d_{1,4}$ $d_{2,3}$ $d_{2,4}$ $d_{3,4}$; $d_{1,2,3}$ $d_{1,2,4}$ $d_{1,3,4}$ $d_{2,3,4}$; $d_{1,2,3,4}$

If the RM coefficients is given in a decimal labelling form, $d_j^k$, the order of the coefficient is equal to the order of the subscript j. The order of an integer, j, denoted by |j|, is the number of one's in the binary expansion of j. For example, if n = 4 and j = 11, the binary expansion of j is 1011, then |j| = 3.

The RM and MRM coefficients are associated with the parity and syndrome of a Boolean function f(X) directly. Consider the 0th polarity RM spectral coefficients. Each such coefficient, for instance, $d_{ij}$ , is the parity of a subset function of f(X), denoted by f'(X). The last coefficient $d_{1,...,n}$ is the parity of the function f(X). The entries in f'(X) are the values of f(w), where w is obtained by setting all of the input variables $x_p$ ( p≠i and p≠j ) to zero, while $x_i$ and $x_j$ take all the possible binary combinations. For example, for n=4 function f( $x_4x_3x_2x_1$ ), $d_{12}$ is the parity of f(0000),f(0001),f(0010),and f(0011), obtained by setting $x_3$, $x_4$ to zero, while $x_1$ and $x_2$ take all the possible binary combinations. For the kth RM coefficient, $d_j^k$ is also a parity of a subfunction of f(X). The entries of f'(X) are the values of f(w), where w is obtained by setting all of the input variables $x_p$ ( p≠i and p≠j ) to $k_p$ ( $k_1 \cdots k_p \cdots k_n$ is the binary representation of k ), while $x_i$ and $x_j$ take all possible binary combinations. For example, let n = 4, k = 5. The binary representation of k is $k_4k_3k_2k_1 = 0101$. $d_{1,2}^5$ is the parity of f(0100),f(0101),f(0110),and f(0111), obtained by setting $x_3$ to one and $x_4$ to zero, while $x_1$ and $x_2$ take all the possible binary combinations. The coefficient $d_{i,j}$ provides some information about $x_i$ and $x_j$, so that $d_{i,j}$ can be used to detect the s-a-f on $x_i$ or $x_j$, or some internal s-a-f related to these two variables.

### 3.4. Reed-Muller Transformation and Transform Matrix

In the previous sections, we introduced the generalized Reed-Muller expansion. A polarity-output (PO) matrix and a polarity-coefficient (PC) matrix have been defined. It is obvious that the PO matrix is in the Boolean domain and the PC matrix is in the spectral domain. We can use (3.4b) and (3.5b) as a transformation to and from the spectral domain. However, this is not very convenient.

Besslich [62] has shown that the transformation can be represented by a matrix. The matrix is called a Reed-Muller transform matrix in this research. By using a RM transform matrix, we can establish a very simple relationship between the PO and PC matrices.

### 3.4.1. Reed-Muller Transform Matrix

The RM transform matrix is a square matrix with row and column entries $\in \{0,1\}$. It has a recursive structure as follows :

$$RM^n = \begin{bmatrix} RM^{n-1} & 0 \\ RM^{n-1} & RM^{n-1} \end{bmatrix} \quad , \quad RM^0 = 1 \tag{3.10}$$

The dimension of the matrix is $2^n \times 2^n$ for any n. For increasing n, we have the real integer values:

$$RM^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$RM^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$RM^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We may alternatively express this recursive structure by

$$RM^n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \circledast RM^{n-1} \tag{3.11}$$

where the operator $\circledast$ here denotes the Kronecker product operator[49].

**Theorem 3.3**

The inverse of RM transform matrix $(RM^n)^{-1}$ has a recursive form :

$$(RM^n)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circledast (RM^{n-1})^{-1} \tag{3.12a}$$

**Proof :**

For k=1 we have

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

where I is the order 2 identity matrix. The inductive step follows directly, given the recursive form of the transform.

**Theorem 3.4**

In the Galois Field of order 2, denoted by GF(2), we have :

$$(RM^n)^{-1} = RM^n. \tag{3.12b}$$

**Proof :**

The proof is very simple because in (3.12a) 1 is equivalent to -1 on GF(2).

**Definition 3.4**

Let A be an $m \times n$ matrix and B be an $n \times l$ matrix. The matrix multiplication over GF(2) of A and B, denoted by $\otimes$, is defined as follows :

The entries of $C = A \otimes B$ are given by

$$c_{ij} = \bigoplus_{r=1}^{n} a_{ir} b_{rj}$$

where $\oplus$ denotes mod 2 sum. Instead of the arithmetic addition "+" used in ordinary matrix multiplication, we use exclusive-OR in the matrix multiplication over GF(2).

From Theorem 3.4, we get an interesting result that

$$RM^n \otimes RM^n = I^{2^n}$$

where $I^{2^n}$ is an identity matrix of size $2^n \times 2^n$.

**Theorem 3.5**

The matrix multiplication over GF(2), denoted by $\otimes$ , is associative.

$$(A\otimes B)\otimes C = A\otimes(B\otimes C).$$

**Proof :**

$$[A\otimes(B\otimes C)]_{ij} = \bigoplus_{r=1}^{n} a_{ir}(B\otimes C)_{rj}$$

$$= \bigoplus_{r=1}^{n} a_{ir}(B\otimes C)_{rj}$$

$$= \bigoplus_{r=1}^{n} a_{ir} \bigoplus_{s=1}^{n} b_{rs}c_{sj}$$

$$= \bigoplus_{s=1}^{n} (\bigoplus_{r=1}^{n} a_{ir}b_{rs})c_{sj}$$

$$= \bigoplus_{s=1}^{n} (A\otimes B)_{is}c_{sj}$$

$$= [(A\otimes B)\otimes C]_{ij}.$$

### 3.4.2. The Reed-Muller Transformation

Let $\tilde{D}$ be the polarity coefficient (PC) matrix, and $\tilde{Z}$ be the polarity output (PO) matrix of a given switching function f(X), the kth column of $\tilde{D}$ is the kth polarity RM coefficient column vector corresponding to the kth polarity output column vector in $\tilde{Z}$. The relationship between $\tilde{D}$ and $\tilde{Z}$ is given by

$$\tilde{D} = RM^n \otimes \tilde{Z} \tag{3.13}$$

$$\left[ D^0\, D^1 \cdots D^{2^n-1} \right] = RM^n \otimes \left[ Z^0\, Z^1 \cdots Z^{2^n-1} \right]$$

For each $D^k$ and $Z^k$, $k\in \{0, 1, \dots, N\}$, we have

$$D^k = RM^n \otimes Z^k \tag{3.14}$$

## Example 3.4

Consider the function $f = x_1 + \overline{x}_2 x_3$

| | $x_3x_2x_1$ | f | $Z^0Z^1Z^2Z^3Z^4Z^5Z^6Z^7$ |
|---|---|---|---|
| 0 | 0 0 0 | 0 | 0 1 0 0 1 1 1 1 |
| 1 | 0 0 1 | 1 | 1 0 0 0 1 1 1 1 |
| 2 | 0 1 0 | 0 | 0 0 0 1 1 1 1 1 |
| 3 | 0 1 1 | 0 | 0 0 1 0 1 1 1 1 |
| 4 | 1 0 0 | 1 | 1 1 1 1 0 1 0 0 |
| 5 | 1 0 1 | 1 | 1 1 1 1 1 0 0 0 |
| 6 | 1 1 0 | 1 | 1 1 1 1 0 0 0 1 |
| 7 | 1 1 1 | 1 | 1 1 1 1 0 0 1 0 |

$$\underset{RM^n}{\begin{bmatrix} 10000000 \\ 11000000 \\ 10100000 \\ 11110000 \\ 10001000 \\ 11001100 \\ 10101010 \\ 11111111 \end{bmatrix}} \cdot \underset{\tilde{Z}}{\begin{bmatrix} 01001111 \\ 10001111 \\ 00011111 \\ 00101111 \\ 11110100 \\ 11111000 \\ 11110001 \\ 11110010 \end{bmatrix}} = \underset{\tilde{D}}{\begin{bmatrix} 01001111 \\ 11000000 \\ 01010000 \\ 11110000 \\ 10111011 \\ 11001100 \\ 01010101 \\ 11111111 \end{bmatrix}}$$

Consider the 0th polarity and 2nd polarity RM coefficients column vectors.

$$D^0 = [\,0\,1\,0\,1\,1\,1\,0\,1\,]^t \text{ and } D^2 = [\,0\,0\,0\,1\,1\,0\,0\,1\,]^t$$

the function $f = x_1 + \overline{x}_2 x_3$ can be written in the 0th and 2nd polarity RM expansions, $f = x_1 \oplus x_1x_2 \oplus x_3 \oplus x_1x_3 \oplus x_1x_2x_3$ and $f = \overline{x}_1\overline{x}_2 \oplus x_3 \oplus x_3\overline{x}_2\overline{x}_1$, respectively.

From example 3.2 we can see that the different polarity GRM expansions may have different numbers of terms. In example 3.2, the 2nd GRM expansion has three terms, while the 1st has seven terms. In fault detection of arbitrary switching function realizations based upon GRM expansions, it has been shown that different expansions may lead to different sizes of test sets because of the resulting network topology. In particular, the selection of an GRM expansion that has a minimal number of literals appearing in an even number of product terms will give rise to switching function realizations requiring still fewer tests [52, 53, 57]. This presents a solution to the problem of selecting the GRM expansion of a given function possessing the smallest test set. Most of the earlier work on GRM expansions concentrated on deriving minimal GRM expansions in the sense of having the smallest number of terms [50, 57, 58, 59, 60]. In this research, we pay more attention to the 0th polarity GRM (PRM) expansions for arbitrary realization of arbitrary switching functions, not only the AND-EXOR logic circuits. Consequently, the problem of minimizing GRM expansions is not discussed here.

### 3.5. Modified Reed-Muller Coefficients

The RM transform is defined over GF(2). Each RM coefficient is either 0 or 1. Using them in fault detection, the aliasing probability is higher than other spectra defined in the integer fields. All of the R-W coefficients range from $-2^{n-1}$ to $+2^{n-1}$ except the first one, $r_0$, which is in the range from 0 to $+2^n$. Using ordinary matrix multiplication instead of matrix multiplication over GF(2), $\otimes$, in (3.13) and (3.14), we can get the *modified Reed-Muller polarity coefficient (MPC) matrix*, denoted by $\underset{\sim}{D}$, where

$$\underset{\sim}{D} = RM^n \,.\, Z \tag{3.15}$$

The jth entry in the kth column, $\underset{\sim}{D}^k$, is called Modified Reed-Muller coefficient (MRM coefficient) denoted by $\underset{\sim}{d}_j^k$, where $0 \leq \underset{\sim}{d}_j^k \leq 2^n - 1$, not only the Boolean values (0 or 1). For both of the R-W and MRM coefficients, the range of size are up to $2^n$, so give better discrimination and lead to less aliasing.

**Example 3.4**

Consider the function in example (3.2) $f(x_1,x_2,x_3) = x_1 + \overline{x}_1 x_3$, we have

$$\begin{bmatrix} 10000000 \\ 11000000 \\ 10100000 \\ 11110000 \\ 10001000 \\ 11001100 \\ 10101010 \\ 11111111 \end{bmatrix} \cdot \begin{bmatrix} 01001111 \\ 10001111 \\ 00011111 \\ 00101111 \\ 11110100 \\ 11111000 \\ 11110001 \\ 11110010 \end{bmatrix} = \begin{bmatrix} 01001111 \\ 11002222 \\ 01012222 \\ 11114444 \\ 12111011 \\ 33223322 \\ 23232323 \\ 55555555 \end{bmatrix}$$

$$RM^n \qquad \tilde{Z} \qquad \underset{\sim}{D}$$

It is obvious that

$$\tilde{D} = [\, \underset{\sim}{\tilde{D}} \,]_{mod\,2} \text{ and } d_j^k = [\, \underset{\sim}{d}_j^k \,]_{mod2}$$

But the transformation from MRM to RM coefficients is not directly reversible because of the mod 2 sum. However, no information has been lost since, in either cases, the original Boolean function can be recovered.

From (3.2b) we can see that the general form of modified generalized RM (MGRM) expansion is

$$f(x_1 \cdots x_n) = \bigoplus_{j=0}^{2^n-1} [\, \underset{\sim}{d}_j^k \,]_{mod2} [\prod_{All\ i,\ j_i=1} x_i^{k_i}] \tag{3.16}$$

The inverse of the RM transform matrix for modified RM coefficients is given by (3.12a).

$$(RM^n)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circledast (RM^{n-1})^{-1}$$

It is just the Arithmetic transform matrix [61].

From (3.15) and (3.12a), we can get the general expression to calculate the kth MRM coefficient $\underset{\sim}{d}_j^k$ of a Boolean function f(X).

**Theorem 3.6**

Let $0 \le k \le 2^n-1$ and let the kth polarity MRM expansion for a Boolean function f be

$$f(x_1 \cdots x_n) = \bigoplus_{j=0}^{2^n-1} [\, \underset{\sim}{d}_j^k \,]_{mod2} [\prod_{All\ i,\ j_i=1} x_i^{k_i}]$$

For the 0th polarity MRM expansion, (3.3b) becomes

$$\underline{d}_j^0 = \sum_{m \supseteq_q j} f(m) \tag{3.17}$$

For the kth polarity MRM expansion, (3.5b) becomes

$$\underline{d}_j^k = \sum_{m \supseteq_q j} f(m \,\overline{\oplus}\, k) \tag{3.18}$$

**Example 3.5**

Consider the 0th polarity MRM coefficients in example 3.4. From (3.17) we have

$$\begin{bmatrix} \underline{d}_0 \\ \underline{d}_1 \\ \underline{d}_2 \\ \underline{d}_3 \\ \underline{d}_4 \\ \underline{d}_5 \\ \underline{d}_6 \\ \underline{d}_7 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(0)+f(1) \\ f(0)+f(2) \\ f(0)+f(1)+f(2)+f(3) \\ f(0)+f(4) \\ f(0)+f(1)+f(4)+f(5) \\ f(0)+f(2)+f(4)+f(6) \\ f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}$$

The interpretation and order of the modified RM coefficients are similar to those of the RM coefficients. The difference is that each MRM coefficient is the syndrome of a subfunction of f(X) rather than its parity. The last MRM coefficient is the syndrome of the given function f(X) ($r_0$ of the Rademacher-Walsh spectrum).

Because the Reed-Muller and modified RM coefficients are the parity and syndrome of subfunctions of f(X), their verification does not require exhaustive testing except for the last one. The number of output values required for each RM or MRM coefficient, $d_j^k$ or $\underline{d}_j^k$, is $q_j$, where $1 \le q_j \le 2^n$. $q_j$ is determined by the order of the coefficient $d_j$, or, simply, the order of j, $q_j = 2^{|j|}$. Suppose there are m 1's in the binary representation of j, $d_j^k$ is the mod 2 sum of $2^{|j|}$ entries in the kth polarity output column vector $Z^k$, while $\underline{d}_j^k$ is the sum of $q_j$ entries in the kth polarity output column vector $Z^k$.

**Definition 3.5**

A *size vector* of the MRM coefficient column vector is a $2^n \times 1$ column vector, denoted by Q. The jth entry of Q, $q_j = 2^{|j|}$. Q is obtained by

$$Q = RM^n \cdot I \tag{3.19}$$

where I is an $2^n \times 1$ column vector. The entries of I are all one's. For example, let n = 4, we have

$$Q = RM^n \cdot I = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&0&0&0&1&0&0&0 \\ 1&1&0&0&1&1&0&0 \\ 1&0&1&0&1&0&1&0 \\ 1&1&1&1&1&1&1&1 \end{bmatrix} \cdot \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 1\\2\\2\\4\\2\\4\\4\\8 \end{bmatrix}$$

## 3.6. Shannon Decomposition of RM Coefficient Column Vector

The Shannon decompositions of the RM and MRM spectral coefficients are very useful. In this section, our only concern is the decompositions of the 0th polarity RM and MRM coefficient column vector D and $\underset{\sim}{D}$. Consider the Shannon expansion of f(X) with respect to variable $x_n$, as given in (2.10),

$$f(X) = \bar{x}_n f(x_1, \cdots, x_{n-1}, 0) + x_n f(x_1, \cdots, x_{n-1}, 1)$$

$$= \bar{x}_n f_0(x_1, \cdots, x_{n-1}) + x_n f_1(x_1, \cdots, x_{n-1})$$

The output column vector of f(X) can be halved as:

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix}$$

where $Z_0$ and $Z_1$ are the vectors for $f_0(x_1 \cdots x_{n-1}) = f(x_1 \cdots x_{n-1}, 0)$ ,and $f_1(x_1 \cdots x_{n-1}) = f(x_1 \cdots x_{n-1}, 1)$ respectively.

Then the 0th polarity RM coefficient vector D can be halved as:

$$D = \begin{bmatrix} D^0 \\ D^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} D_0 \\ D_1 \end{bmatrix} = \begin{bmatrix} RM^{n-1} & 0 \\ RM^{n-1} & RM^{n-1} \end{bmatrix} \otimes \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} \qquad \textbf{(3.20)}$$

where

$$D_0 = RM^{n-1} \otimes Z_0 \text{ and } D_1 = RM^{n-1} \otimes Z_1 \qquad (3.21)$$

combining (3.20) and (3.21), we have:

$$D^0 = D_0 \text{ and } D^1 = D_0 \oplus D_1$$

$$D_0 = D^0 \text{ and } D_1 = D_0 \oplus D^1 \qquad (3.22)$$

If we repeat applications of Shannon's expansion theorem with respect to n-m variables $x_n$, $x_{n-1}$, ...., $x_{m+1}$, the output vector Z and RM coefficient vector D of f(X) can be partitioned into $2^{n-m}$ parts

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \cdot \\ \cdot \\ Z_t \end{bmatrix}, \text{ and } D = \begin{bmatrix} D^0 \\ D^1 \\ \cdot \\ \cdot \\ D^t \end{bmatrix} \quad \text{where } t = 2^{n-m} - 1. \qquad (3.23)$$

Each $Z_i$, $0 \le i \le t$, has $2^m$ elements. The spectrum of $Z_i$ is

$$D_i = RM^m \otimes Z_i \qquad (3.24)$$

(3.14) can be rewritten as

$$D = \begin{bmatrix} D^0 \\ D^1 \\ \cdots \\ D^j \\ \cdots \\ D^t \end{bmatrix} = RM^{n-m} \otimes \begin{bmatrix} D_0 \\ D_1 \\ \cdots \\ D_j \\ \cdots \\ D_t \end{bmatrix} \quad \begin{matrix} t = 2^{n-m} - 1 \\ 0 \le j \le t \end{matrix} \qquad (3.25)$$

Since the inverse of RM transform matrix is the RM transform matrix itself in GF(2), we have

$$\begin{bmatrix} D_0 \\ D_1 \\ \cdots \\ D_j \\ \cdots \\ D_t \end{bmatrix} = RM^{n-m} \otimes \begin{bmatrix} D^0 \\ D^1 \\ \cdots \\ D^j \\ \cdots \\ D^t \end{bmatrix} \qquad \begin{matrix} t = 2^{n-m}-1 \\ 0 \le j \le t \end{matrix} \tag{3.26}$$

The partition of the modified Reed-Muller coefficient column vector, $\underset{\sim}{D}$, are similar to those of the RM coefficients. Consider the partition of $\underset{\sim}{D}$ with respect to variable $x_n$, the 0th polarity MRM coefficient vector $\underset{\sim}{D}$ can be halved as:

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \end{bmatrix} = \begin{bmatrix} RM^{n-1} & 0 \\ RM^{n-1} & RM^{n-1} \end{bmatrix} \cdot \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} \tag{3.27}$$

where

$$\underset{\sim}{D}_0 = RM^{n-1} \otimes Z_0 \quad \text{and} \quad \underset{\sim}{D}_1 = RM^{n-1} \otimes Z_1 \tag{3.28}$$

combining (3.27) and (3.28), we have:

$$\underset{\sim}{D}^0 = \underset{\sim}{D}_0 \text{ and } \underset{\sim}{D}^1 = \underset{\sim}{D}_0 + \underset{\sim}{D}_1$$

$$\underset{\sim}{D}_0 = \underset{\sim}{D}^0 \text{ and } \underset{\sim}{D}_1 = \underset{\sim}{D}^1 - \underset{\sim}{D}^0 \tag{3.29}$$

Consider the partition of $\underset{\sim}{D}$ with respect to n-m variables $x_n, x_{n-1}, ...., x_{m+1}$, the output vector Z and MRM coefficient vector $\underset{\sim}{D}$ of f(X) can be partitioned into $2^{n-m}$ parts

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \cdot \\ \cdot \\ Z_t \end{bmatrix}, \quad \text{and} \quad \underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdot \\ \cdot \\ \underset{\sim}{D}^t \end{bmatrix} \qquad \text{where } t = 2^{n-m}-1. \tag{3.30}$$

Each $Z_i$, $0 \le i \le t$, has $2^m$ elements. The spectrum of $Z_i$ is

$$\underset{\sim}{D}_i = RM^m \ . \ Z_i \tag{3.31}$$

(3.15) can be rewritten as

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdots \\ \underset{\sim}{D}^j \\ \cdots \\ \underset{\sim}{D}^t \end{bmatrix} = RM^{n-m} \cdot \begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \\ \cdots \\ \underset{\sim}{D}_t \\ \cdots \\ \underset{\sim}{D}_t \end{bmatrix} \quad \begin{matrix} t = 2^{n-m}-1 \\ 0 \leq j \leq t \end{matrix} \tag{3.32}$$

