The Sitnikov Problem for Manev Type Potentials

by

Bogdan Verjinschi
B.Sc., Universitatea Babes Bolyai, Cluj Romania 1978

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We accept this thesis as conforming
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[redacted]

Dr. F.N. Diacu, Supervisor (Department of Mathematics & Statistics)

[redacted]

Dr. R. Edwards, Departmental Member (Department of Mathematics & Statistics)

[redacted]

Dr. P. van den Driessche, Departmental Member (Department of Mathematics & Statistics)

[redacted]

Dr. J. Navarro, External Examiner (Department of Physics & Astronomy))

Supervisor: Dr. F.N. Diacu.

Abstract

This thesis focuses on how the Sitnikov problem changes if the classical Newtonian potential is replaced by a relativist potential of the Manev type. Considering initially the movement of the two primaries only, we study how replacing the potential will affect their trajectories. We establish the condition for closed trajectories for the movement of the primaries. The Sitnikov problem for Manev-type potential is a relativistic perturbation of the classical Newtonian potential, a particular case of the Manev three-body problem with two primaries moving on precessional ellipses and a third negligible mass that oscillates on an axis passing through their centre of mass, perpendicularly to their plane of motion. We use Melnikov's theory to prove the existence of transverse homoclinic orbits for at least a discrete set of initial conditions, showing that in those cases the motion is very complicated, thus extending a result obtained by Dankowikz and Holmes in the classical Newtonian case.

Examiners: [redacted]

Dr. F.N. Diacu, Supervisor (Department of Mathematics & Statistics)

[redacted]

Dr. R. Edwards, Departmental Member (Department of Mathematics & Statistics)

[redacted]

Dr. P. van den Driessche, Departmental Member (Department of Mathematics & Statistics)

[redacted]

Dr. J. Navarro, External Examiner (Department of Physics & Astronomy)

Contents

Abstract **ii**

Contents **iii**

List of Figures **iv**

Acknowledgements **v**

1 Introduction **1**

2 The Manev Two-Body Problem **6**

2.1 Closed Orbits 6

2.1.1 Relationship between eccentricity and the total energy . . . 11

2.1.2 Relationship between the quasi-period and the semi- major axis 14

2.1.3 Relationship between time and polar angle 19

2.1.4 Relationship between time and the distance between particles 19

2.2 Stability of Circular Orbits of the Primaries 20

3 The Sitnikov-Manev Problem **28**

3.1 Equations of Motion 28

3.2 Circular Case 32

3.3 The Perturbed Case 45

3.4 The Melnikov method 47

3.4.1 Lemma 48

3.4.2 The Melnikov Theorem 51

4 Conclusions **56**

Bibliography **58**

List of Figures

Figure 1.1 The Sitnikov problem. 2
Figure 1.2 The trajectory of the planet Mercury is a precessional ellipse. The figure exaggerates the advance of the perihelion by a factor of about 10^4. 4

Figure 2.1 Graph of $r(\theta) = (1 + 0.88\cos 0.6\theta)^{-1}$ with $p = 3$, q=5 and $B > 0$ 12
Figure 2.2 Graph of $r(\theta) = (1 + 0.80\cos 0.5\theta)^{-1}$ with $p = 1$, q=2 and $B > 0$ 13
Figure 2.3 Graph of $r(\theta) = (1 + 0.80\cos \frac{14}{15}\theta)^{-1}$ with $p = 14$, q=15 and $B > 0$ 14
Figure 2.4 Graph of $r(\theta) = (1 + 0.80\cos 7\theta)^{-1}$ with $p = 7$, q=1 and $B < 0$ 15
Figure 2.5 Graph of $r(\theta) = (1 + 0.80\cos \frac{7}{2}\theta)^{-1}$ with $p = 7$, q=2 and $B < 0$ 16
Figure 2.6 Graph of $r(\theta) = 1 + 0.05\cos 0.5\theta$ 24
Figure 2.7 Graph of $r(\theta) = 1 + 0.05\cos \frac{2}{3}\theta$ 25
Figure 2.8 Graph of $r(\theta) = 1 + 0.05\cos 11\theta$ 26
Figure 2.9 Graph of $r(\theta) = 1 + 0.05\cos 5.5\theta$ 27

Figure 3.1 The Sitnikov-Manev Problem. 29
Figure 3.2 The phase portrait for the circular case. 34
Figure 3.3 The vector field of the non-linear system (3.15). 37
Figure 3.4 The vector field of the linear system (3.18). 38
Figure 3.5 The vector field of the system (3.28). 41
Figure 3.6 The vector field of the linear system (3.31). 42
Figure 3.7 The vector field of the non-linear system (3.30). 43
Figure 3.8 The phase portrait of the system (3.33) 44
Figure 3.9 The vector field of the system (3.33) 45
Figure 3.10 Graph of $f(t(z_0))$ versus z_0. 54

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Chapter 1

Introduction

The Sitnikov problem is a special case of the restricted three-body problem in which two of the bodies, named primaries, have equal masses and move on elliptic orbits under Newton's law of attraction, around their common center of mass, considered at rest and at the origin. The third body has zero mass and moves along an axis perpendicular to the primaries' plane of motion and passing through their center of mass (see Figure 1.1). It moves under the combined attraction of the primaries and it has no effect on their motion. The relatively simpler case where the primaries describe circular motions was studied by W.D.MacMillan in 1913 (see[8]). In this integrable case the particle having zero mass oscillates and the period of oscillations is a function of the amplitude.

In 1961 Sitnikov proved that, under some initial conditions, the distance between the primaries remains bounded and the distance between either of them and the third body is unbounded but does not go monotonically to infinity (oscillatory motion with arbitrary large amplitude) (see[14],[16]). When the distance between the primaries increases, the intensity of their combined gravitational field decreases and the opposite happens when the distance decreases. For certain initial conditions, the third body can experience smaller gravitational pull when receding from the pri-

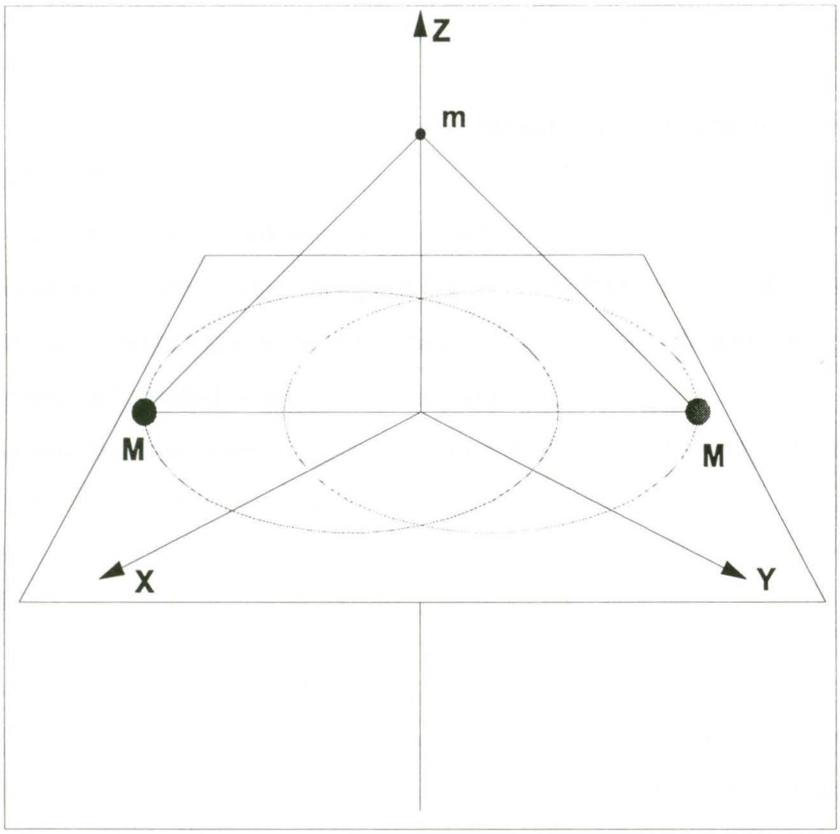

Figure 1.1: The Sitnikov problem.

maries than when passing through the same positions when approaching them, in which case it will pass through the origin with higher speed than at the previous passing. With every oscillation it will go farther and farther away from the primaries and the amplitude of the oscillations increases without bound. Alekseev showed that this problem has many kinds of solution and that this is also true for the case where the third body has finite mass (see for example [3] and [11]). Assume that at $t = 0$

the coordinate z of the third body is $z(0) = 0$, its speed is $v(0) = v_0$, the relative distance between the primaries is $r(0)$ and the critical escape velocity is $v_{esc}(r(0))$. If :

a)$v_0 \ll v_{esc}(r(0))$, then the third body will oscillate and the amplitude of the oscillation is bounded.

b)$v_0 > v_{esc}(r(0))$, then it will escape to infinity.

c)v_0 is close to $v_{esc}(r(0))$, then the third body can oscillate periodically with an arbitrary period, it can escape to infinity after a finite number of oscillations or it can oscillate with a bounded or an unbounded amplitude.

The problem becomes even more complicated if the three bodies move under an attraction law that is no longer of the Newtonian type.

A potential of the type

$$U(r) = \frac{A}{r} + \frac{B}{r^2} \tag{1.1}$$

was first used by Newton in an attempt to describe the apsidal motion of the Moon, that he could not explain within the framework of the inverse-square-force model. He showed that two bodies moving under central forces derived from a potential of this type describe precessional ellipses. A precessional ellipse is an ellipse that rotates in its plane. The planet Mercury moves on such a trajectory and the observed advance of its perihelion is 574 seconds of arc per century (see Figure 1.2). Classical mechanics could not explain the perihelion advance of the planet Mercury. In 1865 Leverier calculated the theoretical value of 531 seconds of arc per century taking into account the perturbation due to other planets (see[15]). The difference of 43 seconds of arc per century was explained by Einstein's general theory of relativity in 1916. In 1924, Manev had another approach to explaining the perihelion advance of Mercury. He showed that the problem can be solved in the classical mechanics

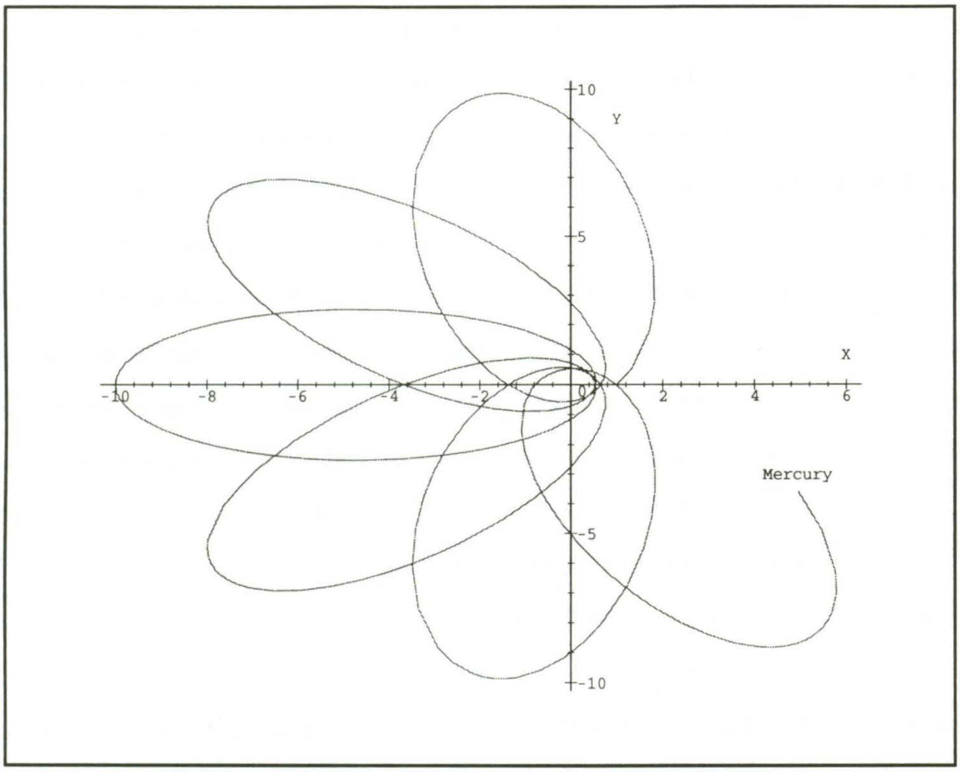

Figure 1.2: The trajectory of the planet Mercury is a precessional ellipse. The figure exaggerates the advance of the perihelion by a factor of about 10^4.

framework. Manev introduced the potential

$$U(r) = \frac{A}{r} + \frac{B}{r^2} = \frac{GMm}{r}\left(1 + \frac{3GM}{2c^2 r}\right).$$

The constants A and B were found using general physical principles (see [9]).

$$A = GMm > 0 \qquad B = \frac{3G^2M^2m}{2c^2} > 0,$$

where G is the gravitational constant, c is the speed of light, m and M are the masses of the planet and the Sun respectively.

A potential of the form (1.1) with A and B real is called the Manev type potential (see [2] and [4]). We consider only the case $A > 0$.

The Sitnikov-Manev problem is a Sitnikov problem where the Newtonian potential is replaced by a Manev type potential. In this case the trajectories of the primaries are precessional ellipses. The ellipses can be closed or not, depending on the values of the constant B that appears in the Manev type potential. A trajectory is closed if the particle eventually retraces its own footsteps. We restrict our study of the motion of the primaries to bounded orbits, both circular and elliptic, with special attention given to the case of small-eccentricity elliptic orbits. The distance between the primaries is always a periodic function of time even when their motion is not periodic. In Chapter 2 we investigate the motion of the primaries ignoring the third massless body. The Manev two body problem has been studied extensively in the literature (see [2], [4] and [13]). In section 2.2,which contains new material, we consider an extension of Bertrand's Theorem.

In Chapter 3, we investigate the motion of the third massless body. We start this chapter with a simplified Sitnikov-Manev problem, where the primaries move around their common center of mass in circular orbits. As far as we are aware the Sitnikov problem for Manev type potentials has not been studied before. The study is undertaken using both analytical and qualitative methods as used to study the classical Sitnikov problem (see [1],[6] and [10]). The case where the primaries move on precessional ellipses is considered a perturbation of the circular case. For this case we are able to prove the existence of transverse homoclinic orbits using the Melnikov method. In Chapter 4 we summarize the results.

Chapter 2

The Manev Two-Body Problem

2.1 Closed Orbits

Consider a system of two particles of masses m_1 and m_2 moving under the influence of a mutual central force due to an interaction potential $U(r)$ of the form:

$$U(r) = \frac{A}{r} + \frac{B}{r^2}, \quad A > 0, \tag{2.1}$$

where r is the distance between the two particles. The magnitude of the mutual force is

$$F = \frac{A}{r^2} + \frac{2B}{r^3}. \tag{2.2}$$

Let the position vector of the first body relative to the second be $\mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2$ where $\mathbf{r}_1$ is the position vector of the body of mass m_1 and $\mathbf{r}_2$ is the position vector of the body of mass m_2.

