Statistical Methods in Auditing

by

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Abstract

In performing an audit, an auditor typically employs statistical sampling methods. Information from the sample is used to form an estimate of the total error in an audit population. This estimate is often expressed as a one-sided or two-sided confidence bound. By computing a one-sided upper bound for total audit error, an auditor has a certain level of assurance that the total error does not exceed the upper confidence bound. Most research has centered on finding suitable upper bounds for the error in an audit. Several such methods will be reviewed in this report.

A suitable bound is one which is both reliable and efficient. For example, a 95% upper bound is reliable if, when used repeatedly, the bound exceeds the true audit error 95% of the time. Efficiency measures the size of the bound; the smaller the bound is, the more efficient it is said to be. A method which yields reliable, efficient bounds is most suitable for use by auditors in order to avoid costly errors or overauditing.

The results of an extensive simulation study comparing various methods are presented. Both real and simulated data are used for this purpose. In each case measures of reliability and efficiency are provided. The performance of the various methods de-

pend on the distribution of the population. No one method was found to be superior. The multinomial-Dirichlet method of Tsui, Matsamura and Tsui [50] demonstrated the best reliability for a variety of populations. Other Bayesian methods, such as the Bayesian normal bound [31] and the Cox and Snell bound [12] are reliable and more efficient than the multinomial-Dirichlet bound for particular populations.

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$\boldsymbol{y_i}$ book value of the i'th line item

$\boldsymbol{x_i}$ audit value of the i'th line item

$\boldsymbol{e_i = y_i - x_i}$ error of the i'th line item

$\boldsymbol{Y}$ population book amount

$\boldsymbol{X}$ population audit amount

$\boldsymbol{E}$ total error amount

$\boldsymbol{t_i = e_i / y_i}$ taint of the i'th line item

$\boldsymbol{\mu_t}$ true mean taint

$\boldsymbol{\mu_z}$ true mean nonzero taint

$\boldsymbol{\bar{t}}$ sample mean taint

$\boldsymbol{\bar{z}}$ sample mean nonzero taint

$\boldsymbol{\pi}$ error rate; either for line items or dollar-units

$\boldsymbol{m}$ sample number of errors

$\boldsymbol{m'}$ sample number of understatement errors

$\hat{\boldsymbol{X}}_{\boldsymbol{m}}$ mean-per-unit estimator of X

$\boldsymbol{s^2(\hat{X}_m)}$ estimated variance of $\hat{X}_m$

$\hat{\boldsymbol{X}}_{\boldsymbol{ms}}$ mean-per-unit estimator of X based on stratified random sampling

$\boldsymbol{s^2(\hat{X}_{ms})}$ estimated variance of $\hat{X}_{ms}$

$\hat{\boldsymbol{X}}_{\boldsymbol{d}}$ difference estimator of X

$\hat{\boldsymbol{X}}_{\boldsymbol{ds}}$ difference estimator of X based on stratified random sampling

$\hat{\boldsymbol{E}}_{\boldsymbol{d}}$ difference estimator of E

$\boldsymbol{s^2(\hat{E}_d)}$ estimated variance of $\hat{E}_d$

$\hat{\boldsymbol{X}}_{\boldsymbol{r}}$ ratio estimator of X

$\hat{\boldsymbol{X}}_{\boldsymbol{rs}}$ ratio estimator of X based on stratified random sampling

$\hat{\boldsymbol{E}}_{\boldsymbol{r}}$ ratio estimator of E

$\boldsymbol{s^2(\hat{E}_r)}$ estimated variance of $\hat{E}_r$

$\hat{\boldsymbol{X}}_{\boldsymbol{pps}}$ probability-proportional-to-size estimator of X

$\hat{\boldsymbol{E}}_{\boldsymbol{pps}}$ probability-proportional-to-size estimator of E

$\boldsymbol{s^2(\hat{E}_{pps})}$ estimated variance of $\hat{E}_{pps}$

$\hat{\boldsymbol{X}}_{\boldsymbol{w,d}}$ combined mean-per-unit difference estimator of X

$\hat{\boldsymbol{E}}_{\boldsymbol{w,d}}$ combined mean-per-unit difference estimator of E

$\boldsymbol{s^2(\hat{E}_{w,d})}$ estimated variance of $\hat{E}_{w,d}$

$\hat{\boldsymbol{X}}_{\boldsymbol{w,r}}$ combined mean-per-unit ratio estimator of X

$\hat{\boldsymbol{E}}_{\boldsymbol{w,r}}$ combined mean-per-unit ratio estimator of E

$\boldsymbol{s^2(\hat{E}_{w,r})}$ estimated variance of $\hat{E}_{w,r}$

$\hat{\boldsymbol{X}}_{\boldsymbol{m,dus}}$ dollar-unit mean-per-unit estimator of X

$\hat{\boldsymbol{E}}_{\boldsymbol{m,dus}}$ dollar-unit mean-per-unit estimator of E

$\boldsymbol{s^2(\hat{E}_{m,dus})}$ estimated variance of $\hat{E}_{m,dus}$

$\boldsymbol{\pi_u(m; 1-\alpha)}$ $1-\alpha$ upper confidence bound for the population error rate π when m errors are found in the sample

$\boldsymbol{\pi_l(m'; 1-\alpha)}$ $1-\alpha$ lower confidence bound for the population proportion of understatements when m' understatements are found in the sample

$\boldsymbol{z_{(1-\alpha)}}$ $(1-\alpha)100$'th percentile of the standard normal distribution

ST Stringer bound

ST-meik Stringer bound with Meikle's adjustment for understatements

ST-lta Stringer bound with the LTA adjustment for understatements

MM Modified moment bound

BN Bayesian normal bound

CS Cox and Snell bound

CS-lta Cox and Snell bound with the LTA adjustment for understatements

MD Multinomial-Dirichlet bound

MD-lta Multinomial-Dirichlet bound with the LTA adjustment for understatements

PP Parametric power bound

PP-lta Parametric power bound with the LTA adjustment for understatements

CL Clayton's combined bootstrap Hoeffing/bootstrap-t bound

OS overstatement

US understatement

OS100 100% overstatement

p_{OS} proportion of nonzero taints which are overstatements

p_{US} proportion of nonzero taints which are understatements

p_{OS100} proportion of nonzero taints which are 100% overstatements

μ_{OS} mean overstatement taint

μ_{US} mean understatement taint

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Chapter 1

Introduction

An audit includes an examination and check of the internal controls, accounts and financial statements of an organization. This allows the auditor to express an opinion on the validity of the financial statements, i.e. whether the statements fairly reflect the organization's financial position. Such financial information is required by banks, government departments and investors to make decisions concerning loans, taxes and investments.

Checking the balances of certain accounting populations, such as accounts receivable and inventory, is an essential step in the audit process. An audit population consists of two types of elements: those containing errors of varying amounts and those which are error free. One objective of an audit is to provide a statistical bound for the total error amount in the population. By computing an upper confidence bound for the total error, an auditor has a certain assurance that the total error does not exceed the computed bound.

Because large organizations process such an enormous volume of financial transactions, it is far too costly and time-consuming for an auditor to examine each individual transaction in order to determine the degree of error. By employing statistical sampling, the number of transactions which are actually audited or checked is greatly reduced. The end result of an audit is the expression of the auditor's *opinion* regarding the accuracy of the financial statements. To express his opinion, the auditor does not require absolute assurance that his findings are accurate. Thus every transaction need not be examined. Statistical sampling methods which provide an acceptable degree of assurance will suffice.

A financial statement is considered to be *materially in error* if knowledge of the error would reasonably affect a decision of a reader of the statement, i.e. the correction of the error would yield a different opinion on the part of the reader. Prior to performing an audit, an auditor determines the *level of materiality*, the largest amount of error which will be tolerated. Upon completion of an audit, an auditor must decide between two conclusions: (1) the financial statements present fairly or (2) the financial statements are materially in error. Unless all the transactions are examined and correctly assessed there is some risk that the auditor will come to an incorrect conclusion.

If an auditor concludes that a financial statement could be materially in error, he must request that the client adjust its figures or perform additional audit procedures to confirm his conclusion. Falsely concluding that a material error exists can be costly to the client and the auditor.

Alternatively, the auditor could wrongly conclude that a financial statement is correct. In this situation, the error may not be detected before the incorrect statements are released. This can be costly for both the auditor and the client and possibly those who read the false statement.

The amount of error in a stated balance is often assessed via a confidence bound. Depending on the situation, either an upper or lower bound is of interest. If the objective is to determine whether a firm's actual assets are materially smaller than as stated, an upper bound for the total error is calculated. If this upper bound exceeds the level of materiality, the assets are considered to be misstated. If an individual is being audited for income tax purposes, the objective is to determine whether the individual's income is larger than stated. A lower bound for the error in the stated income may be used for adjustment purposes. In order to avoid costly errors, an auditor requires reliable methods for forming such confidence bounds.

Most research has centered on finding suitable upper bounds for the error in an audit. As is explained in this report, traditional methods are often inappropriate for application to auditing due to characteristics of the audit population. Many alternative methods for finding an upper bound for the error in an audit are reviewed. Following the presentation of the methods, the results of an extensive simulation analysis are presented. Several proposed methods are compared using both real and simulated audit data in order to determine which methods are suitable for use by auditors in different circumstances. Though this is not the first time such a simulation has been performed, none has been quite as extensive both in terms of the number of methods included and the variety of populations studied.

Chapter 2

Early Methods

2.1 Terminology

The first known research concerning statistical sampling methods in auditing was published in 1933. Lewis A. Carman [8], in his paper entitled "The Efficacy of Tests", demonstrated the possible application of statistics to the field of auditing. It was nine years before another article appeared on the subject. In 1942 *The Journal of Accountancy* published an article by Robert H. Prytherch [43]. Prytherch's research carried on from that done by Carman. He looked at auditing materiality, types of errors (sampling and non-sampling) and the necessity of examining all large items in an audit population. This article seemed to stimulate thought on the subject, resulting in an increased number of articles and books.

Lawrence L. Vance [52] wrote the first book on the topic. *Scientific Method for*

Auditing: Application of Statistical Sampling Theory to Auditing Procedures was published in 1950. To this point the literature dealt primarily with sampling in which the characteristics of interest were attributes [37]. *Attributes sampling* is a statistical method used to estimate the proportion of items in a population containing a characteristic or attribute of interest [2]. This unknown proportion is called the *occurrence rate*. For example, attributes sampling could be applied to estimate the occurrence rate of errors in a population of accounts receivable balances. The estimated occurrence rate is the ratio of the number of balances in the sample having errors to the total number of balances. The estimate is commonly expressed as an interval, providing a range of probable values for the occurrence rate.

On the other hand, when a quantitative characteristic is of interest, such as a dollar amount, the procedure is called *sampling by variables*. If, for example, an auditor wishes to estimate the total value of a company's inventory, variables sampling would apply. Several methods fall into this category and they will be explained further in this review.

Although sampling by attributes posed few statistical difficulties, the usefulness of this technique was questioned. Consider, for example, that an auditor estimates the error rate for a population of accounts. He could then determine an upper limit for the error rate but this does not provide any information as to the seriousness of the error in terms of monetary value. For this reason the direction of research changed. Variables sampling seemed to offer more useful conclusions.

Vance worked with John Neter to produce the first book on sampling by variables in auditing [53]. *Statistical Sampling for Auditors and Accountants* was published in 1956. The relevant theory and terminology from this text will be introduced via an example.

Consider an accounts receivable population which consists of N individual accounts or *line items.* The auditor wishes to judge the accuracy of the total of the recorded line items. An error in the statement of this total is considered *material* if knowledge of the misstatement would reasonably affect a decision of a reader of the statement. The auditor must judge the size of an error to determine if it is in fact a material error and consequently whether to reject the population total as being correct.

The recorded values of the line items are called the *book amounts.* The auditor takes a sample, of size n, of the line items and for each of these determines the correct value, called the *audit amount.* Let y_i denote the book amount and x_i the audit amount of the i'th line item. For the sample, the *errors* of the individual line items can be calculated by $e_i = y_i - x_i$.

Using all the book values in the population, the book balance, called the *population book amount*, can be determined as $Y = \sum_{i=1}^{N} y_i$. If audit values were available for the entire population of line items the audit balance, called the *population audit amount* could be found through $X = \sum_{i=1}^{N} x_i$, where N represents the total number of line items. The *total error amount*, the error of the book balance, is $E = Y - X = \sum_{i=1}^{N} e_i$.

Because x_i, the audit amount, is determined only for a sample of n line items, X and E are unknown.

The auditor wishes to judge the accuracy of Y; this can be done in two ways. X can be estimated, using the sample of x_i, and compared with Y, or E can be estimated, from the sample of e_i. Whether this value is sufficiently small as to be immaterial can then be determined.

2.2 Classical Bounds

Each of the estimators described in this section involve both a point estimate and an associated standard error of the estimate. The two components of the estimator can be combined to form an interval estimate based on large sample normal theory.

A two-sided confidence interval for X is given by

$$\hat{X} \pm z_{(1-\alpha/2)} s(\hat{X}), \tag{2.1}$$

where $\hat{X}$ is a point estimate of X and $s(\hat{X})$ is the estimated standard error of $\hat{X}$. $z_{(1-\alpha/2)}$ is the $(1-\alpha/2)100$'th percentile of the standard normal distribution. Similarly, the one-sided lower confidence limit

$$\hat{X} - z_{(1-\alpha)} s(\hat{X}) \tag{2.2}$$

or the one-sided upper confidence limit

$$\hat{X} + z_{(1-\alpha)} s(\hat{X}) \tag{2.3}$$

can be constructed.

The *mean-per-unit (mpu) estimator* of X is a straight expansion of the sample total audit amount,

$$\hat{X}_m = \frac{N}{n}\sum_{i=1}^{n} x_i = N\overline{x}, \tag{2.4}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. For a simple random sample chosen without replacement, the estimated variance of the mean-per-unit estimator is

$$s^2(\hat{X}_m) = \frac{N^2(N-n)}{nN(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2. \tag{2.5}$$

The mean-per-unit estimator is *unbiased* when random sampling is performed without replacement but it is *imprecise* due to the large variability in most audit populations. An estimator $\hat{X}$ is unbiased if its expected value is the true value X. An estimator is imprecise if it has a large standard error. The concept of precision is most easily understood in terms of confidence intervals. The larger the standard error of an estimate, the wider the confidence interval. For a given confidence level, a smaller interval actually reveals better information than a larger interval since the range of probable values is narrower. The estimated standard error of the mean-per-unit estimator is based on the audit amounts and, as a result, may be very large. The size of the standard error decreases as the sample size increases. To obtain precise estimates using this method, large sample sizes are required.

This problem can be alleviated to some degree by employing *stratified random sampling.* To obtain a stratified random sample the population is divided into nonoverlapping subpopulations, called strata, and then a separate random sample of line items is selected from each stratum. The appropriate statistic is computed for each sample and these values are properly weighted to form an overall estimate. The sample standard errors are combined in the same manner. If the stratification is done in such a manner that the measurements in a particular stratum are fairly homogeneous, then the overall error bound will be smaller than if a simple random sampling technique were used.

The mean-per-unit estimator using stratified random sampling based on book amount is

$$\hat{X}_{ms} = \sum_{h=1}^{L} N_h \overline{x}_h, \tag{2.6}$$

where L is the number of strata, N_h is the number of population elements in the h'th stratum and $\overline{x}_h$ is the sample mean audit amount for the h'th stratum. The estimated variance of this estimator is

$$s^2(\hat{X}_{ms}) = \sum_{h=1}^{L} \frac{N_h^2(N_h - n_h)}{N_h n_h} s_h^2, \tag{2.7}$$

where s_h^2 is the sample variance for the h'th stratum.

Although stratification can alleviate the imprecision problem to some degree, there is another fundamental difficulty with this estimator. Since accounting populations often have low error rates, it is quite common for a sample to contain no errors. For this situation it is fairly likely that $\hat{X}$ will differ from Y since $\hat{X}$ is based only on the sample. Thus, even if no errors are found in the sample, use of the mean-per-unit estimator may indicate an error in the total book amount.

This problem led to the estimation of the audit error (E) from the sample of line item errors (e_i). As $e_i = y_i - x_i$, auxiliary information is incorporated in the sample, and so precision can be increased. *Techniques in Accounting* by Robert M. Trueblood and Richard M. Cyert [50] presented techniques for this method. Several estimators have been proposed by these authors and others.

The following *auxiliary information estimators* are commonly used: the *difference estimator*

$$\hat{E}_d = \frac{N}{n}\sum_{i=1}^{n}(y_i - x_i) = \frac{N}{n}\sum_{i=1}^{n} e_i = N\overline{e}; \tag{2.8}$$

the *ratio estimator*

$$\hat{E}_r = Y\left(\frac{\sum_{i=1}^{n} e_i}{\sum_{i=1}^{n} y_i}\right) = Y\left(\frac{\overline{e}}{\overline{y}}\right); \tag{2.9}$$

and the *estimator with probability-proportional-to-size (pps) sampling*

$$\hat{E}_{pps} = \frac{1}{n}\sum_{i=1}^{n}\frac{e_i}{y_i/Y} = Y\sum_{i=1}^{n}\frac{e_i/y_i}{n}. \tag{2.10}$$

Based on simple random sampling without replacement, the estimated variances of the difference and ratio estimators are

$$s^2(\hat{E}_d) = \frac{N^2(N-n)}{Nn(n-1)}\sum_{i=1}^{n}(e_i - \overline{e})^2 \tag{2.11}$$

and

$$s^2(\hat{E}_r) = \frac{N^2(N-n)}{Nn(n-1)}\sum_{i=1}^{n}\left(e_i - \frac{\overline{e}}{\overline{y}}y_i\right)^2. \tag{2.12}$$

Based on probability–proportional–to–size sampling, the estimated variance of the pps estimator is

$$s^2(\hat{E}_{pps}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \left(\frac{e_i}{y_i/Y} - \hat{E}_{pps} \right)^2 . \tag{2.13}$$

For each of the above estimators, the two-sided confidence interval

$$\hat{E} \pm z_{(1-\alpha/2)} s(\hat{E}), \tag{2.14}$$

the one-sided lower confidence limit

$$\hat{E} - z_{(1-\alpha)} s(\hat{E}) \tag{2.15}$$

or the one-sided upper confidence limit

$$\hat{E} + z_{(1-\alpha)} s(\hat{E}) \tag{2.16}$$

can be constructed; where $z_{(1-\alpha)}$ is the $(1-\alpha)100$'th percentile of the standard normal distribution.

Consider again the accounts receivable example in which the auditor wishes to judge the accuracy of the book total by estimating the total error amount and determining if it is small enough to be acceptable as immaterial.

In difference estimation, the population total error amount is estimated by an expanded sample total error amount. Allowing for the fact that the sample consists of the errors (e_i) rather than the audit amounts (x_i), the difference estimator is calculated in exactly the same manner as the mean-per-unit estimator (2.4). This technique has an intuitive appeal and is relatively easy to use. For these reasons, difference estimation is employed extensively by auditors.

Ratio estimation involves multiplying the proportion of sample dollars in error by the total recorded book value.

In sampling with probability–proportional–to–size, larger account balances have a greater probability of being selected in the sample than smaller account balances. This appeals to an auditor since larger accounts can potentially contain larger errors. The probability–proportional–to–size estimator is essentially the average proportion of error per book amount multiplied by the total book amount. This estimator will later be shown to be related to another type of sampling called dollar-unit sampling.

Probability–proportional–to–size sampling involves sampling with replacement, meaning the same item can appear more than once in a sample. This method provides an unbiased estimate of the total error amount, E. When random sampling is performed without replacement, the difference estimator provides an unbiased estimate of the total error amount. The ratio estimator is slightly biased, but *consistent*, meaning it is equivalent to the population value E when $n = N$. The bias can be shown to be no greater than the reciprocal of the sample size. Wilburn [55] advises that if the sample size is at least 100, one can essentially ignore the bias.

It is possible to combine the mean-per-unit estimator with either the ratio estimator, the difference estimator or both. Consider $\hat{E}_m = Y - \hat{X}_m$. Then a weighted average of $\hat{E}_m$ and $\hat{E}_d$ or $\hat{E}_r$ can be used to estimate the total error. The estimators take the following form

$$\hat{E}_{w,d} = w\hat{E}_m + (1-w)\hat{E}_d \tag{2.17}$$

$$\hat{E}_{w,r} = w\hat{E}_m + (1-w)\hat{E}_r \tag{2.18}$$

where $0 \leq w \leq 1$ and are denoted by combined mpu-difference estimator and combined mpu-ratio estimator, respectively.