Since the inverse of the MRM transform matrix for modified RM coefficients is given by (3.12a).

$$(RM^n)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \circledast (RM^{n-1})^{-1}$$

From (3.32) we have

$$\begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \\ \cdots \\ \underset{\sim}{D}_j \\ \cdots \\ \underset{\sim}{D}_\alpha \end{bmatrix} = (RM^{n-m})^{-1} \cdot \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdots \\ \underset{\sim}{D}^j \\ \cdots \\ \underset{\sim}{D}^\alpha \end{bmatrix} \quad \begin{matrix} \alpha = 2^{n-m}-1 \\ 0 \leq j \leq \alpha \end{matrix} \tag{3.33}$$

From (3.12a) we can see that the ith row and jth column entry of $RM^{-1}$ is

$$\begin{matrix} (-1)^q, & \text{if } i \supseteq j, \\ 0 & \text{if } i \not\supseteq j. \end{matrix}$$

where $q = |i| + |j|$, $|i|$ and $|j|$ are the order of i and j, respectively. So that for each $\underset{\sim}{D}_j$ in (3.33), we have

$$\underset{\sim}{D}_j = \sum_{i \supseteq} (-1)^q \underset{\sim}{D}^i \tag{3.34}$$

**Example 3.6**

Consider the 4 variable function $f(x_1x_2x_3x_4)$.

$$f(X) = x_4\bar{x}_3\bar{x}_2 + x_4x_2\bar{x}_1 + \bar{x}_4x_3\bar{x}_2 + \bar{x}_4x_2x_1$$

The Shannon decomposition of Z, D with respect to the variables $x_4$ and $x_3$ is given in Table 3.2.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | | D | | $D_i$ | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | | 0 | | 0 | |
| 0 | 0 | 0 | 1 | 0 | $Z^0$ | 0 | $D^0$ | 0 | $D_0$ |
| 0 | 0 | 1 | 0 | 0 | | 0 | | 0 | |
| 0 | 0 | 1 | 1 | 1 | | 1 | | 1 | |
| 0 | 1 | 0 | 0 | 1 | | 1 | | 1 | |
| 0 | 1 | 0 | 1 | 1 | $Z^1$ | 0 | $D^1$ | 0 | $D_1$ |
| 0 | 1 | 1 | 0 | 0 | | 1 | | 1 | |
| 0 | 1 | 1 | 1 | 1 | | 0 | | 1 | |
| 1 | 0 | 0 | 0 | 1 | | 1 | | 1 | |
| 1 | 0 | 0 | 1 | 1 | $Z^2$ | 0 | $D^2$ | 0 | $D_2$ |
| 1 | 0 | 1 | 0 | 1 | | 0 | | 0 | |
| 1 | 0 | 1 | 1 | 0 | | 0 | | 1 | |
| 1 | 1 | 0 | 0 | 0 | | 0 | | 0 | |
| 1 | 1 | 0 | 1 | 0 | $Z^3$ | 0 | $D^3$ | 0 | $D_3$ |
| 1 | 1 | 1 | 0 | 1 | | 0 | | 1 | |
| 1 | 1 | 1 | 1 | 0 | | 0 | | 1 | |

**Table 3.2** The Shannon Decomposition of RM Spectrum

Consider $D^3$. From (3.25) we have

$$D^3 = D_0 \oplus D_1 \oplus D_2 \oplus D_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \oplus \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \oplus \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \oplus \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Consider $D_3$. From (3.26) we have

$$D_3 = D^0 \oplus D^1 \oplus D^2 \oplus D^3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \oplus \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \oplus \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \oplus \begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Similarly, the Shannon decomposition of the MRM coefficient column vector $\underset{\sim}{D}$ with respect to $x_4$ and $x_3$ is given in Table 3.3.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | | $\underset{\sim}{D}$ | | $\underset{\approx}{D}$ | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | | 0 | | 0 | |
| 0 | 0 | 0 | 1 | 0 | $Z^0$ | 0 | $\underset{\sim}{D}^0$ | 0 | $\underset{\approx}{D}_0$ |
| 0 | 0 | 1 | 0 | 0 | | 0 | | 0 | |
| 0 | 0 | 1 | 1 | 1 | | 1 | | 1 | |
| 0 | 1 | 0 | 0 | 1 | | 1 | | 1 | |
| 0 | 1 | 0 | 1 | 1 | $Z^1$ | 2 | $\underset{\sim}{D}^1$ | 2 | $\underset{\approx}{D}_1$ |
| 0 | 1 | 1 | 0 | 0 | | 1 | | 1 | |
| 0 | 1 | 1 | 1 | 1 | | 4 | | 3 | |
| 1 | 0 | 0 | 0 | 1 | | 1 | | 1 | |
| 1 | 0 | 0 | 1 | 1 | $Z^2$ | 2 | $\underset{\sim}{D}^2$ | 2 | $\underset{\approx}{D}_2$ |
| 1 | 0 | 1 | 0 | 1 | | 2 | | 2 | |
| 1 | 0 | 1 | 1 | 0 | | 4 | | 3 | |
| 1 | 1 | 0 | 0 | 0 | | 2 | | 0 | |
| 1 | 1 | 0 | 1 | 0 | $Z^3$ | 4 | $\underset{\sim}{D}^3$ | 0 | $\underset{\approx}{D}_3$ |
| 1 | 1 | 1 | 0 | 1 | | 4 | | 1 | |
| 1 | 1 | 1 | 1 | 0 | | 8 | | 1 | |

**Table 3.3** The Shannon Decomposition of MRM Spectrum

Consider $\underset{\sim}{D}^3$. From (3.32) we have

$$\underset{\sim}{D}^3 = \underset{\approx}{D}_0 + \underset{\approx}{D}_1 + \underset{\approx}{D}_2 + \underset{\approx}{D}_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\1\\3\end{bmatrix} + \begin{bmatrix}1\\2\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\\4\\8\end{bmatrix}$$

Consider $\underset{\approx}{D}_3$. From (3.33) and (3.34) we have

$$\underset{\approx}{D}_3 = \underset{\sim}{D}^0 - \underset{\sim}{D}^1 - \underset{\sim}{D}^2 + \underset{\sim}{D}^3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\2\\1\\4\end{bmatrix} - \begin{bmatrix}1\\2\\2\\4\end{bmatrix} + \begin{bmatrix}2\\4\\4\\8\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

The decomposition of Z, D and $\underset{\sim}{D}$ are frequently used in chapter 5.

### 3.7. The Computation of RM Spectral Coefficients

Besslich [63] introduced a Fast Reed-Muller Transform method used to get D from Z by an iterative strategy. At stage i of the iteration, copy the first half of a previous subfunction of n-i variables, and then obtain the bit-wise exclusive sum of the two halves of that same subfunction . Fig 3.2 shows the method for three variables example.

Compared with the Hadamard transform, the iterative RM transform is computationally simpler. The latter requires $n \times 2^n$ additions and subtractions, where the former requires only $n \times 2^{n-1}$ mod 2 additions.

For the MRM coefficients, the fast transform procedure is the same except using arithmetic additions instead of mod 2 additions.

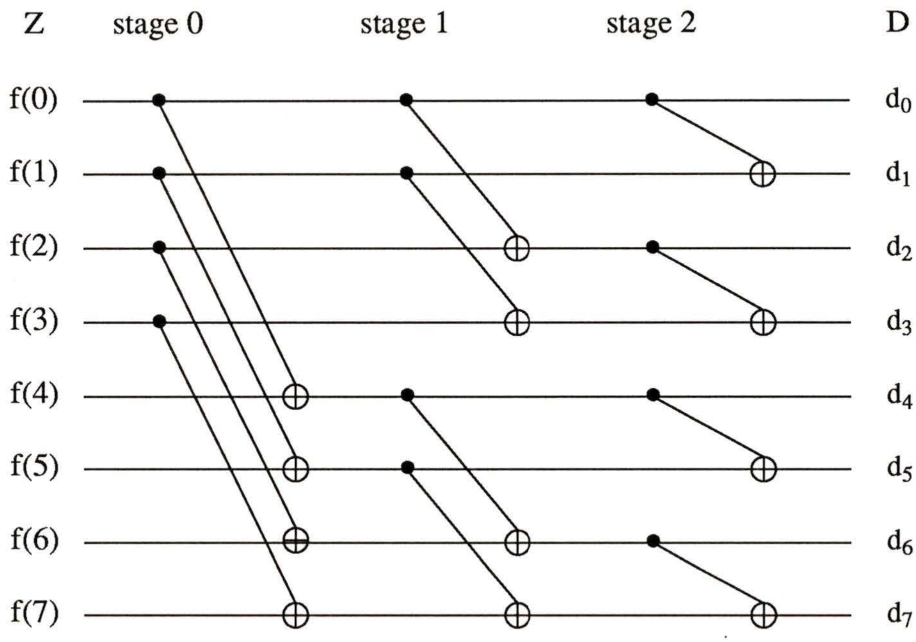



**Figure 3.2** Fast RM transform.

### 3.8. Conclusion

In this chapter, we described the GRM expansion. The RM coefficients, RM transform matrix, and RM transformation have been introduced. Instead of considering the transformation between the RM coefficients and the output values of a Boolean function individually as shown in (3.3) and (3.4), we defined the polarity output and RM coefficient matrices, and a matrix method was used to establish the relationship between the Boolean and the RM spectral domain. It provides a better way to analyse the properties of the Boolean functions and RM spectrum. A set of modified RM coefficients is also proposed. Some more properties of the RM and the MRM spectra will be discussed in next chapter.

# CHAPTER 4

# SOME PROPERTIES OF THE REED-MULLER AND MODIFIED REED-MULLER SPECTRA

## 4.1. Introduction

The Reed-Muller expansion has been discussed in chapter 3. A new set of spectral coefficients, the modified Reed-Muller spectral coefficients was proposed. Up to now, four spectral techniques have been introduced. They are Rademacher-Walsh, arithmetic, Reed-Muller, and modified Reed-Muller spectra. Some of the spectral techniques have been widely used for network analysis, synthesis, and more recently test purposes [34, 38, 40, 47, 52]. In this chapter, we give the relationships between the different spectral coefficients.

Some properties of RM and MRM coefficient are also given in this chapter. Damarla has given a complete list of Reed-Muller coefficient properties in [68]. In this chapter, we give these properties in matrix form. It is shown below that the matrix method is more convenient for analysis and synthesis purposes.

## 4.2. The Relationships

By using different transformations, we can get a variety of spectral coefficients. In order to analyse and compare them for testing purpose, we show the mathematical relationships which exist between these alternative spectral coefficients.

Hurst [61] has shown the relationships between R-W, arithmetic, and RM coefficients.

**Theorem 4.1** [61]

Let R, B be the R-W and arithmetic spectral coefficients column vectors, respectively. The conversion from R to B is given by

$$B = Trb^n \, . \, R \tag{4.1}$$

where $Trb^n$ is called a conversion matrix from R to B, defined by

$$Trb^n = \begin{bmatrix} Trb^{n-1} & Trb^{n-1} \\ 0 & -2Trb^{n-1} \end{bmatrix} \qquad Trb^0 = +1 \tag{4.2}$$

The inverse of $Trb^n$ provides the conversion matrix from B to R, i.e.

$$R = [\, Trb^n \,]^{-1} \, . \, B = Tbr^n \, . \, B \tag{4.3}$$

where

$$Tbr^n = \begin{bmatrix} 2Tbr^{n-1} & Tbr^{n-1} \\ 0 & -Tbr^{n-1} \end{bmatrix} \qquad Trb^0 = +1 \tag{4.4}$$

The proof of Theorem 4.1 is in [61].

**Example 4.1**

Consider an example for n=3, the conversion matrices from R to B and B to R are

$$Trb^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \text{ and } Tbr^3 = \begin{bmatrix} 8 & 4 & 4 & 2 & 4 & 2 & 2 & 1 \\ 0 & -4 & 0 & -2 & 0 & -2 & 0 & -1 \\ 0 & 0 & -4 & -2 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Now let's look at the relationship between the R-W and MRM coefficients. From chapter 2 we saw that there are $2^n$ different polarity RM and MRM expansions for a Boolean function. In this research, we concentrate on the 0th polarity coefficients. From now on we use D and $\underset{\sim}{D}$ for the 0th polarity Reed-Muller and Modified Reed-Muller coefficient column vectors, respectively.

**Theorem 4.2**

Let R, $\underset{\sim}{D}$ be the R-W and MRM spectral coefficient column vectors, respectively. The conversion from $\underset{\sim}{D}$ to R is given by

$$R = T\underset{\sim}{d}r^n \ . \ \underset{\sim}{D} \tag{4.5}$$

where $T\underset{\sim}{d}r^n$ is called a conversion matrix from $\underset{\sim}{D}$ to R, defined by

$$T\underset{\sim}{d}r^n = \begin{bmatrix} 0 & T\underset{\sim}{d}r^{n-1} \\ 2T\underset{\sim}{d}r^{n-1} & -T\underset{\sim}{d}r^{n-1} \end{bmatrix} \qquad T\underset{\sim}{d}r^0 = +1 \tag{4.6}$$

The inverse of $T\underset{\sim}{d}r^n$ will provide the conversion matrix from R to $\underset{\sim}{D}$, i.e.

$$\underset{\sim}{D} = \frac{1}{2^n}[\, T\underset{\sim}{d}r^n \,]^{-1} \ . \ R = Tr\underset{\sim}{d}^n \ . \ R \tag{4.7}$$

where

$$Tr\underset{\sim}{d}^n = \begin{bmatrix} Tr\underset{\sim}{d}^{n-1} & 2Tr\underset{\sim}{d}^{n-1} \\ Tr\underset{\sim}{d}^{n-1} & 0 \end{bmatrix} \qquad Tr\underset{\sim}{d}^0 = +1 \tag{4.8}$$

**Proof:**

Let $R = T^n \, . \, Z$, and $\underset{\sim}{D} = RM^n \, . \, Z$. We have

$$R = T^n \, . \, (\, (RM^n)^{-1} \,) \, . \, \underset{\sim}{D} \tag{4.9}$$

Substituting (2.2) and (3.12a) into (4.9), we have

$$R = \begin{bmatrix} T^{n-1} & T^{n-1} \\ T^{n-1} & -T^{n-1} \end{bmatrix} \cdot \begin{bmatrix} (RM^{n-1})^{-1} & 0 \\ -(RM^{n-1})^{-1} & (RM^{n-1})^{-1} \end{bmatrix} \cdot \underset{\sim}{D}$$

$$= \begin{bmatrix} 0 & T^{n-1}(RM^{n-1})^{-1} \\ 2T^{n-1}(RM^{n-1})^{-1} & -T^{n-1}(RM^{n-1})^{-1} \end{bmatrix} . \underset{\sim}{D}$$

$$= \begin{bmatrix} 0 & T\underset{\sim}{d}r^{n-1} \\ 2T\underset{\sim}{d}r^{n-1} & -T\underset{\sim}{d}r^{n-1} \end{bmatrix} . \underset{\sim}{D}$$

The proof of (4.8) is similar to that of (4.7).

**Example 4.2**

Consider the function $f(X) = \overline{x}_1\overline{x}_2x_3 + \overline{x}_1x_2\overline{x}_3 + x_1x_2x_3$.

| $x_3$ | $x_2$ | $x_1$ | f | $\underset{\sim}{D}$ | D | R |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 0 | 0 | 1 | 1 | 1 | 1 | -1 |
| 0 | 1 | 0 | 1 | 1 | 1 | -1 |
| 0 | 1 | 1 | 0 | 2 | 0 | -1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 3 | 1 | -3 |

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & -2 & -2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 4 & -2 & 0 & 0 & -2 & 1 \\ 0 & 4 & 0 & -2 & 0 & -2 & 0 & 1 \\ 8 & -4 & -4 & 2 & -4 & 2 & 2 & -1 \end{bmatrix}}_{T\underset{\sim}{d}r} \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_{\underset{\sim}{D}} = \underbrace{\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}}_{R}$$

$$\frac{1}{8} \cdot \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{Tr\underset{\sim}{d}} \cdot \underbrace{\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}}_{R} = \frac{1}{8} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 16 \\ 0 \\ 8 \\ 8 \\ 24 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_{\underset{\sim}{D}}$$

The relationship between R-W and RM coefficients are given by Theorem 4.3.

**Theorem 4.3**

Let D and R be the RM and R-W coefficient column vectors, respectively, of a Boolean function f(X), then

$$D = \left[ \underset{\sim}{D} \right]_{\text{mod } 2} = \left[ \frac{1}{2^n} \,.\, \mathrm{Tr}\underset{\sim}{\mathrm{d}}^n \,.\, R \right]_{\text{mod } 2} \tag{4.10}$$

$$R = T^n . Z = T^n . (RM^n \otimes D) \tag{4.11}$$

The proof is obvious from Theorem 3.4 and Theorem 4.2.

Theorem 4.4 provides the relationship between arithmetic and MRM coefficients.

**Theorem 4.4**

Let B, $\underset{\sim}{D}$ be the arithmetic and MRM spectral coefficients column vectors, respectively. The conversion from $\underset{\sim}{D}$ to B is given by

$$B = \mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^n \,.\, \underset{\sim}{D} \tag{4.12}$$

where $\mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^n$ is called a conversion matrix from $\underset{\sim}{D}$ to B, defined by

$$\mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^n = \begin{bmatrix} \mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^{n-1} & 0 \\ -2\mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^{n-1} & \mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^{n-1} \end{bmatrix} \qquad \mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^0 = +1 \tag{4.13}$$

The inverse of $\mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^n$ will provide the conversion matrix from B to $\underset{\sim}{D}$, i.e.

$$\underset{\sim}{D} = [\, \mathrm{T}\underset{\sim}{\mathrm{d}}\mathrm{b}^n \,]^{-1} \,.\, B = \mathrm{Tb}\underset{\sim}{\mathrm{d}}^n \,.\, B \tag{4.14}$$

where

$$\mathrm{Tb}\underset{\sim}{\mathrm{d}}^n = \begin{bmatrix} \mathrm{Tb}\underset{\sim}{\mathrm{d}}^{n-1} & 0 \\ 2\mathrm{Tb}\underset{\sim}{\mathrm{d}}^{n-1} & \mathrm{Tb}\underset{\sim}{\mathrm{d}}^{n-1} \end{bmatrix} \qquad \mathrm{Tb}\underset{\sim}{\mathrm{d}}^0 = +1 \tag{4.15}$$

The proof is similar to that of Theorem 4.2.

The relationship between arithmetic and RM coefficients is given by Theorem 4.5.

**Theorem 4.5** Let B, D be the arithmetic and RM spectral coefficient column vectors, respectively. The conversion from B to D is given by

$$D = \left[ Tb\underset{\sim}{d}^{n} . B \right]_{mod2} \tag{4.16}$$

and

$$B = RM^{n} \ . \ Z = RM^{n} . ( RM^{n} \otimes D ) \tag{4.17}$$

**Proof:**

(4.16) is obtained by substituting (4.14) into $D = \left[ \underset{\sim}{D} \right]_{mod\ 2}$.

(4.17) is obvious from Theorem 3.4.

**Summary of The Relationships For n = 1**

**(a) Rademacher-Walsh and arithmetic**

$$Trb = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} , \quad \text{and} \quad Tbr = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

**(b) Rademacher-Walsh and Modified Reed-Muller**

$$T\underset{\sim}{d}r = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} , \quad \text{and} \quad Tr\underset{\sim}{d} = \frac{1}{2^{n}} . \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

**(c) Arithmetic and Modified Reed-Muller**

$$T\underset{\sim}{d}b = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} , \quad \text{and} \quad Tb\underset{\sim}{d} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Because Reed-Muller coefficients are defined on GF(2), while other spectral coefficients are defined over the integer field, it is not possible to get an easy conversion matrix. The conversions between Reed-Muller and other spectral coefficients are given in Theorem 4.3 and Theorem 4.5.

## 4.2. Some Properties of RM and MRM Spectral Coefficients

In this section, we present some properties of RM and MRM spectral coefficients. Only 0th polarity RM and MRM spectral coefficients are considered.

### 4.2.1. Properties of The Reed-Muller Transform

Damarla has given a complete list of Reed-Muller transform properties and their proofs in [67, 68]. The composite RM spectra of two functions f and g, namely, f ⊗ g, f ∩ g, and f ∪ g, are given as follows:

**Theorem 4.6** [68]

(a) Let $F(X) = \overline{f}(X)$, then

$$d_j^F = \begin{cases} d_j^f, & \text{if } j \neq 0; \\ 1 \oplus d_j^f, & \text{if } j = 0. \end{cases} \tag{4.18}$$

where $d_j^F$ and $d_j^f$ are the jth RM coefficients for functions F(X) and f(X), respectively.

(b) Linearity: Let $X = (x_1 x_2 \ldots xn)$, and $d_j^F$, $d_j^f$, and $d_j^g$ be the jth coefficient for function F(X), f(X), and g(X), respectively. If $F(X) = f(x) \oplus g(X)$, then

$$d_j^F = d_j^f \oplus d_j^g \tag{4.19}$$

(c) Let $F(X) = f(X) \cap g(X)$, then

$$d_j^F = \bigoplus_{i \cup k = j} d_i^f \cdot d_k^g \tag{4.20}$$

(d) Let $F(X) = f(X) \cup g(X)$, then

$$d_j^F = d_j^f \oplus d_j^g \oplus \bigoplus_{i \cup k = j} d_i^f \cdot d_k^g \tag{4.21}$$

where ∪, ∩, and ⊕ denote componentwise OR, AND, and EXCLUSIVE-OR operations.

Now, instead of considering the individual coefficient, we use a matrix method to give the properties of RM coefficient column vectors corresponding to Theorem 4.6.