Consider the following notation: $\mathbf{r}_{1,2} \equiv \mathbf{r}$ and

$$f(r) = -\frac{A}{r^2} - \frac{2B}{r^3}. \tag{2.3}$$

The equations of motion of the two bodies are

$$m_1\ddot{\mathbf{r}}_1 = f(r)\frac{\mathbf{r}_{1,2}}{|\mathbf{r}_{1,2}|} \tag{2.4}$$

and

$$m_2\ddot{\mathbf{r}}_2 = -f(r)\frac{\mathbf{r}_{1,2}}{|\mathbf{r}_{1,2}|}, \tag{2.5}$$

where double dot means second derivative with respect to time. Subtracting (2.5) from (2.4) we obtain

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\frac{\mathbf{r}_{1,2}}{|\mathbf{r}_{1,2}|} \tag{2.6}$$

But $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$ with μ the reduced mass of the system. The equation (2.6) becomes

$$\ddot{\mathbf{r}} = \frac{f(r)}{\mu}\frac{\mathbf{r}_{1,2}}{|\mathbf{r}_{1,2}|}. \tag{2.7}$$

In this way the motion can be reduced to the problem of the motion of one fictitious body of mass equal to the reduced mass and having the position vector $\mathbf{r}$. We now use polar coordinates (r, θ) and take into account that the mutual force is central i.e. its direction is along $\mathbf{r}$. The components of acceleration in (r, θ) coordinates are $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ (see for example [7]). For a central force $\mu a_r = f(r)$ and $a_\theta = 0$. Identifying the components we obtain the radial equation

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{\mu}. \tag{2.8}$$

and the transversal equation

$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0. \tag{2.9}$$

From the last equation it follows that

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{2.10}$$

and that $r^2\dot{\theta} = constant = C$, where

$$C = \frac{L}{\mu} \tag{2.11}$$

is the angular momentum per unit mass. The magnitude of the angular momentum is $|\mathbf{L}| = L = |\mathbf{r} \times \mu\mathbf{v}| = \mu|\mathbf{r} \times \mathbf{v}| = \mu r v \sin\alpha$, where α is the angle between $\mathbf{v}$ and $\mathbf{r}$. But because $v \sin\alpha = v_\theta = r\dot{\theta}$, it follows that

$$L = \mu r^2 \dot{\theta}. \tag{2.12}$$

The constancy of the angular momentum means that the trajectory of the body is planar. An important result that follows from the constancy of the angular momentum is the constancy of the areal velocity. The areal velocity is the ratio between the area, $|d(\overrightarrow{Area})|$, swept out by the radius vector in a given time interval, dt, and the length of that time interval.

$$\left| \frac{d(\overrightarrow{Area})}{dt} \right| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| = \frac{1}{2} r^2 \dot{\theta}. \tag{2.13}$$

The vector $d\overrightarrow{Area}$ has the orientation of $\mathbf{r} \times \mathbf{v}$ i.e. the orientation of $\mathbf{L}$. The constancy of the areal velocity is the only Kepler's Law that holds for the two-body Manev problem.

We want now to rewrite the radial equation (2.8) in terms of derivatives with respect to θ. For this we can first write $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$ and $\dot{\theta} = \frac{C}{r^2}$. This gives

$$\frac{dr}{dt} = \frac{dr}{d\theta}(\frac{C}{r^2}) \tag{2.14}$$

and

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta}(\frac{C}{r^2}\frac{dr}{d\theta})\frac{d\theta}{dt}, \tag{2.15}$$

$$\frac{d^2r}{dt^2} = \frac{C}{r^2}\frac{d}{d\theta}(\frac{C}{r^2}\frac{dr}{d\theta}). \tag{2.16}$$

The radial equation (2.8) becomes

$$\frac{d}{d\theta}\left(\frac{C}{r^2}\frac{dr}{d\theta}\right)\frac{C}{r^2} - r\left(\frac{C}{r^2}\right)^2 = \frac{f(r)}{\mu}, \tag{2.17}$$

with $f(r) = -\frac{A}{r^2} - \frac{2B}{r^3}$. The above equation can be rewritten as $-\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{1}{r} = \frac{r^2}{C^2\mu}f(r)$. Let $\frac{1}{r} = u$. Then (2.1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{A + 2Bu}{C^2\mu}, \tag{2.18}$$

or, after rearranging the terms, we can write a Binet type equation,

$$\frac{d^2u}{d\theta^2} + (1 - \frac{2B}{C^2\mu})u = \frac{A}{C^2\mu}. \tag{2.19}$$

It is convenient to introduce the following notation

$$\frac{2B}{C^2\mu} = \alpha, \tag{2.20}$$

and for $A > 0$,

$$\frac{A}{C^2\mu} = k > 0. \tag{2.21}$$

Equation (2.19) then becomes: $\frac{d^2u}{d\theta^2} + (1 - \alpha)u = k$ or, for $1 - \alpha \neq 0$,

$$\frac{d^2u}{d\theta^2} + (1 - \alpha)(u - \frac{k}{1 - \alpha}) = 0.$$

Let $y = u - \frac{k}{1-\alpha}$ and let us write the above as

$$\frac{d^2y}{d\theta^2} + (1 - \alpha)y = 0. \tag{2.22}$$

For $1 - \alpha > 0$ the solution of (2.22) is bounded and of the form

$$y = \Gamma \cos[\sqrt{1 - \alpha}(\theta - \theta_0)], \tag{2.23}$$

where Γ and θ_0 are constants depending on the initial conditions. By convention at the initial time t_0, $\theta(t = t_0) = 0$. Taking into account the previous notations, the solution of the equation of motion is

$$r(\theta) = \frac{1}{\Gamma \cos[\sqrt{1 - \alpha}(\theta - \theta_0)] + \frac{k}{1-\alpha}}. \tag{2.24}$$

Since $\frac{k}{1-\alpha} = \frac{A}{C^2\mu-2B}$, the solution can be rewritten as

$$r(\theta) = \frac{\frac{C^2\mu-2B}{A}}{1+\frac{\Gamma(C^2\mu-2B)}{A}\cos[\sqrt{1-\alpha}(\theta-\theta_0)]}, \tag{2.25}$$

If $r(\theta_0) = r_0$, then $\Gamma = \frac{1}{r_0} - \frac{A}{C^2\mu-2B}$. For circular motion $r = r_0 = constant$. If we substitute this in the radial equation (2.8) we obtain

$$-r_0\dot{\theta}^2 = \frac{f(r_0)}{\mu}, \tag{2.26}$$

which gives

$$r_0 = \frac{C^2\mu - 2B}{A} \tag{2.27}$$

and therefore $\Gamma = 0$ for circular motion. From (2.20) and $1 - \alpha > 0$, it follows that $C^2\mu - 2B > 0$. Using the same notation as for the classical case we take the semilatus rectum to be

$$P = \frac{C^2\mu - 2B}{A} \tag{2.28}$$

and consider

$$\beta = \sqrt{1 - \frac{2B}{C^2\mu}}. \tag{2.29}$$

The eccentricity is

$$e = \frac{\Gamma(C^2\mu - 2B)}{A}, \quad 0 \le e < 1. \tag{2.30}$$

With this notation, (2.25) becomes

$$r(\theta) = \frac{P}{1 + e\cos[\beta(\theta - \theta_0)]}. \tag{2.31}$$

The trajectory is a precessional ellipse. The distance between the two bodies varies between the minimum value $r_{min} = \frac{P}{1+e}$ at the pericenter and the maximum value $r_{max} = \frac{P}{1-e}$ at the apocenter. Defining the semi-major axis as $a = \frac{r_{min}+r_{max}}{2}$ we obtain

$$P = a(1 - e^2). \tag{2.32}$$

Unlike the case of the Kepler problem the direction of the pericenter is not constant. It advances if $\beta > 1$ i.e. if $B < 0$. It is delayed if $\beta < 1$ i.e. if $0 < B < \frac{C^2\mu}{2}$. The advance (or delay) angle is $\varphi = 2\pi - 2\pi/\beta$. The trajectory is closed if β is rational and the motion is periodic. For the case of β irrational the trajectory covers densely the annulus $r_{min} < r < r_{max}$ and the motion is quasi-periodic. The angle between the pericenter and the successive apocenter is $\gamma = \frac{\pi}{\beta}$.

For the classical case, $B = 0$, $\beta = 1$ and $\varphi = 0$. The direction of the pericenter is constant. For the case of a positive rational $\beta = p/q$ the trajectory closes after p "oscillations" of r and q "rotations"(i.e. the change in θ is $\Delta\theta = 2\pi q$). This situation is illustrated in the set of Figures 2.1- 2.5 for some particular values of p and q. From (2.31) we choose $P = 1$ and $\theta_0 = 0$.

2.1.1 Relationship between eccentricity and the total energy

The total energy is given by

$$E = \frac{\mu v^2}{2} - \frac{1}{r}\left(A + \frac{B}{r}\right) \tag{2.33}$$

Taking into account that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{C^2}{r^2}$, we can write the total energy,

$$E = \frac{\mu \dot{r}^2}{2} + \frac{\mu C^2}{2r^2} - \frac{1}{r}\left(A + \frac{B}{r}\right). \tag{2.34}$$

From (2.31) and (2.14), it follows that $\dot{r} = \frac{\beta e C}{P}\sin[\beta(\theta - \theta_0)]$. Substituting this in (2.34), the expression for E is

$$E = \frac{\beta^2 e^2 C^2 \mu}{2P^2}\sin^2[\beta(\theta - \theta_0)] + \frac{C^2\mu}{2P^2}\{1 + e\cos[\beta(\theta - \theta_0)]\}^2 -$$

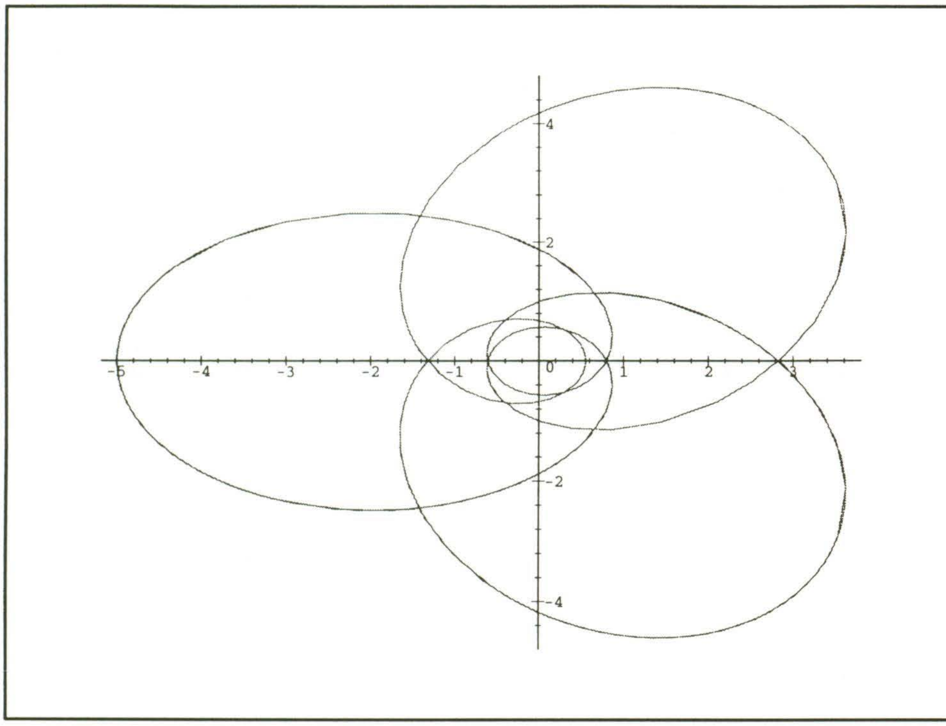

Figure 2.1: Graph of $r(\theta) = (1 + 0.88\cos 0.6\theta)^{-1}$ with $p = 3$, q=5 and $B > 0$

$$-\frac{A}{P}\{1 + e\cos[\beta(\theta - \theta_0)]\} - \frac{B}{P^2}\{1 + e\cos[\beta(\theta - \theta_0)]\}^2. \tag{2.35}$$

Substituting $\beta^2 = \frac{C^2\mu - 2B}{C^2\mu}$, the expression of P given by (2.28), and performing the necessary algebra we obtain

$$E = \frac{A^2(e^2 - 1)}{2(C^2\mu - 2B)}, \tag{2.36}$$

or

$$e^2 = 1 + \frac{2E(C^2\mu - 2B)}{A^2}. \tag{2.37}$$

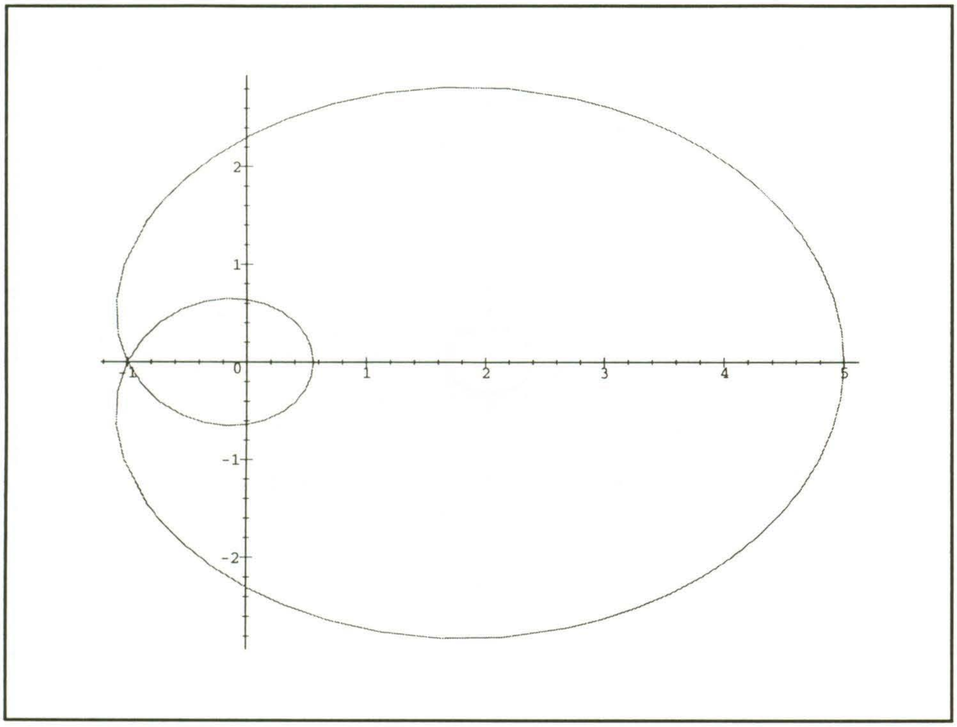

Figure 2.2: Graph of $r(\theta) = (1 + 0.80 \cos 0.5\theta)^{-1}$ with $p = 1$, q=2 and $B > 0$

The relative trajectory of the two bodies is a circle if $e = 0$ and it is a precessional ellipse if $0 < e < 1$. In terms of energy, this conditions can be written as follows

$$-\frac{A^2}{2(C^2\mu - 2B)} \leq E < 0. \tag{2.38}$$

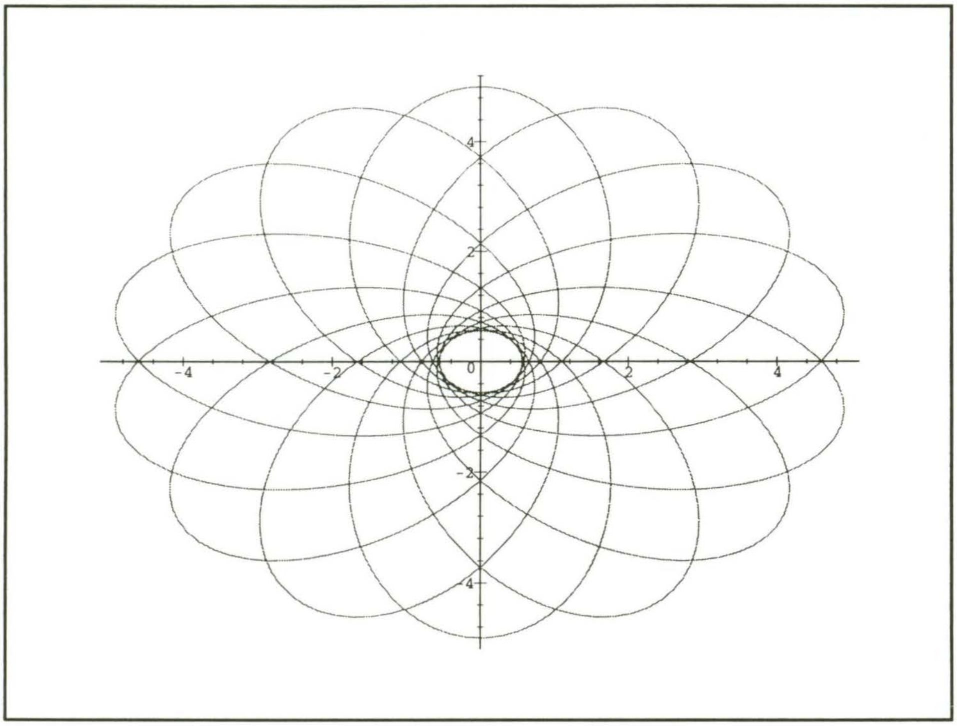

Figure 2.3: Graph of $r(\theta) = (1 + 0.80 \cos \frac{14}{15}\theta)^{-1}$ with $p = 14$, q=15 and $B > 0$