The estimated variances of these estimators are

$$\begin{aligned} s^2(\hat{E}_{w,d}) &= \frac{N^2(N-n)}{Nn(n-1)}\left[\sum_{i=1}^{n}(x_i-\overline{x})^2 + (1-w)^2\sum_{i=1}^{n}(y_i-\overline{y})^2 \right. \\ &\qquad \left. -2(1-w)\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})\right] \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} s^2(\hat{E}_{w,r}) &= \frac{N^2(N-n)}{Nn(n-1)}\left[\sum_{i=1}^{n}(x_i-\overline{x})^2 + (1-w)^2\frac{\overline{x}}{\overline{y}}\sum_{i=1}^{n}(y_i-\overline{y})^2 \right. \\ &\qquad \left. -2(1-w)\frac{\overline{x}}{\overline{y}}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})\right]. \end{aligned} \tag{2.20}$$

Aside from the mean-per-unit estimator, all the estimators presented so far are for the total error amount E. Each of these estimators can lead to an estimate of X, the total audit amount, through the following equations:

$$\hat{X}_d = Y - \hat{E}_d = Y - n\overline{e} \tag{2.21}$$

$$\hat{X}_r = Y - \hat{E}_r = Y\left(\frac{\overline{x}}{\overline{y}}\right) \tag{2.22}$$

$$\hat{X}_{pps} = Y - \hat{E}_{pps} = \frac{Y}{n}\sum_{i=1}^{n}\left(\frac{x_i}{y_i}\right) \tag{2.23}$$

$$\hat{X}_{w,d} = Y - \hat{E}_{w,d} = w\hat{X}_m + (1-w)\hat{X}_d \tag{2.24}$$

$$\hat{X}_{w,r} = Y - \hat{E}_{w,r} = w\hat{X}_m + (1-w)\hat{X}_r. \tag{2.25}$$

2.2.1 Empirical Study of Classical Estimators

An extensive study by Neter and Loebbecke ([33] & [34]) revealed the inadequacies of the above estimators for populations with low error rates. The study populations in the experiment were constructed from real audit data, representing two areas: accounts receivable and inventory. Below are the brief descriptions of the populations provided by Neter and Loebbecke. A more detailed explanation can be found in Chapter 7.

Population 1 Accounts receivable of a freight company. The actual error rate is high, and the errors tend to be balanced between overstatements and understatements. The size of errors is small.

Population 2 Inventory of a medium-size manufacturer. The actual error rate is very high. Both overstatement and understatement errors are present, with understatement errors outweighing overstatement errors. The size of errors is large.

Population 3 Accounts receivable of a medium-size manufacturer. The actual error rate is moderate, with all errors being overstatements. The size of errors is moderate.

Population 4 Accounts receivable of a large manufacturer. The actual error rate is moderate, with all errors being overstatements. The size of errors is large.

The populations differed according to number of accounts, book values, error values and error direction, i.e. overstatements only or overstatements and understatements. Neter and Loebbecke [33] created several study populations from each of the four populations reflecting a variety of error rates. To form a study population with a particular error rate, an appropriate number of line items were selected at random to contain an error. Error values were estimated from evidence obtained in an actual audit. The audit values were obtained by combining the book values and the errors. This process is described extensively in Section 7.3. From each study population, random samples of size 100 and 200 were drawn 600 times.

For the combined estimators a weight of $w = .1$ was utilized. The mean-per-unit, difference and ratio estimators with stratified sampling, denoted $\hat{X}_{ms}$, $\hat{X}_{ds}$ and $\hat{X}_{rs}$, were also included in the study. Twenty strata, based on book amounts, were employed. The Dalenius-Hodges [13] procedure was used to determine the strata boundaries. The total sample size was allocated to the strata optimally according to the Neyman allocation [11]. Consider the mean-per-unit estimator based on stratified random sampling (2.6). The precision of the estimator depends on the sample sizes n_h. Neyman allocation distributes the sample sizes to the strata in such a way as to minimize the variance of the estimator. The n_h's are selected so that they are proportional to $N_h \sigma_h$, where σ_h^2 is the variance of the h'th stratum.

For each estimator, the reliability of the nominal confidence coefficient $(1 - \alpha)$ was assessed through an examination of the associated two-sided confidence interval and the upper and lower one-sided bounds. An estimation method is considered reliable if the actual level of coverage of the interval is at least as large as the nominal confidence coefficient, which is the assumed level of coverage. The achieved coverage is simply the percentage of times a particular bound covers the true audit error E for the 600 samples. If, for example, a 95% confidence interval for E is found to contain the true value only 85% of the time, the interval is unreliable. The relative standard errors were calculated to judge the precision of the estimators. The results for sample size $n = 100$ are reproduced in Table 2.1 and Table 2.2. For each population the skewness of the audit amounts and the sign of the errors are given in parenthesis.

Consider first the coverage of the two-sided intervals for a nominal 95.4 percent confidence coefficient as illustrated in Table 2.1. For low error rates (0.5 or 1.0 percent) the actual percentage of intervals with correct coverage was well below the nominal confidence level among the auxiliary estimators. For higher error rates (5, 10 and 30 percent) the reliability of the nominal confidence coefficient varied among the different populations. The actual coverage was fairly close to the nominal amount for populations 1 and 2 which had both understatement and overstatement errors.

The mean-per-unit estimator performed well in comparison with the auxiliary estimators. The level of coverage did not seem to be affected by the population error percentage; however this coverage was affected by the skewness. Stratification improved the reliability of the nominal confidence coefficient for this estimator. In fact, among the estimators examined, the mean-per-unit estimator with stratification

	Error Percentage						Error Percentage				
	.5	1	5	10	30		.5	1	5	10	70/30
	Population 1 (22, +/-)[1]						Population 2 (3.5, +/-)				
$\hat{X}_m$	81.8	81.8	-[2]	81.7	81.7	$\hat{X}_m$	93.7	93.7	93.7	93.7	93.0
$\hat{X}_{ms}$	95.3	95.0	94.3	93.8	95.2	$\hat{X}_{ms}$	96.0	96.8	96.5	97.3	95.8
$\hat{X}_d$	30.5	37.3	96.8	94.0	96.3	$\hat{X}_d$	31.2	41.8	82.3	97.2	95.5
$\hat{X}_r$	29.8	36.2	94.3	91.5	92.8	$\hat{X}_r$	30.7	41.7	82.2	96.5	94.2
$\hat{X}_{pps}$	17.5	49.7	-	90.3	92.2	$\hat{X}_{pps}$	-	-	80.2	94.5	-
$\hat{X}_{ds}$	22.5	51.8	95.3	94.0	95.0	$\hat{X}_{ds}$	21.3	47.7	83.3	96.8	95.3
$\hat{X}_{rs}$	22.8	52.2	97.0	96.2	97.8	$\hat{X}_{rs}$	23.0	49.8	86.8	98.0	97.8
$\hat{X}_{w,d}$	82.3	82.0	-	83.3	84.3	$\hat{X}_{w,d}$	93.8	94.2	94.3	94.5	93.8
$\hat{X}_{w,r}$	82.3	82.0	-	83.8	85.0	$\hat{X}_{w,r}$	93.8	93.8	94.2	94.5	93.8
	Population 3 (7.9, +)						Population 4 (3.2, +)				
$\hat{X}_m$	82.5	82.5	82.5	82.5	82.5	$\hat{X}_m$	92.7	92.7	92.3	92.8	93.2
$\hat{X}_{ms}$	96.7	96.8	96.3	96.3	94.8	$\hat{X}_{ms}$	95.8	95.8	91.8	89.2	94.2
$\hat{X}_d$	23.3	36.8	73.7	80.3	90.8	$\hat{X}_d$	21.2	30.0	58.2	62.0	74.8
$\hat{X}_r$	24.2	35.2	72.8	79.5	86.2	$\hat{X}_r$	21.0	31.0	59.3	63.3	76.0
$\hat{X}_{pps}$	5.2	-	31.5	44.8	77.0	$\hat{X}_{pps}$	30.7	-	69.7	87.0	95.5
$\hat{X}_{ds}$	7.5	15.7	46.2	60.7	88.7	$\hat{X}_{ds}$	31.3	41.2	71.8	85.8	93.2
$\hat{X}_{rs}$	7.5	15.7	46.7	62.3	91.3	$\hat{X}_{rs}$	31.3	42.8	73.0	87.7	95.7
$\hat{X}_{w,d}$	82.7	82.5	83.0	82.8	82.3	$\hat{X}_{w,d}$	93.7	94.0	94.7	94.8	78.7
$\hat{X}_{w,r}$	82.7	82.5	83.0	82.8	82.5	$\hat{X}_{w,r}$	93.7	94.0	94.7	93.7	81.8

[1] (skewness, error direction)

[2] indicates that the estimator was not computed for the particular study population

Table 2.1: Actual Coverages (in %) for Nominal 95.4 Percent Confidence Intervals (Source: Neter, J., Loebbecke, J. (1975))

performed best in this area. Stratification did not lead to significant improvements in reliability for the difference and ratio estimators.

In general, a similar pattern of problems resulted when the one-sided confidence bounds with a nominal 93.3 percent confidence coefficient were examined. The actual coverage of the two-sided confidence interval and the one-sided bounds was examined for several other values of the nominal confidence coefficient. The results were similar to those listed above.

The relative standard errors, reported in Table 2.2, were calculated for each estimator in order to judge precision. The relative standard error is the standard deviation of the estimator over the random samples as a percentage of the population total audit amount (X). From the results, it is evident that high reliability of the nominal confidence coefficient and good precision do not necessarily go hand in hand. In fact, the ratio and difference estimators, which performed poorly with respect to nominal confidence level, had the smallest relative standard errors.

Since the variance of the population of errors is unknown it is necessary to use the sample variance as an estimate. In many applications this is believed to provide a good estimate, but this is not necessarily true in an audit population with rare errors. The problem was put concisely in an article by Anderson and Teitlebaum [1]

> "To assume that the variability of the two or three small errors discovered is typical of the population is precisely to assume that there are no large errors in the population — which is assuming at the start exactly

	Error Percentage						Error Percentage				
	.5	1	5	10	30		.5	1	5	10	70/30
	Population 1 (22, +/-)[1]						Population 2 (3.5, +/-)				
$\hat{X}_m$	24.1	24.0	-[2]	24.0	23.9	$\hat{X}_m$	18.2	18.2	18.3	18.2	18.3
$\hat{X}_{ms}$	0.9	0.9	1.1	1.2	1.4	$\hat{X}_{ms}$	0.7	0.8	1.1	1.3	2.5
$\hat{X}_d$	0.1	0.2	0.4	0.6	0.9	$\hat{X}_d$	0.2	0.4	1.0	1.3	3.0
$\hat{X}_r$	0.1	0.2	0.4	0.6	0.9	$\hat{X}_r$	0.2	0.4	1.0	1.3	3.1
$\hat{X}_{pps}$	0.2	0.2	-	0.6	1.0	$\hat{X}_{pps}$	-	-	1.0	1.5	-
$\hat{X}_{ds}$	0.2	0.2	0.5	0.7	1.1	$\hat{X}_{ds}$	0.1	0.3	0.9	1.1	2.4
$\hat{X}_{rs}$	0.2	0.2	0.5	0.7	1.1	$\hat{X}_{rs}$	0.1	0.3	0.9	1.1	2.4
$\hat{X}_{w,d}$	2.4	2.4	-	2.5	2.5	$\hat{X}_{w,d}$	1.8	1.9	2.1	2.2	3.6
$\hat{X}_{w,r}$	2.4	2,4	-	2.4	2.5	$\hat{X}_{w,r}$	1.8	1.9	2.1	2.2	3.5
	Population 3 (7.9, +)						Population 4 (3.2, +)				
$\hat{X}_m$	35.7	35.7	35.7	35.8	36.1	$\hat{X}_m$	20.2	20.2	20.4	20.7	21.4
$\hat{X}_{ms}$	0.9	0.9	0.9	1.0	1.1	$\hat{X}_{ms}$	0.9	0.9	1.2	1.9	3.9
$\hat{X}_d$	0.1	0.1	0.1	0.2	0.4	$\hat{X}_d$	1.1	1.1	1.6	3.5	9.2
$\hat{X}_r$	0.1	0.1	0.2	0.3	0.7	$\hat{X}_r$	1.1	1.1	1.6	3.5	8.4
$\hat{X}_{pps}$	0.1	-	0.3	0.5	0.9	$\hat{X}_{pps}$	0.5	-	1.0	1.8	3.9
$\hat{X}_{ds}$	0.1	0.1	0.2	0.3	0.6	$\hat{X}_{ds}$	0.6	0.7	1.0	1.8	3.9
$\hat{X}_{rs}$	0.1	0.1	0.2	0.3	0.6	$\hat{X}_{rs}$	0.6	0.7	1.0	1.8	3.9
$\hat{X}_{w,d}$	3.6	3.6	3.6	3.6	3.6	$\hat{X}_{w,d}$	2.2	2.3	2.5	3.8	8.5
$\hat{X}_{w,r}$	3.6	3.6	3.6	3.7	4.1	$\hat{X}_{w,r}$	2.3	2.3	2.7	4.2	8.5

[1] (skewness, error direction)

[2] indicates that the estimator was not computed for the particular study population

Table 2.2: Relative Standard Errors of Estimators in Terms of the Percentage of the Total Audit Amount (Source: Neter, J., Loebbecke, J. (1975))

what the auditor should be setting out to prove."

If in fact no errors are found in the sample, then $\hat{E} = 0$ and the estimated standard error of this estimate is 0. This would suggest no variability in the estimate, i.e. perfect precision, but this interpretation is invalid because it is based on one sample and from this we cannot conclude that the same holds true for the population.

The study by Neter and Loebbecke provided conclusive evidence that commonly used estimators are not appropriate for auditing populations with low error rates. The mean-per-unit estimator is imprecise due to the large variation in the line item audit amounts and the confidence intervals and bounds based on the asymptotic normality of the auxiliary information estimators fail to ensure the desired level of coverage. Additionally, in the case of error-free samples, these estimators break down, providing no means of inference.

Following the release of this study, several additional auxiliary estimation techniques were examined using the same data sets. These included the regression estimator [4], the stratified regression estimator [3] and the jackknifed ratio estimator [19]. These studies resulted in the same conclusion as the Neter and Loebbecke study, namely auxiliary estimation, though precise, yields confidence bounds with unreliable coverage.

This has serious implications for auditors. If for example an auditor uses an auxiliary information method to construct a 95% upper confidence bound for the total population error E, the actual level of coverage may be far less than 95%. This means the upper bound may considerably underestimate the true error and may cause the auditor to falsely conclude that no material error exists.

2.3 Attributes Bound

Bounds for the total amount of error have been developed based on attributes theory. This sidesteps the problem with the large-sample normal confidence limits discussed in the previous section. Fienberg et al. [18] outline several such methods for obtaining bounds on the total error amount. The authors examine the case where all errors are overstatements and the error in a line item cannot exceed the book amount of that line item; i.e., $0 \leq e_i \leq y_i$ where $e_i = y_i - x_i$.

Based on simple random sampling, the authors suggest

$$\hat{E}_u = NY_M \pi_u(m; 1-\alpha) \tag{2.26}$$

as the upper confidence bound for E, the total error amount. The quantity Y_M is the maximum book amount in the population and $\pi_u(m; 1-\alpha)$ is the $1-\alpha$ upper confidence bound for the population proportion π when m errors are found in the sample. The binomial distribution can be used to find this confidence bound when the size of the sample (without replacement) n is small compared to the population size N. In this case, $\pi_u(m; 1-\alpha)$ is the value of π satisfying the equation [39]

$$P(X \leq m|\pi) = \sum_{x=0}^{m} \binom{n}{x} \pi^x (1-\pi)^{n-x} = \alpha. \tag{2.27}$$

The Poisson approximation to the binomial, which is slightly conservative, can be used when the sample size is not too small and the population error rate is not too large. As a rule of thumb, this approximation should be employed only if $n > 50$ and $n\pi \leq 5$ [21]. Based on the Poisson distribution $\pi_u(m; 1-\alpha)$ is the solution to the equation

$$P(X \leq m|\pi) = \sum_{x=0}^{m} \frac{e^{-n\pi}(n\pi)^x}{x!} = \alpha. \tag{2.28}$$

Tables 2.3 and 2.4, excerpted from Goodfellow, Loebbecke and Neter [21], give the upper confidence limits for the binomial parameter π and the Poisson parameter $\lambda = n\pi$ for 90% and 95% reliability.

	Sample Proportion m/n			
Sample Size n	0.00	0.01	0.02	0.03
	90% Reliability			
100	.0228	.0383	.0524	.0656
200	.0114	.0264	.0396	.0520
300	.0077	.0221	.0348	.0469
	95% Reliability			
100	.0295	.0466	.0616	.0757
200	.0149	.0311	.0452	.0583
300	.0099	.0256	.0391	.0518

Table 2.3: Upper Precision Limits for the Binomial Parameter π (Source: Goodfellow, J. L., Loebbecke, J. K., Neter, J. (1974))

To understand how the confidence limit in (2.26) was obtained, consider

$$B = N\pi Y_M. \tag{2.29}$$

By treating every observed error as if it was the maximum Y_M, B is a conservative

	Sample Number of Errors m						
Reliability Level	0	1	2	3	4	5	6
90%	2.303	3.890	5.323	6.681	7.994	9.275	10.532
95%	2.996	4.744	6.296	7.754	9.154	10.513	11.842

Table 2.4: Upper Precision Limits for the Poisson Parameter $\lambda = n\pi$ (Source: Goodfellow, J. L., Loebbecke, J. K., Neter, J. (1974))

estimate of E. The parameter π, however, is unknown. It can be replaced by an upper confidence limit. The quantity

$$NY_M \pi_u(m; 1-\alpha) \tag{2.30}$$

is the $1-\alpha$ upper bound for B, so

$$E \leq NY_M \pi_u(m; 1-\alpha) \tag{2.31}$$

with confidence coefficient at least $1-\alpha$.

This bound is conservative meaning it is generally higher than the total error E. Thus, the nominal confidence level associated with the bound is reliable. For populations with large variability in the book values, the attributes bound may grossly overestimate the true error. This *inefficiency* can be costly. For example, if the true error amount is below the materiality limit, but the calculated bound is above the limit, the auditor will conclude that the book total is in error. In this situation, the auditor will either request that the client change the stated balance or undertake further sampling, both of which can be costly to the client.

Chapter 3

Alternative Methods

Chapter 2 reviewed some of the early methods available to auditors for finding bounds on the total error E. The classical methods which rely on normal sampling theory were shown to be unreliable. The attributes bound, which requires no distributional assumptions, was reliable but inefficient. In terms of an upper bound for total error, unreliability may lead an auditor to falsely conclude that a book total is not in error; inefficiency may cause an auditor to falsely conclude that a book total is in error. To this point there was no method which could be depended upon to regularly avoid such false conclusions.

3.1 Audit Distribution

Kaplan [27] recognized that the particular distribution of the audit population contributed to the difficulty in finding appropriate methods and concluded that "entirely new approaches may be required for statistical sampling in auditing."

A typical audit population has a large proportion of error-free items. An audit sample may reveal very few or possibly no errors. For analysis of such data, in which most observations are zero, the classical interval estimation of the total error amount based on the asymptotic normality of the sampling distribution is not reliable [24].

Line items in an audit population can be divided into two groups: those which are recorded correctly and those containing nonzero errors of differing amounts. In this light, the error distribution can be represented as a nonstandard mixture of two underlying distributions: one degenerate (a point mass at zero) and one continuous [24]. The error e of an item chosen randomly from an auditing population can be modelled as

$$e = \begin{cases} z & \text{with probability } \pi \\ 0 & \text{with probability } (1-\pi) \end{cases} \tag{3.1}$$

where $z \neq 0$ is a random variable representing the error amount and π is the rate of occurrence of errors in the audit population.

Mixtures of distributions have been researched extensively. Karl Pearson [40] pioneered research in this area. In 1894 he modelled the heights of 16 year olds as a mixture of two normal distributions, one representing females and one males. Robbins and Pitman [46] studied mixtures of χ^2 distributions as probability models for quadratic forms of normal random variables. Many more papers have appeared on the subject; the majority, like those examples given above, dealing with mixtures of like distributions. Little has been published on "nonstandard" mixtures combining discrete and continuous distributions which are paramount in many auditing applications.