**Theorem 4.7**

(a) Let $F(X) = f(X) \cap g(X)$, and the Reed-Muller spectral coefficient column vector of F, f, and g be $D_F$, $D_f$, and $D_g$, respectively. Then we have

$$D_F = [\,(D_f * RM^t) \odot RM\,] \otimes D_g$$

or

$$D_F = [\,(\hat{D}_f \otimes RM^t) \odot RM\,] \otimes D_g \tag{4.22}$$

where " * " is the convolution operator [34], and the ith row and jth column entry of $\hat{D}_f$ is equal to $d^f_{i \oplus \bar{j}}$. The operator " $\odot$" is called the Hadamard product. An entry in the Hadamard product of two matrices is just the product of the corresponding entries in the two matrices, i.e., $[\,A \odot B\,]_{ij} = a_{ij} \cdot b_{ij}$ for two matrices A and B.

(b) If $F(X) = f(X) \oplus g(X)$, then

$$D_F = D_f \oplus D_g \tag{4.23}$$

(c) If $F(X) = f(X) \cup g(X)$, then

$$D_F = D_f \oplus D_g \oplus D_{f \cap g} \tag{4.24}$$

(d) If $F(X) = \bar{g}(X)$, then

$$D_F = J^n \oplus D_g \tag{4.25}$$

where $J^n$ is a $2^n \times 1$ matrix. All of the entries of J equal to "0" except the first entry. The first entry of J is "1".

**Proof**

The proofs for (4.23), (4.24), and (4.25) come from Theorem 4.6 directly.

The proof of (4.22) is given as follows.

From (4.20) we have

$$d^F_j = \bigoplus_{i \cup k = j} d^f_i \cdot d^g_k$$

Since $a(b \oplus c \oplus d \oplus \ldots\ldots) = ab \oplus ac \oplus ad \oplus \ldots\ldots$, (4.20) can be written in another form

$$d^F_j = \bigoplus_{k \supseteq j} [\bigoplus_{\text{All } i, i \cup k = j} d^f_i\,] \cdot d^g_k \tag{4.26}$$

As an example, let n = 3 and j = 011.

$$d_3^F = \bigoplus_{\text{All } i,k;\, i\cup k=3} d_i^f \,.\, d_k^g$$

$$= d_0^f d_3^g \oplus d_2^f d_1^g \oplus d_3^f d_1^g \oplus d_3^f d_2^g \oplus d_1^f d_2^g \oplus d_3^f d_0^g \oplus d_3^f d_1^g \oplus d_3^f d_2^g \oplus d_3^f d_3^g$$

$$= d_0^g d_3^f \oplus d_1^g (d_2^f \oplus d_3^f) \oplus d_2^g (d_1^f \oplus d_3^f) \oplus d_3^g (d_0^f \oplus d_1^f \oplus d_2^f \oplus d_3^f)$$

$$= \bigoplus_{\text{All } k,\, k\supseteq 3} \Big[ \bigoplus_{\text{All } i,\, i\cup k=3} d_i^f \Big] \,.\, d_k^g$$

Let $\hat{d}_{jk}^f$ be the jth row and kth column entry in $\hat{D}_f$, we have

$$\hat{d}_{jk}^f = d_{\overline{j\oplus k}}^f. \tag{4.27}$$

Let $U = \hat{D}_f \otimes RM^t$ and $u_{jk}$ be the jth row and kth column entry in U.

$$u_{jk} = \bigoplus_{\text{all } i\supseteq k} \hat{d}_{ji}^f = \bigoplus_{\text{all } i\supseteq k} d_{\overline{j\oplus i}}^f$$

Let $W = U \odot RM$ and $w_{jk}$ be the jth row and kth column entry in W, then $w_{jk} = 0$, if $k \not\supseteq j$,

$$\text{i.e.} \qquad w_{ji} = \begin{cases} \bigoplus_{\text{all} i\supseteq k} d_{\overline{j\oplus i}}^f & \text{if } k\supseteq j; \\ 0 & \text{else.} \end{cases}$$

Let $V = W \otimes D_g$. V is an $2^n \times 1$ column vector. Let vsuj be the jth entry of V, then

$$v_j = \bigoplus_{\text{all} k} w_{jk} \,.\, d_k^g$$

but $w_{jk} = 0$ unless $k\supseteq j$, so

$$v_j = \bigoplus_{\text{all } k} w_{jk} \,.\, d_k^g = \bigoplus_{\text{all } k\supseteq j} \Big[ \bigoplus_{\text{all } i\supseteq k} d_{\overline{j\oplus i}}^f \Big] d_k^g$$

Now the question is to prove, for any $k\supseteq j$,

$$\bigoplus_{\text{all } i\supseteq k} d_{\overline{j\oplus i}}^f = \bigoplus_{\text{all } i,\, i\cup k=j} d_i^f \tag{4.28}$$

Without lossing generality, let

$$j = \underbrace{\overline{0 \cdots \cdots 0}}_{} \ \overline{1 \cdots \cdots 1} \quad \text{with } \overset{\text{a bits}}{0 \cdots \cdots 0}\ \overset{\text{n-a bits}}{1 \cdots \cdots 1} \tag{4.29}$$

Consider some $k \supseteq j$, we have

$$k = \overset{\text{b bits}}{\overline{0 \cdots \cdots 000 \cdots 0}}\ \overset{\text{n-b bits}}{\overline{1 \cdots 1}} \qquad \text{where } b \geq a.$$

Then the set of all i, $i \cup k = j$ is given by

$$\overset{\text{a bits}}{\overline{0 \cdots \cdots 0}}\ \overset{\text{b-a bits}}{\overline{1 \cdots 1}}\ \overset{\text{n-b bits}}{\overline{X \cdots X}} \tag{4.30}$$

The set of all i, $i \supseteq k$ is given by

$$\overset{\text{b bits}}{\overline{0 \cdots \cdots 0}}\ \overset{\text{n-b bits}}{\overline{X \cdots X}} \tag{4.31}$$

From (4.29) and (4.31) we can see that the set of all $j\overline{\oplus}i$ for these i is given by

$$\overset{\text{a bits}}{\overline{0 \cdots \cdots 0}}\ \overset{\text{b-a bits}}{\overline{1 \cdots 1}}\ \overset{\text{n-b bits}}{\overline{X \cdots X}} \tag{4.32}$$

where is the same set as (4.30). Then we have

$$\bigoplus_{\text{all } i \supseteq k} d^f_{j\overline{\oplus}i} = \bigoplus_{\text{all } i, i \cup k = j} d^f_i$$

So that (4.26) is proved.

**Example 4.3**

Let $F(X) = f(X) \cap g(X)$, and $D_F$, $D_f$, and $D_g$ be the RM coefficient column vectors for F, f, and g, respectively. For n = 3, we have

$$\hat{D}_f = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 2 & 3 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}.$$

Where the ith row and jth column entry of $\hat{D}_f$ is $d^f_{\overline{i \oplus j}}$. In the example shown above, the integer entry "k" stands for $d^f_k = d^f_{\overline{i \oplus j}}$.

$$\hat{D}_f \otimes RM^t = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 2 & 3 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (0) & (0,1) & (0,2) & (0,1,2,3) & (0,4) & (0,1,4,5) & (0,2,4,6) & (0,1,2,..,7) \\ (1) & (1,0) & (1,3) & (0,1,2,3) & (1,5) & (0,1,4,5) & (1,3,5,7) & (0,1,2,..,7) \\ (2) & (2,3) & (2,0) & (0,1,2,3) & (2,6) & (2,3,6,7) & (0,2,4,6) & (0,1,2,..,7) \\ (3) & (3,2) & (3,1) & (0,1,2,3) & (3,7) & (2,3,6,7) & (1,3,5,7) & (0,1,2,..,7) \\ (4) & (4,5) & (4,6) & (4,5,6,7) & (4,0) & (0,1,4,5) & (0,2,4,6) & (0,1,2,..,7) \\ (5) & (5,4) & (5,7) & (4,5,6,7) & (5,1) & (0,1,4,5) & (1,3,5,7) & (0,1,2,..,7) \\ (6) & (6,7) & (6,4) & (4,5,6,7) & (6,2) & (2,3,6,7) & (0,2,4,6) & (0,1,2,..,7) \\ (7) & (7,6) & (7,5) & (4,5,6,7) & (7,3) & (2,3,6,7) & (1,3,5,7) & (0,1,2,..,7) \end{bmatrix}$$

The matrix entry (i,j,k,..) stands for $d^f_i \oplus d^f_j \oplus d^f_k \oplus \cdots$.

$$D_F = [(\hat{D}_f \otimes RM^t) \odot RM] \otimes D_g =$$

$$\begin{bmatrix} (0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1) & (1,0) & 0 & 0 & 0 & 0 & 0 & 0 \\ (2) & 0 & (2,0) & 0 & 0 & 0 & 0 & 0 \\ (3) & (3,2) & (3,1) & (0,1,2,3) & 0 & 0 & 0 & 0 \\ (4) & 0 & 0 & 0 & (4,0) & 0 & 0 & 0 \\ (5) & (5,4) & 0 & 0 & (5,1) & (0,1,4,5) & 0 & 0 \\ (6) & 0 & (6,4) & 0 & (6,2) & 0 & (0,2,4,6) & 0 \\ (7) & (7,6) & (7,5) & (4,5,6,7) & (7,3) & (2,3,6,7) & (1,3,5,7) & (0,1,2,...,7) \end{bmatrix} \otimes \begin{bmatrix} d^g_0 \\ d^g_1 \\ d^g_2 \\ d^g_3 \\ d^g_4 \\ d^g_5 \\ d^g_6 \\ d^g_7 \end{bmatrix}$$

Then, we have $d^F_j = \bigoplus_{\text{All } i,\, i \supseteq j} [\bigoplus_{\text{All } k,\, k \cup i = j} d^f_k] \,.\, d^g_i$. This is just (4.26).

### 4.2.2. Properties of The Modified Reed-Muller Transform

For the modified Reed-Muller spectral coefficients, we have the following results.

**Theorem 4.8**

(a) Let $F(X) = \overline{f}(X)$, and $\underset{\sim}{d}_j^F$, and $\underset{\sim}{d}_j^g$ be the jth MRM coefficient of F, f, and g, respectively. Then

$$\underset{\sim}{d}_j^F = q_j - \underset{\sim}{d}_j^f \qquad 0 \leq j \leq 2^n - 1. \tag{4.33}$$

where $q_j$ is determined by the order of the coefficient, $q_j = 2^{|j|}$.

(b) If $F(X) = f(x) \oplus g(X)$, then

$$\underset{\sim}{d}_j^F = \underset{\sim}{d}_j^f + \underset{\sim}{d}_j^g - 2\,\underset{\sim}{d}_j^{f \cap g} \tag{4.34}$$

(c) If $F(X) = f(X) \cup g(X)$, then

$$\underset{\sim}{d}_j^F = \underset{\sim}{d}_j^f + \underset{\sim}{d}_j^g - \underset{\sim}{d}_j^{f \cap g} \tag{4.35}$$

(d) If $F(X) = f(X) \cap g(X)$, then

$$\underset{\sim}{d}_j^F = \underset{\sim}{d}_j^f + \underset{\sim}{d}_j^g - \underset{\sim}{d}_j^{f \cup g} \tag{4.36}$$

**Proof**

Karp [45] has given the following results between binary operations and those over the integer field.

Let x and y be binary values, $x, y \in \{0,1\}$, then

$$x \oplus y = x + y - 2xy \tag{4.37}$$

$$x \cup y = x + y - xy \tag{4.38}$$

$$x \cap y = xy \tag{4.39}$$

If $F(X) = f(X) \oplus g(X)$, from (3.17) and (4.37) we have

$$\underset{\sim}{d}_j^F = \sum_{i \supseteq j} F(i) = \sum_{i \supseteq j} \left[ f(i) \oplus g(i) \right]$$

$$= \sum_{i \supseteq j} \Big[ f(i) + g(i) - 2\, f(i)\, g(i) \Big]$$

$$= \sum_{i \supseteq j} f(i) + \sum_{i \supseteq j} g(i) - 2 \sum_{i \supseteq j} \Big[ f(i)\, g(i) \Big]$$

$$= \underset{\sim}{d}_j^{f} + \underset{\sim}{d}_j^{g} - 2\underset{\sim}{d}_j^{f \cap g}$$

If $F(X) = f(X) \cup g(X)$, From (3.17) and (4.38) we have

$$\underset{\sim}{d}_j^{F} = \sum_{i \supseteq j} F(i) = \sum_{i \supseteq j} \Big[ f(i) \cup g(i) \Big]$$

$$= \sum_{i \supseteq j} f(i) + \sum_{i \supseteq j} g(i) - \sum_{i \supseteq j} \Big[ f(i)\, g(i) \Big]$$

$$= \underset{\sim}{d}_j^{f} + \underset{\sim}{d}_j^{g} - \underset{\sim}{d}_j^{f \cap g}$$

If $F(X) = f(X) \cap g(X)$, substituting (4.39) into (4.38), then

$$x \cap y = x + y - x \cup y \tag{4.40}$$

From (3.17) and (4.40) we have

$$\underset{\sim}{d}_j^{F} = \sum_{i \supseteq j} F(i) = \sum_{i \supseteq j} [\, f(i) \cap g(i) \,]$$

$$= \sum_{i \supseteq j} \Big[ f(i) + g(i) - f(i) \cup g(i) \Big]$$

$$= \sum_{i \supseteq j} f(i) + \sum_{i \supseteq j} g(i) - \sum_{i \supseteq j} [\, f(i) \cup g(i) \,]$$

$$= \underset{\sim}{d}_j^{f} + \underset{\sim}{d}_j^{g} - \underset{\sim}{d}_j^{f \cup g}$$

In section 3.5 we have seen that each MRM coefficient, $\underset{\sim}{d}_j$, is the sum of $q_j$ output values, where $q_j = 2^{|j|}$. So that if $F(X) = \overline{f}(X)$, $\underset{\sim}{d}_j^{F} = q_j - \underset{\sim}{d}_j^{f}$.

Theorem 4.9 gives some properties of the modified Reed-Muller coefficients in matrix form.

**Theorem 4.9**

(a) Let $F(X) = \overline{f}(X)$, $D_F$ and $D_f$ be the MRM coefficient column vectors for F(X) and f(X), respectively, then

$$\underset{\sim}{D}_F = Q - \underset{\sim}{D}_f \tag{4.41}$$

where Q is the size vector defined in (3.19), and each entry, $q_j$, in Q is determined by the order of j, $q_j = 2^{|j|}$.

(b) Let $F(X) = f(X) \oplus g(X)$, $\underset{\sim}{D}_F$, $\underset{\sim}{D}_f$, $\underset{\sim}{D}_g$, and $\underset{\sim}{D}_{f\cap g}$ be the MRM coefficient column vectors for F, f, g, and $f \cap g$, respectively. We have

$$\underset{\sim}{D}_F = \underset{\sim}{D}_f + \underset{\sim}{D}_g - 2\,\underset{\sim}{D}_{f\cap g} \tag{4.42}$$

(c) Let $F(X) = f(X) \cup g(X)$, we have

$$\underset{\sim}{D}_F = \underset{\sim}{D}_f + \underset{\sim}{D}_g - \underset{\sim}{D}_{f\cap g} \tag{4.43}$$

(d) Let $F(X) = f(X) \cap g(X)$, we have

$$\underset{\sim}{D}_F = \underset{\sim}{D}_f + \underset{\sim}{D}_g - \underset{\sim}{D}_{f\cup g} \tag{4.44}$$

**Example 4.4**

Consider the function $F(X) = \overline{x}_1\overline{x}_2 + \overline{x}_1\overline{x}_3 + x_1x_2x_3$, Let $f(X) = \overline{x}_1\overline{x}_2$, and $g(X) = \overline{x}_1\overline{x}_3 + x_1x_2x_3$. F(X) can be written in the form of $F(X) = f(X) \cup g(X)$. The MRM coefficient column vectors for F, f, g, and $f \cap g$ are shown in Table 4.1.

| $F(X) = f(X) \cup g(X)$ | | | $f(X) = \overline{x}_1\overline{x}_2$ | | | $g(X) = \overline{x}_1\overline{x}_3 + x_1x_2x_3$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_3$ | $x_2$ | $x_1$ | F | f | g | $f\cap g$ | $\underset{\sim}{D}_f$ | $\underset{\sim}{D}_g$ | $\underset{\sim}{D}_{f\cap g}$ | $\underset{\sim}{D}_f+\underset{\sim}{D}_g-\underset{\sim}{D}_{f\cap g}$ |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 2 | 1 | 2 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 2 | 1 | 2 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 1 | 1 | 2 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 2 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 1 | 3 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 2 | 3 | 1 | 4 |

**Table 4.1** Example for Theorem 4.9(c)

## 4.3. Conclusion

In this chapter, we have shown the relationships between several alternative spectral representations. The conversion matrices for R-W, arithmetic, and MRM spectral coefficients all have recursive forms. One can derive a spectral representation from another one easily using the conversion matrices. Because the RM coefficients are defined on GF(2), it is not possible to get a conversion matrix as we had for the other coefficients. Since RM coefficients can be derived from MRM coefficients by mod 2 operation, the conversion from other coefficients to RM coefficients is usually based on MRM coefficients.

Some properties of RM and MRM coefficients are also discussed in this chapter. We give the properties in matrix form, which is very convenient for analysing the RM and MRM coefficient column vectors under some fault effects.

# CHAPTER 5

# TESTABILITY BY USING REED-MULLER AND MODIFIED REED-MULLER SPECTRA

## 5.1. Introduction

The classical approach to the design of logic circuits is based primarily upon the criterion of minimizing hardware complexity. With the rapidly declining cost of semiconductor devices, interest began to focus upon the design of easily testable networks. In the classical approach, we have to consider the "testability" of the devices and the network structure or topology, since the test for a network depends on both the network topology and the devices used. Recently, a number of researchers have shown considerable interest in the problem of designing tests to detect multiple faults in realizations of an arbitrary function. A method for implementing such combinational switching functions so that the resulting network is easily testable was proposed by Reddy [52].

In Chapter 3, we saw that an arbitrary switching function f(X) can be represented by $2^n$ different polarity generalized Reed-Muller (GRM) expansions, and any GRM expansion can be realized by an AND-EXOR logic network as shown in Fig 3.1. Most of the early work concerning the testability of AND-EXOR logic networks concentrated on detecting the faults in the networks by using test sets [52, 53, 57].

Reddy and Saluja [52, 53, 54] concentrate on the test set method for a AND-EXOR logic network realized by the 0th polarity GRM expansion. Reddy showed [52] that switching function realizations based on GRM expansions could be tested for all single stuck-at faults by the application of only $n + 2n_e + 4$ input vectors where n is the number of input variables and $n_e$ is the number of variables appearing in an even number of product terms in the GRM expansion. The test-input vectors are easily derived and, the $2n_e$ input vectors are not required if the network is provided with an extra observation point, thus reducing the number of required tests to n + 4. A test set $T = T_1 \cup T_2$ is sufficient to detect all single stuck-at faults, where

$$T_1 = \begin{array}{c} x_0 x_1 x_2 \cdots x_n \\ 0\ 0\ 0 \cdots 0 \\ 0\ 1\ 1 \cdots 1 \\ 1\ 0\ 0 \cdots 0 \\ 1\ 1\ 1 \cdots 1 \end{array} \quad \text{4 input vectors} \tag{5.1}$$

and

$$T_2 = \begin{array}{c} x_1 x_2 x_3 \cdots x_n \\ 0\ 1\ 1 \cdots 1 \\ 1\ 0\ 1 \cdots 1 \\ 1\ 1\ 0 \cdots 1 \\ \cdot\ \cdot\ \cdot \cdots \cdot \\ \cdot\ \cdot\ \cdot \cdots \cdot \\ 1\ 1\ 1 \cdots 0 \end{array} \quad \text{n input vectors}$$

In (5.1) $x_0$ is an extra input corresponding to the RM coefficient $d_0$ as shown in Fig 3.1. The test set T is independent of the function f(X).

Saluja and Reddy [53] extended the earlier result of Reddy [52] and obtained a fault detecting test set for multiple faults in GRM networks, realizing an n-variable logic function. They showed that, to detect *t* stuck-at faults in GRM networks, one needs to apply a predetermined test set, independent of the function being realized, whose cardinality is

$$4 + \sum_{i=1}^{\lceil \log_2 2t \rceil} \binom{n}{i} \tag{5.2}$$

The number of tests sufficient to detect all single and multiple faults can be reduced to n + 4 by adding extra observable outputs [55].

Pradhan [56] extends the previous network model to implement different polarity RM expansions. Both variables and inverted variables are allowed to appear in the expression. Pradhan also allows realizations using shared logic. In this way, the number of product terms and the implementation cost can be significantly reduced.

Compared with conventional Boolean realizations using AND/OR/NAND/NOR gates, logic design using AND-EXOR networks improves the testability of circuits significantly. Unfortunately, the AND-EXOR realizations are usually more costly than the equivalent vertex form, and, because AND-EXOR networks are composed of a cascade of EXOR gates driven by AND gates connected to the circuit inputs, the circuit delay is so large as to make the method of only theoretical interest.

Damarla and Karpovsky [67] present a new approach using Reed-Muller spectral coefficients. They show that fault detection can be carried out by verification of a few RM coefficients. The advantage of this technique is that every RM spectral coefficient depends on the local behaviour of a Boolean function, and, for almost all coefficients, does not require exhaustive testing. The computation of RM coefficients is over GF(2) (Exclusive-OR operation) and only one bit for storing an RM coefficient is required, giving better data compaction compared with R-W coefficients, which may require n bits for storage.

Our interest is in fault detection for random logic circuits using Reed-Muller and modified Reed-Muller spectral coefficients. Only irredundant combinational circuits are considered. The fault models used are stuck-at fault and non-feedback bridging fault.

