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OBSERVATIONAL CONSTRAINTS ON HIGHER-DIMENSIONAL AND VARIABLE-Λ COSMOLOGIES

BY

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B. Sc., University of Waterloo, 1989
M. Sc., University of Waterloo, 1992

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN THE DEPARTMENT OF PHYSICS AND ASTRONOMY.

WE ACCEPT THIS DISSERTATION AS CONFORMING
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Abstract

Nonstandard cosmological models of two broad classes are examined: those in which there are more than four spacetime dimensions, and those in which there is a variable cosmological "constant" Λ. We test claims that a number of higher-dimensional models give rise to inflation. New constraints are placed on such models, and a number of them are ruled out. We then investigate the potential of variable-Λ theories to address the problem of the initial singularity. We consider a number of different phenomenological representations for this parameter, assessing their implications for the evolution of the cosmological scale factor as well as a range of observational data. In several cases we find nonsingular models which are compatible with observation.

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Acknowledgements

I would like to extend my gratitude to Fred Cooperstock for his steady encouragement, Valerio Faraoni and Luis de Menezes for advice with calculations, F. D. A. Hartwick for discussions, and the University of Victoria for financial support during my first three years as a doctoral student. I would also like to thank ex-Salty Seaman Dave Balam for his patience as an officemate, Pearl Dawn Duerksen for all her help, and the rest of the astrograds for comic relief.

Dedication

To Ayumi

Chapter 1

Introduction

The standard model of cosmology is extremely successful as it stands. Among other things it explains why the sky is dark at night, why galaxies recede according to Hubble's Law, why we are immersed in an isotropic bath of microwave radiation, and why the light elements exist in the ratios that they do.

However, there are still pieces missing from the puzzle. Some of these will be common to any theory that has to straddle the gulf between classical and quantum physics: the relic abundance problem, the cosmological constant problem, and the problem of explaining the observed asymmetry between matter and antimatter in the universe. Other defects of the standard model are classical in origin: the horizon and flatness problems, the problems of explaining structure formation and discerning the identity of the dark matter, and — perhaps most troubling of all — the existence of the big bang singularity, which in any field *other* than cosmology would be generally regarded as proof that the theory was fundamentally incomplete.

In this thesis, we will be concerned with three subsets of the issues just mentioned: those related to *inflation* (the horizon, flatness, relic abundance and structure formation problems), those related to the cosmological "constant" Λ, and those related to the initial singularity. In addressing them, we will consider two kinds of departures from the standard model: theories in which spacetime has *more than four dimensions*, and theories in which Λ is actually *variable*.

To begin with, we will assess the ability of higher-dimensional theories to give rise to inflation, thereby solving the first class of problems. This is not to say that inflation is the only way to accomplish this. The relic problem could, for instance, resolve itself within particle physics (eg. Dvali *et al* 1997), or might not actually arise at all if our understanding of grand unified theories is incorrect. Nonsingular "oscillating" models like those discussed later in this thesis can provide alternative explanations for the flatness (Landsberg *et al* 1992) and smoothness (Durrer and Laukenmann 1996) of the universe; and structure formation can be successfully attributed to topological defects instead of preinflationary quantum fluctuations (Brandenberger 1994). Inflation has defects of its own, too, including questions of initial conditions (Penrose 1989) and falsifiability (Brandenberger 1996). Nevertheless the inflationary scenario, or something very close to it, remains at present the simplest way to address all four problems simultaneously (Hu *et al* 1994, Liddle 1994).

In the second part of the thesis, we will investigate the impact of a variable cosmological term on the question of the initial singularity. The Λ-term has a venerable history in cosmology; excellent reviews are found in Zel'dovich (1968), Rindler (1977, §§ 9.2, 9.9) and Weinberg (1989). The "cosmological constant problem" is the problem of explaining why the effective value of this parameter is over a hundred orders of magnitude smaller than one would expect based on the energy density in the vacuum "zero-point" field (Weinberg 1989, 1996; Wesson 1991, Dolgov 1997). A proper explanation of this discrepancy will undoubtedly be of central importance in any future union of general relativity and quantum theory. The cosmological constant problem will not, however, be the primary focus of our investigation here, entering the discussion only indirectly. Rather than asking why Λ has the value it does, we wish to determine *how large it can be*, within empirical constraints.

It has long been known that the initial singularity can be averted with a sufficiently large cosmological *constant* (Robertson 1933, Harrison 1967). The required values of Λ have traditionally been thought incompatible with observation (Ellis 1984, Felten and Isaacman 1986, Börner and Ehlers 1988, Lahav *et al* 1991). It has however recently been suggested in the context of the cosmological constant problem that Λ might *vary* rather than being a constant of nature (eg., Adler 1982, Hawking 1984, Barr 1987, Weinberg 1989, Dolgov 1997). This opens up the possibility, so far largely unmentioned even by the authors just listed, of avoiding the initial singularity with a Λ-term which was large at early times but has subsequently decayed down to more modest levels. We will confirm that the prospects for nonsingular cosmology are greatly improved in a wide variety of variable-Λ scenarios.

Our two main themes are connected in many ways. Besides giving rise to inflation, extra dimensions are also frequently invoked to solve the cosmological constant problem, and indeed we will find on several occasions that they are helpful in this regard. Similarly, nonsingular oscillating models have been found in higher-dimensional theories by several authors (eg., Yoshimura 1984, Sato 1984, Tomimatsu and Ishihara 1986, Deruelle and Madore 1987). The cosmological constant, of course, provided the original mechanism for inflation, in de Sitter's model. More recently, inflationary theories have been proposed as solutions to the cosmological constant problem (eg. Tsamis and Woodard 1996, Brandenberger and Zhitnitsky 1997). And the possibility of an "inflationary equation of state" itself is intimately bound up with the question of singularity avoidance.

In the course of constructing and evaluating the viability of a number of nonstandard cosmological models, we will take care to stay within sight of *observational constraints* at all times. Despite the relative wealth of experimental data currently becoming available to cosmologists, it appears that many choose to focus on theories which are not and may never be falsifiable. We avoid this by concentrating on models which, though occasionally speculative, can also be tied to specific predictions. In some cases we find surprising new connections between theory and experiment — models, for example, in which data from the *Cosmic Background Explorer (COBE)* satellite can set useful limits on the dimensionality of spacetime. In other cases we find that seemingly well-established constraints (as, for instance, that a large cosmological term cannot simultaneously be reconciled with both the observed matter density and quasar redshifts) turn out to be a good deal less compelling than commonly thought.

The remainder of the thesis is organized as follows: In chapter 2 we show that a simple five-dimensional cosmology with conformal rescaling and a cosmological term can be ruled out on observational grounds. This is extended to higher dimensions in chapter 3, where we examine the more sophisticated models proposed by Berezin *et al* (1989), Cho (1990, 1992), Cho and Yoon (1993), and Yoon and Brill (1990). Broadly speaking we find that it is more difficult to build viable inflation into these models than their authors have appreciated. We then turn to variable-Λ cosmology, which has so far been studied comprehensively only as a means of addressing the cosmological constant problem. In chapter 4 we show that it can also provide a basis for singularity-free cosmology, and introduce the relevant equations and definitions. These are then applied to four different classes of variable-Λ models in chapters 5 through 8, and it is discovered that in several cases the big bang singularity can be removed without excessive fine-tuning, in a manner that does not come into conflict with any known observational constraints. We make some predictions about the present values of Λ and Ω (the matter density of the universe) that would be necessary to achieve this. Our conclusions are summarized in chapter 9.

Chapter 2

Cosmology in Five Dimensions

2.1 Kaluza-Klein Cosmology

Kaluza-Klein theory is the extension of Einstein's general theory of relativity to higher dimensions. The primary motivation for this is aesthetic, for it can be shown that Maxwell's laws of electrodynamics, together with the Klein-Gordon equation for a massless scalar field, are all contained in Einstein's field equations, if these are assumed to hold in a manifold of $4+1$, instead of $3+1$ dimensions (Kaluza 1921). When extended even further, the same procedure can in principle encompass all the interactions of the standard model of particle physics. This led in the 1980s to $D = 11$ supergravity (Duff *et al* 1986) and $D = 10$ superstrings (Green *et al* 1987). Today it forms the basis of the latest candidate "theory of everything," M-theory (Witten 1995). The experimental fact that we do not perceive the extra dimensions is usually explained by assuming that they are very small (Klein 1926) — on the order of the Planck length — although other mechanisms have been explored too (Schmutzer 1988, Wesson *et al* 1996). Kaluza-Klein theories have been reviewed by Bailin and Love (1987) and Collins *et al* (1989).

We are interested here in Kaluza-Klein *cosmology*; that is, those higher-dimensional models in which the universe is Robertson-Walker-like on four-dimensional spacetime sections. Solutions of this form were first found by Chodos and Detweiler (1980) and Freund (1982); reviews of the field may be found in Coley (1994) and Overduin and Wesson (1997a). Among other things, we want to see whether such models can help resolve the problems of the standard model (as mentioned in Chapter 1), such as the cosmological constant problem and the problems of relic abundance, flatness, smoothness and structure formation. The latter four in particular imply that we are interested in models which give rise to *inflation* in 4D spacetime (although, as noted in Chapter 1, there are also other ways to address these issues).

2.2 The Field Equations

Let us begin by concentrating on the simplest five-dimensional case, with metric:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\mu\nu} & \\ & \phi \end{pmatrix}, \tag{2.1}$$

where we have neglected off-diagonal (vector) terms, partly because these would effectively pick out preferred directions, in contradiction to our hypothesis of isotropy, and partly because we assume that in the context of cosmology, dynamics can be taken to be largely dominated by the influence of gravity and the scalar field. In all the calculations that follow, we will use the conventions $(A, B, ...) = (0, 1, 2, 3, 4)$, $(\mu, \nu, ...) = (0, 1, 2, 3)$, and $(i, j, ...) = (1, 2, 3)$. The hat refers to five-dimensional (5D) quantities. It is crucial to note that, although we work in five dimensions, we assume that all physical quantities are functions only of the standard 3+1 space-time variables, with *no* x^4*-dependence*. In other words we assume that the fifth dimension itself does not actually enter into the physics. This was called the "cylinder condition" by Kaluza (1921), who imposed it *a priori*. Klein (1926) showed that a plausible justification for it could be found in compactification of the extra dimension, and this is still standard practice in most Kaluza-Klein theories today (Bailin and Love 1987, Collins *et al* 1989, Overduin and Wesson 1997a).

We can define the 5D Christoffel symbol and Ricci tensor:

$$\hat{\Gamma}^{C}_{AB} = \frac{1}{2}\hat{g}^{CD}\left(\frac{\partial \hat{g}_{DB}}{\partial x^{A}} + \frac{\partial \hat{g}_{DA}}{\partial x^{B}} - \frac{\partial \hat{g}_{AB}}{\partial x^{D}}\right) \tag{2.2}$$

$$\hat{R}_{AB} = (\hat{\Gamma}^{C}_{AB})_{,C} - (\hat{\Gamma}^{C}_{AC})_{,B} + \hat{\Gamma}^{C}_{AB}\hat{\Gamma}^{D}_{CD} - \hat{\Gamma}^{C}_{AD}\hat{\Gamma}^{D}_{BC}, \tag{2.3}$$

exactly as in Einstein's theory, simply replacing all the usual (4D) quantities with hatted ones. After some work, we find (imposing the cylinder condition $\partial/\partial x^4 = 0$) that:

$$\begin{aligned}
\hat{\Gamma}^{\mu}_{\nu\lambda} &= \Gamma^{\mu}_{\nu\lambda} \\
\hat{\Gamma}^{4}_{\nu 4} &= \frac{1}{2}\partial_\nu(\ln\phi) \\
\hat{\Gamma}^{\mu}_{44} &= -\frac{1}{2}\partial^\mu\phi \\
\hat{\Gamma}^{\mu}_{\nu 4} &= \hat{\Gamma}^{4}_{\nu\lambda} \;=\; \hat{\Gamma}^{4}_{44} \;=\; 0 \\
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\frac{\nabla_\mu\nabla_\nu\phi}{\phi} + \frac{1}{4}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2} \\
\hat{R}_{\mu 4} &= 0 \\
\hat{R}_{44} &= -\frac{1}{2}\nabla^\mu\nabla_\mu\phi + \frac{1}{4}\frac{\nabla^\mu\phi\nabla_\mu\phi}{\phi}.
\end{aligned} \tag{2.4}$$

The 5D Ricci curvature scalar is found as usual by contracting $\hat{R}_{AB}$ with $\hat{g}^{AB}$:

$$\hat{R} = R + \frac{2}{\phi}\hat{R}_{44}. \tag{2.5}$$

Finally, the 5D Einstein tensor $\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}$, turns out to be:

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - \frac{g_{\mu\nu}\hat{R}_{44}}{\phi} - \frac{1}{2}\frac{\nabla_\mu\nabla_\nu\phi}{\phi} + \frac{1}{4}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2}. \tag{2.6}$$

The next step is to interpret these expressions physically; ie., in four dimensions.

2.3 Radiation-Like Equation of State

Let us make the economical assumption that the universe in higher dimensions (here five) is *empty*. In other words, we begin with the *vacuum* field equations in 5D, and sees what happens in 4D as a result. In the 5D vacuum, both $\hat{G}_{AB}$ and $\hat{R}_{AB}$ vanish; therefore, $\hat{G}_{\mu\nu} = 0$ and $\hat{R}_{44} = 0$. Let us call these the "$\mu\nu$" and "44" components of the 5D field equations, respectively. In eq. (2.6), they imply:

$$G_{\mu\nu} = \frac{1}{2\phi}\left(\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi}\right). \tag{2.7}$$

From Einstein's field equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we therefore infer the existence of an *effective 4D energy momentum tensor* (even though the 5D universe is empty), given by:

$$T_{\mu\nu} = \frac{1}{16\pi G\phi}\left(\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi}\right). \tag{2.8}$$

To understand what kind of matter $T_{\mu\nu}$ represents, we have to choose a particular set of coordinates. We make the traditional cosmological assumptions of homogeneity and isotropy; this means using a Robertson-Walker (RW) metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{2.9}$$

where $a(t)$ is the scale factor. The convenient transformation $d\chi \equiv dr/\sqrt{1-kr^2}$ allows us to write $g_{\mu\nu}$ in the form:

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 f^2 & \\ & & & a^2 f^2 \sin^2\theta \end{pmatrix}, \tag{2.10}$$

where f is defined by:

$$f = r(\chi) = \begin{cases} \sin\chi & \text{if } k=+1 \\ \chi & \text{if } k=0 \\ \sinh\chi & \text{if } k=-1. \end{cases} \tag{2.11}$$

The covariant derivatives $\nabla_\mu\nabla_\nu\phi = \partial_\mu(\partial_\nu\phi) - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi$ in eq. (2.8) can be computed in terms of the Christoffel symbols for this metric:

$$\begin{aligned} \Gamma^0_{00} &= \Gamma^0_{0i} = \Gamma^0_{ij} = 0 \\ \Gamma^0_{11} &= a\dot{a} \\ \Gamma^0_{22} &= a\dot{a}f^2 \\ \Gamma^0_{33} &= a\dot{a}f^2\sin^2\theta, \end{aligned} \tag{2.12}$$

where a dot means $\partial/\partial x^0$ (note that isotropy implies $\partial/\partial x^i = 0$). The results are:

$$\begin{aligned} T_{00} &= \frac{1}{16\pi}\left(\frac{\ddot\phi}{\phi} - \frac{\dot\phi^2}{2\phi^2}\right) \\ T_{0i} &= T_{i0} = 0 \\ T_{ij} &= -\frac{1}{16\pi}\left(\frac{\dot a}{a}\frac{\dot\phi}{\phi}\right) g_{ij}. \end{aligned} \tag{2.13}$$

These can be identified with the elements of the energy-momentum tensor of a perfect fluid with pressure p, density ρ, and comoving fluid velocity $u^\mu = \delta^{\mu 0}$:

$$T^{PF}_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu. \tag{2.14}$$

Since $u_\mu = g_{\mu 0}$ one obtains the relations $T^{PF}_{00} = \rho$, $T^{PF}_{0i} = T^{PF}_{i0} = 0$, and $T^{PF}_{ij} = pg_{ij}$. Therefore the matter "induced" by the presence of the extra dimension x^4 behaves like a perfect fluid of pressure and density given by:

$$\begin{aligned} \rho &= \frac{1}{16\pi G}\left(\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{2\phi^2}\right) \\ p &= -\frac{1}{16\pi G}\left(\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}\right). \end{aligned} \tag{2.15}$$

To find the equation of state obtained by this fluid, we make use of the "44" component of the 5D Einstein equations ($\hat{R}_{44} = 0$), which, with the definition of eq. (2.4), gives:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{2}\frac{\dot{\phi}^2}{\phi} = 0. \tag{2.16}$$

Together with eqs. (2.15) this differential equation reveals that:

$$p = \frac{\rho}{3}. \tag{2.17}$$

This simple model, then, can only be useful — if it is useful at all — to describe radiation-dominated conditions, such as those thought to have prevailed in the early universe. Others have reached the same conclusion (Mann and Vincent 1985).

2.4 Relationship to Brans-Dicke Theory

Let us see what constraints can be placed on the model so far. The action of 5D Kaluza-Klein theory is exactly the same as the one that leads to Einstein's equations in four dimensions, except that all quantities take hats and the integration is over one extra dimension $y \equiv x^4$:

$$S = -\frac{1}{16\pi\hat{G}}\int \hat{R}\sqrt{-\hat{g}}\, d^4x\, dy, \tag{2.18}$$

where $\hat{g}$ is the determinant of $\hat{g}_{AB}$. Eq. (2.1) tells us that $\hat{g} = g\phi$, and eq. (2.5) (with the "44" equation again) implies that $\hat{R} = R$. Therefore eq. (2.18) reduces to:

$$S = -\frac{1}{16\pi G_0}\int R\psi\sqrt{-g}\,d^4x, \tag{2.19}$$

where we have pulled the dy out of the integral, absorbed it into a new constant $G_0 \equiv \hat{G}/\int dy$, and defined a new scalar $\psi \equiv \sqrt{\phi}$ for convenience.

Eq. (2.19) is a special case ($\omega = 0$) of the well-known *Brans-Dicke* action (Brans and Dicke 1961):

$$S_{BD} = -\frac{1}{16\pi G_0}\int\left[R\psi - \omega\left(\frac{\partial^\mu\psi\partial_\mu\psi}{\psi}\right)\right]\sqrt{-g}\,d^4x, \tag{2.20}$$

where ω is the Brans-Dicke parameter, and G_0 is the present value of Newton's gravitational "constant" (which varies with time in this theory). Brans-Dicke theory reduces to standard general relativity in the limit as $\omega \to \infty$, and Viking radar ranging observations to Mars currently require $\omega > 500$ (Will 1981), so our simple model cannot describe the universe at present. (This could have been expected anyway since it is thought by most cosmologists that we live in matter-, not radiation-dominated conditions.) One can however evade the lower bound on ω by adding a nonzero potential $V(\phi)$ to the above action, as in extended inflation (La *et al* 1989) and other theories (Wetterich 1988a, Soleng 1991), or by allowing the Brans-Dicke parameter ω to *vary* as a function of ϕ, as in hyperextended (Steinhardt and Accetta 1990) and other inflationary models (McDonald 1993). Could our model apply to the earlier radiation-dominated era?

The best constraint on the theory at early times comes from looking at the variation of G with time. The ratio $\dot{G}/G$ has been experimentally determined to be very nearly constant; if it has been varying with time as $t^{-\chi}$ since the time of primordial nucleosynthesis, then χ lies in the range $-0.009 < \chi < 0.008$ (Accetta *et al* 1990). In Brans-Dicke theory, $G(t) \propto 1/\psi(t)$, so we have:

$$\frac{\dot{G}}{G} = \sqrt{\phi}\frac{d}{dt}\frac{1}{\sqrt{\phi}} = -\frac{1}{2}\frac{\dot{\phi}}{\phi}. \tag{2.21}$$

Einstein's field equations with a RW metric produce this well-known differential equation

(Weinberg 1972, p. 472) for the scale factor $a(t)$:

$$\dot{a}^2 + k = \frac{8\pi}{3}\rho a^2. \tag{2.22}$$

If we insert into eq. (2.22) the fact that $\rho = 3p$ where p is defined by eq. (2.15), and take $k = 0$ for simplicity, then we find that:

$$\frac{\dot{\phi}}{\phi} = -2\frac{\dot{a}}{a}. \tag{2.23}$$

Since $a(t) \propto t^{1/2}$ under radiation-dominated conditions (if $k = 0$), we have:

$$\frac{\dot{G}}{G} = \frac{1}{t}. \tag{2.24}$$

Therefore our model predicts $\chi = 1$, which is ruled out by experiment.

The simple $D = 5$ Kaluza-Klein model is thus of limited usefulness, even during the radiation-dominated era. We would like to generalize the model, to see if other equations of state for the induced matter are possible. One option might be to add in explicit *higher-dimensional matter fields*, as is done in most Kaluza-Klein theories in order to induce "spontaneous compactification" of the extra dimensions (Cremmer and Scherk 1976, Bailin and Love 1987). Another idea might be to relax the cylinder condition, introducing a limited dependence on the fifth coordinate; this can lead to a wide variety of equations of state without upsetting the experimental successes of 4D general relativity (Overduin and Wesson 1997a). We would prefer here, however, to stay with traditional thinking on the nature of extra dimensions. A third possibility is to carry out a *conformal* (or Weyl) *rescaling* of the 4D metric.

2.5 Conformal Rescaling of the Metric

A number of authors, going back to Pauli, have commented on the "conformal ambiguity" in the process of dimensional reduction (ie., in choosing how to define the physical, 4D metric in terms of the higher-dimensional one). In our metric, eq. (2.1) above, we implicitly set $g_{\mu\nu} = \hat{g}_{\mu\nu}$. It would have been equally valid, however, to consider a *conformally rescaled* 4D metric:

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \tag{2.25}$$

where $\Omega^2 > 0$ is the rescaling factor. Sokołowski and Golda (1987) and Cho (1987) have argued that without such a rescaling any Kaluza-Klein cosmology with more than five dimensions is unstable, because the kinetic energies of the scalar fields could be negative and unbounded from below. Cho (1992) claims in addition that the rescaling is necessary if one wants to identify the metric responsible for gravitation as that of a massless spin-2 graviton.

Many aspects of effective (4D) physics are not affected by the conformal rescaling. For instance, it does not affect the gravitational energy of the 5D analogue of a black hole, the "Kaluza-Klein soliton" (Bombelli *et al* 1987), or the form of the matter-free Brans-Dicke Lagrangian (Cho 1992). In general, though, physics is dependent on choice of conformal frame. When matter fields are included in the Brans-Dicke Lagrangian, for example, their couplings to the metric depend on the rescaling factor. Such nonstandard couplings constitute violations of the weak equivalence principle that could manifest themselves as a "fifth force" of nature. These effects would be small, however, and at any rate, the coupling of ordinary matter to the metric may be unimportant compared to the coupling of the *ϕ-field*, if in fact ϕ does comprise the bulk of the dark matter in the universe, as discussed by Damour *et al* (1990) and Cho (1990, 1992). [The earliest suggestion we have found of scalar fields as dark matter candidates was made by Ratra and Peebles (1988).]

A recent review of the conformal question is found in Magnano and Sokołowski (1994). For our purposes, the important issue is whether or not it will lead, in general, to a different equation of state. The question arises: is there a preferred choice for the rescaling factor? And if so, what is it? Sokołowski (1989) has proved that there is a unique choice of Ω^2 which will guarantee positivity of 4D kinetic energy for any number of extra dimensions. The factor depends in general on the number of extra dimensions and the Ricci curvature of the compact extra-dimensional space. For one extra dimension, which gives rise to one scalar field ϕ, it takes the simple form:

$$\Omega^2 = \sqrt{\phi}. \tag{2.26}$$

This is the same as the factor singled out by Cho (1987).

We therefore transform our 4D metric according to:

$$g_{\mu\nu} \to g'_{\mu\nu} = \sqrt{\phi}\, g_{\mu\nu}. \tag{2.27}$$

The old (unrescaled) metric $g_{\mu\nu} = \hat{g}_{\mu\nu}$ is often referred to in the literature as the "Jordan metric." It is to be emphasized that the *physical* metric (ie., the one responsible for Einstein's gravitation), often called the "Pauli metric," is the rescaled metric $g'_{\mu\nu}$.

The Lagrangian density corresponding to the action of our theory, eq. (2.19), is:

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi G} R \sqrt{\phi}. \tag{2.28}$$

The conformal transformation will change this Lagrangian in two ways. Firstly, $\sqrt{-g} \to \sqrt{-g'} = \phi\sqrt{-g}$. Secondly, the Ricci scalar transforms (Wald 1984, p. 446) as:

$$R \to R' = \Omega^{-2}\left[R + 6\frac{\Box\Omega}{\Omega}\right]. \tag{2.29}$$

With $\Omega^2 = \sqrt{\phi}$, one can show (using the "44" field equation) that this means:

$$R = \sqrt{\phi}\, R' + \frac{3}{8}\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi^2}. \tag{2.30}$$

Noting that $\nabla_\mu\phi\nabla^\mu\phi = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = (\sqrt{\phi}g'^{\mu\nu})\nabla_\mu\phi\nabla_\nu\phi = \sqrt{\phi}\nabla'_\mu\phi\nabla'^\mu\phi$, we can then rewrite the Lagrangian density, eq. (2.28), in terms of the transformed quantities:

$$\mathcal{L} = -\frac{\sqrt{-g'}}{16\pi G}\left(R' + \frac{3}{8}\frac{\nabla'_\mu\phi\nabla'^\mu\phi}{\phi^2}\right). \tag{2.31}$$

Now, this Lagrangian does not have the correct form for the kinetic energy of the scalar field ϕ, but that is easily remedied by *redefining* the scalar field as follows:

$$\sigma \equiv \frac{\sqrt{3}}{2}\ln\phi. \tag{2.32}$$

This new scalar field σ is often termed the "dilaton." The Lagrangian density becomes:

$$\mathcal{L} = -\frac{\sqrt{-g'}}{16\pi G}\left(R' + \frac{1}{2}\nabla'_\mu\sigma\nabla'^\mu\sigma\right). \tag{2.33}$$

This has the correct form for the kinetic energy of the scalar field.

2.6 "Stiff" Equation of State

We are interested in finding the equation of state that governs this new, conformally transformed situation. Pressure and density can be found in terms of the field equations and the equation of energy conservation. The first has already been given; it is eq. (2.22), which reads (for $k = 0$):

$$\rho = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 . \tag{2.34}$$

Energy conservation for perfect fluids can be expressed (Weinberg 1972, p. 472):

$$\frac{\partial}{\partial a}\left(\rho a^3\right) = -3pa^2. \tag{2.35}$$

Inserting eq. (2.34), we get:

$$p = -\frac{1}{4\pi G}\left[\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2\right] . \tag{2.36}$$

Density and pressure are thus known in terms of the ratios $\dot{a}/a$ and $\ddot{a}/a$. It remains to find the time-dependence of the scale factor. In the radiation-dominated case this was proportional to $t^{1/2}$, but that is no longer necessarily the case.

With the conformal factor, the metric (2.9) can be written:

$$ds^2 \rightarrow ds'^2 = \sqrt{\phi}\left(-dt^2 + a^2 dx^2\right) , \tag{2.37}$$

where x is shorthand for the spatial part of the metric. From eq. (2.23), we know that $\phi \propto a^{-2}$, which means we can write this metric as:

$$ds'^2 = \kappa a^{-1}\left(-dt^2 + a^2 dx^2\right) , \tag{2.38}$$

where κ is some constant. To cast this back into RW form, ($ds'^2 = -dt'^2 + a'^2 dx'^2$), we need to *rescale* the quantities dx, dt, and a as follows:

$$\begin{aligned} dx &\rightarrow dx' = dx \\ dt &\rightarrow dt' = \sqrt{\kappa a^{-1}}\, dt \\ a &\rightarrow a' = \sqrt{\kappa a}. \end{aligned} \tag{2.39}$$

(Only two out of the three need to be rescaled; we have chosen dt and a.) Since $a \propto t^{1/2}$, we have $dt' \propto t^{-1/4}dt$, so that $t' \propto t^{3/4}$. Also, since $a' \propto a^{1/2}$, we have $a' \propto t^{1/4}$. Therefore $a' \propto t'^{1/3}$. In other words, the conformally rescaled scale factor varies as time to the power *one-third*, rather than one-half as before. This implies:

$$\frac{\dot{a}'}{a'} = \frac{1}{3t'} \quad , \quad \frac{\ddot{a}'}{a'} = -\frac{2}{9t'^2}. \tag{2.40}$$

These results into eqs. (2.34) and (2.36) above then yield:

$$\rho = p = \frac{1}{24\pi G t'^2}. \tag{2.41}$$

The conformal transformation has thus led to a new equation of state, as desired, but the change is not a welcome one — it is doubtful that this "stiff" equation of state is relevant to *any* epoch in the history of the universe (see § 4.7 for discussion). There is only one way, short of adding higher-dimensional matter or relaxing the cylinder condition, to obtain a more realistic equation of state, and that is to graft a higher-dimensional cosmological term $\hat{\Lambda}$ onto the theory.

2.7 Addition of a Cosmological Term

A positive cosmological term behaves like a repulsive force between the elements of the cosmological fluid, acting against their mutual gravitational attraction and thus altering the prevailing equation of state. It has to be added to the theory carefully, however, beginning again at the stage of dimensional reduction. We assume as before that the five-dimensional universe is empty ($\hat{T}_{AB} = 0$), so eqs. (2.4), (2.5), and (2.6) for $\hat{\Gamma}^C_{AB}$, $\hat{R}_{AB}$, $\hat{R}$, and $\hat{G}_{\mu\nu}$ are not altered. However, eq. (2.7) no longer holds because the 5D vacuum field equations no longer imply that $\hat{G}_{\mu\nu}$ and $\hat{R}_{44}$ vanish. Instead the "$\mu\nu$" components now read $\hat{G}_{\mu\nu} + \hat{\Lambda}\hat{g}_{\mu\nu} = 0$ or, with eq. (2.6):

$$G_{\mu\nu} = \frac{1}{2\phi}\left(\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi}\right) + \left(\frac{\hat{R}_{44}}{\phi} - \hat{\Lambda}\right) g_{\mu\nu}. \tag{2.42}$$

To find the new expression for $\hat{R}_{44}$, we begin with the full 5D field equations:

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} + \hat{\Lambda}\hat{g}_{AB} = 0. \tag{2.43}$$

Contracting with $\hat{g}^{AB}$ gives:

$$\hat{R} = \frac{10}{3}\hat{\Lambda}. \tag{2.44}$$

This expression, back into eq. (2.43), gives:

$$\hat{R}_{AB} = \frac{2}{3}\hat{\Lambda}\hat{g}_{AB}. \tag{2.45}$$

Therefore the "44" field equation is now:

$$\hat{R}_{44} = \frac{2}{3}\hat{\Lambda}\phi. \tag{2.46}$$

Inserting this result into eq. (2.42) gives the replacement for eq. (2.7):

$$G_{\mu\nu} + \frac{1}{3}\hat{\Lambda}g_{\mu\nu} = \frac{1}{2\phi}\left(\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi}\right). \tag{2.47}$$

By comparing this with the field equations in four dimensions, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$, we can see that the effective 4D energy-momentum tensor $T_{\mu\nu}$ is just as before, eq. (2.8). In addition, however, there is now an *effective 4D cosmological term* Λ, given by:

$$\Lambda = \frac{1}{3}\hat{\Lambda}. \tag{2.48}$$

To find the new equation of state, we make use of the "44" component of the 5D field equations, as before. Because $T_{\mu\nu}$ is unchanged, eqs. (2.9) through (2.15) are still valid. But eq. (2.16) must change to reflect the fact that $\hat{R}_{44}$ no longer vanishes. The new "44" equation, eq. (2.46), with the definition (2.4), gives:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{2}\frac{\dot{\phi}^2}{\phi} = 4\Lambda\phi. \tag{2.49}$$

This result, together with the expressions for ρ and p in eq. (2.15), gives:

$$p = \frac{\rho}{3} - \frac{\Lambda}{12\pi G}. \tag{2.50}$$

This equation of state appears more general. Besides radiation-dominated conditions ($\Lambda \ll \rho$), one could attempt to model matter-dominated ones ($p \sim 0$) with $\Lambda \sim 4\pi G\rho$, and *inflationary* ones ($p < -\rho/3$) with $\Lambda > 8\pi G\rho$ (see § 4.7 for discussion). However, assuming that Λ = constant, differentiation gives $dp/d\rho = 1/3$, which implies that the fluid under consideration remains radiation-like in all cases.

As in § 2.4, we consider also the action of the theory. In the presence of a cosmological term, eq. (2.18) is modified (Landau and Lifshitz 1975, p. 358) to read:

$$S = -\frac{1}{16\pi\hat{G}} \int \left(\hat{R} - 2\hat{\Lambda}\right) \sqrt{-\hat{g}}\, d^4x\, dy, \tag{2.51}$$

From eq. (2.1), $\hat{g} = g\phi$, as before. However, eq. (2.5), with the new "44" eq. (2.46), now no longer gives $\hat{R} = R$. Instead we get (with $\hat{\Lambda} = 3\Lambda$):

$$\hat{R} = R + \frac{2}{\phi}\left(\frac{2}{3}\hat{\Lambda}\phi\right) = R + 4\Lambda. \tag{2.52}$$

If we then define the 4D gravitational constant by $G_0 \equiv \hat{G}/\int dy = \hat{G}/2\pi r_4$, where r_4 is the radius of the compact extra dimension, then the action in 4D reads:

$$S = -\frac{1}{16\pi G_0} \int (R - 2\Lambda) \sqrt{\phi}\sqrt{-g}\, d^4x. \tag{2.53}$$

As we found before in eq. (2.19), the 4D action has exactly the same form as the 5D one, except for the factor of $\sqrt{\phi}$. It is, as before, a $\omega = 0$ version of the Brans-Dicke action, eq. (2.20), except that the Ricci curvature R is modified by the 2Λ-term. There is no kinetic energy term for the scalar field.

2.8 Conformal Rescaling of the Metric

To get a more interesting theory, we again introduce the conformal rescaling, eq. (2.27). As before, this means $\sqrt{-g} \to \sqrt{-g'} = \phi\sqrt{-g}$ and $R \to R' = \Omega^{-2}[R + 6\Box\Omega/\Omega]$, where $\Omega^2 = \sqrt{\phi}$. However, eq. (2.30) (which depends on the "44" equation of state) becomes:

$$R = \sqrt{\phi}\, R' + \frac{3}{8}\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi^2} + 6\Lambda. \tag{2.54}$$

The Lagrangian density corresponding to the action (2.53) is given by:

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi G}\,(R - 2\Lambda)\sqrt{\phi}, \tag{2.55}$$

where we have put $G_0 \to G$ for convenience. In terms of the conformally rescaled quantities g' and R', this reads:

$$\mathcal{L} = -\frac{\sqrt{-g'}}{16\pi G}\left(R' + \frac{3}{8}\frac{\nabla'_\mu\phi\nabla'^\mu\phi}{\phi^2} + \frac{4\Lambda}{\sqrt{\phi}}\right). \tag{2.56}$$

This equation is the $\Lambda \neq 0$ version of eq. (2.31). As before, the kinetic energy of the scalar field ϕ is not in standard form. Re-introducing the dilaton σ via $\sigma \equiv \frac{\sqrt{3}}{2}\ln\phi$, we find that the Lagrangian density becomes:

$$\mathcal{L} = -\frac{\sqrt{-g'}}{16\pi G}\left(R' + \frac{1}{2}\nabla'_\mu\sigma\nabla'^\mu\sigma + 4\Lambda e^{-\sigma/\sqrt{3}}\right), \tag{2.57}$$

which has the correct form for kinetic energy, and is the $\Lambda \neq 0$ version of eq. (2.33).

We can go further and compare this expression with the *general form* of the Lagrangian density for a minimally coupled scalar field ϕ (Madsen 1988):

$$\mathcal{L} = -\sqrt{-g}\left[\frac{R}{16\pi G} + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right]. \tag{2.58}$$

Comparison reveals that if we make one trivial additional rescaling:

$$\bar{\sigma} \equiv \frac{\sigma}{\sqrt{16\pi G}}, \tag{2.59}$$

then eq. (2.57) becomes:

$$\mathcal{L} = -\sqrt{-g'}\left[\frac{R'}{16\pi G} + \frac{1}{2}\nabla'_\mu\bar{\sigma}\nabla'^\mu\bar{\sigma} + \left(\frac{\Lambda}{4\pi G}\right)\exp\left(-\sqrt{\frac{16\pi}{3}}\frac{\bar{\sigma}}{m_{pl}}\right)\right], \tag{2.60}$$

where we have chosen units so that $\hbar = c = 1$, so that $m_{pl} = G^{-1/2}$ (m_{pl} being the Planck mass). By introducing a cosmological term into the theory, we have endowed the scalar field with an *effective potential:*

$$V_\Lambda(\phi) = -\left(\frac{\Lambda}{4\pi G}\right)\exp\left(-\sqrt{\frac{16\pi}{3}}\frac{\phi}{m_{pl}}\right). \tag{2.61}$$

In fact, apart from giving rise to this potential, the cosmological term does not appear in the theory. This might be explored further as a means to address the *cosmological constant problem*, as mentioned in Chapter 1. Higher-dimensional models appear to offer several ways to approach this problem (§§ 3.1.2, 3.2.6).

2.9 Inflationary Equation of State

To find the equation of state for the theory, we can appeal to work that has been done already by Starkovich and Cooperstock (1992). These authors assumed a cosmological equation of state of the form:

$$p = (\gamma - 1)\rho, \tag{2.62}$$

where the value of γ is positive and constant in any given epoch, but can change discontinuously between epochs; eg., at the transition from radiation-dominated ($\gamma = 4/3$) to matter-dominated ($\gamma = 1$) conditions. Inflation (ie., $\ddot{a} > 0$) arises for any value of γ between 0 and 2/3. (We will return to this equation of state in later chapters; see § 4.7 for discussion.) Starkovich and Cooperstock (1992) found that this equation of state implied the existence of a minimally coupled scalar field ϕ with a potential of the following form:

$$V_{sc}(\phi) = C\left(e^{-\alpha\phi} + \beta e^{\alpha\phi}\right)^{1/b}, \tag{2.63}$$

where:

$$\alpha \equiv \sqrt{24\pi\gamma}\left(\frac{b}{m_{pl}}\right) \quad , \quad b \equiv \frac{3\gamma - 2}{6\gamma}, \tag{2.64}$$

and where C and β are parameters that can, like γ, take different constant values in each era of cosmic history.

Our potential, V_Λ, is a special case of V_{sc} with $\beta = 0$ and $C = -\Lambda/4\pi G$. By comparing eq. (2.61) to eqs. (2.63) and (2.64), we can see immediately that $\gamma_\Lambda = 2/9$. This value lies between 0 and 2/3, which means that the theory describes *inflationary conditions*, such as those often thought to precede the radiation-dominated era.

Inflationary models can be tested by considering the spectrum of density perturbations that they produce; these will show up as anisotropies in the cosmic microwave background (CMB) and can be constrained by observations such as those of the *Cosmic Background Explorer (COBE)* or the radio telescopes at Tenerife. For example, perturbations of the density $\rho(\boldsymbol{x}, t)$ can be expanded as a series of Fourier modes $\rho_{\boldsymbol{k}}(t)$ of wave vector $\boldsymbol{k}$. The "density contrast" $\delta \equiv (\delta\rho_{\boldsymbol{k}})/\rho_{\boldsymbol{k}}$ obeys $\delta^2 \propto k^{n-1}$ at the moment when the perturbations enter the horizon after inflation. The parameter n is known as the "spectral index" of the density perturbations.

Our potential $V_A(\phi)$ turns out to belong to the class of "power-law inflationary" (PLI) potentials, whose general form can be written (Lucchin and Matarrese 1985):

$$V_p(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{pl}}\right), \tag{2.65}$$

where p is related to the expansion of the scale factor via $a(t) \propto t^p$ (hence the term "power-law"), and must be greater than unity for inflation. By comparing eqs. (2.61) and (2.65) we see that, in our theory, $p_A = 2/3\gamma_A = 3$. The PLI parameter p is simply related to the spectral index of density perturbations (Liddle and Lyth 1993) by:

$$n = 1 - \frac{2}{p}, \tag{2.66}$$

which means that in our theory we have $n_A = 1/3$. This is far from a flat spectrum of density perturbations ($n = 1$). In fact, current 1σ lower limits on n from combined *COBE* and Tenerife observations (Hancock *et al* 1994, Bennett *et al* 1996) are:

$$n_{obs} \geq 0.9. \tag{2.67}$$

Our model, based on 5D Kaluza-Klein theory with a cosmological term and a conformal transformation, is thus ruled out by observation.

It is of some interest to determine whether the *general* Starkovich-Cooperstock potential, eq. (2.63), can be constrained in the same manner. We have carried out a preliminary study of this, but it lies somewhat out of the main line of development of the thesis, so we present the results separately in Appendix A. The main results are as follows: while a potential of this form appears to be compatible with the observational limits on n, the density contrast itself is several orders of magnitude larger than that observed by *COBE*, and the theory also violates certain constraints on the "energy at the end of inflation" (Liddle and Lyth 1993). We conclude that the model of Starkovich and Cooperstock (1992) is probably not viable as a theory of inflation. [We cannot make any definite statements about the modified version of this theory proposed by Bayin *et al* (1994).] Prospects for improving the model are briefly discussed at the end of Appendix A.

Chapter 3

Cosmology in Higher Dimensions

3.1 Extension to Higher Dimensions

3.1.1 Model of Cho

It is natural to wonder whether the negative result of the last chapter continues to hold for generalizations of our simple model to *more* than five dimensions. Do they give rise to inflation? If so, is it compatible with observations from *COBE* and *Tenerife*? Cho's new "unified cosmology" (Cho 1990, 1992) is essentially an extension to $(4+d)$ dimensions of our simple 5D model, and it is claimed to solve the same problems that inflation does (Cho 1990), so it makes a perfect candidate for analysis[1]. In this theory the metric of eq. (2.1) is replaced by:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\mu\nu} & \\ & \phi_{ab} \end{pmatrix}, \tag{3.1}$$

where the indices (a, b) run over $4, 5, \ldots, (4+d)$, ϕ_{ab} is a d-dimensional Riemannian metric with all its dimensions spacelike, and hatted quantities are now understood to be $(4+d)$-dimensional. We have dropped Cho's off-diagonal gauge field B_μ in eq. (3.1), both for simplicity's sake and because we expect that during inflation the universe is dominated by the scalar field. This field is introduced via:

$$\phi \equiv |\det(\phi_{ab})|. \tag{3.2}$$

[1]§§ 3.1 and 3.2 are based in part on work published in Faraoni, Cooperstock and Overduin (1995) and presented by J. M. Overduin at the 6th Canadian Conference on General Relativity and Relativistic Astrophysics in Fredericton (Faraoni, Cooperstock and Overduin 1997).