2.1.2 Relationship between the quasi-period and the semi- major axis

The relative motion of the two bodies is periodic only in the case the trajectory is closed. Let us define the quasi-period τ of the motion as the time interval elapsed between passing through two successive pericenters (or two successive apocenters). The area swept out in this time interval by the radius vector $\mathbf{r}$ is $Area = \int_0^{\gamma} r^2 d\theta$. Taking into account the trajectory equation (2.31), the integral becomes

$$Area = \int_0^{\gamma} \frac{P^2}{[1 + e \cos \beta(\theta - \theta_0)]^2} d\theta. \tag{2.39}$$

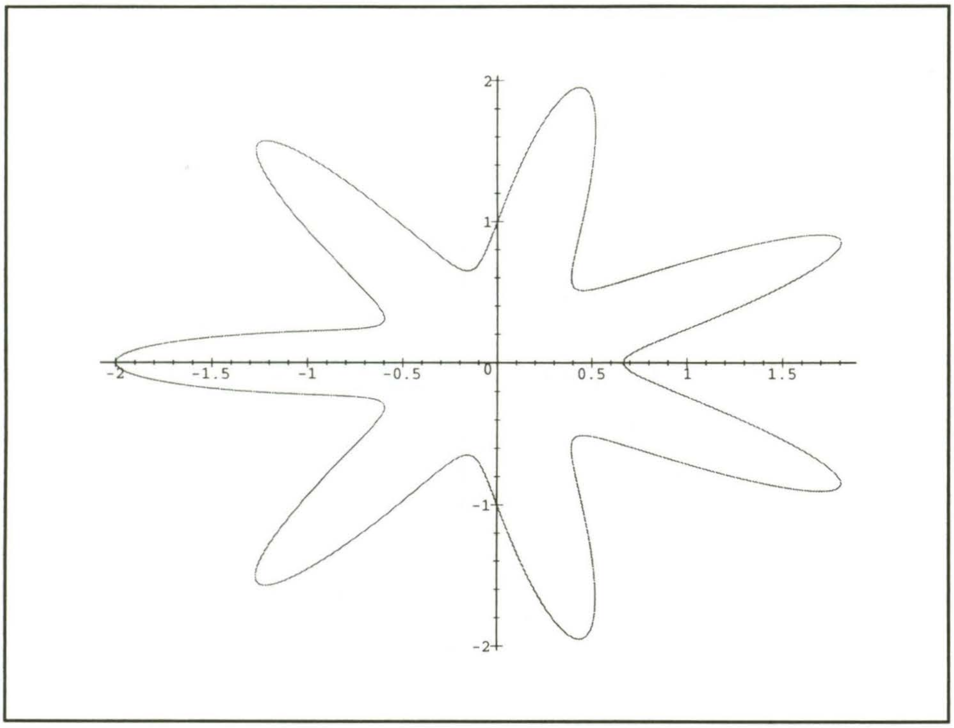

Figure 2.4: Graph of $r(\theta) = (1 + 0.80 \cos 7\theta)^{-1}$ with $p = 7$, q=1 and $B < 0$

Using the formulas 354 and 324 (pg. 381, 382) from Zwillinger (see [17]),

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{b \sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2} \int \frac{dx}{a + b\cos x} \tag{2.40}$$

and

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2} \tan(x/2)}{a + b}, \tag{2.41}$$

we obtain

$$Area = \frac{\pi P^2}{\beta(1 - e^2)^{3/2}}.$$

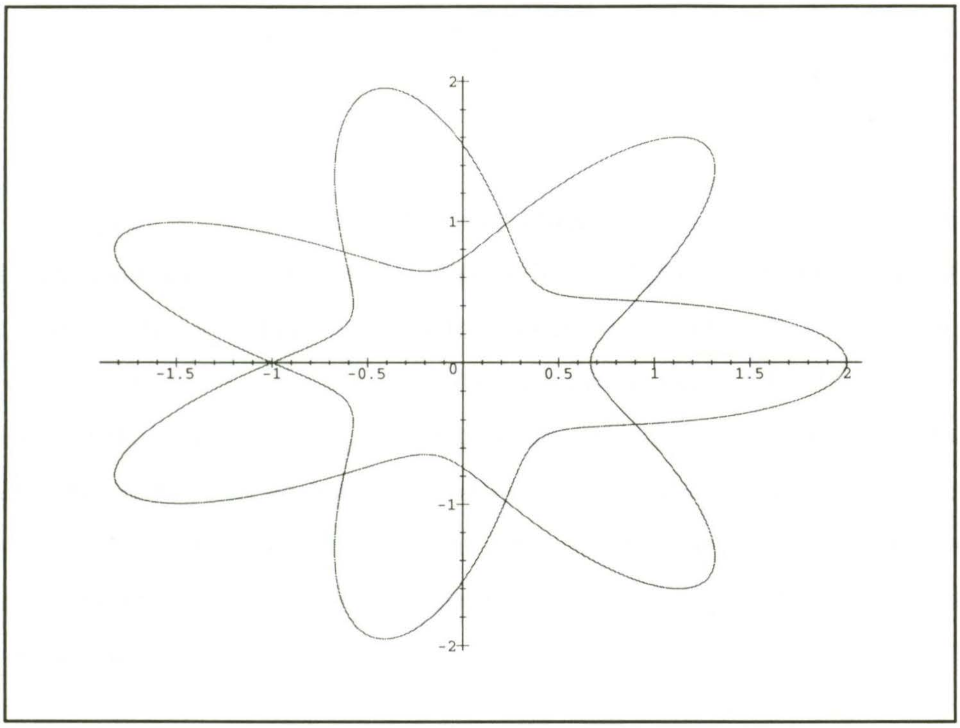

Figure 2.5: Graph of $r(\theta) = (1 + 0.80 \cos \frac{7}{2}\theta)^{-1}$ with $p = 7$, q=2 and $B < 0$

The areal velocity is constant and equal to $\frac{dArea}{dt} = \frac{C}{2} = \frac{Area}{\tau}$ so $Area$=$\tau \frac{C}{2}$ and this yields

$$\tau = \frac{2\pi P^2}{C\beta(1 - e^2)^{3/2}}. \tag{2.42}$$

Recalling that the semi-major axis is $a = \frac{r_{min}+r_{max}}{2}$ and taking into account the expressions for P and β,

$$P = \frac{C^2\mu - 2B}{A} = a(1 - e^2),$$

$$\beta = \sqrt{\frac{C^2\mu - 2B}{C^2\mu}},$$

we obtain

$$\tau = 2\pi\sqrt{\frac{\mu a^3}{A}}, \tag{2.43}$$

which implies that

$$\tau^2 = 4\pi^2\frac{\mu a^3}{A}. \tag{2.44}$$

The square of the quasi-period is proportional to the cube of the semi-major axis. The expression (2.43) has the value of B included in the term a and has the same form as Kepler's Third Law for the classical case. The period of motion, T, is defined only if the precessional ellipse trajectory is closed. In this case it is a multiple of the quasi-period, $T = p\tau$, p being the number of complete "oscillations" of the relative distance, r , between the two bodies. The distance $r(t)$ between the two bodies is a periodic function of time $r(t) = |\mathbf{r}(t)| = r(t+\tau)$ but their relative position vector, $\mathbf{r}$(t), is a periodic function with period T only if the trajectory is closed. From

$$a = \frac{P}{1-e^2} = \frac{C^2\mu - 2B}{A(1-e^2)},$$

it follows that

$$C^2\mu = aA(1-e^2) + 2B, \tag{2.45}$$

which allows us to write (2.44) in the form

$$\tau^2 = \frac{4\pi^2\mu a^4}{aA} = \frac{4\pi^2\mu a^4(1-e^2)}{C^2\mu - 2B}.$$

Taking into account the expression for β and (2.45) we obtain

$$\tau^2 = \frac{4\pi^2 a^4\mu(1-e^2)}{\beta^2 C^2\mu}, \tag{2.46}$$

or

$$\tau^2 = \frac{4\pi^2 a^4\mu(1-e^2)}{\beta^2[aA(1-e^2)+2B]}. \tag{2.47}$$

The expression (2.44) is equivalent to (2.47).

We can obtain the same relation (2.44) starting with (2.34),

$$E = \mu\frac{\dot{r}^2}{2} + \mu\frac{C^2}{2r^2} - \frac{1}{r}(A + \frac{B}{r}).$$

We can now express $\dot{r}^2$ as

$$\dot{r}^2 = \frac{2}{\mu}\left[E + \frac{1}{r}\left(A + \frac{B}{r}\right) - \frac{C^2\mu}{2r^2}\right] \tag{2.48}$$

and therefore

$$t = \sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{E + \frac{1}{r}(A + \frac{B}{r}) - \frac{C^2\mu}{2r^2}}}, \tag{2.49}$$

which can be rewritten as

$$t = \sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{rdr}{\sqrt{Er^2 + r(A + \frac{B}{r}) - \frac{C^2\mu}{2}}}. \tag{2.50}$$

This can be integrated using the change of variable $r = a(1 - e\cos\Psi)$. The eccentric anomaly Ψ varies between 0 and 2π when θ varies between 0 and 2π for a complete revolution. Using $P = \frac{C^2\mu - 2B}{A} = a(1 - e^2)$ we find that $C^2 = \frac{Aa(1-e^2)+2B}{\mu}$ and that $E = -\frac{A}{2a}$. Plugging these into (2.50) and performing the necessary computations, we obtain the same type of relation as for the classical Kepler problem (see [5]),

$$t = \sqrt{\frac{\mu a^3}{A}}\int_{\Psi_0}^{\Psi}(1 - ecos\Psi)d\Psi\ , \tag{2.51}$$

$$t = \sqrt{\frac{\mu a^3}{A}}(\Psi - \Psi_0 - e\sin\Psi + e\sin\Psi_0). \tag{2.52}$$

The above results allow us to compute the quasi-period τ of the motion. For $\Psi = 2\pi$ and $\Psi_0 = 0$, it follows that $t = \tau = 2\pi\sqrt{\frac{\mu a^3}{A}}$. We thus obtain the relation (2.43) using a different approach.

2.1.3 Relationship between time and polar angle

The expression of the angular momentum $L = \mu r^2 \dot{\theta}$ can be rewritten as

$$dt = \frac{\mu r^2}{L} d\theta. \tag{2.53}$$

Substituting the equation of the trajectory (2.31) and integrating (2.53), we can express time as a function of the polar angle given by the integral

$$t(\theta) = \frac{\mu P^2}{L} \int_{\theta_0}^{\theta} \frac{d\theta}{[1 + e\cos\beta(\theta - \theta_0)]^2}. \tag{2.54}$$

Using the integration formulas (2.40) and (2.41) we obtain the following relationship between time and polar angle:

$$t(\theta) = \frac{\mu P^2}{L\beta} \left[\frac{e \sin\beta(\theta - \theta_0)}{(e^2 - 1)(1 + e\cos\beta(\theta - \theta_0))} + \frac{2}{(1 - e^2)^{3/2}} \tan^{-1} \frac{\sqrt{1 - e^2} \tan \frac{\beta(\theta - \theta_0)}{2}}{1 + e} \right]. \tag{2.55}$$

2.1.4 Relationship between time and the distance between particles

We can integrate (2.49) directly, using the formulas 234 and 226 (pg. 375, 374) from Zwillinger (see [17]), and rewriting the previous integral as

$$t = \sqrt{\mu} \int_{r_0}^{r} \frac{r dr}{\sqrt{2Er^2 + 2Ar + 2B - C^2\mu}}. \tag{2.56}$$

For $X = a + bx + cx^2$ and $q = 4ac - b^2$, these formulas are

$$\int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}, \tag{2.57}$$

and for $c < 0$

$$\int \frac{dx}{\sqrt{X}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}. \tag{2.58}$$

For our case $a = 2B - C^2\mu < 0$, $b = 2A$, $2E = c < 0$, $q = 8E(2B - C^2\mu) - 4A^2$, but $E = \frac{A^2(e^2-1)}{2(C^2\mu-2B)}$ therefore $q = -4A^2e^2 < 0$. Using the above relations, we obtain

$$t(r) = \sqrt{\mu}\left[\frac{\sqrt{2Er^2 + 2Ar - 2B + C^2\mu}}{2E} - \frac{A}{(-2E)^{3/2}}\sin^{-1}\frac{2Er + A}{Ae}\right] -$$

$$-\sqrt{\mu}\left[\frac{\sqrt{2Er_0^2 + 2Ar_0 - 2B + C^2\mu}}{2E} - \frac{A}{(-2E)^{3/2}}\sin^{-1}\frac{2Er_0 + A}{Ae}\right], \tag{2.59}$$

where $r \in (r_{min} = \frac{P}{1+e}, r_{max} = \frac{P}{1-e})$ and $e \neq 0$. For the circular case $e = 0$ and $r(t) = r_0 =$ constant and the inverse function $t = t(r)$ does not exist. If solved numerically the relation (2.59) can give r as function of time.

2.2 Stability of Circular Orbits of the Primaries

From the literature, it is known that circular orbits are possible under any attractive central force but that the circular orbits are not stable in all cases. Stability of the circular orbit is understood here in the following way: if the particle moving in a circular orbit suffers a small perturbation, then the new trajectory will remain close to the original circular orbit. We start by considering the radial equation (2.8) and rewrite it as

$$\ddot{r} - \frac{C^2}{r^3} = \frac{f(r)}{\mu}. \tag{2.60}$$

For the case of a circular trajectory $r = constant$. Let r_0 be the radius of a circular orbit. Then equation (2.60) becomes

$$-\frac{C^2}{r_0^3} = \frac{f(r_0)}{\mu}. \tag{2.61}$$

Substitution of (2.12) and (2.3) into (2.61) gives

$$\mu r_0\dot{\theta}^2 = \frac{Ar_0 + 2B}{r_0^3} \tag{2.62}$$

which expresses the equality of centripetal force to $f(r_0)$. A circular trajectory of radius r_0 is possible only if

$$Ar_0 + 2B > 0 \tag{2.63}$$

Now let $x = r - r_0$ and rewrite (2.60) as

$$\mu\ddot{x} - \mu\frac{C^2}{(x+r_0)^3} = f(x+r_0). \tag{2.64}$$

Expanding the terms containing $x + r_0$ as power series in x, equation (2.64) is transformed into

$$\mu\ddot{x} - \mu\frac{C^2}{r_0^3}(1 - \frac{3x}{r_0} + \frac{6}{r_0^2}x^2 + h.o.t.) = f(r_0) + f'(r_0)x + \frac{f''}{2}(r_0)x^2 + h.o.t., \tag{2.65}$$

where prime means derivative with respect to r. The series converges for $|\frac{x}{r_0}| < 1$. Taking into account only the linear terms in x, the relation (2.61) and rearranging the terms, we obtain

$$\ddot{x} + \frac{1}{\mu}\left[-\frac{3}{r_0}f(r_0) - f'(r_0)\right]x = 0. \tag{2.66}$$

We can compare this equation with the equation of motion of the harmonic oscillator $\ddot{x} + \omega^2 x = 0$. If in the equation (2.66) the coefficient of x is positive, then the perturbed circular equation is stable. Thus, we obtain a condition of stability for the perturbed circular equation

$$-\frac{3}{r_0}f(r_0) - f'(r_0) > 0. \tag{2.67}$$

Taking into account that

$$f'(r_0) = \frac{2A}{r_0^3} + \frac{6B}{r_0^4},$$

the condition (2.67) becomes $\frac{A}{r_0^3} > 0$, which is true for any $A > 0$, and therefore the particle performs a harmonic oscillation with the angular frequency

$$\omega_{osc} = \sqrt{\frac{A}{\mu r_0^3}}. \tag{2.68}$$

The solution of the equation (2.66) with the initial condition $x(0) = \delta$ is

$$x(t) = \delta \cos(\omega_{osc} t). \tag{2.69}$$

From the condition $f(r) = F_{centripetal}$ which is written as $\frac{A}{r_0^2} + \frac{2B}{r_0^3} = \mu r_0 \omega_{rotation}^2$, where $\omega_{rotation} = \dot{\theta} = Constant$ is the constant angular velocity, it follows that

$$\omega_{rotation} = \frac{1}{r_0^2}\sqrt{\frac{Ar_0 + 2B}{\mu}}. \tag{2.70}$$

For $\delta \ll r_0$ we can write $\theta \approx \omega_{rotation} t$, and $r(t) \approx r_0 + \delta \cos\left(\theta \frac{\omega_{osc}}{\omega_{rotation}}\right)$ or, from(2.68),(2.70) and the above,

$$r(\theta) = r_0 + \delta \cos\left(\theta\sqrt{\frac{Ar_0}{Ar_0 + 2B}}\right), \tag{2.71}$$

or

$$r(t) = r_0 + \delta \cos\left(t\sqrt{\frac{A}{\mu r_0^3}}\right). \tag{2.72}$$

If $\beta = \sqrt{\frac{Ar_0}{Ar_0+2B}}$ is a (positive) rational number $\frac{p}{q}$, p and q natural numbers, then the trajectory is closed; it retraces itself after p oscillations and q rotations. If β is irrational the particle has a dense trajectory inside the annulus $(r_0 - \delta, r_0 + \delta)$. Substituting relation (2.27) into the above expression of β we obtain

$$\beta = \sqrt{\frac{C^2\mu - 2B}{C^2\mu}} \tag{2.73}$$

which is exactly the same as the one used in the case of precessional ellipses in the previous section. Using (2.45) we obtain

$$\beta^2 = \frac{aA(1-e^2)}{aA(1-e^2) + 2B} = \frac{p^2}{q^2}, \tag{2.74}$$

with p and q natural numbers. The precessional ellipse with a given e and a is closed if the constants A and B satisfy the condition (2.74).