The criteria for RM and MRM techniques to detect the stuck-at and bridging faults are derived below. Because a RM coefficient is only a special case of a MRM coefficient (taken over GF(2)), the results below are derived for the MRM coefficients.

## 5.2. Testability for Input Stuck-at Faults

### 5.2.1. Single Input Stuck-at Faults

Consider a single stuck-at fault (s-a-f) on the input line $x_i$, i.e., $x_i$ stuck-at 0 (s-a-0) or $x_i$ stuck-at 1 (s-a-1), denoted by $x_i\,/0$ or $x_i\,/1$, respectively. By using RM and MRM coefficients, we have the following testing criteria.

**Theorem 5.1**

For a single s-a-f on input line $x_i$ of a combinational network implementing $f(x_1 \cdots x_n)$,

$x_i/0$ and $x_i/1$ are $\underset{\sim}{d}_{i,\alpha}$ (or $d_{i,\alpha}$)-detectable if and only if $\underset{\sim}{d}_{i,\alpha} \neq 2\underset{\sim}{d}_{\alpha}$ (or $d_{i,\alpha} = 1$ ).

$x_i/1$ is $\underset{\sim}{d}_{\alpha}$ (or $d_{\alpha}$)-detectable if and only if $\underset{\sim}{d}_{i,\alpha} \neq 2\underset{\sim}{d}_{\alpha}$ (or $d_{i,\alpha} = 1$).

where $\alpha \subseteq \{1, 2,..., i\text{-}1, i\text{+}1, ..., n\}$ or $\alpha = \varnothing$.

**Proof:**

Without loss of generality, consider a s-a-f on $x_n$. Using the Shannon decompositions for the output vector Z of function f(X) and the MRM coefficient column vector $\underset{\sim}{D}$ with respect to $x_n$ as shown in section 2.4.3.3 and 3.6, we have

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix}$$

and

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \end{bmatrix} = \begin{bmatrix} RM^{n-1} & 0 \\ RM^{n-1} & RM^{n-1} \end{bmatrix} \cdot \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix}$$

If $x_n$ /0, $Z_0^* = Z_0$, and $Z_1^* = Z_0$. From (3.27), (3.28), and (3.29) we have

$$\underset{\sim}{D}^* = \begin{bmatrix} \underset{\sim}{D}^{0^*} \\ \underset{\sim}{D}^{1^*} \end{bmatrix} = \begin{bmatrix} RM^{n-1} & 0 \\ RM^{n-1} & RM^{n-1} \end{bmatrix} \cdot \begin{bmatrix} Z_0 \\ Z_0 \end{bmatrix} \tag{5.3}$$

so that

$$\underset{\sim}{D}^{0^*} = \underset{\sim}{D}_0 = \underset{\sim}{D}^0 \quad \text{and} \quad \underset{\sim}{D}^{1^*} = \underset{\sim}{D}_0 + \underset{\sim}{D}_0 = 2\underset{\sim}{D}_0 = 2\underset{\sim}{D}^0 \tag{5.4}$$

Note that $\underset{\sim}{d}_{\alpha}$ and $\underset{\sim}{d}_{n,\alpha}$ are the entries of $\underset{\sim}{D}^0$ and $\underset{\sim}{D}^1$, respectively. The fault $x_n/0$ can be detected by $\underset{\sim}{d}_{n,\alpha}$ if $\underset{\sim}{d}^*_{n,\alpha} \neq \underset{\sim}{d}_{n,\alpha}$. So from (5.4), $x_n/0$ can be detected by verification of $\underset{\sim}{d}_{n,\alpha}$ if $\underset{\sim}{d}_{n,\alpha} \neq 2\underset{\sim}{d}_{\alpha}$ for the fault free network.

For the case of the RM coefficients, (5.4) taken over GF(2) becomes

$$D^{0^*} = D_0 \quad \text{and} \quad D^{1^*} = D_0 \oplus D_0 = \dot{0} \tag{5.5}$$

where $\dot{0}$ is a column vector in which all entries are zero. So $x_n/0$ can be detected by of $d_{n,\alpha}$ if and only if $d_{n,\alpha} = 1$ in $D^1$ for the fault free network ( $n \notin \alpha$ ).

For $x_n/1$, $Z_0^* = Z_1$, and $Z_1^* = Z_1$. For the MRM coefficients,

$$\underset{\sim}{D}^{0^*} = \underset{\sim}{D}_1 = \underset{\sim}{D}^1 - \underset{\sim}{D}^0 \tag{5.6}$$

$$\underset{\sim}{D}^{1^*} = \underset{\sim}{D}_1 + \underset{\sim}{D}_1 = 2\underset{\sim}{D}_1 = 2\,(\,\underset{\sim}{D}^1 - \underset{\sim}{D}^0\,)$$

giving

$$\underset{\sim}{D}^{1^*} - \underset{\sim}{D}^1 = \underset{\sim}{D}^1 - 2\underset{\sim}{D}^0 \tag{5.7}$$

From (5.7) we see that $x_n/1$ can be detected by $\underset{\sim}{d}_{n,\alpha}$ if $\underset{\sim}{d}_{n,\alpha} \neq 2\underset{\sim}{d}_{\alpha}$ for the fault free network.

For the RM coefficients, (5.6) and (5.7) become

$$D^{0^*} = D_1 = D^1 \oplus D^0 \quad \text{and} \quad D^{1^*} = D_1 \oplus D_1 = \dot{0} \tag{5.8}$$

where $d_\alpha$ is an entry of $D^0$, and $d_{n,\alpha}$ is an entry of $D^1$ ( $n \notin \alpha$ ). So $x_n/1$ can be detected by $d_{n,\alpha}$ if and only if $d_{n,\alpha} = 1$ for the fault free network, where $n \notin \alpha$.

**Example 5.1**

Consider the circuit shown in Fig. 5.1. This circuit realizes the Boolean function

$$f(X) = x_4\bar{x}_3\bar{x}_2 + x_4x_2\bar{x}_1 + \bar{x}_4x_3\bar{x}_2 + \bar{x}_4x_2x_1$$

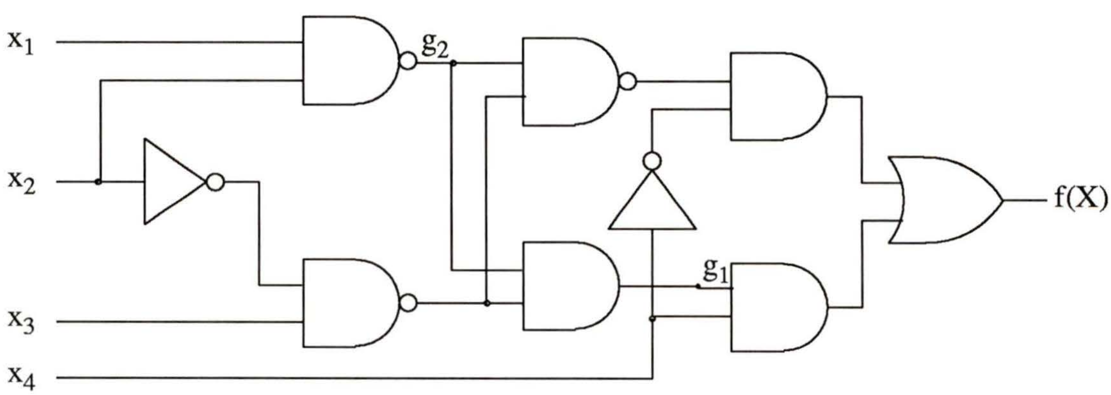



**Figure 5.1.** An Example Circuit.

For the RM testing, the output column vector, fault free PRM vector, and faulty RM coefficients corresponding to all of the single input s-a-f are given in Table 5.1.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | D | $D^*_{x_1/0}$ | $D^*_{x_1/1}$ | $D^*_{x_2/0}$ | $D^*_{x_2/1}$ | $D^*_{x_3/0}$ | $D^*_{x_3/1}$ | $D^*_{x_4/0}$ | $D^*_{x_4/1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | $\underline{1}$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | 0 | 0 | $\underline{1}$ | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 5.1** The Fault Free and Faulty RM Spectrum of the Network

**Note:** In Table 5.1 and the following tables, the coefficient in the faulty column vector which is different from the fault free one is marked by the under line.

For example, consider the s-a-f on input line $x_3$. The fault free RM coefficient column vector is

| $d_0$ | $d_1$ | $d_2$ | $d_{1,2}$ | $d_3$ | $d_{1,3}$ | $d_{2,3}$ | $d_{1,2,3}$ | $d_4$ | $d_{1,4}$ | $d_{2,4}$ | $d_{1,2,4}$ | $d_{3,4}$ | $d_{1,3,4}$ | $d_{2,3,4}$ | $d_{1,2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$x_3/0$ and $x_3/1$ can be detected using $d_3$ and $d_{2,3}$ because $d_3 = d_{2,3} = 1$ for the fault free network. $x_3/1$ can also be detected by verifications of $d_0$ (since $d_3 = 1$) and $d_2$ ( since $d_{2,3} = 1$).

For the MRM testing, the output column vector, fault free modified PRM vector, and faulty modified PRM coefficients corresponding to all of the single input s-a-f are given in Table 5.2.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | $\underset{\sim}{D}$ | $\underset{\sim}{D}^*_{x_1/0}$ | $\underset{\sim}{D}^*_{x_1/1}$ | $\underset{\sim}{D}^*_{x_2/0}$ | $\underset{\sim}{D}^*_{x_2/1}$ | $\underset{\sim}{D}^*_{x_3/0}$ | $\underset{\sim}{D}^*_{x_3/1}$ | $\underset{\sim}{D}^*_{x_4/0}$ | $\underset{\sim}{D}^*_{x_4/1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | $\underline{1}$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | $\underline{2}$ | 0 | $\underline{2}$ |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | 0 | 0 | $\underline{1}$ | 0 | $\underline{2}$ |
| 0 | 0 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{2}$ | $\underline{0}$ | $\underline{2}$ | 1 | $\underline{3}$ | 1 | $\underline{3}$ |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{2}$ | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | $\underline{0}$ | $\underline{4}$ | 2 | 2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | $\underline{3}$ | $\underline{2}$ | $\underline{0}$ | $\underline{0}$ | $\underline{2}$ | 1 | $\underline{3}$ |
| 0 | 1 | 1 | 1 | 1 | 4 | $\underline{2}$ | $\underline{6}$ | 4 | 4 | $\underline{2}$ | $\underline{6}$ | 4 | 4 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{2}$ |
| 1 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $\underline{0}$ | $\underline{4}$ |
| 1 | 0 | 1 | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $\underline{0}$ | $\underline{4}$ |
| 1 | 0 | 1 | 1 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | $\underline{2}$ | $\underline{6}$ |
| 1 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 1 | 1 | 0 | 1 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 1 | 1 | 1 | 0 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | $\underline{2}$ | $\underline{6}$ |
| 1 | 1 | 1 | 1 | 0 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |

**Table 5.2** The Fault Free and Faulty MRM Spectrum of the Network

For example, consider the s-a-f's on input line $x_3$. the fault free MRM coefficient column vector is

| $\underset{\sim}{d}_0$ | $\underset{\sim}{d}_1$ | $\underset{\sim}{d}_2$ | $\underset{\sim}{d}_{1,2}$ | $\underset{\sim}{d}_3$ | $\underset{\sim}{d}_{1,3}$ | $\underset{\sim}{d}_{2,3}$ | $\underset{\sim}{d}_{1,2,3}$ | $\underset{\sim}{d}_4$ | $\underset{\sim}{d}_{1,4}$ | $\underset{\sim}{d}_{2,4}$ | $\underset{\sim}{d}_{1,2,4}$ | $\underset{\sim}{d}_{3,4}$ | $\underset{\sim}{d}_{1,3,4}$ | $\underset{\sim}{d}_{2,3,4}$ | $\underset{\sim}{d}_{1,2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 2 | 1 | 4 | 1 | 2 | 2 | 4 | 2 | 4 | 4 | 8 |

$x_3/0$ and $x_3/1$ can be detected using $\underset{\sim}{d}_3$, $\underset{\sim}{d}_{1,3}$, $\underset{\sim}{d}_{2,3}$, or $\underset{\sim}{d}_{1,2,3}$ since for these coefficients, $\underset{\sim}{d}_{3,\alpha} \neq \underset{\sim}{d}_\alpha$ for the fault free network. $x_3/1$ can also be detected by verification of $\underset{\sim}{d}_0$, $\underset{\sim}{d}_1$, $\underset{\sim}{d}_2$, or $\underset{\sim}{d}_{1,2}$. We cannot use $\underset{\sim}{d}_{3,4}$, $\underset{\sim}{d}_{1,3,4}$, $\underset{\sim}{d}_{2,3,4}$, and $\underset{\sim}{d}_{1,2,3,4}$ to detect the s-a-f on $x_3$ because for these coefficients, $\underset{\sim}{d}_{i,\alpha} = 2\underset{\sim}{d}_\alpha$ for the fault free network.

Let $DS_0^{\ i}$ and $D\underset{\sim}{S}_0^{\ i}$ be the sets of all RM and MRM coefficients which can be used to detect a s-a-0 fault on the line $x_i$, respectively. Similarly, let $D\underset{\sim}{S}_1^{\ i}$ and $D\underset{\sim}{S}_1^{\ i}$ be the sets of all RM and MRM coefficients which can be used to detect a s-a-1 fault on the line $x_i$, respectively. From Theorem 5.1, we have

$$DS_0^{\ i} \subset DS_1^{\ i} \quad \text{and} \quad D\underset{\sim}{S}_0^{\ i} \subset D\underset{\sim}{S}_1^{\ i} \tag{5.9}$$

Any coefficient which can be used to detect a s-a-0 fault on the line $x_i$ can also cover the s-a-1 fault on the same line. So that in the detection of input s-a-f's, we can restrict our attention to the s-a-0 faults. A set of RM or MRM coefficients which can cover all of the single input s-a-0 faults can also cover all of the s-a-1 faults.

Let $DS_0^i$, $\underset{\sim}{DS}{}_0^i$ and $DS_1^i$, $\underset{\sim}{DS}{}_1^i$ be the sets of RM and MRM coefficients which can detect input s-a-0 and s-a-1 faults on line $x_i$, respectively. Then we have

$$DS_0^i \subseteq \underset{\sim}{DS}{}_0^i \quad \text{and} \quad DS_1^i \subseteq \underset{\sim}{DS}{}_1^i \tag{5.10}$$

Because for a RM coefficient $d_j$, if a fault can be detected by the RM coefficient $d_j$, then it can also be detected by $\underset{\sim}{d}_j$. If a fault changes an even number of the output values involved in $d_j$, the fault cannot be detected by $d_j$. But it can be detected by the corresponding MRM coefficient $\underset{\sim}{d}_j$.

Consider the input s-a-f's on line $x_4$ in Example 5.1. Table 5.3 gives a contrast of testability for s-a-0 and s-a-1 faults by using RM or MRM coefficients.

| $D$ | $\underset{\sim}{D}$ | $D^*_{x_4/0}$ | $\underset{\sim}{D}{}^*_{x_4/0}$ | $D^*_{x_4/1}$ | $\underset{\sim}{D}{}^*_{x_4/1}$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\underline{1}$ | $\underline{1}$ |
| 0 | 0 | 0 | 0 | 0 | $\underline{2}$ |
| 0 | 0 | 0 | 0 | 0 | $\underline{2}$ |
| 1 | 1 | 1 | 1 | 1 | $\underline{3}$ |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 2 | 0 | 2 | 0 | 2 |
| 1 | 1 | 1 | 1 | 1 | $\underline{3}$ |
| 0 | 4 | 0 | 4 | 0 | 4 |
| 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{2}$ |
| 0 | 2 | 0 | $\underline{0}$ | 0 | $\underline{4}$ |
| 0 | 2 | 0 | $\underline{0}$ | 0 | $\underline{4}$ |
| 0 | 4 | 0 | $\underline{2}$ | 0 | $\underline{6}$ |
| 0 | 2 | 0 | 2 | 0 | 2 |
| 0 | 4 | 0 | 4 | 0 | 4 |
| 0 | 4 | 0 | $\underline{2}$ | 0 | $\underline{6}$ |
| 0 | 8 | 0 | 8 | 0 | 8 |

**Table 5.3** The Contrast Between RM and MRM Coefficients.

**Corollary 5.1**

If a s-a-f on input line $x_i$ is not detected by any of the conditions in Theorem 5.1, $x_i$ is redundant.

**Proof:**

Without loss of generality, consider a s-a-f on $x_n$. If such a fault cannot be detected by any of the conditions in Theorem 5.1, then, by considering the RM coefficient for the fault free network, we have $D^1 = \dot{0}$.

$$D^1 = D_0 \oplus D_1 = \dot{0} \quad \text{if and only if} \quad D_0 = D_1.$$

Since the RM spectrum is unique, we have $Z_0 = Z_1$ for the fault free network, so that $x_n$ is redundant.

The result follows similarly for MRM coefficients.

### 5.2.2. Multiple Input Stuck-at Faults

Now we present some results concerning multiple input s-a-f's.

Consider a multiple s-a-f on input lines $x_{m+1} \cdots x_n$ , where $x_{m+i}/u_i$ , $1 < i < n\text{-}m$ , $m < n$. $U = \sum_{i=1}^{n-m} u_i 2^{i-1}$ . Such a fault is referred to as $x_{m+1} \cdots x_n$ stuck-at U , denoted by $x_{m+1} \cdots x_n / U$. It is sufficient to consider this class of multiple input s-a-f's.

**Definition 5.1**

We use <i> to indicate the set corresponding to the binary representation of i, so, for example, if i = 13 has a binary representation of 1101, then <i> = { 1, 3, 4 }.

Using RM and modified RM coefficients to detected the multiple s-a-f, we have the following theorem.

**Theorem 5.2**

In a combinational network, a multiple input s-a-f ( $x_{m+1} \cdots x_n$ )/U is (i). $d_{\alpha\beta}$-detectable if $d_{\alpha\beta} = 1$.

(ii). $\underset{\sim}{d}_{\alpha\beta}$-detectable if $\underset{\sim}{d}_{\alpha\beta} \neq 2^{|\alpha|} \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha}$, $q = |i| + |u|$.

If U is not zero, the fault is

(iii). $d_\alpha$-detectable if $d_\alpha \neq \bigoplus_{i \subseteq u} d_{<i>\alpha}$.

(iv). $\underset{\sim}{d}_\alpha$-detectable if $\underset{\sim}{d}_\alpha \neq \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha}$, $q = |i| + |u|$.

where $\alpha \subseteq \{1, 2, ..., m\}$ or $\alpha = \varnothing$, and $\beta \subseteq \{ m+1, ..., n\}$, $( \beta \neq \varnothing )$.

**Proof**

Consider the Shannon decomposition of the output column vector Z and MRM column vector $\underset{\sim}{D}$ with respect to $x_{m+1}, ..., x_n$ as shown in section 3.6. We have

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \cdots \\ Z_t \end{bmatrix} \quad \text{and} \quad \underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdots \\ \underset{\sim}{D}^t \end{bmatrix} = RM^{n-m} \cdot \begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \\ \cdots \\ \underset{\sim}{D}_t \end{bmatrix} \qquad t = 2^{n-m}-1$$

If $x_{m+1} \cdots x_n / U$, we have

$$Z_0^* = Z_1^* = \cdots = Z_t^* = Z_u \text{ and}$$

$$\underset{\sim}{D}_0^* = \underset{\sim}{D}_1^* = \cdots = \underset{\sim}{D}_t^* = \underset{\sim}{D}_u \qquad (5.11)$$

where u is the integer representation of U. For each $\underset{\sim}{D}^{j^*}$ in the faulty MRM spectrum vector $\underset{\sim}{D}^*$, we have

$$\underset{\sim}{D}^{j^*} = \sum_{i \subseteq j} \underset{\sim}{D}_i^* = \sum_{i \subseteq j} \underset{\sim}{D}_u = 2^p \underset{\sim}{D}_u \qquad (5.12)$$

where $p = |j|$. From (3.34) we can see that

$$\underset{\sim}{D}_u = \sum_{i \subseteq u} (-1)^q \underset{\sim}{D}^i$$

So that

$$\underset{\sim}{D}^{j^*} = 2^p \underset{\sim}{D}_u = 2^p \sum_{i \subseteq u} (-1)^q \underset{\sim}{D}^i$$

Let $\underset{\sim}{d}_{\alpha\beta}$ and $\underset{\sim}{d}_{<i>\alpha}$ be the MRM coefficient in $\underset{\sim}{D}^j$ and $\underset{\sim}{D}^i$, respectively. $x_{m+1} \cdots x_n / U$ can be detected by verification of $\underset{\sim}{d}_{\alpha\beta} \neq 2^p \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha}$ for the fault free network. Moreover, if U is not zero, we have

$$\underset{\sim}{D}^{0^*} = \underset{\sim}{D}_u = \sum_{i \subseteq u} (-1)^q \underset{\sim}{D}^i$$

Let $\underset{\sim}{d}_\alpha$ be a coefficient in $\underset{\sim}{D}^0$. Then $x_{m+1} \cdots x_n / U$ can also be detected by verification of $\underset{\sim}{d}_\alpha \neq \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha}$ for the fault free network.

For the RM coefficients, let $D_i = RM^m \otimes Z_i$.