Then, assuming (as we did in our 5D model) vacuum general relativity in the higher-dimensional universe, we have the Lagrangian density:

$$\mathcal{L} = -\frac{m_{pl}^2}{16\pi}\left(\hat{R}+\hat{\Lambda}\right)\sqrt{-\det(\hat{g}_{AB})}, \tag{3.3}$$

where $\hat{R}$ and $\hat{\Lambda}$ are the Ricci curvature and cosmological term of the $(4+d)$-dimensional space respectively (Here $\hat{\Lambda}$ is defined slightly differently than the one in our 5D theory, the two being related by $\hat{\Lambda}_{Cho} = -2\hat{\Lambda}_{5D}$). Cho then makes the same conformal transformation as in eq. (2.27), and defines the dilaton via:

$$\sigma \equiv \frac{1}{2}\sqrt{\frac{d+2}{d}}\ln\phi. \tag{3.4}$$

The Lagrangian density in terms of 4D quantities then turns out to be:

$$\mathcal{L} = -\frac{m_{pl}^2}{16\pi}\left[R+\frac{1}{2}\nabla_\mu\sigma\nabla^\mu\sigma + V_C(\sigma) + \lambda\left(|\det(\rho_{ab})|-1\right)\right]\sqrt{-g}, \tag{3.5}$$

where $\rho_{ab} \equiv \phi^{-1/d}\phi_{ab}$, λ is a Lagrange multiplier [to enforce $|\det(\rho_{ab})| = 1$] and:

$$V_C(\sigma) = R_\rho \exp\left(-\sqrt{\frac{d+2}{d}}\sigma\right) + \hat{\Lambda}\exp\left(-\sqrt{\frac{d}{d+2}}\sigma\right) \tag{3.6}$$

is Cho's potential, with R_ρ corresponding to the Ricci curvature of ρ_{ab}.

3.1.2 Renormalization of the Dilaton

This Lagrangian has the required kinetic term for σ. But — as we noticed in our 5D theory — it is still *not* in the proper form for the Lagrangian of a massless scalar field coupled minimally to general relativity, eq. (2.58). To make this interpretation, one has to *renormalize* the dilaton according to:

$$\bar{\sigma} \equiv \frac{\sigma}{\sqrt{16\pi G}}. \tag{3.7}$$

We will find shortly that this seemingly modest point is crucial because it modifies the arguments of the exponentials in the potential, eq. (3.6). Introducing several more renormalized quantities:

$$\bar{R} \equiv \frac{R_\rho}{16\pi G} \quad , \quad \bar{\Lambda} \equiv \frac{\hat{\Lambda}}{16\pi G} \quad , \quad \bar{\lambda} \equiv \frac{\lambda}{16\pi G}, \tag{3.8}$$

we find that the Lagrangian becomes:

$$\mathcal{L} = -\left[\frac{m_{pl}^2}{16\pi}R + \frac{1}{2}\nabla_\mu\bar{\sigma}\nabla^\mu\bar{\sigma} + V_C(\bar{\sigma}) + \bar{\lambda}\left(|\det(\rho_{ab})| - 1\right)\right]\sqrt{-g}, \tag{3.9}$$

where the *correct* form for the Cho potential — ie., the one consistent with $\bar{\sigma}$ as a massless, minimally coupled scalar field — reads:

$$V_C(\bar{\sigma}) = \bar{R}\exp\left(-\sqrt{\frac{16\pi(d+2)}{d}}\frac{\bar{\sigma}}{m_{pl}}\right) + \bar{\Lambda}\exp\left(-\sqrt{\frac{16\pi d}{d+2}}\frac{\bar{\sigma}}{m_{pl}}\right). \tag{3.10}$$

It is interesting to note that, as in the 5D theory, the cosmological term has been absorbed into the inflationary potential. This is a result of the conformal transformation, and it means that $\Lambda = 0$ in the observed (4D) universe. We now use the potential to constrain the model in each of seven possible cases.

3.1.3 Coasting Models

Let us consider first the case $d \gg 1$. This limit, although it may seem somewhat inelegant, is of interest since some Kaluza-Klein theorists (Alvarez and Gavela 1983, Abbott *et al* 1984) have proposed that a large number (~ 40) of extra dimensions could help explain the high degree of entropy in the observed four-dimensional universe. With $d \gg 1$ eq. (3.10) becomes:

$$V_C(\phi) \approx (\bar{R} + \bar{\Lambda})\exp\left(-\sqrt{16\pi}\frac{\phi}{m_{pl}}\right). \tag{3.11}$$

This potential has a power-law form, eq. (2.65). Comparison of V_C with V_{PLI}, however, reveals immediately that the power-law parameter $p = 1$, which corresponds to a "coasting universe," $a \propto t$, and not an inflationary one as desired.

3.1.4 Negative or Vanishing Potential

We now return to general d and consider all possible values for the other free parameters of the theory, $\bar{R}$ and $\bar{\Lambda}$. To begin with, if both these quantities are negative, or if either one is negative while the other vanishes, then $V_C(\bar{\sigma}) < 0$. This is incompatible with inflation, as we now prove. The "canonical energy-momentum tensor" $T_{\mu\nu}$ may be defined (Wald 1984, p. 457) by:

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{3.12}$$

We have $\mathcal{L} = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)$, so that:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{3.13}$$

which may be compared with the perfect-fluid expression for $T_{\mu\nu}$, eq. (2.14), to yield:

$$\begin{aligned} \rho &= \frac{1}{2}\dot\phi^2 + V(\phi) \\ p &= \frac{1}{2}\dot\phi^2 - V(\phi). \end{aligned} \tag{3.14}$$

Now, in addition to the two central equations of RW cosmology, eqs. (2.22) and (2.35), one sometimes finds the third one (Weinberg 1972, p. 472):

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{3.15}$$

(This is not functionally independent of the other two, but related to them via the Bianchi identities.) Putting the expressions (3.14) into eq. (3.15), one obtains:

$$\frac{\ddot a}{a} = -\frac{8\pi G}{3}\left(\dot\phi^2 - V\right). \tag{3.16}$$

Since inflation requires positive $\ddot a$, the present case (with $V_C \leq 0$ at all times) cannot be inflationary.

3.1.5 Power-Law Expansion with $p < 1$

If, on the other hand, $\bar R$ is *positive* while $\bar\Lambda$ vanishes, then Cho's potential reduces to the PLI one, eq. (2.65). Comparison of V_C with V_{PLI} shows immediately that the power-law parameter is given in this case by:

$$p_C = \frac{d}{d+2}. \tag{3.17}$$

This is also incompatible with inflation, as can be simply shown by twice differentiating $a \propto t^p$ to obtain $\ddot a \propto p(p-1)t^{p-2}$. The present case has (for all finite $d \geq 1$) $p_C < 1$ and consequently $\ddot a < 0$.

3.1.6 Power-Law Inflation with $n \leq 0.9$

If it is $\bar{R}$ that vanishes, while $\bar{\Lambda}$ is positive, then we again find a potential of the PLI form. (When $d = 1$, this case reduces to our 5D model, since a one-dimensional metric ϕ_{ab} must have vanishing Ricci curvature.) Comparison of V_C with V_{PLI} shows immediately that the power-law parameter for this theory is:

$$p_C = \frac{d+2}{d}, \tag{3.18}$$

and hence, from eq. (2.66), that its spectral index of inflationary perturbations is:

$$n_C = 1 - \frac{2d}{d+2}. \tag{3.19}$$

When $d = 1$ we get $n_C = 1/3$ (as expected from our 5D result). With *more* than one extra dimension, $d > 1$ and n_C drops *below* 1/3. Hence this version of the theory is incompatible with the *COBE* and Tenerife constraint ($n \geq 0.9$).

3.1.7 Easther Models with $\xi^2 > 0.15$

The combination $\bar{R} < 0$ with $\bar{\Lambda} > 0$ turns out to be a special case of the scenario studied by Easther (1994), with a potential of the form:

$$V_B(\phi) = \mathcal{A} \exp\left(-\xi\sqrt{8\pi}\,\frac{\phi}{m_{pl}}\right) - \mathcal{B} \exp\left(-\xi m\sqrt{8\pi}\,\frac{\phi}{m_{pl}}\right), \tag{3.20}$$

where $\mathcal{A}, \mathcal{B}, \xi > 0$, and $m > 1$ are all free parameters. Comparison with $V_C(\phi)$ gives:

$$\mathcal{A} = \bar{\Lambda} \quad , \quad \mathcal{B} = -\bar{R} \quad , \quad \xi = \sqrt{\frac{2d}{d+2}} \quad , \quad m = \frac{d+2}{d}. \tag{3.21}$$

Easther combines observations of CMB anisotropy from the *COBE* satellite with constraints on the dark matter biasing parameter b_8 from the *QDOT* survey (assuming "standard cold dark matter," ie., $k = 0$) to infer that any viable model must satisfy $\xi^2 \leq 0.15$ (see Appendix B). The present case is therefore ruled out, since $\xi^2 \geq 2/3$ (assuming $d \geq 1$).

3.1.8 Modified Easther Models with $m^2\xi^2 > 0.15$

The opposite combination, where $\bar{R} > 0$ and $\bar{\Lambda} < 0$, is not eliminated quite as easily. Comparison of V_C with V_B produces:

$$\mathcal{A} = \bar{R} \quad , \quad \mathcal{B} = -\bar{\Lambda} \quad , \quad \xi = \sqrt{\frac{2(d+2)}{d}} \quad , \quad m = \frac{d}{d+2}. \tag{3.22}$$

This case is not covered by Easther (1994), on account of his restriction $m > 1$. However it is not too difficult to extend Easther's procedure so that it includes values in the range $0 < m < 1$. When this is done (see Appendix B, § B.1) we find that observation requires $m^2\xi^2 \leq 0.15$. Assuming that $d \geq 1$, we have $m^2\xi^2 \geq 2/3$, so this scenario does not work either.

3.1.9 Noncompact Extra Dimensions

The final possibility, where *both* $\bar{R}$ and $\bar{\Lambda}$ are positive, turns out to have exactly the same form as a special case discussed by Berezin *et al* (1989, § 4.1). These authors argue that, given a potential of this form, the system will tend to "decompactify" regardless of initial conditions; that is, the extra dimensions will begin to appear in low-energy physics as inflation progresses. Most Kaluza-Klein theorists would consider this grounds for ruling out this model as firmly as the others.

3.1.10 Summary of Constraints

In *no* case, then, does Cho's theory (1990, 1992) lead to inflation compatible with observation. It should be emphasized that, without the renormalization given by eq. (3.7) this would not have been noticed, and in fact several of the above scenarios would have appeared viable. These pessimistic results are summarized in Table 3.1, where we have switched from $\bar{R}$ and $\bar{\Lambda}$ back to the original R_ρ and $\hat{\Lambda}$ for ease of reference.

The lesson to be drawn is probably that a more sophisticated model is needed to obtain realistic inflation from extra dimensions. Similar conclusions have been reached in other theories based on a higher-dimensional vacuum (Levin 1995). One could attempt

Table 3.1: Constraints on the Model of Cho

	$R_\rho < 0$	$R_\rho = 0$	$R_\rho > 0$
$\hat{\Lambda} < 0$	No Inflation ($V < 0$)	No Inflation ($V < 0$)	Inflation Violates *COBE+QDOT* (mod. Easther; $m^2\xi^2 > 0.15$)
$\hat{\Lambda} = 0$	No Inflation ($V < 0$)	No Inflation ($V = 0$)	No Inflation ($p < 1$)
$\hat{\Lambda} > 0$	Inflation Violates *COBE+QDOT* (Easther; $\xi^2 > 0.15$)	Inflation Violates *COBE* ($n < 0.9$)	Decompactification (Berezin *et al*)

to overcome this by inducing inflationary behaviour with an appropriately defined $(4+d)$-dimensional energy-momentum tensor. In fact, this was the first mechanism suggested for Kaluza-Klein inflation (Dereli and Tucker 1983) and it is still the one used most widely today. Various species of matter that have been pressed into service this way include higher-dimensional perfect fluids (Sahdev 1984, Ishihara 1984, Szydłowski and Biesiada 1990, Beloborodov *et al* 1994, Fabris and Sakellariadou 1995), tensor fields derived from supergravity (Moorhouse and Nixon 1985), non-minimally coupled scalar fields (Sunahara *et al* 1990), strings (Gasperini *et al* 1991), and others (Chatterjee and Sil 1993, Burakovsky and Horwitz 1995, Carugno *et al* 1995).

3.2 Inclusion of Torsion

3.2.1 Model of Cho and Yoon

One difficulty with higher-dimensional matter is that there is no consensus on how it should be defined (as the number of candidates listed above amply demonstrates). An alternative idea is to modify the *action* of general relativity in higher dimensions. Inflation was obtained early on with the addition of extra terms in the curvature, for example (Shafi and Wetterich 1983, 1985, 1987; Linde 1990 § 9.5).

In an interesting recent development, Yoon and Brill (1990) and Cho and Yoon (1993) have reported that the introduction of torsion can accomplish the same thing. We do not concern ourselves with the details of their procedure; for our purposes the important thing is that a number of new terms appear in the potential. We focus in particular on the model of Cho and Yoon (1993), in which in which a *new scalar field* φ appears (essentially as part of the definition of the metric of the compact subspace) in addition to the dilaton σ already discussed in the context of Cho's theory. There is no higher-dimensional matter, apart from the $\hat{\Lambda}$-term. After dimensional reduction and a conformal rescaling, as in Cho's theory, the Lagrangian density is given by:

$$\mathcal{L} = -\frac{m_{pl}^2}{16\pi}\left[R + \frac{1}{2}\nabla_\mu\sigma\nabla^\mu\sigma + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi + V_{CY}(\sigma,\varphi)\right]\sqrt{-g}, \tag{3.23}$$

where the Cho-Yoon "cosmological potential" $V_{CY}(\sigma,\varphi)$ is in general a complicated function of many parameters. We concentrate here on one model discussed at some length by Cho and Yoon (1993), in which it simplifies to:

$$\begin{aligned} V_{CY}(\sigma,\varphi) &= \frac{A}{4}\exp\left(-\sqrt{\frac{d+2}{d}}\,\sigma\right)\left\{\exp\left[-(d+1)\sqrt{\frac{2}{d(d-1)}}\,\varphi\right]\right. \\ &- \left.(d+1)\exp\left[-\sqrt{\frac{2}{d(d-1)}}\,\varphi\right]\right\} - \hat{\Lambda}\exp\left(-\sqrt{\frac{d}{d+2}}\,\sigma\right), \end{aligned} \tag{3.24}$$

where A is a coefficient of the theory (with the same dimensions as V_{CY} and $\hat{\Lambda}$). This potential is of interest since it is said to give rise to inflation in the case where $\varphi = 0$, and possibly also in the case where $\sigma = 0$. We will examine these two cases separately. But first we note that, as in Cho's theory, the Lagrangian density (3.23) does not have the canonical form for a massless, minimally coupled scalar field. To interpret the theory this way we first have to *renormalize* both of the scalar fields, as in eq. (3.7):

$$\bar{\sigma} \equiv \frac{\sigma}{\sqrt{16\pi G}} \quad , \quad \bar{\varphi} \equiv \frac{\varphi}{\sqrt{16\pi G}}. \tag{3.25}$$

Introducing also the renormalized quantities:

$$\bar{A} \equiv \frac{-A}{64\pi G} \quad , \quad \bar{\Lambda} \equiv \frac{-\hat{\Lambda}}{16\pi G}, \tag{3.26}$$

we find that the *correct* form for the Cho-Yoon potential is:

$$\begin{aligned} V_{CY}(\sigma, \varphi) &= \bar{A} \exp\left(-\sqrt{\frac{16\pi(d+2)}{d}} \frac{\bar{\sigma}}{m_{pl}}\right) \left\{ (d+1) \exp\left[-\sqrt{\frac{32\pi}{d(d-1)}} \frac{\bar{\varphi}}{m_{pl}}\right] - \right. \\ &- \left. \exp\left[-(d+1)\sqrt{\frac{32\pi}{d(d-1)}} \frac{\bar{\varphi}}{m_{pl}}\right] \right\} + \bar{\Lambda} \exp\left(-\sqrt{\frac{16\pi d}{d+2}} \frac{\bar{\sigma}}{m_{pl}}\right). \end{aligned} \tag{3.27}$$

We now consider the two special cases $\bar{\varphi} = 0$ and $\bar{\sigma} = 0$ in turn.

3.2.2 The Case $\bar{\varphi} = 0$

When $\bar{\varphi} = 0$, we obtain:

$$V_{CY}(\bar{\sigma}, 0) = \bar{A} d \exp\left(-\sqrt{\frac{16\pi(d+2)}{d}} \frac{\bar{\sigma}}{m_{pl}}\right) + \bar{\Lambda} \exp\left(-\sqrt{\frac{16\pi d}{d+2}} \frac{\bar{\sigma}}{m_{pl}}\right). \tag{3.28}$$

This is exactly the same as Cho's potential, eq. (3.10), except $\bar{R} \longrightarrow \bar{A}d$. Consequently we can bring over all the results of §§ 3.1.4 – 3.1.9, concluding immediately that this model cannot be both inflationary and compatible with observation. Results are summarized in Table 3.2, where we have switched from $\bar{A}$ and $\bar{\Lambda}$ back to the original A and $\hat{\Lambda}$ for ease of reference.

3.2.3 The Case $\bar{\sigma} = 0$

The second case, when $\bar{\sigma} = 0$, proves to be more interesting:

$$\begin{aligned} V_{CY}(0, \bar{\varphi}) &= \bar{A} \left\{ (d+1) \exp\left[-\sqrt{\frac{32\pi}{d(d-1)}} \frac{\bar{\varphi}}{m_{pl}}\right] \right. \\ &- \left. \exp\left[-(d+1)\sqrt{\frac{32\pi}{d(d-1)}} \frac{\bar{\varphi}}{m_{pl}}\right] \right\} + \bar{\Lambda}. \end{aligned} \tag{3.29}$$

Table 3.2: Constraints on the Model of Cho and Yoon with $\bar{\varphi} = 0$

	$A > 0$	$A = 0$	$A < 0$
$\hat{\Lambda} > 0$	No Inflation ($V < 0$)	No Inflation ($V < 0$)	Inflation Violates *COBE+QDOT* (mod. Easther; $m^2\xi^2 > 0.15$)
$\hat{\Lambda} = 0$	No Inflation ($V < 0$)	No Inflation ($V = 0$)	No Inflation ($p < 1$)
$\hat{\Lambda} < 0$	Inflation Violates *COBE+QDOT* (Easther; $\xi^2 > 0.15$)	Inflation Violates *COBE* ($n < 0.9$)	Decompactification (Berezin *et al*)

For one thing, we see that the cosmological term $\bar{\Lambda}$ is no longer absorbed into the potential, but now gives rise in general to an effective *four-dimensional* cosmological term $\Lambda = 16\pi G\bar{\Lambda} = -\hat{\Lambda}$. This distinguishes it from the other models studied so far.

We now proceed, as in §§ 3.1.4 – 3.1.9, to consider all possible values for the free parameters of the theory, $\bar{A}$ and $\bar{\Lambda}$. To begin with, we notice that if $\bar{A} = 0$ and $\bar{\Lambda} \leq 0$, or if $\bar{\Lambda} = 0$ and $\bar{A} < 0$, then $V_{CY}(0, \bar{\varphi}) \leq 0$ and there can be no inflation (§ 3.1.4).

3.2.4 de Sitter Expansion

If $\bar{A} = 0$ and $\bar{\Lambda} > 0$, on the other hand, then we have the very simple potential:

$$V_{CY}(0, \bar{\varphi}) = \bar{\Lambda}. \tag{3.30}$$

Neglecting $\dot{\bar{\varphi}}^2$ in comparison with $V_{CY}(0, \bar{\varphi})$ (this is known in inflationary theory as the "slow roll" approximation), eq. (3.14) gives:

$$\rho = V_{CY}(0, \bar{\varphi}) = \bar{\Lambda} = \frac{|\hat{\Lambda}|}{16\pi G}. \tag{3.31}$$

(Note that $\hat{\Lambda}$ is negative in this case.) Then, assuming $k = 0$ (which, although not required *per se* by inflation, is the most natural way to implement it), we obtain from eq. (2.22):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{|\hat{\Lambda}|}{6}. \tag{3.32}$$

Integration yields:

$$a(t) = e^{\sqrt{|\hat{\Lambda}|/6}\,t}, \tag{3.33}$$

which we recognize as a case of *de Sitter* expansion (Weinberg 1972, p. 615).

3.2.5 *COBE* Constraints on Dimensionality

The richest case is the one in which $\bar{A} > 0$ and $\hat{\Lambda} = 0$. This results in a potential which has the Easther form, § 3.1.7. Comparing eqs. (3.20) and (3.29), we find for the Easther parameters:

$$\mathcal{A} = \bar{A}(d+1) \quad , \quad \mathcal{B} = \bar{A} \quad , \quad \xi = \frac{2}{\sqrt{d(d-1)}} \quad , \quad m = (d+1). \tag{3.34}$$

Easther's (1994) constraints (see Appendix B) lead to the following necessary and sufficient conditions for viability, respectively:

$$\begin{aligned} \xi^2 &= \frac{4}{d(d-1)} \le 0.15 &\Longrightarrow\quad & d \ge 6 \\ m\xi^2 &= \frac{4(d+1)}{d(d-1)} \le 0.15 &\Longrightarrow\quad & d \ge 29. \end{aligned} \tag{3.35}$$

Combining these two results, we see that the model is certainly viable for $d \ge 29$, and it *may* additionally be viable for $6 \le d \le 28$. For models with d in this range, each case must be evaluated individually. Details are given in § B.2 of Appendix B. It turns out that all of them (except $d = 6$) are viable, but only when the scalar field $\bar{\varphi}$ exceeds a certain minimum value $\bar{\varphi}_*(d)$, values of which are listed in § B.2.

To summarize, then, this scenario cannot be viable if $d < 7$. It is viable (for sufficiently large values of $\bar{\varphi}$) when $7 \le d \le 28$, and for $d \ge 29$ it is viable for all values of $\bar{\varphi}$.

It is remarkable that we can put limits on the dimensionality of spacetime in this way. While connections have previously been made between the number of compact dimensions and the amount of entropy in the universe (Alvarez and Gavela 1983, Abbott *et al* 1984), this appears to be the first time that they have actually been constrained by satellite and radio telescope observations.

3.2.6 Radiation-Dominated Models

For all other cases, $\bar{A} \neq 0$ and $\bar{\Lambda} \neq 0$, the potential (3.29) is a combination of two exponentials and a cosmological term, and cannot be simply constrained. We can, however, test the model using the approximation $d \gg 1$ discussed in § 3.1.3. We find:

$$V_{CY}(0, \bar{\varphi}) \approx (\bar{A}d + \bar{\Lambda}) - \bar{A}\exp\left(-\sqrt{32\pi}\,\frac{\bar{\varphi}}{m_{pl}}\right). \tag{3.36}$$

In these models an effective cosmological term:

$$\Lambda = 16\pi G\left(\bar{A}d + \bar{\Lambda}\right) = -\left(\hat{\Lambda} + \frac{Ad}{4}\right) \tag{3.37}$$

is induced in 4D spacetime. This is interesting because it implies that in a universe with the right number of dimensions:

$$d = -\frac{4\hat{\Lambda}}{A}, \tag{3.38}$$

the higher dimensional $\hat{\Lambda}$-term would be exactly cancelled by the A-term, so that we would be left with no 4D cosmological constant at all — a possible resolution of the "cosmological constant problem." Using the special value $\hat{\Lambda} = -8$ favoured on independent grounds by Cho and Yoon (1993), for example, and assuming that A is of order unity, we find that the right number of dimensions is $d \sim 32$, which is not only consistent with the assumption $d \gg 1$, but agrees almost exactly with the number picked out by Kaluza-Klein theorists attempting to explain the origin of entropy on dimensional grounds (Alvarez and Gavela 1983, Abbott *et al* 1984).

Pursuing this case further, however, we note that if d does take on this special value, then the universe cannot be inflationary. This can be seen by considering the possible

Table 3.3: Constraints on the Model of Cho and Yoon with $\bar{\sigma} = 0$

	$A > 0$	$A = 0$	$A < 0$
$\hat{\Lambda} > 0$	Inconclusive (No Inflation if $d \gg 1$ and $\Lambda = 0$)	No Inflation ($V < 0$)	Inconclusive (No Inflation if $d \gg 1$ and $\Lambda = 0$)
$\hat{\Lambda} = 0$	No Inflation ($V < 0$)	No Inflation ($V = 0$)	Inflation if $d \geq 29$ (or $d \geq 7$ and $\bar{\varphi} > \bar{\varphi}_*$)
$\hat{\Lambda} < 0$	Inconclusive (No Inflation if $d \gg 1$ and $\Lambda = 0$)	de Sitter Expansion	Inconclusive (No Inflation if $d \gg 1$ and $\Lambda = 0$)

values of A. If $A \leq 0$ then $V_{CY}(0, \bar{\varphi}) \leq 0$, which rules out inflation (§ 3.1.4). If $A > 0$, on the other hand, then the potential (3.36) reduces to a simple PLI form, eq. (2.65), with power-law parameter $p = 1/2$. This is also incompatible with inflation (§ 3.1.5); in fact, it implies $a(t) \propto \sqrt{t}$, which describes a *radiation-dominated* universe.

3.2.7 Summary of Constraints

The model of Cho and Yoon (1993) fares slightly better than that of Cho (1990, 1992) — at least in the case where the dilaton $\bar{\sigma}$ vanishes, leaving the second scalar field $\bar{\varphi}$ to act as the inflaton instead. Inflation is possible and, in at least one case ($A < 0$ with $\hat{\Lambda} = 0$), explicitly compatible with observation. (In several other cases, it cannot be ruled out.) These findings are summarized in Table 3.3, where we have once again switched from $\bar{A}$ and $\bar{\Lambda}$ back to A and $\hat{\Lambda}$ for ease of comparison with the original paper.

3.3 Multiple Compact Subspaces

3.3.1 Model of Berezin *et al*

As an alternative to higher-dimensional matter, extra curvature terms, and torsion, one can also consider generalizing the vacuum Kaluza-Klein cosmology to incorporate *multiple compact subspaces*. Ezawa *et al* (1991) have calculated that the probability of inflation (defined in this case as an expansion of the scale factor in the macroscopic 4D spacetime by at least a factor of 10^{30}) is significantly enhanced when there are two compact subspaces. However these authors did not address the question of observational constraints.

We examine here the theory of Berezin *et al* (1989), in which the d extra dimensions are divided among m compact subspaces of dimension d_ℓ, where $\sum_{\ell=1}^{m} d_\ell = d$. This is essentially a generalization of the metric (3.1), but with ϕ_{ab} broken up as follows:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\mu\nu} & & & & \\ & h_1^2\phi^{(1)}_{a_1b_1} & & & \\ & & h_2^2\phi^{(2)}_{a_2b_2} & & \\ & & & \ddots & \\ & & & & h_m^2\phi^{(m)}_{a_mb_m} \end{pmatrix}, \tag{3.39}$$

where $(a_\ell, b_\ell, ...) = (1, \ldots, d_\ell)$, and $\phi^{(\ell)}_{a_\ell b_\ell}$ and h_ℓ are the metric and scale factor of the ℓ-th submanifold respectively. Dimensional reduction and a conformal rescaling (as usual) then lead to the "Einstein frame" in which the 4D metric $g_{\mu\nu}$ is a solution of the field equations with a *series* of minimally coupled dilatons φ_ℓ defined by:

$$h_\ell \equiv \ln(\varphi_\ell), \tag{3.40}$$

as the only material sources. The resulting Lagrangian is quite complex, as may be imagined, and in fact Berezin *et al* (1989) were compelled to focus on two special cases in studying the dynamics of the theory: the cases of one and two dilaton fields. Of these, they concluded that the latter was qualitatively the same as the former, as far as the behaviour of the potential was concerned. So it suffices here to evaluate the case of a single dilaton φ.

The energy-momentum tensor of this dilaton is given by Berezin *et al* (1989) as:

$$T_{\mu\nu} = \frac{d(d+2)}{2}\left[\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}\nabla_\lambda\varphi\nabla^\lambda\varphi g_{\mu\nu}\right] + V_B(\varphi)g_{\mu\nu}, \tag{3.41}$$

with the potential:

$$V_B(\varphi) = -\frac{R_\phi}{2}\exp[-(d+2)\varphi] + \hat{\Lambda}\exp(-d\varphi), \tag{3.42}$$

where R_ϕ is the Ricci curvature of the compact submanifold.

Berezin *et al* (1989) go on to draw a number of conclusions from the form of the potential (3.42). However, this procedure is invalidated by two deceptively trivial mistakes in eq. (3.41). Firstly, it is missing a factor of $1/8\pi G$; and secondly, it is not in the canonical form (3.13) for a minimally coupled scalar field. To put it in this form we once again have to *renormalize*, as follows:

$$\bar{\varphi} \equiv \frac{\varphi}{\sqrt{16\pi G}} \quad , \quad \bar{R} \equiv \frac{-R_\phi}{32\pi G} \quad , \quad \bar{\Lambda} \equiv \frac{\hat{\Lambda}}{16\pi G}. \tag{3.43}$$

In terms of $\bar{\varphi}$, $\bar{R}$ and $\bar{\Lambda}$ we obtain the *correct* dilaton potential:

$$V_B(\bar{\varphi}) = \bar{R}\exp\left(-\sqrt{\frac{16\pi(d+2)}{d}}\,\frac{\bar{\varphi}}{m_{pl}}\right) + \bar{\Lambda}\exp\left(-\sqrt{\frac{16\pi d}{d+2}}\,\frac{\bar{\varphi}}{m_{pl}}\right). \tag{3.44}$$

This is exactly the same as Cho's potential, eq. (3.10), which is perhaps not surprising since they are both higher-dimensional vacuum theories with a single conformal scalar field.

3.3.2 Summary of Constraints

Bringing over all the results of §§ 3.1.4 – 3.1.9, we conclude immediately that this model cannot give rise to inflation in a manner compatible with observation. Results are summarized in Table 3.4, where we have switched from $\bar{R}$ and $\bar{\Lambda}$ back to the original R_ϕ and $\hat{\Lambda}$ as usual.

This completes our survey of higher-dimensional vacuum cosmologies. Among other things we have looked particularly at the potential for some models to provide a natural mechanism for inflation, thereby solving the horizon, flatness, and relic abundance

Table 3.4: Constraints on the Model of Beresin *et al*

	$R_\phi > 0$	$R_\phi = 0$	$R_\phi < 0$
$\hat{\Lambda} < 0$	No Inflation ($V < 0$)	No Inflation ($V < 0$)	Inflation Violates *COBE+QDOT* (mod. Easther; $m^2\xi^2 > 0.15$)
$\hat{\Lambda} = 0$	No Inflation ($V < 0$)	No Inflation ($V = 0$)	No Inflation ($p < 1$)
$\hat{\Lambda} > 0$	Inflation Violates *COBE+QDOT* (Easther; $\xi^2 > 0.15$)	Inflation Violates *COBE* ($n < 0.9$)	Decompactification (Berezin *et al*)

problems. Contrary to the impression one receives from the literature, this turns out to be difficult. Of the models we have examined, only one — the torsion-based theory of Cho and Yoon (1993) — explicitly gave rise to inflation in a manner compatible with observation, and this occurred only for certain (generally large) numbers of compact dimensions.

We have however found that in some cases the extra dimensions appear to provide a framework for addressing the *cosmological constant* problem (§ 3.2.6). We now leave Kaluza-Klein cosmology behind and turn our attention fully to the cosmological "constant" in the remainder of the thesis.

Chapter 4

Cosmology With a Cosmological Term Λ

4.1 The Singularity Problem

Among the shortcomings of the standard model described in Chapter 1, the problems of smoothness, flatness, structure formation and dark matter have received the most attention. The fact that the theory predicts its own demise in a singularity, which would generally be regarded as catastrophic in any other field, has met with comparatively little resistance from most cosmologists[1]. Many have become so accustomed to the big bang singularity that they no longer think of it as a liability. Some have even come to regard it as essential, dismissing *a priori* any nonsingular cosmology as unphysical (Olson and Jordan 1987, Gott *et al* 1989, Martel and Wasserman 1990, Perlmutter *et al* 1997).

This widespread acceptance may stem partly from the success of the singularity theorems (Hawking and Ellis 1973). As these authors acknowledge, however, there are important caveats in the application of their arguments to cosmology. Firstly, general relativity may go over to some more complete theory of quantum gravity at early times, and this could well turn out to be nonsingular (eg., Lund 1973, Starobinsky 1980, de Barros *et al* 1997). Secondly, Einstein's theory may itself be incomplete. The initial singularity can be removed by considering extra terms in the curvature (Ruzmaĭkina

[1] With some exceptions, as related for example in the popular account by Lerner (1992). A balancing (and more mainstream) popular treatment may be found in Gribbin (1986).

and Ruzmaĭkin 1970, Barrow and Ottewill 1983, Page 1987, Brandenberger *et al* 1993), torsion (Kopczyński 1972, Trautman 1973, Yu 1989), a nonsymmetric metric (Moffat 1979), or *two* metrics (Rosen 1979).

Thirdly, even within general relativity, non-Robertson-Walker solutions of the field equations can be nonsingular. This is true, for example, of certain inhomogeneous models (Senovilla 1990, 1996). Finally, relaxation of the *energy conditions* assumed by Hawking and Ellis can lead to singularity avoidance. This is the route taken, for example (either explicitly, or implicitly via an unconventional equation of state), in nonsingular theories based on "vacuum-like" matter (Gliner 1970, Blome and Priester 1991), properties of matter at high densities (Bahcall and Frautschi 1971; Markov 1982, 1983), scalar fields (Bekenstein 1974, Rosen 1985, Israelit and Rosen 1989, Starkovich and Cooperstock 1992, Abreu *et al* 1994, Bayin *et al* 1994, Mimoso and Wands 1995, Rama 1997a), matter creation (Brout *et al* 1978, Wesson 1985, Gunzig *et al* 1987, Prigogine *et al* 1989, Petry 1990, Lima and Abramo 1996), tensor fields (Petry 1981, Tangherlini 1993), inflation (Vilenkin 1982, Dymnikova 1986, Linde *et al* 1994, Borde and Vilenkin 1997, Garriga and Vilenkin 1997), particle interactions (Rose 1986, Parker 1991), supergravity (Barrow and Deruelle 1988, Balbinot *et al* 1990), textures (Hacyan and Sarmiento 1993), strings (Behrndt and Förste 1994, Gregory 1996, Jain 1997, Larsen and Wilczek 1997, Rama 1997b, Maggiore 1997), and walls (Dąbrowski 1996).

The same energy conditions are violated by something arguably much simpler: a sufficiently large *cosmological term* Λ. In fact, it has long been known that high-Λ universes have no big bang singularity (Robertson 1933, Harrison 1967). These models have, however, been ruled out as incompatible with observation (Ellis 1984, Felten and Isaacman 1986, Lahav *et al* 1991). We will return to the observational constraints later on, merely noting here that *proofs* of this incompatibility have so far assumed $\Lambda = \text{constant}$ (Crilly 1968, Börner and Ehlers 1988, Ehlers and Rindler 1989). What if Λ were allowed to vary? Could one postulate a cosmological term which was initially very large — perhaps even large enough to avert the big bang singularity — but which has subsequently decayed down to some observationally acceptable size? As we will see, precisely this type of behaviour has been proposed in recent years in the context of another of the problems described in Chapter 1, the *cosmological constant problem*. As far as we are aware, however, there has not been a comprehensive effort to assess the impact of such decaying-Λ models on the question of the initial singularity. This will be our objective in the remainder of the thesis.

4.2 The Cosmological "Constant"

To begin with, we remind ourselves why the cosmological term has often been treated as a constant of nature. This approach has its roots in the structure of the Einstein field equations, which read (with the Λ-term):

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.1}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - R g_{\mu\nu}/2$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter. (We will use units such that $c = 1$, but retain G throughout the rest of the thesis.) Taking the covariant divergence of eq. (4.1), recalling that the vanishing covariant divergence of the Einstein tensor is guaranteed by the Bianchi identities, and assuming that the energy-momentum tensor satisfies the conservation law $T^{\mu\nu}{}_{;\nu} = 0$, we see that the covariant divergence of $\Lambda g_{\mu\nu}$ must vanish also. Since Λ is a scalar, and the metric tensor has a vanishing covariant derivative, it follows that $\partial\Lambda/\partial x^{\mu} = 0$, or $\Lambda =$ constant. This argument, which situates Λ firmly on the left-hand side of the field equations, constitutes a "geometrical interpretation" of the cosmological term.

By contrast, in modern formulations of general relativity (eg. Weinberg 1972, p. 613, Misner *et al* 1973, p. 411; Peebles 1993, p. 65) it is increasingly common to move the cosmological term to the *right*-hand side of eq. (4.1):

$$G_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu}, \tag{4.2}$$

where:

$$\tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}; \tag{4.3}$$

that is, to interpret Λ as part of the *matter content* of the universe, rather than a purely geometrical entity [this idea goes back at least to Gliner (1966) and Zel'dovich (1968)]. Once this is done, there are no *a priori* reasons why Λ should not vary as it likes — as long as the *effective* energy-momentum tensor $\tilde{T}_{\mu\nu}$ satisfies energy conservation. This possibility seems first to have been noticed by Linde (1974). The first explicit functional form for a variable Λ term was put forward by Kazanas (1980), who suggested (before the advent of inflation) that a power-law dependence of Λ on temperature below some critical phase transition temperature T_c [ie., $\Lambda \propto (T/T_c)^{\beta}$, with $\beta = \text{constant}$] could lead to exponential expansion of the scale factor and help explain the observed isotropy of the universe.

4.3 Variable-Λ Theories

Many have seen in such variation the potential for resolving the *cosmological constant problem*, by means of a negative counter-term Λ_c which could grow with time and eventually "screen" or even cancel out the large positive cosmological constant Λ_v expected from zero-point quantum fluctuations of the vacuum (Adler 1982, Hawking 1984, Barr 1987, Weinberg 1989, Moffat 1995, Dolgov 1997). The sum of these two terms would be driven toward zero at late times (or low energies):

$$\Lambda = (\Lambda_c + \Lambda_v) \to 0. \tag{4.4}$$

The first actual mechanism for Λ_c (based on a nonminimally coupled scalar field) was proposed by Dolgov (1983), who suggested a decay rate $\dot{\Lambda}$ of order H^{-1} (where $H \equiv \dot{a}/a$ is the Hubble parameter)[2]. Perhaps the best-known descendant of this work is the

[2]The reader may have noticed that the pioneering papers in this field appear to have been written almost entirely by Soviet scientists: from Gliner's (1966) suggestion that the vacuum be treated as physical matter with $p = -\rho$, to Zel'dovich's (1968) extension of this treatment to "Λ-matter," to Linde's (1974) idea of allowing Λ to vary, and finally to the mechanism of Dolgov (1983). It is hard not to draw a connection with *dialectical materialism*, the official state philosophy of the Soviet Union, two fundamental premises of which may be rendered as follows: (1) that basic reality is *material*; and (2) that it is in a state of eternal change or *variability*. (The term "dialectic" refers to the endless Hegelian cycle of evolution from thesis through antithesis to synthesis, and back to thesis again.) Perhaps this is an example of the oft-unappreciated way in which societal beliefs influence scientific ones, even in those

model of Peebles and Ratra (1988) and Ratra and Peebles (1988), in which the universe is permeated by a self-interacting scalar field φ, coupled only weakly to matter, that "rolls" continuously down a potential gradient toward some minimum. (This could be the same field as that responsible for inflation, and/or the dark matter.) The role of the cosmological term is played by the total energy density of the scalar field, $\Lambda \propto (\dot{\varphi}^2 + \kappa\varphi^{-\beta})$, which drops as its potential energy is converted to kinetic energy (κ and β are constants). The case $\Lambda =$ constant is recovered as $\beta \to 0$. Observational constraints have been placed on models of this kind from number counts of faint galaxies (Yoshii and Sato 1992), CMB anisotropies (Sugiyama and Sato 1992), and gravitational lensing (Ratra and Quillen 1992). They tend to favour small values of β (of order unity), whereas large values (~ 10) are required to solve the cosmological constant problem. Variations on the scalar field theme have been explored by a number of subsequent workers (Wetterich 1988b, Fujii and Nishioka 1991, Frieman *et al* 1995, Moffat 1995, Fujii 1997). In particular, the observationally important question of *density perturbations* in the presence of a decaying cosmological term of this kind is the subject of recent studies by Coble *et al* (1997) and Viana and Liddle (1997).

These models illustrate that variation in the cosmological term can be associated quite generically with the existence of scalar fields, and — by extension — with most versions of *inflation*. It is therefore somewhat surprising that the large majority of cosmologists who have embraced the inflationary scenario should continue to treat Λ as constant. It should be noted, however, that while Λ may decay in these theories, it

realms of inquiry which seem furthest from human affairs, by delineating the scope of the questions that may be asked.

One could go further and speculate whether western scientists, perhaps under the sway of German idealism, may have been unconsciously reluctant to dislodge Λ from its place on the left-hand side of the field equations [the "pure marble of geometry," as Einstein is said to have called it (Salam 1980)] and join it to the "base wood of matter," there to be subject to change and decay. In this connection it should be mentioned that some authors have followed the mathematically equivalent but philosophically very different path of letting Λ vary *while keeping it on the left-hand side* of the field equations. Eq. (4.1) then implies that matter satisfies the usual conservation law $T^{\mu\nu}{}_{;\nu} = 0$. However, the resulting theory is not general relativity, since the Einstein tensor $G_{\mu\nu}$ can no longer have a vanishing covariant divergence. In fact one obtains what is best described as a "modified Brans-Dicke" theory. This possibility was first considered in the context of the cosmological constant problem by Endō and Fukui (1977), and it has subsequently been studied by Bertolami (1986a,b), Berman *et al* (1989) and Berman (1990a,b;1991a).

does not usually do so quickly enough to solve the cosmological constant problem. [In one early model (Abbott 1985) the relaxation process was estimated to require at least 10^{450} years.] Weinberg (1989, 1996) has in fact argued on general grounds that this problem cannot be solved by any mechanism which relies solely on scalar fields.