Relation (2.74) resembles the result obtained by Bertrand for potentials of the form

$$U(r) = \frac{k}{r^n} \qquad k < 0$$

(see [5], Bertrand's Theorem). For such potentials, the orbits remain closed when slightly disturbed from circularity if $n = 2 - \frac{p^2}{q^2}$, where p and q are natural numbers. When deviations from circularity are considerable, the orbits remain closed if $n = 1$ (Newtonian potential) or $n = -2$ (elastic type potential). Interestingly enough, the stability condition (2.67) resembles the one in Bertrand's Theorem. We dealt with Manev type potentials and obtained the condition (2.74) for the constants A and B. This condition is valid under any type of perturbation, small or not.

Taking $A = 1$, $r_0 = 1$ in $\beta = \sqrt{\frac{Ar_0}{Ar_0+2B}} = \frac{p}{q}$ it follows that $B = \frac{1}{2}(\frac{q^2}{p^2} - 1)$. If $B > 0$ then $\beta < 1$ and for β rational $p < q$. This situation is illustrated in the following figures where we plot (2.71):

i)Figure 2.6 for $\beta = 1/2$, $A = 1$, $B = 1.5$, $r_0 = 1$ and $\delta = 0.05$.

ii)Figure 2.7 for $\beta = 2/3$, A=1, B=5/8, $r_0 = 1$ and $\delta = 0.05$.

If $B < 0$ then $\beta > 1$ and for β rational $p > q$. This situation is illustrated in the following figures where we also plot (2.71):

i) Figure 2.8 for $\beta = 11$, $A = 1$, $B = -\frac{60}{121}$, $r_0 = 1$ and $\delta = 0.05$.

ii)Figure 2.9 for $\beta = 11/2$, $A = 1$, $B = -\frac{117}{242}$, $r_0 = 1$ and $\delta = 0.05$.

Note For $\delta \ll r_0$ and $e \ll 1$ the equations

$$r(t) = r_0 + \delta \cos t \sqrt{\frac{A}{\mu r_0^3}}$$

and

$$r(\theta) = \frac{P}{1 + e\cos[\beta(\theta - \theta_0)]}$$

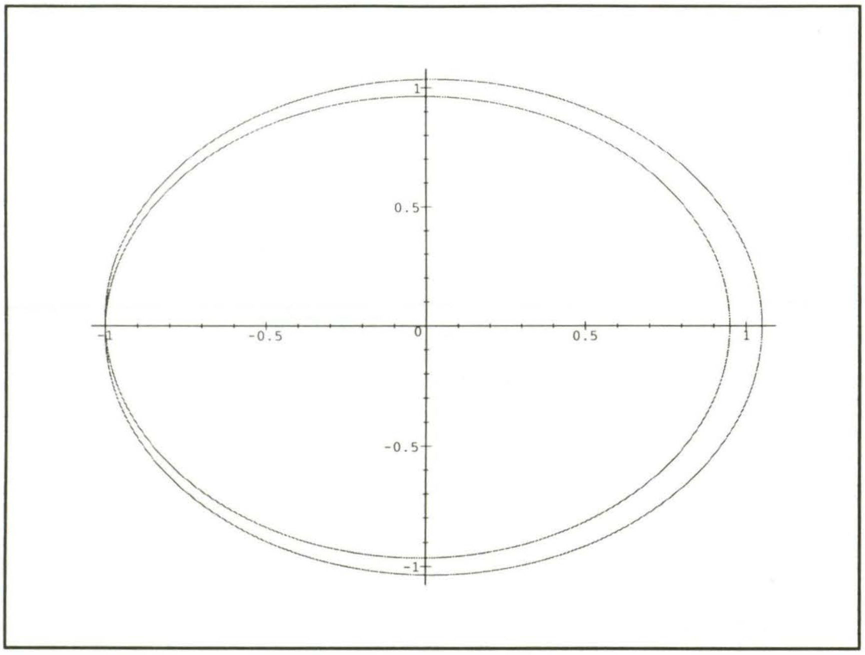

Figure 2.6: Graph of $r(\theta) = 1 + 0.05 \cos 0.5\theta$

are equivalent. We use this at the beginning of section 3.3.

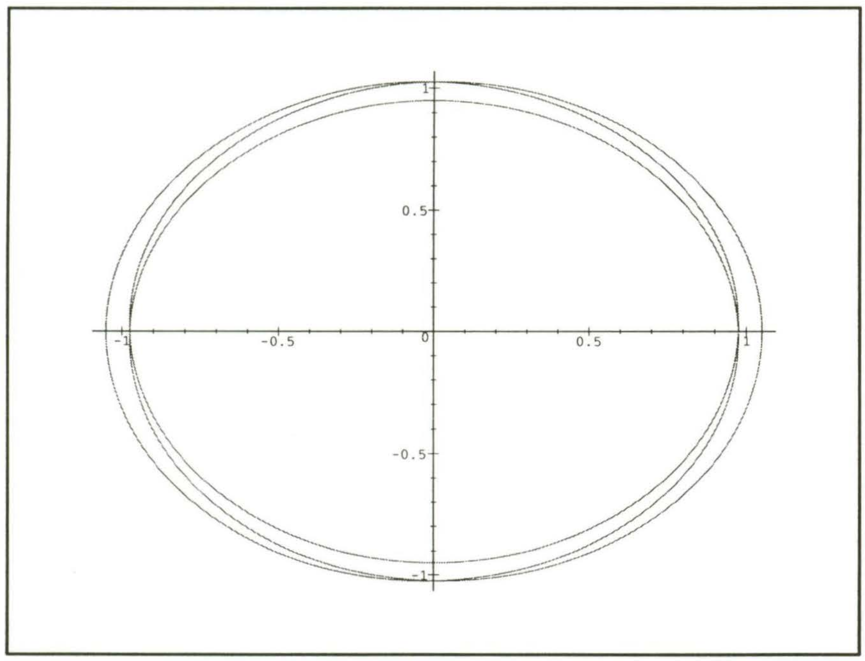

Figure 2.7: Graph of $r(\theta) = 1 + 0.05 \cos \frac{2}{3}\theta$

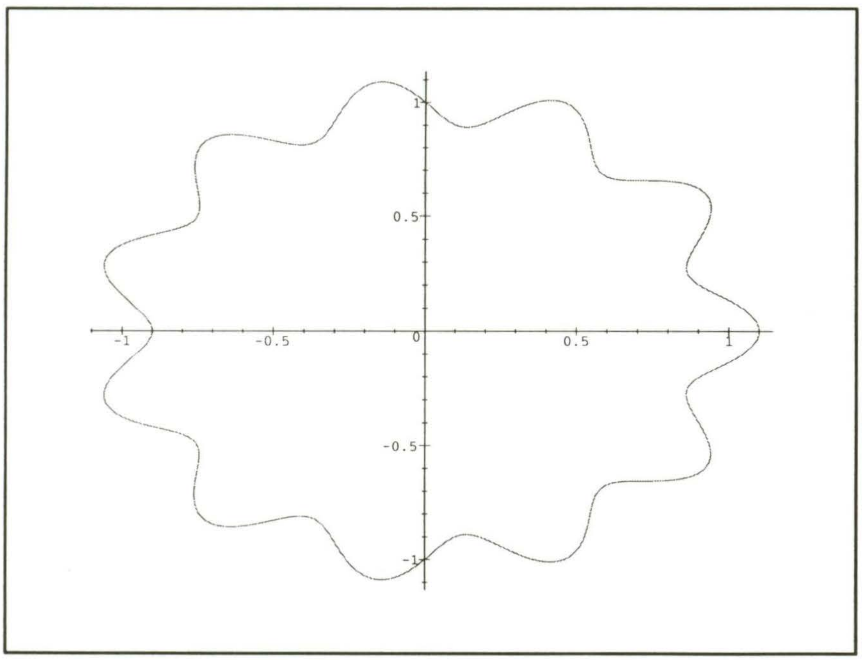

Figure 2.8: Graph of $r(\theta) = 1 + 0.05\cos 11\theta$

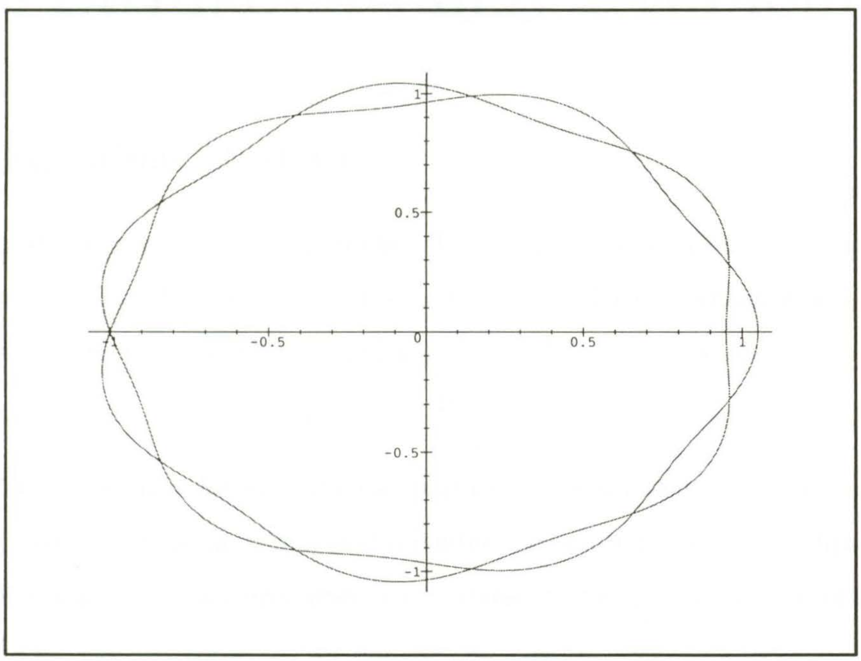

Figure 2.9: Graph of $r(\theta) = 1 + 0.05 \cos 5.5\theta$

Chapter 3

The Sitnikov-Manev Problem

3.1 Equations of Motion

Consider a system of three particles. Two of them, called primaries, have equal masses $m_1 = m_2 = M$ and move in the plane (xy) under the influence of a mutual central force described by an interaction potential $U(r)$ of the Manev type,

$$U(r) = \frac{A}{r} + \frac{B}{r^2}, \quad A > 0, \tag{3.1}$$

where r is the distance between the two particles. The primaries move around their common center of mass on orbits that are either circles or precessional ellipses. The center of mass of the two primaries is considered to be situated at the origin (see Figure 3.1).

A third body of zero mass $m_3 = m = 0$ is moving on a trajectory passing through the center of mass of the primaries and perpendicular to their plane of motion (z axis). It moves under the combined attraction of the primaries and it has no effect on their motion. The potential energy of the third body, V_3, due to the interaction with the primaries is the sum of two terms that describe the interaction with each of the primaries. In terms of the potential $U_3 = -V_3$, we have

$$U_3 = U_{1,3} + U_{2,3}, \tag{3.2}$$

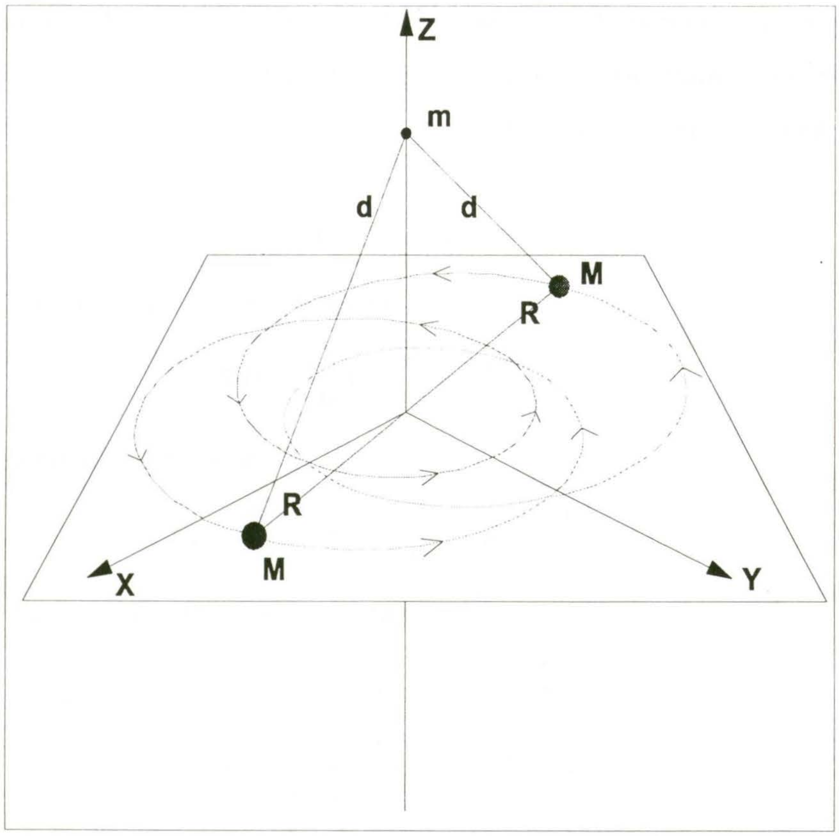

Figure 3.1: The Sitnikov-Manev Problem.

$$U_{3,1} = \frac{A}{d_{3,1}} + \frac{B}{d_{3,1}^2}, \tag{3.3}$$

$$U_{3,2} = \frac{A}{d_{3,2}} + \frac{B}{d_{3,2}^2}. \tag{3.4}$$

Due to the symmetry of the configuration, the distances between the third body and each of the primaries are $d_{3,2} = d_{3,1} = d$, and therefore $U_3 = 2(\frac{A}{d} + \frac{B}{d^2})$ with

$d(t) = (R(t)^2 + z(t)^2)^{\frac{1}{2}}$. $R(t)$ is the distance between the center of mass and each of the primaries. This distance is half of the distance $r(t)$ between the primaries given in the previous chapter, and $z(t)$ is the coordinate of the third body. We can now compute the total force acting on the third particle as being the gradient of the potential,

$$\mathbf{F} = \nabla U(r).$$

The gradient of a function U is defined by

$$\nabla U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}.$$

The components of the force are

$$F_x = \frac{\partial U}{\partial x}, \quad F_y = \frac{\partial U}{\partial y}, \quad F_z = \frac{\partial U}{\partial z}.$$

Computing the partial derivatives we obtain

$$F_x = F_y = 0,$$

$$F_z = -\frac{2Az}{(R^2 + z^2)^{\frac{3}{2}}} - \frac{4Bz}{(R^2 + z^2)^2}.$$

If the mass of the third body is taken to be small but non-zero then the equation of motion along the z-axis, $F_z = ma_z$, can be written in the form

$$\ddot{z} = -\frac{2z}{m}\left[\frac{A}{(R^2 + z^2)^{\frac{3}{2}}} + \frac{2B}{(R^2 + z^2)^2}\right], \tag{3.5}$$

where the dot denotes the derivative with respect to time. The derivative of z with respect to time is the speed, v, of the third particle along the z-axis, and the derivative of this speed is the acceleration of the third particle along the z-axis. Thus equation (3.5) can be written as a system of two first order differential equations:

$$\begin{cases} \dot{z} = v \\ \dot{v} = -\frac{2z}{m}\left[\frac{A}{(R(t)^2+z^2)^{\frac{3}{2}}} + \frac{2B}{(R(t)^2+z^2)^2}\right] \end{cases} \cdot \tag{3.6}$$

We have

$$R = R(t) = \frac{1}{2}\frac{P}{1 + e\cos[\beta(\theta(t) - \theta_0)]}, \tag{3.7}$$

where

$P = \frac{C^2M-4B}{2A}$ with

$C = \frac{2L}{M}$,

$\beta = \sqrt{1 - \frac{4B}{C^2M}}$ and

the eccentricity

$$e = \sqrt{1 + \frac{E(C^2M - 4B)}{A^2}}.$$

The above formulas for P, C, β and e are obtained from (2.28), (2.11), (2.29), and (2.30) with the substitution $\mu = \frac{M}{2}$.

