A NOTE ON EIGENVALUES OF FIXED RANK PERTURBATIONS OF DIAGONAL MATRICES

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An $n \times n$ real matrix $T \in M_k$ if $T = D + A$ where D is diagonal and rank $A = k$. For $0 \leq k \leq n-1$ and A diagonally symmetrizable, we prove that all but k eigenvalues of T lie in the closed interval between the minimum and maximum diagonal entry of D; but show that no such result holds for general A. This answers an open problem posed by Furth and Sierksma. We also correct their proof of the result that $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} = M_n$.

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Our purpose is to resolve the first open problem in section 7 of [1] and to give a correct proof of Theorem 3 of [1]. A real $n \times n$ matrix T belongs to the class M_k if and only if there are real numbers $d_1, \ldots, d_n$ such that

$$\text{rank}(T - \text{diag}(d_1,\ldots,d_n)) = k.$$

Here $\text{diag}(d_1,\ldots,d_n)$ is the $n \times n$ diagonal matrix with entries $d_1, \ldots, d_n$. Letting $N \equiv \{1,2,\ldots,n\}$, the first open problem in [1] is:

> Let $0 \le k \le n-1$, let $T \in M_k$ have real eigenvalues, and suppose that $\text{rank}\,(T-\text{diag}(d_1,\ldots,d_n)) = k$. Is it true that all except k of the eigenvalues of T are in the interval $\left[\min_{i\in N} d_i, \max_{i\in N} d_i\right]$?

A partial affirmative answer was provided in [1] for the case $k = 1$ by the following.

Theorem 8 [1]. *Let* $T = \text{diag}(d_1,\ldots,d_n) + A \in M_1$ *be such that* $A = (a_{ij})$ *is of rank* 1. *Let* $n \ge 2$ *and suppose that either i)* $a_{ii} \ge 0$ *for each* $i \in N$ *or ii)* $a_{ii} \le 0$ *for each* $i \in N$. *Then the eigenvalues* $\lambda_i = \lambda_i(T)$ *of* T *are real for each* $i \in N$. *Assuming that* $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, *the following holds.*

Case i) $d_1 \le \lambda_1 \le d_2 \le \lambda_2 \le \ldots \le d_n \le \lambda_n \le d_n + \text{trace } A$

Case ii) $d_1 + \text{trace } A \le \lambda_1 \le d_1 \le \ldots \le \lambda_n \le d_n$.

We first point out that the answer to the problem for $k = 1$ is negative in all other cases except for one trivial possibility. If $d_1 = \dots = d_n = d$, then $D = dI$ is a scalar matrix, and because A is of rank 1, d is an eigenvalue of $T = D + A$ of multiplicity at least $n-1$; hence all but possibly one eigenvalue of T are in the interval $[d,d] = \{d\}$. Now suppose that not all diagonal entries of D are equal and that the diagonal entries of A are allowed to have opposite sign. Without loss of generality, assume that D has the two distinct diagonal entries 0, 1. Consider first the case $n = 2$. Let $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a^2 & a^2 \\ -(a-1)^2 & -(a-1)^2 \end{bmatrix}$. Then $T = D + A$ has eigenvalues $\{a, a\}$ which may be arbitrarily far from the interval $[0, 1]$. We may achieve the same result for arbitrary n as follows: Suppose $1 \le i < j \le n$, $d_i = 0$, $d_j = 1$, and define the rank 1 matrix A by $a_{ii} = a_{ij} = a^2$, $a_{ji} = a_{jj} = -(a-1)^2$ and $a_{k\ell} = 0$ otherwise. Then the eigenvalues of $D + A$ are $\{a, a, d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{j-1}, d_{j+1}, \dots, d_n\}$. Thus, the answer to the problem for $k = 1$ is, in general, negative except for the cases discussed above.

The example just cited limits consideration for $k > 1$ to the case where all $a_{ii} \ge 0$ (or $a_{ii} \le 0$). That this sign restriction alone is not sufficient for the problem to have an affirmative answer is illustrated by the following example. Let $T = D + A$ where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Note that A has rank 2 and a_{ii} are all of the same sign. Matrix A has eigenvalues $\{0, 0, 3\}$ and $D + A$ has eigenvalues which are approximately $\{-0.214, 0.158, 2.956\}$, all of which are outside the interval $[-0.1, 0]$. The above A is not diagonalizable but by changing the $(1,1)$ entry of A to 1.01 we have a diagonalizable A for which the eigenvalues of $D + A$ (approximately $\{-0.211, 0.162, 2.960\}$) still lie outside $[-0.1, 0]$.

The proof of Theorem 8 [1] succeeds because a rank 1 matrix with diagonal entries all positive (or all negative) is diagonally similar to a symmetric matrix. We now extend Theorem 8 to diagonally symmetrizable A of rank $k > 1$.