Tensor (eg. Hawking 1984) and vector fields (Dolgov 1997) can also contribute nontrivially to the vacuum energy, and hence to the effective value of Λ, reinforcing the point that the case in which this quantity is exactly constant is a special one. Similar effects have been obtained with a wide variety of other mechanisms. Some give rise to simple decay laws, such as Hiscock's (1986) model based on the effective stress-energy of quantum fields in de Sitter space, for which $\dot{\Lambda} \propto \Lambda^{\beta}$ ($\beta =$ constant). Others, like the model of Tsamis and Woodard (1996) based on infrared processes in quantum gravity, are more complicated. In most variable-Λ papers, specific decay laws for Λ are not actually derived at all; the intent is merely to demonstrate that decay (and preferably near-cancellation) of the effective cosmological term is possible in principle. New matter fields and quantum effects are not necessary; even *cosmological perturbations* can drive the effective value of the cosmological term toward zero (during inflation; Abramo *et al* 1997).

4.4 Phenomenological Λ-Decay Laws

Inspired by these ideas, a number of other authors have constructed purely *phenomenological* models, in which some specific decay law is postulated for Λ and investigated for its cosmological consequences. Those proposed so far may be divided into four classes: (1) theories in which $\dot{\Lambda} = \dot{\Lambda}(\Lambda, H)$; (2) theories in which $\Lambda = \Lambda(t)$; (3) theories in which $\Lambda = \Lambda(a)$; and (4) theories in which $\Lambda = \Lambda(a, H)$. The following is intended as a brief but reasonably comprehensive review of these theories.

The first class consists of decay laws with forms similar to those of the Dolgov (1983) and Hiscock (1986) mechanisms mentioned above. Specific examples include the case $\dot{\Lambda} \propto H^3$ discussed at some length by Reuter and Wetterich (1987), and the model with

$\dot{\Lambda} \propto \beta\Lambda - \Lambda^2$ proposed by Moffat (1996a,b).

The second class of theories appears to have been introduced by Lau (1985), who presented a variable-G model with $\Lambda \propto t^{-2}$. The same time-dependence has been assumed by Berman (1991b), who adds the supplementary conditions $\rho \propto t^{-2}$ and $q =$ constant in his theory (where $q \equiv -\ddot{a}a/\dot{a}^2$ is the deceleration parameter); and also by Lopez and Nanopoulos (1996), who assume in addition that $\Lambda \propto a^{-2}$ (so that $a \propto t$). These latter authors make the important claim that a Λ-decay ansatz of this kind could follow from certain versions of *string theory*. Other, more complicated decay laws of the form $\Lambda = \Lambda(t)$ have been obtained by Barr (1987), Kalligas *et al* (1992), Beesham (1993) and Elizalde and Odintsov (1994).

Theories of the third class originate with a paper of Rajeev (1983), who argued that a decay law with the interesting form $\Lambda \propto e^{-\beta a}$ could follow from certain versions of quantum gravity. Power-law decay in a was first proposed by Özer and Taha (1986, 1987), who described a universe with critical density $\rho = \rho_{crit}$ of (ordinary) matter along with a cosmological term decaying as $\Lambda \propto a^{-2}$. This has been investigated further by Abdel-Rahman (1990, 1992, 1995), John and Joseph (1996, 1997) and Özer (1997). A $k = 0$ variant of the idea, in which the two components *together* make up the critical density, was proposed by Chen and Wu (1990), and generalized to include dissipative terms by Calvao *et al* (1992). Freese *et al* (1987) proposed a different scheme in which Λ scales with a at nearly the same rate as *radiation*, $\Lambda \propto a^{-4(1-\epsilon)}$, where $\epsilon \ll 1$. The implications of this law for nucleosynthesis were studied by Sato *et al* (1990) in the limit $\epsilon \rightarrow 0$, and Overduin *et al* (1993) used the CMB intensity to constrain ϵ.

The generalization to *arbitrary* powers of the scale factor, $\Lambda \propto a^{-m}$, was considered briefly by Gasperini (1987), who argued on thermal grounds that this is a plausible way to model the vacuum; and also by Olson and Jordan (1987), who confined themselves to calculating the age of the universe. Pavón (1991) has examined the models of Özer and Taha (1986, 1987), Freese *et al* (1987), Gasperini (1987) and Chen and Wu (1990) from a thermodynamical standpoint. Decays of this kind have been studied in the context of higher-dimensional embedded models by Maia and Silva (1994). Several kinds of observational constraints have been considered, including those from CMB anisotropies (Silveira and Waga 1994, 1997) and lensing statistics (Torres and Waga 1996). The class has also been extended by the introduction of a second free parameter ($\Lambda \propto \Lambda_0 + a^{-m}$) in a variable-$G$ theory of Sisteró (1991) and a survey of observational constraints by Matyjasek (1995).

The fourth class, finally, was ushered in by Carvalho *et al* (1992), who postulated $\Lambda \propto \beta a^{-2} + H^2$, where β is an adjustable constant. This has been investigated further by Arbab and Abdel-Rahman (1994), and extended (Waga 1993) to flat models with an additional free parameter, $\Lambda \propto \Lambda_0 + \beta a^{-2} + H^2$. Pavón's method of thermodynamical analysis, meanwhile, has been applied to models with $\Lambda \propto \beta a^{-m} + H^2$ by Salim and Waga (1993). There was also a variable-G theory of Lima and Carvalho (1994) in which Λ depended *solely* on H via $\Lambda \propto H^2$; this has been studied further by Arbab (1997). [The same H-dependence also appears in a scalar field-based model of Wetterich (1995).] More complicated decay laws in which Λ depends on both a and H have been discussed by Lima and Maia (1994), Lima and Trodden (1996) and Nesteruk *et al* (1997).

The primary *defect* of these phenomenological proposals (besides the somewhat *ad hoc* form of the decay laws) lies in accounting physically for the energy lost in the decay process. In some models (Özer and Taha 1986, 1987; Reuter and Wetterich 1987) this is not addressed at all; in others the energy is channelled into what is effectively "spontaneous" production of baryonic matter (eg., Chen and Wu 1990, Lima and Carvalho 1994), radiation (eg., Freese *et al* 1987, Gasperini 1987, Lima 1996) or both (Abdel-Rahman 1992, 1995; Calvao *et al* 1992, Carvalho *et al* 1992, Lima and Maia 1994, Matyjasek 1995). The former case can be constrained by observations of the diffuse gamma-ray background, since (assuming the decay process does not violate baryon number) one would expect equal amounts of matter and antimatter to be formed (Freese *et al* 1987). The latter case can be constrained by limits on CMB anisotropies, assuming the spectrum of the created radiation is non-Planckian (Freese *et al* 1987); or if that fails, simply by the absolute CMB intensity (Overduin *et al* 1993).

In any case, the fact that that no explicit physical mechanism is included to mediate the conversion of vacuum energy into its decay products is a drawback. [This is not true of theories such as that of Peebles and Ratra (1988), in which the desired mechanism is provided by the scalar field as it rolls down the potential gradient, picking up kinetic energy.] Ultimately the difficulty can be traced to the fact that these theories have not been derived from complete Lagrangians of their own, but are simply *a posteriori* modifications of general relativity. [Thus, as pointed out by Ratra and Peebles (1988), while they all involve a variable Λ-term, there are no derivatives of Λ in their Lagrangians, which are identical to the Lagrangian of Einstein's theory.] Nevertheless, until it is proved that a self-consistent (and observationally acceptable) decay mechanism is impossible, they remain feasible in principle. We take the point of view here that they are worth examining, in light of their potential to address some of the most pressing problems in cosmology.

4.5 Nonsingular Cosmology with Λ Decay

All the authors discussed in the previous section cite the cosmological constant problem as the primary motivation for Λ decay. Very few of them, by contrast, consider the potential relevance of their models to the question of the *initial singularity*. Exceptions are Özer and Taha (1986, 1987), Abdel-Rahman (1990, 1992, 1995), Arbab and Abdel-Rahman (1994), Lima and Maia (1994), Lima and Trodden (1996), and John and Joseph (1996, 1997), who claim explicitly nonsingular solutions. Three other authors mention that nonsingular solutions are possible in their models under certain circumstances (Sisteró 1991, Beesham 1993, Matyjasek 1995). The others either do not comment on the issue at all, or — as in one case (Olson and Jordan 1987) — go out of their way to state explicitly that *no* singularity-free cases will be considered. This constitutes a large and tempting gap in variable-Λ theory as it stands. We will attempt to close part of the gap in this thesis.

One way to do this would be to evaluate the above-mentioned theories one by one. As we have seen, however, these are based on decay laws which tend to suffer not only from a lack of physical motivation, but generality as well. Most are of very specific power-law forms, with the exponents (and sometimes even the coefficients) specified from within the theory. Even then additional conditions (such q = constant, $\rho = \rho_{crit}$, or $k = 0$) are often assumed. We would like to take a slightly more broadminded approach, but one that is still simple enough to be tractable. To this end, we will consider scenarios of the following four types:

$$\Lambda = \mathcal{A}t^{-\ell} \tag{4.5}$$

$$\Lambda = \mathcal{B}a^{-m} \tag{4.6}$$

$$\Lambda = \mathcal{C}H^{n} \tag{4.7}$$

$$\Lambda = \mathcal{D}q^{r}, \tag{4.8}$$

where t is a (suitably defined) time variable, a, H, q are the usual scale factor, Hubble and deceleration parameters respectively, and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ell, m, n$ and r serve as adjustable constants. These decay laws and their cosmological consequences will be the subjects of Chapters 5 – 8. For each scenario we will attempt to determine (1) whether nonsingular homogeneous and isotropic solutions are *possible*; and (2) whether they are *compatible with observation.*

Of course, eqs. (4.5) – (4.8) still suffer from a lack of physical motivation. However, they are more general than most of the Λ-decay laws proposed so far, and one may therefore hope that they will be more likely to accommodate a realistic decay mechanism, should one be found. One might also argue that any scheme which alleviates the abovementioned problems of the standard model and agrees with observations would be a strong basis for *picking out* some physical mechanisms over other ones. We will return to this issue at the end of Chapter 8.

4.6 Dynamical Equations

Let us begin by assembling the required dynamical equations. We will continue to assume a Robertson-Walker line element (ie., a homogeneous and isotropic universe), which means that we can bring over the relevant equations from Chapters 2 and 3, merely modifying them to incorporate a new (and, in general, variable) cosmological term. Since we are regarding this term as part of the *matter* content of the universe (§ 4.2), the required modifications will involve only the source terms p and ρ.

We will also continue to assume that the matter contents of the universe in the absence of the cosmological term are well-described by a perfect fluid; ie., that the energy-momentum tensor $T_{\mu\nu}$ in eq. (4.3) has the form (2.14). Eq. (4.3) then implies that the effective energy-momentum tensor (including both ordinary and Λ-matter) *also* has the perfect fluid form. In particular we find:

$$\bar{T}_{\mu\nu} = \bar{p}g_{\mu\nu} + (\bar{p} + \bar{\rho})u_\mu u_\nu, \tag{4.9}$$

where:

$$\bar{p} \equiv p - \frac{\Lambda}{8\pi G} \quad , \quad \bar{\rho} \equiv \rho + \frac{\Lambda}{8\pi G}. \tag{4.10}$$

We can therefore employ the same dynamical equations as in Chapters 2 and 3, provided that we replace the quantities p and ρ with the modified pressure and density given by eq. (4.10). [This argument is due to Weinberg (1972), p. 614.]

The first things we need are the *field equation* (2.22) and the *energy conservation* equation (2.35). Replacing p and ρ in these equations, as specified above, we find that the first one takes the form:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{3}a^2 - k, \tag{4.11}$$

while the second is written:

$$\frac{d}{da}\left[\left(\rho + \frac{\Lambda}{8\pi G}\right)a^3\right] = -3\left(p - \frac{\Lambda}{8\pi G}\right)a^2. \tag{4.12}$$

Eq. (4.11) is sometimes misnamed the "Einstein equation," although Einstein did not derive it (Felten and Isaacman 1986).

We also require an *equation of state* relating p and ρ, the pressure and density of ordinary matter. For this we can bring over eq. (2.62) from Chapter 2:

$$p = (\gamma - 1)\rho, \tag{4.13}$$

where γ is the *adiabatic index*, possible values for which will be discussed in the following section. Note that we do not replace p and ρ in eq. (4.13) with their modified expressions (4.10). This is because we want *ordinary* matter, not the "effective matter" obtained by including the Λ-term, to obey this simple equation of state. Some authors (eg. Madsen and Ellis 1988, Madsen *et al* 1992) take the opposite approach and absorb Λ into their equation of state; we prefer to keep it separate so that its effects on the dynamics can be more readily distinguished.

Cosmological models derived using the fundamental equations (4.11) – (4.13) are often called "Friedmann-Robertson-Walker" (FRW) or "Friedmann models" (eg. Weinberg 1972, p. 473). This term, however, is misleading because Friedmann solved only the special case $\gamma = 1$. In fact it was Lemaître who first derived eqs. (4.11) and (4.12), so these are properly termed the *Lemaître equations*, and models based all three equations (4.11) – (4.13) with a general value of γ should be called Lemaître models. Friedmann models form the subset of Lemaître models with $\gamma = 1$, and the appellation "Robertson-Walker" (RW) should be reserved for the *line element* (2.9). [These clarifications of terminology have been emphasized by Felten and Isaacman (1986).]

Substituting eq. (4.13) into eq. (4.12), we find:

$$\frac{d}{da}\left(\rho a^{3\gamma}\right) = -\frac{a^{3\gamma}}{8\pi G}\frac{d\Lambda}{da}. \tag{4.14}$$

This is a *generalized energy conservation equation*, valid whether Λ varies or not. When $\Lambda =$ constant, eq. (4.14) returns the well-known result that density ρ scales as a^{-3} in a pressure-free universe ($\gamma = 1$) and a^{-4} in a radiation-dominated universe ($\gamma = 4/3$).

Differentiating eq. (4.11) with respect to time and inserting the energy conservation equation (4.14) produces the following expression for the acceleration of the scale factor:

$$\ddot{a} = \frac{8\pi G}{3}\left(1 - \frac{3\gamma}{2}\right)\rho a + \frac{\Lambda}{3}\,a. \tag{4.15}$$

When $\Lambda = 0$ this reduces [with the help of the equation of state (4.13)] to eq. (3.15), as one would expect. Eqs. (4.11) and (4.15), which can be derived directly from the field equations (4.1), are sometimes presented as a pair of independent differential equations.

As we have shown here, they can be derived from each other using the equation of energy conservation. [This is a consequence of the fact that the field equations are related to the conservation equation via the Bianchi identities (Weinberg 1972, p. 473).]

Eq. (4.15) already tells us a great deal about cosmic evolution. From it we can confirm, for example, that a positive density ρ acts to decelerate the expansion, as expected — *but only if* $\gamma > 2/3$. If, on the other hand, the cosmological fluid is such that $\gamma < 2/3$, then its density can actually *accelerate* the expansion, $\ddot{a} > 0$. This is the phenomenon commonly known as inflation. Eq. (4.15) also confirms that a positive cosmological term will always contribute positively to acceleration, "propping up" the scale factor against the deceleration caused by the matter term ρ (when $\gamma > 2/3$). This, of course, is why a positive cosmological term helps to extend the lifetime of the universe. A *negative* Λ-term, on the other hand, always acts in the opposite direction and brings about recollapse more quickly.

Combining eqs. (4.11) and (4.15), finally, we can eliminate ρ and obtain a differential equation for the scale factor in terms of the cosmological term alone:

$$\frac{\ddot{a}}{a} = \left(1 - \frac{3\gamma}{2}\right)\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) + \frac{\gamma}{2}\Lambda. \tag{4.16}$$

To find the behaviour of $a(t)$ in an "extended Lemaître" model characterized by some expression for $\Lambda(t, a, \dot{a}, \ddot{a})$, one need only substitute it into eq. (4.16) and solve. This differential equation will therefore be of central importance in the chapters that follow.

4.7 Equation of State

We digress in this section to discuss possible values for γ in eq. (4.13). As we have noted, Friedmann assumed that $\gamma = 1$, corresponding to a dust-like universe with zero pressure. The majority of cosmologists since Friedmann's time have *retained* this assumption, at least in analytical and numerical studies of the evolution of the scale factor with time (Stabell and Refsdal 1966, Refsdal *et al* 1967, Robertson and Noonan 1968, Edwards 1972, Campusano *et al* 1975, Rindler 1977, Heidmann 1980, Felten and Isaacman 1986). Felten and Isaacman (1986), for example, argue that pressure is negligible for nearly the entire history of the universe, becoming significant only just after the big bang (on cosmic scales), and can therefore be neglected in numerical approximations. It is true that the present universe is probably well-described as dust-like. It should also be borne in mind, however, that this perception is not based on a lack of observed higher-pressure matter — after all, some ninety percent of the mass density of the universe has not been identified, and could well be composed of light neutrinos (Sciama 1994, Overduin and Wesson 1997b) — but simply on the fact that higher pressures lead to *lifetimes* too short to be reconciled with observations (Peebles 1993, p. 312). This means that it would be prudent to consider at least the case of radiation-dominated conditions ($\gamma = 4/3$, *viz.* § 2.3), as has been done by several authors (McIntosh 1968, Agnese *et al* 1970).

We would also like to be able to model a variety of physical situations besides dust and radiation. *Inflation* (ie., $\ddot{a} > 0$), for example, occurs for any $\gamma < 2/3$, as noted above. In particular, we saw in § 2.9 that γ takes the value $2/3p$ in *power-law* inflation, where p is the power-law parameter, which can in principle take any value greater than one. (In § 2.9, $p_A = 3$ meant that $\gamma_A = 2/9$. The limiting case $p \longrightarrow \infty$ would correspond to pure de Sitter expansion, $\gamma = 0$.) During inflation of other kinds, γ does not stay strictly constant — but it does stay *nearly* constant whenever the slow-roll conditions $\dot{\phi}^2 \ll V(\phi)$ hold (that is, in nearly all realistic models of inflation), since eqs. (3.14) then imply that $p \approx -\rho$, or $\gamma \approx 0$. [This equation of state was appears to have been suggested initially by Gliner (1966) as a means of modelling *vacuum-like* matter.]

The same equations also show that a "fast-rolling" scalar field would behave like *stiff matter* ($\gamma \approx 2$, *viz.* § 2.6), as discussed by Madsen *et al* (1992). [The stiff equation of state was already used in a cosmological context by Zel'dovich (1972), to describe a universe filled with "cold baryons."]

Caldwell *et al* (1997), meanwhile, have proposed a new type of cosmological matter, the "Q-component" (or "quintessence"), with γ between zero and one (the cases $\gamma = 5/6$ and $2/3$ are cited as specific examples). A fluid with $\gamma = 2/3$ ($p = -\rho/3$) would have special properties since, from eq. (4.15), it would not contribute to the deceleration; this has been used elsewhere to model string-dominated conditions (Gott and Rees

1987), so-called "K-matter" (Kolb 1989), and matter in low-density closed universes (Kamionkowski and Toumbas 1996). In a similar vein, Turner and White (1997) have speculated on the possible existence of a new "x-component" of cold dark matter with $\gamma \approx 0.4$.

With all these possible applications in mind, we will relax Friedmann's assumption in this thesis and *consider any (constant) value of γ in the range* $0 \leq \gamma \leq 2$. (These limits come from the fact that the speed of sound in the fluid is given by $v_s = \sqrt{p/\rho} = \sqrt{\gamma - 1}$, which we require to be both real and less than c, or one in our units.) It is worth pointing out that negative values of γ have also been contemplated in cosmology (Harrison 1967). An interesting discussion of the implications of different values of γ for astrophysical problems such as galaxy formation is given in Visser (1997).

4.8 Generalizations

A generalization of this treatment could include *multiple* fluids. If, for example, radiation and dust-like matter were present with comparable energy densities, then one might want to "average over" both their contributions to the total pressure and density of the cosmological fluid. This is discussed by Madsen *et al* (1992), who show that the effective adiabatic index is then:

$$\gamma = \rho^{-1} \sum_i \gamma_i \rho_i, \tag{4.17}$$

where γ_i and ρ_i are the adiabatic index and density of the i-th matter component, and ρ is the combined density of all the components. Strictly speaking, this equation assumes that the various fluid components are *noninteracting*; that is, not exchanging energy at a significant rate. We will not adopt eq. (4.17) here, leaving this for future work. In practice this means we assume the matter of the universe is well-approximated by a single fluid component at any given time in its history.

One could in principle go further and generalize to *variable* γ. This might allow for smooth modelling of the evolution of the universe as it moves from, say, an inflationary to a radiation-dominated stage. In their study of density evolution as a function of the scale factor, for instance, Madsen *et al* (1992) used the expressions:

$$\gamma_{ri}(a) = \frac{4/3}{1+C_{ri}a^4} \quad \longrightarrow \quad \begin{cases} 4/3 & \text{for } a \ll C_{ri}^{-1/4} \\ 0 & \text{for } a \gg C_{ri}^{-1/4} \end{cases} ; \tag{4.18}$$

$$\gamma_{rm}(a) = \frac{(4/3)+C_{rm}a}{1+C_{rm}a} \quad \longrightarrow \quad \begin{cases} 4/3 & \text{for } a \ll C_{rm}^{-1} \\ 1 & \text{for } a \gg C_{rm}^{-1} \end{cases} \tag{4.19}$$

to model the transitions from radiation-dominated to inflationary conditions (in the very early universe) and radiation- to matter-dominated conditions (at decoupling) respectively. Similarly, Caldwell (1996) used an expression of the form:

$$\gamma_{ir}(a) = \frac{(4C_{ir}/3)a^4}{1+C_{ir}a^4} \quad \longrightarrow \quad \begin{cases} 0 & \text{for } a \ll C_{ir}^{-1/4} \\ 4/3 & \text{for } a \gg C_{ir}^{-1/4} \end{cases} \tag{4.20}$$

to track metric perturbations across the inflation-to-radiation transition. (Here C_{ri}, C_{ir} and C_{rm} are constants, with subscripts "r," "i" and "m" denoting radiation, inflation and matter respectively). Other authors have expressed γ as a function of density or time instead of the scale factor: via a polytropic equation of state $p = \rho^m$, for example (McVittie 1965, p. 154), or more complicated relationships such as those given by McIntosh (1968), Rosen (1985), Israelit and Rosen (1989) Stornaiolo (1994), Mangano *et al* (1995), or Capozziello *et al* (1995).

In theory, any or all of these expressions for variable γ could be used in eqs. (4.11) through (4.15) to solve for the scale factor as a function of time. This would certainly oblige us to "resort at once to numerical integration," as McVittie (1965) puts it. For most purposes, however, it is probably adequate to adopt the simplifying assumption that the transition periods between the different epochs of cosmic evolution are unimportant in comparison to the epochs themselves. One can then make a "multistage" approximation in which γ takes on a succession of different (constant) values; eg., from $\gamma \approx 0$ during inflation, to $\gamma = 4/3$ during the radiation era, to $\gamma = 1$ at present. (The discontinuous changes in γ would presumably correspond to phase transitions in the fluid.) This is the approach taken by Starkovich and Cooperstock (1992) in their classical scalar fluid model of the universe. It is supported by at least two independent arguments. Firstly, the transitions between the inflationary, radiation- and matter-dominated eras *are* believed on theoretical grounds to have been very short. The time-scale characterizing the transition from inflationary to radiation-dominated conditions (commonly known as "reheating"), for example, is usually taken to be much less than the expansion time-scale H^{-1} in both standard (Kolb and Turner 1990, p. 274) and chaotic (Kofman 1996) inflationary scenarios. And the transition from radiation- to matter-dominated conditions, which occurred between the epochs of matter-radiation equality (at $t_{eq} \sim 10^{11}$ s) and recombination ($t_{rec} \sim 10^{13}$ s; Kolb and Turner 1990, p. 74) was also very short in relative terms. The second piece of support is the fact that Madsen *et al* (1992) obtained the same qualitative results using eqs. (4.17) – (4.19) as they had in a similar study of density evolution four years earlier (Madsen and Ellis 1988) employing the multi-stage approximation. We have, therefore, both theoretical and "experimental" grounds for believing that our assumption $\gamma =$ constant is not unreasonably restrictive.

4.9 Definitions

We conclude this chapter by introducing the terms and definitions which will be needed to connect our solutions to observation. Chief among these are the "lambda" and density parameters given by:

$$\lambda \equiv \frac{\Lambda}{3H^2} \quad , \quad \Omega \equiv \frac{\rho}{\rho_{crit}} \quad , \quad \rho_{crit} \equiv \frac{3H_0^2}{8\pi G}. \tag{4.21}$$

We will be particularly interested in the values of these parameters *at the present time*:

$$\lambda_0 = \frac{\Lambda_0}{3H_0^2} \quad , \quad \Omega_0 = \frac{8\pi G\rho_0}{3H_0^2}. \tag{4.22}$$

These will constitute our primary free parameters throughout the following chapters.

The value of the quantity λ_0, which is widely used in numerical (Campusano *et al* 1975) and observational studies of the cosmological term (Loh 1986, Lahav *et al* 1991, Moles 1991, Futamase 1992, Kochanek 1992, Krauss and Schramm 1993, Hoell *et al* 1994, Kochanek 1996, Totani *et al* 1997), rests in the fact that it is the Λ-*equivalent* of the more familiar Ω_0. This can be verified by using the definitions (4.21) to rewrite the Lemaître equation (4.11) in the shorthand form:

$$H^2 = H_0^2(\Omega + \lambda) - k/a^2, \tag{4.23}$$

or, at the present time $\tau = \tau_0$:

$$k = a_0^2 H_0^2(\Omega_0 + \lambda_0 - 1), \tag{4.24}$$

The parameter λ_0 is sometimes denoted Ω_Λ (eg. Perlmutter *et al* 1997), for obvious reasons. From the form of eq. (4.24) it is clear that:

$$\begin{aligned} \Omega_0 + \lambda_0 > 1 \quad &\Longrightarrow \quad k > 0 \\ \Omega_0 + \lambda_0 = 1 \quad &\Longrightarrow \quad k = 0 \\ \Omega_0 + \lambda_0 < 1 \quad &\Longrightarrow \quad k < 0. \end{aligned} \tag{4.25}$$

Most cosmologists implicitly choose units for a_0 such that the value of k is normalized to either 0 or ± 1. We will follow Felten and Isaacman (1986) in refraining from this, because it is more convenient for our purposes to choose units such that $a_0 \equiv 1$. In the remainder of this thesis, then, the value of k is specified by eq. (4.24) and should not be assumed to be 0 or ± 1.

Finally, we mention that some authors (eg. Stabell and Refsdal 1966, Refsdal *et al* 1967) employ the quantities q_0 and σ_0, rather than λ_0 and Ω_0. We give here the relationships between them. The density parameter σ is defined by:

$$\sigma \equiv \frac{4\pi G\rho}{3H^2}, \tag{4.26}$$

so we have (at the present time):

$$\sigma_0 = \frac{1}{2}\Omega_0. \tag{4.27}$$

The deceleration parameter q is defined by:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}. \tag{4.28}$$

If we substitute this into eq. (4.15) along with the definitions (4.21), we obtain the relation $q = (3\gamma/2 - 1)\Omega - \lambda$, or (at the present time):

$$q_0 = \left(\frac{3\gamma}{2} - 1\right)\Omega_0 - \lambda_0. \tag{4.29}$$

From this result we can confirm the well-known rule that a large (positive) cosmological term tends to drive q_0 toward negative values. In the case $\gamma = 1$, these expressions reduce to those given by Felten and Isaacman (1986).

Chapter 5

Models With $\Lambda \propto t^{-\ell}$

5.1 Interpretation of the Time Co-ordinate

We begin with what is probably the most straightforward implementation of the variable-Λ idea; namely, one in which the cosmological term is a simple power-law function of time t, as set out in eq. (4.5):

$$\Lambda = \mathcal{A} t^{-\ell}, \tag{5.1}$$

where $\mathcal{A}$ and ℓ are, for the time being, arbitrary constants. As mentioned in § 4.3, the case $\ell = 2$ has previously been considered by several authors. In each case, however, supplementary conditions are present as well. Lau (1985) assumes a time-varying gravitational "constant" G. Berman (1991b) imposes the requirements that the density ρ of ordinary matter *also* scale as t^{-2}, and that the deceleration parameter q be a constant. Lopez and Nanopoulos (1996) take Λ to have the same dependence on the *scale factor* as on time ($\Lambda \propto a^{-2}$), for late times at least. None of these authors address the question of singularity avoidance. We wish to study the properties of models with the property (5.1) in a more comprehensive way.

Several conceptual issues should be dealt with before we proceed. Firstly, we are at least as interested in *nonsingular* models as we are in those which began in a singularity. Therefore we are obliged to broaden the traditional definition of cosmic time, in which it is set to zero at the moment when $a = 0$; ie., at the big bang (eg. Weinberg, p. 473). We choose here to measure t from the point where $a = 0$ in singular universes (the big bang, as usual), *or from the point where* $da/dt = 0$ *in nonsingular ones* (the "big bounce"). (Strictly speaking, in the latter case we choose the *most recent* past event $da/dt = 0$.) In either case we refer to this as the "initial moment" and denote it by $t = t_* \equiv 0$. We reserve the subscript "0" for the present epoch, $t = t_0 \geq 9.6 \times 10^9$ yr (Chaboyer *et al* 1997). Similar conventions have been adopted by others in working with nonsingular models (Özer and Taha 1987, Abdel-Rahman 1990, 1992; Starkovich and Cooperstock 1992, Bayin *et al* 1994).

Secondly, it should not necessarily be assumed that the parameter t in eq. (5.1) *is* in fact cosmic time, as defined in the standard model of cosmology (eg. Weinberg 1972, p. 410; Narlikar 1983, p. 94; Wald 1984, p. 95). For example, eq. (5.1) implies that $\Lambda \to \infty$ in the limit $t \to 0$, which may not be realistic. Λ may go to infinity at some *other* time (t_∞, say) than the initial moment. Or, more likely, its divergent behaviour may be cut off at some critical temperature (at time t_c, say) by a phase transition or similar mechanism, above which Λ is effectively constant (Kazanas 1980). So a more plausible formulation of eq. (5.1) in terms of cosmic time t would take the form:

$$\Lambda = \begin{cases} \Lambda_c & \text{when } t < t_c \\ \mathcal{A}(t - t_\infty)^{-\ell} & \text{when } t \geq t_c \end{cases}, \tag{5.2}$$

where continuity across the cutoff time t_c implies that $\Lambda_c \equiv \mathcal{A}(t_c - t_\infty)^{-\ell}$.

Eq. (5.2) for $\Lambda(t)$ could, if desired, be substituted into eq. (4.16) and the other dynamical equations of § 4.6, which could then be solved for the scale factor $a(t)$. All these equations now use the same time coordinate t. However, we can save ourselves a great deal of unnecessary work by noting that the dynamical equations in themselves contain no explicit t-dependence, but only *derivatives* with respect to t. We can therefore shift our time co-ordinate as follows:

$$t \to \tilde{t} \equiv (t - t_\infty), \tag{5.3}$$

without changing their form. Eqs. (4.11) – (4.16) remain valid, with the overdot now signifying differentiation with respect to $\tilde{t}$ instead of t. And the decay law (5.2) goes back to its original, simpler form:

$$\Lambda = \mathcal{A}\bar{t}^{-\ell}, \tag{5.4}$$

for all $\bar{t} \geq t_c - t_\infty$ (ie., $t \geq t_c$). In practice we will assume that eq. (5.4) holds at all times of interest. For earlier times, $\bar{t} < (t_c - t_\infty)$, the standard (Lemaître) solution holds, with $\Lambda = \text{constant} = \Lambda_c$. If $t_c \approx t_\infty$, then eq. (5.4) holds all the way down to $\bar{t} \approx 0$.

It is interesting to note that according to some accounts (eg. Weinberg 1972, p. 409), expressions of the form (5.4), in which some cosmic scalar field decays monotonically everywhere, are essential in attaching physical significance to the *definition* of cosmic time. One assumes in the standard model of cosmology that four-dimensional spacetime can be "foliated" into a series of spacelike hypersurfaces, each a surface of simultaneity with respect to cosmic time. [This is part of what is known as Weyl's postulate; Narlikar (1983), p. 91.] But the concept remains largely esoteric until these surfaces are "tagged" with successive values of some physical quantity such as proper density or black-body radiation temperature. In the theory we are considering here, this role could equally well be played by successive values of the *cosmological "constant."*

5.2 Riccati's Equation

For later convenience, we will make one last change to our independent variable, choosing to measure it in units of *Hubble times* H_0^{-1}, where H_0 is the current value of Hubble's parameter. In other words we define a dimensionless new parameter:

$$\tau \equiv H_0\bar{t}, \tag{5.5}$$

whereupon the decay law (5.4) takes its final form:

$$\Lambda = \mathcal{A}\tau^{-\ell}, \tag{5.6}$$

for all $\tau \geq H_0(t_c - t_\infty)$, where we have absorbed a factor of H_0^ℓ into the proportionality constant $\mathcal{A}$.

We are now in a position to study the evolution of the scale factor. This is governed by the differential equation (4.16), which can be written as follows in terms of τ:

$$\frac{1}{a}\frac{da'}{d\tau} = \left(1 - \frac{3\gamma}{2}\right)\left(\frac{a'^2}{a^2} + \frac{k}{H_0^2 a^2}\right) + \frac{\gamma}{2}\frac{\Lambda}{H_0^2}. \tag{5.7}$$

where primes denote differentiation with respect to τ, and we have used the definition (5.5) to put $\dot{a} \to H_0 a'$.

Eq. (5.7) is a nonlinear first-order differential equation in a', with variable coefficients in a. Let us try changing dependent variables via:

$$H \equiv \frac{\dot{a}}{a} = H_0\left(\frac{a'}{a}\right), \tag{5.8}$$

whereupon:

$$\frac{1}{a}\frac{da'}{d\tau} = \frac{1}{H_0^2}\left(H_0\frac{dH}{d\tau} + H^2\right). \tag{5.9}$$

Eq. (5.7) then takes the form:

$$\frac{dH}{d\tau} = \left(\frac{-3\gamma}{2H_0}\right)H^2 + \left(\frac{\gamma}{2H_0}\right)\Lambda + \left(1 - \frac{3\gamma}{2}\right)\left(\frac{k}{H_0 a^2}\right). \tag{5.10}$$

If we restrict ourselves to spatially *flat universes* ($k = 0$), then the last term drops off and we are left with a special case of *Riccati's equation*:

$$H' = \mathcal{P}(\tau)\,H^2 + \mathcal{Q}(\tau)\,H + \mathcal{R}(\tau), \tag{5.11}$$

where:

$$\mathcal{P}(\tau) \equiv -\frac{3\gamma}{2H_0} \quad , \quad \mathcal{Q}(\tau) \equiv 0 \quad , \quad \mathcal{R}(\tau) \equiv \left(\frac{\gamma}{2H_0}\right)\Lambda(\tau). \tag{5.12}$$

We will adopt this restriction for the remainder of Chapter 5. The same thing was done by Lau (1985), Berman (1991b), and Lopez and Nanopoulos (1997) in their solutions.

Riccati's equation is solved (Spiegel 1981, p. 217) by changing dependent variables from H to x via:

$$H \equiv -\frac{1}{\mathcal{P}x}\frac{dx}{d\tau} = \left(\frac{2H_0}{3\gamma}\right)\frac{1}{x}\frac{dx}{d\tau}. \tag{5.13}$$

Making this change, we find that eq. (5.10) with $k = 0$ takes the form:

$$\frac{1}{x}\frac{d^2x}{d\tau^2} = \left(\frac{3\gamma}{2H_0}\right)\left(\frac{\gamma}{2H_0}\right)\Lambda(\tau). \tag{5.14}$$

Substituting eq. (5.6) for $\Lambda(\tau)$ into this result, we finally obtain:

$$\tau^\ell \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{5.15}$$

where:

$$\alpha \equiv \frac{3\gamma^2 \mathcal{A}}{4H_0^2}. \tag{5.16}$$

This is now a linear second-order differential equation, albeit still one with a variable coefficient. We will solve for $x(\tau)$ in the cases $\ell = 1, 2, 3, 4$.

Once $x(\tau)$ is found, the Hubble parameter $H(\tau)$ is given immediately from the definition (5.13). Moreover, the *scale factor* $a(\tau)$ is also given to us, as may be verified by comparing eqs. (5.13) and (5.8) to yield:

$$a(\tau) = [x(\tau)]^{2/3\gamma}. \tag{5.17}$$

The constant α given by eq. (5.16) can be fixed in terms of observational quantities as follows. Applying the decay law (5.6) to the present epoch $\tau = \tau_0$, and using the definition (4.22) of λ_0, we find that:

$$\mathcal{A} = 3H_0^2\lambda_0\tau_0^\ell, \tag{5.18}$$

which can be substituted into eq. (5.16) to yield:

$$\alpha = \left(\frac{3\gamma}{2}\right)^2 \lambda_0\tau_0^\ell. \tag{5.19}$$

It should be mentioned that, since we are operating under the assumption that $k = 0$ in this chapter, Ω_0 is always given by $(1 - \lambda_0)$.

5.3 The Case $\ell = 1$

We now proceed to the first case. When $\ell = 1$ the differential equation (5.15) reads:

$$\tau \frac{d^2 x}{d\tau^2} - \alpha x = 0, \tag{5.20}$$

where α is given by eq. (5.19):

$$\alpha = \left(\frac{3\gamma}{2}\right)^2 \lambda_0 \tau_0. \tag{5.21}$$

The general solution can be obtained (Murphy 1960, p. 329) by changing independent variables from x to z, defined by:

$$z \equiv 2\sqrt{-\alpha\tau}. \tag{5.22}$$

In terms of the new variable z, the differential equation (5.20) takes the form:

$$z^2 \frac{d^2 x}{dz^2} - z\frac{dx}{dz} + z^2 x = 0. \tag{5.23}$$

This is transformable to *Bessel's equation*, with general solution (Spiegel 1974, p. 125):

$$x(z) = c_1 z J_1(z) + c_2 z Y_1(z), \tag{5.24}$$

where $J_1(z)$ and $Y_1(z)$ are Bessel and Neumann functions of order one. Eq. (5.17) then gives us the scale factor:

$$a(\tau) = \tau^{1/3\gamma} \left[c_1 J_1(\psi) + c_2 Y_1(\psi)\right]^{2/3\gamma}, \tag{5.25}$$

where we have absorbed a factor of $\sqrt{-4\alpha}$ into c_1, c_2, and $\psi(\tau)$ is defined by:

$$\psi(\tau) \equiv \sqrt{-4\alpha\tau}. \tag{5.26}$$

The Hubble parameter is found by putting $x(z)$ into eq. (5.13), which yields:

$$H(\tau) = H_0 \sqrt{-\lambda_0 \left(\frac{\tau_0}{\tau}\right)} \left[\frac{c_1 J_0(\psi) + c_2 Y_0(\psi)}{c_1 J_1(\psi) + c_2 Y_1(\psi)}\right], \tag{5.27}$$

where $J_0(z)$ and $Y_0(z)$ are Bessel and Neumann functions of order zero. We note from the definition (5.21) that $\psi(\tau)$, and hence $a(\tau)$ and $H(\tau)$, can only be real (for positive times) if $\lambda_0 \leq 0$, which would imply a *negative cosmological constant*. While it is true that some authors have discussed such a possibility (Banks 1984, Galicki 1991, Fujiwara *et al* 1991, Guendelman and Kaganovich 1994, Coussaert *et al* 1995, Paternoga and Graham 1996, Gibbons 1996), we will see shortly that it can probably be ruled out, in this model at least, because it implies an unrealistically short *age* for the universe (Fig. 5.1). Therefore, since eq. (5.24) represents the general solution for the differential equation (5.20), the case $\ell = 1$ is probably not realized in nature.

We also note from the expressions (5.25) and (5.27) that, as $\tau \to 0$, the scale factor $a \to 0$ and the Hubble parameter $H \to \infty$. So it appears that a decaying cosmological term of the form (5.6) with $\ell = 1$ does not lead to singularity avoidance in a flat universe. Let us make this more precise by considering the boundary conditions on the theory.

If the age τ_0 of the universe is given to us, then we can impose the following boundary conditions at the present epoch:

$$a(\tau_0) = 1 \qquad , \qquad H(\tau_0) = H_0. \tag{5.28}$$

Substituting these expressions into eqs. (5.25) and (5.27), it is straightforward to solve for c_1 and c_2:

$$c_1 = \frac{\sqrt{-\lambda_0}\, Y_0(\psi_0) - Y_1(\psi_0)}{\sqrt{-\lambda_0}\tau_0 \left[J_1(\psi_0)Y_0(\psi_0) - J_0(\psi_0)Y_1(\psi_0)\right]} \tag{5.29}$$

$$c_2 = \frac{-\sqrt{-\lambda_0}\, J_0(\psi_0) + J_1(\psi_0)}{\sqrt{-\lambda_0}\tau_0 \left[J_1(\psi_0)Y_0(\psi_0) - J_0(\psi_0)Y_1(\psi_0)\right]}, \tag{5.30}$$

where:

$$\psi_0 \equiv 3\gamma\tau_0\sqrt{-\lambda_0}. \tag{5.31}$$

Unfortunately, τ_0 is *not* given to us. In fact, in a sense this is precisely what we are trying to determine; a nonsingular model by definition would have $\tau_0 \to \infty$.

We require an additional boundary condition to specify the value of τ_0 along with c_1 and c_2. According to the discussion in § 5.1, a nonsingular model should satisfy $H(0) = 0$. Is this possible in the present case? We can show that it is impossible, as follows.

From eq. (5.27), the Hubble parameter $H(\tau) \propto c_1 J_0(\sqrt{-4\alpha\tau}) + c_2 Y_0(\sqrt{-4\alpha\tau})$. We must have $c_2 = 0$ in order to keep this finite at $\tau = 0$, since $Y_0(x)$ diverges logarithmically at $x = 0$. Then, in order to satisfy the condition $H(0) = 0$, we are obliged to set $c_1 = 0$ *as well,* since $J_0(0) = 1$. With $c_1 = c_2 = 0$, however, $a(\tau) = 0$ for all τ, which is unacceptable. The only way to avoid this conclusion is to lift the condition $H(0) = 0$, which is another way of saying that the model is necessarily singular at $\tau = 0$.