The total energy of the third particle is E=Kinetic Energy (K.E.) + Potential Energy (V) with $V = -U_3 = -2(\frac{A}{d} + \frac{B}{d^2})$. In the case that the distance between the primaries depends on time the total energy of the third particle is not constant.

$$E(z, v, t) = \frac{mv^2}{2} - 2\left[\frac{A}{(R(t)^2 + z^2)^{\frac{1}{2}}} + \frac{B}{R(t)^2 + z^2}\right]. \tag{3.8}$$

The Hamiltonian is

$$H(z, v, t) = \frac{v^2}{2} - \frac{2}{m}\left[\frac{A}{(R(t)^2 + z^2)^{\frac{1}{2}}} + \frac{B}{R(t)^2 + z^2}\right]. \tag{3.9}$$

It is easy to check that

$$\begin{cases} \frac{\partial H}{\partial v} = v \\ \frac{\partial H}{\partial z} = \frac{2z}{m}\left[\frac{A}{(R(t)^2+z^2)^{\frac{3}{2}}} + \frac{2B}{(R(t)^2+z^2)^2}\right] \end{cases} \tag{3.10}$$

and the system (3.6) is of the form

$$\begin{cases} \dot{z} = \frac{\partial H}{\partial v} \\ \dot{v} = -\frac{\partial H}{\partial z}. \end{cases} \tag{3.11}$$

It follows that the system (3.6) is Hamiltonian. Because the Hamiltonian (3.9), is time dependent the system (3.6) is not conservative.

3.2 Circular Case

Equations of Motion Let us consider now the case in which the primaries describe circular obits of radius $R_0 = r_0/2$ around their common center of mass. Recall that the radial equation (2.8) is

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{\mu}.$$

In the case of circular obits $r = r_0 = \textit{constant}$ and the radial equation is just

$$-r_0\dot{\theta}^2 = \frac{f(r_0)}{\mu}, \tag{3.12}$$

with $f(r_0) = -\frac{A}{r_0^2} - \frac{2B}{r_0^3}$ and, by definition, $\dot{\theta} = \frac{2\pi}{T}$, where T is the period of the circular motion. Substituting the last two relations into the radial equation allows us to write

$$r_0\frac{4\pi^2}{T^2} = \frac{1}{\mu}\left(\frac{A}{r_0^2} + \frac{2B}{r_0^3}\right), \tag{3.13}$$

which gives

$$T^2 = \frac{4\pi^2\mu r_0^4}{Ar_0 + 2B}.$$

We can obtain the relationship between the period T and the radius R_0 of the circular orbit of the primaries in the system of the center of mass, where $\mu = M/2$ and $r_0 = 2R_0$. Thus,

$$T^2 = \frac{16\pi^2 MR_0^4}{AR_0 + B}. \tag{3.14}$$

The system of differential equations that describes the motion of the third body has the same form as (3.6), with the difference that in this case $R = R_0$ and does not depend on time any more. Thus (3.6) becomes

$$\begin{cases} \dot{z} = v \\ \dot{v} = -\frac{2z}{m}\left[\frac{A}{(R_0^2+z^2)^{\frac{3}{2}}} + \frac{2B}{(R_0^2+z^2)^2}\right]. \end{cases} \tag{3.15}$$

This system is Hamiltonian with

$$H_0(z,v) = \frac{v^2}{2} - \frac{2}{m}\left[\frac{A}{(R_0^2+z^2)^{\frac{1}{2}}} + \frac{B}{R_0^2+z^2}\right]. \tag{3.16}$$

We use the subscript 0 to denote the circular case. This Hamiltonian function does not depend on time. The system (3.15) is Hamiltonian and conservative. It can be written in the form

$$\begin{cases} \dot{z} = \frac{\partial H_0}{\partial v} \\ \dot{v} = -\frac{\partial H_0}{\partial z}. \end{cases} \tag{3.17}$$

The Hamiltonian is a constant of motion. The level curves $H_0(z,v) = Constant$ represent all the possible solutions of the system (3.15) and are shown in Figure 3.2.

For $H_0 < 0$, there is a family of closed solutions, which corresponds to periodic solutions. In analogy to the two-body problem, these periodic solutions are called elliptic solutions. There are two parabolic orbits represented by the level curves $H_0 = 0$, in which case the particle tends to $z = \pm\infty$ with zero velocity. For hyperbolic orbits corresponding to $H_0 > 0$, the third massless body tends to $z = \pm\infty$ with a non-zero velocity. The parabolic orbit acts as separatrix between the other two classes of orbits.

The equilibria of the system (3.15) are the points $(z = 0, v = 0)$ and $(z = \pm\infty, v = 0)$. Let us now study the nature of these equilibria.

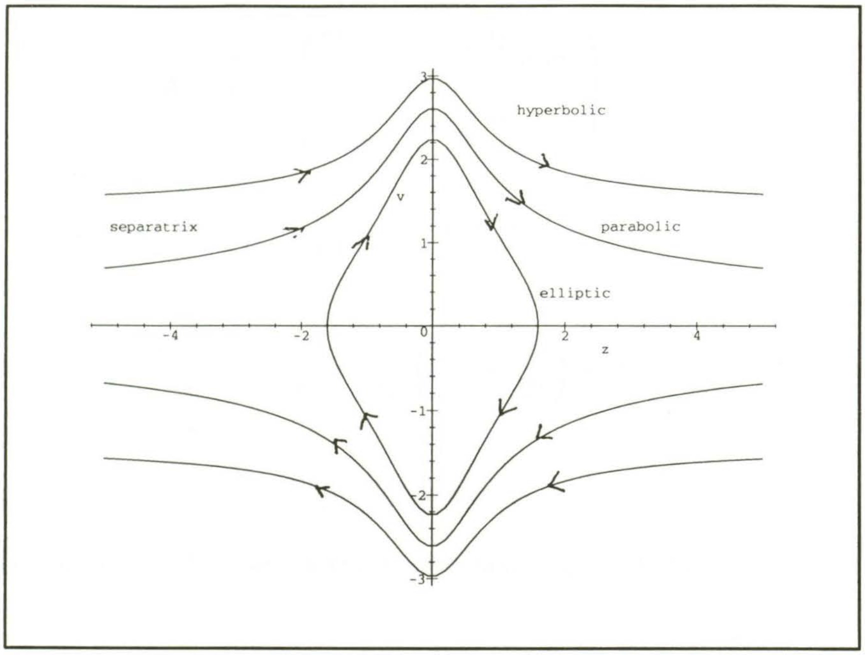

Figure 3.2: The phase portrait for the circular case.

Nature of the $(z = 0, v = 0)$ equilibrium.

Consider the following notation

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} F(z,v) \\ G(z,v) \end{pmatrix}$$

with $F(z,v) = v$ and $G(z,v) = -\frac{2z}{m}\left[\frac{A}{(R^2+z^2)^{\frac{3}{2}}} + \frac{2B}{(R^2+z^2)^2}\right]$, and

$$\mathbf{x} = \begin{pmatrix} z \\ v \end{pmatrix}.$$

With this notation the system (3.15) can be written as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The linearized system is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with $\mathbf{A} = D\mathbf{f}(\mathbf{0})$. Performing the following computation,

$$\mathbf{A} = \begin{pmatrix} \frac{\partial F(z,v)}{\partial z}\big|_{(0,0)} & \frac{\partial F(z,v)}{\partial v}\big|_{(0,0)} \\ \frac{\partial G(z,v)}{\partial z}\big|_{(0,0)} & \frac{\partial G(z,v)}{\partial v}\big|_{(0,0)} \end{pmatrix},$$

with

$$\frac{\partial F(z,v)}{\partial z}\big|_{(0,0)} = 0, \qquad \frac{\partial F(z,v)}{\partial v}\big|_{(0,0)} = 1,$$

$$\frac{\partial G(z,v)}{\partial z}\big|_{(0,0)} = -\frac{2}{m}\left(\frac{A}{R^3} + \frac{2B}{R^4}\right) < 0,$$

and

$$\frac{\partial G(z,v)}{\partial v}\big|_{(0,0)} = 0,$$

we obtain the matrix of the linear system corresponding to (3.15)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix},$$

with $\alpha = -\frac{2}{m}\left(\frac{A}{R^3} + \frac{2B}{R^4}\right) < 0$. Thus the linearized system near $(z = 0, v = 0)$ is

$$\begin{pmatrix} \dot{z} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} z \\ v \end{pmatrix}. \tag{3.18}$$

The eigenvalues of $\mathbf{A}$ are the solutions of the characteristic equation $det\mathbf{A}_\lambda = 0$, where $\mathbf{A}_\lambda = \mathbf{A} - \lambda\mathbf{I}$, and $\mathbf{I}$ is the (2 x 2) identity matrix. Here

$$\mathbf{A}_\lambda = \begin{pmatrix} -\lambda & 1 \\ \alpha & -\lambda \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - \alpha = 0$ and its solutions are $\lambda_{1,2} = \pm i\sqrt{-\alpha}$. Because *Real* $(\lambda) = 0$, the origin $(z = 0, v = 0)$ is a center for the linearized system (3.18), and, by definition [1] a non-hyperbolic, or degenerate, equilibrium point for

[1] If $\mathbf{x}_0$ is an equilibrium point for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ i.e. $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ and one of the eigenvalues λ of $\mathbf{A} = D\mathbf{f}(\mathbf{x}_0)$ is zero or has zero real part then $\mathbf{x}_0$ is called a non-hyperbolic, or degenerate, equilibrium point.

the nonlinear system (3.15). Thus the Hartman-Grobman Theorem does not apply. According to Theorem 5 pg. 142 from Perko ([12]), included below, the origin is either a center, a center-focus or a focus for the nonlinear system (3.15).

Theorem 3.1. *[12]*

Let E be an open subset of $\boldsymbol{R}^2$ and let $\boldsymbol{f} \in C^1(E)$. Suppose that the origin is a center for the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = D\boldsymbol{f}(0)$. Then the origin is either a center, a center-focus or a focus for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

The system (3.15) is invariant under the transformation $(t, v) \to (-t, -v)$ and therefore, by definition, is symmetric with respect to the z-axis. It is also symmetric with respect to the v-axis because it is invariant under the transformation $(t, z) \to (-t, -z)$. We can reach the same conclusion by examining the phase-plane portrait of this system, shown in Figure 3.2. This allows us to apply Theorem 6 pg. 144 from Perko ([12]), included bellow.

Theorem 3.2. *[12]*

Let E be an open subset of $\boldsymbol{R}^2$, $\boldsymbol{x} = (x, y)^T$, and let $\boldsymbol{f} \in C^1(E)$. If the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is symmetric with respect to the x-axis or the y-axis, and if the origin is a center for the linear system $\mathbf{x} = \mathbf{A}\mathbf{x}$ with $A = D\boldsymbol{f}(0)$, then the origin is a center for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

We thus conclude that the origin ($z = 0, v = 0$) is a center for the nonlinear system (3.15). The similar behaviour of the flow near the origin is shown in Figure 3.3 and Figure 3.4.

In the physical space, whenever its speed is smaller than the escape velocity, v_{esc}, the third particle will oscillate along the z-axis on one side or the other of $z = 0$. Let us consider the initial conditions $z(t = 0) = 0$ and $v(t = 0) = v_0 < v_{esc}$. Then the amplitude z_{max} of the oscillations depends only on the value of the initial velocity v_0, for a given R_0, the radius of the circular orbit of the primaries.

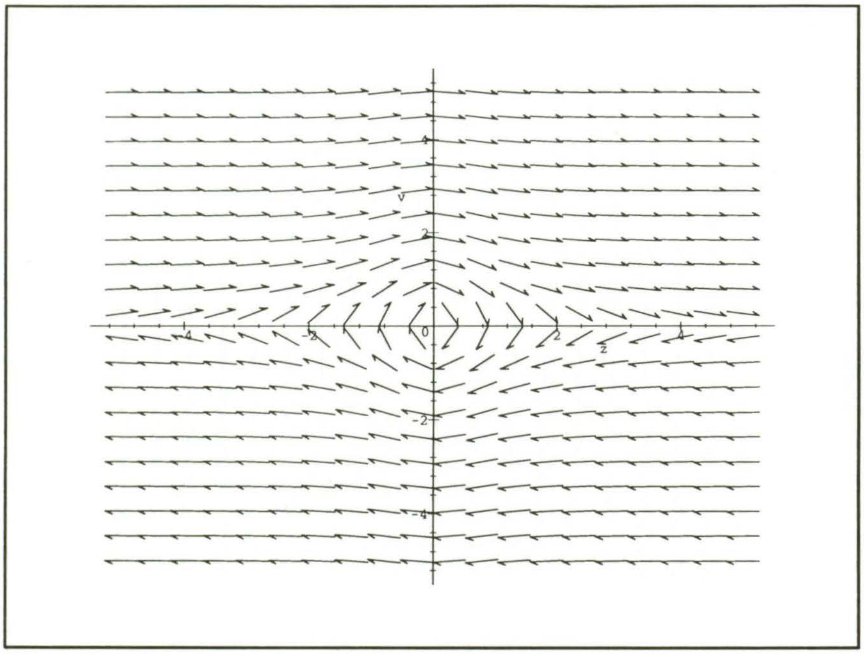

Figure 3.3: The vector field of the non-linear system (3.15).

The condition

$$H_0(z=0, v=v_0) = H_0(z=z_{max}, v=0)$$

gives the following equation in z_{max}

$$\frac{v_0^2}{2} - \frac{2}{m}\left[\frac{A}{R_0} + \frac{B}{R_0^2}\right] = -\frac{2}{m}\left[\frac{A}{(R_0^2+z_{max}^2)^{1/2}} + \frac{B}{R_0^2+z_{max}^2}\right], \tag{3.19}$$

with the solution

$$z_{max} = \sqrt{\left(\frac{A+\sqrt{A^2-2BmH_0}}{-mH_0}\right)^2 - R_0^2}, \tag{3.20}$$

where

$$H = \frac{v_0^2}{2} - \frac{2}{m}\left[\frac{A}{R_0} + \frac{B}{R_0^2}\right]. \tag{3.21}$$

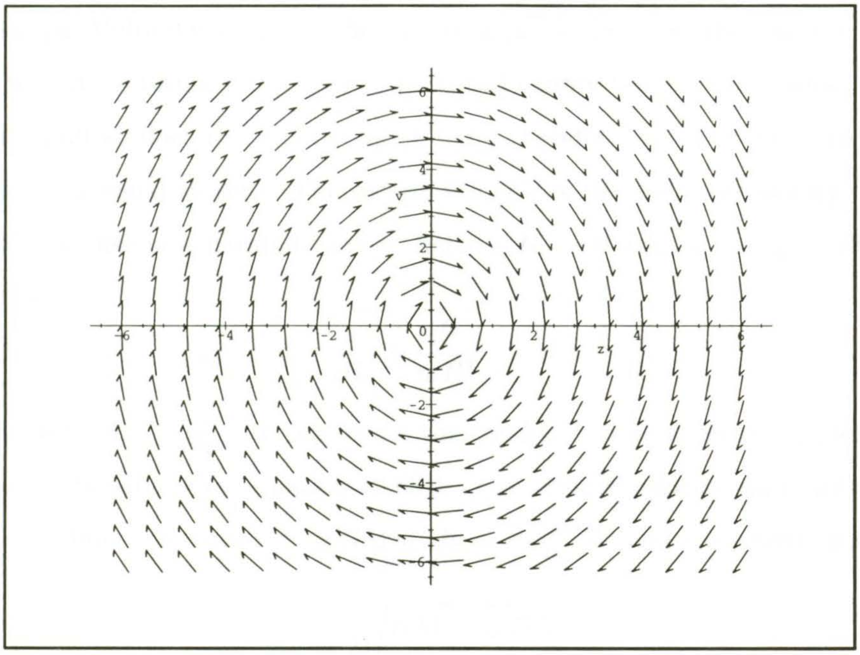

Figure 3.4: The vector field of the linear system (3.18).