3.2 Dollar-Unit Sampling

The search for alternative procedures that do not depend on large-sample theory has frequently involved monetary-unit sampling. Monetary-unit sampling (MUS), commonly referred to as dollar-unit sampling (DUS), received its first real public exposure through an article by Anderson and Teitlebaum [1] in a 1973 issue of *Canadian Chartered Accountant.* The authors acknowledged that their research developed from work by Stringer [48]. In fact, van Heerden [54] was probably the first to develop this sampling technique. He developed the guilder-unit method in The Netherlands.

Essentially dollar-unit sampling treats individual dollars as sampling units rather than individual line items. The auditor draws a random sample of n individual dollars. For each of the dollars in the sample, he looks at the line item in which that dollar falls and prorates any error found to the dollar units belonging to the line item. The prorated errors, called taints, are calculated as follows:

$$t_i = e_i/y_i. \tag{3.2}$$

For each dollar in the sample, the taint of the line item in which it falls is recorded.

DUS can be modelled by a mixture distribution as follows:

$$t = \begin{cases} z & \text{with probability } \pi \\ 0 & \text{with probability } (1-\pi) \end{cases} \tag{3.3}$$

where t is an independent observation of tainting of a dollar unit, $z \neq 0$ is a random variable representing the tainting amount and π is the probability that a dollar unit is in error [24]. This is analogous to (3.1); thus, the mixture model applies for sampling both individual items and individual dollars. It should be noted, however, that the probability of a dollar-unit being in error is not necessarily equivalent to the probability of a line item being in error.

Unless specified otherwise, any reference to sampling in the remainder of the paper implies dollar-unit sampling. Thus any errors are dollar-unit errors or taints.

There are several methods of selecting a random sample of dollar-units. The four methods employed most often are simple random sampling, systematic sampling, cell sampling and varying-interval sampling. The following example, adapted from Leslie, Teitlebaum and Anderson [29], will be used to illustrate the methods. Each of the rectangles in Figure 3.1 represent a population of \$50,000. Each of the four methods are applied to select a sample of four dollar-units.

For *systematic sampling* (A.), also called *fixed-interval sampling*, the population is divided into four intervals since four dollar-units are to be selected. Each interval has width \$12,500 (\$50,000 $\div$ 4). Within the first interval, a start point is selected randomly. In this example, dollar number 3,098 is selected. This is the first dollar-unit

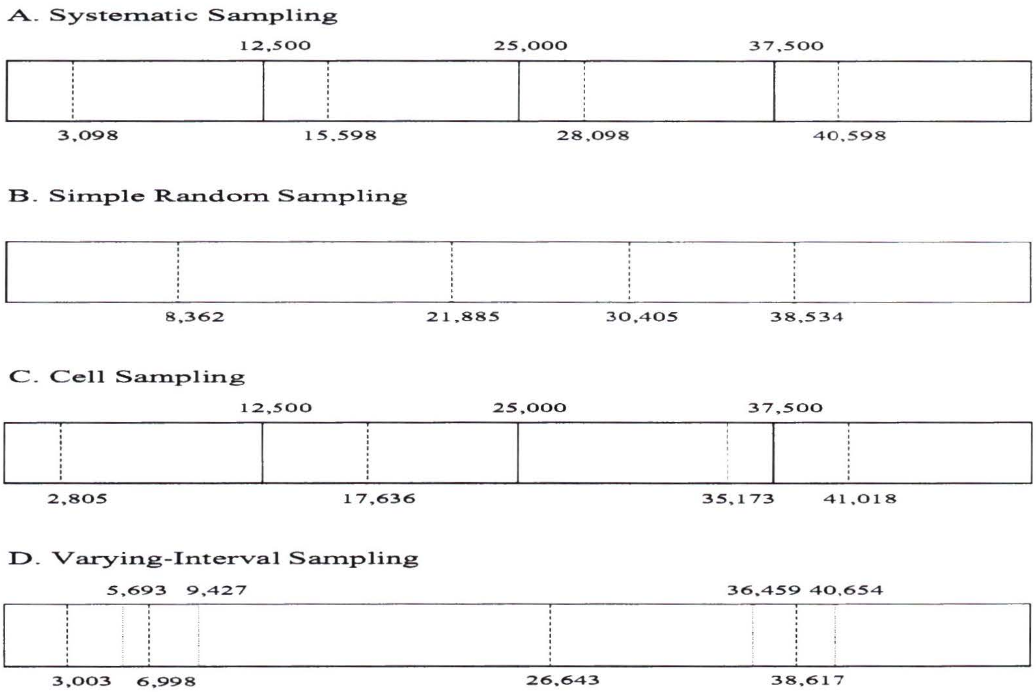


Figure 3.1: Techniques for selecting dollar-unit samples

in the sample. Every 12,500'th dollar is then selected until the sample is complete. Thus the sample would consist of dollar units 3,098, 15,598, 28,098 and 40,598. This method is fairly easy to implement but many auditors avoid it due to the possibility of patterns or trends.

Simple random sampling (B.), the simplest method, involves the random selection of points throughout the population. In this example the four dollar units selected are 8,362, 21,885, 30,405 and 38,534. These units could have been chosen in any order. Since each dollar-unit is selected randomly from the other dollar-units in the sample, this method is unaffected by any patterns and trends in the data, unlike the fixed-interval method.

Cell sampling (C.) involves fixed intervals, as in systematic sampling, with a dollar unit being selected randomly in each interval or cell. In the example, there would be four cells, each containing 12,500 dollars. Four numbers between 1 and 12,500 are selected randomly. The first number chosen is 2,805 so the unit \$2,805 into the first cell is selected. The second random number is 5,136 so the unit \$5,136 into the second cell (\$12,500 + \$5,136 = \$17,636) is chosen. This process continues until the entire sample is selected.

Varying-interval sampling (D.) is similar to cell sampling except that the interval boundaries are chosen randomly. In many cases, this form of sampling will be identical to random sampling.

Cell sampling is generally the preferred method. The problems of systematic sampling are avoided while the sample is fairly evenly spaced throughout the population.

When monetary units are selected at random with replacement and any errors found are prorated to the individual monetary units, the process can be viewed equivalently as sampling with probability proportional to book amount with replacement [24]. Let $\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i$, the average tainting for a sample of size n. The dollar-unit mean-per-unit estimator for the total error E,

$$\hat{E}_{m,dus} = Y\bar{t} \tag{3.4}$$

is equivalent to the probability-proportional-to-size estimator (2.10) since

$$\hat{E}_{m,dus} = Y\bar{t} = Y\sum_{i=1}^{n}\frac{t_i}{n} = Y\sum_{i=1}^{n}\frac{e_i/y_i}{n} = \hat{E}_{pps}. \tag{3.5}$$

The estimated variance of the dollar-unit mean-per-unit estimator is

$$s^2(\hat{E}_{m,dus}) = \frac{Y^2}{n(n-1)} \sum_{i=1}^{n} (t_i - \bar{t})^2. \tag{3.6}$$

Consider

$$\hat{X}_{m,dus} = Y - \hat{E}_{m,dus} = Y - Y\bar{t}, \tag{3.7}$$

the estimated total audit value, with sample variance equivalent to (3.6). Note that this estimated variance does not include the finite population correction factor $\frac{N-n}{N}$, as does the estimated variance (2.5) of the mean-per-unit estimator based on line item sampling. This is because dollar-unit sampling is random sampling with replacement.

Classical methods, which rely on asymptotic normal distribution theory, can be employed to find an upper bound for the population total error based on the dollar-unit mean-per-unit estimator. Thus, a $100(1-\alpha)\%$ upper bound for E is given by

$$\hat{E}_{m,dus} + z_{(1-\alpha)} s(\hat{E}_{m,dus}). \tag{3.8}$$

This estimator, and the corresponding bound, were included in the study by Neter and Loebbecke [33], discussed earlier. The dollar-unit mean-per-unit estimator proved to be at least as precise as the mean-per-unit estimator for all populations studied. The reliability of the nominal confidence coefficient was low for all populations, particularly when the error rate was low.

3.3 DUS Attributes Bound

Dollar-unit sampling provides the basis for another method of finding bounds for the total error amount without relying on asymptotic normal theory. This will be similar to the development of the attributes bound (2.26) which was based on sampling for individual line item attributes. Again, all errors are assumed to be overstatements with the maximum size of an error equivalent to the book amount. This means the taint, the amount of overstatement as a proportion of the book amount, must lie between 0 and 1.

Now the maximum taint, t_M, is \$1. A conservative upper bound for E, assuming all taints are at the maximum, is

$$B = Y\pi t_M = Y\pi \tag{3.9}$$

where π is the proportion of dollar units in error. Replace π by $\pi_u(m; 1-\alpha)$, the $1-\alpha$ upper confidence bound for the population proportion π when m nonzero taints are found in the sample. Thus,

$$\hat{E}_u = Y\pi_u(m; 1-\alpha) \tag{3.10}$$

is a $1-\alpha$ upper confidence bound for E. Like (2.26), this is a conservative bound, but since $Y \leq NY_M$, the attributes bound obtained through dollar-unit sampling (3.10) is tighter than the attributes bound obtained through line item sampling (2.26).

3.4 DUS CAV Bounds

The two attributes methods suggested so far for finding an upper bound for the total error depend only on the number of errors discovered in a sample. In each case it was assumed that all errors were maximum overstatements. This bound can be tightened by incorporating information about the magnitude of the errors in the sample.

Procedures which use both attributes and variables information are collectively referred to as combined attributes and variables (CAV) estimation [21]. Various methods have been developed to cope with audit issues, namely to minimize the cost of sampling while maintaining reliability in the findings. The most widely used such method was developed by K. Stringer, though he did not publish an explanation of his work.

3.4.1 Stringer Bound

The *Stringer bound* heuristically reduces the conservative bound (3.10) just discussed, accounting for the likelihood that many of the observed taints may not be at a maximum of 1.0. Let m be the number of nonzero taints found in the sample of size n. These taints are ordered, $0 < t_m \leq \ldots \leq t_1 \leq 1$. The Stringer bound is

$$Y\pi_u(m; 1-\alpha) - Y\sum_{j=1}^{m}[\pi_u(j; 1-\alpha) - \pi_u(j-1; 1-\alpha)](1-t_j) \tag{3.11}$$

or, equivalently

$$Y\{\pi_u(0; 1-\alpha) + \sum_{j=1}^{m}[\pi_u(j; 1-\alpha) - \pi_u(j-1; 1-\alpha)]t_j\} \tag{3.12}$$

where $\pi_u(m; 1-\alpha)$ is the $1-\alpha$ upper bound for the error rate π when the sample contains m errors, based on the binomial distribution. When all the nonzero taints are at the maximum of 1.0 the Stringer bound (3.12) is equivalent to the attributes bound based on dollar-unit sampling (3.10).

In practice, the Stringer bound is almost always calculated with Poisson limits in place of binomial limits because they are far easier to tabulate given that they depend on only one parameter (Meikle [30]; Anderson and Teitlebaum [1]). However, the extra effort of obtaining binomial limits for the Stringer method does result in a slightly tighter bound. The rationale for this bound will be illustrated with an example taken from Goodfellow, Loebbecke and Neter [21]. Binomial confidence limits are used, the values for which can be found in Table 2.3.

Consider an audit population consisting of Y = 10,000,000 dollars. A sample of size n = 100 is taken and one taint, t_1 = \$.10, is discovered. Had no errors been discovered, a 95% confidence bound for the total error would have been

$$Y\pi_u(0; .95)(\$1) = 10,000,000(.0295)(\$1) = \$295,000,$$

according to (3.10), where $\pi_u(0; .95)$ is tabulated in Table 2.3.

By considering the one error discovered to be a maximum, the 95% confidence bound would be

$$Y\pi_u(1; .95)(\$1) = 10,000,000(.0466)(\$1) = \$466,000.$$

A bound for the total error given one error in the sample and accounting for less-than-maximum errors should lie somewhere between \$295,000 and \$466,000.

The maximum number of dollars in error given no errors in the sample, $Y\pi_u(0;.95)$ is considered to consist of maximum \$1 taints. So the maximum number of dollars in error given one sample error $Y\pi_u(1;.95)$ is considered to consist of $Y\pi_u(0;.95)$ maximum \$1 taints and $Y[\pi_u(1;.95) - \pi_u(0;.95)]$ \$.10 taints. The difference in probabilities, $[\pi_u(1;.95) - \pi_u(0;.95)]$, is termed the "precision adjustment factor" by Meikle and the "incremental factor" by Anderson and Teitlebaum. Based on this, the modified upper confidence bound for the total error E is given by

$$\begin{aligned}
\hat{E}_u &= Y\pi_u(0;.95)(\$1) + Y[\pi_u(1;.95) - \pi_u(0;.95)](\$.10) \\
&= 10,000(.0295)(\$1) + 10,000,000[.0466 - .0295](\$.10) \\
&= \$295,000 + \$17,100 \\
&= \$312,100
\end{aligned}$$

If for this population an error of \$500,000 was considered material, the auditor would conclude that the population book total was not materially in error. If the materiality limit was below \$312,000, the reverse conclusion would be made.

This method can also be employed for more than one sample error. For example, if two taints t_1 and t_2 are discovered,

$$\hat{E}_u = Y\pi_u(0;.95)(\$1) + Y[\pi_u(1;.95) - \pi_u(0;.95)]t_1 \quad (3.13)$$

$$+Y[\pi_u(2;.95) - \pi_u(1;.95)]t_2 \quad (3.14)$$

where $t_1 \geq t_2$ is a 95% upper confidence bound for the total error E. The ranking of the taints allows for a more conservative bound since

$$[\pi_u(1;.95) - \pi_u(0;.95)] > [\pi_u(2;.95) - \pi_u(1;.95)]. \tag{3.15}$$

If for the audit population of 10,000,000 dollars the sample of size 100 had revealed one taint of \$.10 and a second taint of \$.20, the upper confidence bound for the error would be given by

$$\begin{aligned}\hat{E}_u &= 10,000,000(.0295)(\$1)\\ &\quad +10,000,000[.0466 - .0295](\$.20)\\ &\quad +10,000,000[.0616 - .0466](\$.10)\\ &= \$344,200\end{aligned}$$

One strike against the Stringer method is the lack of statistical theory to support the confidence level ascribed to the bound. It is difficult to determine the sampling distributions of CAV estimators which are heuristic. Simulation studies (Reneau [45]; Leitch, Neter, Plante and Sinha [28]) have provided strong evidence that the actual confidence level of the Stringer bound is at least the nominal level and in fact, often close to 100%. The high coverage comes at a price: the Stringer bound is not very tight and the bound values are often much higher than the population total error amount. This can lead to incorrect conclusions by the auditor; i.e., that a material error exists when in reality the total book amount is accurate.

3.4.2 Meikle's Adjustment

Meikle [30] describes a CAV method which can handle understatement, as well as overstatement errors. This is an appropriate modification of the Stringer bound. Essentially, a lower confidence limit for the total population understatement is subtracted from an upper confidence limit for the total population overstatement.

An understatement error occurs when $X_i > Y_i$. The smallest understatement, when $X_i = Y_i$, results in a taint of \$0. Assuming all understatement errors are at the minimum, a lower confidence bound for the population total understatement is given by

$$Y\pi_l(m';1-\alpha)(\$0) = \$0 \tag{3.16}$$

where $\pi_l(m';1-\alpha)$ is the $1-\alpha$ lower bound for the population proportion of understatement errors when the sample contains m' understatement errors. The calculation of $\pi_l(m';1-\alpha)$ is similar to the calculation of $\pi_u(m;1-\alpha)$. Based on the binomial distribution $\pi_l(m';1-\alpha)$ is the value of π satisfying the equation [39]

$$P(X \geq m'|\pi) = \sum_{x=m'}^{n} \binom{n}{x} \pi^x (1-\pi)^{n-x} = \alpha. \tag{3.17}$$

Using the Poisson approximation to the binomial distribution $\pi_l(m';1-\alpha)$ is the solution to the equation

$$P(X \geq m'|\pi) = \sum_{x=m'}^{\infty} \frac{e^{-n\pi}(n\pi)^x}{x!} = \alpha. \tag{3.18}$$

Table 3.1, taken from Goodfellow, Loebbecke and Neter [21], gives lower precision limits for the binomial parameter π for 90% and 95% reliability. Table 3.2, adapted from Odeh et al. [38] gives analogous limits for the Poisson parameter $\lambda = n\pi$.

	Sample Proportion m'/n			
Sample Size n	0.00	0.01	0.02	0.03
	90% Reliability			
100	.0000	.0011	.0053	.0111
200	.0000	.0027	.0088	.0158
300	.0000	.0037	.0105	.0182
	95% Reliability			
100	.0000	.0005	.0036	.0082
200	.0000	.0018	.0069	.0132
300	.0000	.0027	.0088	.0157

Table 3.1: Lower Precision Limits for the Binomial Parameter π (Source: Goodfellow, J. L., Loebbecke, J. K., Neter, J. (1974))

	Sample Number of Errors m'						
Reliability Level	0	1	2	3	4	5	6
90%	0.000	0.105	0.532	1.102	1.745	2.433	3.152
95%	0.000	0.051	0.355	0.818	1.366	1.970	2.613

Table 3.2: Lower Precision Limits for the Poisson Parameter $\lambda = n\pi$ (Source: Odeh, R. E., Owen, D. B., Birnbaum, Z. W., Fisher, L. (1977))

This limit can be modified, taking into account the values of the sample taints. If one understatement error is found in the sample, the CAV lower confidence bound for the total understatement error is given by

$$\hat{E}_l = \$0 + Y\pi_l(1; 1-\alpha)|t_1|. \tag{3.19}$$

This is similar to (3.12) since $\pi_l(0; 1-\alpha)$, the lower confidence bound when no understatements are found in the sample, is always zero.

Similarly, if two understatement errors are found in the sample

$$\hat{E}_l = \$0 + Y\pi_l(1; 1-\alpha)|t_1| + Y[\pi_l(2; 1-\alpha) - \pi_l(1; 1-\alpha)]|t_2| \qquad (3.20)$$

where $|t_1| \geq |t_2|$. This pattern can be expanded to more than two errors as follows:

$$\hat{E}_l = \$0 + Y\pi_l(1; 1-\alpha)|t_1| + Y\sum_{j=2}^{m}[\pi_l(j; 1-\alpha) - \pi_l(j-1; 1-\alpha)]|t_j|. \qquad (3.21)$$

The two bounds $\hat{E}_l$ and $\hat{E}_u$ can be combined to find an overall upper precision limit for E, the total net error as follows

$$\hat{E}'_u = \hat{E}_u - \hat{E}_l. \qquad (3.22)$$

Consider the Y = 10,000,000 dollar population from which n =100 dollar units are sampled. If two overstatement taints (\$.20 and \$.10) and two understatement taints (\$.15 and \$.25) are discovered, the overall 95% upper bound would be computed as follows:

$$\begin{aligned}
\hat{E}_u &= \$344,200 \\
\hat{E}_l &= \$0 + 10,000,000(.0005)(\$.25) \\
&\quad + 10,000,000[.0036 - .0005](\$.15) \\
&= \$5,900 \\
\hat{E}'_u &= \hat{E}_u - \hat{E}_l = \$344,200 - \$5,900 = \$338,300
\end{aligned}$$

3.4.3 LTA Adjustment

Leslie, Teitlebaum and Anderson [29] discuss another modification of the Stringer bound to allow for understatement errors, which will be denoted the *LTA bound.* They suggest reducing the Stringer bound by the mean understatement error, estimated from the sample. Thus,

$$\hat{E}'_u = \hat{E}_u - Y\frac{m'}{n}\frac{1}{m'}\sum_{t_i<0}|t_i| \tag{3.23}$$

where m' is the number of understatement errors discovered in the sample.

Consider the above example with $Y = \$10,000,000$, $n = 10$, and the taints $.2, .1, -.15, -.25$.

$$\begin{aligned}\hat{E}'_u &= \$344,200 - \$10,000,000\left(\frac{2}{100}\right)\frac{.15+.25}{2}\\ &= \$344,200 - \$40,000\\ &= \$304,200\end{aligned}$$

The LTA bound yields a smaller result than Meikle's modification to the Stringer bound. Grimlund and Schroeder [23] found the *LTA adjustment* to the Stringer bound to be reliable and uniformly more efficient than *Meikle's adjustment.* This was expected since Meikle's method subtracts off a lower bound for the mean absolute understatement which will always be less than the sample mean absolute understatement.