We turn next to the scale factor, which according to eq. (5.25) goes as $a(\tau) \propto c_1 J_1(\sqrt{-4\alpha\tau}) + c_2 Y_1(\sqrt{-4\alpha\tau})$. In order to keep this finite at $\tau = 0$, we must have:

$$c_2 = 0, \tag{5.32}$$

since $Y_1(x)$ also diverges logarithmically at $x = 0$. This then constitutes our third boundary condition. In conjunction with eq. (5.30) it implies:

$$J_1(\psi_0) - \sqrt{-\lambda_0}\, J_0(\psi_0) = 0. \tag{5.33}$$

This equation can be solved numerically for τ_0 [with the help of the definition (5.31)], which can then be substituted back into eq. (5.29) to fix the value of c_1. With c_1 and c_2 both known, $a(\tau)$ and $H(\tau)$ are given by eqs. (5.25) and (5.27) respectively. This completes our solution for the case $\ell = 1$.

The evolution of the scale factor for $\gamma = 1$ and various values of λ_0 is illustrated in Fig. 5.1, which is modelled after similar diagrams in the review of Felten and Isaacman (1986). In particular, we have followed these authors in plotting the scale factor a as a function of $(\tau - \tau_0)$, rather than τ for each curve. [Note that τ_0 depends on the choice of λ_0 via eqs. (5.33) and (5.31) above.] This has the effect of shifting all the curves so that they intersect at $(0, 1)$, which marks the present epoch. [Recall that we have chosen units such that $a_0 = 1$, eq. (4.24).]

We have plotted for four Hubble times into the future, and one Hubble time into the past. It may be seen that the $\lambda_0 = 0$ curve (solid line) intersects the time axis at $(\tau - \tau_0) = -2/3$, which confirms the well-known rule that the age of a flat universe with no cosmological constant is $\tau_0 = 2/3$. The models with $\lambda_0 < 0$ (dashed lines) are

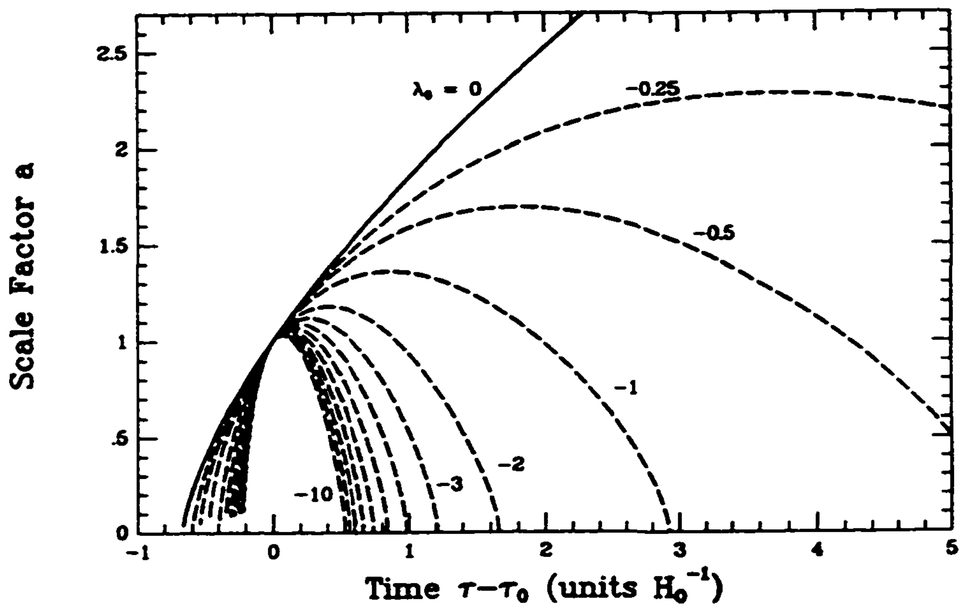


Figure 5.1: Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-1}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

younger than this, which considerably diminishes their attractiveness. The $\lambda_0 = -1$ model, for instance, has $\tau_0 = 0.48$, while the $\lambda_0 = -3$ model has $\tau_0 = 0.35$. If we use a current widely-accepted value of $H_0 = 73 \pm 10$ km s^{-1} Mpc^{-1} (Freedman 1996) for the Hubble parameter, then (recalling that $t = \tau/H_0$) we see that the age of the universe in these models can be no more than 7.4 and 5.4 billion years old respectively. This conflicts badly with estimated globular cluster ages, which are thought to be at least 9.6 billion years old in some cases (Chaboyer *et al* 1997). The situation improves slightly if one switches to the lower values for H_0 which are reported by some authors. If $H_0 = 55 \pm 10$ km s^{-1} Mpc^{-1} (Sandage and Tammann 1996), then the maximum possible age for these two models increases to 10.4 and 7.6 billion years respectively. One can safely rule out models with $\lambda_0 < -1$ on this basis, while models in the range $-1 < \lambda_0 \leq 0$ remain marginally viable at best.

Fig. 5.1 tells us that, while flat models with $\lambda_0 = 0$ will continue to expand indefinitely as usual, *those with negative values of λ_0 will experience eventual recollapse.* This can be understood by looking once again at eq. (4.15), which shows that there are two contributions to the deceleration: one (which goes as ρa) due to ordinary matter and the other (which goes as $|\Lambda| a$) due to the negative cosmological constant. Because the density ρ of matter rapidly thins out with expansion, the first contribution alone is not enough to close the universe when $k = 0$. The second contribution, however, is diluted much more slowly ($|\Lambda|$ drops off as only τ^{-1} in this case), and is therefore enough to bring about eventual recollapse, no matter how small its value at the present time. Thus, models with $\lambda_0 = -1$ and $\lambda_0 = -3$ encounter the "big crunch" after only 2.94 and 1.21 Hubble times respectively.

Note that we have not plotted any curves for positive values of λ_0 since (as discussed above) the scale factor is imaginary in these cases.

5.4 The Case $\ell = 2$

We proceed next to the second case, $\ell = 2$, for which the differential equation (5.15) takes the form:

$$\tau^2 \frac{d^2 x}{d\tau^2} - \alpha x = 0, \tag{5.34}$$

with α given by eq. (5.19) as follows:

$$\alpha = \left(\frac{3\gamma}{2}\right)^2 \lambda_0 \tau_0^2. \tag{5.35}$$

This is a special case of *Euler's differential equation*. The general solution (Spiegel 1981, p. 212) can be obtained by changing independent variables from τ to y, defined by:

$$y \equiv \ln \tau, \tag{5.36}$$

whereupon eq. (5.34) takes the form:

$$\frac{d^2 x}{dy^2} - \frac{dx}{dy} - \alpha x = 0. \tag{5.37}$$

This now has constant coefficients, as desired. The roots of the "auxiliary equation" $m^2 - m - \alpha = 0$ are given by $m = [1 \pm \sqrt{1 + (3\gamma\tau_0)^2 \lambda_0}]/2$, so there are three subcases to consider, according as λ_0 is greater than, equal to, or less than $-1/(3\gamma\tau_0)^2$.

5.4.1 The Subcase $\lambda_0 > -1/(3\gamma\tau_0)^2$

Since we expect on observational grounds that λ_0 is probably positive, this is the most physical of the subcases. The general solution of eq. (5.37) is:

$$x(y) = c_1 e^{m_1 y} + c_2 e^{m_2 y}, \tag{5.38}$$

where c_1, c_2 are arbitrary constants and:

$$m_1 \equiv \frac{1}{2}\left[1 + \sqrt{1 + (3\gamma\tau_0)^2\lambda_0}\right] \quad , \quad m_2 \equiv \frac{1}{2}\left[1 - \sqrt{1 + (3\gamma\tau_0)^2\lambda_0}\right]. \tag{5.39}$$

Using the definition (5.36), we obtain:

$$x(\tau) = c_1 \tau^{m_1} + c_2 \tau^{m_2}. \tag{5.40}$$

The scale factor and Hubble parameter are then found from eqs. (5.17) and (5.13):

$$a(\tau) = \tau^{1/3\gamma}\left(c_1\tau^{m_0} + c_2\tau^{-m_0}\right)^{2/3\gamma} \tag{5.41}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{m_1 c_1 \tau^{m_0} + m_2 c_2 \tau^{m_0}}{\tau\left(c_1\tau^{m_0} + c_2\tau^{-m_0}\right)}\right], \tag{5.42}$$

where:

$$m_0 \equiv \frac{1}{2}\sqrt{1 + (3\gamma\tau_0)^2\lambda_0}. \tag{5.43}$$

In this form it is seen that $a(\tau)$ diverges at $\tau = 0$ for $\lambda_0 > 0$, *unless*:

$$c_2 = 0, \tag{5.44}$$

which we consequently adopt as our first boundary condition. [This is the same as our earlier argument in the $\ell = 1$ case, eq. (5.32).] The scale factor (5.41) and Hubble parameter (5.42) simplify to:

$$a(\tau) = \left(c_1\tau^{m_1}\right)^{2/3\gamma} \tag{5.45}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{m_1}{\tau}\right]. \tag{5.46}$$

We then apply the boundary conditions (5.28) at $\tau = \tau_0$. Eq. (5.46) with $H(\tau_0) = H_0$ implies that:

$$\tau_0 = \frac{2m_1}{3\gamma}. \tag{5.47}$$

Substituting this result into eq. (5.45), with $a(\tau_0) = 1$, we obtain:

$$c_1 = \left(\frac{3\gamma}{2m_1}\right)^{m_1}, \tag{5.48}$$

which can be put back into eq. (5.45) to yield this expression for $a(\tau)$:

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{2m_1/3\gamma}. \tag{5.49}$$

The scale factor expands as a simple power-law function of time. This is consistent with the special solutions $a \propto t^{1/3}$, $a \propto t^{2/3}$ and $a \propto t$ obtained respectively by Lau (1985), Berman (1991b) and Lopez and Nanopoulos (1996) in their $\ell = 2$ models.

In conjunction with the definition (5.39), eq. (5.47) fixes the age of the universe at:

$$\tau_0 = \frac{2}{3\gamma(1-\lambda_0)}, \tag{5.50}$$

from which we draw two important conclusions: Firstly, that all models satisfying the boundary conditions obey:

$$\lambda_0 < 1; \tag{5.51}$$

and secondly, that the age of the universe $\tau_0 \to \infty$ as $\lambda_0 \to 1$. The initial singularity can thus be pushed back arbitrarily far into the past. This is reminiscent of models proposed by Lima and Maia (1994) and Lima and Trodden (1996), which also have $a \to 0$ as $\tau \to 0$. Interestingly, these authors term their solutions nonsingular, even though a vanishes, because $\rho \to 0$ in the same limit (de Sitter-like initial conditions). This is accomplished by having Λ decay in such a way that the Hubble parameter H rises to a finite maximum value H_I in the early universe[1]. In the present case, by

[1] In a similar spirit, Israelit and Rosen (1993) use a special equation of state to keep ρ finite even as $a \to 0$, arguing that the resulting singularity is not physical but merely "geometrical." This however is an uncertain argument at best, since density is conventionally defined as a limit of the form $\lim_{\Delta V \to 0}(\Delta m/\Delta V)$, and it is difficult to make sense of the concept of a volume element if $a = 0$. Ultimately it is probably safest to rely on the notion of *geodesic incompleteness* in deciding whether to call a given solution singular or not (Wald 1984, § 9.1).

contrast, eq. (5.46) shows that $H \to \infty$ at $\tau \to 0$. (Refinements of the model, such as the introduction of the cutoff value of Λ mentioned in § 5.1, might alter the situation.)

We also find a *lower* limit on the age of the universe in these models by noting that $m_1 = (m_0 + 1/2) > 1/2$. Inserting this into eq. (5.47) produces the result:

$$\tau_0 > \frac{1}{3\gamma}. \tag{5.52}$$

In other words, assuming zero pressure ($\gamma = 1$), all models in this case have survived for at least one-third of a Hubble time. Adopting the upper limit of 83 km s^{-1} Mpc^{-1} reported by Freedman (1996), this implies a minimum age of at least 3.9 billion years. Finally, putting eq. (5.52) back into the expression (5.50) for τ_0, we find that:

$$\lambda_0 > -1, \tag{5.53}$$

which defines the critical value of λ_0 separating this subcase from the other two.

5.4.2 The Subcase $\lambda_0 = -1/(3\gamma\tau_0)^2$

For this subcase the auxiliary equation of Euler's differential equation (5.37) has only one root $m = 1/2$, and the general solution is:

$$x(y) = (c_3 + c_4\, y)\, e^{\, y/2}, \tag{5.54}$$

where c_3, c_4 are arbitrary constants. Using the definition (5.36), we obtain:

$$x(\tau) = \sqrt{\tau}\,(c_3 + c_4 \ln \tau). \tag{5.55}$$

The scale factor and Hubble parameter are then found from eqs. (5.17) and (5.13):

$$a(\tau) = \tau^{1/3\gamma}\,(c_3 + c_4 \ln \tau)^{2/3\gamma} \tag{5.56}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{(c_3 + 2c_4) + c_4 \ln \tau}{2\tau(c_3 + c_4 \ln \tau)}\right]. \tag{5.57}$$

We can see from the form of eq. (5.56) that $a(\tau)$ diverges at $\tau = 0$ unless:

$$c_4 = 0, \tag{5.58}$$

which we consequently assume. Eqs. (5.56) and (5.57) then simplify to:

$$a(\tau) = (c_3\sqrt{\tau})^{2/3\gamma} \quad (5.59)$$

$$H(\tau) = \frac{H_0}{3\gamma\tau}. \quad (5.60)$$

Inserting $H(\tau_0) = H_0$ into eq. (5.60), we find for the age of the universe in this model:

$$\tau_0 = \frac{1}{3\gamma}, \quad (5.61)$$

which is exactly the limit $\lambda_0 \to -1$ in eq. (5.50), confirming that the two subcases are separated by this value for the current value of the cosmological term. Eq. (5.61) also corresponds to the lower limit allowed by eq. (5.52), as one might expect.

Substituting the age (5.61) into eq. (5.59), meanwhile, and imposing $a(\tau_0) = 1$ as usual, we find:

$$c_3 = \sqrt{3\gamma}. \quad (5.62)$$

Inserted back into eq. (5.59), this leads to the following expression for the scale factor:

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{1/3\gamma}, \quad (5.63)$$

which joins smoothly onto the result (5.49) from the previous subcase.

5.4.3 The Subcase $\lambda_0 < -1/(3\gamma\tau_0)^2$

For this final subcase Euler's differential equation (5.37) has the general solution:

$$x(y) = e^{y/2}[c_5 \sin(m_3 y) + c_6 \cos(m_3 y)], \quad (5.64)$$

where c_5, c_6 are arbitrary constants and:

$$m_3 \equiv \frac{1}{2}\sqrt{-(3\gamma\tau_0)^2\lambda_0 - 1}. \quad (5.65)$$

Using the definition (5.36), we obtain:

$$x(\tau) = \sqrt{\tau}\,[c_5 \sin(m_3 \ln \tau) + c_6 \cos(m_3 \ln \tau)], \quad (5.66)$$

The scale factor and Hubble parameter are then found from eqs. (5.17) and (5.13):

$$a(\tau) = \tau^{1/3\gamma}\left[c_5\sin(m_3\ln\tau) + c_6\cos(m_3\ln\tau)\right]^{2/3\gamma}, \quad (5.67)$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left\{\frac{(c_5 - 2m_3c_6)\sin(m_3\ln\tau) + (c_6 + 2m_3c_5)\cos(m_3\ln\tau)}{2\tau[c_5\sin(m_3\ln\tau) + c_6\cos(m_3\ln\tau)]}\right\}. \quad (5.68)$$

Application of the boundary conditions (5.28) gives c_5 and c_6 in terms of τ_0:

$$c_5 = \frac{1}{\sqrt{\tau_0}}\sin(m_3\ln\tau_0) + \frac{1}{m_3}\left[\left(\frac{3\gamma}{2}\right)\sqrt{\tau_0} - \frac{1}{2\sqrt{\tau_0}}\right]\cos(m_3\ln\tau_0) \quad (5.69)$$

$$c_6 = \frac{1}{\sqrt{\tau_0}}\cos(m_3\ln\tau_0) - \frac{1}{m_3}\left[\left(\frac{3\gamma}{2}\right)\sqrt{\tau_0} - \frac{1}{2\sqrt{\tau_0}}\right]\sin(m_3\ln\tau_0). \quad (5.70)$$

As usual, we need a third boundary condition to fix the value of τ_0. This subcase, however, is not like the other two, for *both* terms in the scale factor diverge at $\tau = 0$. We cannot keep $a(\tau)$ finite by setting either of c_5 or c_6 to zero. It may appear that it *is* possible to keep $a(\tau)$ finite by setting $m_3 = 0$. This, however, is too strong a condition, since putting $m_3 = 0$ into eqs. (5.67) and (5.68) gives:

$$a(\tau) = (c_6\sqrt{\tau})^{2/3\gamma} \quad (5.71)$$

$$H(\tau) = \frac{H_0}{3\gamma\tau}, \quad (5.72)$$

which is *exactly the same* as the solution of the previous subcase, eqs. (5.59) and (5.60). In conjunction with the boundary condition $H(\tau_0) = H_0$, this result would imply that $\tau_0 = 1/3\gamma = $ constant *for all* $\lambda_0 < -1$, which is unrealistic.

In fact, numerical experiments with different values of τ_0 demonstrate that the shape of $a(\tau)$ does depend on the value of λ_0, and furthermore that $a(\tau)$ does approach zero smoothly as $\tau \to 0$. As the quickest route to solving for τ_0 for a given value of λ_0, therefore, we have opted for a simple iterative procedure, as follows. We know that for this subcase, the age τ_0 will be *less than* the lower limit $1/3\gamma$ given by eq. (5.61), because larger negative values of λ_0 lead to faster recollapse. Let us therefore define:

$$\tau_0^{(0)} \equiv 1/3\gamma \quad (5.73)$$

as our zeroth-order *approximation* to the true age. Setting $\tau_0 = \tau_0^{(0)}$, we then solve (numerically) for the roots of the equation $a(\tau) = 0$ using eq. (5.67). We select the largest root which is less than $\tau_0^{(0)}$, and denote it by $\tau_*^{(0)}$. The first-order approximation to the true age τ_0 is then given by:

$$\tau_0^{(1)} \equiv \tau_0^{(0)} - \tau_*^{(0)}. \tag{5.74}$$

We then repeat the process, with $\tau_0 = \tau_0^{(1)}$. This produces a *new* root, which we denote $\tau_*^{(1)}$. The procedure is repeated k times:

$$\tau_0^{(k)} = \tau_0^{(k-1)} - \tau_*^{(k-1)}, \tag{5.75}$$

until the solution eventually converges onto the correct value of τ_0. This occurs when:

$$\tau_0^{(k)} \approx \tau_0^{(k-1)}, \tag{5.76}$$

to within some specified tolerance. At this point we conclude that $\tau_0 \approx \tau_0^{(k)}$ is the age of the universe for the model in question. The values of c_5 and c_6 then follow from eqs. (5.70) and (5.70) respectively. This completes our solution for the case $\ell = 2$.

The evolution of the scale factor for $\gamma = 1$ and various values of λ_0 is illustrated in Fig. 5.2, which has exactly the same format as Fig. 5.1, except that we have plotted for three Hubble times into the past instead of one. Fig. 5.2 exhibits a considerably richer variety of solutions than Fig. 5.1. The most noticeable difference is the existence of solutions for positive λ_0 (short-dashed lines). Of particular interest is the limiting case $\lambda_0 = 1$, which levels off in the infinite past, putting off the initial singularity indefinitely. This case is not very realistic, however, as it has zero density (since $\Omega_0 = 1 - \lambda_0$). It is in fact the empty de Sitter model. The same solution is found for these values of λ_0 and Ω_0 in *conventional* Lemaître cosmology with $\Lambda =$ constant (Felten and Isaacman 1986).

Fig. 5.2 therefore shows that we cannot remove the initial singularity in a theory with $\Lambda \propto \tau^{-2}$. We can, however, *significantly extend the age of the universe*. Suppose we choose $\lambda_0 = 0.5$, for example; a value compatible with the tightest observational bounds so far (Perlmutter *et al* 1997). Fig. 5.2 shows that this model would have come into being 1.33 Hubble times ago [see also eq. (5.50) above]. Even if we adopt the upper limit of 83 km s^{-1} Mpc^{-1} reported by Freedman (1996) for H_0, this translates into an age

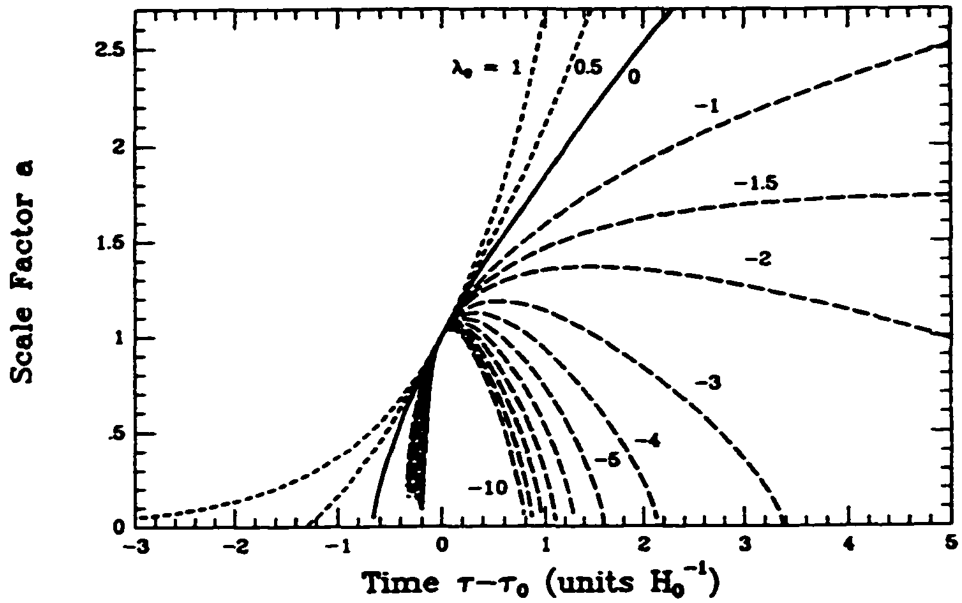


Figure 5.2: Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-2}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

of 15.7 billion years — more than enough time for the oldest globular clusters to form (Chaboyer *et al* 1997). By way of comparison, a *constant*-Λ model with $\lambda_0 = \Omega_0 = 0.5$ has an age of only 0.83 Hubble times. [We will show how to calculate this quantity in the next chapter (§ 6.4.2).]

Flat models with *no* cosmological constant, meanwhile, are represented in Fig. 5.2 by the curve labelled $\lambda_0 = 0$ (solid line). As in the $\ell = 1$ case, these have an age of 2/3 Hubble times. Models with a negative λ_0 (long-dashed lines) all have shorter ages, as in the $\ell = 1$ case. The difference is that they are now *even shorter*, because the negative cosmological term is driven to high negative values *more quickly* in the past direction when $\ell = 2$. Thus, the age of the $\lambda_0 = -1$ model has dropped from 0.48 to 0.33 Hubble times, while that of the $\lambda_0 = -3$ model is now only $\tau_0 = 0.30$, down from 0.35 in Fig. 5.1.

Fig. 5.2 indicates that the $\lambda_0 < 0$ models tend toward eventual recollapse, as they did in the $\ell = 1$ case. However, this process now takes much longer. In other words, while the $\ell = 2$ models are younger, their *life expectancies* are considerably greater.

This can be understood by means of the same argument as before (§ 5.3). The fact that ℓ is now larger means that the contribution of the cosmological term to the deceleration drops off more quickly in the future direction, thereby postponing recollapse for a longer period of time. Thus, models with $\lambda_0 = -1$ and $\lambda_0 = -3$ now survive for $t \gg 5$ and $t = 3.37$ Hubble times respectively.

5.5 The Case $\ell = 3$

We proceed to the third case. When $\ell = 3$ the differential equation (5.15) reads:

$$\tau^3 \frac{d^2x}{d\tau^2} - \alpha x = 0, \tag{5.77}$$

where α is given by eq. (5.19):

$$\alpha = \left(\frac{3\gamma}{2}\right)^2 \lambda_0 \tau_0^3. \tag{5.78}$$

The solution can be obtained (Murphy 1960, p. 368) by changing independent variables from τ to z, defined by:

$$z \equiv 1/\tau. \tag{5.79}$$

In terms of the new variable z, the differential equation (5.77) takes the form:

$$z \frac{d^2x}{dz^2} + 2\frac{dx}{dz} - \alpha x = 0. \tag{5.80}$$

This is now in a similar form to the starting equation for the $\ell = 1$ case, eq. (5.20), and it can be solved in the same manner (Murphy 1960, p. 329). We change independent variables from z to y, defined by:

$$y \equiv 2\sqrt{-\alpha z}, \tag{5.81}$$

whereupon eq. (5.80) takes the form [*cf.* eq. (5.23)]:

$$y^2 \frac{d^2x}{dy^2} + 3y\frac{dx}{dy} + y^2 x = 0. \tag{5.82}$$

This is again transformable to *Bessel's equation*, but with a different general solution (Spiegel 1974, p. 101) given by:

$$x(y) = y^{-1}[c_1 J_1(y) + c_2 Y_1(y)], \tag{5.83}$$

where c_1, c_2 are arbitrary constants, as before. Eq. (5.17) then gives $a(\tau)$, as usual:

$$a(\tau) = \tau^{1/3\gamma} \left[c_1 J_1(\psi) + c_2 Y_1(\psi)\right]^{2/3\gamma}, \tag{5.84}$$

where we have absorbed a factor of $\sqrt{-4\alpha}$ into c_1, c_2, and $\psi(\tau)$ is now defined by:

$$\psi(\tau) \equiv \sqrt{\frac{-4\alpha}{\tau}}. \tag{5.85}$$

The Hubble parameter is found by putting $x(y)$ into eq. (5.13), as usual:

$$H(\tau) = H_0 \sqrt{-\lambda_0 \left(\frac{\tau_0}{\tau}\right)^3} \left[\frac{c_1 J_2(\psi) + c_2 Y_2(\psi)}{c_1 J_1(\psi) + c_2 Y_1(\psi)}\right], \tag{5.86}$$

where $J_2(\psi)$ and $Y_2(\psi)$ are Bessel and Neumann functions of order two. As for the case $\ell = 1$, these solutions are in general real only if the cosmological term is *negative*. The same comments we made in that case apply here; this model is probably interesting only in an academic sense. It might however be worth speculating on why this property appears to characterize only models in which Λ goes as an *odd* power of time.

Let us follow our usual procedure and impose the boundary conditions (5.28). In conjunction with eqs. (5.84) and (5.86), this yields for c_1 and c_2:

$$c_1 = \frac{\sqrt{-\lambda_0}\, Y_2(\psi_0) - Y_1(\psi_0)}{\sqrt{-\lambda_0}\tau_0 \left[J_1(\psi_0)Y_2(\psi_0) - J_2(\psi_0)Y_1(\psi_0)\right]} \tag{5.87}$$

$$c_2 = \frac{-\sqrt{-\lambda_0}\, J_2(\psi_0) + J_1(\psi_0)}{\sqrt{-\lambda_0}\tau_0 \left[J_1(\psi_0)Y_2(\psi_0) - J_2(\psi_0)Y_1(\psi_0)\right]}, \tag{5.88}$$

where ψ_0 is given exactly as before, eq. (5.31). We require one additional boundary condition to fix τ_0. Unfortunately, as in the last subcase to be considered, the procedure is complicated by the fact that *both* terms in eq. (5.84) diverge at $\tau = 0$, whereas we expect that the scale factor as a whole should behave smoothly there.

We can make this more precise by employing the *asymptotic expressions* for $J_1(\psi)$ and $Y_1(\psi)$ at large ψ, which read (Spiegel 1974, p. 101):

$$\begin{aligned} J_1(\psi) &\sim \frac{1}{\sqrt{\pi\psi}}(\sin\psi - \cos\psi) \qquad (\psi \gg 1) \\ Y_1(\psi) &\sim \frac{-1}{\sqrt{\pi\psi}}(\sin\psi + \cos\psi) \qquad (\psi \gg 1). \end{aligned} \tag{5.89}$$

If we define two new quantities ω_0 and B_0 via:

$$\omega_0 \equiv 3\gamma\tau_0\sqrt{-\lambda_0\tau_0} \quad , \quad B_0 \equiv 1/\sqrt{\pi\omega_0}, \tag{5.90}$$

then eqs. (5.89) take the form:

$$\begin{aligned} J_1(\psi) &\sim B_0\tau^{1/4}\left[\sin(\omega_0\tau^{-1/2}) - \cos(\omega_0\tau^{-1/2})\right] \qquad (\tau \ll 1) \\ Y_1(\psi) &\sim -B_0\tau^{1/4}\left[\sin(\omega_0\tau^{-1/2}) + \cos(\omega_0\tau^{-1/2})\right] \qquad (\tau \ll 1), \end{aligned} \tag{5.91}$$

where we have used eqs. (5.19) and (5.85) for α and ψ. Substituting these expressions into eq.(5.84) for the scale factor, we find (for $\tau \ll 1$):

$$a(\tau) \sim \tau^{1/2\gamma}\left[C_1\sin(\omega_0\tau^{-1/2}) + C_2\cos(\omega_0\tau^{-1/2})\right]^{2/3\gamma}, \tag{5.92}$$

where:

$$C_1 \equiv B_0^{2/3\gamma}(c_1 - c_2) \quad , \quad C_2 \equiv -B_0^{2/3\gamma}(c_1 + c_2). \tag{5.93}$$

As $\tau \to 0$, we expect that eq. (5.92) should become exact. Therefore, since $\sin\theta$ and $\cos\theta$ remain finite as $\theta \to \infty$, we can see that in fact the scale factor does go smoothly to zero as $\tau \to 0$, as expected. However, these expressions do not help us to fix the value of τ_0 in conjunction with eqs. (5.87) and (5.88). We have therefore opted to use the same numerical method described in the $\ell = 2$ case, eqs. (5.73) – (5.75). The only thing we do differently this time is to tailor our zeroth-order estimate so that:

$$\tau_0^{(0)} \equiv 2/3\gamma, \tag{5.94}$$

corresponding to the age of the longest-lived models in the present case (those with $\lambda_0 \to 0$). [We know that models with zero cosmological constant must have an age of

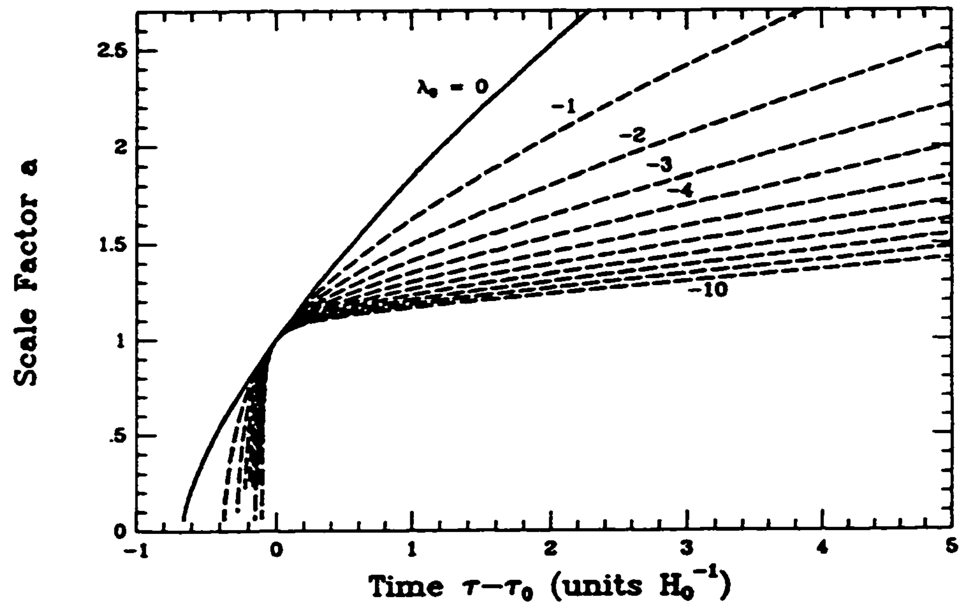


Figure 5.3: Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-3}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

2/3 Hubble times.] Once τ_0 is found from the numerical procedure, the values of c_1 and c_2 follow from eqs. (5.87) and (5.88) respectively. This completes our solution for the case $\ell = 3$.

The evolution of the scale factor for $\gamma = 1$ and various values of λ_0 is illustrated in Fig. 5.3, which has exactly the same format as Fig. 5.1. The $\lambda_0 = 0$ model (solid line) has an age of 2/3 Hubble times, while those with negative values of λ_0 (long-dashed lines) all have shorter ages, as usual. The age of the $\lambda_0 = -3$ model, for example, has dropped from 0.30 to just 0.27 Hubble times. It is unlikely that any of these models could describe the real universe, given the observational constraints on H_0 and τ_0 (§ 5.3).

The main difference between Fig. 5.3 and its predecessors occurs at large times, where we observe that the curves all straighten out, and show no sign of leading to a recollapse of the scale factor. The explanation for this is that the cosmological term is now decaying so quickly with time that it is, like ordinary matter, no longer sufficient to turn the expansion around.

5.6 The Case $\ell = 4$

We proceed finally to the fourth case, for which the differential equation (5.15) takes the form:

$$\tau^4 \frac{d^2 x}{d\tau^2} - \alpha x = 0, \tag{5.95}$$

with α given by eq. (5.19) as follows:

$$\alpha = \left(\frac{3\gamma}{2}\right)^2 \lambda_0 \tau_0^4. \tag{5.96}$$

As in the previous case, this equation can be solved (Murphy 1960, p. 371) by changing independent variables from τ to z via $z \equiv 1/\tau$, whereupon eq. (5.95) takes a form similar to eq. (5.80):

$$z \frac{d^2 x}{dz^2} + 2\frac{dx}{dz} - \alpha z x = 0. \tag{5.97}$$

If we then make a change of *dependent* variables from x to y, where:

$$\frac{dy}{dz} \equiv z\, x(z), \tag{5.98}$$

then it may be verified by differentiation that $y(z)$ satisfies the familiar equation:

$$\frac{d^2 y}{dz^2} \equiv \alpha y. \tag{5.99}$$

The solutions of eq. (5.99) are well-known; there are three subcases to consider, according as α [and hence λ_0, eq. (5.96)] is positive, zero, or negative.

5.6.1 The Subcase $\lambda_0 > 0$

Since we expect on observational grounds that λ_0 is probably positive, this is the most physical of the three. The general solution of eq. (5.99) is:

$$y(z) = c_1 \exp(\sqrt{\alpha}\, z) + c_2 \exp(-\sqrt{\alpha}\, z), \tag{5.100}$$

where c_1, c_2 are constants as usual. Differentiating and applying eq. (5.98), we find:

$$x(\tau) = \sqrt{\alpha}\,\tau \left[c_1 \exp\left(\frac{\sqrt{\alpha}}{\tau}\right) + c_2 \exp\left(\frac{-\sqrt{\alpha}}{\tau}\right)\right]. \tag{5.101}$$

This into eqs. (5.17) and (5.13) gives for the scale factor and Hubble parameter:

$$a(\tau) = \tau^{2/3\gamma}\left[c_1 \exp\left(\frac{\sqrt{\alpha}}{\tau}\right) + c_2 \exp\left(\frac{-\sqrt{\alpha}}{\tau}\right)\right]^{2/3\gamma} \quad (5.102)$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left[\frac{c_1(1-\sqrt{\alpha}/\tau)\exp(\sqrt{\alpha}/\tau) + c_2(1+\sqrt{\alpha}/\tau)\exp(-\sqrt{\alpha}/\tau)}{c_1\tau\exp(\sqrt{\alpha}/\tau) + c_2\tau\exp(-\sqrt{\alpha}/\tau)}\right] \quad (5.103)$$

where we have absorbed a factor of $\sqrt{\alpha}$ into c_1 and c_2. From eq. (5.102) it can be seen that $a(\tau)$ diverges at $\tau = 0$ unless:

$$c_1 = 0, \quad (5.104)$$

which we consequently assume. Eqs. (5.102) and (5.103) then simplify:

$$a(\tau) = \left[c_2\tau \exp\left(\frac{-\sqrt{\alpha}}{\tau}\right)\right]^{2/3\gamma} \quad (5.105)$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left(1 + \frac{\sqrt{\alpha}}{\tau}\right)\frac{1}{\tau}. \quad (5.106)$$

We then apply the boundary conditions (5.28) at $\tau = \tau_0$, as usual. Eq. (5.106) with $H(\tau_0) = H_0$ fixes the age at:

$$\tau_0 = \frac{2}{3\gamma(1-\sqrt{\lambda_0})}. \quad (5.107)$$

Substituting this result into eq. (5.105) with $a(\tau_0) = 1$, we obtain:

$$c_2 = \frac{1}{\tau_0}\exp\left(\frac{\sqrt{\alpha}}{\tau_0}\right). \quad (5.108)$$

Putting this back into eq. (5.105) produces this simple expression for the scale factor:

$$a(\tau) = \left[\left(\frac{\tau}{\tau_0}\right)\exp\left(\frac{\sqrt{\alpha}}{\tau_0} - \frac{\sqrt{\alpha}}{\tau}\right)\right]^{2/3\gamma}. \quad (5.109)$$

This differs in two ways from the power-law expansion we obtained in the case $\ell = 2$, eq. (5.49). Firstly, it is modified by the exponential factor, which acts primarily to flatten out the expansion at small times $\tau < \tau_0$; and secondly, the exponent is now independent of λ_0.

We can draw a number of conclusions from eq. (5.107). Firstly, we infer that:

$$\lambda_0 < 1; \quad (5.110)$$

and secondly, that the age of the universe $\tau_0 \to \infty$ as $\lambda_0 \to 1$. This is reminiscent of the case $\ell = 2$ and, in fact, eq. (5.107) is almost identical to eq. (5.47), the only difference being that λ_0 in the denominator has been replaced by $\sqrt{\lambda_0}$. Therefore, for the same value of λ_0, the $\ell = 4$ models are *longer-lived*, by a factor of $(1 - \lambda_0)/(1 - \sqrt{\lambda_0})$. This is due to the fact that, for *positive* Λ, the higher value of ℓ means that the cosmological term exerts a *greater repulsive* force in the past direction, and is consequently able to push the big bang back more effectively. As in the $\ell = 2$ case, we also find a lower limit on the age of these models. From eq. (5.107) we see that:

$$\tau_0 > \frac{2}{3\gamma} \tag{5.111}$$

This is twice as long as in the $\ell = 2$ case, but here the reason is simply that this lower limit corresponds to the case $\lambda_0 = 0$ (not $\lambda_0 = -1$ as before). With dust-like conditions ($\gamma = 1$) and the upper bound $H_0 \leq 83$ km s^{-1} Mpc^{-1} from Freedman (1996), we now find a minimum age of at least 7.9 billion years.

5.6.2 The Subcase $\lambda_0 = 0$

For this subcase the solution of eq. (5.99) is trivial; we see immediately that:

$$z(\tau) = c_3 + c_4\tau, \tag{5.112}$$

with c_3, c_4 arbitrary constants as usual. This in eqs. (5.17) and (5.13) gives:

$$a(\tau) = (c_3 + c_4\tau)^{2/3\gamma} \tag{5.113}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left(\frac{c_4}{c_3 + c_4\tau}\right). \tag{5.114}$$

For the first time we have a scale factor with the potential to go to smoothly to some finite value *other than* zero at $\tau = 0$. Let us pursue this possibility and see if a nonsingular solution is possible. Instead of the boundary condition $a(\tau_0) = 1$, we impose:

$$a(0) = a_*, \tag{5.115}$$

where a_* is the minimum value of the scale factor. In eq. (5.113) this implies:

$$c_3 = a_*^{3\gamma/2}. \tag{5.116}$$

Inserting this back into eq. (5.113) and applying the usual condition $a(\tau_0) = 1$, we find:

$$c_4 = \frac{1 - c_3}{\tau_0}. \tag{5.117}$$

Substituting this into eq. (5.114) and applying the third boundary condition $H(\tau_0) = H_0$, we obtain for the age of the universe:

$$\tau_0 = \frac{2}{3\gamma}(1 - c_3). \tag{5.118}$$

At this point we note that the present subcase only matches onto the previous one, eq. (5.107), if:

$$c_3 = 0, \tag{5.119}$$

which from eq. (5.116) implies that:

$$a_* = 0. \tag{5.120}$$

The present case is therefore singular at $\tau = 0$, like all the others studied in this chapter. The age of the universe is given by eq. (5.118) as:

$$\tau_0 = \frac{2}{3\gamma}. \tag{5.121}$$

Putting these results back into eq. (5.113), we find:

$$a(\tau) = \left(\frac{\tau}{\tau_0}\right)^{2/3\gamma}. \tag{5.122}$$

This is just the standard $k = 0$ solution with no cosmological term, as might have been expected.

5.6.3 The Subcase $\lambda_0 < 0$

For our final subcase, eq. (5.99) has the general solution:

$$y(z) = c_5 \cos(\sqrt{-\alpha}\, z) + c_6 \sin(\sqrt{-\alpha}\, z), \tag{5.123}$$

where c_5, c_6 are constants as usual. Differentiating and applying eq. (5.98), we find:

$$x(\tau) = \sqrt{-\alpha}\,\tau \left[c_5 \sin\left(\frac{\sqrt{-\alpha}}{\tau}\right) + c_6 \cos\left(\frac{\sqrt{-\alpha}}{\tau}\right)\right]. \tag{5.124}$$

This into eqs. (5.17) and (5.13) gives for the scale factor and Hubble parameter:

$$a(\tau) = \tau^{2/3\gamma}\left[c_5 \sin\left(\frac{\sqrt{-\alpha}}{\tau}\right) + c_6 \cos\left(\frac{\sqrt{-\alpha}}{\tau}\right)\right]^{2/3\gamma} \tag{5.125}$$

$$H(\tau) = \frac{2H_0}{3\gamma}\left\{\frac{[c_5 + (\sqrt{-\alpha}/\tau)c_6]\sin(\sqrt{-\alpha}/\tau) + [c_6 - (\sqrt{-\alpha}/\tau)c_5]\cos(\sqrt{-\alpha}/\tau)}{c_5\tau\sin(\sqrt{-\alpha}/\tau) + c_6\tau\cos(\sqrt{-\alpha}/\tau)}\right\},$$

where we have absorbed a factor of $\sqrt{-\alpha}$ into c_5 and c_6. Application of the boundary conditions (5.28) fixes these constants in terms of τ_0:

$$c_5 = \frac{1}{\sqrt{-\alpha}}\left[\beta_0 \sin\beta_0 + \cos\beta_0\left(1 - \frac{3\gamma}{2}\tau_0\right)\right] \tag{5.126}$$

$$c_6 = \frac{1}{\sqrt{-\alpha}}\left[\beta_0 \cos\beta_0 - \sin\beta_0\left(1 - \frac{3\gamma}{2}\tau_0\right)\right], \tag{5.127}$$

where:

$$\beta_0 \equiv \left(\frac{3\gamma}{2}\right)\tau_0\sqrt{-\lambda_0}. \tag{5.128}$$

As usual, we require one additional boundary condition to fix the value of τ_0. The situation is again complicated by the fact that both terms in eq. (5.125) diverge at $\tau = 0$, whereas $a(\tau)$ itself goes smoothly to zero there. [In fact, it is interesting to see that eq. (5.125) has *exactly* the same form as the asymptotic expression (5.92) in the $\ell = 3$ case, except that the exponent on τ outside the square brackets has increased from $1/2\gamma$ to $2/3\gamma$.] We therefore solve the problem numerically once again, using eqs. (5.73) – (5.75), with $\tau_0^{(0)} \equiv 2/3\gamma$ as in the case $\ell = 3$. Once τ_0 is obtained in this way, the values of c_5 and c_6 are fixed by eqs. (5.126) and (5.127). This completes our solution for the case $\ell = 4$.