Theorem 1. *Let* $0 \le k \le n-1$, *and assume that* $T \in M_k$ *may be written as* $T = D + A$ *where* $D = \operatorname{diag}(d_1, \ldots, d_n)$ *and* A *is a diagonally symmetrizable matrix of rank* $k = p + q$ *with* $p \ge 0$ *positive eigenvalues and* $q \ge 0$ *negative eigenvalues. If* $d_1 \le d_2 \le \cdots \le d_n$ *and* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ *where* $\lambda_i = \lambda_i(T)$, *then*

$$d_j \le \lambda_{j+q}(T) \le d_{j+k}, \qquad j = 1, \ldots, n-k.$$

In particular, all but k *eigenvalues of* T *are in the interval*

$$\left[\min_{i \in N} d_i, \max_{i \in N} d_i\right].$$

Proof. Let $\lambda_1^+, \ldots, \lambda_p^+$ and $\lambda_1^-, \ldots, \lambda_q^-$ be the p positive and q negative eigenvalues of A, and let E be a diagonal matrix for which $EAE^{-1} = S$ is symmetric. Then $EDE^{-1} = D$, so $D + S$ and T are similar and have the same eigenvalues. By the spectral theorem for real symmetric matrices,

$$S = \sum_{i=1}^{p} \lambda_i^+ u_i u_i^t + \sum_{i=1}^{q} \lambda_i^- v_i v_i^t$$

where $\{u_i\}_{i=1}^p$, $\{v_i\}_{i=1}^q$ are the corresponding orthonormal eigenvectors. By Corollary 4.3.3 and Theorem 4.3.6 of [2] we have (for $1 \le j \le n-k$)

$$\begin{aligned} d_j &\le \lambda_j(D + \sum_{i=1}^{p} \lambda_i^+ u_i u_i^t) \\ &\le \lambda_{j+q}(D + \sum_{i=1}^{p} \lambda_i^+ u_i u_i^t + \sum_{i=1}^{q} \lambda_i^- v_i v_i^t) \\ &= \lambda_{j+q}(D+S) \\ &\le \lambda_{j+q+p}(D + \sum_{i=1}^{q} \lambda_i^- v_i v_i^t) \\ &\le \lambda_{j+q+p}(D) = d_{j+k}. \end{aligned}$$

Thus $d_j \le \lambda_{j+q}(D+S) = \lambda_{j+q}(T) \le d_{j+k}$ for $j = 1, \ldots, n-k$. ∎

We next wish to correct two errors in the proof of Theorem 3 in [1]. The statement following the proof that any square matrix with rank $k \le n-1$ can be changed into a matrix of rank $k+1$ by changing precisely one element on the main diagonal is already incorrect for $n = 2$; for example take $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. As this was the idea used to demonstrate that $M_k \subset M_{k+1}$, that proof is invalid. Furthermore, in showing that $M_{n-1} = M_n$, the authors claim that an invertible $n \times n$ matrix can be changed to a matrix of rank $n-1$ by perturbing its (n,n) entry. However, the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible under any perturbation of the $(2,2)$ entry. The Gaussian elimination argument given in [1] can be used to show that <u>after row permutations</u> the (n,n) entry may be changed to reduce the rank to $n-1$.

We now restate and prove the theorem which shows that $\{M_k\}$ interpolates between the sets M_0, the set of all $n \times n$ diagonal matrices, and M_n, the set of all $n \times n$ matrices.

THEOREM 2. (Theorem 3 [1]) $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} = M_n$.

Proof. Let E_{jj} be the $n \times n$ matrix with the (j,j) entry equal to 1 and all other entries equal to 0. Let $1 \le k \le n-1$ and take any $n \times n$ matrix $A \in M_k$; without loss of generality assume that rank $A = k$. Choose a real number d which is not an eigenvalue of A. Then rank $A = k$, rank $(A-dI) = n$ and in the sequence of matrices A, $A-dE_{11}$, $A-dE_{11}-dE_{22}$, ..., $A-dI$ each consecutive pair of matrices differs in rank by at most 1. Therefore, for some integer $j \in N$, $\text{rank}(A - d \sum_{i=1}^{j} E_{ii}) = k+1$ and hence $A \in M_{k+1}$.

The $n \times n$ matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

given in the proof in [1] shows that $M_{k-1} \neq M_k$ for $k = 1, \ldots, n-1$.

To prove $M_{n-1} = M_n$, we show that they are both equal to the set of all real $n \times n$ matrices. Let T be any real $n \times n$ matrix. If d is not an eigenvalue of T, then rank $(T-dI) = n$; thus, $T \in M_n$. Now by choosing $d_1, \ldots, d_n$ sufficiently far apart we can separate the Geršgorin disks of $T_1 = T - \text{diag}(d_1,\ldots,d_n)$. Therefore T_1 has distinct eigenvalues. Let λ

be any eigenvalue of T_1; then $T_1 - \lambda I$ has rank $n-1$. Since $T_1 - \lambda I$ is a diagonal perturbation of T, $T \in M_{n-1}$. ■

References

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