3.4.4 Neter and Loebbecke Bound

Two alternative CAV bounds, the *Neter and Loebbecke bound* and the *average error bound*, both of which assume all errors are overstatements, have received little use. Neter and Loebbecke [33] included the first of these bounds in their study of classical estimators, hence the name. This upper $(1-\alpha)$ confidence bound for E has the following form:

$$\hat{E}_u = Y\pi_u(m; 1-\alpha) - \frac{Y}{n}\sum_{i=1}^{n}(1-t_i)v_i \tag{3.24}$$

where

$$v_i = \begin{cases} 0 & \text{if } t_i = 0 \\ 1 & \text{if } t_i \neq 0 \end{cases} \tag{3.25}$$

and $\pi_u(m; 1-\alpha)$ is the upper $1-\alpha$ confidence interval for the population proportion π when m nonzero taints are found in the sample.

This method adjusts the attributes bound (3.10) downwards according to the errors found in the sample. $\sum(1-t_i)v_i$ is the sum of the "non-error" proportions of those dollar units in error. Hence $\frac{1}{n}\sum(1-t_i)v_i$ is the average proportion of a dollar unit which is not in error. Multiplying this value by Y gives the projected value of dollars in the population which are not in error. The attributes bound, which assumes all taints have a value of \$1, is adjusted downwards by this amount. In their simulation study, Neter and Loebbecke found this method to be conservative.

3.4.5 Average Error Bound

Both the Neter and Loebbecke bound and the average error bound were included in a simulation study by Reneau [45]. The average error bound is simply

$$\hat{E}_u = Y\pi_u(m; 1-\alpha)\overline{z} \tag{3.26}$$

where $\overline{z}$ is the average of the nonzero taints. If $m = 0$, $\overline{z}$ is taken to be 1, assuming maximum errors, and (3.26) is the same as the attributes bound (3.10). The results of the simulation study showed this bound to be unreliable for error rates greater than .005; i.e. the actual coverage fell below the nominal confidence level.

3.4.6 GOLOS Bound

The final CAV method covered here is the *greater of load or spread (GOLOS) bound.* This bound was developed for cell sampling but can also be applied to simple random sampling of dollar units [41]. Smieliauskas [47] describes this as an iterative method, based on two factors termed load and spread. As with the Neter and Loebbecke bound and the average error bound, it is assumed that all errors are overstatements. The taints are ranked such that $t_1 \geq t_2 \geq \ldots \geq t_m$. For the k'th sample taint ($k = 1, 2, \ldots, m$), an upper error factor is determined as follows:

$$UE_k = \max\{UE_{k-1} + \frac{t_k}{n}, \pi_u(k; 1-\alpha)t_k\} \quad \text{for } k \geq 1 \tag{3.27}$$

where $UE_0 = \pi_u(0; 1-\alpha)$. After m iterations, the bound is

$$\hat{E}_u = Y\,UE_m. \tag{3.28}$$

Smieliauskas [47] found the reliability of this bound to decrease as the number of sample errors increased, especially at higher sample sizes.

For samples containing understatements as well as overstatements, Leslie, Teitlebaum and Anderson [29] suggest handling this as two separate samples. Ignoring any overstatements in the sample, the lower GOLOS bound for the total understatement error can be computed. Then, ignoring understatements, the upper GOLOS bound for the total overstatement is derived. Assuming the net error is positive, the lower understatement bound is subtracted from the upper overstatement bound to give an upper bound for net error. In fact, this method of offsetting the upper bound by the lower limit of understatements can also be employed with the Neter and Loebbecke bound and the average error bound [45].

3.4.7 Cell Bound

Plante et al. [41] use a slight variation of the GOLOS bound, called the *cell bound*, in their simulation study. The difference comes in the second factor of the maximum. For the cell bound, the k'th taint is replaced by the average of the first k recorded taints, so the upper error factor is

$$UE_k = \max\{UE_{k-1} + \frac{t_k}{n}, \pi_u(k; 1-\alpha)\frac{1}{n}\sum_{i=1}^{k} t_k\} \quad \text{for } k \geq 1. \tag{3.29}$$

In their simulation study, the authors found that the cell bound was tighter than the Stringer bound but had actual coverage percentages well above the nominal confidence level.

The introduction of dollar-unit sampling provided significant advancements in the search for dependable methods for finding a bound on the total error in an audit population. CAV methods based on DUS offer considerable improvements over earlier methods but the disadvantage of these sometimes ad hoc techniques is the lack of knowledge of their distributional properties. In the next chapter methods which combine some distributional assumptions with the dollar-unit sampling model will be explored.

Chapter 4

Improved DUS Methods

Combined attributes and variables (CAV) estimation involves information about the nonzero taints and requires no assumptions about the distribution of the population of taints. Though these bounds are straightforward to employ, theory to support their use is sketchy and they tend to be inefficient. Recognizing these constraints, Fienberg, Neter and Leitch [18] developed a bound for total audit error based on the multinomial distribution.

4.1 Multinomial Bound

To develop the *multinomial bound*, dollar-unit sampling is viewed as multinomial sampling. Each observed taint is rounded to some desired degree and then classified. For example, the taints may be rounded to the nearest cent, resulting in 101 classes

(00 to 100 cents). The errors can now be modelled by

$$t = \frac{i}{100} \quad \text{with probability } p_i, \quad i = 0, \ldots, 100 \tag{4.1}$$

where p_i is the probability that an error is classified in the i'th category (i cents). Then

$$E(t) = \mu_t = \sum_{i=0}^{100} \frac{i}{100} p_i, \tag{4.2}$$

so

$$E = Y\mu_t = Y \sum_{i=0}^{100} \frac{i}{100} p_i. \tag{4.3}$$

The p_i are unknown of course. Denote the number of sample observations which fall in the i'th category by w_i; $\sum_{i=0}^{100} w_i = n$. If the sample is done with replacement $(w_0, w_1, \ldots, w_{100})$ has exactly a multinomial distribution with the parameters $(n, p_0, p_1, \ldots, p_{100})$. This distribution becomes approximate if the sampling is done without replacement.

E, the total error, can be estimated by $Y\hat{\mu}_t$, where

$$\hat{\mu}_t = \sum_{i=0}^{100} \frac{i}{100} \frac{w_i}{n}. \tag{4.4}$$

In order to develop an upper confidence bound for μ_t, and thus E, let S be a set of outcomes $\boldsymbol{v} = (v_0, v_1, \ldots, v_{100})$ that are no more extreme than the observed results $(w_0, w_1, \ldots, w_{100})$. Fienberg, Neter and Leitch [18] propose a set called "step-down S". This includes all possible outcomes such that (1) the number of outcome errors does not exceed the observed number of errors; i.e. $\sum_{i=1}^{100} v_i \leq m$ and

(2) each error amount does not exceed the corresponding observed error amount. This second constraint can be expressed mathematically as follows: $\sum_{i=k}^{100} v_i \le \sum_{i=k}^{100} w_i$ for $k = 0, \ldots, 100$. This set is used to develop a joint $(1-\alpha)$ multinomial "confidence" region for $\boldsymbol{p} = (p_0, p_1, \ldots, p_{100})$, specifically those values of $\boldsymbol{p}$ that satisfy

$$\sum_{\boldsymbol{v} \in S} \frac{n!}{v_0! \ldots v_{100}!} \prod_{i=0}^{100} p_i^{v_i} = \alpha, \quad \text{where} \sum_{i=0}^{100} v_i = n. \tag{4.5}$$

A $(1-\alpha)$ upper bound for μ_t is then obtained by maximizing (4.2) over those $\boldsymbol{p} = (p_0, p_1, \ldots, p_{100})$ constrained by (4.5). To obtain the upper confidence bound for E, this bound is simply multiplied by Y, the total book amount.

The "confidence" region defined by (4.5) depends on the observed counts (w_i) and the definition of the S-set. This may result in a confidence level less than $(1-\alpha)$. Leitch et al. [28] have demonstrated that the achieved confidence levels with the step-down S-set usually exceed the nominal confidence level.

Consider a dollar-unit sample which contains no nonzero errors. Then $w_0 = n$ and $w_i = 0$ for $i = 1, \ldots, 100$. In this case S consists of only one possible outcome $(n, 0, \ldots, 0)$. Equation (4.5) thus reduces to $p_0^n = \alpha$ and the problem becomes maximizing (4.2), or similarly $\sum_{i=1}^{100} ip_i$, subject to this. The maximum will be achieved by setting $\hat{p}_0 = \alpha^{1/n}$, $\hat{p}_i = 0$ for $i = 1, \ldots, 99$ and $\hat{p}_{100} = 1 - \hat{p}_0 = 1 - \alpha^{1/n}$.

The upper confidence bound for E takes the form

$$\frac{Y}{100}[0\hat{p}_0 + 100\hat{p}_{100}] = Y\hat{p}_{100} \tag{4.6}$$

Since $\hat{p}_{100} = 1-\alpha^{1/n}$ is the $1-\alpha$ upper confidence bound for the population proportion

π when 0 nonzero taints are found in the sample, the upper confidence bound for E can be written

$$\hat{E}_u = Y\pi_u(0; 1-\alpha), \tag{4.7}$$

which is identical to the attributes bound (3.10) in the case of no errors.

Now consider the case of one error, with a tainting of d cents. S is chosen so that the number of errors does not exceed the observed number of errors and each error amount does not exceed any observed error amount. These restrictions can be represented mathematically as follows:

$$\sum_{i=1}^{100} v_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{100} iv_i \leq d. \tag{4.8}$$

S consists of $d+1$ possible outcomes: $(n, 0, \ldots, 0)$, $(n-1, 1, 0 \ldots, 0)$, $(n-1, 0, 1, 0 \ldots, 0), \ldots, (n-1, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 falling in the d'th category. Based on this set S, the problem is to maximize

$$\sum_{i=1}^{100} ip_i \tag{4.9}$$

subject to

$$p_0^n + np_0^{n-1}\sum_{i=1}^{d} p_i = \alpha. \tag{4.10}$$

If the discovered error is a maximum, i.e. $d = 100$, the constraint (4.10) becomes

$$p_0^n + np_0^{n-1}(1-p_0) = \alpha. \tag{4.11}$$

The solution to this equation is $\hat{p}_0 = \pi_l(n-1; 1-\alpha)$, the $1-\alpha$ lower confidence bound for the population proportion π when $n-1$ nonzero errors are found in the sample. Thus, to maximize (4.9) set $\hat{p}_0 = \pi_l(n-1; 1-\alpha)$, $\hat{p}_i = 0$ for $i = 1, \ldots, 99$ and $\hat{p}_{100} = 1 - \hat{p}_0 = 1 - \pi_l(n-1; 1-\alpha) = \pi_u(1; 1-\alpha)$. Then the upper confidence bound for E can be written

$$\hat{E}_u = \frac{Y}{100}\sum_{i=1}^{100} ip_i = \frac{Y}{100}[100\pi_u(1; 1-\alpha)] = Y\pi_u(1; 1-\alpha). \tag{4.12}$$

In the case of a maximum error, i.e. a taint of 100 cents, the multinomial bound (4.12) is equivalent to (3.10), the bound developed from attributes theory.

Now consider the case where the observed error results in a taint of less than 100 cents. Then the problem is to maximize (4.9) subject to (4.10) where $d < 100$. This is achieved by setting $\hat{p}_1 = \ldots = \hat{p}_{d-1} = \hat{p}_{d+1} = \ldots \hat{p}_{99} = 0$ and $\hat{p}_{100} = 1 - \hat{p}_0 - \hat{p}_d$. Thus, (4.9), now given by

$$\begin{aligned} dp_d + 100p_{100} &= dp_d + 100(1 - p_0 - p_d) \\ &= 100(1 - p_0) - (100 - d)p_d \end{aligned} \tag{4.13}$$

must be maximized subject to

$$p_0^{n-1}(p_0 + np_d) = \alpha \tag{4.14}$$

where $p_0 + p_d + p_{100} = 1$ and $p_0, p_d, p_{100} \geq 0$.

Now consider the constraint (4.14). The estimate $\hat{p}_0$ will be largest when $\hat{p}_d = 0$; i.e. when $\hat{p}_0^n = \alpha$, so $\hat{p}_0 = \pi_l(n; 1-\alpha)$. The estimate $\hat{p}_0$ will be smallest when $\hat{p}_d = 1 - \hat{p}_0$ ($\hat{p}_{100} = 0$); i.e., $\hat{p}_0^n + n\hat{p}_0^{n-1}(1 - \hat{p}_0) = \alpha$, so $\hat{p}_0 = \pi_l(n-1; 1-\alpha)$. Thus, the following interval for $\hat{p}_0$ is obtained:

$$\pi_l(n-1; 1-\alpha) \leq \hat{p}_0 \leq \pi_l(n; 1-\alpha). \tag{4.15}$$

Equation (4.13) can be maximized subject to (4.14) using the Lagrange multiplier technique. Differentiate

$$L = 100(1 - p_0) - (100 - d)p_d + \lambda(p_0^n + np_0^{n-1}p_d - \alpha) \tag{4.16}$$

with respect to p_0, p_d and λ respectively and set the resulting equations equal to zero to obtain:

$$\frac{\partial L}{\partial p_0} = -100 + \lambda n\hat{p}_0^{n-1}\left[1 + \frac{(n-1)\hat{p}_d}{\hat{p}_0}\right] = 0 \tag{4.17}$$

$$\frac{\partial L}{\partial p_d} = -(100 - d) + \lambda n\hat{p}_0^{n-1} = 0 \tag{4.18}$$

$$\frac{\partial L}{\partial \lambda} = \hat{p}_0^n + n\hat{p}_0^{n-1}\hat{p}_d - \alpha = 0. \tag{4.19}$$

To solve for $\hat{p}_0$ and $\hat{p}_d$, first combine (4.17) and (4.18).

$$\begin{aligned} -100 + (100 - d)\left[1 + \frac{(n-1)\hat{p}_d}{\hat{p}_0}\right] &= 0 \\ 1 + (n-1)\frac{\hat{p}_d}{\hat{p}_0} &= \frac{100}{100 - d} \end{aligned} \tag{4.20}$$

Solve (4.19) for $\hat{p}_d$.

$$\hat{p}_d = \frac{1}{n}\left[\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0\right] \tag{4.21}$$

Combine (4.20) and (4.21) to find $\hat{p}_0$.

$$\begin{aligned} 1 + \frac{n-1}{\hat{p}_0}\frac{1}{n}\left[\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0\right] &= \frac{100}{100 - d} \\ \frac{(n-1)\alpha}{n\hat{p}_0^n} &= \frac{100n + (n-1)(100 - d) - n(100 - d)}{n(100 - d)} \\ \frac{\hat{p}_0^n}{(n-1)\alpha} &= \frac{(100 - d)}{100n + 100n - 100 - nd + d - 100n + nd} \end{aligned}$$

$$\begin{aligned}
\hat{p}_0^n &= \frac{(100-d)(n-1)\alpha}{(100n-100+d)} \\
\hat{p}_0^n &= \frac{\alpha}{\frac{100n-100+d}{(100-d)(n-1)}} \\
\hat{p}_0^n &= \frac{\alpha}{\frac{(100-d)(n-1)+nd}{(100-d)(n-1)}} \\
\hat{p}_0^n &= \frac{\alpha}{1+\frac{nd}{(100-d)(n-1)}} && (4.22)
\end{aligned}$$

Now, looking at (4.21), $\hat{p}_d$ must be greater than zero. This requires the following

$$\begin{aligned}
\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0 &\geq 0 \\
\alpha - \hat{p}_0^n &\geq 0 \\
\alpha &\leq \hat{p}_0^n \\
\hat{p}_0 &\leq \pi_l(n; 1-\alpha) && (4.23)
\end{aligned}$$

which means we only have to be concerned with the lower bound for $\hat{p}_0$ in (4.15). Therefore,

$$\hat{p}_0 = \max\left[\left(\frac{\alpha}{1+\frac{nd}{(100-d)(n-1)}}\right)^{1/n}, \pi_l(n-1; 1-\alpha)\right] \tag{4.24}$$

and the $1-\alpha$ upper bound for E is given by

$$\hat{E}_u = \frac{Y}{100}[d\hat{p}_d + 100\hat{p}_{100}] \tag{4.25}$$

where

$$\hat{p}_d = \frac{1}{n}\left[\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0\right] \tag{4.26}$$

and

$$\hat{p}_{100} = 1 - \hat{p}_0 - \hat{p}_d. \tag{4.27}$$

Note that $\pi_l(n-1; 1-\alpha)$ is equivalent to $1 - \pi_u(1; 1-\alpha)$, which may be easily read from Table 2.3.

To illustrate this bound consider the following example. A sample of size $n = 100$ is selected from a population of 1 million dollars and one error with a taint of 0.4 ($d = 40$ cents) is found. A 95% upper confidence bound would be calculated as follows:

$$\left(\frac{\alpha}{1+\frac{nd}{(100-d)(n-1)}}\right)^{1/n} = \left(\frac{.05}{1+\frac{100*40}{60*99}}\right)^{1/100} = .9655$$

$$\pi_l(99; .95) = 1 - \pi_u(1; .95) = 1 - .0466 = .9534$$

$$\hat{p}_0 = \max(.9655, .9534) = .9655$$

$$\hat{p}_d = \frac{1}{n}\left[\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0\right] = \frac{1}{100}\left[\frac{.05}{.9655^{99}} - .9655\right] = .0065$$

$$\hat{p}_{100} = 1 - \hat{p}_0 - \hat{p}_d = 1 - .9655 - .0065 = .028$$

$$\hat{E}_u = \frac{Y}{100}[d\hat{p}_d + 100\hat{p}_{100}] = \frac{1,000,000}{100}[40(.0065) + 100(.028)] = \$30,600$$

For the case of zero errors, the maximization of (4.2) led to all zero p_i's, except p_0 and p_{100}. For one sample error, all the p_i's were set equal to zero except p_0, p_{100} and p_d. In fact, it can be shown that for "step-down S" the maximization process sets all p_i's equal to zero except p_0, p_{100} and the p_i's corresponding to the observed sample error amounts [35]. This leads to a simplified S set. Consider the case of two sample taints, with values $d_1 = 25$ and $d_2 = 40$, in cents. Table 4.1 illustrates the simplified "step-down S" for this example, giving the number of possible outcomes for the relevant v_i.

	v_0	v_{25}	v_{40}	v_{100}
0 errors	n	0	0	0
1 error	n-1	1	0	0
	n-1	0	1	0
2 errors	n-2	1	1	0
	n-2	2	0	0

Table 4.1: Step-down S for two overstatement errors

Neter, Leitch and Fienberg [35] suggest using an analogous procedure to find a bound for the total understatement error in order to form joint confidence bounds for understatement and overstatement errors. After setting the maximum possible understatement per dollar unit, finding a lower confidence bound for the total understatement error is analogous to finding an upper confidence bound for the total overstatement error. In fact, if the maximum understatement is set at 100 cents, the understatement bound would be calculated using the exact formulas above.

The upper multinomial bound for overstatement errors is calculated assuming there are no understatement errors. The lower multinomial bound for understatement errors is calculated assuming there are no overstatement errors. Together, these two bounds can reveal information regarding the overall error. According to the Bonferroni inequality, a 95% lower bound for understatement errors and a 95% upper bound for overstatement errors have at least a 90% joint confidence level.

Consider the example given above with a book amount of \$1,000,000 and a sample of size $n = 100$. Now, suppose the sample reveals one understatement error with a taint of $-.20$ and one overstatement error with a taint of $.40$. The 95% multinomial upper confidence bound for overstatement errors was shown above to be

$$\hat{E}_u = \$30,600.$$

Assuming the maximum understatement error per dollar unit is 100, a 95% lower confidence bound for understatement errors would be calculated as follows:

$$\left(\frac{\alpha}{1+\frac{nd}{(100-d)(n-1)}}\right)^{1/n} = \left(\frac{.05}{1+\frac{100*20}{80*99}}\right)^{1/100} = .9683$$

$$\pi_l(99; .95) = 1 - \pi_u(1; .95) = 1 - .0466 = .9534$$

$$\hat{p}_0 = \max(.9683, .9534) = .9683$$

$$\hat{p}_d = \frac{1}{n}\left[\frac{\alpha}{\hat{p}_0^{n-1}} - \hat{p}_0\right] = \frac{1}{100}\left[\frac{.05}{.9683^{99}} - .9683\right] = .0025$$

$$\hat{p}_{100} = 1 - \hat{p}_0 - \hat{p}_d = 1 - .9683 - .0025 = .0292$$

$$\hat{E}_l = \frac{Y}{100}[d\hat{p}_d + 100\hat{p}_{100}] = \frac{1,000,000}{100}[20(.0025) + 100(.0292)] = \$29,700$$

Because this is for understatements, it may be considered -\$29,700. If the auditor considered an error of \$50,000 to be material, the above bounds, which have a joint confidence of 90%, will lead him to assume a material error does not exist.