The evolution of the scale factor for $\gamma = 1$ and various values of λ_0 is illustrated in Fig. 5.4, which has exactly the same format as Fig. 5.2. Several features may be noted. To begin with, we see that models with $\Lambda \propto \tau^{-4}$ are qualitatively the same as those with $\Lambda \propto \tau^{-2}$ for *positive* λ_0, and qualitatively similar to those with $\Lambda \propto \tau^{-3}$ for *negative* λ_0.

There are important quantitative differences, however. Models with positive λ_0 have become significantly older. With $\lambda_0 = 0.5$, for example, τ_0 is now 2.28 Hubble times — older by a factor of 1.71 times than the equivalent $\ell = 2$ model, exactly as predicted

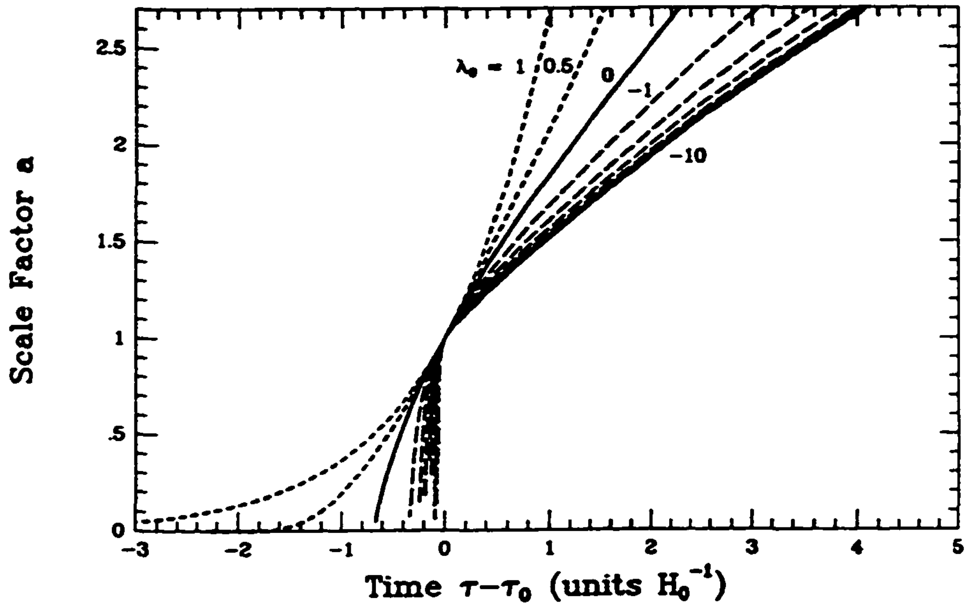


Figure 5.4: Evolution of the scale factor for flat models with $\Lambda \propto \tau^{-4}$ and $\gamma = 1$. Values of λ_0 are labelled beside each curve, and $\Omega_0 = 1 - \lambda_0$ in each case.

in the discussion following eq. (5.107). This is once again due to the fact that, with $\ell = 4$, a positive Λ-term increases in size very rapidly in the past direction. Negative-λ_0 models, on the other hand, have once again become younger. The age of the $\lambda_0 = -3$ model, for instance, has dropped from 0.27 to just 0.25 Hubble times. And finally, in the future direction, we see that there is no longer very much distinction between the $\lambda_0 < 0$ and $\lambda_0 = 0$ models, compared to Fig. 5.3. This is because the contribution of the cosmological term to the deceleration of the scale factor now drops off so quickly that it rapidly becomes irrelevant.

Chapter 6

Models with $\Lambda \propto a^{-m}$

6.1 Previous Work

We have seen in Chapter 5 that variable-Λ theories in which $\Lambda \propto t^{-\ell}$ do not give rise to singularity avoidance, at least for $k = 0$ models with $\ell = 1, 2, 3$ or 4. In this chapter[1] we move on to consider decay laws of the form set out in eq. (4.6):

$$\Lambda = \mathcal{B}\, a^{-m}, \tag{6.1}$$

where $\mathcal{B}$ and m are, for the time being, arbitrary constants. The scale factor may be more natural than time as an independent variable, to the extent that many physical quantities (such as temperature) depend more simply on a than t (Chen and Wu 1990, Madsen *et al* 1992). As discussed in § 4.3, most variable-Λ theories to date have been based on laws of the form (6.1). The case $m = 2$ has been singled out for the most attention (Özer and Taha 1986, 1987; Abdel-Rahman 1990, Chen and Wu 1990, Abdel-Rahman 1992, Calvao *et al* 1992, Abdel-Rahman 1995; John and Joseph 1996, 1997). Chen and Wu (1990) have offered some interesting dimensional arguments in support of this value; otherwise its choice appears to be more or less *ad hoc* in these models. A second group of papers has concentrated on values of $m \approx 4$ (Freese *et al* 1987, Sato *et al* 1990, Overduin *et al* 1993), for which the Λ-term behaves like ordinary *radiation*.

[1]Chapter 6 is based in part on a presentation by J. M. Overduin at the 7th Canadian Conference on General Relativity and Relativistic Astrophysics in Calgary (to be published in Overduin and Cooperstock 1997).

Pavón (1991) has shown that, for certain kinds of Λ decay, the lower of these two values of m is thermodynamically more stable.

There are also some studies in which the value of m is not fixed *a priori*. Gasperini (1987) was concerned with establishing the plausibility of a thermal interpretation of Λ, and found that m should satisfy $3.6 < m < 4$ on this basis. Olson and Jordan (1987) calculated the ages of models with general m, and concluded that observations require $m < 3$. Silveira and Waga (1994) found that the evolution of matter density perturbations is not strongly affected by the decaying Λ-term, at least for $0 \leq m \leq 2$. The same authors have since expanded their investigation to include constraints from supernovae and lensing statistics, and recently reported that these observational data are best-fit by values of $m \geq 1.6$ for flat models (Silveira and Waga 1997). Lensing statistics have also been employed in a study by Torres and Waga (1996), who find that models with $m \geq 1$ are best able to reproduce the observations. Phenomena such as age, lensing probability and background radiation intensity have been discussed in the context of models with $\Lambda \propto \Lambda_0 + a^{-m}$ by Matyjasek (1995), who however did not obtain specific numerical bounds on m.

Of the authors mentioned above, only Özer and Taha (1986, 1987), Abdel-Rahman (1990, 1992, 1995), Matyjasek (1995) and John and Joseph (1996, 1997) have drawn a connection between the variable-Λ scenario and the singularity problem. All but Matyjasek (1995) have confined their attention to specific variants of the $m = 2$ case, while the latter has merely noted that, in some circumstances, the presence of an initial singularity would require $0 < m < 4$. We wish to take the opposite approach here and determine the conditions under which it is realistically possible to have *no* initial singularity. Moreover we will extend the discussion, not only to general m, but to general γ as well [where γ is the adiabatic index characterizing the equation of state of matter, eq. (4.13).]

If m and γ are thought of as defining a parameter space, then we wish to discover, firstly, how much of the space is free of the initial singularity; and secondly, how much of it is observationally viable. As we have emphasized, previous studies of nonsingular models have been restricted to the points defined by $m = 0, 2$ and $\gamma = 1, 4/3$. We will find that, contrary to the prevailing view in this subject, there is significant overlap between the two regions.

6.2 Evolution of the Scale Factor

We begin by bringing over the dynamical equations (4.11) – (4.16) derived in Chapter 4. In particular, we are interested in the energy conservation law (4.14), which, with the decay law (6.1), becomes:

$$\frac{d}{da}\left(\rho a^{3\gamma}\right) = \left(\frac{mB}{8\pi G}\right) a^{3\gamma-(m+1)}. \tag{6.2}$$

Integrating, we find for the matter energy density:

$$\rho(a) = \rho_0 a^{-3\gamma} f(a), \tag{6.3}$$

where we have chosen units such that $a(t_0) = a_0 = 1$ (§ 4.9), applied the boundary condition $\rho(a_0) = \rho_0$, and defined:

$$f(a) \equiv 1 + \kappa_0 \times \begin{cases} m(a^{3\gamma-m} - 1)/(3\gamma - m) & \text{if } m \neq 3\gamma \\ 3\gamma \ln(a) & \text{if } m = 3\gamma. \end{cases} \tag{6.4}$$

When $m = 0$, then $f(a) = 1$ and eq. (6.3) returns the well-known result that ρ scales as a^{-3} in a pressure-free universe ($\gamma = 1$) and a^{-4} in a radiation-dominated universe ($\gamma = 4/3$).

The new parameter κ_0 is defined by:

$$\kappa_0 \equiv \frac{B}{8\pi G \rho_0}. \tag{6.5}$$

This can be fixed in terms of observable quantities as follows. Since we choose units such that $a_0 = 1$, it follows from the decay law (6.1) that:

$$B = \Lambda_0 = 3H_0^2 \lambda_0, \tag{6.6}$$

where λ_0 is defined as usual by eq. (4.22). Substituting this result into eq. (6.5), together with the definition (4.22) of Ω_0, we find:

$$\kappa_0 = \frac{\lambda_0}{\Omega_0}. \tag{6.7}$$

The parameter κ_0 is simply the ratio of energy density in the cosmological term to that in ordinary matter at the present epoch.

Substitution of eqs. (6.1) and (6.3) into the Lemaître equation (4.11) yields:

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi G}{3}\right)\rho_0 a^{-3\gamma} f(a) - ka^{-2} + \frac{B}{3}a^{-m}. \tag{6.8}$$

If we introduce the *Hubble time* $\tau \equiv H_0 t$ (§ 5.2) then:

$$\frac{\dot{a}}{a} = \frac{H_0}{a}\frac{da}{d\tau}, \tag{6.9}$$

so that eq. (6.8) takes the simple form:

$$\frac{da}{d\tau} = a\left[\Omega_0 a^{-3\gamma} f(a) - (\Omega_0 + \lambda_0 - 1)a^{-2} + \lambda_0 a^{-m}\right]^{1/2}, \tag{6.10}$$

where we have made use of the definitions (4.22) and (6.6). [Note that we have chosen the positive root of eq. (6.8), corresponding to the fact that *redshifts*, not blueshifts, are observed.]

Eq. (6.10) is essentially a definition of the Hubble parameter in these models. At this point we could choose integer values of m and integrate this expression to obtain analytic solutions for $a(\tau)$, as we did in the previous chapter. Analyses of this kind have been carried out already for the case when the cosmological term is constant, $m = 0$. McVittie (1965) and Heidmann (1980), for instance, have shown how to express some special cases (with $\gamma = 1$) as elliptic integrals. Agnese *et al* (1970) have included radiation-like, as well as dust-like matter, and obtained solutions in terms of hypergeometric functions by expanding in powers of Λ (assumed to be small). Edwards (1972) derived exact expressions for *general* values of Λ by means of Jacobian elliptic functions, but restricted himself to singular solutions with $\gamma = 1$.

It is doubtful that analytic approaches such as these can be usefully extended to the general situation in which $m \neq 0$. On the other hand, values in the range $0 \leq m \leq 4$ have now been considered by a number of authors (§ 6.1), so it is clearly of interest to obtain solutions for this case. We opt here to solve the problem numerically, following

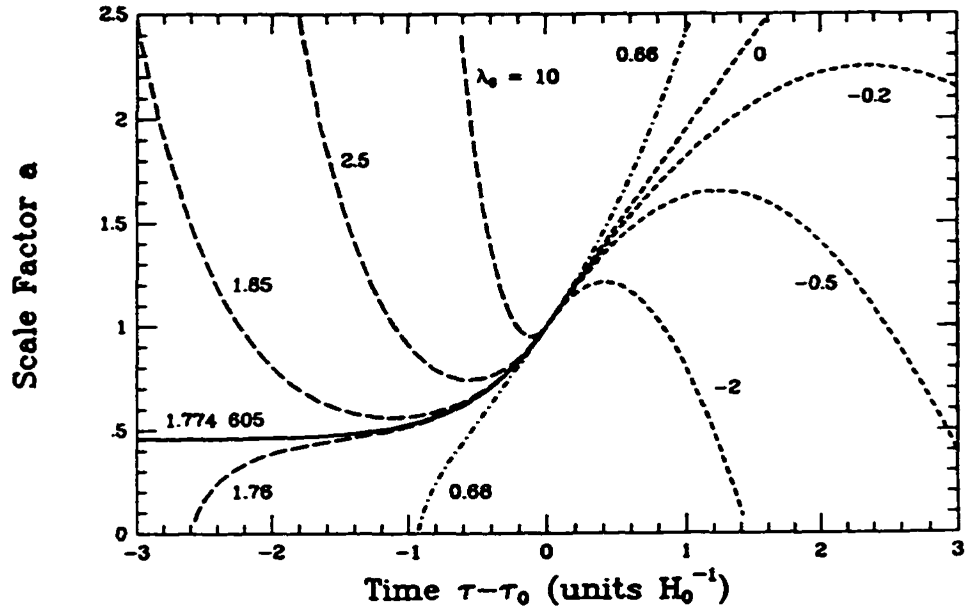


Figure 6.1: Evolution of the scale factor for models with $m = 0, \gamma = 1$, $\Omega_0 = 0.34$, and values of λ_0 labelled beside each curve (after Felten and Isaacman 1986).

the lead of Felten and Isaacman (1986). The time-derivative of eq. (6.10) is:

$$\frac{d^2a}{d\tau^2} = \left(1 - \frac{3\gamma}{2}\right) \Omega_0 a^{1-3\gamma} f(a) + \lambda_0 a^{1-m}. \tag{6.11}$$

[This could equally well have been obtained from eq. (4.15).] We then substitute eqs. (6.10) and (6.11) into a *Taylor expansion* for the scale factor:

$$a_k \approx a_{k-1} + \left(\frac{da}{d\tau}\right)_{k-1} \Delta\tau + \frac{1}{2}\left(\frac{d^2a}{d\tau^2}\right)_{k-1} (\Delta\tau)^2. \tag{6.12}$$

This can be integrated numerically backwards in time to determine whether or not a model with given values of $\{m, \gamma, \Omega_0, \lambda_0\}$ eventually reaches $a = 0$ or not; ie., whether it is nonsingular. We have tested the procedure for the case of a constant cosmological term ($m = 0$) and dust-like equation of state ($\gamma = 1$), and our results in this case confirm those of Felten and Isaacman (1986). Fig. 6.1 depicts a group of examples with $\Omega_0 = 0.34$ (typical of large-scale observations, eg. Bahcall *et al* 1997). and various values of λ_0 (labelled beside each curve). Note that the difference between this figure and the ones in the previous chapter is that we now show models of all *three* kinds:

closed (long-dashed lines), flat (dash-dotted lines) and open (short dashes). (To keep the diagram from being too crowded, we show only models with the same value Ω_0 of the matter density.) Fig. 6.1 shows that negative values of λ_0 can lead to recollapse even in *open* universes (*cf.* Chapter 5). Of greater interest in Fig. 6.1, however, is the fact that models with λ_0 greater than a *critical value* λ_* (=1.774 605 in this case) avoid the big bang singularity, undergoing a finite "big bounce" instead. Models with slightly *less* than this critical value (eg., $\lambda_0 = 1.76$ in this case) are of the "coasting Lemaître" kind; they begin in a singular state but go through an extended phase in which the scale factor is nearly constant. Models with *exactly* $\lambda_0 = \lambda_*$ (shown in Fig. 6.1 with a solid line) are perhaps the most interesting of all. As time $\tau \to -\infty$, they neither plunge to zero size nor bounce back up to infinite size; instead, they level off indefinitely at some value a_* (=0.46 in this case). These are nonsingular Eddington-Lemaître models, asymptotic to the static Einstein universe in the infinite past.

All these features are discussed at greater length by Felten and Isaacman (1986). (It should be noted that these authors employ a slightly different free parameter than we do; namely, $\Lambda_0/H_0^2 = 3\lambda_0$.) Our purpose here is to generalize the problem to arbitrary values of m and γ.

6.3 Critical Values of λ_0

In particular, we wish to obtain a general expression for the critical value λ_* of the lambda parameter λ_0 (and the limiting size a_* of the scale factor a), given any class of models $\{m, \gamma, \Omega_0\}$. As discussed in § 5.1, we are interested in models for which $da/d\tau \to 0$ at some time in the past. This occurs, for example, at the moment of the "bounce" in all the oscillating models shown in Fig. 6.1. The *critical* case is distinguished by that fact that not only $da/d\tau$, but also $d^2a/d\tau^2$ vanishes at this point (Felten and Isaacman 1986, Börner and Ehlers 1988). We therefore set $da/d\tau = d^2a/d\tau^2 = 0$ in eqs. (6.10) and (6.11). This yields:

$$\lambda_* = \frac{(3\gamma-2)(3\gamma-m)\Omega_0}{3\gamma(2-m)a_*^{3\gamma-m} + (3\gamma-2)m}, \tag{6.13}$$

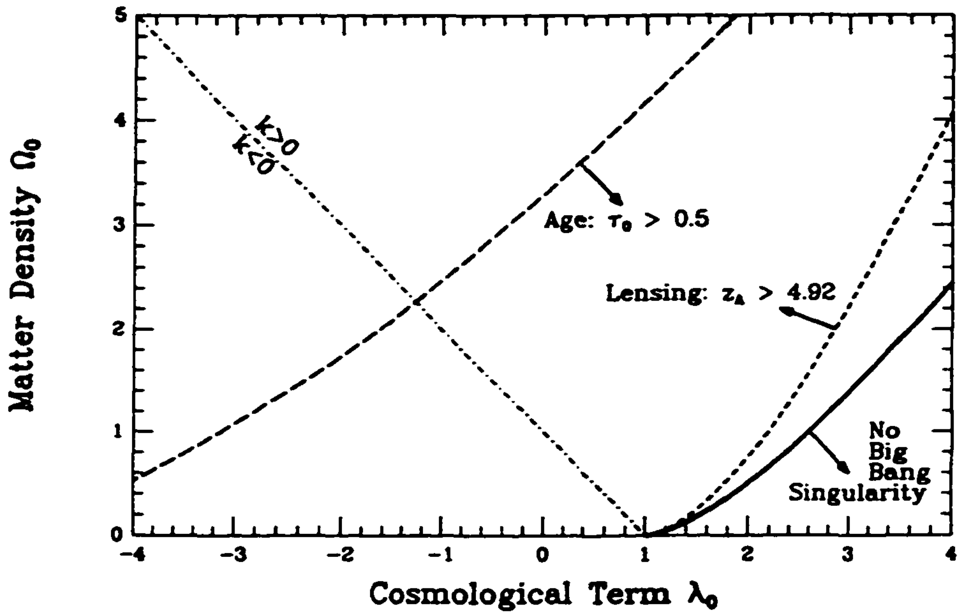


Figure 6.2: Phase space diagram showing constraints on models with $m = 0$ and $\gamma = 1$ (after Lahav *et al* 1991).

where a_*, the minimum value of the scale factor, is found by solving:

$$3\gamma(3\gamma - m)\Omega_0 a_*^2 + 3\gamma(2 - m)(1 - \Omega_0)a_*^{3\gamma} - (3\gamma - 2)(3\gamma\Omega_0 - m)a_*^m = 0 . \tag{6.14}$$

In general eq. (6.14) has to be solved numerically, but in the case $m = 0, \gamma = 1$ it reduces to the cubic equation given by Felten and Isaacman (1986). Fig. 6.2 is a phase space portrait of this case, with each point on the diagram corresponding to a choice of Ω_0 and λ_0 (after Lahav *et al* 1991). The critical values λ_* are represented in this figure by a heavy solid line. The region to the right of this curve corresponds to universes with $\lambda_0 > \lambda_*$; that is, with no big bang singularity. Also shown in Fig. 6.2 is a straight dash-dotted line representing the boundary between open and closed universes; models on this line have $k = 0$ ($\Omega_0 = 1 - \lambda_0$) while those on the left and right have $k < 0$ and $k > 0$ respectively (§ 4.9).

We now have the tools we need to investigate models with arbitrary values of m and γ. The idea will be to use phase space diagrams like Fig. 6.2 in order to determine how much parameter space is available for nonsingular models. We can then focus on specific values of Ω_0 and λ_0 inside the nonsingular region, carry out the numerical integration described in § 6.2, and check our results with evolution diagrams like Fig. 6.1.

6.4 Observational Constraints

6.4.1 Upper Bounds on λ_0

We pause first to take stock of some of the observational constraints that have been placed on nonsingular and nonzero-Λ models. Most immediate are *direct upper bounds on* λ_0 from a variety of methods, most of them assuming that $\Omega_0 + \lambda_0 = 1$. Until recently, these have typically been of order ~ 1 (Lahav *et al* 1991). An exception was that of Loh (1986), who found $\lambda_0 \leq 0.3$ based on number counts of faint galaxies, but this result is now thought to be unreliable due to the fact that it neglected the possible effects of the Λ-term on galaxy evolution (Martel 1994). Additional methods have since become available. Cosmic microwave background (CMB) fluctuations, for instance, have produced an upper limit of $\lambda_0 \leq 0.86$ (Bunn and Sugiyama 1995). Gravitational lens statistics give a tighter bound of $\lambda_0 < 0.66$ (Kochanek 1996), and observations of Type Ia supernovae appear to reduce this still further, to $\lambda_0 < 0.51$ (Perlmutter *et al* 1997). All of the above are described as 95% confidence level measurements. On the other hand, a *lower* limit of $\lambda_0 > 0.53$ has been obtained from the galactic luminosity density — *also* at 95% confidence (Totani *et al* 1997).

Complicating the picture somewhat is the fact that several other observational data (including the lack of observed small-scale dark matter, the expectation that inflation should lead to near-flatness, and especially the high *age* of the universe inferred from models of stellar evolution) are well-explained by large values of λ_0. This "cosmic concordance" (Kochanek 1996) at one time led to calls for $\lambda_0 \sim 0.8$ (Efstathiou *et al* 1990, Krauss and Schramm 1993). The value has since dropped to $0.5 - 0.7$ (Krauss and Turner 1995, Turner 1997), largely under pressure from the constraints quoted above, but there are still occasional estimates as high as $\lambda_0 = 0.87 \pm 0.13$ (Moles 1991), or even 1.08 ± 0.02 (Hoell *et al* 1994).

So the observational situation is not yet settled. It is clear, however, that the non-singular models in Fig. 6.1, which require $\lambda_0 > 1.77$, are almost certainly unphysical. It remains to be seen if the same conclusion applies when $m \neq 0$ and/or $\gamma \neq 1$.

6.4.2 Age of the Universe

A lower limit on λ_0 comes from the *age of the universe*, which can be calculated as follows:

$$t_0 = \int_0^{a_0} \frac{da}{\dot{a}}, \tag{6.15}$$

or, in our units of Hubble times:

$$H_0 t_0 = \tau_0 = \int_0^1 \frac{da}{da/d\tau}. \tag{6.16}$$

If we use the Freedman (1996) value of 73 ± 10 km s^{-1} Mpc^{-1} for H_0 in conjunction with Chaboyer *et al*'s (1997) oldest globular cluster age of > 9.6 Gyr for t_0, then:

$$\tau_0 \geq 0.62; \tag{6.17}$$

that is, the universe is at least 0.62 Hubble times old. Of course, as noted in § 5.3, the true status of these two parameters (especially H_0) is still a subject of some controversy. However, even if we use a lower value of 55 ± 10 km s^{-1} Mpc^{-1} (Sandage and Tammann 1996), we still obtain a minimum of:

$$\tau_0 \geq 0.44, \tag{6.18}$$

or about half a Hubble time. We will use an intermediate lower limit; in conjunction with our expression (6.10) for $da/d\tau$, this implies:

$$\int_0^1 \frac{da}{\left[\Omega_0 a^{2-3\gamma} f(a) - (\Omega_0 + \lambda_0 - 1) + \lambda_0 a^{2-m}\right]^{1/2}} > 0.5. \tag{6.19}$$

This expression agrees with the standard results (Lahav *et al* 1991, Peebles 1993, p. 102) in the case when $m = 0$ and $\gamma = 1$.

We have found that the integral (6.19) is more easily evaluated in practice with a change of independent variables from a to:

$$v \equiv a^{-1} = 1 + z, \tag{6.20}$$

where z is the *redshift*. This produces:

$$\tau_0 = \int_1^\infty \frac{dv}{[\Omega_0 v^{2+3\gamma} f(v) - (\Omega_0 + \lambda_0 - 1)v^4 + \lambda_0 a^{2+m}]^{1/2}} > 0.5, \tag{6.21}$$

where, from eq. (6.4):

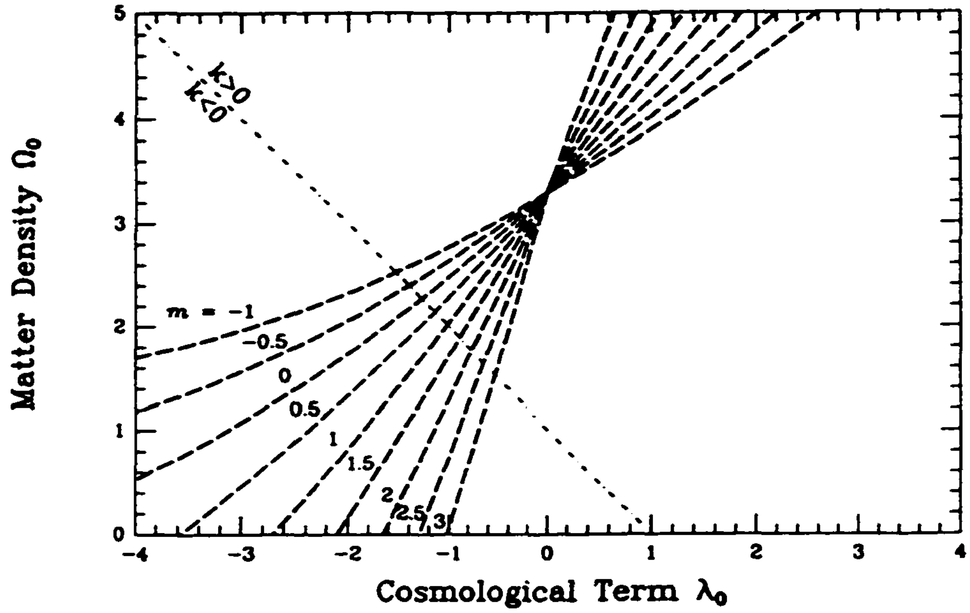
$$f(v) \equiv 1 + \kappa_0 \times \begin{cases} m(v^{m-3\gamma} - 1)/(3\gamma - m) & \text{if } m \neq 3\gamma \\ -3\gamma \ln(v) & \text{if } m = 3\gamma. \end{cases} \tag{6.22}$$

This reduces to the expression given by Refsdal *et al* (1967) when $m = 0$ and $\gamma = 1$. Numerical solution of eq. (6.21) produces a *lower limit* on λ_0 as a function of $\{m, \gamma, \Omega_0\}$. This age constraint is shown in Fig. 6.2 (for the case $m = 0, \gamma = 1$) as a long-dashed line. Its position matches that in a similar plot by Lahav *et al* (1991). The region to the left of this curve corresponds to universes younger than half a Hubble time. Nonsingular models, of course, are not constrained by this; they are infinitely old (by definition), the heavy solid line being precisely the boundary where $\tau_0 \to \infty$. The main impact of the age constraint is to rule out models with a *negative* cosmological term.

We show in Fig. 6.3 the effects of varying the parameters m and γ respectively on this age constraint. It may be seen that altering the value of m changes the *slope* of the curve, but does not otherwise greatly affect the age, even over the range $-1 \leq m \leq 3$. Altering the value of γ, on the other hand, has a larger effect. In particular, the "harder" the equation of state (ie., the larger the value of γ), the further this constraint encroaches on the available parameter space. This is in accord with the well-known fact that a radiation-dominated universe ($\gamma = 4/3$), for example, is a short-lived one.

6.4.3 Gravitational Lensing and the "Antipode"

For closed models, the most stringent constraint on λ_0 comes from *gravitational lensing*, which requires that the "antipode" be further away than the most distant normally



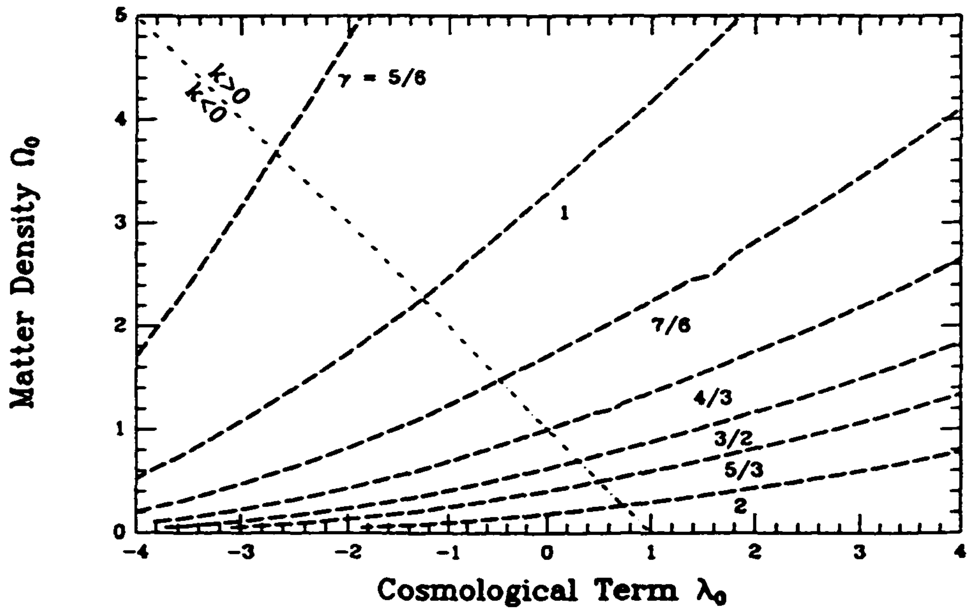


Figure 6.3: The age constraint $\tau_0 > 0.5$ as a function of m (top), assuming $\gamma = 1$; and as a function of γ (bottom), assuming $m = 0$.

lensed object (Gott *et al* 1989, Lahav *et al* 1991). (It is thought difficult to form more than one very faint image of a source beyond this point.) The antipode is the point where the radial coordinate ω, defined by:

$$\omega \equiv \int_t^{t_0} \frac{dt}{a(t)} = \int_a^{a_0} \frac{da}{\dot{a}a} \tag{6.23}$$

takes the value π (McVittie 1965, p. 158). Using our expression (6.10) for $da/d\tau$, together with eq. (4.24), we obtain (taking $a_0 = 1$ as usual):

$$\omega = \sqrt{\Omega_0 + \lambda_0 - 1} \int_1^{1+z} \frac{dv}{[\Omega_0 v^{3\gamma} f(v) - (\Omega_0 + \lambda_0 - 1)v^2 + \lambda_0 a^m]^{1/2}}, \tag{6.24}$$

where we have again switched variables from a to v via (6.20) above. Eq. (6.24) differs from eq. (6.21) in three ways: an extra term in front (which is guaranteed to be real for closed models), one less power of v in the denominator, and a finite upper limit of integration $(1 + z)$. We recover the formula of Refsdal *et al* (1967) when $m = 0$ and $\gamma = 1$.

When $\omega = \pi$, the redshift z in eq. (6.24) refers to the redshift z_A of the antipode. The lensing constraint requires that this be larger than the redshift z_ℓ of the furthest known normally lensed object (Gott *et al* 1989). The current holder of this distinction is a pair of lensed galaxies at $z_\ell = 4.92$ (Franx *et al* 1997). We therefore require:

$$z_A(\Omega_0, \lambda_0, m, \gamma) > 4.92, \tag{6.25}$$

where $z_A(\Omega_0, \lambda_0, m, \gamma)$ is defined by eq. (6.24) with $\omega = \pi$. Numerical solution of this equation yields an *upper limit* on λ_0 as a function of $\{m, \gamma, \Omega_0\}$. This lensing constraint is shown in Fig. 6.2 (for the case $m = 0, \gamma = 1$) as a short-dashed line. Its position is close to that in similar plots by Gott *et al* (1989) and Lahav *et al* (1991), who used a smaller value $z_\ell = 3.27$. (Our constraints are slightly stronger.) The region to the right of this curve corresponds to universes incompatible with the lensing observations. Since this includes the entirety of nonsingular parameter space, we can see that nonsingular pressure-free models with constant Λ are ruled out, as noted previously by these authors.

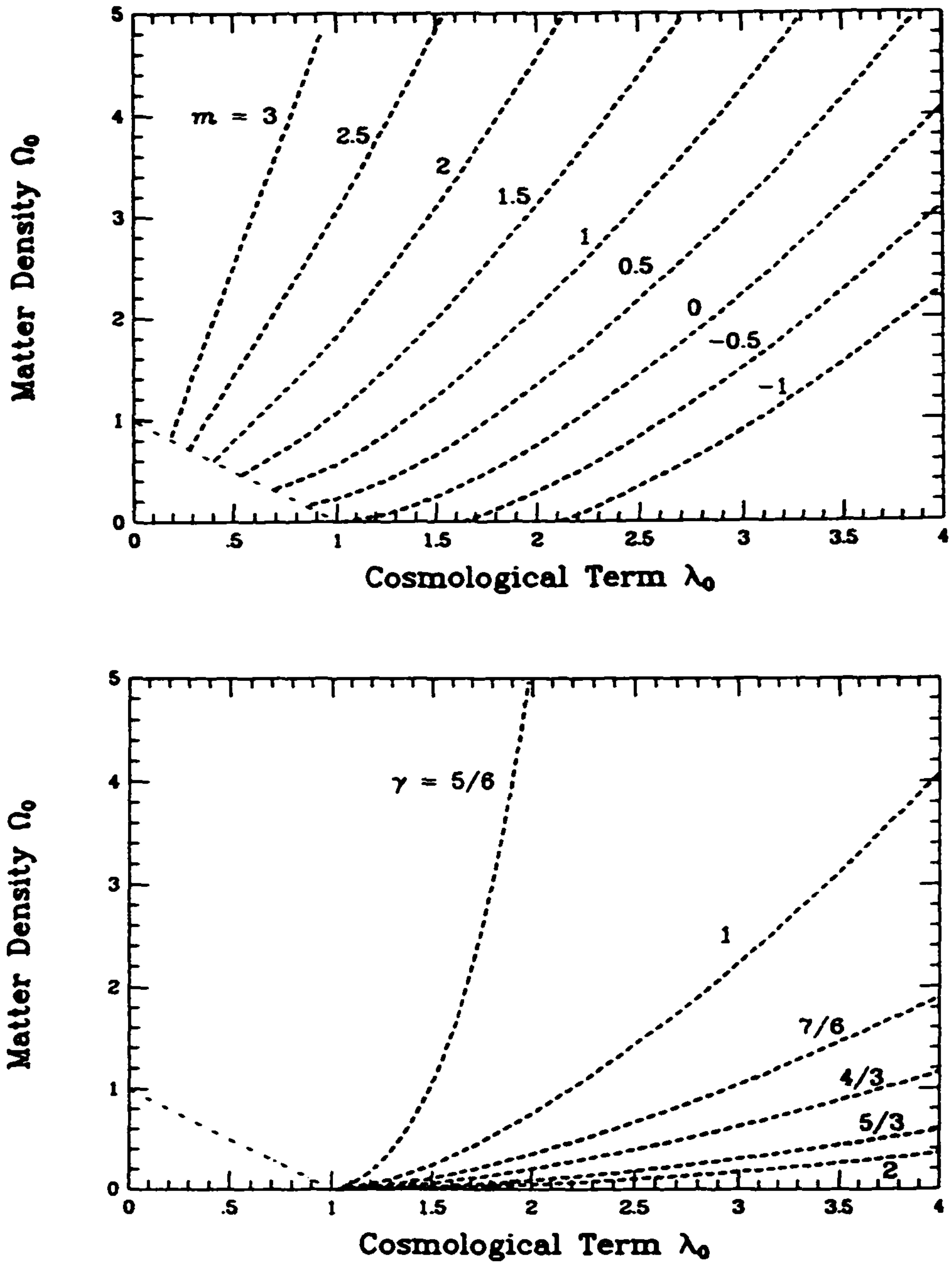


Figure 6.4: The lensing constraint $z_A > 4.92$ as a function of m (top), assuming $\gamma = 1$; and as a function of γ (bottom), assuming $m = 0$.

We show in Fig. 6.4 the effects of varying the parameters m and γ respectively on this lensing constraint. As before, it is seen that harder values of γ lead to tighter constraints on the available parameter space. However, the situation with regard to m is altered quite dramatically. In particular, the higher the value of m, the *weaker* the lensing constraint becomes. As we will see, this significantly improves the prospects for viable nonsingular models.

6.4.4 The Maximum Redshift Constraint

A fourth observational constraint, which must be satisfied *only* by nonsingular models, concerns the *maximum observable redshift*:

$$z_* = a_*^{-1} - 1, \tag{6.26}$$

prevailing in a universe with a minimum scale factor a_*. This must obviously be larger than the greatest redshift z_{obs} actually observed. Thus the nonsingular models in Fig. 6.1, which never get smaller than $a_* = 0.46$, cannot accommodate redshifts greater than $z_* = 1.2$. This disagrees with observations such as those of the lensed galaxies mentioned above.

Crilly (1968), Börner and Ehlers (1988) and Ehlers and Rindler (1989) have reformulated the condition $z_* > z_{obs}$ as an upper limit on the *matter density* of the universe, as follows. Assuming that $da/d\tau = 0$ and $d^2a/d\tau^2 \geq 0$ at $z = z_*$ for nonsingular models, eqs. (6.10) and (6.11) can be combined to read (assuming $m \neq 3\gamma$):

$$\Omega_0 \leq \frac{2 - m + (m/3\gamma)(3\gamma - 2)(1 + z_*)^{3\gamma - m}}{2 - m - (3\gamma - m)(1 + z_*)^{3\gamma - 2} + (3\gamma - 2)(1 + z_*)^{3\gamma - m}}. \tag{6.27}$$

[This reduces to an earlier expression of Crilly (1968) when $m = 0$ and $\gamma = 1$; and to the formulae of Börner and Ehlers (1988) and Ehlers and Rindler (1989) when $m = 0$ and $\gamma = 1, 4/3$.] The Börner-Ehlers argument then runs as follows: given that quasar redshifts have been observed out to $z_{obs} > 4$, we know that the universe has $z_* > 4$. This into eq. (6.27) then implies (assuming that $m = 0$ and $\gamma = 1$ or $4/3$) that $\Omega_0 \leq 0.018$, which is contrary to observation. Therefore we do not live in a nonsingular universe.

Table 6.1: Börner-Ehlers-type upper limits on matter density Ω_0 for various values of m and γ, assuming $z_* > 6$.

$\gamma =$	2/3	5/6	1	7/6	4/3	5/3	2
$m = 0$	∞	0.033	0.006	0.002	0.000	0.000	0.000
1/4	∞	0.16	0.10	0.079	0.067	0.053	0.044
1/2	∞	0.31	0.20	0.16	0.14	0.11	0.090
3/4	∞	0.49	0.33	0.26	0.22	0.17	0.14
1	∞	0.75	0.48	0.38	0.32	0.25	0.20
5/4	∞	1.1	0.71	0.55	0.46	0.35	0.29
3/2	∞	1.9	1.1	0.87	0.71	0.54	0.44
7/4	∞	4.2	2.4	1.8	1.4	1.0	0.84
2	∞	∞	∞	∞	∞	∞	∞

Let us see how this conclusion changes when we generalize the situation to values of $m \neq 0$. To make things more interesting, we will *strengthen* the argument by noting that some distant galaxies have now been assigned photometric redshifts as high as $z_{obs} > 6$ (Lanzetta *et al* 1996), implying that $z_* > 6$. The resulting upper limits on density Ω_0 are listed in Table 6.1 for various values of m and γ. From this table we see that the new photometric redshifts tighten the Börner-Ehlers constraint noticeably: as long as $m = 0$, a nonsingular universe requires $\Omega_0 \leq 0.006$ (if $\gamma = 1$) or $\Omega_0 < 0.001$ (if $\gamma = 4/3$). These numbers, of course, are too low to describe the real universe. However, Table 6.1 demonstrates that much higher densities are possible in singularity-free models with *variable* Λ. For example, staying with pressure-free conditions ($\gamma = 1$), we see that if $m = 1$, then the matter density in a nonsingular universe must satisfy $\Omega_0 < 0.48$. This value is not unreasonable at all; in fact it is well above most dynamical measurements, which suggest $\Omega_0 \approx 0.3$ (Bahcall *et al* 1997). The constraint is similarly loosened if we move toward softer equations of state, such as the $\gamma = 5/6$ suggested by Caldwell *et al* (1997), or the $\gamma \approx 0.4$ considered by Turner and White (1997).