From the expression of the Hamiltonian

$$H_0(z,v) = \frac{v^2}{2} - \frac{2}{m}\left[\frac{A}{(R_0^2+z^2)^{\frac{1}{2}}} + \frac{2B}{R_0^2+z^2}\right],$$

we obtain

$$\left(\frac{dz}{dt}\right)^2 = v^2 = 2H_0 + \frac{4}{m}\left[\frac{A}{(R_0^2+z^2)^{1/2}} + \frac{B}{R_0^2+z^2}\right], \tag{3.22}$$

and find the period of the oscillations

$$T = 4\int_0^{z_{max}} \frac{dz}{\sqrt{2H_0 + \frac{4}{m}\left[\frac{A}{(R_0^2+z^2)^{1/2}} + \frac{B}{R_0^2+z^2}\right]}} \quad . \tag{3.23}$$

The factor 4 in front of the integral appears because during a period the particles goes from $z(0) = 0$ to $z(T/4) = z_{max}$, back to $z(T/2) = 0$, than to $z(3T/4) = -z_{max}$

and then once again to $z(T) = 0$.

Escape Velocity (v_{esc}) In the case of a parabolic orbit the particle goes to infinity with a vanishing velocity. The potential energy tends to zero when z tends to infinity and so does the kinetic energy, and therefore in this type of orbit, the total energy is equal to zero. The escape velocity is the value of velocity through $z = 0$ for the case of a parabolic orbit. Setting $H_0 = 0$ and $v_0 = v_{esc}$ in (3.21), it follows that

$$v_{esc} = 2\sqrt{\frac{A}{mR_0} + \frac{B}{mR_0^2}} \qquad A > -\frac{B}{R_0}.$$

The inequality $A > -\frac{B}{R_0}$ also represents the condition that a circular trajectory of radius R_0 is possible (see equation (2.63)). The escape velocity does not depend on m. For Manev potential $A = GMm$, $B = \frac{3G^2M^2m}{2c^2}$, and the above expression becomes

$$v_{esc} = 2\sqrt{\frac{GM}{R_0} + \frac{3G^2M^2}{2c^2R_0^2}}.$$

Relationship between time and the z coordinate for parabolic orbits. Taking $H_0 = 0$ in (3.22), we find

$$v = \frac{dz}{dt} = \frac{2}{\sqrt{m}}\left[\frac{A}{(R_0^2+z^2)^{1/2}} + \frac{B}{R_0^2+z^2}\right]^{1/2} \tag{3.24}$$

from which, by integration, it follows that

$$t = \frac{\sqrt{m}}{2}\int \frac{dz}{\sqrt{\frac{A}{\sqrt{R_0^2+z^2}} + \frac{B}{R_0^2+z^2}}}, \tag{3.25}$$

which can be integrated between any two values of z, namely z_1 and z_2, to give the time interval in which the third massless body covers the distance $|z_2 - z_1|$.

For $R_0 = 1$, $A = 1$ and $B = 1$, we obtain using MATHEMATICA

$$t(z) = \frac{\sqrt{m}}{2}\frac{1}{3z\sqrt{1+z^2}}\sqrt{\frac{1+z^2}{1+\sqrt{1+z^2}}}\left(2z^2c + 3\sqrt{2c}(1+\sqrt{1+z^2})\tan^{-1}\sqrt{\frac{c}{2}}\right) \tag{3.26}$$

with $c = -1 + \sqrt{1+z^2}$. The expression of the integral for any given values of the constant involved is too long to be written here (about two pages).

Nature of equilibria $(-\infty, 0)$ **and** $(\infty, 0)$.

We now want to investigate the equilibria $(-\infty, 0)$ and $(\infty, 0)$. In order to do this we will apply a technique introduced by R. McGehee ([10]). Consider the coordinate transformation

$$\begin{cases} q = \frac{1}{\sqrt{z}} \\ p = v. \end{cases} \tag{3.27}$$

This transformation will map the equilibria $(z \to +\infty, v = 0)$ into the origin of the (q, p) coordinate system. The system in (z, v) coordinates, (3.15),

$$\begin{cases} \dot{z} = v \\ \dot{v} = -\frac{2z}{m}\left[\frac{A}{(R_0^2+z^2)^{\frac{3}{2}}} + \frac{2B}{(R_0^2+z^2)^2}\right] \end{cases}$$

becomes

$$\begin{cases} \dot{q} = -\frac{q^3 p}{2} \\ \dot{p} = -\frac{2q^4}{m}\left[\frac{A}{(R_0^2 q^4+1)^{3/2}} + \frac{2Bq^2}{(R_0^2 q^4+1)^2}\right]. \end{cases} \tag{3.28}$$

The above system is subjected to the condition $q > 0$ and has infinity many, non-isolated, equilibria on the p-axis (see Figure 3.5). The equilibrium point $(z = \infty, v = 0)$ is a degenerate one.

Re-scale time by a factor of q^{-n} with $n \in N, \tau = tq^n$, $d\tau = q^n dt$. The system (3.28) can be rewritten as

$$\begin{cases} \frac{dq}{d\tau} = -\frac{q^3 p}{2} q^{-n} \\ \frac{dp}{d\tau} = -\frac{2q^4}{m}\left[\frac{A}{(R_0^2 q^4+1)^{3/2}} + \frac{2Bq^2}{(R_0^2 q^4+1)^2}\right] q^{-n}, \end{cases} \tag{3.29}$$

and choosing $n = 3$ we obtain a simpler form of (3.29)

$$\begin{cases} \frac{dq}{d\tau} = -\frac{p}{2} \\ \frac{dp}{d\tau} = -\frac{2q}{m}\left[\frac{A}{(R_0^2 q^4+1)^{3/2}} + \frac{2Bq^2}{(R_0^2 q^4+1)^2}\right]. \end{cases} \tag{3.30}$$

We now linearize (3.30) around the equilibrium $(q = 0, p = 0)$. Let $F(q, p) = -\frac{p}{2}$ and $G(q, p) = -\frac{2q}{m}\left[\frac{A}{(R_0^2 q^4+1)^{3/2}} + \frac{2Bq^2}{(R_0^2 q^4+1)^2}\right]$. The matrix $\mathbf{A}$ of the linearized system at the point $(q = 0, p = 0)$ is

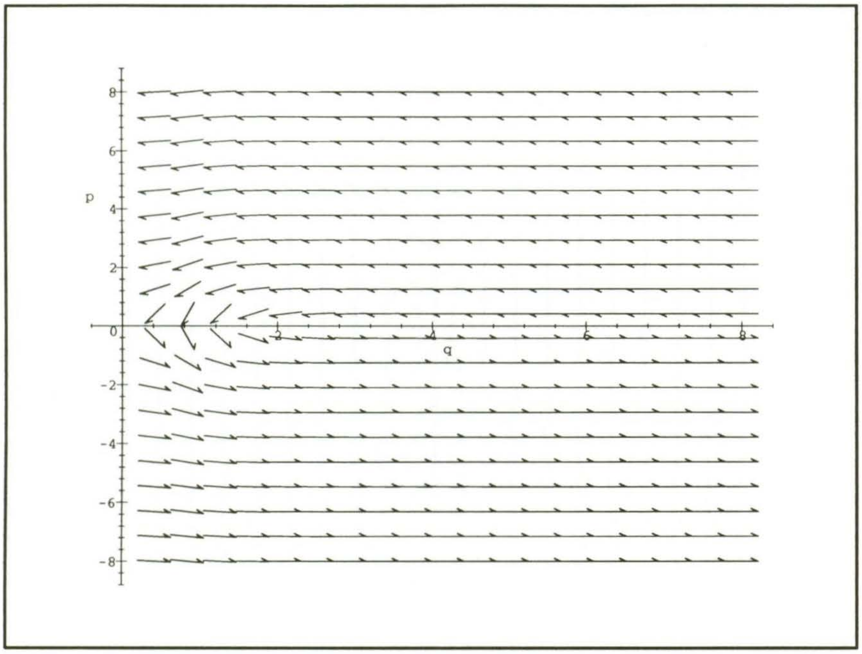

Figure 3.5: The vector field of the system (3.28).

$$\mathbf{A} = \begin{pmatrix} \frac{\partial F(q,p)}{\partial q}\Big|_{(0,0)} & \frac{\partial F(q,p)}{\partial p}\Big|_{(0,0)} \\ \frac{\partial G(q,p)}{\partial q}\Big|_{(0,0)} & \frac{\partial G(q,p)}{\partial p}\Big|_{(0,0)} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2A}{m} & 0 \end{pmatrix}$$

and the linearized system near the equilibrium $(q = 0, p = 0)$ is

$$\begin{cases} \frac{dq}{d\tau} = -\frac{p}{2} \\ \\ \frac{dp}{d\tau} = -\frac{2Aq}{m}. \end{cases} \tag{3.31}$$

The characteristic equation $\lambda^2 - \frac{A}{m} = 0$ yields the eigenvalues $\lambda_1 = \sqrt{\frac{A}{m}} > 0$ and $\lambda_2 = -\sqrt{\frac{A}{m}} < 0$. The equilibrium $(q = 0, p = 0)$ is a saddle for the linearized system

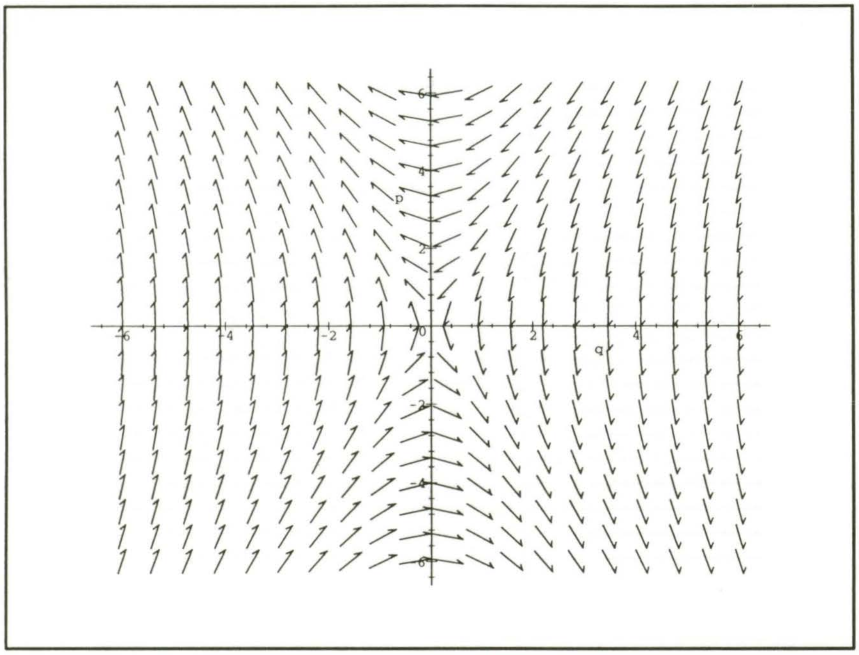

Figure 3.6: The vector field of the linear system (3.31).

(3.31) and this allows us to conclude, based on the Hartman-Grobman Theorem, that this equilibrium point is a hyperbolic saddle for the non-linear system (3.30). The local behavior of the flow in the two cases is shown in Figure 3.6 and Figure 3.7, respectively.

The transformation $(z, v) \rightarrow (q, p)$ maps the origin to infinity and so the unperturbed homoclinic orbit, which passes through $z = 0$, remains non-compact. To remedy this we introduce a new transformation

$$\begin{cases} z = \tan u \\ v = w, \end{cases} \tag{3.32}$$

$u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $w \in \mathbf{R}$, which applied to the system (3.15) transforms it into the system

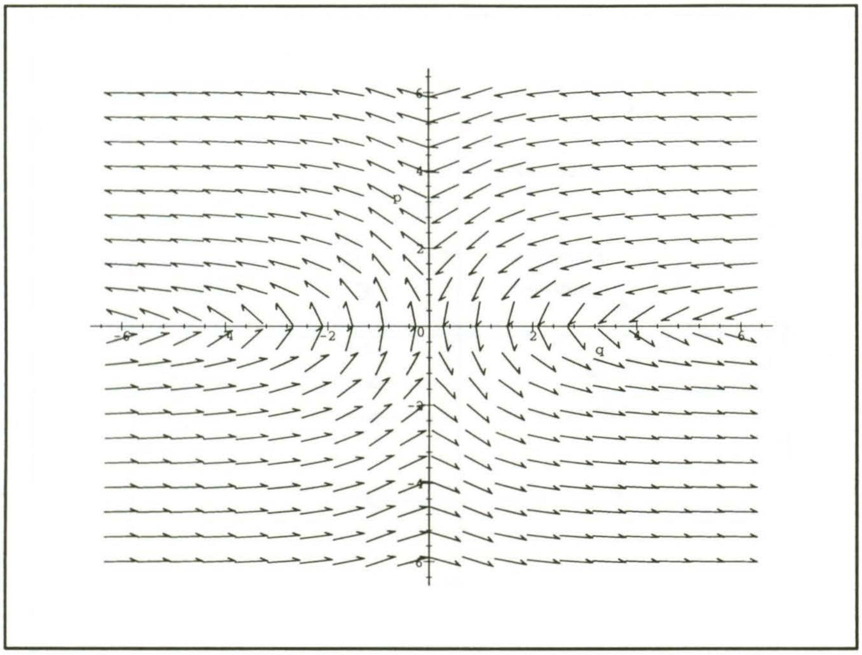

Figure 3.7: The vector field of the non-linear system (3.30).

$$\begin{cases} \dot{u} = w\cos^2 u \\ \dot{w} = -\frac{2}{m}\left[\frac{A\tan u}{(\tan^2 u + R_0^2)^{3/2}} + \frac{2B\tan u}{(\tan^2 u + R_0^2)^2}\right]. \end{cases} \tag{3.33}$$

The above system is not Hamiltonian since $F_u(u,w) \neq -G_w(u,w)$ where $F(u,w) = w\cos^2 u$ and $G(u,w) = -\frac{2}{m}\left[\frac{A\tan u}{(\tan^2 u + R_0^2)^{3/2}} + \frac{2B\tan u}{(\tan^2 u + R_0^2)^2}\right]$, but it has $H_0(u,w)$ as an integral of motion. Applying the transformation (3.32) to the Hamiltonian in (z,v) coordinates given by (3.16), we have

$$H_0(u,w) = \frac{w^2}{2} - \frac{2}{m}\left[\frac{A}{(\tan^2 u + R_0^2)^{1/2}} + \frac{B}{\tan^2 u + R_0^2}\right]. \tag{3.34}$$

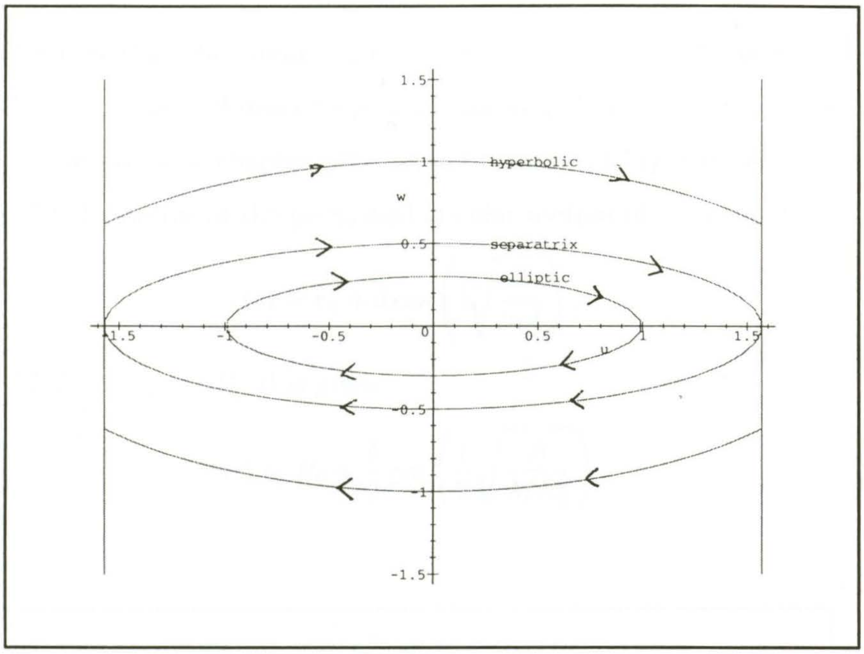

Figure 3.8: The phase portrait of the system (3.33)

The derivative with respect to time of $H_0(u, w)$ is

$$\dot{H}_0 = w\dot{w} + \frac{2\dot{u}\tan u}{m\cos^2 u}\left[\frac{A}{(\tan^2 u + R_0^2)^{3/2}} + \frac{2B}{(\tan^2 u + R_0^2)^2}\right]. \tag{3.35}$$

Substituting in (3.35) the expressions for $\dot{u}$ and $\dot{w}$ given by (3.33) we obtain that $\dot{H}_0(u, w) = 0$, and therefore $H_0(u, w)$ is a constant of motion and the solution curves of the system (3.33) are given by the equations $H_0(u, w)$=constant. The phase portrait of this system is restricted to the region between the lines $u = -\frac{\pi}{2}$ and $u = \frac{\pi}{2}$ and it is shown in Figure 3.8. The vector field is shown in Figure 3.9.