The exact level of confidence of the multinomial bound is unknown. In a comparison study conducted by Plante, Neter and Leitch [41], the multinomial bound outperformed both the Stringer and the cell bound. It was tighter than these two

CAV bounds and the observed confidence level was not significantly different than the nominal level.

Analytic solutions to the multinomial bound are only possible for the case of zero or one sample error. Beyond this, numerical search routines are necessary for the maximization. Neter, Leitch and Fienberg [35] employ a gradient search method. As the number of sample errors increases, so too does the size of the step-down S-set. For more than 8 sample errors, the computational costs become great due to the size of the S-set, making this method impractical.

4.2 Modified Multinomial Bound

Leitch et al. [28] present a modification of the multinomial bound, based on clustering of the sample taintings, which can be employed when the number of sample errors is as high as 25 or 30. Following the clustering of the sample taintings, all taintings in a particular cluster are set equal to the maximum tainting in that cluster. This reduces the size of the S-set and the multinomial bound can then be computed more easily. Table 4.2, adapted from Leitch et al., shows the step-down S-set for the case of three sample taintings of 20, 50 and 60 cents. The table also shows the S-set when two clusters, (20) and (50, 60) are employed. This involves treating the three taintings as 20, 60 and 60. Without clustering there are 14 possible outcomes in the S-set; with clustering, only 9. This example, which only involves three sample errors, is chosen for explanatory purposes. The difference in the sizes of the two sets increases with the number of sample errors.

unclustered	v_0	v_{20}	v_{50}	v_{60}	v_{100}
0 errors	n	0	0	0	0
1 error	n-1	1	0	0	0
	n-1	0	1	0	0
	n-1	0	0	1	0
2 errors	n-2	1	1	0	0
	n-2	0	1	1	0
	n-2	1	0	1	0
	n-2	2	0	0	0
	n-2	0	2	0	0
3 errors	n-3	1	1	1	0
	n-3	2	1	0	0
	n-3	2	0	1	0
	n-3	1	2	0	0
	n-3	3	0	0	0

clustered	v_0	v_{20}	v_{50}	v_{60}	v_{100}
0 errors	n	0	-	0	0
1 error	n-1	1	-	0	0
	n-1	0	-	1	0
2 errors	n-2	1	-	1	0
	n-2	2	-	0	0
	n-2	0	-	2	0
3 errors	n-3	1	-	2	0
	n-3	2	-	1	0
	n-3	3	-	0	0

Table 4.2: Step-down S for three overstatement errors

The procedure for choosing clusters is obviously important. Ideally, one would like to use the clustering method which leads to the tightest possible bound. The authors employ a method developed by R. A. Fisher. For a chosen number of clusters J, this involves allocating observations x_{ij} to cluster j by minimizing

$$\sum_{j}^{J}\sum_{i}^{n_j}(L_j - x_{ij}) \tag{4.28}$$

where L_j is the largest observation in the j'th cluster, x_{ij} is the i'th observation in the

j'th cluster and n_j is the number of observations in the j'th cluster. This clustering procedure is fairly simple to program.

Leitch et al. [28] conducted a simulation study to examine the performance of the *modified multinomial bound* based on clustering. Study populations were chosen to provide a range of tainting distributions, error rates and sample sizes. Four types of tainting distributions were employed. A reversed J-shaped distribution, labelled J, was modelled by the χ^2 distribution with 1 degree of freedom. J-100, a reversed J-shaped distribution with a point mass at 1.0 representing 100% overstatements was also modelled by a χ_1^2 distribution. Neter et al. [36], in their empirical study of dollar-unit taint distributions for accounts receivable and inventory populations, found the existence of a point mass at 1.0 to be a common characteristic for such populations. A χ^2 distribution with 3 degrees of freedom was used to model a unimodal distribution, which would reflect higher average taintings than the reverse J-shaped distributions. Finally, a uniform(0,1) distribution was included in the study to represent an extreme audit environment. The χ^2 distributions were scaled by 1/10 in all cases to yield means between 0 and 1.

To examine the tightness of the modified multinomial bound, solitary samples were set up as replicates of the above distributions with varying number of errors to reflect different error rates. For each such sample the modified multinomial bound and the Stringer bound were computed. From the study of these typical tainting patterns it was observed that the modified multinomial bound was substantially tighter than the

Stringer bound in all cases. The tightness of the bound, with respect to the Stringer bound, depended to some degree on the tainting distribution and the error rate. For each tainting pattern, the modified multinomial bound was computed with cluster sizes ranging from five to ten. Though the bound became tighter as the cluster size increased, computing costs tended to increase with number of clusters. The authors found five or six clusters to provide sufficient results for a wide range of tainting distributions.

The study of typical tainting patterns was augmented with a simulation study based on the same tainting distributions with various error rates. Five hundred samples of size 100 were generated from each study population. The modified multinomial bound with six clusters and the Stringer bound were calculated for each study. From this simulation, the reliability of the modified multinomial bound could be examined. The actual level of coverage for the modified multinomial bound was above the nominal level in three of the four cases presented. Only in the case of the uniform tainting distribution did the achieved coverage fall slightly below the nominal level.

4.3 Moment Bound

Dworin and Grimlund [14] proposed a parametric bound which, unlike the multinomial bound, is simple to compute. They approximate the mean tainting sampling distribution by a three-parameter gamma distribution. The parameter values are estimated using the method of moments. Like previous CAV methods, the error rate is analysed separately from the observed taints. The sample moments of the error rate likelihood distribution are computed, as are the sample moments of the error tainting distribution. These moments are then combined, yielding the first three moments of the sampling distribution for the mean tainting. An upper confidence bound for the mean taint, and eventually the total error (E), is constructed based on the percentiles of the gamma distribution. This moment bound has the advantage of easily allowing for both overstatements and understatements.

It is necessary to introduce some notation for the moments of a distribution. In the following section, the r'th moment about the origin, called the noncentral moment, of the random variable X is denoted $\upsilon_r'(X)$. The r'th moment about the mean, called the central moment, is denoted $\upsilon_r(X)$.

Let $f_{\overline{T}}(\bar{t})$ be the sampling distribution for the sample mean error tainting $\bar{t}$, then a $100(1-\alpha)\%$ upper bound for μ_t is $\hat{U}_u'$ such that

$$\int_{-\infty}^{\hat{U}_u'} f_{\overline{T}}(\bar{t})\, d\bar{t} = 1 - \alpha. \tag{4.29}$$

As mentioned above, the authors employ the three-parameter gamma distribution as an approximation for the sampling distribution. Thus, the density is given by

$$f_{\overline{T}}(\bar{t}) = \frac{(\bar{t} - G)^{A-1} \exp[-\frac{(\bar{t}-G)}{B}]}{B^A \Gamma(A)}, \tag{4.30}$$

where $G < \bar{t} < \infty$, $B, A > 0$ and $-\infty < G < \infty$. Depending on the values of the sample moments of the error tainting distribution, the estimate for the parameter B

may be negative. For this case, Dworin and Grimlund take the lower α fractile of the gamma distribution as the $100(1-\alpha)\%$ upper bound for μ_t.

The upper confidence bound is approximated using the Wilson-Hilferty approximation for the percentiles of the gamma distribution. Based on this method, the following is an approximation to a $100(1-\alpha)\%$ upper confidence bound for the mean error tainting:

$$\hat{U}'_u = G + AB\left[1 + \frac{z_{(1-\alpha)}}{\sqrt{9A}} - \frac{1}{9A}\right]^3 \tag{4.31}$$

where $z_{(1-\alpha)}$ is the $(1-\alpha)100$'th percentile of the standard normal distribution. A $100(1-\alpha)\%$ upper confidence bound for the total error E is thus given by

$$\hat{E}'_u = Y\left(G + AB\left[1 + \frac{z_{(1-\alpha)}}{\sqrt{9A}} - \frac{1}{9A}\right]^3\right). \tag{4.32}$$

The three parameters A, B and G are estimated by the method of moments. The r'th noncentral moment of the gamma distribution is given by

$$\upsilon'_r = \sum_{i=0}^{r} \binom{r}{i} G^{r-i} B^i \Gamma(i+A)/\Gamma(A). \tag{4.33}$$

From this equation, the first three noncentral moments of the sampling distribution of the mean error tainting are

$$\begin{aligned}
\upsilon'_1(\bar{t}) &= G + BA \\
\upsilon'_2(\bar{t}) &= G^2 + 2GBA + B^2A(A+1) \\
\upsilon'_3(\bar{t}) &= G^3 + 3G^2BA + 3GB^2A(A+1) + B^3A(A+1)(A+2).
\end{aligned} \tag{4.34}$$

Central moments can be derived from noncentral moments by the following relationships:

$$
\begin{aligned}
\upsilon_1 &= 0 \\
\upsilon_2 &= \upsilon_2' - \upsilon_1'^{\,2} \\
\upsilon_3 &= \upsilon_3' - 3\upsilon_2'\upsilon_1' + 2\upsilon_1'^{\,3}.
\end{aligned} \tag{4.35}
$$

Thus, the second and third central moments of the sampling distribution are

$$
\begin{aligned}
\upsilon_2(\overline{t}) &= G^2 + 2GBA + B^2A(A+1) - (G+BA)^2 \\
&= G^2 + 2GBA + B^2A(A+1) - G^2 - 2GBA - B^2A^2 \\
&= B^2A^2 + B^2A - B^2A^2 \\
&= B^2A \\
\upsilon_3(\overline{t}) &= G^3 + 3G^2BA + 3GB^2A(A+1) + B^3A(A+1)(A+2) \\
&\quad - 3(G+BA)(G^2 + 2GBA + B^2A(A+1)) + 2(G+BA)^3 \\
&= G^3 + 3G^2BA + 3GB^2A^2 + 3GB^2A + B^3A^3 + 3B^3A^2 + 2B^3A - 3G^3 \\
&\quad - 6G^2BA - 3GB^2A^2 - 3GB^2A - 3G^2BA - 6GB^2A^2 - 3B^3A^3 - 3B^3A^2 \\
&\quad + 2G^3 + 6G^2BA + 6GB^2A^2 + 2B^3A^3 \\
&= 2B^3A.
\end{aligned} \tag{4.36}
$$

Solving the first non-central moment equation and the second and third central moment equations for A, B and G yields the following relations:

$$
A = \frac{4\upsilon_2(\overline{t})^3}{\upsilon_3(\overline{t})^2}
$$

$$B = \frac{\upsilon_3(\overline{t})}{2\upsilon_2(\overline{t})}$$

$$G = \upsilon_1'(\overline{t}) - \frac{2\upsilon_2(\overline{t})^2}{\upsilon_3(\overline{t})}. \tag{4.37}$$

Now, the problem is reduced to estimating the first three central moments of the sampling distribution $f_{\overline{T}}(\overline{t})$.

A dollar-unit sample of size n is selected from the population. Let m be the number of nonzero taints in the sample. The average tainting ($\overline{z}$) of the nonzero errors is calculated. At this point a hypothetical tainting observation, z^*, is constructed. The formula for z^* varies depending on whether the population of interest is accounts receivable or inventory. For inventory data

$$z^* = .81[1 - .667\tanh(10\overline{z})] \tag{4.38}$$

and for accounts receivable data

$$z^* = .81[1 - .667\tanh(10\overline{z})] * [1 + .667\tanh(n/10)]. \tag{4.39}$$

If $\overline{z} < 0$, $\overline{z}$ is set equal to 0 in order to calculate z^*.

The hypothetical taint z^* is treated as an additional observed taint in calculating the sample moments of the unknown error distribution. The moment bound resulting from this additional observation tends to be larger than had this hypothetical observation not been introduced. The authors feel this conservatism compensates for the lack of information due to the small number of nonzero errors observed. As m, the number of nonzero errors in the sample, increases, the impact of the point z^* lessens.

The hyperbolic tangent function was chosen for its general shape and the various constants were selected numerically through extensive testing of audit populations. An error free sample yields $z^* = .81$. This value was selected so that the moment bound is approximately the same as the Stringer and the multinomial bounds for the case of no errors in the sample. Overall, the form of z^* was selected to ensure the reliability of the nominal confidence level of the moment bound.

Following the addition of z^* to the sample, the noncentral sample moments of the error tainting distribution are calculated as follows:

$$\hat{v}_j'(z) = [(z^*)^j + \sum_{i=1}^{m} z_i^j]/(m+1) \quad j = 1, 2, 3 \tag{4.40}$$

These moments are used to represent the distribution of the nonzero taints, $f(z)$.

The overall tainting distribution is a mixture of zero values and nonzero taintings drawn from $f(z)$ with probability π. This mixture model, shown below, has been discussed previously

$$t = \begin{cases} z & \text{with probability } \pi \\ 0 & \text{with probability } (1-\pi) \end{cases} \tag{4.41}$$

The first three noncentral moments of this distribution, which is conditional on π, can easily be shown to be

$$\begin{aligned} v_1'(t|\pi) &= \pi v_1'(z) \\ v_2'(t|\pi) &= \pi v_2'(z) \\ v_3'(t|\pi) &= \pi v_3'(z) \end{aligned} \tag{4.42}$$

The central moments can then be calculated, using (4.35):

$$\begin{aligned} \upsilon_2(t|\pi) &= \upsilon_2'(t|\pi) - \upsilon_1'(t|\pi)^2 \\ \upsilon_3(t|\pi) &= \upsilon_3'(t|\pi) - 3\upsilon_1'(t|\pi)\,\upsilon_2'(t|\pi) + 2\upsilon_1'(t|\pi)^3 \end{aligned} \tag{4.43}$$

We are interested in the sampling distribution of the mean error tainting ($\bar{t}$). This distribution is conditional on the error rate, π. The parameter π is treated like a nuisance parameter and is eliminated through integration. The authors assume the error rate follows a beta distribution with parameters $m+1$ and $n-m+1$, so

$$f(\pi) = \frac{\Gamma(n+2)}{\Gamma(m+1)\Gamma(n-m+1)}\pi^m(1-\pi)^{n-m} \tag{4.44}$$

The first three noncentral moments of this distribution are given below:

$$\begin{aligned} \upsilon_1'(\pi) &= \int_0^1 \pi \frac{\Gamma(n+2)}{\Gamma(m+1)\Gamma(n-m+1)}\pi^m(1-\pi)^{n-m}\,d\pi \\ &= \frac{m+1}{n+2} \end{aligned}$$

$$\begin{aligned} \upsilon_2'(\pi) &= \int_0^1 \pi^2 \frac{\Gamma(n+2)}{\Gamma(m+1)\Gamma(n-m+1)}\pi^m(1-\pi)^{n-m}\,d\pi \\ &= \frac{(m+2)}{(n+3)}\upsilon_1'(\pi) \end{aligned}$$

$$\begin{aligned} \upsilon_3'(\pi) &= \int_0^1 \pi^3 \frac{\Gamma(n+2)}{\Gamma(m+1)\Gamma(n-m+1)}\pi^m(1-\pi)^{n-m}\,d\pi \\ &= \frac{(m+3)}{(n+4)}\upsilon_2'(\pi) \end{aligned} \tag{4.45}$$

The noncentral moments of the unconditional sampling distribution of $\bar{t}$ can be calculated by integrating out π.

$$\begin{aligned}
\upsilon_i'(\bar{t}) &= \int_{-\infty}^{\infty} \bar{t}^i f_{\overline{T}}(\bar{t})\, d\bar{t} \\
&= \int_{-\infty}^{\infty} \bar{t}^i \left[\int_0^1 f_{\overline{T}}(\bar{t}|\pi) f(\pi)\, d\pi\right] d\bar{t} \\
&= \int_0^1 \int_{-\infty}^{\infty} \bar{t}^i f_{\overline{T}}(\bar{t}|\pi)\, d\bar{t} f(\pi)\, d\pi \\
&= \int_0^1 \upsilon_i'(\bar{t}|\pi) f(\pi)\, d\pi \qquad i = 1, 2, 3
\end{aligned} \tag{4.46}$$

We require the first three noncentral moments of the sampling distribution of $\bar{t}$, conditional on π; $\upsilon_i'(\bar{t}|\pi)$, $i = 1, 2, 3$. Let $T = \sum_{i=1}^{n} t_i$, then

$$\upsilon_i'(\bar{t}|\pi) = E(\bar{t}^i) = E\left(\left[\frac{1}{n}\sum_{i=1}^{n} t_i\right]^i\right) = \frac{1}{n^i} E\left(\left[\sum_{i=1}^{n} t_i\right]^i\right). \tag{4.47}$$

Thus, we can determine the moments for T and divide by the appropriate power of n. The moments of T will be derived via the moment generating function.

The taints, t_i, are independent random variables, each having moment generating function $\phi_t(s) = E(e^{st})$. Hence, the moment generating function of T is

$$\phi_T(s) = [\phi_t(s)]^n = [E(e^{st})]^n. \tag{4.48}$$

Each moment of the distribution of T can now be obtained through differentiation, specifically

$$\phi_T{}^{(r)}(0) = E(T^r) = \upsilon_r'(T|\pi). \tag{4.49}$$

The first three derivatives are

$$\phi_T'(s) = n[E(e^{st})]^{n-1} E(te^{st})$$

$$\phi_T''(s) = n(n-1)[E(e^{st})]^{n-2}E(te^{st})^2 + n[E(e^{st})]^{n-1}E(t^2e^{st})$$

$$\begin{aligned}\phi_T'''(s) &= n(n-1)(n-2)[E(e^{st})]^{n-3}E(te^{st})^3 + 3n(n-1)[E(e^{st})]^{n-2}E(te^{st})E(t^2e^{st}) \\ &\quad + n[E(e^{st})]^{n-1}E(t^3e^{st})\end{aligned} \tag{4.50}$$

Evaluating the above derivatives at $s = 0$ yields the following noncentral moments:

$$\begin{aligned}\upsilon_1'(T|\pi) = \phi_T'(0) &= nE(t) \\ &= n\upsilon_1'(t|\pi)\end{aligned}$$

$$\begin{aligned}\upsilon_2'(T|\pi) = \phi_T''(0) &= n(n-1)E(t)^2 + nE(t^2) \\ &= n(n-1)\upsilon_1'(t|\pi)^2 + n\upsilon_2'(t|\pi)\end{aligned}$$

$$\begin{aligned}\upsilon_3'(T|\pi) = \phi_T'''(0) &= n(n-1)(n-2)E(t)^3 + 3n(n-1)E(t)E(t^2) + nE(t^3) \\ &= n(n-1)(n-2)\upsilon_1'(t|\pi)^3 + 3n(n-1)\upsilon_1'(t|\pi)\,\upsilon_2'(t|\pi) \\ &\quad + n\upsilon_3'(t|\pi)\end{aligned} \tag{4.51}$$

Thus, by (4.47)

$$\begin{aligned}\upsilon_1'(\bar{t}|\pi) &= \frac{1}{n}\upsilon_1'(T|\pi) \\ &= \upsilon_1'(t|\pi)\end{aligned}$$

$$\begin{aligned}
v_2'(\bar{t}|\pi) &= \frac{1}{n^2} v_2'(T|\pi) \\
&= \frac{n-1}{n} v_1'(t|\pi)^2 + \frac{1}{n} v_2'(t|\pi)
\end{aligned}$$

$$\begin{aligned}
v_3'(\bar{t}|\pi) &= \frac{1}{n^3} v_3'(T|\pi) \\
&= \frac{(n-1)(n-2)}{n^2} v_1'(t|\pi)^3 + \frac{3(n-1)}{n^2} v_1'(t|\pi)\, v_2'(t|\pi) \\
&\quad + \frac{1}{n^2} v_3'(t|\pi) \qquad (4.52)
\end{aligned}$$

Using previous results, (4.42) and (4.52), it can be shown that each of the $v_i'(\bar{t}|\pi)$ is a linear combination of π and the $v_i'(z)$.