6.5 Realistic Nonsingular Models

We now demonstrate that models with $m \neq 0$ are capable of satisfying all the constraints

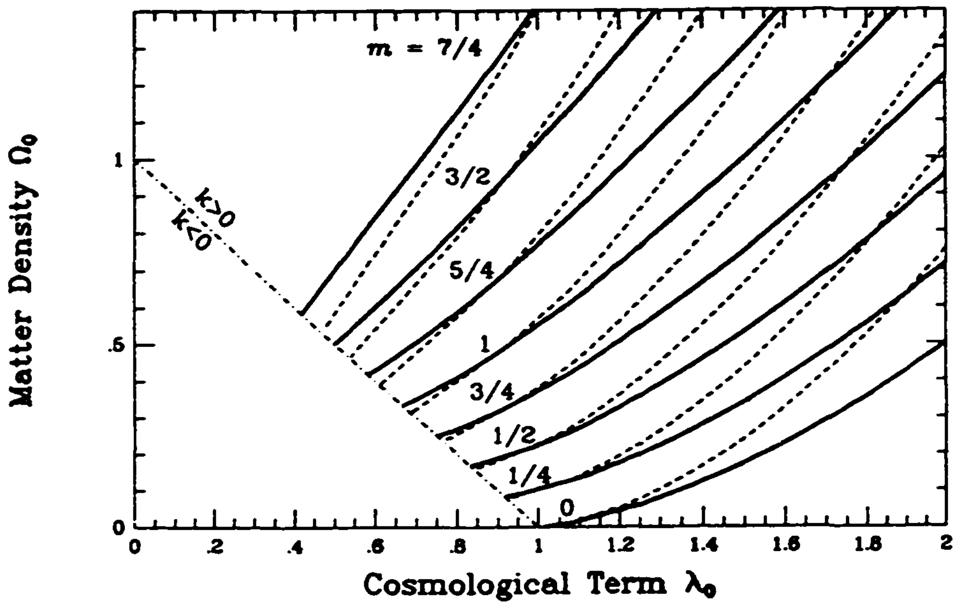


Figure 6.5: Enlarged view of the phase space diagram, Fig. 6.2, now plotted for various values of m between 0 and 2 (labelled beside each pair of curves), assuming $\gamma = 1$.

discussed above. Fig. 6.5 is a phase space diagram like Fig. 6.2, but plotted for a range of nonzero values of m. We have assumed $\gamma = 1$ as before, but there is now a different line of critical values $\lambda_*(m, \gamma, \Omega_0)$ for each value of m. Like Fig. 6.2, Fig. 6.5 shows that much of the nonsingular parameter space (below and to the right of the heavy solid lines) is eliminated because it does not overlap with the region allowed by the lensing constraint (above and to the left of the lighter dashed lines). This is especially true for small values of m. *With increasing* m, however, significant triangle-shaped regions of parameter space appear near $k = 0$. Thus for $m = 1$, there are allowed models with $\Omega_0 \sim 0.3$ and λ_0 between about 0.7 and 0.9. These are very close to the values favoured by observation (Bahcall *et al* 1997), so we focus on this case. The lensing constraint imposes the lower bound $\Omega_0 > 0.31$. If we take $\Omega_0 = 0.34$ as a specific example (as before), then we find by tracing across Fig. 6.5 horizontally that lensing also places an upper bound $\lambda_0 \leq 0.72$ on the cosmological term. The critical value for this case turns out from eq. (6.13) to be $\lambda_* = 0.68$, which is marginally consistent with the upper limits on λ_0 mentioned earlier. Therefore $m = 1$ models with $\Omega_0 = 0.34$ and $0.68 \leq \lambda_0 \leq 0.72$ are both realistic and singularity-free.

For larger values of m, the range of acceptable λ_0-values is broader, but one is also driven to higher values of Ω_0. With $m = 3/2$, for example, we find that viable nonsingular models occur only for $\Omega_0 > 0.46$, and that at $\Omega_0 = 0.51$ they lie in the range $0.51 \leq \lambda_0 \leq 0.57$. At $m = 7/4$ and $\Omega_0 = 0.59$ this broadens to $0.42 \leq \lambda_0 \leq 0.50$ — low enough to satisfy even the supernova constraint. As $m \to 2$, we find that $a_* \to 0$, so that nonsingularity is lost. The same thing happens in the limit $\gamma \to 2/3$ and, interestingly, as $\lambda_* \to 1 - \Omega_0$. (It is for this latter reason that all the nonsingular solutions in Fig. 6.5 have $k > 0$.)

Unusual values of γ, although interesting and possibly relevant at early times, are not required to establish our claim. In practice we find that the parameter space allowed by observation shrinks slightly for harder values of γ, like 7/6 or 4/3, but grows significantly for softer values such as those in the range $2/3 < \gamma < 1$ proposed by Caldwell *et al* (1997), or the $\gamma \approx 0.4$ of Turner and White (1997).

To verify that models with the properties described above can in fact avoid the initial singularity, Fig. 6.6 shows the evolution of the scale factor when $m = 1, \gamma = 1$ and $\Omega_0 = 0.34$, obtained as usual from the Taylor expansion (6.12). Various values of λ_0 are labelled beside the appropriate curves. This plot has exactly the same format as Fig. 6.1, with long dashes corresponding to $k > 0$, short ones to $k < 0$, and the dash-dotted line corresponding to $k = 0$.

Fig. 6.6 confirms that when λ_0 takes on the critical value λ_* (=0.679 807 621 in this case), the scale factor a evolves back to a constant value a_*, as expected (solid line). More importantly, the small *size* of this minimum value, $a_* = 0.0097$, means that we can now accommodate observed redshifts up to $z_* = 102$, well beyond the furthest objects yet seen. The model is thus compatible with all observational data. It cannot, however, accommodate larger redshifts like that attributed to the last scattering surface ($z_{lss} \sim 1100$), let alone the era of nucleosynthesis, which occurred at temperatures $T_{nuc} \sim 10^{10}$ K, or — since $T \propto (1 + z)$ — redshifts $z_{nuc} \sim 10^{10}$ (Kolb and Turner

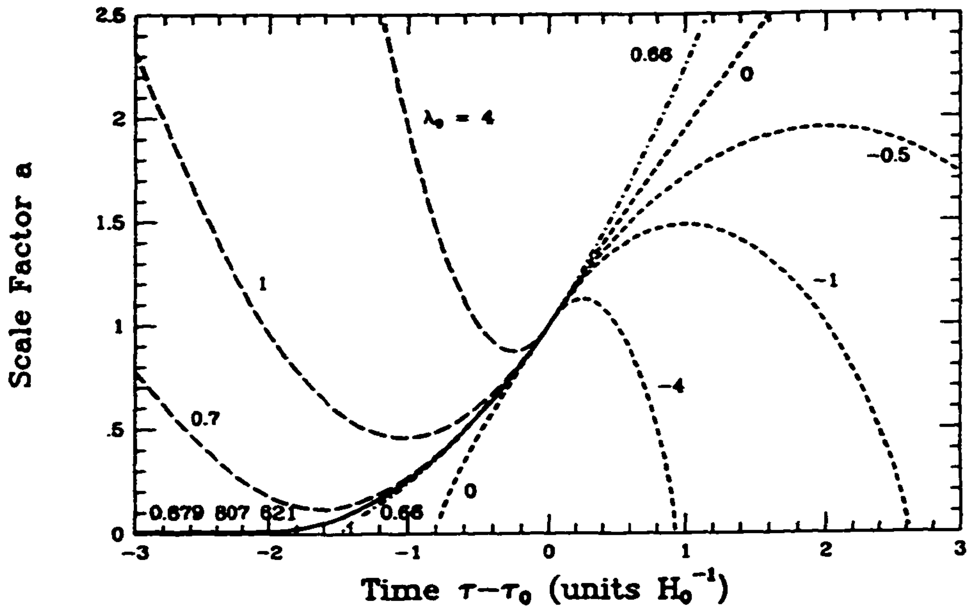


Figure 6.6: Evolution of the scale factor for universes with $m = 1, \gamma = 1$ and $\Omega_0 = 0.34$. Compare Fig. 6.1.

1990, p. 73). In fact, with $z_* = 102$ the $m = 1$ model heats up to no more than about $T_* = 103T_0 = 281$ K, which is just under room temperature (where the measured temperature of the CMB at zero redshift is $T_0 = 2.73$ K, Bennet *et al* 1996). That is to say, if the universe is described by a nonsingular variable-Λ theory with $m = 1$, then the CMB and the abundance of the light elements cannot be accounted for in the conventional way.

Alternative explanations have been proposed for both light element synthesis (eg., Hoyle 1992) and the isotropic microwave background (eg., Lerner 1995). These proposals, however, are not widely accepted (Peebles 1993, § 7). The traditional explanation in terms of a hot, dense early phase can be retained in the context of variable-Λ theory by moving to larger values of m. As discussed above, this leads to smaller values of a_* and hence larger values of z_* and T_*. In this picture the high temperature of the early universe is a result, not of an initial big bang singularity, but of a very deep "big bounce." With $m = 1.5$ and $\Omega_0 = 0.51$, for instance, we find that $a_* = 1.7 \times 10^{-4}$, corresponding to a maximum redshift of 5900 and a "bounce temperature" of 16,000 K

— more than sufficient for recombination. At $m = 1.8$, this latter number climbs to 5.7×10^9 K — hot enough for nucleosynthesis. (An analysis like that below Fig. 6.5 shows that models with $m = 1.8$ and $\Omega_0 = 0.61$ are observationally viable if $0.41 \leq \lambda_0 \leq 0.49$.) With $m = 1.9$, it is possible to entertain bounce temperatures as high as 2.9×10^{13} K (2.5 GeV), approaching the realm of electroweak unification. (Models with $m = 1.9$ and $\Omega_0 = 0.67$ agree with observation if $0.39 \leq \lambda_0 \leq 0.48$.) As $m \rightarrow 2$, in fact, one finds that $a_* \rightarrow 0$ and $T_* \rightarrow \infty$. (Nonsingularity is lost if $m = 2$ exactly.) As remarked above, however, these higher values of m come at the modest observational price of higher *matter densities*. The lensing constraint sets a limit of $\Omega_0 > 0.54$ for $m = 1.8$ models, for example, and $\Omega_0 > 0.57$ for $m = 1.9$ models.

Chapter 7

Models with $\Lambda \propto H^n$

7.1 Previous Work

We have seen in Chapter 6 that variable-Λ theories in which $\Lambda \propto a^{-m}$ appear to offer the possibility of avoiding the initial singularity without violating any observational constraints. The parameter space occupied by viable models, however, remains rather small. One reason for this is that, as discussed in § 6.5, the nonsingular solutions are all closed, whereas observational evidence tends to favour an open universe (Peebles 1993, § 26). In this chapter we will shift our attention to decay laws of the form set out in eq. (4.7):

$$\Lambda = C H^n, \tag{7.1}$$

where H is the Hubble parameter and C and n are, for the time being, arbitrary constants. Among other things we will find that a cosmological term of this kind appears to lead to singularity avoidance in *open* models.

Decay laws of the form (7.1) were introduced by Carvalho *et al* (1992), who used a dimensional argument similar to that of Chen and Wu (1990) to argue that n should take the value two. This value, in fact, has subsequently been adopted by nearly every author who has considered H-dependence in the cosmological term (§ 4.4). Most of these authors have incorporated a-dependence into their models as well (Carvalho *et al* 1992, Salim and Waga 1993, Waga 1993, Arbab and Abdel-Rahman 1994), so their decay laws are not strictly of the form (7.1). As far as we are aware, the only models in which Λ has *exactly* this form are the variable-G theories of Lima and Carvalho (1994) and Arbab (1997), and the scalar field theory of Wetterich (1995). The first two authors introduced some new arguments in favour of $n = 2$ based on Dirac's large number hypothesis.

Phenomenological Λ-decay laws have become considerably more sophisticated with time. One example is the theory of Lima and Maia (1994) and Lima and Trodden (1996), who have set the cosmological term equal to the total energy density of the universe, multiplied by a factor of $(\alpha + \beta H)$:

$$\Lambda = (\alpha + \beta H)\left(H^2 + \frac{k}{a^2}\right), \tag{7.2}$$

where α and β are adjustable parameters. In effect this is a third-order polynomial expansion of Λ in powers of H. Another example is furnished by the matter creation theory of Nesteruk *et al* (1997), in which:

$$\Lambda \propto H\left(a\frac{dH}{da} + 2H\right). \tag{7.3}$$

We will restrict ourselves here to the shorter expression (7.1), reasoning that, as long as we are in the realm of purely phenomenological scenarios, simpler is probably better.

7.2 Riccati's Equation

We can rewrite the left-hand side of the differential equation (4.16) in terms of H by recalling that $H \equiv \dot{a}/a$ and differentiating, whereupon:

$$\frac{\ddot{a}}{a} = aH\frac{dH}{da} + H^2. \tag{7.4}$$

Substituting this into eq. (4.16) along with the decay law (7.1), we obtain the following differential equation:

$$\frac{dH}{da} = \left(\frac{\gamma C}{2a}\right)H^{n-1} - \left(\frac{3\gamma}{2a}\right)H - \left(\frac{3\gamma - 2}{2a}\right)\left(\frac{k}{a^2}\right)H^{-1}. \tag{7.5}$$

This is nonlinear, but bears some resemblance to the Riccati equation (5.11). Let us make a change of dependent variables from H to x, defined by:

$$x \equiv H^r, \tag{7.6}$$

where r is an arbitrary constant, whose value we will choose in a moment. In terms of x, eq. (7.5) takes the form:

$$\frac{dx}{da} = \left(\frac{r\gamma C}{2a}\right) x^{\ell} - \left(\frac{3r\gamma}{2a}\right) x - \left[\frac{r(3\gamma - 2)k}{2a^3}\right] x^m, \tag{7.7}$$

where:

$$\ell \equiv \frac{n + r - 2}{r} \quad , \quad m \equiv \frac{r - 2}{r}. \tag{7.8}$$

Choosing $n = 4$ and $r = 2$, we find that $\ell = 2$ and $m = 0$, whereupon eq. (7.7) can be written:

$$\frac{dx}{da} = \mathcal{P}(a)\, x^2 + \mathcal{Q}(a)\, x + \mathcal{R}(a), \tag{7.9}$$

where:

$$\mathcal{P}(a) \equiv \frac{\gamma C}{a} \quad , \quad \mathcal{Q}(a) \equiv -\frac{3\gamma}{a} \quad , \quad \mathcal{R}(a) \equiv \frac{(2 - 3\gamma)k}{a^3}. \tag{7.10}$$

This is *precisely* the form of Riccati's equation (5.11). So it should be possible to obtain analytic solutions for the Hubble parameter in the case $n = 4$.

On the other hand, choosing $n = 2$ and $r = 2$, we find that $\ell = 1$ and $m = 0$, whereupon eq. (7.7) reduces to:

$$\frac{dx}{da} + \mathcal{S}(a)\, x = \mathcal{T}(a), \tag{7.11}$$

where:

$$\mathcal{S}(a) \equiv \frac{\gamma(3 - C)}{a} \quad , \quad \mathcal{T}(a) \equiv \frac{(2 - 3\gamma)k}{a^3}. \tag{7.12}$$

This is an ordinary linear first-order differential equation. So it is also possible to solve analytically for H when $n = 2$.

A check reveals that no other combinations of n and r will put eq. (7.7) into a simply-integrable form. We therefore concentrate on the cases $n = 2$ and $n = 4$, leaving other choices for future numerical work. Even with this restriction we will find considerable richness in the theory.

7.3 The Case $n = 2$

Let us begin with the linear case first. Eq. (7.11) is put into exact form with the multiplying factor:

$$\exp\left[\int \mathcal{S}(a)da\right] = a^{\gamma(3-C)}, \tag{7.13}$$

whereupon:

$$\frac{d}{da}\left[x\, a^{\gamma(3-C)}\right] = \mathcal{T}(a)\, a^{\gamma(3-C)}. \tag{7.14}$$

This can be integrated to yield:

$$x = \left[\frac{(2-3\gamma)k}{\gamma(3-C)-2}\right] a^{-2} + C_0 a^{-\gamma(3-C)}, \tag{7.15}$$

where C_0 is a constant of integration. The Hubble parameter is then found from eq. (7.6), which (with $r = 2$) yields:

$$H(a) = \left\{\left[\frac{(2-3\gamma)k}{\gamma(3-C)-2}\right] a^{-2} + C_0 a^{\gamma(C-3)}\right\}^{1/2}. \tag{7.16}$$

We can fix the constant C_0 by imposing the boundary condition $H(a_0) = H(1) = H_0$. This produces:

$$C_0 = H_0^2 - \frac{(2-3\gamma)k}{\gamma(3-C)-2}. \tag{7.17}$$

Two other new quantities will become useful:

$$\begin{aligned} \beta &\equiv \left[\frac{2-3\gamma}{\gamma(3-C)-2}\right]\frac{k}{H_0^2} \\ \alpha &\equiv 1-\beta. \end{aligned} \tag{7.18}$$

In terms of these parameters, eq. (7.16) becomes:

$$H(a) = H_0\left[\alpha\, a^{\gamma(C-3)} + \beta a^{-2}\right]^{1/2}. \tag{7.19}$$

We can fix α, β and C in terms of observable quantities at the present time $t = t_0$, as follows. Firstly, the decay law (7.1) gives (with $n = 2$):

$$C = \frac{\Lambda_0}{H_0^2} = 3\lambda_0, \tag{7.20}$$

where we have used the definition (4.22). Substituting this result into eqs. (7.18), together with the expression (4.24) for the curvature parameter k, we find (setting $a_0 = 1$):

$$\begin{aligned} \alpha &= \left[\frac{(3\gamma/2 - 1)\Omega_0 - \lambda_0}{(3\gamma/2)(1-\lambda_0) - 1}\right] \\ \beta &= 1 - \alpha. \end{aligned} \tag{7.21}$$

In fact, it is interesting to see that the numerator in this expression for α is exactly q_0, eq. (4.29).

Switching to Hubble time $\tau \equiv H_0\, t$ (§ 5.2) as usual, we note that:

$$H = \frac{H_0}{a}\frac{da}{d\tau}, \tag{7.22}$$

so that eq. (7.19) takes the simple form:

$$\frac{da}{d\tau} = \left[\alpha\, a^{2-3\gamma(1-\lambda_0)} + \beta\right]^{1/2}, \tag{7.23}$$

where we have used eq. (7.20) for C.

7.4 Evolution of the Scale Factor

At this stage, one could attempt to solve exactly for the scale factor $a(\tau)$ via:

$$\frac{da}{\sqrt{\alpha\, a^s + \beta}} = d\tau, \tag{7.24}$$

where:

$$s \equiv 2 - 3\gamma(1 - \lambda_0). \tag{7.25}$$

This might be soluble in terms of elliptic (or simpler) integrals if λ_0 takes special values such that $s = I$ (where I is an integer); this would require:

$$\lambda_0 = \frac{I + (3\gamma - 2)}{3\gamma}. \tag{7.26}$$

Rather than depending on a coincidence of this form, we will instead take the same approach as in Chapter 6 and evolve the scale factor numerically in terms of its first and second derivatives. The latter of these takes the form:

$$\frac{d^2a}{d\tau^2} = \alpha\left[1 - \frac{3\gamma}{2}(1 - \lambda_0)\right] a^{1-3\gamma(1-\lambda_0)}. \tag{7.27}$$

It remains to substitute this result, together with eq. (7.23), into the Taylor expansion (6.12), integrating backward in time to determine the behaviour of the scale factor.

7.5 Minimum Values of the Scale Factor

In order to have some idea of the best values of Ω_0 and λ_0 to use in this procedure, we do as in the previous chapter and begin by identifying the viable regions of parameter space. As discussed in § 6.3, the nonsingular models of interest evolve backward to some point where $a = a_*$ and $da/d\tau = 0$. Some have $d^2a/d\tau^2 = 0$ there in addition; these are the Eddington-Lemaître solutions. But as a minimum requirement, we can impose:

$$\frac{da}{d\tau} = 0 \qquad (\text{at } a = a_*) \ . \tag{7.28}$$

In conjunction with our solution (7.23), this implies:

$$a_*^{3\gamma(1-\lambda_0)-2} = -\frac{\alpha}{\beta}. \tag{7.29}$$

It is convenient to distinguish two cases, according to whether the exponent on a_* is positive or negative. Using eqs. (7.21), we obtain:

$$a_* = \begin{cases} \left[\dfrac{2\lambda_0 - (3\gamma-2)\Omega_0}{(3\gamma-2)(1-\Omega_0-\lambda_0)}\right]^{1/[3\gamma(1-\lambda_0)-2]} & \text{if } \lambda_0 < \lambda_c \\ \left[\dfrac{(3\gamma-2)(1-\Omega_0-\lambda_0)}{2\lambda_0 - (3\gamma-2)\Omega_0}\right]^{1/[2-3\gamma(1-\lambda_0)]} & \text{if } \lambda_0 > \lambda_c, \end{cases} \tag{7.30}$$

where we have defined:

$$\lambda_c \equiv 1 - \frac{2}{3\gamma}, \tag{7.31}$$

and left the case $\lambda_0 = \lambda_c$ aside for the time being. From eq. (7.30) we draw a number of important conclusions. Firstly, (1) that spatially *flat* solutions ($\Omega_0 + \lambda_0 = 1$) have either $a_* = \infty$ (if $\lambda_0 < \lambda_c$) or $a_* = 0$ (if $\lambda_0 > \lambda_c$). The former case is not interesting. The latter case is singular, with the initial singularity pushed back into the infinite past. We have encountered this kind of solution before (§ 5.4.3).

(2) Secondly, if we make the modest assumption that $\gamma > 2/3$ (ie., normal, non-inflationary matter), then we notice that *closed* solutions ($\lambda_0 > 1 - \Omega_0$) must satisfy:

$$\lambda_0 < \left(\frac{3\gamma}{2} - 1\right)\Omega_0; \tag{7.32}$$

while *open* ones (ie., $\lambda_0 < 1 - \Omega_0$) obey:

$$\lambda_0 > \left(\frac{3\gamma}{2} - 1\right)\Omega_0; \tag{7.33}$$

These conclusions follow from requiring that the terms in square brackets be positive; ie., from the requirement that a_* be a real number.

[The only possible exceptions to this consist of the cases in which the exponent is an even integer:

$$\frac{1}{3\gamma(1-\lambda_0) - 2} = 2I \qquad (\ I \text{ an integer }\ ), \tag{7.34}$$

which occur only for the special values:

$$\lambda_0 = 1 - \frac{2}{3\gamma} - \frac{1}{6\gamma I}. \tag{7.35}$$

While these are possible in principle, we will ignore them for the moment and concentrate on the more general case.]

(3) Thirdly, requiring that a_* lie in the range $0 < a_* < 1$, we learn that both the numerators and denominators in eq. (7.30) must be positive. This may be verified as follows. Comparing their relative absolute magnitudes, we distinguish two possibilities: (a) if $\lambda_0 < \lambda_c$, then both the numerator and denominator must be positive, since otherwise:

$$\begin{aligned}(3\gamma - 2)(1 - \Omega_0 - \lambda_0) &< 2\lambda_0 - (3\gamma - 2)\Omega_0; \qquad \text{or} \\ \lambda_0 &> \lambda_c,\end{aligned} \tag{7.36}$$

which is contrary to the hypothesis. On the other hand, (b) if $\lambda_0 > \lambda_c$, then we *also* find that both numerator and denominator must be positive, since otherwise:

$$\begin{aligned}(3\gamma - 2)(1 - \Omega_0 - \lambda_0) &> 2\lambda_0 - (3\gamma - 2)\Omega_0; \qquad \text{or} \\ \lambda_0 &< \lambda_c,\end{aligned} \tag{7.37}$$

which is again contrary to the hypothesis. Therefore both the numerators and denominators are positive in *all cases*.

(4) Fourthly, it follows from the conclusion (3), in conjunction with the assumption $\gamma > 2/3$, that we must have:

$$\lambda_0 < 1 - \Omega_0. \tag{7.38}$$

In other words, to realistically describe the present universe, nonsingular models in the present theory must be *open*.

While open solutions have previously been found in some nonsingular theories involving involving scalar fields (Bekenstein 1974) and higher-order curvature terms (Page 1987), we are not aware of precedents for this in theories based on a cosmological term. Lima and Maia (1994) and Lima and Trodden (1996) describe their variable-Λ theory (which can be either open, flat or closed) as nonsingular, but this is true only in the sense of de Sitter; the initial singularity is pushed back into the infinite past (*cf.* § 5.4.3). We are referring here to models, like those of Bekenstein (1974) and Page (1987), in which there is a finite and *nonzero* minimum scale factor. Besides these, and one other which is flat in both space *and* time (Petry 1981), all the singularity-free cosmological models of which we are aware are *closed* (see § 4.1). The reason for this is that singularity avoidance in most of these cases requires a large amount of special matter (in particular, matter which violates the energy conditions). Whether this is in the form of Λ-matter or something else, it will in any case contribute to the total energy density of the universe, usually closing it in the process. A cosmological term decaying like eq. (7.1), by contrast, need not (and in fact cannot) contribute this much.

(5) Fifthly, we infer by combining (2) and (4) that:

$$\lambda_0 > \left(\frac{3\gamma}{2} - 1\right)\Omega_0, \tag{7.39}$$

which sets a lower limit on the size of the cosmological term.

(6) Finally, (4) and (5) together impose an upper limit:

$$\Omega_0 < \frac{2}{3\gamma}, \tag{7.40}$$

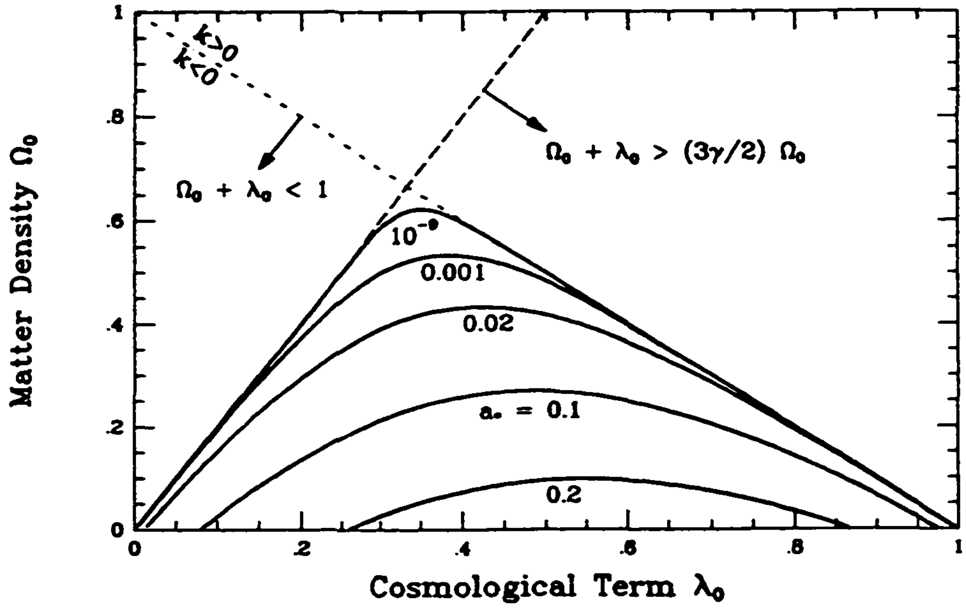


Figure 7.1: Phase space diagram for the case $n = 2$ with $\gamma = 1$, showing contours of equal minimum size a_*.

on the matter density of the universe.

7.6 Realistic Nonsingular Models

The information contained in eqs. (7.30) – (7.40) is summarized in Fig. (7.1), which is a phase space plot like Fig. 6.5, but enlarged to show only the range of interest to us here, $0 \leq \Omega_0 \leq 1$ and $0 \leq \lambda_0 \leq 1$. Models with (λ_0, Ω_0) are represented by points on this diagram, as usual. The critical values of λ_0 in this theory define the upper edges of the triangular region at the base of the diagram; ie., the region bounded by the curves $\Omega_0 + \lambda_0 < 1$ (dash-dotted line) and $\Omega_0 + \lambda_0 > (3\gamma/2)\Omega_0$ (dashed line). All models between these curves are nonsingular, with real values of a_* in the range $0 < a_* < 1$, as we have stipulated. Fig. 7.1 is plotted for $\gamma = 1$.

Using eq. (7.30), we have plotted contours of *equal minimum scale factor* a_* in this region. Any point along one of the heavy solid lines corresponds to a nonsingular model with the labelled value of a_*. Following the discussion in § 6.5, we would like to find models in which a_* is as small as possible, in order to obtain the largest possible maximum observable redshift z_{obs}:

$$z_{obs} \leq \frac{1}{a_*} - 1. \tag{7.41}$$

For instance, to be compatible with observations of quasars ($z_{obs} \approx 10$), a_* must be less than about 0.1. If we want to explain the CMB as relic radiation from the last scattering surface at $z_{lss} \approx 1100$, then we need a smaller minimum scale factor, $a_* < 0.001$. And if we want the early universe to heat up to the temperatures required for nucleosynthesis:

$$T_{nuc} \approx T_0 \left(\frac{a_0}{a_*}\right) \approx 10^{10} \text{ K} , \tag{7.42}$$

then (taking $T_0 = 2.73$ K along with $a_0 = 1$ as usual), our model must satisfy:

$$a_* < 10^{-9}. \tag{7.43}$$

Fig. 7.1 demonstrates that the present theory, based on a cosmological term decaying as $\Lambda \propto H^2$, *has no trouble meeting this constraint.* Any model lying along the curve labelled 10^{-9} will be capable, in principle, of reaching these temperatures near the "big bounce." This confirms the remark made by Blome and Priester (1991) and Dąbrowski (1996) that there is no reason in principle why the oscillations in a nonsingular model cannot be deep enough to account for all the evidence which is usually taken as proof that the universe began in a singularity. Moreover, the only observational constraint which seriously limited the variable-Λ models in the last chapter — the lensing constraint — does not apply in this chapter because there is no antipode in an open universe.

It is true that, to meet the condition (7.43), models in this theory must lie very near the upper edge of the triangular region in Fig. 7.1. In a sense this is "fine-tuning." In another sense, however, it is an advantage, because (as in Chapter 6), it allows us to *predict* the values of λ_0 that would be required in a realistic singularity-free theory. As an example, let us consider the observationally favoured value of $\Omega_0 = 0.3$ (Bahcall *et al* 1997), and let us assume $\gamma = 1$ as usual. Tracing horizontally across the line defined by $\Omega_0 = 0.3$ in Fig. 7.1, we can see that the nucleosynthesis condition (7.43) is met by only *two* values of the cosmological term: $\lambda_0 \approx 0.15$ and $\lambda_0 \approx 0.7$. Of these, the larger value is only marginally viable, being very close to the observational upper bounds described in § 6.4.1. The smaller value, however, is perfectly acceptable from an observational standpoint. (Note that the larger value corresponds to a nearly flat model with $\Omega_0 + \lambda_0$ just under unity, while the smaller value corresponds to $\Omega_0 + \lambda_0 < 0.5$.)

In general, the theory predicts that the most likely value of λ_0 is either *just above*:

$$\left(\frac{3\gamma}{2} - 1\right)\Omega_0, \tag{7.44}$$

(which is to say, just above $\Omega_0/2$ in a dust-like universe with $\gamma = 1$); or else *just below*:

$$1 - \Omega_0. \tag{7.45}$$

The former situation is in better agreement with the increasingly stringent observational upper limits on λ_0 (§ 6.4.1), while the latter one might be preferable to some on theoretical grounds (eg., Krauss and Turner 1995).

To confirm that models with the features we have outlined here really do avoid the initial singularity, we can do as in Chapter 6 and integrate numerically back in time using eqs. (7.23) and (7.27) with the Taylor series expansion (6.12). We have done this in Fig. 7.2 for the case $\Omega_0 = 0.3$ and $\gamma = 1$. This diagram is an enlarged version of the evolution plots in Chapter 6 (Figs. 6.1 and 6.6), showing only the past three Hubble times and none of the future. Values of λ_0 are marked beside the appropriate curves. Several features can be noted.

Firstly, the initial singularity is avoided for any value of λ_0 between 0.15 and 0.7, as expected on the basis of the phase space diagram, Fig. 7.1. In the limiting case where $\lambda_0 = 0.7$ exactly, which is spatially flat, we see that $a_* = 0$ (de Sitter-like behaviour), as expected based on the discussion following eq. (7.30). It is important to note that models with *larger* values of λ_0 than this (eg., 1, 2, 10) no longer avoid the initial singularity with progressively steeper bounces, as they did in the previous chapter, but instead become singular again, as the diagram clearly shows. [Models with smaller values of λ_0 (eg., 0, -0.5, -1) are also singular, as usual.]

Secondly, Fig. 7.2 confirms that the value of a_* is smallest near the critical values of λ_0: 0.064 for the $\lambda_0 = 0.6$ case (just below 0.7), and 0.018 for the $\lambda_0 = 0.2$ case

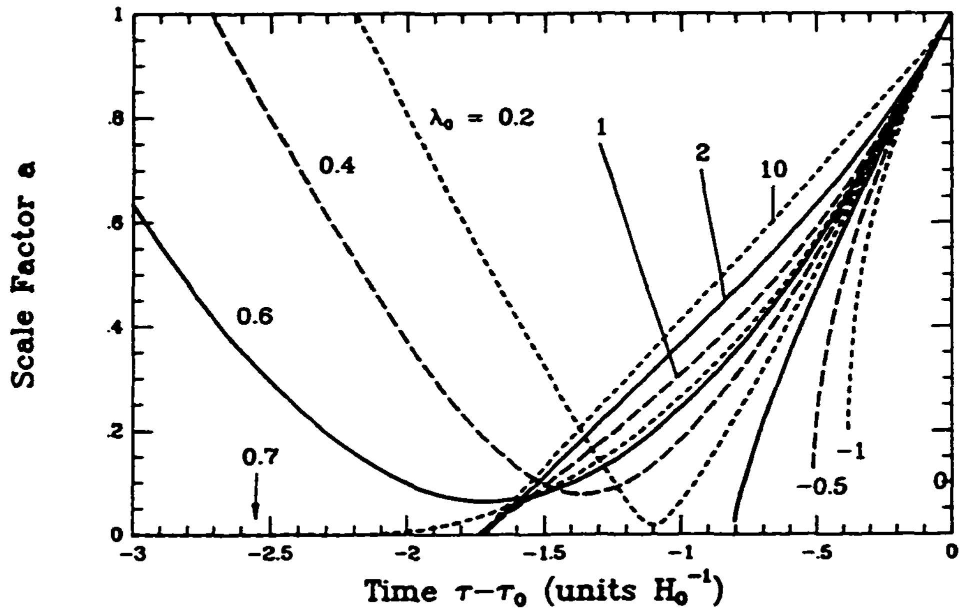


Figure 7.2: Evolution of the scale factor for models with $n = 2, \gamma = 1$, $\Omega_0 = 0.3$, and values of λ_0 labelled beside each curve.

(just above 0.15). [These numbers have been chosen for illustrative purposes; smaller values of a_* (with consequently larger bounce temperatures) are obtained by letting λ_0 approach the critical values more closely.]

Thirdly, this evolution plot gives us some information about the *ages* of the models; that is, the elapsed time from the big bang — *or the big bounce*, as appropriate — to the present epoch, denoted as usual by $\tau = \tau_0$. It may be seen that higher values of λ_0 correspond to older universes, as usual: 1.10 Hubble times for $\lambda_0 = 0.2$, and 1.70 Hubble times for $\lambda_0 = 0.6$. (Strictly speaking, this now applies only for λ_0 less than the critical value 0.7.) Both these numbers are in excellent agreement with the observational requirements discussed in § 6.4.2. Even if H_0 takes on the largest value 83 km s^{-1} Mpc^{-1} allowed by Freedman (1996), the ages of the two models are 13.0 and 20.1 billion years respectively — well above Chaboyer *et al*'s (1997) lower limit of 9.6 Gyr on the age of the oldest globular clusters.

7.7 The Case $n = 4$

We proceed to the other case of interest, $n = 4$, which consists of the Riccati equation (7.9) for $x(a)$. This can be solved (Spiegel 1981, p. 217) by changing dependent variables from x to y, where:

$$x = \frac{-1}{\mathcal{P}y}\frac{dy}{da}. \tag{7.46}$$

The differential equation then takes the new form:

$$\frac{d^2y}{da^2} + \alpha(a)\frac{dy}{da} + \beta(a)\,y = 0, \tag{7.47}$$

where:

$$\alpha(a) \equiv -\left(\frac{1}{\mathcal{P}}\frac{d\mathcal{P}}{da} + \mathcal{Q}\right) = \frac{1+3\gamma}{a} \tag{7.48}$$

$$\beta(a) \equiv \mathcal{P}\mathcal{R} = \frac{\gamma\mathcal{C}(2-3\gamma)k}{a^4}, \tag{7.49}$$

and we have made use of eqs. (7.10). This is now linear as desired, but with variable coefficients. The second of these can be connected to observation as follows. From the decay law (7.1), with $n = 4$:

$$\mathcal{C} = \frac{\Lambda_0}{H_0^4} = \frac{3\lambda_0}{H_0^2}. \tag{7.50}$$

Substituting this result into eq. (7.49), together with the expression (4.24) for the curvature parameter k, we find (with $a_0 = 1$):

$$\beta(a) \equiv \frac{3\gamma(2-3\gamma)\lambda_0(\Omega_0+\lambda_0-1)}{a^4}. \tag{7.51}$$

Let us summarize the situation by writing the differential equation (7.47) in the form:

$$\frac{d^2y}{da^2} + \alpha_0 a^{-1}\frac{dy}{da} + \beta_0 a^{-4}\,y = 0, \tag{7.52}$$

where the constants α_0, β_0 are defined by:

$$\alpha_0 \equiv 1+3\gamma \tag{7.53}$$

$$\beta_0 \equiv 3\gamma(2-3\gamma)\lambda_0(\Omega_0+\lambda_0-1). \tag{7.54}$$

This is a nontrivial problem. We can however put the differential equation in "normal form" (Murphy 1960, p. 90) by changing independent variables from a to z, defined by:

$$z \equiv \int e^{-\phi(a)} da, \tag{7.55}$$

where:

$$\phi(a) \equiv \int (\alpha_0 a^{-1}) da = \alpha_0 \ln a. \tag{7.56}$$

We find that:

$$z = \left(\frac{1}{1-\alpha_0}\right) a^{1-\alpha_0}. \tag{7.57}$$

In terms of z, eq. (7.52) becomes:

$$z^{\alpha_1} \frac{d^2 y}{dz^2} + \beta_1 y = 0, \tag{7.58}$$

where:

$$\alpha_1 \equiv \frac{2(2-\alpha_0)}{1-\alpha_0} \tag{7.59}$$

$$\beta_1 \equiv \beta_0 (1-\alpha_0)^{-\alpha_1}. \tag{7.60}$$

This is now in normal form, as desired. Somewhat surprisingly, it has the same form as the differential equation (5.15) which described all the main solutions of the $\Lambda \propto \tau^{-\ell}$ case in Chapter 5. (In fact, the two are identical if we put $z \to \tau$, $y \to x$, $\alpha_1 \to \ell$, and $\beta_1 \to -\alpha$.) So we could in principle bring over all the results of Chapter 5. However, inspection of the exponent α_1 using eq. (7.53) reveals that it is given by:

$$\alpha_1 = \frac{2}{3\gamma}(3\gamma - 1). \tag{7.61}$$

In Chapter 5 we were able to solve only for the special values $1, 2, 3, 4$ of this exponent. As we can see by inverting eq. (7.61):

$$\gamma = \frac{1}{3(1-\alpha_1/2)}, \tag{7.62}$$

this corresponds to values of $\gamma = \frac{2}{3}, \infty, -\frac{2}{3}$ and $-\frac{1}{3}$ respectively. None of these are reasonable values for the adiabatic index of ordinary matter, at least not in the present universe (see § 4.7). Conversely, values of γ that *are* reasonable (such as $\gamma = 1$ or $\frac{4}{3}$) correspond to non-integral values of α_1 (such as $\frac{4}{3}$ and $\frac{3}{2}$ respectively). It is doubtful that eq. (7.58) can be solved analytically in these cases. We therefore leave the possibility that $n = 4$ for future numerical attention.

Chapter 8

Models with $\Lambda \propto q^r$

8.1 Evolution of the Scale Factor

We turn finally to the last of our phenomenological decay laws, eq. (4.8), which we write in the form:

$$\Lambda = \mathcal{D} \left(\frac{\ddot{a}}{a} \right)^r , \tag{8.1}$$

where $\mathcal{D}$ and r are, for the time being, arbitrary constants. As far as we are aware, no such dependence has previously been considered for the cosmological term. However, it is a natural extension of the other decay scenarios considered so far. There is no fundamental difference between the first and second derivatives of the scale factor that would preclude the latter from acting as an independent variable if the former is acceptable. We discuss several interesting analogies and a possible physical interpretation of the cosmological term at the end of this chapter (§ 8.3).

If we substitute the decay law (8.1) into our central differential equation (4.16), along with the relationship (7.4) between $\ddot{a}$ and H, then we obtain:

$$\mathcal{D} \left(aH\frac{dH}{da} + H^2 \right)^r = \left(3 - \frac{2}{\gamma} \right) \left(H^2 + \frac{k}{a^2} \right) + \frac{2}{\gamma} \left(aH\frac{dH}{da} + H^2. \right) , \tag{8.2}$$

from which it is immediately clear that no analytic solution for H will be forthcoming unless:

$$r = 1. \tag{8.3}$$

We will adopt this value for r for the remainder of the chapter, since we require the Hubble parameter in analytic form in order to make use of of the Taylor expansion (6.12). Eq. (8.2) then takes the form:

$$\frac{dH}{da} = \frac{\gamma}{a}\left(\frac{3-\mathcal{D}}{\gamma\mathcal{D}-2}\right)H + \frac{k}{a^3}\left(\frac{3\gamma-2}{\gamma\mathcal{D}-2}\right)H^{-1}. \tag{8.4}$$

As in § 7.2, let us make a change of dependent variables from H to x, defined by:

$$x \equiv H^s, \tag{8.5}$$

where s is an arbitrary constant. Eq. (8.4) takes the form:

$$\frac{dx}{da} = \frac{s\gamma}{a}\left(\frac{3-\mathcal{D}}{\gamma\mathcal{D}-2}\right)x + \frac{s\,k}{a^3}\left(\frac{3\gamma-2}{\gamma\mathcal{D}-2}\right)x^{(s-2)/s}. \tag{8.6}$$

If we choose:

$$s = 2, \tag{8.7}$$

then we obtain a linear first-order differential equation:

$$\frac{dx}{da} + \mathcal{S}(a)\,x = \mathcal{T}(a), \tag{8.8}$$

where:

$$\mathcal{S}(a) \equiv 2\gamma\left(\frac{\mathcal{D}-3}{\gamma\mathcal{D}-2}\right)a^{-1} \quad , \quad \mathcal{T}(a) \equiv 2k\left(\frac{3\gamma-2}{\gamma\mathcal{D}-2}\right)a^{-3}. \tag{8.9}$$

Multiplying through by a factor of:

$$\exp\left[\int \mathcal{S}(a)da\right] = a^{2\gamma(\mathcal{D}-3)/(\gamma\mathcal{D}-2)}, \tag{8.10}$$

and solving exactly as in § 7.3, we obtain:

$$x = C_0 a^{-2\gamma(\mathcal{D}-3)/(\gamma\mathcal{D}-2)} - ka^{-2}, \tag{8.11}$$

where C_0 is a constant of integration. This is very reminiscent of the situation in the last chapter, eq. (7.15). [The similarity is probably related to the fact that Λ contributes to both $(\dot{a}/a)^2$ and $(\ddot{a}/a)$ in exactly the same manner, eqs. (4.11) and (4.15).] The Hubble parameter is found from eq. (8.5), which yields (with $s = 2$):

$$H(a) = \left[C_0 a^{-2\gamma(\mathcal{D}-3)/(\gamma\mathcal{D}-2)} - ka^{-2}\right]^{1/2}. \tag{8.12}$$

The boundary condition $H(a_0) = H(1) = H_0$ produces:

$$C_0 = H_0^2 + k, \tag{8.13}$$

or, applying eq. (4.24) with $a_0 = 1$:

$$C_0 = H_0^2(\Omega_0 + \lambda_0). \tag{8.14}$$

We can also fix $\mathcal{D}$ in terms of observable quantities, as follows. The decay law (8.1) gives (with $r = 1$):

$$\mathcal{D} = \Lambda_0 \left(\frac{\ddot{a}}{a}\right)_{t=t_0} = -\frac{\Lambda_0}{H_0^2\, q_0} = -\frac{3\lambda_0}{q_0}, \tag{8.15}$$

where q_0 is the present value of the deceleration parameter, and we have used the definitions (4.22) and (4.28). Substituting this result into eq. (8.12), along with the expression (4.24) for k, we find:

$$H(a) = H_0 \left[\alpha\, a^{-3\gamma(q_0+\lambda_0)/[q_0+(3\gamma/2)\lambda_0]} + \beta\, a^{-2}\right]^{1/2}, \tag{8.16}$$

where:

$$\alpha = (\Omega_0 + \lambda_0) \quad , \quad \beta = 1 - \alpha. \tag{8.17}$$

This is very similar to eq. (7.19) in the last chapter.