Then we have

$$D = \begin{bmatrix} D^0 \\ D^1 \\ \cdots \\ D^t \end{bmatrix} = RM^{n-m} \otimes \begin{bmatrix} D_0 \\ D_1 \\ \cdots \\ D_t \end{bmatrix} \tag{5.13}$$

For $x_{m+1} \cdots x_n / U$, (5.11) becomes

$$D_0^* = D_1^* = \cdots = D_t^* = D_u \tag{5.14}$$

For all of the $D^i$, $1 \leq i \leq t$, each of them contains an even number of terms. For example, $D^1 = D_0 \oplus D_1$, $D^3 = D_0 \oplus D_1 \oplus D_2 \oplus D_3$ and so on. For $( x_{m+1} \cdots x_n ) / U$, let u be the integer representation of U, we have

$$D_0^* = D_1^* = \cdots = D_t^* = D_u$$

so that all $D^{i^*}, 1 \leq i \leq t$, are the zero vector. $d_{\alpha\beta}$ is an entry of $D^i$, so $( x_{m+1} \cdots x_n ) / U$ is testable if $d_{\alpha\beta} = 1$ for the fault free network, where $\alpha \subseteq \{1, 2,.., m\}$ or $\alpha = \varnothing$, and $\beta \subseteq \{m+1,..., n\}$, $( \beta \neq \varnothing )$.

If U is not zero, then we have

$$D^{0^*} = D^{0^*} = D_u$$

From (3.25), we have

$$D^{0^*} = D_u = \bigoplus_{i \subseteq u} D^i$$

Let $d_\alpha$ and $d_{<u>\alpha}$ be the entry in $D^0$ and $D_u$, respectively. Then $( x_{m+1} \cdots x_n ) / U$ is testable if $d_\alpha \neq d_{<u>\alpha}$, where $d_{<u>\alpha} = \bigoplus_{i \subseteq u} d_{<i>\alpha}$.

**Example 5.2**

Consider the circuit shown in Example 5.1. The faulty RM spectra of some multiple s-a-f's are given in Table 5.4.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | D | $D^*_{x_2x_1/01}$ | $D^*_{x_3x_1/10}$ | $D^*_{x_4x_2/11}$ | $D^*_{x_3x_2/11}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 5.4** Faulty RM Spectrum of Some Multiple Stuck-at Faults

For example, the fault $x_3x_2/11$ can be detected by verification of $d_{1,2}$, $d_3$, or $d_{2,3}$ since for these coefficients, we have $d_{\alpha\beta} = 1$, $\beta \subseteq \{ 2, 3 \}$ ( $\beta \neq \varnothing$ ) and $\alpha$ $\{ 1, 4 \}$ or ( $\alpha = \varnothing$ ), for the fault free network. $x_3x_2/11$ can also be detected by verification of $d_1$. From Theorem 5.2, for $\alpha = \{ 1 \}$ and u = 3, we have

$$\bigoplus_{i \subseteq u} d_{<i>\alpha} = d_1 \oplus d_{1,2} \oplus d_{1,3} \oplus d_{1,2,3} = 0 \oplus 1 \oplus 0 \oplus 0 = 1 \neq d_1 = 0$$

$d_1$ satisfied the condition $d_\alpha \neq \bigoplus_{i \subseteq u} d_{<i>\alpha}$. So the fault can be detected by this coefficient.

For the MRM testing, The faulty MRM spectrum of some multiple s-a-f's are given in Table 5.5.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | $\underset{\sim}{D}$ | $\underset{\sim}{D}^*_{x_2x_1/01}$ | $\underset{\sim}{D}^*_{x_3x_1/10}$ | $\underset{\sim}{D}^*_{x_4x_2/11}$ | $\underset{\sim}{D}^*_{x_3x_2/11}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | $\underline{1}$ | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | $\underline{2}$ | $\underline{1}$ | $\underline{1}$ |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | $\underline{1}$ | $\underline{2}$ | $\underline{0}$ |
| 0 | 0 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{2}$ | $\underline{2}$ | $\underline{2}$ |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | $\underline{2}$ | $\underline{2}$ | $\underline{0}$ |
| 0 | 1 | 0 | 1 | 1 | 2 | 2 | $\underline{4}$ | 2 | 2 |
| 0 | 1 | 1 | 0 | 0 | 1 | $\underline{2}$ | $\underline{2}$ | $\underline{4}$ | $\underline{0}$ |
| 0 | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 | 4 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | $\underline{2}$ | 1 |
| 1 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| 1 | 0 | 1 | 0 | 1 | 2 | 2 | 2 | $\underline{4}$ | 2 |
| 1 | 0 | 1 | 1 | 0 | 4 | 4 | 4 | 4 | 4 |
| 1 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | $\underline{4}$ | 2 |
| 1 | 1 | 0 | 1 | 0 | 4 | 4 | 4 | 4 | 4 |
| 1 | 1 | 1 | 0 | 1 | 4 | 4 | 4 | $\underline{8}$ | 4 |
| 1 | 1 | 1 | 1 | 0 | 8 | 8 | 8 | 8 | 8 |

**Table 5.5** Faulty MRM Spectrum of Some Multiple Stuck-at Faults

The fault $x_4x_2/11$ can be detected by verification of $\underset{\sim}{d}_2$, $\underset{\sim}{d}_{1,2}$, $\underset{\sim}{d}_{2,3}$, $\underset{\sim}{d}_4$, $\underset{\sim}{d}_{2,4}$, $\underset{\sim}{d}_{3,4}$, and $\underset{\sim}{d}_{2,3,4}$ since for these coefficients we have $\underset{\sim}{d}_{\alpha\beta} \neq 2^p \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha}$, where $\beta \subseteq \{ 2, 4 \}$ ( $\beta \neq \varnothing$ ) and $\alpha \subseteq \{ 1, 3 \}$ or $\alpha = \varnothing$. For example, consider $\underset{\sim}{d}_{3,4}$. For $\underset{\sim}{d}_{3,4}$ we have $\beta = 4$, $\alpha = 3$, and $p = |4| = 1$.

$$2^p \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha} = 2 ( \underset{\sim}{d}_3 - \underset{\sim}{d}_{2,3} - \underset{\sim}{d}_{3,4} + \underset{\sim}{d}_{2,3,4} ) = 2 ( 1 - 1 - 2 + 4) = 4 \neq \underset{\sim}{d}_{3,4}$$

$x_4x_2/11$ can also be detected by verification of $\underset{\sim}{d}_0$, $\underset{\sim}{d}_1$, and $\underset{\sim}{d}_3$ because for these coefficients we have

$$\underset{\sim}{d}_\alpha \neq \sum_{i \subseteq u} \underset{\sim}{d}_{<i>\alpha}$$

For $\underset{\sim}{d}_3$ we have $\alpha = 3$ and

$$\sum_{i \subseteq u} \underset{\sim}{d}_{<i>\alpha} = \underset{\sim}{d}_3 - \underset{\sim}{d}_{2,3} - \underset{\sim}{d}_{3,4} + \underset{\sim}{d}_{2,3,4} = 1 - 1 - 2 + 4 = 2 \neq \underset{\sim}{d}_3$$

The fault cannot be detected by verification of $\underset{\sim}{d}_{1,4}$ because for $\underset{\sim}{d}_{1,4}$, $\beta = 4$, $\alpha = 1$ and $p = 1$, then

$$2^p \sum_{i \subseteq u} (-1)^q \underset{\sim}{d}_{<i>\alpha} = 2\,(\,\underset{\sim}{d}_1 - \underset{\sim}{d}_{1,2} - \underset{\sim}{d}_{1,4} + \underset{\sim}{d}_{1,2,4}\,) = 2\,(\,0 - 1 - 2 + 4) = 2 = \underset{\sim}{d}_{1,4}$$

When considering single s-a-f's, it was only necessary to consider the s-a-0 case. For multiple faults on the input lines $x_{m+1}, \cdots, x_n$, the result generalizes, but is much stronger, in that it is only necessary to consider one out of the $2^{n-m}$ possible multiple s-a-f's. From Theorems 5.2, if $d_{\alpha\beta}$ and $\underset{\sim}{d}_{\alpha\beta}$ ($\beta \neq \varnothing$), are the RM and MRM coefficients which can detect the multiple s-a-f $x_{m+1}, ..., x_n$ / 00...0, respectively, then the coefficient can be used to detect any multiple s-a-f on lines $x_{m+1}, ..., x_n$.

Moreover, if $d_{<i>\alpha}$ can detect the single s-a-0 fault on line $x_i$, then it can also detect any multiple s-a-f which includes $x_i$. On the other hand, if $d_{\alpha\beta}$ can detect the multiple s-a-f $x_{m+1}, ..., x_n$ / U and $\beta = 1\ 1\ \cdots\ 1$, then $d_{\alpha\beta}$ can detect any single s-a-f on line $x_i$, where $m+1 \leq i \leq n$..pp For the modified Reed-Muller coefficients, we have similar results.

**Corollary 5.2**

If a multiple input s-a-f ( $x_{m+1} \cdots x_n$ ) / U cannot be detected by any of the conditions of Theorem 5.2, $x_{m+1}, \cdots, x_n$ are redundant.

The proof is similar to that of Corollary 5.1.

**Corollary 5.3**

All of the single or multiple input s-a-f in a combinational network can be detected by verification of at most n RM or MRM spectral coefficients, where n is the number of input variables.

**Proof**

From Theorem 5.1 and Theorem 5.2 , we see that $d_{i,\alpha} = 1$ can detect all of the single or multiple s-a-f involving $x_i$ . The worst case is when we only have $d_i = 1$ or $d_0 =$

1, for $1 \leq i \leq n$, and all other RM coefficients $d_{i,\alpha}$, $(\alpha \neq \emptyset)$ are zero, that is no coefficient can cover more than one input variable. The corresponding PRM expansion for these types of functions is

$$f_1(X) = x_1 \oplus x_2 \oplus \cdots \oplus x_n \quad (d_0 = 0) \qquad \text{or}$$

$$f_2(X) = \overline{f}_1(X) = 1 \oplus x_1 \oplus x_2 \oplus \cdots \oplus x_n \quad (d_0 = 1)$$

For these cases, n spectral coefficients $d_1, d_2, \cdots, d_n$ are required to detect all of the single or multiple input s-a-f in an irredundant combinational network.

**Corollary 5.4**

All of the single or multiple input s-a-f in a combinational network can be detected by verification of the single coefficient $d_{11...1}$ (or $\underset{\sim}{d}_{11...1}$) if $d_{11...1} = 1$ (or $\underset{\sim}{d}_{11...1}$ is odd) for the fault free network.

The result follows directly from Theorems 5.1 and 5.2. However, $d_{11...1}$ is just the parity of the syndrome $\underset{\sim}{d}_{11...1}$, so $d_{11...1} = 1$ means the parity of the syndrome is odd, and Corollary 5.4 is a well known result [69, 70, 71].

## 5.3. Testability For Internal Stuck-at Faults

Consider a s-a-f on some internal line g in a circuit. We can model the circuit as shown in Fig. 2.9. The function f(X) can be rewritten in the form $f(x_1 \cdots x_n) = h(x_1, x_2, ..., x_n, x_{n+1})$, where $x_{n+1} = g(x_1 \cdots x_n)$. h(X, g(X)) is considered formally as a function of n+1 variables.

Consider the Shannon decomposition of h(X, g(X)) with respect to g(X). We have

$$f(X) = h(X, g(X)) = \overline{g}h_0 + gh_1 \tag{5.15}$$

where $h_0(x_1 \cdots x_n) = h(x_1 \cdots x_n, 0)$, and $h_1(x_1 \cdots x_n) = h(x_1 \cdots x_n, 1)$, respectively.

Let H, $H_0$, and $H_1$ be the column vectors of h, $h_0$, and $h_1$ respectively. For the RM coefficients, we have

$$D = RM^n \otimes Z \quad \text{and} \quad \hat{D} = RM^{n+1} \otimes H$$

Similarly, for the MRM coefficients, we have

$$\underset{\sim}{D} = RM^n . Z \quad \text{and} \quad \hat{\underset{\sim}{D}} = RM^{n+1} . H.$$

Note that since h is a function of n+1 inputs, H, $\hat{D}$, and $\hat{\underset{\sim}{D}}$ contain all $2^{n+1}$ entries.

From (5.15), If g/0, then

$$Z^* = H_0 \;, \quad D^* = D_{h_0} = \hat{D}_0 \quad \text{and} \quad \underset{\sim}{D}^* = \underset{\sim}{D}_{h_0} = \hat{\underset{\sim}{D}}_0$$

If g/1 , then

$$Z^* = H_1 \;, \quad D^* = D_{h_1} = \hat{D}_1 \quad \text{and} \quad \underset{\sim}{D}^* = \underset{\sim}{D}_{h_1} = \hat{\underset{\sim}{D}}_1$$

where $\hat{D}_0 = RM^n \otimes H_0$, $\hat{D}_1 = RM^n \otimes H_1$, $\hat{\underset{\sim}{D}}_0 = RM^n . H_0$ and $\hat{\underset{\sim}{D}}_1 = RM^n . H_1$.

For the RM coefficients, let $d_\alpha$, $\hat{d}^0_\alpha$, and $\hat{d}^1_\alpha$ be corresponding the entries of D, $\hat{D}_0$, and $\hat{D}_1$, respectively, and similarly for the MRM coefficients. For g/0 or g/1, the following results are immediate.

**Theorem 5.3**

The fault g/0 is $d_\alpha$-detectable (or $\underset{\sim}{d}_\alpha$-detectable) if

$$d_\alpha \neq \hat{d}^0_\alpha \quad (\text{or} \;\; \underset{\sim}{d}_\alpha \neq \hat{\underset{\sim}{d}}^0_\alpha)$$

and g/1 is $d_\alpha$-detectable (or $\underset{\sim}{d}_\alpha$-detectable) if

$$d_\alpha \neq \hat{d}^1_\alpha \quad (\text{or} \;\; \underset{\sim}{d}_\alpha \neq \hat{\underset{\sim}{d}}^1_\alpha)$$

**Example 5.3**

For the circuit in Fig. 5.1, the function is

$$f(X) = x_4\bar{x}_3\bar{x}_2 + x_4x_2\bar{x}_1 + \bar{x}_4x_3\bar{x}_2 + \bar{x}_4x_2x_1$$

For the internal line marked $g_1$,

$$h(X,g_1(X)) = x_5x_4 + \bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2 \quad \text{where } x_5 = g_1(X)$$

$$h_0(X) = \bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2 \quad \text{and} \quad h_1(X) = x_4 + x_2x_1 + x_3\bar{x}_2$$

So the function f(X) can be rewritten in the form

$$f(X) = \bar{g}_1(\bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2) + g_1(x_4 + x_2x_1 + x_3\bar{x}_2)$$

The fault free and faulty RM spectrum for g/0 and g/1 are given in Table 5.6.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | D | $g_1/0$ | | $g_1/1$ | | $g_2/0$ | | $g_2/1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $H_0$ | $D_{h_0}$ | $H_1$ | $D_{h_1}$ | $H_0$ | $D_{h_0}$ | $H_1$ | $D_{h_1}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $\underline{1}$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | $\underline{0}$ | 1 | $\underline{0}$ |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | $\underline{0}$ | 1 | 1 | 1 | $\underline{0}$ |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | $\underline{0}$ |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | $\underline{1}$ | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | $\underline{0}$ | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | $\underline{0}$ | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | $\underline{1}$ | 1 | $\underline{1}$ | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | $\underline{1}$ | 1 | $\underline{1}$ | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | $\underline{1}$ | 1 | 0 | 0 | 0 |

**Table 5.6** The Fault Free and Faulty RM Spectrum for the lines $g_1$ and $g_2$.

From Table 5.6 $g_1/0$ is $d_4$, $d_{1,3,4}$, $d_{3,4}$, and $d_{2,3,4}$ testable. $g_1/1$ is $d_3$, $d_{1,2,3}$, $d_{1,2,4}$, $d_{2,3,4}$, and $d_{1,2,3,4}$ testable.

The fault free and faulty MRM spectrum for $g_1/0$ and $g_1/1$ are given in Table 5.7.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | $\underset{\sim}{D}$ | $g_1/0$ | | $g_1/1$ | | $g_2/0$ | | $g_2/1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $H_0$ | $\underset{\sim}{D}_{h_0}$ | $H_1$ | $\underset{\sim}{D}_{h_1}$ | $H_0$ | $\underset{\sim}{D}_{h_0}$ | $H_1$ | $\underset{\sim}{D}_{h_1}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $\underline{1}$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $\underline{2}$ |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $\underline{2}$ |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | $\underline{0}$ | 1 | $\underline{4}$ |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | $\underline{0}$ | 1 | 1 | 1 | $\underline{2}$ |
| 0 | 1 | 0 | 1 | 1 | 2 | 1 | 2 | 0 | $\underline{0}$ | 1 | 2 | 1 | $\underline{4}$ |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | $\underline{4}$ |
| 0 | 1 | 1 | 1 | 1 | 4 | 1 | 4 | 1 | $\underline{3}$ | 0 | $\underline{2}$ | 1 | $\underline{8}$ |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | $\underline{0}$ | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 2 | 0 | $\underline{0}$ | 1 | 2 | 1 | 2 | 0 | 2 |
| 1 | 0 | 1 | 0 | 1 | 2 | 0 | $\underline{0}$ | 1 | 2 | 1 | 2 | 0 | 2 |
| 1 | 0 | 1 | 1 | 0 | 4 | 0 | $\underline{1}$ | 1 | $\underline{5}$ | 1 | 4 | 0 | 4 |
| 1 | 1 | 0 | 0 | 0 | 2 | 0 | $\underline{1}$ | 1 | 2 | 0 | 2 | 0 | 2 |
| 1 | 1 | 0 | 1 | 0 | 4 | 0 | $\underline{2}$ | 1 | 4 | 0 | 4 | 0 | 4 |
| 1 | 1 | 1 | 0 | 1 | 4 | 0 | $\underline{1}$ | 1 | $\underline{5}$ | 1 | 4 | 0 | 4 |
| 1 | 1 | 1 | 1 | 0 | 8 | 0 | $\underline{4}$ | 1 | $\underline{11}$ | 1 | 8 | 0 | 8 |

**Table 5.7** The Fault Free and Faulty MRM Spectrum for the lines $g_1$ and $g_2$.

From Table 5.7 $g_1/0$ is $\underset{\sim}{d}_4$, $\underset{\sim}{d}_{1,4}$, $\underset{\sim}{d}_{2,4}$, $\underset{\sim}{d}_{1,3,4}$, $\underset{\sim}{d}_{3,4}$, $\underset{\sim}{d}_{1,3,4}$, and $\underset{\sim}{d}_{2,3,4}$ testable. $g_1/1$ is $\underset{\sim}{d}_3$, $\underset{\sim}{d}_{1,3}$, $\underset{\sim}{d}_{1,2,3}$, $\underset{\sim}{d}_{1,2,4}$, $\underset{\sim}{d}_{2,3,4}$, and $\underset{\sim}{d}_{1,2,3,4}$ testable.

Theorem 5.3 involves comparing the RM or MRM coefficients for f(X) with the coefficients for $h_0$ or $h_1$. So it is inconvenient to use them to detect an internal s-a-f. For some networks, we have better criteria to detect the faults.

The following results apply only when the Boolean function of the circuit can be decomposed in a particular form. An example follows the theorem.

**Theorem 5.4**

For any boolean function which can be written can be written in the form :

$$f(X) = \bar{g}(X_3)h_0(X_1) + g(X_3)h_1(X_2), \text{ where } X_1 \subset X,\ X_2 \subset X,\ X_3 \subseteq X. \qquad (5.16)$$

(i) $g/0$ is $d_{\alpha\beta}$-detectable if $d_{\alpha\beta} = 1$.

$g/0$ is $\underset{\sim}{d}_{\alpha\beta}$-detectable if $\underset{\sim}{d}_{\alpha\beta} \neq 2^{|\alpha|}\hat{\underset{\sim}{d}}_{\beta_{h_0}}$.

where $X'_1 = X - X_1$. $\alpha \subseteq X'_1$ ($\alpha \neq \emptyset$), and $\beta \not\subseteq X'_1$ or $\beta = \emptyset$.

(ii) g/1 is $d_{\alpha\beta}$-detectable if $d_{\alpha\beta} = 1$.

g/1 is $\underset{\sim}{d}_{\alpha\beta}$-detectable if $\underset{\sim}{d}_{\alpha\beta} \neq 2^{|\alpha|}\hat{\underset{\sim}{d}}_{\beta_{h_1}}$.

where $\alpha \subseteq X'_2$ ($\alpha \neq \emptyset$), $\beta \not\subseteq X'_2$ or $\beta = \emptyset$, and $X'_2 = X - X_2$.

Note that $X'_1 = X - X_1 = \{ x \mid x \in X, x \notin X_1 \}$.