$$v_1'(\bar{t}|\pi) = v_1'(t|\pi) = \pi v_1'(z)$$

$$\begin{aligned}
v_2'(\bar{t}|\pi) &= \frac{n-1}{n} v_1'(t|\pi)^2 + \frac{1}{n} v_2'(t|\pi) \\
&= \frac{n-1}{n} \pi^2 v_1'(z)^2 + \frac{1}{n} \pi v_2'(z)
\end{aligned}$$

$$\begin{aligned}
v_3'(\bar{t}|\pi) &= \frac{(n-1)(n-2)}{n^2} v_1'(t|\pi)^3 + \frac{3(n-1)}{n^2} v_1'(t|\pi)\, v_2'(t|\pi) \\
&\quad + \frac{1}{n^2} v_3'(t|\pi) \\
&= \frac{(n-1)(n-2)}{n^2} \pi^3 v_1'(z)^3 + \frac{3(n-1)}{n^2} \pi^2 v_1'(z)\, v_2'(z) \\
&\quad + \frac{1}{n^2} \pi v_3'(z) \qquad (4.53)
\end{aligned}$$

Now it is straightforward to integrate in order to find the noncentral moments of the unconditional sampling distribution:

$$\begin{aligned}
\upsilon_1'(\bar{t}) &= \int_0^1 \upsilon_1'(\bar{t}|\pi) f(\pi)\, d\pi \\
&= \int_0^1 \pi \upsilon_1'(z) f(\pi)\, d\pi \\
&= \upsilon_1'(z) \int_0^1 \pi f(\pi)\, d\pi \\
&= \upsilon_1'(z)\, \upsilon_1'(\pi)
\end{aligned}$$

$$\begin{aligned}
\upsilon_2'(\bar{t}) &= \int_0^1 \upsilon_2'(\bar{t}|\pi) f(\pi)\, d\pi \\
&= \int_0^1 \left[\frac{n-1}{n} \pi^2 \upsilon_1'(z)^2 + \frac{1}{n} \pi \upsilon_2'(z) \right] f(\pi)\, d\pi \\
&= \frac{n-1}{n} \upsilon_1'(z)^2 \int_0^1 \pi^2 f(\pi)\, d\pi + \frac{1}{n} \upsilon_2'(z) \int_0^1 \pi f(\pi)\, d\pi \\
&= \frac{n-1}{n} \upsilon_1'(z)^2\, \upsilon_2'(\pi) + \frac{1}{n} \upsilon_2'(z)\, \upsilon_1'(\pi)
\end{aligned}$$

$$\begin{aligned}
\upsilon_3'(\bar{t}) &= \int_0^1 \upsilon_3'(\bar{t}|\pi) f(\pi)\, d\pi \\
&= \int_0^1 \left[\frac{(n-1)(n-2)}{n^2} \pi^3 \upsilon_1'(z)^3 + \frac{3(n-1)}{n^2} \pi^2 \upsilon_1'(z)\, \upsilon_2'(z) + \frac{1}{n^2} \pi \upsilon_3'(z) \right] f(\pi)\, d\pi \\
&= \frac{(n-1)(n-2)}{n^2} \upsilon_1'(z)^3 \int_0^1 \pi^3 f(\pi)\, d\pi \\
&\quad + \frac{3(n-1)}{n^2} \upsilon_1'(z)\, \upsilon_2'(z) \int_0^1 \pi^2 f(\pi)\, d\pi + \frac{1}{n^2} \upsilon_3'(z) \int_0^1 \pi f(\pi)\, d\pi \\
&= \frac{(n-1)(n-2)}{n^2} \upsilon_1'(z)^3\, \upsilon_3'(\pi) \\
&\quad + \frac{3(n-1)}{n^2} \upsilon_1'(z)\, \upsilon_2'(z)\, \upsilon_2'(\pi) + \frac{1}{n^2} \upsilon_3'(z)\, \upsilon_1'(\pi) \qquad (4.54)
\end{aligned}$$

Now the central moments required for the calculation of the upper bound can be calculated, again using (4.35):

$$\begin{aligned} \upsilon_2(\bar{t}) &= \upsilon_2'(\bar{t}) - \upsilon_1'(\bar{t})^2 \\ \upsilon_3(\bar{t}) &= \upsilon_3'(\bar{t}) - 3\upsilon_2'(\bar{t})\, \upsilon_1(\bar{t}) + 2\upsilon_1'(\bar{t})^3 \end{aligned} \tag{4.55}$$

The following summary, adapted from Dworin and Grimlund [14] Table 1, serves to highlight the procedure for finding the moment bound. These steps could be combined further. A sample of size n is drawn and the number of nonzero taints in the sample is m.

1. Average and Hypothetical Error Taintings

$$\bar{z} = \frac{1}{m}\sum_{i=1}^{m} z_i$$

$$\begin{aligned} z^* &= .81[1 - .667\tanh(10\bar{z})] \\ \text{or}\quad z^* &= .81[1 - .667\tanh(10\bar{z})][1 + .667\tanh(m/10)] \end{aligned}$$

2. Error Tainting Noncentral Moments

$$\hat{\upsilon}_j'(z) = \frac{(z^*)^j + \sum_{i=1}^{m} {z_i}^j}{m+1} \qquad j = 1, 2, 3$$

3. Error Rate Noncentral Moments

$$\begin{aligned} \hat{\upsilon}_1'(\pi) &= \frac{m+1}{n+2} \\ \hat{\upsilon}_2'(\pi) &= \frac{m+2}{n+3}\upsilon_1'(\pi) \\ \hat{\upsilon}_3'(\pi) &= \frac{m+3}{n+4}\upsilon_2'(\pi) \end{aligned}$$

4. Mean Taint Noncentral Moments

$$
\begin{aligned}
\hat{\upsilon}'_1(\bar{t}) &= \hat{\upsilon}'_1(\pi)\hat{\upsilon}'_1(z) \\
\hat{\upsilon}'_2(\bar{t}) &= \frac{\hat{\upsilon}'_1(\pi)\,\hat{\upsilon}'_2(z) + (n-1)\hat{\upsilon}'_2(\pi)\,\hat{\upsilon}'_1(z)^2}{n} \\
\hat{\upsilon}'_3(\bar{t}) &= \frac{\hat{\upsilon}'_1(\pi)\,\hat{\upsilon}'_3(z) + 3(n-1)\hat{\upsilon}'_2(\pi)\,\hat{\upsilon}'_1(z)\,\hat{\upsilon}'_2(z)}{n^2} \\
&\quad + \frac{(n-1)(n-2)\hat{\upsilon}'_3(\pi)\,\hat{\upsilon}'_1(z)^3}{n^2}
\end{aligned}
$$

5. Mean Taint Central Moments

$$
\begin{aligned}
\hat{\upsilon}_2(\bar{t}) &= \hat{\upsilon}'_2(\bar{t}) - \hat{\upsilon}'_1(\bar{t})^2 \\
\hat{\upsilon}_3(\bar{t}) &= \hat{\upsilon}'_3(\bar{t}) - 3\hat{\upsilon}'_1(\bar{t})\,\hat{\upsilon}'_2(\bar{t}) + 2\hat{\upsilon}'_1(\bar{t})^3
\end{aligned}
$$

6. Gamma Distribution Parameters

$$
\begin{aligned}
\hat{A} &= \frac{4\hat{\upsilon}_2(\bar{t})^3}{\hat{\upsilon}_3(\bar{t})^2} \\
\hat{B} &= \frac{\hat{\upsilon}_3(\bar{t})}{2\hat{\upsilon}_2(\bar{t})} \\
\hat{G} &= \hat{\upsilon}'_1(\bar{t}) - \frac{2\hat{\upsilon}_2(\bar{t})^2}{\hat{\upsilon}_3(\bar{t})}.
\end{aligned}
$$

7. $1 - \alpha$ Upper Confidence Bound for the Mean Error

$$\hat{U}'_u = \hat{G} + \hat{A}\hat{B}\left[1 + \frac{z_{(1-\alpha)}}{\sqrt{9\hat{A}}} - \frac{1}{9\hat{A}}\right]^3$$

8. $1 - \alpha$ Upper Confidence Bound for the Total Error

$$\hat{E}'_u = Y\left(\hat{G} + \hat{A}\hat{B}\left[1 + \frac{z_{(1-\alpha)}}{\sqrt{9\hat{A}}} - \frac{1}{9\hat{A}}\right]^3\right)$$

The above steps are demonstrated through an example. Consider a dollar-unit sample of size 100 drawn from an accounts receivable population of \$1,000,000. Four nonzero errors are discovered with tainting values of $-.16$, .01, .18, and .47. The 95% moment upper confidence bound for the total error is calculated.

1. $$\overline{z} = 1.325 \times 10^{-1}$$

$$z^* = 4.275 \times 10^{-1}$$

2. $\hat{v}_1'(z) = 1.915 \times 10^{-1}$ $\quad$ $\hat{v}_2'(z) = 9.264 \times 10^{-2}$ $\quad$ $\hat{v}_3'(z) = 3.675 \times 10^{-2}$

3. $\hat{v}_1'(\pi) = 4.902 \times 10^{-2}$ $\quad$ $\hat{v}_2'(\pi) = 2.856 \times 10^{-2}$ $\quad$ $\hat{v}_3'(\pi) = 1.922 \times 10^{-4}$

4. $\hat{v}_1'(\overline{t}) = 9.387 \times 10^{-3}$ $\quad$ $\hat{v}_2'(\overline{t}) = 1.491 \times 10^{-4}$ $\quad$ $\hat{v}_3'(\overline{t}) = 2.994 \times 10^{-6}$

5. $\hat{v}_2(\overline{t}) = 6.096 \times 10^{-5}$ $\quad$ $\hat{v}_3(\overline{t}) = 4.502 \times 10^{-7}$

6. $\hat{A} = 4.4714$ $\quad$ $\hat{B} = 3.692 \times 10^{-3}$ $\quad$ $\hat{G} = -7.120 \times 10^{-3}$

7. $$\hat{U}_u' = 2.393 \times 10^{-2}$$

8. $$\hat{E}_u' = 23,930$$

Dworin and Grimlund [14] performed a simulation study to examine the viability of this moment bound. Study populations, reflecting a variety of error rates and tainting distributions, were generated based on results of empirical studies of real accounting populations. The method Dworin and Grimlund used to generate these study populations has since been used in other studies (Grimlund and Felix [22]; Clayton [10]). This method is described extensively in Chapter 7 of this report. Basically, a variety of error rate combinations and tainting models are employed.

Each error rate combination includes an overall error rate and the percentage of errors falling into three categories: understatements, non-100% overstatements and 100% overstatements. Each tainting model includes a distribution for understatements and a distribution for non-100% overstatements. Dworin and Grimlund used 27 error rate combinations to model accounts receivables and 32 error rate combinations to model inventory. They used the same four tainting models in each case, yielding 108 accounts receivable study populations and 128 inventory study populations. The tainting models involved χ^2 distributions with 1, 2 and 3 degrees of freedom and the uniform distribution, following the methods of Leitch et al. [28].

For each study population, a random sample of size 100 was drawn repeatedly 1000 times. The 95% moment bound was calculated for each sample. Since the true mean error tainting is known for each study population, this allows the actual coverage of the bound to be calculated. The Stringer bound was also calculated for each sample in order to compute the relative advantage of the moment bound with respect to the Stringer bound. The relative advantage was defined as (Stringer bound - moment bound)/Stringer bound. The average relative advantage for the 1000 samples was computed as a summary statistic. The moment bound takes into account both overstatement and understatement errors. This is expected to provide a significant enhancement for inventory data which tends to contain a higher percentage of understatement errors than accounts receivable data [36]. Thus, a comparison of the moment bound to the Stringer bound, which assumes all errors are overstatements, would provide little information for the inventory studies. For comparative purposes, the authors calculated two CAV bounds which consider both understatement and overstatement errors. The authors determined the relative advantage of the moment bound to the Stringer bound and compared this with the relative advantage of the dollar-unit mean-per-unit bound and the LTA bound to the Stringer bound.

The simulation indicated that the nominal confidence level of the moment bound is reliable. The average coverage of the moment bound for all test populations was found to be 98.2% for accounts receivable and 98.8% for inventory. These numbers suggest the bound is somewhat conservative. The moment bound was found to be significantly tighter than the Stringer bound and for inventory populations, containing both understatement and overstatement errors, significantly tighter than popular CAV bounds.

The moment bound was also compared to the multinomial bound. This comparison was limited to sets of positive error taintings for which multinomial bounds had been calculated previously by Neter et al. [35] and Leitch et al. [28]. It was found that both the multinomial and moment bound were consistently tighter than the Stringer bound. The moment bound was tighter than the multinomial bound whenever this bound was 4% or less of the total book value. Otherwise, the multinomial bound was tighter.

4.4 Modified Moment Bound

Dworin and Grimlund [15] proposed a modification of the moment bound to reduce its conservatism to some degree. Evidence in the simulation study from their earlier paper suggested the moment bound is overly conservative when the number of understatements is equal to or exceeds the number of overstatements. The modification deals with the formation of the hypothetical tainting observation as in (4.38) and (4.39). In the case where $\bar{z} < 0$, $\bar{z}$ is replaced by $|\bar{z}|$, rather than 0 as for the original bound. Essentially this amounts to replacing $\bar{z}$ by $|\bar{z}|$ in all cases.

The authors replicated the simulation studies of Dworin and Grimlund [14] to examine the reliability of the modified moment bound. The modification caused the reliability level of the bound to decrease, but the actual coverage percentages were still at least as high as the nominal confidence level. The reduction in reliability was greater for the inventory populations than the accounts receivables due to the fact that a greater number of inventory samples would have $\bar{z} < 0$ and subsequently benefit from the modification.

To examine the relative difference in size between the two versions of the moment bound, several examples with $\bar{z} < 0$ were examined. The modified moment bound demonstrated a moderate improvement over the original bound for these cases. As the number of understatements increased and $\bar{z}$ became increasingly negative, the difference between the two bounds grew.

Chan and Smieliauskas [9] used the real Neter and Loebbecke populations 1 and

2 described briefly in Chapter 2 to examine the performance of the two-sided modified moment bound. Each of these populations contained both overstatements and understatements. As with the one-sided bound, the two-sided bounds are calculated using the Wilson-Hilferty approximation for the percentiles of the gamma distribution. Both the two-sided and one-sided bounds were calculated for 500 dollar-unit samples from each of the 10 study populations.

The authors found that the behaviour of the one-sided and two-sided bounds were quite different. The one-sided bound was very reliable, achieving the nominal 95% confidence level for all of the populations examined. In contrast, the two-sided bound was less reliable; in many cases the actual coverage of the two-sided bound was less than the nominal level. Two factors may be causing this effect. First of all, the hypothetical error tainting was formulated for one-sided upper bounds only. The form of the hypothetical tainting may have to be modified for use with two-sided bounds. Secondly, the Wilson-Hilferty approximation is based on the symmetric standard normal distribution though the gamma distribution is positively skewed and this may cause bias in the calculation of the bounds.

Chapter 5

Bayesian Methods

All the methods reviewed so far have been based on the frequentist approach to inference. There have been several methods proposed which rely on the Bayesian approach. The first stages of an audit involve compliance testing to gauge the reliability of a company's internal control system. These tests reveal information to the auditor. Auditors also have empirical evidence regarding accounting populations from studies such as Johnson, Leitch and Neter [26] and Neter, Johnson and Leitch [36]. Bayesian inference provides a means of combining the auditor's prior knowledge with sample information.

Consider a simple case in which the random variable of interest is E, the total overstatement error amount. The density $f(E)$ reflects prior information about E. The likelihood of a certain sample result being obtained given E, $l(s|E)$, can then be combined with the prior density to obtain a posterior density function $f(E|s)$ using

Bayes theorem as follows:

$$f(E|s) = \frac{f(E)l(s|E)}{\int_0^\infty f(E)l(s|E)\,dE}. \tag{5.1}$$

This posterior density can be thought of as an update of the prior density, using information in the sample.

The $(1-\alpha)100$'th percentile of the posterior density is a $100(1-\alpha)\%$ upper Bayesian bound for the total overstatement error. This percentile, say E^*, is the value such that

$$P(E \le E^*|s) = \int_0^{E^*} f(E|s)\,dE = 1-\alpha. \tag{5.2}$$

A Bayesian interval has a different interpretation than a classical confidence interval. In the classical approach we consider the characteristic of interest, here E, to be an unknown constant and the confidence coefficient is interpreted as the relative frequency of correct intervals in repeated sampling from the same population. In contrast, the Bayesian approach considers the characteristic of interest to be a random variable and the coefficient gives the probability, given the sample result, that the interval contains the random variable. In this sense a Bayesian interpretation applies to the particular interval while a classical, frequentist approach does not.

5.1 Normal Bound

Felix and Grimlund [16] developed a parametric Bayesian model for the total error in an accounting population, based on line item sampling. Their approach involves the combination of separate priors for the error rate π and the distribution of the nonzero errors z. Menzefricke and Smieliauskas [31] used this approach to model the mean error per dollar unit for dollar-unit sampling, and this method will be presented here. As in the Felix and Grimlund model, the error distribution and error rate are initially treated separately. One advantage of this method is that no special adjustments are required to handle understatements.

The true population rate of dollars in error, π, is assumed to have a beta prior distribution with parameters $n_0\pi_0$ and $n_0(1-\pi_0)$. The parameter π_0 is the auditor's prior expected value of π and n_0 can be viewed as the prior sample size reflecting the auditor's confidence in the choice for π_0. The prior density is

$$f(\pi) = \frac{1}{B[n_0\pi_0, n_0(1-\pi_0)]}\pi^{n_0\pi_0-1}(1-\pi)^{n_0(1-\pi_0)-1}, \tag{5.3}$$

where

$$B[n_0\pi_0, n_0(1-\pi_0)] = \frac{\Gamma(n_0\pi_0)\Gamma[n_0(1-\pi_0)]}{\Gamma(n_0)}. \tag{5.4}$$

The number of errors, m, in a sample of size n is assumed to be generated by a Bernoulli process. The likelihood of such a sample can be written

$$\begin{aligned} l(m|\pi) &= \prod_{i=1}^{n} \pi^{x_i}(1-\pi)^{1-x_i} \quad x_i = 0, 1 \text{ for all } i \\ &= \pi^m(1-\pi)^{n-m}. \end{aligned} \tag{5.5}$$

The posterior distribution of π can then be calculated.

$$f(\pi|m) = \frac{f(\pi)l(m|\pi)}{\int_0^1 f(\pi)l(m|\pi)\,d\pi} = \frac{1}{B(m_1, n_1-m_1)}\pi^{m_1-1}(1-\pi)^{n_1-m_1-1} \tag{5.6}$$

The posterior distribution of π is again a beta distribution with the parameters $m_1 = n_0\pi_0 + m$ and $n_1 - m_1 = (n_0 + n) - (n_0\pi_0 + m)$. Thus the posterior mean and variance of π are

$$E(\pi) = \frac{m_1}{n_1} \tag{5.7}$$

and

$$Var(\pi) = \frac{m_1(n_1 - m_1)}{n_1^2(n_1 + 1)}, \tag{5.8}$$

where $n_1 = n_0 + n$.

The density of nonzero taintings, z, is assumed to be normal with mean μ_z and variance σ_z^2, both unknown. Prior information is expressed in terms of prior distributions for these parameters. The prior for μ_z conditional on the precision, $h = \sigma_z^{-2}$, is assumed to be normal with mean μ_0 and variance σ_z^2/r_0. The precision is assumed to have a gamma prior distribution with parameters $\theta_0/2$ and $2/\theta_0\phi_0$. μ_0 and $1/\phi_0$ are the auditor's prior expected values of μ_z and h; r_0 and θ_0 are prior sample sizes, reflecting the auditor's confidence in the choice of μ_0 and ϕ_0. This yields a joint normal-gamma prior distribution for (μ_z, h) with density

$$\begin{aligned} f(\mu_z, h) &= f(\mu_z|h)f(h) \\ &= \frac{1}{\sqrt{2\pi/hr_0}} \exp\left[-\frac{1}{2}hr_0(\mu_z - \mu_0)^2\right] \\ &\qquad \times \frac{h^{\theta_0/2-1}(\theta_0\phi_0/2)^{\theta_0/2}}{\Gamma(\theta_0/2)} \exp\left[-h\theta_0\phi_0/2\right] \end{aligned} \tag{5.9}$$

The likelihood for a sample of nonzero errors $(z_1, \ldots, z_m)$ from a normal process with parameters $(\mu_z, \sigma_z^2 = \frac{1}{h})$ is

$$\begin{aligned} l(\boldsymbol{z}|\mu_z, h) &= h^{m/2} \exp\left\{-\frac{h}{2}\sum_{i=1}^{m}(z_i - \mu_z)^2\right\} \\ &= h^{m/2} \exp\left\{-\frac{h}{2}[(m-1)s_z^2 + m(\overline{z} - \mu_z)^2]\right\} \end{aligned} \tag{5.10}$$

where $\overline{z} = \frac{1}{m}\sum_{i=1}^{m} z_i$ and $s_z^2 = \frac{1}{m-1}\sum_{i=1}^{m}(z_i - \overline{z})^2$.