Changing once again to units of Hubble times (§ 5.2) we obtain:

$$\frac{da}{d\tau} = \left[\alpha\, a^{\xi} + \beta\right]^{1/2}, \tag{8.18}$$

where we have defined:

$$\xi \equiv \frac{(2 - 3\gamma)q_0}{q_0 + (3\gamma/2)\lambda_0}. \tag{8.19}$$

This expression, together with its time derivative:

$$\frac{d^2a}{d\tau^2} = \frac{\xi}{2}(\Omega_0 + \lambda_0)a^{\xi-1}, \tag{8.20}$$

is now ready to be substituted into the Taylor expansion (6.12).

8.2 Minimum Values of the Scale Factor

As in the previous chapter (§ 7.5), we require that nonsingular models satisfy:

$$\frac{da}{d\tau} = 0 \qquad (\text{at } a = a_*) \ . \tag{8.21}$$

In conjunction with eq. (8.18), this implies:

$$a_*^\xi = \left(\frac{\Omega_0 + \lambda_0 - 1}{\Omega_0 + \lambda_0} \right) . \tag{8.22}$$

Let us write this out explicitly using eq. (8.19). Results depend on the sign of the exponent, as in the last chapter. As with eq. (7.30), we will find it convenient to distinguish two possible cases:

$$a_* = \begin{cases} \left(\dfrac{\Omega_0 + \lambda_0}{\Omega_0 + \lambda_0 - 1} \right)^{\left[\frac{1+(3\gamma\lambda_0/2q_0)}{3\gamma-2}\right]} & \text{if } q_0 > q_c \\ \left(\dfrac{\Omega_0 + \lambda_0 - 1}{\Omega_0 + \lambda_0} \right)^{\left[\frac{-(3\gamma\lambda_0/2q_0)-1}{3\gamma-2}\right]} & \text{if } q_0 < q_c, \end{cases} \tag{8.23}$$

where we have defined:

$$q_c \equiv -\left(\frac{3\gamma}{2}\right)\lambda_0, \tag{8.24}$$

and left the case $q_0 = q_c$ aside for the time being. We can draw a number of useful conclusions from the form of eq. (8.23). Firstly, (1) that spatially *flat* solutions ($\Omega_0 + \lambda_0 = 1$) again have either $a_* = \infty$ (if $q_0 > q_c$) or $a_* = 0$ (if $q_0 < q_c$). This is just as in the previous chapter (§ 7.5).

(2) Secondly, requiring real values for a_* (subject to the same proviso about even-numbered integer exponents as in § 7.5), we can conclude that *closed* solutions ($\lambda_0 > 1 - \Omega_0$) must satisfy:

$$\lambda_0 > -\Omega_0, \tag{8.25}$$

while *open* ones (ie., $\lambda_0 < 1 - \Omega_0$) obey:

$$\lambda_0 < -\Omega_0. \tag{8.26}$$

It follows that, if λ_0 is a positive quantity, as observations almost certainly indicate (§ 6.4.2), then nonsingular models in the present theory must be *closed*. (We will assume that both these conditions hold in the following.)

(3) Thirdly, requiring that a_* lie in the range $0 < a_* < 1$ as before, we learn that the deceleration parameter q_0 satisfies:

$$q_0 < q_c. \tag{8.27}$$

This follows from the fact that $|\Omega_0 + \lambda_0 - 1|$ cannot be greater than $|\Omega_0 + \lambda_0|$ (assuming both quantities are positive).

(4) Finally, combining the conclusion (4) with the definition (8.24), we infer that:

$$q_0 < -\left(\frac{3\gamma}{2}\right)\lambda_0. \tag{8.28}$$

Assuming as we are that the cosmological term is positive, this implies that the deceleration parameter must be *negative* for a universe filled with normal matter ($\gamma > 2/3$).

The deceleration parameter q_0 is unfortunately among the most poorly-constrained quantities in observational cosmology (Carroll and Ostlie 1996, p. 1278). Some authors have in fact claimed negative measurements [eg., $q_0 = -0.37 \pm 0.13$ (Moles 1991); or even -1.07 ± 0.02 (Hoell *et al* 1994)]. Nevertheless, it is fair to say that the majority opinion among cosmologists holds that q_0 is probably positive (Kolb and Turner 1990, p. 6; Carroll and Ostlie 1996, p. 1278). Loh (1986), for instance, measured a value of $q_0 = 0.35^{+0.3}_{-0.4}$ using number counts of faint galaxies, although it has since been argued that evolutionary effects were not properly taken into account (Martel 1994). Gurvits (1994) has obtained a value of 0.16 ± 0.71 from a study of active galactic nuclei at radio wavelengths.

The most recent, and probably most reliable value can be inferred from Perlmutter *et al*'s (1997) determination of λ_0. Assuming a flat universe, these authors obtained $\lambda_0 = 0.06^{+0.28}_{-0.34}$. Making the same assumption ($\Omega_0 + \lambda_0 = 1$) for simplicity, and also using $\gamma = 1$, we find from eq. (4.29) that:

$$q_0 = \frac{1}{2} - \frac{3}{2}\lambda_0. \tag{8.29}$$

The result reported by Perlmutter *et al* (1997) therefore implies that $q_0 = 0.46 \pm 0.47$. This is (barely) compatible with a negative deceleration, as are all the other limits quoted above. However, the present theory is somewhat disfavoured in comparison with the one studied in Chapter 7 ($\Lambda \propto H^2$), and we judge that this is a reasonable place to halt our investigation for the time being.

8.3 Analogies and Interpretations

We close this chapter, and the main part of the thesis, with some speculations concerning the possible physical interpretations of the decay laws (4.5) – (4.8).

To begin with, we observe from the form of eq. (4.15) that, for normal matter (ie., $\gamma > 2/3$), the Λ-term acts like a "brake" on the deceleration of the cosmic scale factor, resisting the tendency of spacetime to shrink under the influence of its own gravity. Could one push this analogy further and think of a decaying cosmological term as a kind of global *friction force*? At first sight this idea is attractive in the light of a number of suggestive parallels that one can draw between the Λ-decay laws we have considered and the more familiar laws of ordinary mechanics.

In the case of decays like $\Lambda \propto a^{-m}$, for example, we determined in Chapter 6 that, in order to accommodate bounce temperatures high enough for nucleosynthesis, the value of m had to be very close to two. This brings to mind the classic force laws of Newton and Coulomb, in which $F \propto r^{-2}$ (here F is the magnitude of the force between two objects separated by distance r). In Chapter 7 we learned that cosmological observations are perhaps most compatible with nonsingular models in which $\Lambda \propto \dot{a}^2$. This is reminiscent of the phenomenon of *air resistance* in projectile motion ("Newtonian damping"), for which $F \propto v^2$ (where $v = \dot{r}$ is the velocity). Finally, we have noticed in the present chapter that nonsingular models are also possible for $\Lambda \propto \ddot{a}$, which has the form of an *inertial force law*, $F \propto dv/dt$.

We do not suggest, of course, that the universe is filled with an air-like ether. The ideas in this section are put forward as possible Newtonian analogs for the processes which must ultimately govern the behaviour of the cosmological term in the context of some more fundamental quantum field theory. They may possibly stimulate new thought as to the reasons for Λ-decay, or motivate the form of what are otherwise purely phenomenological decay laws. It may be, for instance, that the correct coupling of Λ (and its *derivatives*) to gravity in the Lagrangian of general relativity would give rise to behaviour of this kind in the Newtonian limit.

A somewhat different interpretation of the variable-Λ idea can be made if we approach it from a dimensional standpoint. The cosmological term, after all, influences cosmic dynamics primarily via its *energy density* $\rho_\Lambda = \Lambda/8\pi G$, so it may be inappropriate to draw analogies with force laws. With this in mind, it is intriguing to note that perhaps our most successful decay law (that of Chapter 7) had the form $\Lambda \propto \dot{a}^2$. This is analogous to the ordinary expression for *kinetic energy*, $E_k \propto v^2$ (where $v = \dot{r}$). In the spirit of the foregoing discussion, one could speculate about whether the variable Λ-term might represent a kind of "kinetic energy of the universe," locked up in the "motion" of the scale factor and opposite in sign to its gravitational potential energy. To what would such motion be relative? How could it be defined? We are straying into the realm of metaphysics, and content ourselves with letting matters rest here.

Chapter 9

Conclusions

We have surveyed two kinds of non-standard cosmological models: those in which there are more than four dimensions, and those in which there is a cosmological term Λ which is not constant, but varies with time.

In five dimensions, it was necessary to perform a conformal rescaling of the metric in order to describe anything more interesting than pure radiation in four dimensions. The extra dimension also led to a Brans-Dicke-type scalar field ϕ, and it was necessary to assume a nonzero five-dimensional cosmological term $\hat{\Lambda}$ in order to evade the stringent constraints on this quantity. (This essentially worked because it led to a nontrivial potential for ϕ.) With both a conformal rescaling *and* a cosmological term, the theory did turn out to be inflationary — in fact it was a special case of a nonsingular inflationary model developed previously by Starkovich and Cooperstock (1992) — but it also turned out to violate *COBE* and *Tenerife* constraints on the index of density perturbations.

In higher dimensions, the situation was not much improved. The model of Cho (1990, 1992), with an arbitrary number of compact extra dimensions, was ruled out by *COBE*-type observations in the few cases where it was inflationary in principle. The same thing applied to the model of Berezin *et al* (1989), in which the compact dimensions were further subdivided into multiple compact subspaces. Only the model of Cho and Yoon (1993), incorporating *torsion* into higher-dimensional general relativity, was successful in accounting for inflation. In this case we had the remarkable result that the *dimensionality of spacetime* could be constrained by *COBE*-type experimental data.

In general, we have found that higher-dimensional cosmological models do not lend themselves to inflation (in four dimensions) as easily as their authors have sometimes claimed. We have, however, noticed that higher dimensions might be of help in the *cosmological constant* problem. In several cases, including the five-dimensional one, the cosmological term (in higher dimensions) effectively vanished in four dimensions, being absorbed into the potential of the scalar field instead. We have not investigated this in depth, and it remains to be seen how generic this behaviour is.

Our survey of models with a *variable* cosmological "constant" began with a review of the specific models that have been proposed so far. Most of these are phenomenological in character. We have carried on in the same spirit, and considered Λ-dependency of four possible kinds: power-law in time, in the scale factor, or in the first or second derivative of the scale factor with respect to time. We have also extended the existing treatments by adopting a fairly general equation of state for matter, in contrast to the assumption of dust-like (or sometimes radiation-like) matter that is usually made.

In the case of a *time*-dependent Λ-term, we have obtained a number of explicit solutions for the scale factor, most of which appear not to have been noticed before. Among other things it was surprising to find Bessel functions appearing in a cosmological context. We have concentrated on the case of spatially flat models in which Λ scales as a power-law function of time. While they do not solve the problem of the singularity, these models can significantly lengthen the age of the universe in comparison to the case where $\Lambda =$ constant. We have determined that, if Λ scales as an *odd* power of time, then it must be negative (or zero) if the scale factor is to be real-valued. Our conclusions may not extend to cases in which $k \neq 0$; this will require more detailed numerical analysis.

For a cosmological term that scales as an arbitrary power of the *scale factor*, we have found that it is possible to obtain closed models which satisfy all the observational constraints and *contain no initial singularity*. This is in sharp contrast to the situation where $\Lambda =$ constant, for which observational evidence points very strongly to the existence of a big bang singularity. (The variation effectively allows one to obtain a large

Λ-term where it is most important — near the "big bounce" — without the price of a large cosmological constant at present times.) This appears not to have been widely appreciated, probably because variable cosmological terms have so far been studied almost exclusively as a mechanism to address the cosmological constant problem within the context of particle physics. We have focused on constraints imposed by measured upper limits on Λ at the present time, and by requirements of sufficient age, high-redshift gravitational lensing, and the density of matter in the universe. As specific numerical examples we have shown that closed dust-like models with $\Omega_0 = \{0.34, 0.51, 0.61\}$ and λ_0 in the ranges $\{0.68 - 0.72, 0.51 - 0.57, 0.41 - 0.49\}$ are both nonsingular and observationally viable if $m = 1, 1.5$, and 1.8 respectively, where $\Lambda \propto a^{-m}$. If nucleosynthesis is to be attributed to high temperatures near the big bounce, then $m \geq 1.8$.

In the cases where Λ varies as a power-law function of the first or second *derivative* of the scale factor with respect to time, we have been able to obtain nonsingular solutions for even integer powers (other cases will require more detailed investigation). These models were distinguished from the ones in the previous chapter, both by the fact that they were not necessarily *closed*, and by the relative ease with which they could accommodate very high redshifts (and hence phenomena such as nucleosynthesis). In particular, we have shown that open dust-like models with λ_0 either just below $1 - \Omega_0$ or just above $\Omega_0/2$ are both nonsingular and viable if $\Lambda \propto H^2$.

Broadly speaking, our results indicate that extra dimensions are perhaps not as useful in addressing the problems of the standard model as has sometimes been thought. The idea of a variable cosmological term, by contrast, appears to warrant far more attention than it has so far received.

Chapter 10

Bibliography

Abbott, L. F. 1985, *Physics Letters* **150B**, 427.

Abbott, R. B., Barr, S. M. and Ellis, S. D. 1984, *Physical Review* **D30**, 720.

Abdel-Rahman, A.-M. M. 1990, *General Relativity and Gravitation* **22**, 655.

Abdel-Rahman, A.-M. M. 1992, *Physical Review* **D45**, 3497.

Abdel-Rahman, A.-M. M. 1995, *General Relativity and Gravitation* **27**, 573.

Abramo, L. R. W., Brandenberger, R. H. and Mukhanov, V. F. 1997, *Physical Review* **D56**, 3248.

Abreu, J. P., Crawford, P. and Mimoso, J. P. 1994, *Classical and Quantum Gravity* **11** 1919.

Accetta, F. S., Krauss, L. M. and Romanelli, P. 1990, *Physics Letters* **248B**, 146.

Adler, S. L. 1982, *Reviews of Modern Physics* **54**, 729 (§ VI.C).

Agnese, A., La Camera, M. and Wataghin, A. 1970, *Il Nuovo Cimento* **66B**, 202.

Alvares, E. and Gavela, M. B. 1983, *Physical Review Letters* **51**, 931.

Arbab, A. I. and Abdel-Rahman, A.-M. M. 1994, *Physical Review* **D50**, 7725.

Bahcall, J. N. and Frautschi, S. 1971, *Astrophysical Journal Letters* **170**, L81.

Bahcall, N. A., Fan, X. and Cen, R. 1997, preprint astro-ph/9706018 (Los Alamos National Laboratory).

Bailin, D. and Love, A. 1987, *Reports on Progress in Physics* **50**, 1087.

Balbinot, R., Fabris, J. C. and Kerner, R. 1990, *Classical and Quantum Gravity Letters* **7**, L17.

Banks, T. 1984, *Physical Review Letters* **52**, 1461.

Barr, S. M. 1987, *Physical Review* **D36**, 1691.

Barrow, J. D. and Deruelle, N. 1988, *Nuclear Physics* **B297**, 733.

Barrow, J. D. and Ottewill, A. C. 1983, *Journal of Physics* **A16**, 2757.

Bayin, S. Ş., Cooperstock, F. I. and Faraoni, V. 1994, *Astrophysical Journal* **428**, 439.

Beesham, A. 1993, *Physical Review* **D48**, 3539.

Behrndt, K. and Förste, S. 1994, *Nuclear Physics* **B430**, 441.

Bekenstein, J. D. 1974, *Annals of Physics (New York)* **82**, 535.

Beloborodov, A., Ivanov, P. and Polnarev, A. 1994, *Classical and Quantum Gravity* **11**, 665.

Bennett, C. L. *et al* 1996, *Astrophysical Journal Letters* 464, L1.

Berezin, V. A, Domenech, G., Levinas, M. L., Loustó, C. O. and Umérez, N. D. 1989, *General Relativity and Gravitation* **21**, 1177.

Berman, M. S. 1990a, *International Journal of Theoretical Physics* **29**, 567.

Berman, M. S. 1990b, *International Journal of Theoretical Physics* **29**, 1419.

Berman, M. S. 1991a, *General Relativity and Gravitation* **23**, 465.

Berman, M. S. 1991b, *Physical Review* **D43**, 1075.

Berman, M. S., Som, M. M. and de Mello Gomide, F. 1989, *General Relativity and Gravitation* **21**, 287.

Bertolami, O. 1986a, *Il Nuovo Cimento* **93B**, 36.

Bertolami, O. 1986b, *Fortschritte der Physik* **94**, 829.

Blome, H. J. and Priester, W. 1991, *Astronomy and Astrophysics* **250**, 43.

Bombelli, L., Koul, R. K., Kunstatter, G., Lee, J. and Sorkin, R. D. 1987, *Nuclear Physics* **B289**, 735.

Borde, A. and Vilenkin, A. 1997, *Physical Review* **D56**, 717.

Börner, G. and Ehlers, J. 1988, *Astronomy and Astrophysics* **204**, 1.

Brandenberger, R., Mukhanov, V. and Sornborger, A. 1993, *Physical Review* **D48**, 1629.

Brandenberger, R. H. 1994, *International Journal of Modern Physics* **A9**, 2117.

Brandenberger, R. H. 1996, preprint astro-ph/9609045 (Los Alamos National Laboratory).

Brandenberger, R. H. and Zhitnitsky, A. R. 1997, *Physical Review* **D55**, 4640.

Brans, C. and Dicke, R. H. 1961, *Physical Review* **124**, 925.

Brout, R., Englert, F. and Gunzig, E. 1978, *Annals of Physics* **115**, 78.

Bunn, E. and Sugiyama, N. 1995, *Astrophysical Journal* **446**, 49.

Burakovsky, L. and Horwitz, L. P. 1995, *General Relativity and Gravitation* **27**, 1043.

Burd, A. B. and Barrow, J. D. 1988, *Nuclear Physics* **B308**, 929.

Caldwell, R. R. 1996, *Classical and Quantum Gravity* **13**, 2437.

Caldwell, R. R., Dave, R. and Steinhardt, P. J. 1997, preprint astro-ph/9708069 (Los Alamos National Laboratory).

Calvao, M. O., de Oliveria, H. P., Pavón, D. and Salim, J. M. 1992, *Physical Review* **D45**, 3869.

Campusano, L, Heidmann, J. and Nieto, J. L. 1975, *Astronomy and Astrophysics* **41**, 229.

Capozziello, S., de Ritis, R., Dymnikova, I., Rubano, C. and Scudellaro, P. 1995, *Physics Letters* **203A**, 18.

Carroll, B. W. and Ostlie, D. A. 1996, *An Introduction to Modern Astrophysics* (Reading, Massachusetts: Addison-Wesley).

Carugno, E., Litterio, M., Occhionero, F. and Pollifrone, G. 1995, preprint gr-qc/9510066 (Los Alamos National Laboratory).

Carvalho, J. C., Lima, J. A. S. and Waga, I. 1992, *Physical Review* **D46**, 2404.

Chaboyer, B., Demarque, P., Kernan, P. J. and Krauss, L. M. 1997, preprint CERN-TH/97-121 (Centre Européen de Recherches Nucléaires); astro-ph/9706128 (Los Alamos National Laboratory).

Chatterjee, S. and Sil, A. 1993, *General Relativity and Gravitation* **25**, 307.

Chen, W. and Wu, Y.-S. 1990, *Physical Review* **D41**, 695.

Cho, Y. M. 1987, *Physical Review* **D35**, 2628.

Cho, Y. M. 1990, *Physical Review* **D41**, 2462.

Cho, Y. M. 1992, *Physical Review Letters* **68**, 3133.

Cho, Y. M. and Yoon, J. H. 1993, *Physical Review* **D47**, 3465.

Chodos, A. and Detweiler, S. 1980, *Physics Reviews* **D21**, 2167.

Coble, K., Dodelson, S. and Frieman, J. A. 1997, *Physical Review* **D55**, 1851.

Coley, A. A. 1994, *Astrophysical Journal* **427**, 585.

Collins, P. D. B., Martin, A. D. and Squires, E. J., *Particle Physics and Cosmology* (New York: Wiley).

Coussaert, O., Henneaux, M. and van Driel, P. 1995, *Classical and Quantum Gravity* **12**, 2961.

Cremmer, E. and Scherk, J. 1976, *Nuclear Physics* **B108**, 409.

Crilly, A. J. 1968, *Monthly Notices of the Royal Astronomical Society* **141**, 435.

Dąbrowski, M. P. 1996, *Annals of Physics (New York)* **248**, 199.

Damour, T., Gibbons, G. W. and Gundlach, C. 1990, *Physical Review Letters* **64**, 123.

de Barros, J. A., Pinto-Neto, N. and Sagioro-Leal, M. A. 1997, preprint gr-qc/9710084 (Los Alamos National Laboratory).

Dereli, T. and Tucker, R. W. 1983, *Physics Letters* **125B**, 133.

Deruelle, N. and Madore, J. 1987, *Physics Letters* **186B**, 25.

Dolgov, A. D. 1983, *The Very Early Universe*, ed. G. W. Gibbons, S. W. Hawking and S. T. C. Siklos (Cambridge: Cambridge University Press), p. 449.

Dolgov, A. D. 1997, preprint astro-ph/9708045 (Los Alamos National Laboratory).

Duff, M. J., Nilsson, B. E. W. and Pope, C. N. 1986, *Physics Reports* **130**, 1.

Durrer, R. and Laukenmann, J. 1996, *Classical and Quantum Gravity* **13**, 1069.

Dvali, G., Liu, H. and Vachaspati, T. 1997, preprint Imperial/TP/97-98/3 (Imperial College, London); CERN-TH/97-273 (Centre Européen de Recherches Nucléaires); hep-ph/9710301 (Los Alamos National Laboratory).

Dymnikova, I. G. 1986, *Soviet Physics JETP (Journal of Experimental and Theoretical Physics)* **63**, 1111.

Easther, R. 1994, preprint NZ-CAN-RE-94/1 (University of Canterbury, New Zealand); astro-ph/9405034 (Los Alamos National Laboratory).

Edwards, D. 1972, *Monthly Notices of the Royal Astronomical Society* **159**, 51.

Efstathiou, G., Sutherland, W. J. and Maddox, S. J. 1990, *Nature* **348**, 705.

Ehlers, J. and Rindler, W. 1989, *Monthly Notices of the Royal Astronomical Society* **238**, 503.

Elizalde, E. and Odintsov, S. D. 1994, *Physics Letters* **334B**, 33.

Ellis, G. F. R. 1984, *Annual Review of Astronomy and Astrophysics* **22**, 157.

Ellis, G. F. R. and Madsen, M. S. 1991, *Classical and Quantum Gravity* **8**, 667.

Endō, M. and Fukui, T. 1977, *General Relativity and Gravitation* **8**, 833.

Ezawa, Y., Watanabe, T. and Yano, T. 1991, *Progress of Theoretical Physics* **86**, 89.

Fabris, J. C. and Sakellariadou, M. 1995, preprint gr-qc/9511067 (Los Alamos National Laboratory).

Faraoni, V., Cooperstock, F. I. and Overduin, J. M. 1995, *International Journal of Modern Physics* **D4**, 387.

Faraoni, V., Cooperstock, F. I. and Overduin, J. M. 1997, *Fields Institute Communications* **15**, 237.

Felten, J. E. and Isaacman, R. 1986, *Reviews of Modern Physics* **58**, 689.

Franx, M. *et al.* 1997, *Astrophysical Journal Letters* **486**, L175.

Freedman, W. L. 1996, preprint astro-ph/9612024 (Los Alamos National Laboratory).

Freese, K., Adams, F. C., Frieman, J. A. and Mottola, E. 1987, *Nuclear Physics* **B287**, 797.

Freund, P. G. O. 1982, *Nuclear Physics* **B209**, 146.

Frieman, J. A., Hill, C. T., Stebbins, A. and Waga, I. 1995, *Physical Review Letters* **75**, 2077.

Fujii, Y. 1997, preprint gr-qc/9708010 (Los Alamos National Laboratory).

Fujii, Y. and Nishioka, T. 1991, *Physics Letters* **254B**, 347.

Fujiwara, Y., Higuchi, S., Hosoya, A., Mishima, T. and Siino, M. 1991, *Physical Review* **D44**, 1756.

Futamase, T. 1992, in *Gravitational Lenses*, ed. R. Kayser, T. Schramm and L. Nieser (Berlin: Springer-Verlag), p. 338.

Galicki, K. 1991, *Classical and Quantum Gravity* **8**, 1529.

Garriga, J. and Vilenkin, A. 1997, preprint astro-ph/9707292 (Los Alamos National Laboratory).

Gasperini, M. 1987, *Physics Letters* **194B**, 347.

Gasperini, M., Sánches, N. and Veneziano, G. 1991, *Nuclear Physics* **B364**, 365.

Gibbons, G. W. 1996, preprint hep-th/9601075 (Los Alamos National Laboratory).

Gliner, É. B. 1966, *Soviet Physics JETP (Journal of Experimental and Theoretical Physics)* **22**, 378.

Gliner, É. B. 1970, *Soviet Physics - Doklady* **15**, 559.

Gott III, J. R. and Rees, M. J. 1987, *Monthly Notices of the Royal Astronomical Society* **227**, 453.

Gott III, J. R., Park, M.-G. and Lee, H. M. 1989, *Astrophysical Journal* **338**, 1.

Green, M. B., Schwarz, J. H. and Witten, E. 1987, *Superstring Theory* (Cambridge: Cambridge University Press).

Gregory, R. 1996, *Physical Review* **D54**, 4955.

Gribbin, J. 1986, *In Search of the Big Bang* (New York: Bantam Books).

Guendelman, E. I. and Kaganovich, A. B. 1994, preprint BGU-94/17/June-PH (Ben Gurion University of the Negev); gr-qc/9408031 (Los Alamos National Laboratory).

Gunzig, E., Géhéniau, J. and Prigogine, I. 1987, *Nature* **330**, 621.

Gurvits, L. I. 1994, *Astrophysical Journal* **425**, 442.

Hacyan, S. and Sarmiento, A. 1993, *Physical Review* **D48**, 943.

Hancock, S. *et al.* 1994, *Nature* **367**, 333.

Harrison, E. R. 1967, *Monthly Notices of the Royal Astronomical Society* **137**, 69.

Hawking, S. W. 1984, *Physics Letters* **134B**, 403.

Hawking, S. W. and Ellis, G. F. R. 1973, *The Large Scale Structure of Space-Time*, (Cambridge: Cambridge University Press).

Heidmann, J. 1980, *Relativistic Cosmology* (Berlin: Springer-Verlag), § 12.2.

Hiscock, W. A. 1986, *Physics Letters* **166B**, 285.

Hoell, J., Liebscher, D.-E. and Priester, W. 1994, *Astronomische Nachrichten* **315**, 89.

Hoyle, F. 1992, *Astrophysics and Space Science* **198**, 177.

Hu, Y., Turner, M. S. and Weinberg, E. J. 1994, *Physical Review* **D49**, 3830.

Ishihara, H. 1984, *Progress of Theoretical Physics* **72**, 376.

Israelit, M. 1991, *Astrophysical Journal* **375**, 473.

Israelit, M. 1994, *Astrophysical Journal* **435**, 8.

Israelit, M., and Rosen, N. 1989, *Astrophysical Journal* **342**, 627.

Israelit, M., and Rosen, N. 1991, *Astrophysical Journal* **375**, 463.

Israelit, M., and Rosen, N. 1993, *Astrophysics and Space Science* **204**, 317.

Jain, S. 1997, preprint IISc-CTS-5/97 (Indian Institute of Science); gr-qc/9708018 (Los Alamos National Laboratory).

John, M. V. and Joseph, K. B. 1996, *Physics Letters* **387B**, 466.

John, M. V. and Joseph, K. B. 1997, *Classical and Quantum Gravity* **14**, 1115.

Kalligas, D., Wesson, P. and Everitt, C. W. F. 1992, *General Relativity and Gravitation* **24**, 351.

Kaluza, T. 1921, *Sitzungsberichte Preussische Akademie der Wissenschaften* **K1**, 966.

Kamionkowski, M. and Toumbas, N. 1996, preprint CU-TP-750 (Columbia University); astro-ph/9605100 (Los Alamos National Laboratory).

Kazanas, D. 1980, *Astrophysical Journal Letters* **241**, L59.

Klein, O. 1926, *Zeitschrift für Physik* **37** 895.

Kochanek, C. S. 1992, *Astrophysical Journal* **384**, 1.

Kochanek, C. S. 1996, *Astrophysical Journal* **466**, 638.

Kofman, L. A. 1996, preprint astro-ph/9605155 (Los Alamos National Laboratory).

Kolb, E. W. 1989, *Astrophysical Journal* **344**, 543.

Kolb, E. W. and Turner, M. S. 1990, *The Early Universe* (Reading, Massachusetts: Addison-Wesley).

Krauss, L. M. and Schramm, D. N. 1993, *Astrophysical Journal Letters* **405**, L43.

Krauss, L. M. and Turner, M. S. 1995, preprint CWRU-P6-95 (Case Western Reserve University); astro-ph/9504003 (Los Alamos National Laboratory).

La, D., Steinhardt,P. J. and Bertschinger, E. W. 1989, *Physics Letters* **231B**, 231.

Lahav, O., Lilje, P. B., Primack, J. R. and Rees, M. J. 1991, *Monthly Notices of the Royal Astronomical Society* **251**, 128.

Landau, L. D. and Lifshitz, E. M. 1975, *The Classical Theory of Fields* (Oxford: Pergamon).

Landsberg, P. T., Piggott, K. D. and Thomas, K. S. 1992, *Astrophysical Letters and Communications* **28**, 235.

Lanzetta, K. M., Yahil, A. and Fernández-Soto, A. 1996, *Nature* **381**, 759.

Larsen, F. and Wilczek, F. 1997, *Physical Review* **D55**, 4591.

Lau, Y.-K. 1985, *Australian Journal of Physics* **38**, 547.

Lerner, E. J. 1992, *The Big Bang Never Happened* (New York: Vintage Books).

Lerner, E. J. 1995, *Astrophysics and Space Science* **227**, 61.

Levin, J. 1995, *Physics Letters* **343B**, 69.

Liddle, A. R. 1994, preprint SUSSEX-AST 94/10-2 (University of Sussex); astro-ph/9410083 (Los Alamos National Laboratory).

Liddle, A. R. and Lyth, D. H. 1993, *Physics Reports* **231**, 1.

Lima, J. A. S. 1996, *Physical Review* **D54**, 2571.

Lima, J. A. S. and Abramo, L. R. W. 1996, preprint Brown-HET-1011 (Brown University); gr-qc/9606067 (Los Alamos National Laboratory).

Lima, J. A. S. and Carvalho, J. C. 1994, *General Relativity and Gravitation* **26**, 909.

Lima, J. A. S. and Maia, J. M. F. 1994, *Physical Review* **D49**, 5597.

Lima, J. A. S. and Trodden, M. 1996, *Physical Review* **D53**, 4280.

Linde, A. D. 1974, *JETP (Journal of Experimental and Theoretical Physics) Letters* **19**, 183.

Linde, A. D. 1983, *Physics Letters* **129B**, 177.

Linde, A. D. 1985, *Physics Letters* **162B**, 281.

Linde, A. 1990, *Particle Physics and Inflationary Cosmology* (Chur, Switzerland: Harwood Academic).

Linde, A. 1994, preprint SU-ITP-94-36 (Stanford University); hep-th/9410082 (Los Alamos National Laboratory).

Linde, A. 1996, preprint SU-ITP-96-01 (Stanford University); astro-ph/9601004 (Los Alamos National Laboratory).

Linde, A., Linde, D. and Mezhlumian, A. 1994, *Physical Review* **D49**, 1783.

Loh, E. D. 1986, *Physical Review Letters* **57**, 2865.

Lopez, J. L. and Nanopoulos, D. V. 1996, *Modern Physics Letters* **A11**, 1.

Lucchin, F. and Matarrese, S. 1985, *Physical Review* **D32**, 1316.

Lund, F. 1973, *Physical Review* **D8**, 3253.

Madsen, M. S. 1988, *Classical and Quantum Gravity* **5**, 627.

Madsen, M. S. and Ellis, G. F. R. 1988, *Monthly Notices of the Royal Astronomical Society* **234**, 67.

Madsen, M. S., Mimoso, J. P., Butcher, J. A. and Ellis, G. F. R. 1992, *Physical Review* **D46**, 1399.

Maggiore, M. 1997, preprint CERN-TH/97-228 (Centre Européen de Recherches Nucléaires); gr-qc/9709004 (Los Alamos National Laboratory).

Magnano, G. and Sokołowski, L. M. 1994, *Physical Review* **D50**, 5039.

Maia, M. D. and Silva, G. S. 1994, *Physical Review* **D50**, 7233.

Mangano, G., Miele, G. and Stornaiolo, C. 1995, *Modern Physics Letters* **A10**, 1977.

Mann, R. B. and Vincent, D. E. 1985, *Physics Letters* **107A**, 75.

Markov, M. A. 1982, *JETP (Journal of Experimental and Theoretical Physics) Letters* **36**, 265.

Markov, M. A. 1983, *The Very Early Universe*, ed. G. W. Gibbons, S. W. Hawking and S. T. C. Siklos (Cambridge: Cambridge University Press), p. 353.

Martel, H. 1994, *Astrophysical Journal Letters* **421**, L67.

Martel, H. and Wasserman, I. 1990, *Astrophysical Journal* **348**, 1.

Matyjasek, J. 1995, *Physical Review* **D51**, 4154.

McDonald, J. 1993, *Physical Review* **D48**, 2462.

McIntosh, C. B. G. 1968, *Monthly Notices of the Royal Astronomical Society* **140**, 461.

McVittie, G. C. 1965, *General Relativity and Cosmology* (Urbana: University of Illinois Press).

Mimoso, J. P. and Wands, D. 1995, *Physical Review* **D51**, 477.

Misner, C. W., Thorne, K. S. and Wheeler, J. A. 1973, *Gravitation* (New York: W. H. Freeman).

Moffat, J. W. 1979, *Physical Review* **D19**, 3554.

Moffat, J. W. 1995, *Physics Letters* **357B**, 526.

Moffat, J. W. 1996a, preprint UTPT-96-10 (University of Toronto); astro-ph/9606071 (Los Alamos National Laboratory).

Moffat, J. W. 1996b, preprint UTPT-96-13 (University of Toronto); astro-ph/9608202 (Los Alamos National Laboratory).

Moles, M. 1991, *Astrophysical Journal* **382**, 369.

Moorhouse, R. G. and Nixon, J. 1985, *Nuclear Physics* **B261**, 172.

Murphy, G. M. 1960, *Ordinary Differential Equations and Their Solutions* (Princeton: Van Nostrand).

Narlikar, J. V. 1983, *Introduction to Cosmology*, (Boston: Jones & Bartlett).

Nesteruk, A. V., Maartens, R. and Gunsig, E. 1997, preprint PU-RCG 97/3 (University of Portsmouth); astro-ph/9703137 (Los Alamos National Laboratory).

Olson, T. S. and Jordan, T. F. 1987, *Physical Review* **D35**, 3258.

Overduin, J. M. and Cooperstock, F. I. 1997, in *proceedings of the 7th Canadian Conference on General Relativity and Relativistic Astrophysics* (University of Calgary Press, in press).

Overduin, J. M. and Wesson, P. S. 1997a, *Physics Reports* **283**, 303.

Overduin, J. M. and Wesson, P. S. 1997b, *Astrophysical Journal* **483**, 77.

Overduin, J. M., Wesson, P. S. and Bowyer, S. 1993, *Astrophysical Journal* **404**, 1.

Özer, M. and Taha, M. O. 1986, *Physics Letters* **171B**, 363.

Özer, M. and Taha, M. O. 1987, *Nuclear Physics* **B287**, 776.

Özer, M. 1997, *Physics Letters* **404B**, 20.

Page, D. N. 1987, *Physical Review* **D36**, 1607.

Parker, L. 1991, in *Annals of the New York Academy of Sciences* **631**, 31.

Paternoga, R. and Graham, R. 1996, *Physical Review* **D54**, 4805.

Pavón, D. 1991, *Physical Review* **D43**, 375.

Peebles, P. J. E. 1993, *Principles of Physical Cosmology* (Princeton: Princeton University Press).

Peebles, P. J. E. and Ratra, B. 1988, *Astrophysical Journal Letters* **325**, L17.

Penrose, R. 1989, *Annals of the New York Academy of Sciences* **571**, 249.

Perlmutter, S. *et al.* 1997, *Astrophysical Journal* **483**, 565.

Petry, W. 1981, *General Relativity and Gravitation* **13**, 1057.

Petry, W. 1990, *General Relativity and Gravitation* **22**, 1045.

Prigogine, I., Geheniau, J., Gunzig, E. and Nardone, P. 1989, *General Relativity and Gravitation* **21**, 767.

Rama, S. K. 1997a, *Physical Review Letters* **78**, 1620.

Rama, S. K. 1997b, *Physics Letters* **408B**, 91.

Ratra, B. and Peebles, P. J. E. 1988, *Physical Review* **D37**, 3406.

Ratra, B. and Quillen, A. 1992, *Monthly Notices of the Royal Astronomical Society* **259**, 738.

Refsdal, S., Stabell, R. and de Lange, F. G. 1967, *Memoirs of the Royal Astronomical Society* **71**, 143.

Reuter, M. and Wetterich, C. 1987, *Physics Letters* **188B**, 38.

Rindler, W. 1977, *Essential Relativity*, 2nd edition (New York: Springer-Verlag).

Robertson, H. P. 1933, *Reviews of Modern Physics* **5**, 62.

Robertson, H. P. and Noonan, T. W. 1968, *Relativity and Cosmology* (Philadelphia: W. B. Saunders), § 17.2.

Rose, B. 1986, *Classical and Quantum Gravity* **3**, 975.

Rosen, N. 1979, *Lettere Al Nuovo Cimento* **25**, 266.

Rosen, N. 1985, *Astrophysical Journal* **297**, 347.

Rusmaĭkina, T. V. and Rusmaĭkin, A. A. 1970, *Soviet Physics JETP (Journal of Experimental and Theoretical Physics)* **30**, 372.

Sahdev, D. 1984, *Physics Letters* **137B**, 155.

Salam, A. 1980, *Reviews of Modern Physics* **52**, 525.

Salim, J. M. and Waga, I. 1993, *Classical and Quantum Gravity* **10**, 1767.

Sandage, A. and Tammann, G. A. 1996, preprint astro-ph/9611170 (Los Alamos National Laboratory).

Sato, H. 1984, *Progress of Theoretical Physics* **72**, 98.

Sato, K., Terasawa, N. and Yokoyama, J. 1990, in *The Quest for the Fundamental Constants in Cosmology*, ed. J. Andouze and J. Tran Thanh Van (Gif-sur-Yvette: Éditions Frontières), p. 193.

Schmutzer, E. 1988, *Annalen der Physik (Leipzig)* **45**, 578.

Sciama, D. W. 1994, *Philosophical Transactions of the Royal Society (London)* **A346**, 137.

Senovilla, J. M. M. 1990, *Physical Review Letters* **64**, 2219.

Senovilla, J. M. M. 1996, *Physical Review* **D53**, 1799.

Shafi, Q. and Wetterich, C. 1983, *Physics Letters* **129B**, 387.

Shafi, Q. and Wetterich, C. 1985, *Physics Letters* **152B**, 51.

Shafi, Q. and Wetterich, C. 1987, *Nuclear Physics* **B289**, 787.

Silveira, V. and Waga, I. 1994, *Physical Review* **D50**, 4890.

Silveira, V. and Waga, I. 1997, preprint astro-ph/9703185 (Los Alamos National Laboratory).

Sisteró, R. F. 1991, *General Relativity and Gravitation* **23**, 1265.

Sokołowski, L. M. 1989, *Classical and Quantum Gravity* **6**, 59.

Sokołowski, L. M. and Golda, Z. A. 1987, *Physics Letters* **195B**, 349.

Soleng, H. H. 1991, *General Relativity and Gravitation* **23**, 313.

Spiegel, M. R. 1974, *Fourier Analysis With Applications to Boundary Value Problems* (New York: McGraw-Hill).

Spiegel, M. R. 1981, *Applied Differential Equations*, 3rd ed. (Englewood Cliffs: Prentice-Hall).

Stabell, R. and Refsdal, S. 1966, *Monthly Notices of the Royal Astronomical Society* **132**, 379.

Starkovich, S. P. and Cooperstock, F. I. 1992, *Astrophysical Journal* **398**, 1.