**Proof**

If $X_1 = x_1 \cdots x_m$, then $X'_1 = x_{m+1} \cdots x_n$. Let $X_2 = x_1 \cdots x_t$. then $X'_2 = x_{t+1} \cdots x_n$ . Consider the Shannon decompositions of $h_0$ and $h_1$ with respect to $x_{m+1} \cdots x_n$ and $x_{t+1} \cdots x_n$, respectively, as follows :

$$H_0 = \begin{bmatrix} H_{0_0} \\ H_{0_1} \\ \cdots \\ H_{0_i} \\ \cdots \\ H_{0_\alpha} \end{bmatrix} \qquad H_1 = \begin{bmatrix} H_{1_0} \\ H_{1_1} \\ \cdots \\ H_{1_j} \\ \cdots \\ H_{1_\beta} \end{bmatrix} \qquad \underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdots \\ \underset{\sim}{D}^{i \text{ or } j} \\ \cdots \\ \underset{\sim}{D}^{\alpha \text{ or } \beta} \end{bmatrix} \qquad \begin{array}{c} \alpha = 2^m - 1 \\ \beta = 2^t - 1 \\ 0 \leq i \leq \alpha \\ 0 \leq j \leq \beta \end{array} \tag{5.17}$$

The decompositions for $\underset{\sim}{D}_{h_0}$ and $\underset{\sim}{D}_{h_1}$ with respect to $x_{m+1} \cdots x_n$ and $x_{t+1} \cdots x_n$ are given as follows

$$\underset{\sim}{D}_{h_0} = \begin{bmatrix} \underset{\sim}{D}_{h_0}^0 \\ \underset{\sim}{D}_{h_0}^1 \\ \cdots \\ \underset{\sim}{D}_{h_0}^i \\ \cdots \\ \underset{\sim}{D}_{h_0}^\alpha \end{bmatrix} = RM^m . \begin{bmatrix} \underset{\sim}{D}_{0_{h_0}} \\ \underset{\sim}{D}_{1_{h_0}} \\ \cdots \\ \underset{\sim}{D}_{i_{h_0}} \\ \cdots \\ \underset{\sim}{D}_{\alpha_{h_0}} \end{bmatrix} \tag{5.18}$$

Where

$$\underset{\sim}{D}_{i_{h_0}} = RM^{n-m} . H_{0_i} \tag{5.19}$$

$$\underset{\sim}{D}_{h_1} = \begin{bmatrix} \underset{\sim}{D}_{h_1}^{0} \\ \underset{\sim}{D}_{h_1}^{1} \\ \cdots \\ \underset{\sim}{D}_{h_1}^{j} \\ \cdots \\ \underset{\sim}{D}_{h_1}^{\beta} \end{bmatrix} = RM^{m} . \begin{bmatrix} \underset{\sim}{D}_{0_{h_1}} \\ \underset{\sim}{D}_{1_{h_1}} \\ \cdots \\ \underset{\sim}{D}_{j_{h_1}} \\ \cdots \\ \underset{\sim}{D}_{\beta_{h_1}} \end{bmatrix} \tag{5.20}$$

Where

$$\underset{\sim}{D}_{j_{h_1}} = RM^{n-t} . H_{1_j} \tag{5.21}$$

If g/0, then $\underset{\sim}{D}^{*} = \underset{\sim}{D}_{h_0}$. Because $X_1 \subset X$ but $X_1 \neq X$, $h_0$ is independent of any variable in $X_1^{*}$, So that

$$H_{0_0}^{*} = H_{0_1}^{*} = \cdots = H_{0_\alpha}^{*} = H_{0_0}$$

then we have

$$\underset{\sim}{D}_{0_{h_0}}^{*} = \underset{\sim}{D}_{1_{h_0}}^{*} = \cdots = \underset{\sim}{D}_{\alpha_{h_0}}^{*} = \underset{\sim}{D}_{0_{h_0}} = \underset{\sim}{D}_{h_0}^{0}$$

From (5.18), for each $\underset{\sim}{D}_{h_0}^{j^*}$ in $\underset{\sim}{D}_{h_0}$, we have

$$\underset{\sim}{D}_{h_0}^{j^*} = \sum_{i \subseteq j} \underset{\sim}{D}_{i_{h_0}}^{*} = 2^{p} \underset{\sim}{D}_{h_0}^{0} \tag{5.22}$$

where p is the number of one's in the binary expansion of j. Let $\underset{\sim}{d}_{\alpha\beta}$ and $\hat{\underset{\sim}{d}}_{\beta}$ be the entry in $\underset{\sim}{D}^{j}$ and $\underset{\sim}{D}^{0_{h_0}}$, respectively. Then g/0 is testable if $\underset{\sim}{d}_{\alpha\beta} \neq 2^{p}\hat{\underset{\sim}{d}}_{\beta}$ for the fault free network.

For RM testing, $\underset{\sim}{d}_{\alpha\beta} \neq 2^{p}\hat{\underset{\sim}{d}}_{\beta}$ is equivalent to $d_{\alpha\beta} = 1$ because of mod 2 sum.

The proof for g/1 is similar to that for g/0.

**Corollary 5.5**

If $X'_1 \cap X'_2 \neq \varnothing$, the s-a-f on line g can be detected by $d_{\alpha\beta}$ if $d_{\alpha\beta} = 1$ for fault free network, where $\alpha \subseteq X'$, $(X' = X'_1 \cap X'_2)$, and $\beta \not\subseteq X'$ or $\beta = \varnothing$.

**Example 5.4**

Consider the internal line $g_2$ in the circuit shown in Fig. 5.1. The function f(X) can be written in the form of

$$f(X) = \bar{g}_2(x_4 \oplus \bar{x}_2 x_3) + g_2\bar{x}_4$$

Then we have

$$h_0(X_1) = h_0(x_2, x_3, x_4) = x_4 \oplus \bar{x}_2 x_3 \quad \text{and} \quad h_1(X_2) = h_1(x_4) = \bar{x}_4$$

From Theorem 5.4, we have

$$X'_1 = X - X_1 = (x_1, x_2, x_3, x_4) - (x_2, x_3, x_4) = x_1$$

$$X'_2 = X - X_2 = (x_1, x_2, x_3, x_4) - (x_4) = (x_1, x_2, x_3)$$

The fault free RM coefficient column vector D is

| $d_0$ | $d_1$ | $d_2$ | $d_{1,2}$ | $d_3$ | $d_{1,3}$ | $d_{2,3}$ | $d_{1,2,3}$ | $d_4$ | $d_{1,4}$ | $d_{2,4}$ | $d_{1,2,4}$ | $d_{3,4}$ | $d_{1,3,4}$ | $d_{2,3,4}$ | $d_{1,2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The fault $g_2/0$ can be detected by $d_{1,2}$, and $g_2/1$ can be detected by $d_{1,2}$, $d_3$, and $d_{2,3}$. The fault free and faulty RM coefficient column vectors are given in Table 5.8.

For the MRM coefficient, The fault free MRM coefficient column vector $\underset{\sim}{D}$ is

| $\underset{\sim}{d}_0$ | $\underset{\sim}{d}_1$ | $\underset{\sim}{d}_2$ | $\underset{\sim}{d}_{1,2}$ | $\underset{\sim}{d}_3$ | $\underset{\sim}{d}_{1,3}$ | $\underset{\sim}{d}_{2,3}$ | $\underset{\sim}{d}_{1,2,3}$ | $\underset{\sim}{d}_4$ | $\underset{\sim}{d}_{1,4}$ | $\underset{\sim}{d}_{2,4}$ | $\underset{\sim}{d}_{1,2,4}$ | $\underset{\sim}{d}_{3,4}$ | $\underset{\sim}{d}_{1,3,4}$ | $\underset{\sim}{d}_{2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 4 | 2 | 4 | 4 |

The fault $g_2/0$ can be detected by $\underset{\sim}{d}_1$, $\underset{\sim}{d}_{1,2}$, and $\underset{\sim}{d}_{1,2,3}$ because for these MRM coefficients $\underset{\sim}{d}_{\alpha\beta} \neq 2^p \hat{\underset{\sim}{d}}_{\beta_{h_0}}$. For example, consider the MRM coefficient $\underset{\sim}{d}_{1,2,3}$. $p = 1$ since $\alpha = x_1$.

$$\underset{\sim}{d}_{1,2,3} = 1 \neq 2\underset{\sim}{d}_{2,3h_0} = 2 \,.\, 1 = 2$$

$g_2/0$ cannot be detected by $\underset{\sim}{d}_{1,3,4}$ since

$$\underset{\sim}{d}_{1,3,4} = 0 = 2\underset{\sim}{d}_{3,4h_0} = 2 \,.\, 0 = 0$$

The fault $g_2/0$ can be detected by $\underset{\sim}{d}_{1,2}$ and $\underset{\sim}{d}_{1,2,3}$ because for both of the MRM coefficients $\underset{\sim}{d}_{\alpha\beta} \neq 2^p \hat{\underset{\sim}{d}}_{\beta_{h_1}}$. The fault free and faulty MRM coefficient column vectors for $g_2$ are given in Table 5.7.

## 5.4. Testability For Some Special Networks

From section 5.3 it is clear that testability for internal s-a-f's is more difficult than that for the input s-a-f's. But for some special networks, there are simple results.

### 5.4.1. Testability For Fan-Out Free Networks

If the circuit is fan-out free (FOF), the testing procedure is very simple. For any internal line g in an FOF network, $f(x_1 \cdots x_n)$ can be written in this form : $f(X) = h(X_1, g(X_2))$, where $X_1 \cap X_2 = \phi$, and $X_1 \cup X_2 = X$.

The single s-a-f on internal line $g(X_2)$ is equivalent to some multiple input s-a-f $X_2/u$, where $g(u) = 0$ ( or $g(u) = 1$ ). From Theorem 5.2, we have the following Theorem :

**Theorem 5.5**

In a FOF network, the internal s-a-f on line $g(X_2)$ is $d_\alpha$-detectable if and only if $d_{\alpha\beta} = 1$ for the fault free network, where $\alpha \subseteq X2$.

### 5.4.2. Testability For Internally Unate Networks

If f(X) is a positive unate function with respect to the internal line g, then the function f(X) can be written in the form

$$f(X) = g(X)B(X) + C(X)$$

Then, from (5.15) and (5.16) we have

$$h_0(X) = C(X), \quad \text{and} \quad h_1(X) = B(X) + C(X).$$

If f(X) is a negative unate function, then the function f(X) can be written in the form

$$f(X) = \bar{g}(X)A(X) + C(X)$$

Then, from (5.15) and (5.16) we have

$$h_0(X) = A(X) + C(X), \quad \text{and} \quad h_1(X) = C(X).$$

Then, the faults can be detected by using the conditions in Theorem 5.3 and Theorem 5.4.

## 5.5. Constrained RM and MRM Testing

Recall the partitions of the output vector and the RM and MRM coefficient vectors of a function f(X), with respect to n-m input variables $x_n, x_{n-1}, \cdots, x_{m+1}$, from section 3.6.

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ . \\ . \\ Z_u \end{bmatrix}, \quad D = \begin{bmatrix} D^0 \\ D^1 \\ . \\ . \\ D^u \end{bmatrix}, \quad \text{and} \quad \underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ . \\ . \\ \underset{\sim}{D}^u \end{bmatrix}$$

where

$$D^j = \bigoplus_{i \subseteq j} D_i \, , \quad \underset{\sim}{D}^j = \sum_{i \subseteq j} \underset{\sim}{D}_i \, ,$$

and

$$D_j = RM^m \oplus Z_j \, , \quad \text{and} \quad \underset{\sim}{D}_j = RM^m \, . \, Z_j \, .$$

Each $Z_j$, $0 \le j \le 2^{m-n}-1$, is a column vector for $f_j$, a subfunction of f(X), obtained by setting $x_n, x_{n-1}, \ldots, x_{m-1}$ to j while the remaining inputs are exhaustively exercised. $D_j$ and $\underset{\sim}{D}_j$ are the RM and MRM spectra for $Z_j$.

In the previous sections, we use the RM and MRM coefficients of f(X) to detect the faults. In Theorem 5.3, we have to compute the spectral coefficients of $h_0(X)$ and $h_1(X)$, and compare them with the spectral coefficients of function f(X). If $h_0$ or $h_1$ is not a function of all of the input variables, the fault will be detected by using the conditions in Theorem 5.4 easily. Unfortunately, for a lot of functions, we cannot use the conditions in these theorems directly.

Instead of using the coefficients of function f(X) for fault detection, we can also use the coefficients of $f_j$, a subfunction of f(X). This is constrained RM or MRM testing.

The subfunction $f_j$ results from constraining n-m input variables, $x_n, x_{n-1}, \cdots, x_{m+1}$, to j, where $j = \sum_{i=1}^{n-m} j_i 2^{i-1}$. For $f_j$, the corresponding $h_0$, or $h_1$ may be independent of some unconstrained input variables, so that Theorem 5.4 can be applied to it.

Let $X_c = x_{m+1}, \cdots, x_n$, and $X_u = x_1, ..., x_m$. Without loss of generality, we only consider the constraint which fixes the value of $X_c$ to j. The associated subfunction $f_j(x_1, x_2, \ldots, x_m) = f(x_1, ..., x_m, j_1, j_2, \cdots, j_{n-m})$, has the spectrum $D_j$, which from (3.32) is given by

$$D_j = \bigoplus_{i \subseteq j} D^i \tag{5.23}$$

The MRM spectrum for $Z_j$ is given by (3.34)

$$\underset{\sim}{D}_j = \sum_{i \subseteq j} (-1)^q \underset{\sim}{D}^{i}, \qquad q = | \, i \, | + | \, j \, | .$$

let $\hat{d}_{<j>\alpha}$ be an entry in $D_j$, and similarly for $\underset{\sim}{D}_j$. Let $\left[ h_0 \right]_j$ and $\left[ h_1 \right]_j$ be the $h_0$ and $h_1$ required for $f_j$, i.e.

$$f_j(x_1, x_2, \ldots, x_m) = \left[ h_0 \right]_j \bar{g} + \left[ h_1 \right]_j g.$$

From Theorem 5.4, we have the following theorem.

**Theorem 5.6**

If $\left[ h_0 \right]_j$ is independent of $x_i$ (some $x_i \in X_u$), g/0 is (i). $\hat{d}_{<j>\alpha}$-detectable if $\hat{d}_{<j>\alpha} = 1 \qquad \alpha \subseteq X_u$ and $x_i \in \alpha$ .

(ii). $\underset{\sim}{\hat{d}}_{<j>\alpha}$-detectable if $\underset{\sim}{\hat{d}}_{<j>\alpha} \neq 2^p \underset{\sim}{\hat{d}}_{<j>}$.

If $\left[ h_1 \right]_j$ is independent of $x_i$ (some $x_i \in X_u$), g/1 is

(iii). $\hat{d}_{<j>\alpha}$-detectable if $\hat{d}_{<j>\alpha} = 1 \qquad \alpha \subseteq X_u$ and $x_i \in \alpha$ .

(iv). $\underset{\sim}{\hat{d}}_{<j>\alpha}$-detectable if $\underset{\sim}{\hat{d}}_{<j>\alpha} \neq 2^p \underset{\sim}{\hat{d}}_{<j>}$.

**Example 5.5**

Consider the internal line $g_1$ in the circuit shown in Fig. 5.1.

$$f(X) = x_4\bar{x}_3\bar{x}_2 + x_4x_2\bar{x}_1 + \bar{x}_4x_3\bar{x}_2 + \bar{x}_4x_2x_1$$

For the internal line marked $g_1$,

$$h(X,g_1(X)) = x_5x_4 + \bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2 \quad \text{where } x_5 = g_1(X)$$

$$h_0(X) = \bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2 \quad \text{and} \quad h_1(X) = x_4 + x_2x_1 + x_3\bar{x}_2$$

So the function f(X) can be rewritten in the form

$$f(X) = \bar{g}_1(\,\bar{x}_4x_2x_1 + \bar{x}_4x_3\bar{x}_2\,) + g_1(\,x_4 + x_2x_1 + x_3\bar{x}_2\,)$$

The fault on line $g_1$ cannot be detected by applying Theorem 5.6 directly. By constraining $x_4$ to "1", we have

$$h_{01}(X) = \dot{0} \qquad \text{and} \qquad h_{11}(X) = \dot{1}$$

where $\dot{0}$ and $\dot{1}$ are two constant vectors with entries all "0" or "1", respectively. $h_{01}(X)$ and $h_{11}(X)$ are independent of $x_1$, $x_2$, and $x_3$. The fault free and faulty constrained RM spectra for $g_1/0$ and $g_1/1$ are given in Table 5.8.

The fault can be detected by $\hat{d}_{1,2,4}$, $\hat{d}_{3,4}$, and $\hat{d}_{2,3,4}$ because these constrained RM coefficients are equal to "1".

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | $Z_1$ | $D_1$ | $g_1/0$ | | $g_1/1$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $H_{01}$ | $D_{h_{01}}$ | $H_{11}$ | $D_{h_{11}}$ |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | $\underline{0}$ | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | $\underline{0}$ | 1 | $\underline{0}$ |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | $\underline{0}$ | 1 | $\underline{0}$ |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | $\underline{0}$ | 1 | $\underline{0}$ |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |

**Table 5.8.** The Fault Free and Faulty Contrained RM Spectrum for $g_1/0$ and $g_1$.

## 5.6. Testability For Input Bridging Faults

### 5.6.1. Testability For a Two Input Bridging Fault

The first type of bridging fault considered is the fault between two primary input lines, $x_i$ and $x_j$. Without loss of generality, consider an AND-Bridging fault between two primary input lines $x_n, x_{n-1}$, denoted by AND-BF( $x_n, x_{n-1}$ ). The function $f(x_1 \cdots x_n)$ can be decomposed with respect to inputs $x_n$ and $x_{n-1}$ as :

$$f = \bar{x}_n \bar{x}_{n-1} f_0 + \bar{x}_n x_{n-1} f_1 + x_n \bar{x}_{n-1} f_2 + x_n x_{n-1} f_3 \tag{5.24}$$

where

$$f_0(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 0, 0)$$

$$f_1(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 1, 0)$$

$$f_2(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 0, 1)$$

$$f_3(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 1, 1)$$

This divides the truth value vector, Z, into four parts. named $Z_0, Z_1, Z_2, Z_3$. The corresponding RM spectra, $D_0, D_1, D_2,$ and $D_3$ are given by

$$D_i = RM^{n-2} \otimes Z_i \qquad 0 \leq i \leq 3$$

If the spectrum D of f(X) is partitioned as

$$D = RM^n \otimes F = \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix} = RM^2 \otimes \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

then we have

$$D^0 = D_0 \qquad D^1 = D_0 + D_1 \tag{5.25}$$

$$D^2 = D_0 + D_2 \qquad D^3 = D_0 + D_1 + D_2 + D_3$$

Similarly, for the modified Reed-Muller spectra, we have

$$\utilde{D}^0 = \utilde{D}_0 \qquad \utilde{D}^1 = \utilde{D}_0 + \utilde{D}_1 \tag{5.26}$$

$$\utilde{D}^2 = \utilde{D}_0 + \utilde{D}_2 \qquad \utilde{D}^3 = \utilde{D}_0 + \utilde{D}_1 + \utilde{D}_2 + \utilde{D}_3$$

We use some symmetry conditions to analyse the bridging faults (BF's) between two input lines $x_i$ and $x_j$. More details about symmetry conditions in Boolean functions are given in [34], but we include the necessary definitions here.

**Definition 5.1**

A function of $f(x_1, ..., x_i, ..., x_j, ..., x_n)$ possesses an equivalence symmetry in $\{x_i, x_j\}$, written $E\{x_i, x_j\}$, if $f(x_1, ..., 0, ..., 0, ..., x_n) = f(x_1, ..., 1, ..., 1, ..., x_n)$.

**Definition 5.2**

A function of $f(x_1, ..., x_i, ..., x_j, ..., x_n)$ possesses an non-equivalence symmetry in $\{x_i, x_j\}$, written $N\{x_i, x_j\}$, if $f(x_1, ..., 0, ..., 1, ..., x_n) = f(x_1, ..., 1, ..., 0, ..., x_n)$.

**Theorem 5.7**

For input lines $x_i,x_j$, $j > i$, the AND-BF( $x_i,x_j$ ) is

(i). $d_{i,\beta}$ (or $d_{j,\beta}$)-detectable if $d_{i,\beta}$ (or $d_{j,\beta}$) $= 1$.

(ii). $\utilde{d}_{i,\beta}$ (or $\utilde{d}_{j,\beta}$) -detectable if $\utilde{d}_{i,\beta}$ (or $\utilde{d}_{j,\beta}$) $\neq 2\utilde{d}_{\beta}$.

(iii). $d_{i,j,\beta}$-detectable if $d_{i,\beta} \neq d_{j,\beta}$.

(iv). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,\beta} + \underset{\sim}{d}_{j,\beta} \neq 4\underset{\sim}{d}_{\beta}$.

where i and j $\notin \beta$.

**Proof**

For AND-BF( $x_n$,$x_{n-1}$ ), we have $f_0 = f_1 = f_2$ under the fault effect. Since, for a faulty circuit, $\underset{\sim}{D}_0 = \underset{\sim}{D}_1 = \underset{\sim}{D}_2$ , it follows from (5.26):

$$\underset{\sim}{D}^{0^*} = \underset{\sim}{D}_0 \qquad \underset{\sim}{D}^{1^*} = \underset{\sim}{D}_0 + \underset{\sim}{D}_0 = 2\underset{\sim}{D}_0 \qquad \underset{\sim}{D}^{2^*} = \underset{\sim}{D}_0 + \underset{\sim}{D}_0 = 2\underset{\sim}{D}_0$$

$$\begin{aligned}\underset{\sim}{D}^{3^*} &= \underset{\sim}{D}_0 + \underset{\sim}{D}_0 + \underset{\sim}{D}_0 + \underset{\sim}{D}_3 = 3\underset{\sim}{D}_0 + \underset{\sim}{D}_3 \\ &= 3\underset{\sim}{D}^0 + (\underset{\sim}{D}^0 - \underset{\sim}{D}^1 - \underset{\sim}{D}^2 + \underset{\sim}{D}^3) \\ &= 4\underset{\sim}{D}^0 - \underset{\sim}{D}^1 - \underset{\sim}{D}^2 + \underset{\sim}{D}^3\end{aligned} \qquad (5.27)$$

Note that $\underset{\sim}{d}_{\beta}$, $\underset{\sim}{d}_{n-1,\beta}$, $\underset{\sim}{d}_{n,\beta}$, and $\underset{\sim}{d}_{n,n-1,\beta}$ are the corresponding MRM coefficients in $\underset{\sim}{D}^0$, $\underset{\sim}{D}^1$, $\underset{\sim}{D}^2$, and $\underset{\sim}{D}^3$, respectively. The AND-BF( $x_n$,$x_{n-1}$ ) can be detected by $\underset{\sim}{d}_{n-1,\beta}$ (or $\underset{\sim}{d}_{n,\beta}$) if $\underset{\sim}{d}_{n-1,\beta}$ (or $\underset{\sim}{d}_{n,\beta}$) $\neq 2\underset{\sim}{d}_{\beta}$ for the fault free network, respectively. The fault is also $\underset{\sim}{d}_{n,n-1,\beta}$-detectable if $\underset{\sim}{d}_{n,\beta} + \underset{\sim}{d}_{n-1,\beta} \neq 4\underset{\sim}{d}_{\beta}$.