Now the posterior distribution of (μ_z, h) can be derived:

$$\begin{aligned} f(\mu_z, h|\boldsymbol{z}) &\propto l(\boldsymbol{z}|\mu_z, h) f(\mu_z, h) \\ &\propto h^{m/2} \exp\left\{-\frac{h}{2}[(m-1)s_z^2 + m(\overline{z} - \mu_z)^2]\right\} (hr_0)^{1/2} \exp\left[-\frac{hr_0}{2}(\mu_z - \mu_0)^2\right] \\ &\quad \times h^{\theta_0/2-1} \exp\left[-\frac{h}{2}\theta_0\phi_0\right] \\ &= h^{\frac{\theta_0+m-1}{2}} r_0^{1/2} \exp\left\{-\frac{h(r_0+m)}{2}\left[\mu_z^2 - 2\mu_z\frac{(r_0\mu_0 + m\overline{z})}{r_0+m}\right]\right\} \\ &\quad \times \exp\left\{-\frac{h}{2}\left[r_0\mu_0^2 + m\overline{z}^2 + \theta_0\phi_0 + (m-1)s_z^2\right]\right\} \\ &= h^{\frac{\theta_0+m-1}{2}} r_0^{1/2} \exp\left\{-\frac{h(r_0+m)}{2}\left[\mu_z - \frac{r_0\mu_0 + m\overline{z}}{r_0+m}\right]^2\right\} \\ &\quad \times \exp\left\{-\frac{h}{2}\left[r_0\mu_0^2 + m\overline{z}^2 + \theta_0\phi_0 + (m-1)s_z^2 - \frac{(r_0\mu_0 + m\overline{z})^2}{r_0+m}\right]\right\} \\ &\propto [h(r_0+m)]^{1/2} \exp\left\{-\frac{h(r_0+m)}{2}\left[\mu_z - \frac{r_0\mu_0 + m\overline{z}}{r_0+m}\right]^2\right\} \\ &\quad \times h^{\frac{\theta_0+m}{2}-1} \exp\left\{-\frac{h}{2}\left[\frac{r_0 m(\mu_0 - \overline{z})^2}{r_0+m} + \theta_0\phi_0 + (m-1)s_z^2\right]\right\} \end{aligned} \tag{5.11}$$

For the special case of $r_0 = 0$, the normal prior for μ_z is not valid. Assuming h and μ_z are independent a noninformative prior for μ_z can be employed, i.e., $f(\mu_z) = c$. Then following a similar procedure the posterior distribution of (μ_z,h) is derived as follows:

$$
\begin{aligned}
f(\mu_z, h|\boldsymbol{z}) &\propto c\,h^{m/2} \exp\left\{-\frac{h}{2}[(m-1)s_z^2 + m(\overline{z}-\mu_z)^2]\right\} \\
&\quad \times h^{\theta_0/2-1} \exp\left[-\frac{h}{2}\theta_0\phi_0\right] \\
&= c\,h^{\frac{\theta_0+m}{2}-1} \exp\left\{-\frac{h}{2}[m\mu_z^2 - 2m\mu_z\overline{z}]\right\} \\
&\quad \times \exp\left\{-\frac{h}{2}[m\overline{z}^2 + (m-1)s_z^2 + \theta_0\phi_0]\right\} \\
&\propto (hm)^{1/2} \exp\left\{-\frac{hm}{2}(\mu_z - \overline{z})^2\right\} \\
&\quad \times h^{\frac{\theta_0+m-1}{2}-1} \exp\left\{-\frac{h}{2}[(m-1)s_z^2 + \theta_0\phi_0]\right\} \qquad (5.12)
\end{aligned}
$$

In both cases the joint posterior distribution of (μ_z, σ_z^{-2}) is again normal-gamma. Combining results, the posterior distribution of μ_z given σ_z^{-2} is normal with mean μ_1 and variance σ_z^2/r_1 where

$$\mu_1 = \frac{r_0\mu_0 + m\overline{z}}{r_1} \qquad (5.13)$$

and

$$r_1 = r_0 + m. \qquad (5.14)$$

The posterior distribution of the precision, σ_z^{-2}, is gamma with parameters $\theta_1/2$ and $2/\theta_1\phi_1$ where

$$\theta_1 = \theta_0 + \delta(r_0) + m - 1 \qquad (5.15)$$

and

$$\theta_1\phi_1 = \theta_0\phi_0 + (m-1)s_z^2 + \frac{r_0 m(\mu_0 - \overline{z})^2}{r_1}. \qquad (5.16)$$

Here

$$\delta(r_0) = \begin{cases} 0 & \text{if } r_0 = 0 \\ 1 & \text{if } r_0 > 0 \end{cases}.$$

We actually require the distribution of $\mu_t = \pi\mu_z$, the mean portion of error per dollar unit over the entire population. The first step in determining this distribution is finding the marginal distribution of μ_z as follows:

$$\begin{aligned}
f(\mu_z|\boldsymbol{z}) &= \int_0^\infty f(\mu_z, \sigma_z^{-2}|\boldsymbol{z})\, d\sigma_z^{-2} \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_z^2/r_1}} \exp\left[-\frac{(\mu_z-\mu_1)^2}{2\sigma_z^2/r_1}\right] \frac{(\sigma_z^{-2})^{\frac{\theta_1}{2}-1}(\frac{\theta_1\phi_1}{2})^{\frac{\theta_1}{2}}}{\Gamma(\frac{\theta_1}{2})} \exp\left[-\sigma_z^{-2}\frac{\theta_1\phi_1}{2}\right] d\sigma_z^{-2} \\
&= \frac{(\frac{\theta_1\phi_1}{2})^{\frac{\theta_1}{2}}}{(2\pi/r_1)^{\frac{1}{2}}\Gamma(\frac{\theta_1}{2})} \frac{\Gamma(\frac{\theta_1+1}{2})}{\{\frac{1}{2}[r_1(\mu_z-\mu_1)^2+\theta_1\phi_1]\}^{\frac{\theta_1+1}{2}}} \int_0^\infty A\, d\sigma_z^{-2} \\
&= \frac{(\frac{\theta_1\phi_1}{2})^{\frac{\theta_1}{2}}}{(2\pi/r_1)^{\frac{1}{2}}\Gamma(\frac{\theta_1}{2})} \frac{\Gamma(\frac{\theta_1+1}{2})}{\{\frac{1}{2}[r_1(\mu_z-\mu_1)^2+\theta_1\phi_1]\}^{\frac{\theta_1+1}{2}}} \\
&= \frac{1}{(\pi/r_1)^{\frac{1}{2}}} \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{(\theta_1\phi_1)^{\frac{\theta_1}{2}}}{(\theta_1\phi_1)^{\frac{\theta_1+1}{2}}} \left[\frac{r_1(\mu_z-\mu_1)^2}{\theta_1\phi_1}+1\right]^{-\frac{\theta_1+1}{2}} \\
&= \frac{1}{(\pi/r_1)^{\frac{1}{2}}} \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{(\theta_1\phi_1)^{\frac{1}{2}}} \left[1+\frac{(\mu_z-\mu_1)^2}{\theta_1\phi_1/r_1}\right]^{-\frac{\theta_1+1}{2}} \qquad (5.17)
\end{aligned}$$

where

$$A = (\sigma_z^{-2})^{\frac{\theta_1+1}{2}-1} \exp\left[-\frac{r_1(\mu_z-\mu_1)^2+\theta_1\phi_1}{2\sigma_z^2}\right] \frac{[\frac{1}{2}(r_1(\mu_z-\mu_1)^2+\theta_1\phi_1)]^{\frac{\theta_1+1}{2}}}{\Gamma(\frac{\theta_1+1}{2})}. \qquad (5.18)$$

The transformed random variable $T_z = \frac{\mu_z-\mu_1}{\sqrt{\phi_1/r_1}}$ has a t distribution with θ_1 degrees of freedom since

$$f(T_z) = \frac{1}{(\pi/r_1)^{\frac{1}{2}}} \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{(\theta_1\phi_1)^{\frac{1}{2}}} \left[1+\frac{T_z^2}{\theta_1}\right]^{-\frac{\theta_1+1}{2}} \left(\frac{\phi_1}{r_1}\right)^{\frac{1}{2}}$$

$$= \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{\sqrt{\pi\theta_1}} \left[1 + \frac{T_z^2}{\theta_1}\right]^{-\frac{\theta_1+1}{2}} \tag{5.19}$$

The distribution of μ_z will be denoted $t(\mu_1, \phi_1/r_1, \theta_1)$; giving the mean, variance parameter and degrees of freedom respectively. Thus,

$$E(\mu_z) = \mu_1 \tag{5.20}$$

and

$$Var(\mu_z) = \frac{\phi_1}{r_1} \frac{\theta_1}{\theta_1 - 2} \tag{5.21}$$

since $E(T_z) = 0$ and $Var(T_z) = \frac{\theta_1}{\theta_1 - 2}$. Now, μ_t is a transformation of μ_z, conditional on π. The conditional distribution of μ_t can be derived from (5.17) as follows:

$$\begin{aligned} f(\mu_t|\pi) &= \frac{1}{(\pi/r_1)^{\frac{1}{2}}} \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{(\theta_1\phi_1)^{\frac{1}{2}}} \left[1 + \frac{(\mu_t/\pi - \mu_1)^2}{\theta_1\phi_1/r_1}\right]^{-\frac{\theta_1+1}{2}} \frac{1}{\pi} \\ &= \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{(\pi/r_1)^{\frac{1}{2}}} \frac{1}{(\theta_1\phi_1\pi^2)^{\frac{1}{2}}} \left[1 + \frac{(\mu_t - \pi\mu_1)^2}{\theta_1\phi_1\pi^2/r_1}\right]^{-\frac{\theta_1+1}{2}} \end{aligned} \tag{5.22}$$

The transformed random variable $T_t = \frac{\mu_t - \pi\mu_1}{\sqrt{\phi_1\pi^2/r_1}}$ has a t distribution with θ_1 degrees of freedom conditional on π since

$$f(T_t|\pi) = \frac{\Gamma(\frac{\theta_1+1}{2})}{\Gamma(\frac{\theta_1}{2})} \frac{1}{\sqrt{\pi\theta_1}} \left[1 + \frac{T_t^2}{\theta_1}\right]^{-\frac{\theta_1+1}{2}}. \tag{5.23}$$

The distribution of μ_t conditional on π will be denoted $t(\pi\mu_1, \phi_1\pi^2/r_1, \theta_1)$. Thus,

$$E(\mu_t|\pi) = \pi\mu_1 \tag{5.24}$$

and

$$Var(\mu_t|\pi) = \frac{\phi_1\pi^2}{r_1} \frac{\theta_1}{\theta_1 - 2}. \tag{5.25}$$

The unconditional posterior distribution of μ_t can be obtained by integrating out π.

$$\begin{aligned} f(\mu_t) &= \int_0^1 f(\mu_t, \pi)\, d\pi \\ &= \int_0^1 f(\mu_t|\pi) f(\pi)\, d\pi \end{aligned} \tag{5.26}$$

This distribution cannot be written explicitly but the mean and variance can be derived, using (5.7), (5.8), (5.24) and (5.25).

$$\begin{aligned} E(\mu_t) &= E[E(\mu_t|\pi)] \\ &= E[\pi\mu_1] \\ &= \mu_1 E[\pi] \\ &= \mu_1 \frac{m_1}{n_1} \end{aligned} \tag{5.27}$$

$$\begin{aligned} Var(\mu_t) &= E[Var(\mu_t|\pi)] + Var[E(\mu_t|\pi)] \\ &= E\left[\frac{\phi_1\pi^2}{r_1}\frac{\theta_1}{\theta_1 - 2}\right] + Var[\pi\mu_1] \\ &= \frac{\phi_1}{r_1}\frac{\theta_1}{\theta_1 - 2}E[\pi^2] + \mu_1^2 Var[\pi] \\ &= \frac{\phi_1}{r_1}\frac{\theta_1}{\theta_1 - 2}\frac{(m_1+1)m_1}{n_1(n_1+1)} + \mu_1^2\frac{m_1(n_1 - m_1)}{n_1^2(n_1+1)} \end{aligned} \tag{5.28}$$

Because the unconditional distribution of μ_t is so difficult to evaluate, Menzefricke and Smieliauskas [31] approximate it using Student's t distribution. The standardized statistic

$$T_t = \frac{\mu_t - E(\mu_t)}{\sqrt{Var(\mu_t)\frac{\theta_1 - 2}{\theta_1}}} \tag{5.29}$$

has a Student's t distribution with θ_1 degrees of freedom. The numerator is weighted to ensure that the variance of the t distribution is $\theta_1/(\theta_1 - 2)$. The $100(1-\alpha)$ percentile of this distribution serves as the upper Bayesian bound for the population mean proportion of error per dollar, μ_t. Using the standard t distribution, this bound is

$$\mu_1 \frac{m_1}{n_1} + t_{\theta_1,(1-\alpha)} \sqrt{\left(\frac{\phi_1}{r_1}\frac{\theta_1}{\theta_1 - 2}\frac{(m_1+1)m_1}{n_1(n_1+1)} + \mu_1^2 \frac{m_1(n_1 - m_1)}{n_1^2(n_1+1)}\right)\frac{\theta_1 - 2}{\theta_1}} \tag{5.30}$$

where $t_{\theta_1,(1-\alpha)}$ is the $100(1-\alpha)$ percentile of the Student's t distribution with θ_1 degrees of freedom. Multiplication of this bound by Y, the total book amount yields the $100(1-\alpha)\%$ upper Bayesian bound for E, the total error.

The construction of the normal bound requires the assessment of six prior values; three prior expected values and three corresponding prior sample sizes. Menzefricke and Smieliauskas [31] suggest $\pi_0 = .1$, $\mu_0 = .5$, $\phi_0 = .3$, $n_0 = 1$, $r_0 = 3$ and $\theta_0 = 1$. The value 1 of n_0 is small relative to the sample size. This constitutes a vague prior for π, i.e. the prior information is given a much smaller weight than the sample information. The priors for σ_z^2 and μ_z are not necessarily vague since r_0 may not be small compared to m. This indicates that the choice of values for μ_0 and ϕ_0 may influence the value of the bound.

Smieliauskas [47] suggests employing a diffuse prior, meaning "the effect of prior information is negligible in comparison to information from the sample." He chooses $n_0 = r_0 = 0$, so the choices of π_0 and μ_0 are arbitrary. θ_0 is set at 2 and ϕ_0 at 0.33. The normal bound can only be computed if the total sample sizes $n_0 + n$, $r_0 + m$, and $\theta_0 + m$ exceed minimum values. A pure diffuse prior (i.e., $n_0 = r_0 = \theta_0 = 0$) supplies no prior knowledge, all the information must come from the sample in order to determine the bound. If the sample contains no errors, there is no information from which to compute a bound. To overcome this situation more information must be provided in the prior. For each of the prior parameter settings described above, computation of the normal bound from a sample containing no nonzero taints is impossible. The diffuse prior suggested by Smieliauskas results in a value of 0 for r_1, leaving μ_1 undefined. The prior of Menzefricke and Smieliauskas yields $\theta_1 = 0$ when a sample contains no errors, so neither ϕ_1 or the percentile of the t-distribution can be defined. As mentioned above, a more informative prior could allow for a solution when the number of sample errors is zero but seldom is an auditor able to predict population parameters with any certainty, so the diffuse priors above may be more appropriate.

Menzefricke and Smieliauskas [31] and Smieliauskas [47] included the normal bound, with the prior parameter settings described above, in simulation studies designed to compare several methods. The results which pertain to the normal bound are presented in Section 5.2 following an explanation of the Cox and Snell [12] bound which was also included in the studies.

5.2 Cox and Snell Bound

Cox and Snell [12], like Felix and Grimlund [16], develop a Bayesian bound for the total overstatement error by combining prior knowledge of the error rate and prior knowledge of the tainting distribution of the nonzero errors. Unlike the normal bound, however, the Cox and Snell bound is not constructed to handle understatements. The model which Cox and Snell formulated is important because it provides a theoretical basis for dollar-unit sampling. Infinite population sample theory provides the framework for their approach. This is in contrast to the probability models previously discussed, which assume that the book amounts and the corresponding audit amounts are fixed and finite in number. In an infinite population model, these values are considered random. For an accounting population, this implies viewing the book values, and hence the audit values, as being generated randomly throughout the year according to some probability distribution.

If N, the number of line items is large, the finite population of book values $y_1, y_2, \ldots, y_N$ is treated as effectively an infinite population with density $f(y)$ and mean μ_y. Then, the finite population total Y can be related to μ_y, the mean book value of the population as follows:

$$\frac{Y}{N} = \mu_y = \int_0^\infty y f(y)\, dy. \tag{5.31}$$

The book values and audit values are assumed to have a joint distribution. Similarly, the joint distribution of (y, e) can be considered. The probability of a line item being in error may depend on the book value y, and so is denoted $\pi(y)$. The unconditional density of a **nonzero** error is given by

$$g(e) = \int_0^\infty g(e|y)f(y)\pi(y)\,dy \tag{5.32}$$

where $f(y)$ is the marginal density of y and $g(e|y)$ is the conditional density of e given y. Thus,

$$\frac{E}{N} = \mu_e = \int_0^\infty eg(e)\,de \tag{5.33}$$

is the mean nonzero error of the population, where E is the finite population error total.

Sampling can be viewed as a selection from $f(y)$. According to the sampling plan, the density of y changes from $f(y)$ to $f_s(y)$. In particular, for sampling with probability-proportional-to-size, which is equivalent to dollar-unit sampling,

$$f_s(y) = \frac{yf(y)}{\mu_y}. \tag{5.34}$$

The density $f_s(y)$ represents the sampling population which is again considered infinite. The sample data is a random sample from this sampling distribution. The distribution of nonzero errors in the sampling population for sampling with probability–proportional–to–size, is given by

$$g_s(e) = \int_0^\infty \frac{g(e|y)yf(y)\pi(y)}{\mu_y}\,dy. \tag{5.35}$$

For the sampling population, the probability that an item of book value y is in error is

$$\pi_y = \frac{yf(y)\pi(y)}{\mu_y} = f_s(y)\pi(y) \tag{5.36}$$

and so the unconditional probability that a randomly chosen item is in error for sampling with probability–proportional–to–size is

$$
\begin{aligned}
\pi &= \int_0^\infty \frac{yf(y)\pi(y)}{\mu_y}\,dy \\
&= \frac{Y_e}{N\mu_y} \\
&= \frac{Y_e}{Y}
\end{aligned}
\tag{5.37}
$$

where Y_e is the total book value of the items in the population which are in error. This π is the probability that a dollar unit will be in error [20].

The mean taint in the sampling population, conditional on $e \neq 0$, can be calculated as follows:

$$
\begin{aligned}
\mu_z &= \frac{\int_0^\infty \int_0^\infty \frac{e}{y} g(e|y) \frac{yf(y)}{\mu_y}\pi(y)\,de\,dy}{\int_0^\infty \frac{yf(y)}{\mu_y}\pi(y)\,dy} \\
&= \frac{\int_0^\infty \int_0^\infty eg(e|y)f(y)\pi(y)\,de\,dy}{\int_0^\infty yf(y)\pi(y)\,dy} \\
&= \frac{E}{Y_e}
\end{aligned}
\tag{5.38}
$$

The denominator in the first step of this computation is included to ensure the density for the errors, conditional on $e > 0$, integrates to 1.

Rearranging (5.38) gives

$$E = Y_e\mu_z = Y\pi\mu_z. \tag{5.39}$$

Since Y, the population book total, is known, all that is required is the estimation of the population mean error per dollar unit $\mu_t = \pi\mu_z$. Based on a sample of size n, in which m nonzero errors are discovered having taints $(z_1, \ldots, z_m)$, a natural point estimate for μ_t is

$$\hat{\mu}_t = \hat{\pi}\bar{z} \tag{5.40}$$

where $\hat{\pi} = m/n$ and $\bar{z} = \sum z_i/m$. This value can be multiplied by Y to obtain an estimate for the total error, E. This model provides a motivation for dollar-unit sampling methods. The total error E, shown to be related to the mean error of a line item by $E = N\mu_e$ can also be derived in terms of the mean error of a dollar-unit, μ_t.