Starobinsky, A. A. 1980, *Physics Letters* **91B**, 99.

Steinhardt, P. J. and Accetta, F. S. 1990, *Physical Review Letters* **64**, 2740.

Stornaiolo, C. 1994, *Physics Letters* **189A**, 351.

Sugiyama, N. and Sato, K. 1992, *Astrophysical Journal* **387**, 439.

Sunahara, K., Kasai, M. and Futamase, T. 1990, *Progress of Theoretical Physics* **83**, 353.

Szydłowski, M. and Biesiada, M. 1990, *Physical Review* **D41**, 2487.

Tangherlini, F. R. 1993, *Il Nuovo Cimento* **108B**, 911.

Tomimatsu, A. and Ishihara, H. 1986, *General Relativity and Gravitation* **18**, 161.

Torres, L. F. B. and Waga, I. 1996, *Monthly Notices of the Royal Astronomical Society* **279**, 712.

Totani, T., Yoshii, Y. and Sato, K. 1997, preprint astro-ph/9705014 (Los Alamos National Laboratory).

Tsamis, N. C. and Woodard, R. P. 1996, *Nuclear Physics* **B474**, 235.

Turner, M. S. 1997, preprint astro-ph/9704062 (Los Alamos National Laboratory).

Turner, M. S. and White, M. 1997, preprint FERMILAB-Pub-97/002-A (Fermilab); astro-ph/9701138.

Viana, P. T. P. and Liddle, A. R. 1997, preprint SUSSEX-AST 97/8-2 (University of Sussex); astro-ph/9708247 (Los Alamos National Laboratory).

Vilenkin, A. 1982, *Physics Letters* **117B**, 25.

Visser, M. 1997, *Science* **276**, 88.

Waga, I. 1993, *Astrophysical Journal* **414**, 436.

Wald, S. 1984, *General Relativity*, (Chicago: University of Chicago Press).

Weinberg, S. 1972, *Gravitation and Cosmology*, (New York: Wiley).

Weinberg, S. 1989, *Reviews of Modern Physics* **61**, 1.

Weinberg, S. 1996, preprint UTTG-10-96 (University of Texas); astro-ph/9610044 (Los Alamos National Laboratory).

Wesson, P. S. 1985, *Astronomy and Astrophysics* **151**, 276.

Wesson, P. S. 1991, *Astrophysical Journal* **378**, 466.

Wesson, P. S. *et al.* 1996, *International Journal of Modern Physics* **A11**, 3247.

Wetterich, C. 1988a, *Nuclear Physics* **B302**, 645.

Wetterich, C. 1988b, *Nuclear Physics* **B302**, 668.

Wetterich, C. 1995, *Astronomy and Astrophysics* **301**, 321.

White, M. and Scott, D. 1995, preprint astro-ph/9508157 (Los Alamos National Laboratory).

Will, C. M. 1981, *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press).

Witten, E. 1995, *Nuclear Physics* **B443**, 85.

Yoon, J. H. and Brill, D. R. 1990, *Classical and Quantum Gravity* **7**, 1253.

Yoshii, Y. and Sato, K. 1992, *Astrophysical Journal Letters* **387**, L7.

Yoshimura, M. 1984, *Physical Review* **D30**, 344.

Yu, X. 1989, *Astrophysics and Space Science* **154**, 321.

Zel'dovich, Y. B. 1968, *Soviet Physics Uspekhi* **11**, 381.

Zeldovich, Y. B. 1972, *Monthly Notices of the Royal Astronomical Society* **160**, 1P.

Appendix A

Constraints on the Starkovich-Cooperstock Potential

A.1 Model of Starkovich and Cooperstock

Starkovich and Cooperstock (1992; hereafter "SC") presented a simple but surprisingly rich classical field theory of the universe, inspired by an earlier singularity-free model of Israelit and Rosen (1989). A single cosmological fluid, represented by the scalar field ϕ, describes all matter. This field begins in a cold "prematter" state characterized by the Planck density; there is no initial singularity. The scalar field then passes through three successive phases, corresponding to inflationary, radiation-like, and dust-like conditions. During inflation, a proper choice of the equation of state *heats* the universe rather than cooling it — thus removing the need for the complicated re-heating mechanisms of traditional inflationary theory. When the Planck temperature is attained, a discontinuous change in equation of state (presumably corresponding to a phase transition) occurs, ushering in the radiation era and the rest of the standard model of cosmology (with continued expansion now cooling the cosmological fluid). Besides the idea of a limiting density and temperature [this is due to Markov (1982)], the theory makes only standard assumptions: general relativity with spatial homogeneity, isotropy and no cosmological term. The rest is accomplished by identifying and fitting together the equations of state appropriate to each epoch.

SC theory makes several testable predictions. Firstly, it postulates a closed universe

[This is required by the Lemaître equation (2.22), since there is no cosmological term Λ and the universe begins with a positive density ρ_0 at the moment of the big bounce, when $\dot{a} = 0$.] Secondly, it predicts the present value of the Hubble parameter, which turns out to lie between 33 and 44 km s^{-1} Mpc^{-1} (SC, Table 8). [This is obtained by solving the field equations of the theory for the scale factor a and density ρ, and substituting into eq. (2.22).] Both these predictions go somewhat against the grain of current observational work, which favours an open (or possibly flat) universe (Peebles 1993, § 26) and a value of H_0 in the range (73 ± 10) km s^{-1} Mpc^{-1} (Freedman 1996). The possibility that the universe might be closed is, however, not *ruled out* observationally by any means (eg., White and Scott 1995, Kamionkowski and Toumbas 1996); and some observers continue to report lower values of H_0 in the range (55 ± 10) km s^{-1} Mpc^{-1} (Sandage and Tammann 1996). It should also be mentioned that Bayin *et al* (1994) have modified the SC theory by including a non-minimal coupling of the scalar field to gravity, with one result being a slightly higher prediction of 47 km s^{-1} Mpc^{-1} for the current value of Hubble's parameter. So it is probably premature to dismiss the theory on these grounds alone.

In this appendix, we take up a suggestion by SC (p. 10) to test the theory based on its *inflationary* properties. In particular, we will look at the predicted spectral index of density perturbations, density contrast, and energy scale at the end of inflation. To begin with, however, we perform a consistency check of the theory, integrating the Klein-Gordon equation manually to see if the values of the scalar field reported in SC (Tables 2–7) are reasonable. Along the way we will introduce the various terms and definitions that will be needed in subsequent sections.

A.2 Klein-Gordon Equation

For a minimally-coupled scalar field in a homogeneous and isotropic spacetime, the Klein-Gordon equation reads:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \tag{A.1}$$

where H is the Hubble parameter, ϕ is the scalar field, and $V(\phi)$ is its potential, which is given by eq. (2.63) in the case of SC theory. Eq. (A.1) is identical to eq. (2.13) of SC, and may be found in standard texts, eg. Kolb and Turner (1990, p. 271). [We have also encountered examples of this equation in our five-dimensional theory: eq. (2.16) when $\Lambda = 0$, and eq. (2.49) when $\Lambda \neq 0$.]

The Hubble parameter may be written:

$$H \equiv \frac{da}{adt} = \dot{\phi}\,\frac{d\ln a}{d\phi}. \tag{A.2}$$

In SC theory, the scale factor $a(\phi)$ is related to the potential $V(\phi)$ by the following equation [SC eq. (3.18)]:

$$\frac{d\ln a}{d\phi} = -\frac{1}{3\gamma}\frac{d\ln V}{d\phi}, \tag{A.3}$$

where γ is a constant parameter characterizing the equation of state of the cosmological fluid, eq. (2.62). This parameter takes on very small values during the inflationary era, $1.85 \times 10^{-3} \leq \gamma \leq 2.0153 \times 10^{-3}$ (SC, Table 1).

Putting eqs. (A.2) and (A.3) into the Klein-Gordon equation (A.1), we obtain:

$$\ddot{\phi} + \left(1 - \frac{\dot{\phi}^2}{\gamma V}\right)\frac{dV(\phi)}{d\phi} = 0. \tag{A.4}$$

We can make further progress by noting from eq. (2.12a) of SC that:

$$\gamma = \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V}, \tag{A.5}$$

so that:

$$\frac{\dot{\phi}^2}{\gamma V} = \frac{2}{2-\gamma} = \text{constant}\ . \tag{A.6}$$

Putting this into eq. (A.4) we obtain:

$$\ddot{\phi} = \left(\frac{\gamma}{2-\gamma}\right)\frac{dV(\phi)}{d\phi}. \tag{A.7}$$

This is the most convenient formulation of the Klein-Gordon equation for SC theory. It remains to insert the potential $V(\phi)$ and solve for $\phi(t)$.

The SC potential (2.63) may be rewritten in the form:

$$V(\phi) = C(\beta e^{\alpha\phi})^{1/b}\left(1+\beta^{-1}e^{-2\alpha\phi}\right)^{1/b}, \tag{A.8}$$

where α, β, b and C are parameters to be specified within the theory. [This is equivalent to eq. (3.17) of SC]. Differentiation with respect to ϕ yields:

$$\frac{dV(\phi)}{d\phi} = \frac{\alpha C}{b}(\beta e^{\alpha\phi})^{1/b}\left(1-\beta^{-1}e^{-2\alpha\phi}\right)\left(1+\beta^{-1}e^{-2\alpha\phi}\right)^{1/b-1}. \tag{A.9}$$

Substitution into the Klein-Gordon equation (A.7) produces this differential equation:

$$\ddot{\phi} = \frac{\alpha C}{b}\left(\frac{\gamma}{2-\gamma}\right)(\beta e^{\alpha\phi})^{1/b}\left(1-\beta^{-1}e^{-2\alpha\phi}\right)\left(1+\beta^{-1}e^{-2\alpha\phi}\right)^{1/b-1}. \tag{A.10}$$

Fortunately this can be simplified considerably with some knowledge of the values taken by the various free parameters in the theory.

Our treatment up to this point is applicable to any point in the history of the universe; we have not assumed an inflationary equation of state. We now focus specifically on the *prematter era*, during which inflation takes place in SC theory. Eqs. (2.16), (3.10b), (3.14b), (4.7e) and (4.11ℓ) of SC combine in this case to give:

$$\alpha = b\sqrt{24\pi\gamma}\,\frac{1}{m_{pl}}\,, \quad \beta = e^{-2\alpha\phi_o}\quad, \quad b = \frac{3\gamma-2}{6\gamma}, \tag{A.11}$$

where ϕ_o is the value of the scalar field at the moment when inflation begins. In SC theory, $\phi(t)$ is a monotonically decreasing function of time, so $\phi \leq \phi_o$ at all times. Inserting these expressions into eq. (A.10), we find:

$$\ddot{\phi} = A_p e^{B\phi}\left[1-e^{2\alpha(\phi_o-\phi)}\right]\left[1+e^{2\alpha(\phi_o-\phi)}\right]^{-D}, \tag{A.12}$$

where:

$$A_p \equiv \left(\frac{\gamma BC}{2-\gamma}\right)e^{-2B\phi_o}\quad, \quad B \equiv \sqrt{24\pi\gamma}\,\frac{1}{m_{pl}}\quad, \quad D \equiv \frac{2+3\gamma}{2-3\gamma}, \tag{A.13}$$

and the subscript p denotes the prematter era. With $1.85\times10^{-3} \leq \gamma \leq 2.0153\times10^{-3}$ during this time, as noted above, it follows that α falls between $-67/m_{pl}$ and

$-64/m_{pl}$. Therefore the exponential terms inside the square brackets in eq. (A.12), become insignificant almost immediately after the onset of inflation. [The value of the scalar field ϕ drops by more than a Planck mass during inflation in SC theory (SC, Table 1)]. Neglecting these terms, the Klein-Gordon equation (A.12) simplifies to:

$$\ddot{\phi} = A_p e^{B\phi}, \tag{A.14}$$

where, since $\gamma \ll 2$:

$$A_p \approx \left(\frac{\gamma BC}{2}\right) e^{-2B\phi_o}. \tag{A.15}$$

It turns out that the same simplification can be made during the radiation- and matter-dominated eras, with A_p replaced by $A_r \equiv 2BC\beta^4$ and $A_m \equiv BC\beta^6$ respectively, B given by eq. (A.13) as usual, and C, β and γ taking the values specified in SC for these two periods. We will not make further use of these expressions, as we are interested here in the inflationary prematter era.

To solve the Klein-Gordon equation (A.14), we try defining two new variables:

$$x \equiv e^{B\phi} \quad , \quad y \equiv \dot{x}, \tag{A.16}$$

whereupon:

$$\frac{dy}{dx} = \frac{A_p B x^3 + y^2}{xy}. \tag{A.17}$$

After some experimentation we find that switching to a new dependent variable:

$$z \equiv y/x, \tag{A.18}$$

reduces this to *separable* form:

$$\frac{dz}{dx} = \frac{A_p B}{z}. \tag{A.19}$$

The general solution is:

$$z = \pm\sqrt{2A_p B\, x + E}, \tag{A.20}$$

where E is a constant of integration. Using the definitions (A.16) and (A.18), we obtain:

$$\dot{x} = -x\sqrt{2A_pB\,x + E}, \tag{A.21}$$

selecting a negative sign in order to reflect the fact that ϕ *decreases* with time. If we follow SC and impose the boundary condition $\dot{\phi} = 0$ at $\phi = 0$ (ie., $\dot{x} = 0$ at $x = 1$), then the value of E is given by:

$$E = -2A_pB. \tag{A.22}$$

Proceeding on this assumption, the general solution of eq. (A.21) is:

$$\tan^{-1}\sqrt{x-1} = F - \sqrt{\frac{A_pB}{2}}\,t, \tag{A.23}$$

where F is another integration constant. Inverting and using eq. (A.16), we obtain:

$$\phi(t) = \frac{1}{B}\ln\left[\sec^2\left(F - \sqrt{\frac{A_pB}{2}}\,t\right)\right] \tag{A.24}$$

where the value of F is fixed by the boundary condition $\phi(0) \equiv \phi_0$:

$$F = \tan^{-1}\sqrt{e^{B\phi_0} - 1}. \tag{A.25}$$

This analytic expression for $\phi(t)$ may be employed to estimate the value of ϕ at the end of inflation, and thereby test the self-consistency of SC theory. We have done this using the values of γ, ϕ_o and t (the duration of the prematter era) given in Tables 1–7 of SC. Results are summarized in Table A.1 and compared to the published SC values for this quantity (denoted ϕ_{pr}). It may be seen from this table that our calculated results differ from those of SC by less than one percent in all cases. The small differences may be attributed to the fact that we have replaced the full Klein-Gordon equation (A.10) with the approximation (A.14). Table A.1 thus establishes to within reasonable certainty that the SC theory is self-consistent.

A.3 Spectral Index of Density Perturbations

We proceed to calculate the spectral index n of density perturbations generated during inflation in SC theory. As we saw in § 2.9, the special case $\beta = 0$ gives rise to power-law

Table A.1: Comparison of scalar field values at the end of inflation with those reported by Starkovich and Cooperstock

$\gamma \times 10^3 =$	1.85	1.90	1.95	2.00	2.0153
ϕ_o/m_{pl}	34.29	32.90	31.59	30.34	29.97
t/t_{pl}	26.34	25.60	24.86	24.30	24.11
$\phi(t)/m_{pl}$	33.26	31.89	30.59	29.35	28.99
ϕ_{pr}/m_{pl}	33.16	31.79	30.49	29.25	28.88
% Difference	+0.30	+0.31	+0.33	+0.34	+0.38

inflation (PLI) in a manner inconsistent with the *COBE* and *Tenerife* observations. In the *general case*, β is given by eq. (A.11), and the SC potential (A.8) takes the form:

$$V(\phi) = Ce^{\sqrt{24\pi\gamma}(\phi-2\phi_o)/m_{pl}}\left[1+e^{2\alpha(\phi_o-\phi)}\right]^{1/b}. \tag{A.26}$$

The results of the last section give us some confidence that the exponential term inside the square brackets may be safely neglected during inflation. This approximation, moreover, becomes increasingly exact as inflation progresses and the difference between ϕ and ϕ_o grows. Selecting a typical value of $\gamma = 1.95 \times 10^{-3}$, for example, one finds from eq. (A.24) that the term in question drops to ~ 0.003 after a single Planck time has elapsed (note from Table A.1 that inflation lasts for nearly 25 Planck times in this case). Similar results apply for any value of γ in the range allowed by SC theory. Eq. (A.26) may therefore be well-approximated (especially at late times) by:

$$V(\phi) = V_0 e^{\sqrt{24\pi\gamma}\,\phi/m_{pl}}, \tag{A.27}$$

where:

$$V_0 \equiv Ce^{-\sqrt{96\pi\gamma}\,\phi_o/m_{pl}}. \tag{A.28}$$

This is useful because we are particularly interested in the later stages of the inflationary era, when the "observable universe leaves the horizon," setting the scale for the density perturbations we observe today. [This happens near the end of inflation in most models of inflation (Liddle and Lyth 1993, p. 45)].

The potential (A.27) is of the kind that gives rise to power-law expansion, eq. (2.65), but with the important distinction that its exponent is *positive*. The initial value ϕ_o of the scalar field, in other words, does not correspond to the minimum of its potential, as is the case in standard inflationary scenarios. Eq. (A.27) is in fact an example of a *chaotic* power-law potential (Linde 1985, Ellis and Madsen 1991, § 4.4; Liddle and Lyth 1992, § 5.4.3).

The idea behind chaotic inflation is that, rather than starting off in a local minimum of its potential, the scalar field might initially have taken *any* value. (Table A.1 shows that ϕ_o varies between about 30 and 34 Planck masses in SC theory.) Presumably, in an ensemble of many universes, a wide variety of initial values would be found. In each universe with sufficiently large ϕ_o, inflation would occur as ϕ began to roll down its potential gradient. Of course, only some of these universes — those in which this rolling behaviour was neither too fast nor too slow — would evolve into ones that resemble our own. Chaotic inflation is simpler than the older "standard" scenario, in that it requires no pre-inflationary era, with the potential arranged so that the initial high-temperature local minimum at $\phi = \phi_o$ gives way at lower temperatures to a lower-temperature "effective" minimum somewhere else (thus explaining why inflation begins in the first place). The price for this simplicity is the inherently indeterminate (hence "chaotic") nature of the initial conditions. [Chaotic inflation was first proposed by Linde (1983), and recent reviews by the same author may be found in Linde (1994,1996).]

The power-law behaviour of the scale factor, $a \propto t^p$, depends only on the size of the PLI parameter p in eq. (2.65), and does not distinguish between the standard and chaotic versions of the potential. Viable inflation is obtained as long as two conditions are met (Linde 1985): firstly that $V(\phi)$ grow more slowly than $e^{\sqrt{16\pi}\,\phi/m_{pl}}$; and secondly that the initial value ϕ_o of the scalar field be greater than about a Planck mass (although this latter constraint can be relaxed in some cases). We have already noted that the $\phi_o \sim 30m_{pl}$ in SC theory, and it is straightforward to show that eq. (A.27) meets the first of these criteria, since $\gamma \sim 2 \times 10^{-3}$ in all cases (Table A.1), implying that the potential goes as $\sim e^{\sqrt{0.048\pi}\,\phi/m_{pl}}$.

The value of p in SC theory is therefore obtained by simple comparison of eqs. (2.65) and (A.27), as in § 2.9. This yields:

$$p_{sc} = 2/3\gamma, \tag{A.29}$$

which, from eq. (2.66), implies a spectral index of density perturbations:

$$n_{sc} = 0.994, \tag{A.30}$$

for all values of γ in the allowed range. SC theory therefore satisfies the *COBE* and *Tenerife* constraint on the spectral index of density perturbations ($n_{obs} \geq 0.9$).

It is interesting to note that in many versions of power-law inflation, the strongest constraints on the theory come from demanding that the universe "re-heat" to sufficiently high temperatures for baryogenesis after inflation (Burd and Barrow 1988). This typically requires $p \geq 10$ (Lucchin and Matarrese 1985), a constraint which would easily be met by SC theory since eq. (A.29) shows that $p_{sc} \geq 330$ at all times. This constraint, however, is not relevant to the SC case since the universe heats *during* inflation in this theory, removing the need for reheating mechanisms.

A.4 Density Contrast

We turn next to the *magnitude* of the density perturbations, as defined in § 2.9 [and standard texts, eg. Kolb and Turner (1990, § 9.2); Peebles (1993, § 17)]. According to Liddle and Lyth (1993, § 5.2), this quantity can be specified in terms of the inflationary potential as follows:

$$\delta^2 = \frac{32}{75}\frac{V_*}{m_{pl}^4}\epsilon_*^{-1}, \tag{A.31}$$

where the asterisk denotes a quantity evaluated at the "epoch of horizon crossing;" that is, at the time when the scale of interest left the horizon during inflation.

Eq. (A.31) applies if three *slow-roll conditions* are met (Liddle and Lyth 1993). We now give these constraints and demonstrate that each one is satisfied in SC theory. The first is:

$$\dot{\phi} \sim -\frac{1}{3H}\frac{dV}{d\phi}. \tag{A.32}$$

If we rewrite the Klein-Gordon equation (A.1) in the form:

$$\dot{\phi} = -\frac{1}{3H}\left(\frac{dV}{d\phi} + \ddot{\phi}\right), \tag{A.33}$$

and replace $\ddot{\phi}$ using eq. (A.7), then we obtain:

$$\dot{\phi} = -\frac{1}{3H}\left(1 - \frac{\gamma}{2-\gamma}\right)\frac{dV}{d\phi}, \tag{A.34}$$

which satisfies the required condition since $\gamma \ll 1$.

The second condition reads:

$$\epsilon \equiv \frac{m_{pl}^2}{16\pi}\left(\frac{dV}{V\,d\phi}\right)^2 \ll 1. \tag{A.35}$$

Using eqs. (A.8), (A.9) and (A.11) it is straightforward to show that:

$$\epsilon = \left(\frac{3\gamma}{2}\right)\xi^2, \tag{A.36}$$

where:

$$\xi \equiv \frac{1 - e^{2\alpha(\phi_0 - \phi)}}{1 + e^{2\alpha(\phi_0 - \phi)}}. \tag{A.37}$$

Recalling that $-67/m_{pl}^2 \leq \alpha \leq -64/m_{pl}^2$ in SC theory, it is clear that $0 \leq \xi \leq 1$ at all times, since inflation begins with $\phi = \phi_o$ (ie., $\xi = 0$) and stops at or before the limit in which $\phi \to 0$ (ie., $\xi \to 1$). Eq. (A.35) is therefore satisfied since $\gamma \ll 1$.

The third and final slow-roll condition is given by:

$$|\eta| \ll 1 \quad , \quad \eta \equiv \frac{m_{pl}^2}{8\pi}\frac{d^2V}{V\,d\phi^2}. \tag{A.38}$$

Differentiating eq. (A.9) and using eqs. (A.8) and (A.11) as above, we find that:

$$\eta = -\left(1 - \frac{3\gamma}{2}\right)\left[1 - \left(\frac{1+3\gamma/2}{1-3\gamma/2}\right)\xi^2\right]. \tag{A.39}$$

Since $\gamma \ll 1$ this can be approximated by:

$$\eta \approx -(1-\xi^2). \tag{A.40}$$

As inflation begins, $\xi = 0$ and eq. (A.38) is not satisfied. However, as argued in the previous section, the exponential terms in eq. (A.37) rapidly become negligible as inflation progresses, so that $\xi \approx 1$ for all but the first moments of the prematter era. After ϕ has dropped by one Planck mass from its initial value of ϕ_o, for example, we find that:

$$\xi \approx \frac{1-e^{-130}}{1+e^{-130}} \approx 1, \tag{A.41}$$

to very high precision, so that $|\eta| \ll 1$ as required.

Having satisfied ourselves that SC inflation meets the slow-roll criteria, we return to the density contrast (A.31). With eqs. (A.26) and (A.36), this reads:

$$\delta^2 = \left(\frac{32C}{75m_{pl}^4}\right) e^{-\alpha(2\phi_o-\phi_*)/b}\left[1-e^{2\alpha(\phi_o-\phi)}\right]^2\left[1+e^{2\alpha(\phi_o-\phi)}\right]^{1/b-2}, \tag{A.42}$$

where ϕ_* is the value of the scalar field at the epoch of horizon crossing. We have argued above that the exponential terms inside the square brackets can be neglected throughout most of the inflationary period, and in particular near the *end* of inflation. Fortunately, it is precisely this regime which is of interest to us, since we would like to find the size of the perturbations that were frozen into the CMB when the *currently observable universe* left the horizon (Liddle and Lyth 1993, p. 45). We therefore write:

$$\delta^2 \approx \left(\frac{32C}{75m_{pl}^4}\right) e^{-\alpha(2\phi_o-\phi_*)/b}, \tag{A.43}$$

to very good precision. Let us assume for the moment that the the observable universe crossed the horizon at *precisely* the end of inflation. In this case $\phi_* = \phi_{pr}$ where values of ϕ_{pr} are given by SC. Since these authors also list the values of γ, C and ϕ_o (SC, Table 1), it is simple to compute the size of the density contrast using eq. (A.43) with the definitions (A.11). We have summarized the somewhat surprising results in Table A.2. The predicted matter density contrast is ~ 0.53 for all cases. The magnitude

Table A.2: Size of the density contrast in Starkovich-Cooperstock inflation

$\gamma \times 10^3 =$	1.85	1.90	1.95	2.00	2.0153
α/m_{pl}	-67.11	-66.21	-65.35	-64.53	-64.28
b	-179.7	-174.9	-170.4	-166.2	-164.9
$(C/m_{pl}^4) \times 10^{-5}$	3.66	2.57	1.83	1.31	1.19
ϕ_o/m_{pl}	34.29	32.90	31.59	30.34	29.97
ϕ_{pr}/m_{pl}	33.16	31.79	30.49	29.25	28.88
δ	0.53	0.53	0.53	0.53	0.53

of the corresponding *temperature* fluctuations in the CMB may be found (Kolb and Turner 1990, p. 338) from:

$$\frac{\delta T}{T} = \frac{1}{3}\delta, \tag{A.44}$$

which follows from the fact that density ρ goes as a^{-3} while temperature $T \propto a^{-1}$. The SC prediction is therefore:

$$\frac{\delta T}{T} \sim 0.18, \tag{A.45}$$

which is some 17,000 times larger than the actual fluctuations detected by the *COBE* satellite. [These are reported as being of magnitude $(1.06 \pm 0.04) \times 10^{-5}$ on ten degree scales, Bennett *et al* 1996.] This raises doubts about the viability of the SC inflationary scenario.

One could attempt to evade this constraint by supposing that the observable universe left the horizon some time *before* the end of inflation in SC theory. However, since $\phi(t)$ is a monotonically decreasing function of time, this will push ϕ closer to ϕ_o and *worsen* the discrepancy between theory and experiment. Indeed, in the limit $\phi \to \phi_o$, one obtains:

$$\delta^2_{max} = \left(\frac{32C}{75m_{pl}^4}\right) e^{-\alpha\phi_o/b}, \tag{A.46}$$

which yields $\delta = 0.65$ for all cases, disagreeing with the *COBE* data by a factor of more than $20,000$ times.

A.5 Value of γ

Another suggestion might be to let γ take values outside the range permitted in SC theory. Let us attempt to estimate the value of γ that would be required to bring the theory into line with observation. With the help of the definitions (A.11), eq. (A.43) shows that:

$$\delta \propto e^{-\sqrt{6\pi\gamma}(2\phi_o-\phi_*)/m_{pl}}. \tag{A.47}$$

In SC theory, the values of C, ϕ_o and ϕ_* are dependent on the value of γ. If we nevertheless ignore this dependence for simplicity, we can take the ratio:

$$\frac{\delta}{\delta_{sc}} \approx \left(e^{\sqrt{\gamma}-\sqrt{\gamma_{sc}}}\right)^{\zeta}, \tag{A.48}$$

where the subscript denotes the SC values, and:

$$\zeta \equiv \sqrt{6\pi}\left(\frac{2\phi_o - \phi_*}{m_{pl}}\right). \tag{A.49}$$

Inverting this expression, we see that the desired value of γ is given by:

$$\gamma \approx \left[\sqrt{\gamma_{sc}} + \frac{1}{\zeta}\ln\left(\frac{\delta}{\delta_{sc}}\right)\right]^2. \tag{A.50}$$

Inserting the numerical values for ϕ_o and $\phi_*(=\phi_{pr})$ corresponding to the typical case $\gamma_{sc} = 1.95 \times 10^{-3}$, as usual, and imposing $\delta = \delta_{sc}/16,000$, we obtain:

$$\gamma \approx 1.26 \times 10^{-2}, \tag{A.51}$$

which is almost *seven times* the original SC value. With this value of γ, the PLI parameter p given by eq. (A.29) drops from 330 to ~ 53. (This, by the way, indicates that the problem with SC theory in its original form is that it inflates too quickly.) The drop in p in turn affects the spectral index n of density perturbations via eq. (2.66), with the result that $n = 0.96$. Although this is lower than the original SC value, $n_{sc} = 0.994$, it is still safely above the observational lower limit ($n_{obs} \geq 0.9$). So it might appear feasible to improve the agreement with observation by moving to higher values of γ.

There is, however, a serious side effect to this remedy. The small values of γ in SC are not chosen arbitrarily, but are required in order to lead to realistic values of ρ and H in the present day, subject to the boundary conditions of the theory. If γ is increased by the amount suggested above, then the theory will no longer describe present conditions. The only way to avoid this would be to alter some of the other boundary conditions in the theory, such as the requirement that it begin with precisely the Planck density.

A.6 Energy at the End of Inflation

The results of the previous section indicate that the difficulties with SC theory are, in a sense, inherent in its *boundary conditions*, and this can be confirmed by looking at the energy bound up in the scalar field potential at the end of inflation. According to Liddle and Lyth (1993, § 5.2.4) this cannot exceed:

$$V_{end} < 1.6 \times 10^{16} \text{ GeV} . \qquad \text{(A.52)}$$

To check whether inflation in the SC scenario satisfies this requirement, we could evaluate the SC potential directly, using $\phi = \phi_{pr}$. The approximate expression (A.27) will be nearly exact at the end of inflation, as we have remarked.

However, we can do better in this case by noting from eq. (2.12b) of SC that the potential energy of the scalar field is directly related to the *density of matter in the universe*, according to:

$$V(\phi) = \left(\frac{2-\gamma}{2}\right)\rho. \qquad \text{(A.53)}$$

The boundary conditions of the SC model require that the energy density ρ at the end of the prematter period be given by:

$$\rho(t_{end}) = \frac{\pi^2}{15} m_{pl}^4. \qquad \text{(A.54)}$$

Therefore it is not necessary to evaluate the potential of the theory at all; we conclude immediately from eqs. (A.53) and (A.54) that the potential energy of the scalar field at the end of inflation must be (with $\gamma \ll 2$):

$$V_{end}^{1/4} \approx \left(\frac{\pi^2}{15}\right)^{1/4} m_{pl} = 0.90\, m_{pl} = 1.1 \times 10^{19}\ \mathrm{GeV}\ . \qquad \text{(A.55)}$$

This violates the Liddle-Lyth bound by some 690 times.

One way to repair the problem may be to adopt a different set of boundary conditions at the phase transition between the prematter and radiation era. This is reminiscent, however, of the situation encountered by Israelit (1991, 1994) and Israelit and Rosen (1991) in a similar singularity-free inflationary theory. These authors have also found that initial fluctuations grow too large to be reconciled with observation unless the boundary conditions are fine-tuned. As an alternative remedy, they have introduced intervening *transition periods* with special equations of state between the original prematter and radiation eras. The density fluctuations are then assumed to begin only after inflation has ended. It may be that a similar approach is necessary in SC theory.

Another possibility, which we have not studied here, is that the modified version of SC theory proposed by Bayin *et al* (1994) might prove to be in better agreement with observational data. This theory, however, is considerably harder to work with, as the inflationary potential is given only in numerical form. We leave this task for future work.

Appendix B

Constraints on Double-Exponential Potentials

B.1 Generalization of Easther's Method

In this appendix we study inflationary potentials with two exponential terms, of the form introduced by Easther (1994):

$$V_{\mathcal{B}}(\phi) = \mathcal{A}e^{-\kappa\phi} - \mathcal{B}e^{-m\kappa\phi}, \tag{B.1}$$

where we have defined $\kappa \equiv \sqrt{8\pi}\,\xi/m_{pl}$ for convenience, and where $\xi > 0$ and $0 \leq \phi \leq \infty$. Our first objective is to extend Easther's analysis, which treated only cases in which $m > 1$, to values of m in the range $0 \leq m \leq 1$.

To begin with, we note that the two cases $m = 0$ and $m = 1$ are trivial:

$$V_{\mathcal{B}}(\phi) = \begin{cases} \mathcal{A}e^{-\kappa\phi} - \mathcal{B} & (\text{if } m = 0) \\ (\mathcal{A} - \mathcal{B})e^{-\kappa\phi} & (\text{if } m = 1) \end{cases} . \tag{B.2}$$

The second case is of pure power-law form, eq. (2.65), while the first one just combines a power-law potential with a de Sitter-like Λ-term.

From now on we restrict ourselves to the more interesting cases in which $0 < m < 1$ and $m > 1$. We rewrite eq. (B.1) in the form:

$$V_{\mathcal{B}}(\phi) = \begin{cases} -\mathcal{B}e^{-m\kappa\phi}\left[1 - (\mathcal{A}/\mathcal{B})\,e^{-(1-m)\kappa\phi}\right] & (\text{if } 0 < m < 1) \\ \mathcal{A}e^{-\kappa\phi}\left[1 - (\mathcal{B}/\mathcal{A})\,e^{-(m-1)\kappa\phi}\right] & (\text{if } m > 1) \end{cases} . \tag{B.3}$$

Note that if $\mathcal{A} \leq 0$ and $\mathcal{B} \geq 0$ then $V_{\mathcal{B}} < 0$ and there can be no inflation (§ 3.1.4). In all other cases, inflation is possible depending on the values of $\mathcal{A}, \mathcal{B}, m, \xi$ and ϕ. We assume (with Easther 1994) that $\mathcal{A}, \mathcal{B}$ are positive, but relax Easther's condition on m, considering values in the range $0 < m < 1$ as well as $m > 1$.

Let us follow Easther (1994) in parametrizing ϕ by a new variable y:

$$y = \begin{cases} y_< \equiv 1 - (\mathcal{A}/m\mathcal{B})e^{-(1-m)\kappa\phi} & (\text{if } 0 < m < 1) \\ y_> \equiv 1 - (m\mathcal{B}/\mathcal{A})e^{-(m-1)\kappa\phi} & (\text{if } m > 1) \end{cases}, \tag{B.4}$$

chosen specifically so that the global maximum of the potential is located at $y = 0$ (with $y = 1$ corresponding to the limit $\phi \longrightarrow +\infty$). We can verify this behaviour (for both ranges of m) by noting that:

$$V_B = \begin{cases} -\mathcal{B}e^{-m\kappa\phi}[1 - m(1 - y_<)] & (\text{if } 0 < m < 1) \\ \mathcal{A}e^{-\kappa\phi}[1 - (1 - y_>)/m] & (\text{if } m > 1) \end{cases}, \tag{B.5}$$

and differentiating to obtain:

$$\begin{aligned} V_B' &= \begin{cases} m\kappa\mathcal{B}e^{-m\kappa\phi}y_< & (\text{if } 0 < m < 1) \\ -\kappa\mathcal{A}e^{-\kappa\phi}y_> & (\text{if } m > 1) \end{cases} \\ V_B'' &= \begin{cases} -m^2\kappa^2\mathcal{B}e^{-m\kappa\phi}[1 - (1 - y_<)/2] & (\text{if } 0 < m < 1) \\ \kappa^2\mathcal{A}e^{-\kappa\phi}[1 - m(1 - y_>)] & (\text{if } m > 1) \end{cases}, \end{aligned} \tag{B.6}$$

where the primes denote derivatives with respect to ϕ.

We are then in a position to compute the two inflationary *slow roll parameters* ϵ and η (Liddle and Lyth 1993):

$$\begin{aligned} \epsilon &\equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V_B'}{V_B}\right)^2 \\ &= \frac{1}{2}m^2\xi^2 \times \begin{cases} [y_</(1 - m + my_<)]^2 & (0 < m < 1) \\ [y_>/(m - 1 + y_>)]^2 & (m > 1) \end{cases} \\ \eta &\equiv \frac{m_{pl}^2}{8\pi}\left(\frac{V_B''}{V_B}\right) \\ &= m^2\xi^2 \times \begin{cases} (m - 1 + y_<)/(1 - m + my_<) & (0 < m < 1) \\ (1 - m + my_>)/(m - 1 + y_>) & (m > 1) \end{cases}. \end{aligned} \tag{B.7}$$

The spectral index of density perturbations is given (Liddle and Lyth 1993, p. 28) by:

$$n = 1 + 2\eta - 6\epsilon \tag{B.8}$$

$$= 1 + m\xi^2 \times \begin{cases} \left[2(1-m)^2(y_< - 1) - my_<^2\right]/(1 - m + my_<)^2 \\ \left[2(1-m)^2(y_> - 1) - my_>^2\right]/(m - 1 + y_>)^2 \end{cases}. \tag{B.9}$$

At $y = 0$, when the field point is beginning to roll off the global maximum of the potential, we find:

$$n = 1 - 2m\xi^2 \quad (y = 0, \text{ both cases}), \tag{B.10}$$

while at $y = 1$, ϕ is large and the potential is dominated by one of the two terms in eq. (B.3) — the first term if $m > 1$ (as in Easther 1994), or the second term if $0 < m < 1$. These limiting cases therefore reduce to PLI driven by a single exponential potential. Assuming the slow roll conditions hold (as is nearly universal in realistic theories of inflation; Liddle and Lyth 1993, p. 42), we can write:

$$n = \begin{cases} 1 - m^2\xi^2 & (y = 1, 0 < m < 1) \\ 1 - \xi^2 & (y = 1, m > 1) \end{cases} . \tag{B.11}$$

We then follow Easther (1994) in using experimental data from *COBE* and the *QDOT* survey to infer that $n \geq 0.7$ (for $y = 0$) and $n \geq 0.85$ (for $y = 1$; contributions from tensor perturbations provide a tighter limit). Inserting these numbers into eqs. (B.10) and (B.11) respectively, we obtain the constraints:

$$m\xi^2 \leq 0.15 \quad (\text{ any } m) \tag{B.12}$$

$$m^2\xi^2 \leq 0.15 \quad (0 < m < 1) \tag{B.13}$$

$$\xi^2 \leq 0.15 \quad (m > 1) \tag{B.14}$$

Of these, the first is a ***sufficient*** condition for viable inflation. That is, any model that satisfies eq. (B.12) is automatically compatible with the *COBE* and *QDOT* observations. The second and third, however, are weaker conditions (as can be seen from the ranges of values of m involved), and are therefore merely ***necessary*** conditions for viability. That is, any model that does ***not*** satisfy eq. (B.13) or (B.14) (as appropriate for the value of m at hand) is necessarily ***incompatible*** with the observations. These are our generalized Easther-type constraints on two-termed inflationary potentials of the form (B.1).

B.2 *COBE* and the Dimensionality of Spacetime

We address here the question of whether the Easther-type inflationary potentials are observationally viable when they satisfy the ***necessary*** condition (B.13) or (B.14) as appropriate, but not the ***sufficient*** condition (B.12). Our discussion is specific to the particular model in § 3.2.5, but the same method could be applied to any other potential of the Easther form (3.20). We essentially evaluate the quantities in § B.1 on a case-by-case basis, computing the spectral index n of density perturbations and requiring that $n \geq 0.9$, as implied by *COBE* and *Tenerife* observations, eq. (2.67).

Let us parametrize $\bar{\varphi}$ by the new variable y, as defined in eq. (B.4). Using the Easther parameters $\mathcal{A}, \mathcal{B}, m$ and ξ for the model at hand, as listed in eq. (3.34), we find:

$$y = 1 - \exp\left(-\sqrt{\frac{32\pi d}{d-1}}\frac{\bar{\varphi}}{m_{pl}}\right). \tag{B.15}$$

It is clear that y lies in the range $0 \leq y < 1$. Substitution into eqs. (B.7) gives us the ***slow roll parameters*** for this model:

$$\begin{aligned} \epsilon &= \frac{2y^2(d+1)^2}{d(d-1)d+y)^2} \\ \eta &= \frac{4(d+1)[(d+1)y-d]}{d(d-1)(d+y)}. \end{aligned} \tag{B.16}$$

These in turn can be inserted into eq. (B.8) to obtain the spectral index of density perturbations:

$$n = 1 - \frac{4(d+1)[(d+1)y^2 - 2d^2(y-1)]}{d(d-1)(y+d)^2}. \tag{B.17}$$

For any given value of the dimensionality d we require $n(y) \geq 0.9$. For all $d \geq 4$, $n(y)$ is a monotonically increasing function of y over the interval $0 \leq y < 1$. Therefore it takes on a maximum at $y = 1$, and this must certainly be greater than 0.9, or there will be no viable models at all. We therefore have the ***necessary*** condition:

$$n(1) = 1 - \frac{4}{d(d-1)} \geq 0.9 \quad \Longrightarrow \quad d \geq 7. \tag{B.18}$$

This rules out the models with $d = 6$. For the remaining cases, those with $7 \leq d \leq 28$, we solve for the roots y_* (between 0 and 1) of the equation:

$$n(y_*) \geq 0.9. \tag{B.19}$$

Table B.1: Minimum values of $\bar{\varphi}$ compatible with *COBE* constraints, as a function of the number of compact dimensions

d	≤ 6	7	8	9	10	11	12	13
$\bar{\varphi}_*/m_{pl}$	∞	0.494	0.322	0.273	0.245	0.225	0.210	0.198

d	14	15	16	17	18	19	20	21
$\bar{\varphi}_*/m_{pl}$	0.187	0.178	0.170	0.162	0.156	0.149	0.144	0.138

d	22	23	24	25	26	27	28	≥ 29
$\bar{\varphi}_*/m_{pl}$	0.133	0.128	0.124	0.119	0.115	0.111	0.108	0

Since $n(y)$ increases monotonically over this interval, all $y \geq y_*$ will then satisfy the same constraint. The value of y_* can be converted to a lower limit $\bar{\varphi}_*$ on the value of the scalar field by means of eq. (B.15). Results are presented in Table B.1. The model of Cho and Yoon (1993) discussed in § 3.2.5 is compatible with the *COBE* and *Tenerife* limits if spacetime has d dimensions, as indicated, but *only if* $\bar{\varphi} \geq \bar{\varphi}_*$. To our knowledge this the first use of observations of the cosmic microwave background to put constraints on the dimensionality of spacetime.