From (5.27) , we can see that for $f(x_1 \cdots x_n)$ , if $f_0 = f_1 = f_2$, the AND-BF ( $x_i$,$x_j$ ) can not be detected by the conditions in Theorem 5.7.

From definition 5.1 and 5.2, a function f(X) processes E{$x_i$, $x_j$} which means $\underset{\sim}{D}_0 = \underset{\sim}{D}_3$, and processes N{$x_i$, $x_j$} means $\underset{\sim}{D}_1 = \underset{\sim}{D}_2$. Using the definition 5.1, 5.2, and (5.27), we have the following corollary.

**Corollary 5.6**

(i). If f(X) does not possess E{$x_i$, $x_j$}, but possesses N{$x_i$, $x_j$}, the AND-BF( $x_n$,$x_{n-1}$ ) is

(a). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,\beta}$(or $\underset{\sim}{d}_{j,\beta}$ ) $\neq 2\underset{\sim}{d}_{\beta}$.

(ii). If f(X) possesses E{$x_i$, $x_j$}, but does not possess N{$x_i$, $x_j$}, the AND-BF( $x_n$,$x_{n-1}$ ) is (a). $d_{i,j,\beta}$-detectable if $d_{i,\beta} \neq d_{j,\beta}$.

(b). $\underline{d}_{i,j,\beta}$-detectable if $\underline{d}_{i,j,\beta} \neq \underline{d}_{i,\beta} + \underline{d}_{j,\beta}$.

(iii). If f(X) possesses both of $E\{x_i, x_j\}$ and $N\{x_i, x_j\}$, the AND-BF( $x_n$,$x_{n-1}$ ) is

(a). $\underline{d}_{i,j,\beta}$-detectable if $\underline{d}_{i,j,\beta} \neq 2\underline{d}_{i,\beta}$(or $\underline{d}_{j,\beta}$ ).

**Example 5.6**

Consider the circuit shown in Fig.5.1. The faulty RM coefficients for the six possible two input AND-BF's are given in Table 5.9.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | D | $D^*_{x_1x_2}$ | $D^*_{x_1x_3}$ | $D^*_{x_1x_4}$ | $D^*_{x_2x_3}$ | $D^*_{x_2x_4}$ | $D^*_{x_3x_4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | $\underline{0}$ | 1 | $\underline{0}$ | 1 | $\underline{0}$ |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | $\underline{1}$ | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | $\underline{0}$ | $\underline{0}$ | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ |
| 0 | 1 | 1 | 1 | 1 | 0 | $\underline{1}$ | 0 | 0 | $\underline{1}$ | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | $\underline{0}$ | 1 | $\underline{0}$ | $\underline{0}$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | $\underline{1}$ | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $\underline{0}$ | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | $\underline{1}$ | 0 | $\underline{1}$ | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $\underline{1}$ | $\underline{1}$ |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 5.9** The Fault Free and Faulty RM Spectrum of the Network

The fault free RM coefficient column vector is

| $d_0$ | $d_1$ | $d_2$ | $d_{1,2}$ | $d_3$ | $d_{1,3}$ | $d_{2,3}$ | $d_{1,2,3}$ | $d_4$ | $d_{1,4}$ | $d_{2,4}$ | $d_{1,2,4}$ | $d_{3,4}$ | $d_{1,3,4}$ | $d_{2,3,4}$ | $d_{1,2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

For example, consider the AND-BF ( $x_i$,$x_j$ ). The fault is $d_3$, $d_{2,3}$, and $d_4$ detectable because these coefficients equal "1" for the fault free network. Since f(X) does not possess both of $E\{x_3, x_4\}$ and $N\{x_3, x_4\}$, the fault is also $d_{2,3,4}$ detectable because $d_{2,3} \neq d_{2,4}$ for the fault free network.

For the MRM testing the faulty MRM coefficients for the six possible two input AND-BF's are given in Table 5.10.

| $x_4$ | $x_3$ | $x_2$ | $x_1$ | Z | $\underset{\sim}{D}$ | $\underset{\sim}{D}^*_{x_1x_2}$ | $\underset{\sim}{D}^*_{x_1x_3}$ | $\underset{\sim}{D}^*_{x_1x_4}$ | $\underset{\sim}{D}^*_{x_2x_3}$ | $\underset{\sim}{D}^*_{x_2x_4}$ | $\underset{\sim}{D}^*_{x_3x_4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | $\underline{0}$ | 1 | $\underline{0}$ | 1 | $\underline{0}$ |
| 0 | 1 | 0 | 1 | 1 | 2 | 2 | $\underline{1}$ | 2 | $\underline{0}$ | 2 | $\underline{0}$ |
| 0 | 1 | 1 | 0 | 0 | 1 | $\underline{2}$ | $\underline{2}$ | $\underline{1}$ | $\underline{0}$ | 2 | $\underline{0}$ |
| 0 | 1 | 1 | 1 | 1 | 4 | $\underline{5}$ | $\underline{2}$ | $\underline{2}$ | $\underline{1}$ | 4 | $\underline{2}$ |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | $\underline{0}$ | 1 | $\underline{0}$ | $\underline{0}$ |
| 1 | 0 | 0 | 1 | 1 | 2 | $\underline{1}$ | 2 | $\underline{1}$ | 2 | $\underline{0}$ | $\underline{0}$ |
| 1 | 0 | 1 | 0 | 1 | 2 | $\underline{1}$ | 2 | $\underline{0}$ | 2 | $\underline{1}$ | $\underline{0}$ |
| 1 | 0 | 1 | 1 | 0 | 4 | 4 | 4 | $\underline{1}$ | 4 | $\underline{1}$ | $\underline{2}$ |
| 1 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | $\underline{0}$ |
| 1 | 1 | 0 | 1 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | $\underline{0}$ |
| 1 | 1 | 1 | 0 | 1 | 4 | 4 | 4 | $\underline{2}$ | 4 | $\underline{5}$ | $\underline{1}$ |
| 1 | 1 | 1 | 1 | 0 | 8 | 8 | 8 | $\underline{4}$ | 8 | 8 | $\underline{4}$ |

**Table 5.10** The Fault Free and Faulty MRM Spectrum of the Network

For example, consider the AND-BF ($x_3$,$x_4$). the fault free MRM coefficient column vector is

| $\underset{\sim}{d}_0$ | $\underset{\sim}{d}_1$ | $\underset{\sim}{d}_2$ | $\underset{\sim}{d}_{1,2}$ | $\underset{\sim}{d}_3$ | $\underset{\sim}{d}_{1,3}$ | $\underset{\sim}{d}_{2,3}$ | $\underset{\sim}{d}_{1,2,3}$ | $\underset{\sim}{d}_4$ | $\underset{\sim}{d}_{1,4}$ | $\underset{\sim}{d}_{2,4}$ | $\underset{\sim}{d}_{1,2,4}$ | $\underset{\sim}{d}_{3,4}$ | $\underset{\sim}{d}_{1,3,4}$ | $\underset{\sim}{d}_{2,3,4}$ | $\underset{\sim}{d}_{1,2,3,4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 2 | 1 | 4 | 1 | 2 | 2 | 4 | 2 | 4 | 4 | 8 |

The AND-BF($x_3$,$x_4$) can be detected by $\underset{\sim}{d}_3$, $\underset{\sim}{d}_{1,3}$, $\underset{\sim}{d}_{2,3}$, $\underset{\sim}{d}_{1,2,3}$, $\underset{\sim}{d}_4$, $\underset{\sim}{d}_{1,4}$, $\underset{\sim}{d}_{2,4}$, and $\underset{\sim}{d}_{1,2,4}$, because for all of these coefficients we have $\underset{\sim}{d}_{i\beta} \neq 2\underset{\sim}{d}_{\beta}$ or $\underset{\sim}{d}_{j\beta} \neq 2\underset{\sim}{d}_{\beta}$. Since f(X) dose not possess both of E{$x_3$, $x_4$} and N{$x_3$, $x_4$}, the fault is also $\underset{\sim}{d}_{3,4}$, $\underset{\sim}{d}_{1,3,4}$, $\underset{\sim}{d}_{2,3,4}$, and $\underset{\sim}{d}_{1,2,3,4}$ detectable. For these coefficients we have $4\underset{\sim}{d}_{\beta} \neq \underset{\sim}{d}_{i\beta} + \underset{\sim}{d}_{j\beta}$. For example, consider $\underset{\sim}{d}_{1,2,3,4}$, we have

$$4\underset{\sim}{d}_{1,2} = 4 \;.\; 1 = 4 \neq \underset{\sim}{d}_{1,2,3} + \underset{\sim}{d}_{1,2,4} = 4+4 = 8$$

A similar approach gives the following result for an OR-bridging fault (OR-BF) between the input lines.

**Theorem 5.8**

For input lines $x_i$,$x_j$, the OR-BF( $x_i$,$x_j$ )is

(i). $d_{i,\beta}$-detectable if $d_{i,j,\beta} \neq d_{j,\beta}$.

(ii). $\underset{\sim}{d}_{i,\beta}$-detectable if $\underset{\sim}{d}_{i,\beta} \neq \frac{1}{2}\,( 2\underset{\sim}{d}_{\beta} - \underset{\sim}{d}_{j,\beta} + \underset{\sim}{d}_{i,j,\beta} )$.

(iii). $d_{i,j,\beta}$-detectable if $d_{i,\beta} \neq d_{j,\beta}$.

(iv). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,j,\beta} \neq \frac{1}{2}\,( 3\underset{\sim}{d}_{i,\beta} + 3\underset{\sim}{d}_{j,\beta} - 4\underset{\sim}{d}_{\beta} )$.

where i and j $\notin \beta$.

**Corollary 5.7**

(i). If f(X) does not possess E{$x_i$, $x_j$}, but possesses N{$x_i$, $x_j$}, the OR-BF( $x_n$,$x_{n-1}$ ) is

(a). $\underset{\sim}{d}_{i,\beta}$ (or $\underset{\sim}{d}_{j,\beta}$)-detectable if $\underset{\sim}{d}_{i,\beta}$ (or $\underset{\sim}{d}_{j,\beta}$) $\neq \frac{1}{3}\,( 2\,\underset{\sim}{d}_{\beta} + \underset{\sim}{d}_{i,j,\beta})$.

(b). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,j,\beta} \neq 3\,\underset{\sim}{d}_{j,\beta} - 2\underset{\sim}{d}_{\beta}$.

(ii). If f(X) possesses E{$x_i$, $x_j$}, but does not possess N{$x_i$, $x_j$}, the OR-BF( $x_n$,$x_{n-1}$ ) is

(a). $d_{i,\beta}$ (or $d_{j,\beta}$)-detectable if $d_{i,\beta}$ (or $d_{j,\beta}$) $= 1$.

(b). $d_{i,j,\beta}$-detectable if $d_{i,j,\beta} = 1$.

(c). $\underset{\sim}{d}_{i,\beta}$ (or $d_{j,\beta}$)-detectable if $\underset{\sim}{d}_{i,\beta}$ (or $d_{j,\beta}$) $\neq \frac{1}{2}\,( 3\,\underset{\sim}{d}_{\beta} - \underset{\sim}{d}_{j,\beta}$(or $d_{i,\beta}$ ) ).

(d). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,j,\beta} \neq \frac{1}{2}\,(\underset{\sim}{d}_{i,\beta} + \underset{\sim}{d}_{j,\beta})$.

(iii). If f(X) possesses both of E{$x_i$, $x_j$} and N{$x_i$, $x_j$}, the OR-BF( $x_n$,$x_{n-1}$ ) is

(a). $\underset{\sim}{d}_{i,\beta}$-detectable if $\underset{\sim}{d}_{i,\beta} \neq \underset{\sim}{d}_{i,j,\beta}$.

(b). $\underset{\sim}{d}_{i,j,\beta}$-detectable if $\underset{\sim}{d}_{i,j,\beta} \neq \underset{\sim}{d}_{i,\beta}$(or $\underset{\sim}{d}_{j,\beta}$ ).

### 5.6.2. Testability For Multiple Input Bridging Faults

In this section, we derive a result for multiple input BF's.

Without loss of generality, consider a multiple NFBF on lines $x_{m+1} \cdots x_n$.

The partitions of the output column vector Z, the RM coefficient column vector D, and the MRM coefficient column vector $\underset{\sim}{D}$, with respect to the input variables $x_{m+1} \cdots x_n$, are

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \cdots \\ Z_u \end{bmatrix} \qquad u = 2^{n-m}-1$$

where each $Z_i$, $0 \le i \le u$, has $2^m$ elements.

Let $D_i = RM^m \otimes Z_i$, and $\underset{\sim}{D}_i = RM^m \, . \, Z_i$.

Then we have

$$D = \begin{bmatrix} D^0 \\ D^1 \\ \cdots \\ D^u \end{bmatrix} = RM^{n-m} \otimes \begin{bmatrix} D_0 \\ D_1 \\ \cdots \\ D_u \end{bmatrix} \tag{5.28}$$

and

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}^0 \\ \underset{\sim}{D}^1 \\ \cdots \\ \underset{\sim}{D}^u \end{bmatrix} = RM^{n-m} \otimes \begin{bmatrix} \underset{\sim}{D}_0 \\ \underset{\sim}{D}_1 \\ \cdots \\ \underset{\sim}{D}_u \end{bmatrix} \tag{5.29}$$

For AND-BF on the input lines $x_{m+1} \cdots x_n$, we have

$$Z_0^* = Z_1^* = \cdots = Z_u{}^* = Z_0$$

then

$$D_0^* = D_1^* = \cdots = D_u^* = D_0 \tag{5.30}$$

and

$$\underset{\sim}{D}_0^* = \underset{\sim}{D}_1^* = \cdots = \underset{\sim}{D}_u^* = \underset{\sim}{D}_0 \tag{5.31}$$

**Theorem 5.9**

In a combinational network, the multiple input AND-BF on lines $x_{m+1} \cdots x_n$ is

(i). $d_{\alpha\beta}$-detectable if $d_{\alpha\beta} = 1$.

(ii). $\underset{\sim}{d}_{\alpha\beta}$-detectable if $\underset{\sim}{d}_{\alpha\beta} \neq 2^p \underset{\sim}{d}_{\alpha}$,

where $\alpha \subseteq \{1, 2, ..., m\}$ ( or $\alpha = \varnothing$ ) , and $\beta \subseteq \{ m+1, ..., n\}$ ( $\beta \neq \varnothing$ and $\beta \neq \{ m+1, ..., n\}$ ).

If $\beta = \{ m+1, ..., n\}$, the fault is

(iii). $d_{\alpha\beta}$-detectable if $d_{\alpha\beta} \neq \bigoplus_{i=1}^{u} d_{i\beta}$.

(iv). $\underset{\sim}{d}_{\alpha\beta}$-detectable if $\underset{\sim}{d}_{\alpha\beta} \neq \sum_{i=1}^{u} 2^p(-1)^q \underset{\sim}{d}_{i\beta}$.

**Proof**

From (5.28) and (5.30) we have

$$D^{0^*} = D^0$$

$$D^{i^*} = \dot{0}, \qquad \text{where } 0 \leq i \leq u. \tag{5.32}$$

and

$$D^{u^*} = D_0 \oplus D_u = D^0 \oplus \bigoplus_{i=0}^{u} D^i = \bigoplus_{i=1}^{u} D^i \tag{5.33}$$

If $d_{\alpha\beta}$ is not an entry in $D^0$ or $D^u$, from (5.32), the fault is $d_{\alpha\beta}$-detectable if

$$d_{\alpha\beta} = 1$$

If $d_{\alpha\beta}$ is an entry in $D^u$, from (5.34), the fault is $d_{\alpha\beta}$-detectable if

$$d_{\alpha\beta} \neq \bigoplus_{i=1}^{u} d_{i\beta}$$

## 5.7. Conclusion

In this chapter, the testability for stuck-at or input bridging faults using Reed-Muller and modified Reed-Muller coefficients is considered. For single or multiple input stuck-at or bridging faults, Reed-Muller and modified Reed-Muller spectral tests are very simple and efficient. No more than n RM or MRM coefficients are needed to detect all of the single and multiple input stuck-at or bridging faults in an n input network. A set of RM or MRM coefficients which can detect all of the single input stuck-at faults can also cover all of the multiple input stuck-at faults. A set of RM or MRM coefficients which covers all of the input stuck-at 1 faults can also detect all of the stuck-at 0 faults.

It is more complicated to detect some of the internal stuck-at faults, and in some cases, this requires circuit simulation. However, for some particular network topologies, the RM or MRM tests are very simple. Constrained RM or MRM tests can make the detection of some internal stuck-at faults much easier. The testability of internal bridging faults is still an open question.

Each MRM coefficient can cover more faults than the corresponding RM coefficient, so MRM testing uses fewer coefficients than an RM test to cover all of the faults, but each of the coefficients is more complex.

By considering all of the different polarity MRM coefficients, we see that each MRM coefficient is a syndrome of a subfunction of f(X). So the syndrome and constrained syndrome tests are special cases of MRM testing.

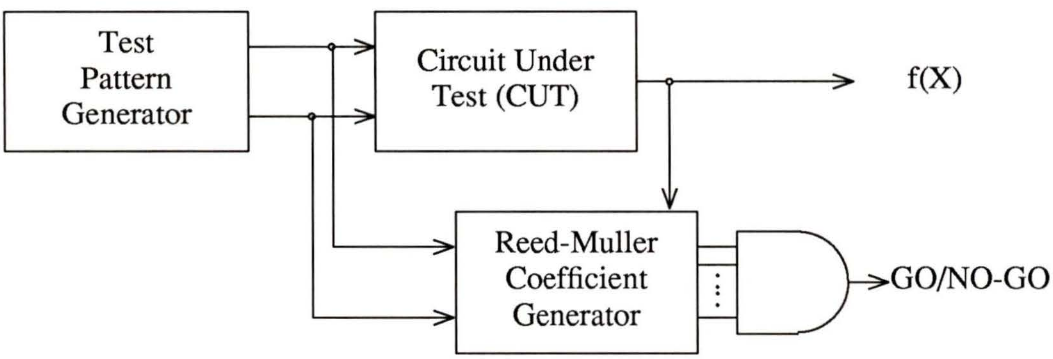
The implementation of RM and MRM testing, and the comparison between different spectral techniques are given in chapter 6.

# CHAPTER 6

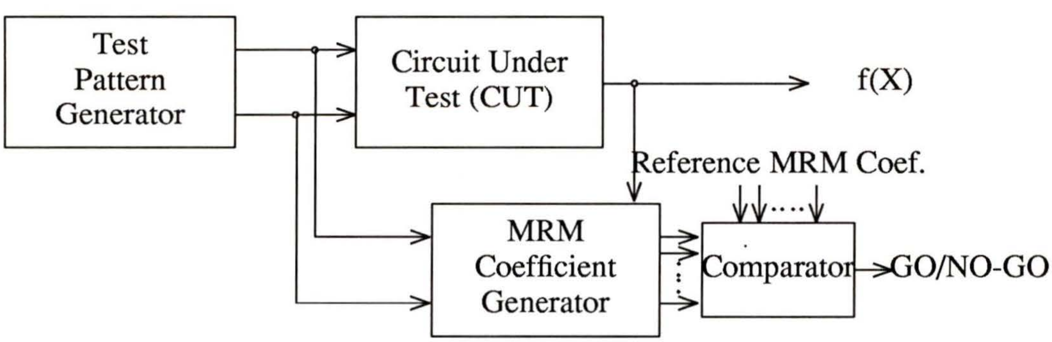
# IMPLEMENTATION AND SIMULATION OF REED-MULLER AND MODIFIED REED-MULLER TESTING

## 6.1. Implementation For Computing The RM and MRM Coefficients

In order to test a network, we have to compute and verify a set of RM or MRM coefficients which can cover all of the possible stuck-at or bridging faults. The structure of RM or MRM testing is shown in Fig. 6.1.




(a) Reed-Muller Testing.




(b) Modified Reed-Muller Testing.

**Figure 6.1** The Block Structure of RM or MRM Testing.

A set of input assignments called test patterns is applied to the input of CUT. The computing of each RM or MRM coefficient, $d_j$, or $\underline{d}_j$, needs $2^{|j|}$ input assignments. Let S be a set of RM or MRM coefficients which can cover all of the possible stuck-at or bridging faults, and B be a set of the subscripts of the coefficients in S. Then the number of test patterns needed, P, is given by

$$P = \sum_{\text{all } j,\ j \subseteq B} 2^{|j|}$$

So the RM and MRM testing is a non-exhaustive testing method except the last coefficient is included in S. The set of coefficients in S is computed by the RM or MRM coefficient generator. To verify a set of RM coefficients, only one AND gate in needed. For MRM testing, we have to store the set of coefficients, and compare with corresponding coefficients expected to give a go/no-go signal.

Damarla and Kapovsky [67] present two schemes to generate the RM spectral coefficients which we discuss below. Note, in this chapter, we only consider generating the 0th polarity RM coefficients.