The matrices of the linearized system at $(u = -\frac{\pi}{2}, w = 0)$ and $(u = \frac{\pi}{2}, w = 0)$ are both equal to the (2×2) zero matrix. Thus the equilibrium points $(u = -\frac{\pi}{2}, w = 0)$ and $(u = \frac{\pi}{2}, w = 0)$ are non-hyperbolic saddles.

3.3 The Perturbed Case

Consider now that the circular orbit of the primaries is perturbed by a small amount. The primaries will describe precessional ellipses with small eccentricity as described in the previous chapter. Recall the relation (2.72), that gives the time dependency of the radius of the perturbed circular motion of the primaries

$$r(t) \approx r_0 + \delta \cos\left(t\sqrt{\frac{A}{\mu r_0^3}}\right).$$

With $\mu = M/2$ and $r_0 = 2R_0$ this gives

$$R(t) \approx R_0 + \frac{\delta}{2} \cos\left(\frac{t}{2}\sqrt{\frac{A}{M R_0^3}}\right).$$

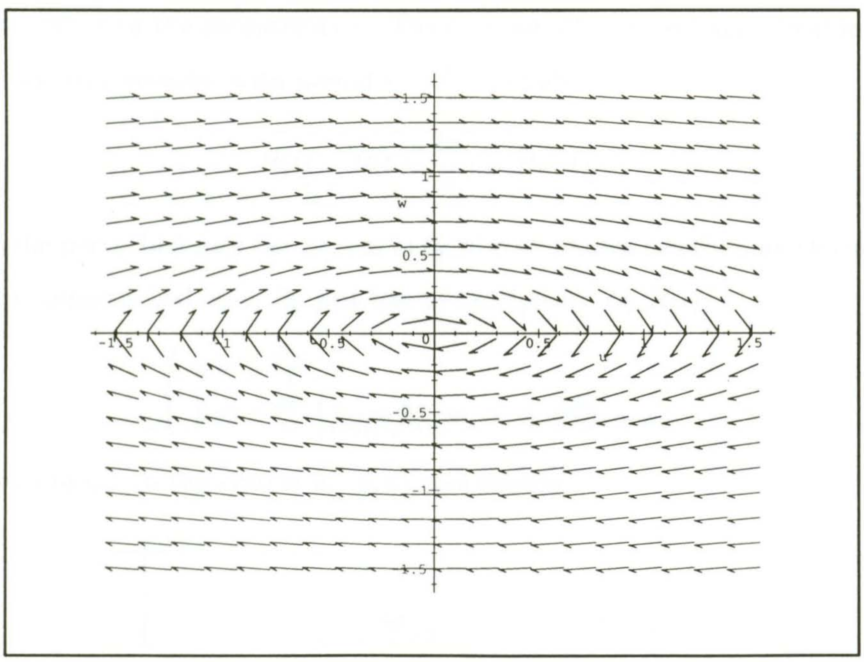

Figure 3.9: The vector field of the system (3.33)

Using the notation $\eta = \frac{1}{2}\sqrt{\frac{A}{MR_0^3}}$, and $0 < \epsilon = \frac{\delta}{2R_0} \ll 1$, we obtain

$$R(t) \approx R_0(1 + \epsilon \cos \eta t). \tag{3.36}$$

From (3.7) it follows that for small eccentricity we have the approximation

$$R(t) = \frac{P}{2}[1 - e\cos\beta(\theta - \theta_0)] + O(e^2).$$

where the terms in e to a power higher than one, are neglected.

From (2.32) and for $e \ll 1$ it follows that $P = a(1 - e^2) \approx a = 2R_0$, and with $\beta = \frac{\omega_{osc}}{\omega_{rotation}}$ and a proper choice for θ_0, we obtain

$$R(t) \approx R_0(1 + e\cos \eta t), \tag{3.37}$$

which is of the same form as (3.36). Comparing the two expressions for $R(t)$ we see that ϵ is related to the eccentricity e. The function $R(t)$ is real analytical in ϵ, and in t, is even and periodic with period $\tau = \frac{2\pi}{\eta}$, namely

$$R(t) = R(t + \frac{2\pi}{\eta}) = R(-t).$$

For the perturbed case the system in (q, p) coordinates has the same form (3.28) as for the unperturbed case, R_0 just has to be replaced by $R(t)$

$$\begin{cases} \dot{q} = -\frac{q^3 p}{2} \\ \dot{p} = -\frac{2q^4}{m}\left[\frac{A}{(R(t)^2q^4+1)^{3/2}} + \frac{2Bq^2}{(R(t)^2q^4+1)^2}\right]. \end{cases} \tag{3.38}$$

The same applies to the system in (u, v) coordinates.

$$\begin{cases} \dot{u} = w\cos^2 u \\ \\ \dot{w} = -\frac{2}{m}\left[\frac{A\tan u}{(\tan^2 u + R(t)^2)^{3/2}} + \frac{2B\tan u}{(\tan^2 u + R(t)^2)^2}\right]. \end{cases} \tag{3.39}$$

The Hamiltonian of the perturbed case can be written as

$$H(u, w) =: H_0(u, w) + \epsilon H_1(u, w)$$

where the subscript 0 stands for the circular, unperturbed case ($\epsilon = 0$).

The equilibrium point ($q = 0, p = 0$) can be seen now as a degenerate periodic solution γ of period $\tau = \frac{2\pi}{\eta}$. According to the McGehee theorem ([10]), the stable and unstable sets of this periodic solution are real analytic manifolds. The set of all the parabolic orbits forms a smooth submanifold of the phase plane. The local stable $W^s_{local}(\gamma)$ and unstable manifolds $W^u_{local}(\gamma)$ are defined as follows

$$W^s_{local}(\gamma) = \{x \in U \,|\, |\Phi_t(x) - \gamma| \to 0 \; as \; t \to \infty, \; and \; \Phi_t(x) \in U \; for \; t \geq 0\},$$

$$W^u_{local}(\gamma) = \{x \in U \,|\, |\Phi_t(x) - \gamma| \to 0 \; as \; t \to -\infty, \; and \; \Phi_t(x) \in U \; for \; t \leq 0\},$$

where U is a neighborhood of γ, and $\Phi_t(x)$ is the flow of the system.

The global stable and unstable manifolds are defined by

$$W^s(\gamma) = \bigcup_{t \leq 0} \Phi_t(W^s_{local}(\gamma)),$$

$$W^u(\gamma) = \bigcup_{t \geq 0} \Phi_t(W^u_{local}(\gamma)).$$

For the unperturbed case, W^s and W^u intersect and their intersection is the escape parabolic orbit. For the perturbed case, choosing a proper initial velocity at $z(t_0) = 0$ the massless body will go to $z \to \infty$ with $v \to 0$ and therefore this orbit is included in W^s. It is included in W^u as well because of the symmetry $(z, v, t) \to (-z, v, -t)$. We conclude that W^s and W^u intersect at $z = 0$, i.e. $q = \infty$. The Melnikov method can be applied to see if the intersection is transversal.

3.4 The Melnikov method

This method is used to detect intersection between the stable and unstable manifolds for time periodic systems in the case that the corresponding unperturbed system has a homoclinic orbit z_0 through a hyperbolic saddle point. The main idea is to make use of the globally computable solution of the unperturbed integrable system in the computation of perturbed solution. The Melnikov method can also be used in the case of a nonhyperbolic saddle point provided that a basic Lemma still holds. This Lemma was introduced by Dankowicz and Holmes in their paper (see [1]). For the system in (u, w) and for Σ^{t_0}, a cross section in the phase plane at $t = t_0$, the following Lemma holds.

3.4.1 Lemma

Lemma.[1] The local stable and unstable manifolds $W^s_{local}(\gamma)$, $W^u_{local}(\gamma)$ of the perturbed orbits are C^∞ close to those of the unperturbed periodic orbits. Orbits $w^s_\epsilon(t, t_0)$, $w^u_\epsilon(t, t_0)$ lying in $W^s_{local}(\gamma)$, $W^u_{local}(\gamma)$ and based on Σ^{t_0} can be expressed as follows, with uniform validity in the indicated time intervals

$$w^s_\epsilon(t, t_0) = w_0(t - t_0) + \epsilon w^s_1(t, t_0) + O(\epsilon^2), \qquad t \in [t_0, \infty),$$

$$w^u_\epsilon(t, t_0) = w_0(t - t_0) + \epsilon w^u_1(t, t_0) + O(\epsilon^2), \qquad t \in (-\infty, t_0],$$

where the vector w equals (u, w) and w^s_1 and w^u_1 are found from the first variational equation of (3.33) along the unperturbed orbit $w_0 = (u_0, w_0)$.

Proof

Consider the system in (q, p) coordinates

$$\begin{cases} \dot{q} = -\frac{q^3 p}{2} \\ \dot{p} = -\frac{2q^4}{m}\left[\frac{A}{(R(t)^2 q^4 + 1)^{3/2}} + \frac{2Bq^2}{(R^2 q^4 + 1)^2}\right] \end{cases} \tag{3.40}$$

and the transformation $(q, p) \to (r, s)$ given by

$$\begin{cases} r = 2\sqrt{\frac{A}{m}}q + p \\ s = 2\sqrt{\frac{A}{m}}q - p. \end{cases} \tag{3.41}$$

We use the above transformation to align the stable manifold along the r axis and the unstable manifold along the s axis. With this transformation, the system (3.40) becomes

$$\begin{cases} \dot{r} + \dot{s} = -\frac{m(r-s)(r+s)^3}{64A} \\ \\ \dot{r} - \dot{s} = -\frac{m(r+s)^4}{64A^2}\left\{A\left[1+\frac{m^2R^2(t)(r+s)^4}{256A^2}\right]^{-3/2} + \frac{mB(r+s)^2}{8A}\left[1+\frac{m^2R^2(t)(r+s)^4}{256A^2}\right]^{-2}\right\}. \end{cases} \tag{3.42}$$

Expanding in Taylor series in variables r, s, and ϵ, we obtain the system

$$\begin{cases} \dot{r} + \dot{s} = -\frac{m(r-s)(r+s)^3}{64A} \\ \\ \dot{r} - \dot{s} = -\frac{m(r+s)^4}{64A} + 0_4 \end{cases} \tag{3.43}$$

with 0_4 a function $f(r, s, \epsilon, t)$ of class C^∞ of period τ in t and such that $f(\lambda r, \lambda s, \epsilon, t)/\lambda^4$ is uniformly bounded in t for $\lambda \to 0$, $\lambda > 0$. The value of B is hidden in the (neglected) higher order terms. Solving (3.43) we obtain the system:

$$\begin{cases} \dot{r} = f_1(r, s, \epsilon, t) = -r\left(\frac{m(r+s)^3}{64A} + 0_4\right) \\ \\ \dot{s} = f_2(r, s, \epsilon, t) = s\left(\frac{m(r+s)^3}{64A} + 0_4\right), \end{cases} \tag{3.44}$$

which is restricted by the condition

$$q = \frac{r+s}{4\sqrt{\frac{A}{m}}} + 0_4 > 0. \tag{3.45}$$

The system (3.44) is the same as the one for the classical Sitnikov problem if we formally set $\frac{A}{m} = \frac{1}{2}$.

In a small neighbourhood of the origin only orbits initially on the r axis will tend to the origin when t tends to infinity, and only orbits initially on the s axis will tend to the origin when t tends to minus infinity. Indeed, if initially $s \neq 0$ then from (3.45) and (3.44) it follows that

$$\frac{\dot{s}}{s} = \frac{m(r+s)^3}{64A} = q^3\sqrt{\frac{A}{m}} + 0_4 > 0,$$

and this implies that if at $t = t_i$, $s(t_i) = s_i$ then $|s(t)| > |s_i|$ for $t > t_i$. If $s_i = 0$ then the equation of motion of a point on the r axis is

$$\dot{r} = -\frac{mr^4}{64A} \tag{3.46}$$

where higher order terms have been neglected. If at $t = t_i$, $r(t_i) = r_i$ then, solving this initial value problem we obtain

$$r(t) = \frac{r_i}{[1 + \frac{3mr_i^3}{64A}(t - t_i)]^{1/3}}, \tag{3.47}$$

and $r(t) \to 0$ as $t \to \infty$. On the other hand, points starting on the s axis will approach the origin in reverse time. For points with $r = 0$, $s(t_i) = s_i$ we have the equation

$$\dot{s} = \frac{ms^4}{64A} \tag{3.48}$$

with the solution

$$s = \frac{s_i}{[1 - \frac{3ms_i^3}{64A}(t - t_i)]^{1/3}}. \tag{3.49}$$

which tends to zero for $t \to -\infty$. Moreover points on the r axis, p and $p + dp_i$, initially separated by $dr_i > 0$ at $t = t_i$, will become closer even if we include the higher order terms. Their distance at a later moment, $t > t_i$, is

$$dr = \frac{dr_i}{[1 + \frac{3mr_i^3}{64A}(t - t_i)]^{4/3}} + O({dr_i}^2) \tag{3.50}$$

and $dr \to 0$ as $t \to \infty$. The rest of the proof goes as in the standard case of application of the Melnikov method as presented by Guckenheimer and Holmes ([6]). □

This allows us to apply the Melnikov theorem as given in the paper of Dankowitz and Holmes ([1]).

3.4.2 The Melnikov Theorem

Theorem 3.3 (Melnikov). *Define the Melnikov function*

$$M(t_0) = \int_{-\infty}^{+\infty} \{\frac{dH_1}{dt}(u_0(t-t_0), w_0(t-t_0), t)\}_{\epsilon=0} dt \tag{3.51}$$

If $M(t_0)$ has simple zeros then, for $\epsilon > 0$ sufficiently small, $W^u(\gamma)$ and $W^s(\gamma)$ intersect transversely.