**Scheme 1:** Computing RM coefficients by clocked T flip-flops and NOR gates.

From (3.3) we can see that each RM coefficient, $d_j$, is a mod 2 sum of a subset of the output responses. So, only one clocked T flip-flop is needed to compute $d_j$. The inputs to this T flip-flop depend on the binary representation of j. From (3.3) we have

$$d_j = \bigoplus_{k \subseteq j} f(k)$$

(3.3) can be written in another form with respect to the binary representation of j. Let $j_1, ..., j_i, ..., j_n$ and $k_1, ..., k_i, ..., k_n$ Let $j = \sum_{i=1}^{n} j_i 2^{i-1}$ and $k = \sum_{i=1}^{n} k_i 2^{i-1}$ be the binary representations of j and k, respectively. we have

$$d_j = \bigoplus_{k=0}^{2^n-1} j_1^{k_1}, ..., j_i^{k_i}, ..., j_n^{k_n} f(k) \tag{6.1}$$

where

$$j_i^{k_i} = j_i \cup \bar{k}_i = \begin{cases} \bar{k}_i, & \text{if } j_i = 0; \\ 1, & \text{if } j_i = 1. \end{cases}$$

So (6.1) can be written in the form of

$$d_j = \bigoplus_{k=0}^{2^n-1} \prod_{\text{All } i,\, j_i=0} \bar{k}_i\, f(k) = \bigoplus_{k=0}^{2^n-1} \overline{\bigcup_{\text{All } i,\, j_i=0} k_i} f(k) \qquad (6.2)$$

From (6.2) we can see that the input to T flip-flop can be controlled by a NOR gate. Since $k_i$ corresponds to the input $x_i$, the inputs of the NOR gate are $x_i$ if $j_i = 0$ in the binary representation of j. For n = 4, consider the RM coefficient $d_{10}$ as an example,

$$d_{10} = d_{1010} = \bigoplus_{x=0}^{2^n-1} \bar{x}_2 \bar{x}_4 f(x) = \bigoplus_{x=0}^{2^n-1} \overline{x_2 \cup x_4} f(x)$$

hence the input to the NOR gate are $x_2$ and $x_4$ . The test circuit to generate $d_{10}$ ($d_{1010}$) is shown in Fig 6.2. ( If more than one spectral coefficient has to be computed, then corresponding NOR gates , AND gates, and T flip-flops have to be added).

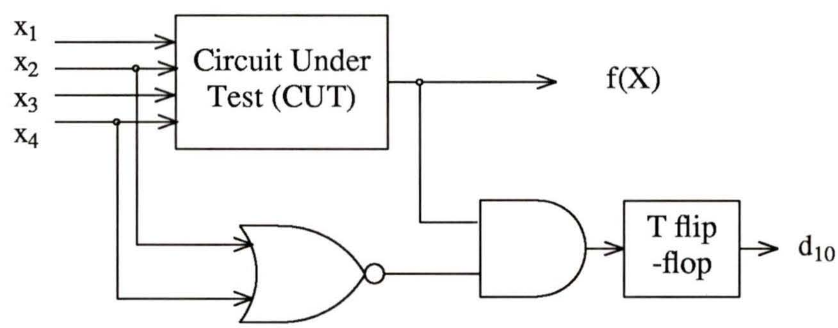



**Figure 6.2** The Circuit Generated the RM Coefficient $d_{10}$ Using Scheme 1.

Let S be a set of RM coefficients which can cover all of the faults and $d_j$ is a coefficient in S. To verify the coefficients we need only one NAND gate. The inputs to this NAND gate are $d_j$ if $d_j = 1$. If $d_j = 0$, another NOT gate is needed. So the output of the NAND gate will be 0 for fault free network. If some fault occurred, some of $d_j$ will be changed, and the output of the NAND gate will be 1. The simple test scheme is illustrated in Fig. 6.2 The input pattern generator may be a high speed counter which cycles through all $2^n$ possible combinations of the n variables, or may generate only those test patterns needed for the set of RM coefficients. The order of input patterns generated is not importment.

Consider the input s-a-f's as an example. From Corollary 5.4, in order to test the input s-a-f's, we need only consider some $d_j$ which equals to 1 for the fault free network. So the RM spectral coefficient generator needs at most n NOR gates, n NAND gates , and n T flip-flops.

From Corollary 5.5, we can see that if $d_{1,1,...,1} = 1$, only one T flip-flop is needed at the output of the circuit under test, which detect all of the single or multiple input stuck-at faults.

**Scheme 2:** Computing RM coefficients by shift registers and EXOR gates

In this scheme all of the output responses for computing the set of RM coefficients are stored in a shift register. Each RM coefficient is computed by hard-wired EXOR gates. This scheme can be shown by an example.

**Example 6.1**

For n = 4, if $d_3 = 1$, $d_4 = 1$, and $d_{10} = 1$ are a set of RM coefficients which can detected all of the s-a-f's in a network, then the RM test using scheme 2 is shown in Fig. 6.3.

In this case the following sequence of test patterns is required

$$T = ( 0000, 0010, 0001, 0011, 0100, 1000, 1010 ),$$

because the test circuit shown in Fig.6.3 is based on the input sequence stored in the shift register.

This scheme can be useful only if the order of the RM coefficients is small.

For the modified Reed-Muller spectral testing which we present in this paper, the MRM coefficient generator is similar to that for the RM coefficients using scheme 1. The difference is the requirement for a mod $2^{|j|}$ counter to substitute for the T flip-flop for each coefficient, $\underset{=}{d}_j$. The fault free MRM coefficient, $\underset{=}{d}_j$, must be stored in a $|j|$ bits register and compared with the coefficient generated. The block structure of MRM testing is shown in Fig. 6.1(b).

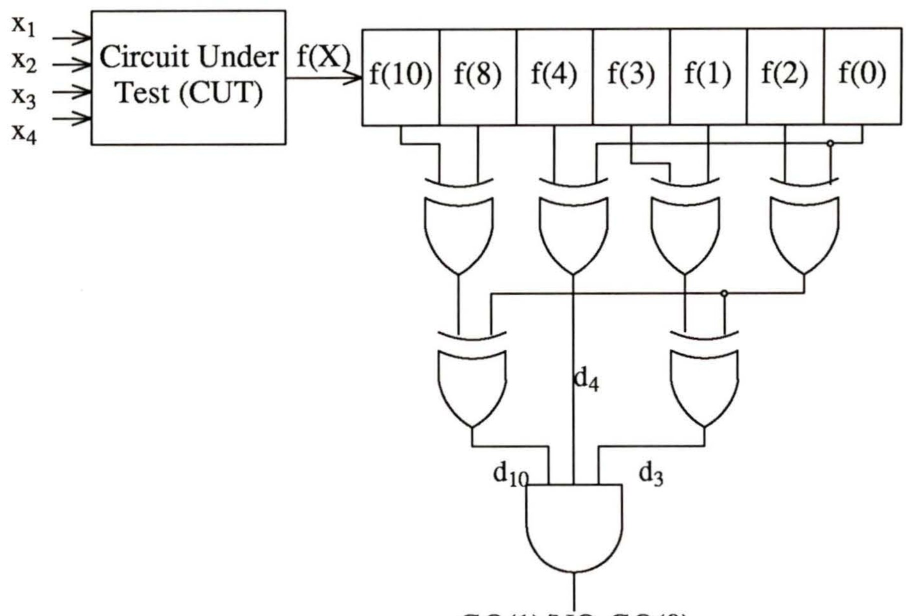



**Figure 6.3** The Circuit Generated the RM Coefficient $d_{10}$ Using Scheme 2.

In Rademacher-Walsh testing, each coefficient needs an n bit counter and an EXOR function network for coefficient selection. The fault free coefficient must be stored in an n bit register and compared with the coefficient generated. In MRM testing, for each coefficient, $\underline{d}_j$, a $|j|$ bit counter and register, $1 \leq |j| \leq n$, a NOR gate, and an AND gate are needed to computer and store the coefficient. For RM testing, the hardware requirement is very small. A T flip-flop, an NOR, and an AND gates are needed for each coefficient. We need not store and compare the fault free and faulty coefficients. Only one NAND gate is needed to verify the set of RM coefficients.

## 6.2. The Simulations of Some Spectral Testing

In order to compare the different test methods, we use a simulation program to simulate some circuits. The aim is to find out the minimum number of coefficients or test patterns used in each spectral testing. The details of all the circuits simulated are given in R. Aitken's M.Sc thesis [72].

The program is written in C and run on Sun UNIX 4.2. It is not intended to be a efficient program for large circuits but to be useful for comparative evaluation. A different simulation would be required for circuits of larger size.

The simulator can simulate the test set, Rademacher-Walsh, arithmetic, Reed-Muller, and modified Reed-Muller testing for single s-a-f's. A full simulation method is used. The output responses or corresponding spectral coefficients for all of the possible single s-a-f's are calculated, and compared with the fault free values.

For the test set method, the simulator finds out the minimum number of test patterns needed to cover all of the single s-a-f's. For most of the cases, only some of the output responses are needed for each of the RM, MRM, or arithmetic coefficient. For these test methods, the simulator can find out the minimum number of coefficients or minimum number of test patterns to cover all the faults. Since all of the possible input assignments must be applied to the inputs for the Rademacher-Walsh testing, the simulator finds out the minimum number of R-W coefficients which can cover all the possible faults. All of the simulation results are given in Table 6.1.

In Table 6.1, "Min C" or "Min P" means the minimum number of coefficients or test patterns, while "#C" or "#P" is the number of coefficients or the number of test patterns, respectively.

From Table 6.1 we can see that the Rademacher-Walsh testing used the minimum number of coefficients to detect all of the possible faults. But the input source must apply all possible input assignments to the CUT. An n-bit counter and a coefficient selection network are required to compute each R-W coefficient. The coefficient generated must be stored in a n-bit register and compared with the fault-free coefficient stored in another n-bit register.

The Reed-Muller test needs more coefficients to cover all the faults. But only one T flip-flop and a NOR gate are needed to compute a RM coefficient. We do not need to store and compare the faulty and fault-free coefficients. Only one NAND gate is required to give a go/no-go signal.

| Name of CUT | No. of Input | Test Set | RM | | | | MRM | | | | R-W | Arithmetic | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Min C | | Min P | | Min C | | Min P | | | Min C | | Min P | |
| | | #P | #C | #P | #C | #P | #C | #P | #C | #P | #C | #C | #P | #C | #P |
| c1 | 3 | 6 | 3 | 7 | 3 | 7 | 2 | 8 | 3 | 7 | 2 | 2 | 8 | 3 | 7 |
| c2 | 4 | 8 | 3 | 16 | 3 | 16 | 1 | 16 | 1 | 16 | 1 | 1 | 16 | 1 | 16 |
| c3 | 3 | 8 | 6 | 8 | 6 | 8 | 4 | 8 | 4 | 8 | 1 | 1 | 8 | 1 | 8 |
| c4 | 3 | 4 | 3 | 6 | 4 | 5 | 2 | 8 | 4 | 5 | 1 | 1 | 8 | 2 | 5 |
| c5 | 3 | 5 | 3 | 8 | 3 | 8 | 3 | 8 | 3 | 8 | 2 | 3 | 8 | 3 | 8 |
| c6 | 3 | 4 | 3 | 8 | 3 | 8 | 2 | 8 | 2 | 8 | 1 | 1 | 8 | 1 | 8 |
| c7 | 3 | 4 | 3 | 8 | 3 | 8 | 1 | 8 | 1 | 8 | 1 | 3 | 8 | 3 | 8 |
| c9 | 4 | 4 | 2 | 12 | 2 | 12 | 2 | 12 | 2 | 12 | 2 | 2 | 12 | 3 | 12 |
| c10 | 4 | 5 | 3 | 10 | 3 | 10 | 1 | 16 | 2 | 10 | 1 | 3 | 10 | 3 | 10 |
| c11 | 4 | 7 | 4 | 10 | 4 | 10 | 3 | 10 | 3 | 10 | 2 | 3 | 16 | 4 | 10 |
| c12 | 4 | 7 | 3 | 10 | 3 | 10 | 2 | 16 | 4 | 10 | 2 | 2 | 16 | 3 | 10 |
| c14 | 4 | 5 | 2 | 10 | 2 | 10 | 1 | 16 | 2 | 10 | 1 | 2 | 16 | 3 | 10 |
| c15 | 4 | 9 | 3 | 16 | 3 | 16 | 2 | 16 | 2 | 16 | 2 | 2 | 16 | 2 | 16 |
| c16 | 4 | 7 | 6 | 12 | 6 | 12 | 3 | 16 | 4 | 12 | 2 | 2 | 16 | 4 | 12 |
| c17 | 4 | 5 | 2 | 16 | 3 | 8 | 1 | 16 | 3 | 8 | 1 | 2 | 16 | 3 | 8 |
| c18 | 5 | 7 | 3 | 20 | 7 | 13 | 1 | 32 | 7 | 13 | 1 | 2 | 20 | 7 | 13 |
| c19 | 5 | 6 | 2 | 18 | 4 | 12 | 1 | 32 | 3 | 12 | 1 | 2 | 18 | 4 | 12 |
| c20 | 5 | 5 | 2 | 32 | 3 | 11 | 1 | 32 | 2 | 11 | 1 | 2 | 32 | 3 | 11 |
| c21 | 5 | 5 | 2 | 20 | 3 | 16 | 2 | 20 | 3 | 16 | 1 | 1 | 32 | 3 | 16 |
| c22 | 5 | 9 | 5 | 24 | 6 | 20 | 2 | 32 | 3 | 20 | 2 | 5 | 24 | 6 | 20 |
| c23 | 5 | 6 | 3 | 14 | 4 | 11 | 1 | 32 | 3 | 11 | 1 | 3 | 14 | 4 | 11 |
| c24 | 6 | 5 | 2 | 48 | 7 | 13 | 1 | 64 | 7 | 13 | 1 | 2 | 48 | 7 | 13 |
| c25 | 6 | 7 | 3 | 64 | 4 | 48 | 1 | 64 | 2 | 48 | 1 | 3 | 64 | 4 | 48 |
| c26 | 6 | 7 | 3 | 22 | 4 | 20 | 1 | 32 | 2 | 20 | 1 | 3 | 22 | 4 | 20 |
| c27 | 6 | 3 | 5 | 8 | 5 | 8 | 1 | 64 | 5 | 8 | 1 | 5 | 8 | 5 | 8 |
| c28 | 6 | 4 | 3 | 22 | 3 | 22 | 1 | 64 | 2 | 22 | 1 | 3 | 22 | 3 | 22 |
| c29 | 6 | 8 | 4 | 34 | 6 | 22 | 4 | 64 | 6 | 22 | 1 | 1 | 64 | 2 | 22 |
| c31 | 7 | 7 | 2 | 30 | 5 | 13 | 1 | 128 | 4 | 13 | 1 | 2 | 30 | 5 | 13 |
| c32 | 7 | 8 | 1 | 128 | 3 | 64 | 1 | 128 | 3 | 64 | 1 | 1 | 128 | 3 | 64 |
| c33 | 7 | 7 | 1 | 128 | 3 | 44 | 1 | 128 | 3 | 44 | 1 | 1 | 128 | 3 | 44 |
| c34 | 8 | 8 | 6 | 28 | 6 | 28 | 3 | 66 | 10 | 28 | 1 | 2 | 88 | 11 | 28 |
| c35 | 8 | 9 | 4 | 42 | 11 | 28 | 2 | 88 | 11 | 28 | 1 | 2 | 88 | 11 | 28 |
| c36 | 8 | 8 | 3 | 256 | 7 | 16 | 1 | 256 | 3 | 16 | 1 | 2 | 256 | 7 | 16 |
| c37 | 8 | 8 | 1 | 256 | 3 | 74 | 1 | 256 | 3 | 74 | 1 | 1 | 256 | 3 | 74 |
| c38 | 8 | 7 | 1 | 256 | 4 | 38 | 1 | 256 | 4 | 38 | 1 | 1 | 256 | 4 | 38 |
| c40 | 9 | 10 | 4 | 134 | 6 | 47 | 1 | 512 | 5 | 47 | 1 | 4 | 512 | 6 | 47 |
| c41 | 9 | 7 | 3 | 512 | 4 | 6 | 1 | 512 | 3 | 56 | 1 | 1 | 512 | 3 | 56 |
| c42 | 10 | 3 | 10 | 11 | 10 | 11 | 10 | 11 | 10 | 11 | 1 | 1 | 1024 | 10 | 11 |
| c43 | 12 | 8 | 4 | 4096 | 6 | 91 | 1 | 4096 | 4 | 91 | 1 | 4 | 4096 | 6 | 91 |
| Average | | 6.4 | 3.2 | 163 | 4.7 | 22.4 | 1.8 | 183.4 | 3.7 | 22.4 | 1.2 | 2.1 | 203.2 | 4.2 | 22.4 |

**Table 6.1** The Simulation Results of Some Circuits.

The hardware requirement for RM test are much smaller than that of R-W test. For all of the circuits simulated, no more than n RM coefficients are needed to cover all of the faults except one circuit, c3.

The simulation results of MRM test are between R-W test and RM test. MRM always uses fewer coefficients againest all the faults than RM test did. The hardware requirement for MRM test is smaller than that of R-W test. Each MRM coefficient, $\underline{d}_j$, needs a $|j|$ bit counter and register to compute and store instead of n bit for the R-W coefficient.

The hardware complexity of MRM and Arithmetic tests are similar. From the simulation results, we can see that for about 81% of the circuits, MRM testing uses less or equal number of coefficients against all of the faults compared with the Arithmetic testing.

The interested result is that the numbers of test patterns in the column "Min P" are identical for RM, MRM, and arithmetic tests. The reason for this may be because each RM, MRM, or arithmetic coefficient involves the same output responses. It still needs further study.

The simulator can also simulate all of the $2^n$ possible polarities RM and MRM tests. Some of the results are very interesting. But we need a further study about the RM and MRM tests using different polarities.

# CHAPTER 7

# CONCLUSION

In this research the problem of fault detection in general combinational networks is addressed, with a concentration on compaction testing based on some spectral techniques. A new spectral technique based on the modified Reed-Muller coefficients is developed. The testability using RM and MRM spectra is examined. A set of criteria for RM and MRM techniques to detect stuck-at and bridging faults is developed.

The RM transform is defined over GF(2). Each RM coefficient is either 0 or 1. The aliasing probability using RM testing is higher than for other spectra defined in the integer fields, so we develop the modified Reed-Muller spectral coefficients, each of which has a larger integer range. MRM testing reduces the aliasing probability compared to RM testing. The iterative MRM transform is computationally simpler than the Hadamard transform, so the hardware requirement for MRM testing is smaller than that for R-W testing. Some of the calculus and properties of MRM transforms are given in Chapter 3 and 4. We give the properties in matrix form, which is very convenient for analysing the RM and MRM coefficient column vectors under some fault effects.

The relationships between several alternative spectral representations are also investigated. The conversion matrices for R-W, arithmetic, and MRM spectral coefficients all have recursive forms. The transformation between representations is straightforward.

The testability for stuck-at faults and non-feedback bridging faults by using Reed-Muller and modified Reed-Muller coefficients is examined. For single or multiple input stuck-at or bridging faults, Reed-Muller and modified Reed-Muller spectral tests are very simple and efficient. No more than n RM or MRM coefficients are needed to detect all of the single and multiple input stuck-at or bridging faults in an n input network. A set of RM or MRM coefficients which can detect all of the single input stuck-at faults can also protect against all of the multiple input stuck-at faults. A set of RM or MRM coefficients which covers all of the input stuck-at 1 faults also detects all of the stuck-at 0 faults.

It is more complex to detect some of the internal stuck-at faults. Some times a fault simulation is required to find the set of RM or MRM coefficients which can cover all of the possible faults. However, for some special internal lines or networks, the RM or MRM tests are very simple. For example, the stuck-at faults in a fan-out free network are very easy to detect using RM or MRM spectral techniques. Constrained RM or MRM testing can make the detection of some internal stuck-at faults much easier.

Several alternative spectral techniques for fault detection are compared in terms of the number of test patterns, the number of coefficients needed to cover all of the single stuck-at faults, and the hardware implementation complexity. A simulation program is developed to analyse a set of example circuits. This shows that Rademacher-Walsh based signatures are shorter than others, but often take longer for the testing to be carried out.

The Reed-Muller test needs more coefficients to cover all the faults. But only one T flip-flop and a NOR gate are required to compute each RM coefficient. We do not need to store and compare the faulty and fault-free coefficients. Only one NAND gate is required to give a go/no-go signal. The hardware requirement for the RM test is much smaller than that for a R-W test.

The simulation results for MRM testing are between those for R-W testing and RM testing in terms of the size of the coefficient set required and the hardware requirement. MRM testing uses fewer coefficients to cover all the possible faults than the RM testing. The hardware requirement for MRM test is smaller than that of R-W test. The hardware complexity of MRM testing and Arithmetic testing is similar.

A number of problems remain to be addressed by further research, namely:

(1) To find a general procedure to derive a set of RM or MRM coefficients to cover all of the possible faults in a network.

(2) Reducing the number of coefficients and test patterns needed in RM or MRM testing by using different polarity RM or MRM coefficients. In this research, we only

investigated the testability by using the 0th polarity RM and MRM coefficients. From the simulation results, it is clear that different polarity coefficients give better coverage in some cases.

(3) To find an upper bound on the number of coefficients required for the testing for general networks. From the simulation results, we can see that for more than 90% of the circuits, no more than n RM or MRM coefficients are needed to cover all of the single stuck-at faults.

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(1) **V&H: A Switchbox Router**, (with D.M. Miller), *Canadian Conference on Very Large Scale Integration*, Oct, 1988.

**FAULT DETECTION IN COMBINATIONAL NETWORKS USING REED-MULLER AND MODIFIED REED-MULLER SPECTRAL TECHNIQUES**

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