In order to apply this theorem we need the expression of $\frac{dH_1}{dt}$. Recall that the Hamiltonian for the system in (u, w) coordinates, is

$$H(u,w) = \frac{w^2}{2} - \frac{2}{m}\left[\frac{A}{(\tan^2 u + R(t)^2)^{1/2}} + \frac{B}{\tan^2 u + R(t)^2}\right], \tag{3.52}$$

and $H(u,w) = H_0(u,w) + \epsilon H_1(u,w,\epsilon,t)$. Thus

$$\frac{dH}{dt} = w\dot{w} + \frac{2A}{m}\frac{\tan u \frac{\dot{u}}{\cos^2 u} + R(t)\dot{R}(t)}{(\tan^2 u + R(t)^2)^{3/2}} + \frac{4B}{m}\frac{\tan u \frac{\dot{u}}{\cos^2 u} + R(t)\dot{R}(t)}{(\tan^2 u + R(t)^2)^2}. \tag{3.53}$$

Substituting the expressions of $\dot{w}$ and $\dot{u}$ given by (3.33),

$$\begin{cases} \dot{u} = w\cos^2 u \\ \dot{w} = -\frac{2}{m}\left[\frac{A\tan u}{(\tan^2 u + R_0^2)^{3/2}} + \frac{2B\tan u}{(\tan^2 u + R_0^2)^2}\right], \end{cases}$$

we obtain

$$\frac{dH}{dt} = \frac{2A}{m}\frac{R(t)\dot{R}(t)}{(\tan^2 u + R(t)^2)^{3/2}} + \frac{4B}{m}\frac{R(t)\dot{R}(t)}{(\tan^2 u + R(t)^2)^2}, \tag{3.54}$$

where $R(t) = R_0(1+\epsilon\cos\eta t)$ and $\dot{R}(t) = -\epsilon R_0 \eta \sin \eta t$. In (3.54) there are two functions of the form

$$f(\epsilon) = \frac{R(t)\dot{R}(t)}{(\tan^2 u + R(t)^2)^n} \tag{3.55}$$

where $n = 3/2$ or $n = 2$. Substituting the expressions for $R(t)$ and $\dot{R}(t)$ we obtain

$$f(\epsilon) = -\frac{\epsilon R_0^2 \eta (1+\epsilon\cos\eta t)\sin\eta t}{[\tan^2 u + R_0^2(1+\epsilon\cos\eta t)^2]^n}. \tag{3.56}$$

Expanding in power series in ϵ and neglecting the nonlinear terms we can approximate

$$f(\epsilon) \approx -\frac{\epsilon R_0^2 \eta \sin \eta t}{(\tan^2 u + R_0^2)^n} \tag{3.57}$$

where $n = 3/2$ or $n = 2$. Now we substitute the last expressions in (3.54) and obtain

$$\frac{dH}{dt} = -\frac{2A}{m}\frac{\epsilon R_0^2 \eta \sin \eta t}{(\tan^2 u + R_0^2)^{3/2}} - \frac{4B}{m}\frac{\epsilon R_0^2 \eta \sin \eta t}{(\tan^2 u + R_0^2)^2}. \tag{3.58}$$

On the other hand we have

$$\frac{dH(u,w)}{dt} = \frac{dH_0(u,w)}{dt} + \epsilon \frac{dH_1}{dt}.$$

H_0 is the Hamiltonian in the unperturbed circular case and $\frac{dH_0}{dt} = 0$. It follows that

$$\frac{dH_1}{dt} = -\frac{2A}{m}\frac{R_0^2 \eta \sin \eta t}{(\tan^2 u + R_0^2)^{3/2}} - \frac{4B}{m}\frac{R_0^2 \eta \sin \eta t}{(\tan^2 u + R_0^2)^2}. \tag{3.59}$$

Using the change of variables $t \to t + t_0$, we obtain a form of the Melnikov function which is more convenient in calculations

$$M(t_0) = \int_{-\infty}^{+\infty} \{\frac{dH_1}{dt}(u_0(t), w_0(t), t + t_0)\}_{\epsilon=0} dt. \tag{3.60}$$

From the above and (3.59), we can write the Melnikov function

$$M(t_0) = -R_0^2 \eta \int_{-\infty}^{+\infty} \left[\left(\frac{2A}{m}\right)\frac{\sin \eta(t+t_0)}{(\tan^2 u_0 + R_0^2)^{3/2}} + \left(\frac{4B}{m}\right)\frac{\sin \eta(t+t_0)}{(\tan^2 u_0 + R_0^2)^2}\right] dt \tag{3.61}$$

and from this,

$$M(0) = -R_0^2 \eta \int_{-\infty}^{+\infty} \left[\left(\frac{2A}{m}\right)\frac{\sin \eta t}{(\tan^2 u_0 + R_0^2)^{3/2}} + \left(\frac{4B}{m}\right)\frac{\sin \eta t}{(\tan^2 u_0 + R_0^2)^2}\right] dt. \tag{3.62}$$

The above integrand is odd and therefore $M(0) = 0$. What we now have to see is if this zero is a simple one, i.e. to see if the derivative $M'(0) \neq 0$. Taking the derivative with respect to time of (3.61) we obtain

$$M'(t_0) = -R_0^2 \eta^2 \int_{-\infty}^{+\infty} \left[\left(\frac{2A}{m}\right)\frac{\cos \eta(t+t_0)}{(\tan^2 u_0 + R_0^2)^{3/2}} + \left(\frac{4B}{m}\right)\frac{\cos \eta(t+t_0)}{(\tan^2 u_0 + R_0^2)^2}\right] dt. \tag{3.63}$$

which gives

$$M'(0) = -R_0^2\eta^2 \int_{-\infty}^{+\infty} \left[\left(\frac{2A}{m}\right) \frac{\cos \eta t}{(\tan^2 u_0 + R_0^2)^{3/2}} + \left(\frac{4B}{m}\right) \frac{\cos \eta t}{(\tan^2 u_0 + R_0^2)^2} \right] dt. \tag{3.64}$$

or

$$M'(0) = -R_0^2\eta^2 \int_{-\infty}^{+\infty} \left[\left(\frac{2A}{m}\right) \frac{\cos \eta t}{(z_0^2 + R_0^2)^{3/2}} + \left(\frac{4B}{m}\right) \frac{\cos \eta t}{(z_0^2 + R_0^2)^2} \right] dt. \tag{3.65}$$

Recall the relationship (3.25) between time and z coordinate for parabolic orbits

$$dt = \frac{\sqrt{m}}{2} \frac{dz}{\sqrt{\frac{A}{\sqrt{R^2+z_0^2}} + \frac{B}{R^2+z_0^2}}} \quad ,$$

which substituted in (3.65), gives

$$M'(0) = -\frac{R_0^2\eta^2\sqrt{m}}{2} \int_{-\infty}^{+\infty} \left(\frac{2A}{m}\right) \frac{\cos(\eta t(z_0))}{(z_0^2 + R_0^2)^{3/2}} \sqrt{\frac{z_0^2 + R_0^2}{A\sqrt{z_0^2 + R_0^2} + B}} dz_0$$

$$-\frac{R_0^2\eta^2\sqrt{m}}{2} \int_{-\infty}^{+\infty} \left(\frac{4B}{m}\right) \frac{\cos(\eta t(z_0))}{(z_0^2 + R_0^2)^2} \sqrt{\frac{z_0^2 + R_0^2}{A\sqrt{z_0^2 + R_0^2} + B}} dz_0. \tag{3.66}$$

The integration is done along the parabolic escape trajectory $z(t) = z_0(t)$ for the circular case ($\epsilon = 0$).

Now, let us show that $M'(0) \neq 0$ for same particular case. Setting $R_0 = 1$, $M = 1$, and the values of the constants A and B such that $\frac{2A}{m} = 1$ and $\frac{4B}{m} = 2$ then $\eta = \frac{1}{2}\sqrt{\frac{A}{MR_0^3}} = \frac{1}{2}$ and the integral (3.66) becomes

$$M'(0) = -\frac{\sqrt{2}}{4} \int_{-\infty}^{+\infty} \left[\frac{\cos \frac{1}{2}t(z_0)}{(z_0^2 + 1)^{3/2}} + \frac{2\cos \frac{1}{2}t(z_0)}{(z_0^2 + 1)^2} \right] \sqrt{\frac{z_0^2 + 1}{\sqrt{z_0^2 + 1} + 1}} dz_0, \tag{3.67}$$

where $t(z)$ is given by (3.26)

$$t(z) = \frac{\sqrt{m}}{2} \frac{1}{3z\sqrt{1+z^2}} \sqrt{\frac{1+z^2}{1+\sqrt{1+z^2}}} \left(2z^2c + 3\sqrt{2c}(1+\sqrt{1+z^2}) \tan^{-1}\sqrt{\frac{c}{2}}\right)$$

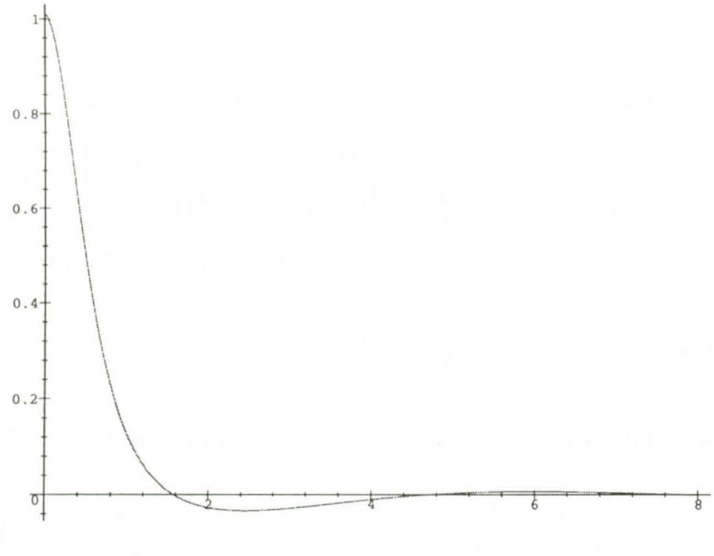

Figure 3.10: Graph of $f(t(z_0))$ versus z_0.

and $c = -1 + \sqrt{1+z^2}$. The integrand

$$f(t(z_0)) = \left[\frac{\cos\frac{1}{2}t(z_0)}{(z_0^2+1)^{3/2}} + \frac{2\cos\frac{1}{2}t(z_0)}{(z_0^2+1)^2}\right]\sqrt{\frac{z_0^2+1}{\sqrt{z_0^2+1}+1}}$$

of the above integral decreases with z_0 (see Figure 3.10), and

$$\lim_{z_0\to\infty}\left[\frac{\cos\frac{1}{2}t(z_0)}{(z_0^2+1)^{3/2}} + \frac{2\cos\frac{1}{2}t(z_0)}{(z_0^2+1)^2}\right]\sqrt{\frac{z_0^2+1}{\sqrt{z_0^2+1}+1}} = 0.$$

The integrand $f(t(z_0))$ is an even function and therefore

$$M'(0) = -\frac{\sqrt{2}}{2}\int_0^{+\infty} f(t(z_0))dz_0. \tag{3.68}$$

Let $\xi_1,\ \xi_2,\ \xi_3, ...\xi_j, ...$ be the zeros of the integrand. These values are solutions of the equation $t(z_0) = (2k+1)\pi$. From $d^2z_0/dt^2 < 0$ we conclude that the sequence

$\xi_j - \xi_{j-1}$ is strictly decreasing with j. On the other hand, consider the alternating sequence,

$$A_1 = \int_0^{\xi_1} \left[\frac{\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^{3/2}} + \frac{2\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^2} \right] \sqrt{\frac{z_0^2+1}{\sqrt{z_0^2+1}+1}} dz_0,$$

$$A_k = \int_{\xi_{k-1}}^{\xi_k} \left[\frac{\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^{3/2}} + \frac{2\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^2} \right] \sqrt{\frac{z_0^2+1}{\sqrt{z_0^2+1}+1}} dz_0, \qquad k \geq 2,$$

with the property $|A_{k+1}| < |A_k|$. Thus

$$M'(0) = \left(-\frac{\sqrt{2}}{4} \right) 2(A_0 + A_1 + A_2 + ... + A_k + ...) \neq 0$$

if $I = A_1 + A_2 > 0$. Using numerical integration methods we have obtained

$$I = \int_0^{\xi_2 = 7.731} \left[\frac{\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^{3/2}} + \frac{2\cos \frac{1}{2} t(z_0)}{(z_0^2+1)^2} \right] \sqrt{\frac{z_0^2+1}{\sqrt{z_0^2+1}+1}} dz_0 \approx 1.371.$$

$I > 0$ for other sets of values of the constants involved. Numerical integrations for discrete values of $B > -0.6A$ yield positive values of I as well. We can conclude the following result

Theorem 3.4.

For some discrete values of the ratio $\frac{B}{A} > -0.6$ and for all but possibly a finite number of values of ϵ in any bounded region, the stable and unstable manifolds $W^s(\gamma)$ and $W^u(\gamma)$ intersect transversally.

The Poincare-Birkhoff-Smale Theorem does not apply in the case of the Sitnikov problem for Manev type potential because of non-hyperbolicity, therefore we cannot conclude the existence of chaos. Nonetheless, Theorem 3.4 shows a complicated behaviour of the orbits.

Chapter 4

Conclusions

The relative trajectories of two bodies that move under the influence of a mutual central force described by an interaction potential of the Manev type, can be precessional ellipses or circles. This happens if their total energy is in an interval $[E_{min}, 0)$ given by (2.38). In these cases the relative distance between the two bodies is a periodic function of time. The relative position vector is a periodic function of time only if the trajectory is closed.

A circular trajectory of radius r_0 is possible only if $Ar_0 + 2B > 0$. The condition for stability for such a circular orbit was obtained considering small deviations from circularity. We have shown that the distance between the two bodies has an harmonic term added to their initial relative distance r_0 and therefore the movement can be seen as a combination of rotations and oscillations.

If both terms in the Manev type potential are of attractive type $(A > 0,\ B > 0)$ then there are more rotation than oscillations. The situation is reverse when the second term is of repulsive type $(B < 0)$.

Circular orbits are stable for all $A > 0$ and for all B. The perturbed trajectories are precessional ellipses of small eccentricity. For these cases, we have found an approximate function of time for the distance between the two bodies (2.72).

The trajectories are closed if the constants A and B, the eccentricity e, and the semi-major axis a satisfy the relation (2.74). For the Sitnikov-Manev problem we have shown that in the case when the primaries are moving on circular trajectories around their common center of mass, the third massless body oscillates with a period that is a function of amplitude (3.23). This is possible if the initial speed is less than the escape velocity. In the perturbed case, primaries are moving on precessional ellipses of small eccentricity and the movement of the third body becomes very complicated. We used Melnikov theory to show the existence of transverse homoclinic orbits.

Bibliography

[1] Dankowitz H., Holmes Ph. (1995) *The existence of transverse homoclinic points in the Sitnikov problem*, Journal of Differential Equations **116**, 468-483.

[2] Delgado J., Diacu F. Lacomba E. , Mingarelli A. , Mioc V., Perez E., Stoica C. (1996) *The global flow of the Manev problem,* Journal of Mathematical Physics, **37**,no 6, 2748-2761.

[3] Diacu F., Holmes Ph. (1996) *Celestial Encounters -The Origins of Chaos and Stability,* Princeton University Press Princeton, New Jersey.

[4] Diacu F., Mioc V., Stoica C. (2000) *Phase-space structure and regularization of Manev-type problems,* Nonlinear Analysis **41**, 1029-1055.

[5] Golstein H. (1980) *Classical Mechanics, 2nd edition,* Addison-Wesley Publishing Company, MA.

[6] Guckenheimer J., Holmes Ph. (1983) *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields,* New York: Springer Verlag.

[7] Konopinski E. J. (1969) *Classical Descriptions of Motion,* W. H. Freeman and Company, San Francisco.

[8] MacMillan W. D. (1913) *An integrable case in the restricted problem of three bodies*, Astronomy Journal, **27**, 11-13.

[9] Maneff M. G. (1924) *La gravitation et le principe de l'egalité de l'action et de la réaction* , Comptes Rendus **178**, 2159-2161.

[10] McGehee R. (1973) *A stable manifold theorem for degenerate fixed points with application to celestial mechanics*, Journal of Differential Equations **14**, 70-78.

[11] Möser J. (1973) *Stable and Random Motions in Dynamical Systems,* Princetown University Press.

[12] Perko L. (1991) *Differential Equations and Dynamical Systems,* New York: Springer Verlag.

[13] Stoica C. (2001). *A proper set of orbital elements and perturbation equations in Maneff-type fields*, preprint

[14] Sitnikov K. (1961) *The existence of oscillatory motion in the three-body problem* , Soviet Physics Dokl. **5**, 647-650.

[15] Szebehely V.G. (1989) *Adventures in Celestial Mechanics-A First Course in the Theory of Orbits,* University of Texas Press, Austin.

[16] Woddnar K. (1993) *The original Sitnikov article-new insights*, Celestial Mechanics and Dynamical Astronomy **56**, 99-101.

[17] Zwillinger D. (1996) *Standard Mathematical Tables and Formulae*, 30th edition, CRC Press.

Vita

Surname:	Verjinschi
Given Names:	Bogdan
Place of Birth:	Bucharest, Romania
Date of Birth:	May, 27 ,1954

Educational Institutions Attended

University of Victoria	1998 to 2003
Universitatea Babes -Bolyai, Cluj, Romania	1974 to 1978

Degrees Awarded

B. Sc. (Nuclear Physics) Universitatea Babes -Bolyai, Cluj, Romania	1978

Honours and Awards

Andy Faquharson Award for Excellence in Graduate Student Teaching	2002

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Author: [redacted]

BOGDAN VERJINSCHI

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