Since π is small and constant, the probability of obtaining m errors in a sample of size n from the sampling population can be approximated by the Poisson distribution with parameter $n\pi$. Thus, an upper bound for π can be found from a Poisson table, leading to an upper bound for μ_t and E. Problems arise, however, when a sample reveals all error-free items.

Cox and Snell [12] propose a Bayesian approach which is applicable even if no errors are found in the sample. They assume that π, the probability that a dollar unit is in error, has a gamma prior distribution with parameters a and π_0/a. The parameter π_0 is the prior expected value of π and a controls the variance of the prior distribution, that is $\sigma_\pi^2 = \pi_0^2/a$. The number of errors observed in a sample, m, is assumed to be generated from a Poisson process with mean $n\pi$. Thus, the prior density of π is

$$f(\pi) = \frac{(a/\pi_0)^a \pi^{a-1} e^{-a\pi/\pi_0}}{\Gamma(a)}, \qquad \pi > 0 \tag{5.41}$$

and the likelihood of m is

$$l(m|\pi) = \frac{e^{-n\pi}(n\pi)^m}{m!}. \tag{5.42}$$

Using this, the posterior distribution of π can be derived:

$$\begin{aligned} f(\pi|m) &= c\, l(m|\pi) f(\pi) \\ &= c\, \frac{e^{-n\pi}(n\pi)^m}{m} \frac{(a/\pi_0)^a \pi^{a-1} e^{-a\pi/\pi_0}}{\Gamma(a)} \\ &= c\, \frac{(a/\pi_0)^a n^m \pi^{a+m-1} e^{-\pi(n+a/\pi_0)}}{m\Gamma(a)} \\ &= \frac{(n+a/\pi_0)^{m+a} \pi^{a+m-1} e^{-\pi(n+a/\pi_0)}}{\Gamma(m+a)} \end{aligned} \tag{5.43}$$

where

$$c = \frac{m\Gamma(a)}{(a/\pi_0)^a n^m} \frac{(n+a/\pi_0)^{m+a}}{\Gamma(m+a)}. \tag{5.44}$$

Thus, the posterior distribution of π is again a gamma distribution with parameters $m+a$ and $(n+a/\pi_0)^{-1}$.

The authors assume an inverse gamma distribution for the mean taint of the nonzero errors, μ_z. This density, with parameters b and $1/\mu_0(b-1)$, is

$$f(\mu_z) = \frac{[\mu_0(b-1)]^b \mu_z^{-(b+1)} e^{[-(b-1)\mu_0/\mu_z]}}{\Gamma(b)}. \tag{5.45}$$

The parameter μ_0 is the prior expected value of μ_z and b specifies the variance of the prior distribution through $\sigma^2_{\mu_z} = \mu_0^2/(b-2)$. The observed nonzero errors yield taints $z_1, z_2, \ldots, z_m$ which are assumed to be m random observations from an exponential distribution with mean μ_z. Thus, the likelihood can be written in terms of

$\overline{z} = \sum z_i/m$, as follows:

$$l(\overline{z}|\mu_z) = \mu_z^{-m} e^{-m\overline{z}/\mu_z}. \tag{5.46}$$

The posterior distribution of μ_z is derived below:

$$\begin{aligned} f(\mu_z|\overline{z}) &= c\, l(\overline{z}|\mu_z) f(\mu_z) \\ &= c\, \mu_z^{-m} e^{-m\overline{z}/\mu_z} \frac{[\mu_0(b-1)]^b \mu_z^{-(b+1)} e^{[-(b-1)\mu_0/\mu_z]}}{\Gamma(b)} \\ &= c\,[\mu_0(b-1)]^b \frac{\mu_z^{-(b+m+1)} e^{-[(b-1)\mu_0 + m\overline{z}]/\mu_z}}{\Gamma(b)} \\ &= \frac{[\mu_0(b-1) + m\overline{z}]^{b+m} \mu_z^{-(b+m+1)} e^{-[\mu_0(b-1)+m\overline{z}]/\mu_z}}{\Gamma(b+m)} \end{aligned} \tag{5.47}$$

where

$$c = \frac{\Gamma(b)}{[\mu_0(b-1)]^b} \frac{[\mu_0(b-1) + m\overline{z}]^{b+m}}{\Gamma(b+m)}. \tag{5.48}$$

Again, the posterior distribution is an inverse-gamma distribution, this time with parameters $b+m$ and $[\mu_0(b-1) + m\overline{z}]^{-1}$.

We actually require the posterior distribution of

$$\mu_t = \pi \mu_z \tag{5.49}$$

in order to find a Bayesian bound for μ_t or E. Cox and Snell assume that π and μ_z are independent random variables. This assumption has been questioned, for example by Godfrey and Neter [20], as will be discussed. Based on this assumption the posterior distribution of μ_t is simply the product of the individual posterior distributions of π and μ_z. It can be shown that this combined posterior distribution is a multiple of the F distribution.

The individual posterior densities can be represented as follows:

$$\begin{aligned}\pi &\sim \text{Gamma}(m+a,(n+a/\pi_0)^{-1})\\ \mu_z &\sim \text{Inverse Gamma}(m+b,(\mu_0(b-1)+m\overline{z})^{-1})\end{aligned}$$

or, equivalently

$$\mu_z^{-1} \sim \text{Gamma}(m+b,(\mu_0(b-1)+m\overline{z})^{-1})$$

Making use of the relationship between the gamma distribution and the chi-square distribution:

$$\begin{aligned}2\pi(n+a/\pi_0) &\sim \chi^2_{2(m+a)}\\ 2\mu_z^{-1}[\mu_0(b-1)+m\overline{z}] &\sim \chi^2_{2(m+b)}\end{aligned}$$

Now,

$$\mu_t = \pi\mu_z = \frac{\pi}{\mu_z^{-1}}$$

is actually a fraction of χ^2 random variables, and it is obvious that this will lead to the F distribution.

$$f\left(\frac{\pi}{\mu_z^{-1}}\right) = \frac{[m+a]}{[m+b]}\frac{[\mu_0(b-1)+m\overline{z}]}{[n+a/\pi_0]}F_{[2(m+a),2(m+b)]} \tag{5.50}$$

where $F_{[v_1,v_2]}$ represents the density of the F distribution with v_1 degrees of freedom in the numerator and v_2 degrees of freedom in the denominator.

The $1-\alpha$ percentile of this distribution can easily be obtained and so a $1-\alpha$ upper Bayesian bound for the overall mean tainting of the population is calculated as follows:

$$\frac{[m+a]}{[m+b]}\frac{[\mu_0(b-1)+m\overline{z}]}{[n+a/\pi_0]}F_{[2(m+a),2(m+b)],(1-\alpha)}. \tag{5.51}$$

Cox and Snell suggest a between 1/2 and 2 and $b \geq 3$ to be reasonable considerations for these parameters.

This bound can be calculated even if the number of overstatement errors discovered in the sample, m, is zero. In this case the posterior distribution of μ_t is simply taken to be the prior distribution, since no sample information is available. Since π and μ_z are assumed to be independent, the prior distribution of μ_t is simply the product of the two prior distributions of π and μ_z. This product is equivalent to (5.50), with m, n and $\overline{z}$ set equal to zero.

Godfrey and Neter [20] point out some possible shortfalls of the Cox and Snell bound. As mentioned earlier, they question the assumed independence of π and μ_z. Little is known about the process which generates errors and it is quite possible that the size of an error is related to the likelihood of the error occuring. They also question the use of a gamma prior for the error rate. The gamma distribution allows for arbitrarily large values while the error rate is restricted between 0 and 1.0. The individual taints are assumed to be generated from an exponential distribution, allowing for higher overstatement taints than is reasonable. An additional problem with this assumed tainting distribution is that it does not allow for the frequent occurrence of taints having value 1.0. Empirical studies, Johnson, Leitch and Neter [26] and Neter, Johnson and Leitch [36], have shown the existence of a point mass a

t =1.0 to be very common, especially among accounts receivable populations. This can be caused in part by accounts which are paid without the payment being recorded. Godfrey and Neter also question the use of the Poisson distribution to model the number of errors, m, as this may only apply for situations with low error rates.

To test the sensitivity of the Cox and Snell bound to the above model assumptions, Godfrey and Neter [20] developed four modifications of the bound, each adjusting one or more of the above assumptions. They then compared the five bounds, their four modifications and the original Cox and Snell bound, for a variety of populations and parameter settings. Suprisingly enough, they found that the Cox and Snell bound was much less sensitive to its parametric assumptions than to varying parameter values.

They found that Bayesian bounds with different prior distributions can differ substantially. Their next step was to perform a simulation study to investigate the probability of correct coverage of the Cox and Snell bound. The study was based on 21 simulated populations, representing a variety of tainting distributions and error rates. Nine sets of prior parameters were employed, giving 9 bounds. For each population, 500 independent samples of size 100 were generated and the 9 Cox and Snell bounds were calculated for each sample. The Stringer bound was also calculated for each sample for comparative purposes. The results of this study suggested the possibility of finding "conservative priors" which lead to a robust Cox and Snell bound while remaining significantly tighter than the Stringer bound. Here, robust means that the probability the bound is correct is near the nominal confidence level over a wide range of different populations.

Neter and Godfrey [32] extended this study, including 9 more populations and several new prior parameter settings. Many of these populations were more adverse than what is typically expected in an accounting population. As a result of this study, the authors note three particular examples of prior settings which yield robust bounds. These are given below, using the authors' notation.

	μ			π		
Bound	μ_0	σ_μ	b	π_0	σ_π	a
CS10	.4	.2	6	.1	.1	1
CS11	.3	.4	2.5625	.15	.2	.5625
CS23	.2	.15	3.3333	.2	.2	1

Table 5.1: Prior parameter values for the Cox and Snell bound

If an auditor is uncertain about the choice of prior parameter values, any of the above bounds could be used. The auditor can then be assured of sufficient coverage probabilities over a wide range of error populations. Additionally, the robust Cox and Snell bounds are significantly tighter then the benchmark Stringer bound.

Consider, for example, calculation of the CS10 bound, which has prior values as given above. If a sample of size 100 contains one overstatement error, with a taint of 0.25, the 95% upper bound for the mean error per dollar, μ_t, is

$$\frac{.4(5) + .25}{100 + (1/0.10)} \frac{1+1}{1+6} \times 3.112 = 0.0182$$

since $F_{[2(1+1),2(1+6)],.95} = 3.112$. Thus, for a population of Y = \$1,000,000, the 95% upper bound for the total error, E, is \$18,200.

Menzefricke and Smieliauskas [31] included both the normal bound, with parameters $\pi_0 = .1$, $n_0 = 1$, $\mu_0 = .5$, $r_0 = 3$, $\phi_0 = .3$, $\theta_0 = 1$, and the Cox and Snell bound, with parameters $\pi_0 = .1$, $a = .1$, $\mu_0 = .5$, $b = 3$, in their simulation study. The Cox and Snell bound was calculated with the LTA adjustment for understatements, as in (3.23), so the two bounds could be compared more fairly.

They examined the performance of these bounds for four tainting distributions, modelled by various beta distributions, each with five error rates. From each study population 600 random samples of size $n = 120$ were drawn. The two Bayesian bounds were calculated for each sample as were the Stringer bound and the GOLOS bound. For the first two populations, which were centered close to zero with a small spread of taints, the bounds were all conservative. The coverage of the normal bound was closest to the nominal 95% confidence level. The Cox and Snell bound had the better performance for the third population which contained a high percentage of 100% overstatements. The fourth population was included to simulate an extreme audit environment, with a high proportion of items in error. In this situation, the normal bound was very unreliable. The normal bound, however, was deemed "the most attractive" by the authors because it demonstrated achieved confidence levels close to the nominal level and requires no special adjustment for understatements.

Smieliauskas [47] studied the performance of these two Bayesian bounds with different prior values. He chose diffuse priors: $n_0 = r_0 = 0$, $\phi_0 = .33$, $\theta_0 = 2$ for the normal bound and $a = 0$, $b = 1$ for the Cox and Snell bound. In each case, the values of π_0 and μ_0 were arbitrary.

Smieliauskas followed the method of Leitch et al. [28], using the J, J-100, unimodal and uniform distributions to model the tainting distribution. To judge the relative tightness of the bounds he computed each bound for typical tainting patterns of the various distributions with error rates of 6, 10, 15, 20 and 25 percent. In each case the normal and Cox and Snell bounds were computed at the 95% confidence level based on a sample size of $n = 100$. The Stringer and GOLOS bounds were also calculated in order to provide a comparison between Bayesian and non-Bayesian methods. Smieliauskas also included results for the modified multinomial bound and the moment bound, taken from Leitch et al. [28] and Dworin and Grimlund [14].

In order to examine the reliability of the four bounds a simulation study was performed with 500 random samples of size $n = 100$ being drawn from each population. The error rates included in this part of the study were 3, 6, 10, 14, 20, 25 and 60 percent. The bounds were calculated for each sample, and since the true mean taint was known for each population, the actual coverage percentage could be calculated.

Again, Smieliauskas included results taken from Leitch et al. [28] and Dworin and Grimlund [14] regarding the coverage levels of the modified multinomial bound and the moment bound. Though all results are based on the same tainting distributions and error rates, care must be exercised in comparing these "second-hand" results with those from the present simulation.

The Stringer and GOLOS bounds appeared overly conservative, having an achieved confidence of 100% for many of the study populations and always obtaining at least the nominal confidence level. Between the Bayesian methods, the normal bound achieved the best reliability. Its lowest coverage was 93.2% for the J-100 distribution with 10% errors. The coverage of the Cox and Snell bound was poor in comparison; the nominal level was achieved for only 6 of the 28 study populations.

The Cox and Snell bound tended to be the tightest of the four for the J and J-100 distributions. The normal bound was the tightest for the unimodal distribution and the uniform distribution with low error rates. The GOLOS bound had the best performance for the uniform distribution with high error rates. In terms of combining reliability and efficiency, the normal Bayesian bound was superior.

5.3 Multinomial-Dirichlet Bound

Tsui, Matsamura and Tsui [51] propose a nonparametric Bayesian approach based on the multinomial sampling model used by Fienberg, Neter and Leitch [18]. This model for dollar-unit taints was introduced in Chapter 4. It is assumed that all errors are overstatements with a maximum taint of 1.0. Each taint is rounded and classified according to its value in cents (00 to 100 cents), so the following model applies:

$$t = \frac{i}{100} \quad \text{with probability } p_i, \quad i = 0, \ldots, 100 \tag{5.52}$$

where $\sum_{i=0}^{100} p_i = 1$ and $p_i \geq 0$. p_i is the probability that a taint is classified in the i'th category (i cents). Then

$$E(t) = \mu_t = \sum_{i=0}^{100} \frac{i}{100} p_i, \tag{5.53}$$

so

$$E = Y\mu_t = Y \sum_{i=0}^{100} \frac{i}{100} p_i. \tag{5.54}$$

Denoting the number of sample observations which fall in the i'th category w_i, the set of counts $\boldsymbol{w}=(w_0, w_1, \ldots, w_{100})$ has a multinomial distribution with the parameters n and $(p_0, p_1, \ldots, p_{100})$ if sampling is done with replacement. This is an approximate distribution if sampling is done without replacement, as long as n is small relative to Y.

Tsui, Matsamura and Tsui [51] assume a Dirichlet prior distribution for $\boldsymbol{p} = (p_0, p_1, \ldots, p_{100})$, specifically $\mathrm{Dir}(K\alpha_0, \ldots, K\alpha_{100})$, where $\alpha_i > 0$, $i = 0, \ldots, 100$, and $\sum \alpha_i = 1$. The α_i's are the auditor's best predictions for the p_i's and K gives a relative weight to these predictions. K is usually chosen to be considerably less than n, giving more weight or reliability to the sample information than the prior information. The probability density function for $\boldsymbol{p}$ has the following form:

$$f(\boldsymbol{p}|K\alpha_0, \ldots, K\alpha_{100}) = \frac{\Gamma(K)}{\Gamma(K\alpha_0) \ldots \Gamma(K\alpha_{100})} \prod_{i=0}^{100} p_i^{K\alpha_i - 1} \tag{5.55}$$

for $p_i > 0$, $i = 0, \ldots, 100$, and $\sum p_i = 1$. Since this is a probability density,

$$\int_\Omega \frac{\Gamma(K)}{\Gamma(K\alpha_0) \ldots \Gamma(K\alpha_{100})} \prod_{i=0}^{100} p_i^{K\alpha_i - 1} \, d\boldsymbol{p} = 1 \tag{5.56}$$

where $\Omega = (\boldsymbol{p}\ |p_i > 0, \sum p_i = 1, i = 0, \ldots, 100)$. Thus,

$$\int_\Omega \prod_{i=0}^{100} p_i^{K\alpha_i - 1}\, d\boldsymbol{p} = \frac{\prod_{i=0}^{100} \Gamma(K\alpha_i)}{\Gamma(K)}. \tag{5.57}$$

This equality can be used to derive the moments of the Dirichlet distribution, which will be required for the development of the bound for μ_t.

The mixed moments $\upsilon'_{r_0,\ldots,r_{100}}$ are given by

$$\begin{aligned}
\upsilon'_{r_0,\ldots,r_{100}} &= E\left[\prod_{j=0}^{100} p_j^{r_j}\right] \\
&= \frac{\Gamma(K)}{\prod_{j=0}^{100}\Gamma(K\alpha_j)} \int_\Omega \prod_{j=0}^{100} p_j^{K\alpha_j + r_j - 1}\, d\boldsymbol{p} \\
&= \frac{\Gamma(K)\prod_{j=0}^{100}\Gamma(K\alpha_j + r_j)}{\prod_{j=0}^{100}\Gamma(K\alpha_j)\Gamma(K + \sum_{j=0}^{100} r_j)}
\end{aligned} \tag{5.58}$$

To find $E(p_i)$, solve $\upsilon'_{r_0,\ldots,r_{100}}$ for $r_i = 1$, $r_j = 0, j \neq i$. So,

$$\begin{aligned}
E(p_i) &= \frac{\Gamma(K)\Gamma(K\alpha_i + 1)\prod_{\substack{j=0\\ j\neq i}}^{100}\Gamma(K\alpha_j)}{\Gamma(K+1)\prod_{j=0}^{100}\Gamma(K\alpha_j)} \\
&= \alpha_i
\end{aligned} \tag{5.59}$$

$E(p_i^2)$ can be derived by solving $\upsilon'_{r_0,\ldots,r_{100}}$ for $r_i = 2$, $r_j = 0, j \neq i$.

$$\begin{aligned}
E(p_i^2) &= \frac{\Gamma(K)\Gamma(K\alpha_i + 2)\prod_{\substack{j=0\\ j\neq i}}^{100}\Gamma(K\alpha_j)}{\Gamma(K+2)\prod_{j=0}^{100}\Gamma(K\alpha_j)} \\
&= \frac{\alpha_i(K\alpha_i + 1)}{K+1}
\end{aligned} \tag{5.60}$$

Solve $v'_{r_0,\ldots,r_{100}}$ for $r_i = 1$, $r_j = 1$, $r_l = 0$, $i \neq j \neq l$ to find $E(p_i p_j)$.

$$\begin{aligned}
E(p_i p_j) &= \frac{\Gamma(K)\Gamma(K\alpha_i + 1)\Gamma(K\alpha_j + 1) \prod\limits_{\substack{l=0 \\ l \neq i,j}}^{100} \Gamma(K\alpha_l)}{\Gamma(K+2) \prod\limits_{l=0}^{100} \Gamma(K\alpha_l)} \\
&= \frac{\Gamma(K) K\alpha_i K\alpha_j \prod\limits_{l=0}^{100} \Gamma(K\alpha_l)}{(K+1)K\Gamma(K) \prod\limits_{l=0}^{100} \Gamma(K\alpha_l)} \\
&= \frac{K\alpha_i\alpha_j}{K+1} \qquad \text{for } i \neq j,
\end{aligned} \tag{5.61}$$

The authors give three examples to illustrate the choice of parameters for the Dirichlet prior distribution. These examples were alluded to throughout the study and so will be briefly mentioned here. Table 5.2 lists the parameter values for each example.

Example	α_0	α_i[1]	α_{100}	K
B1	$\frac{1}{101}$	$\frac{1}{101}$	$\frac{1}{101}$	.2525
B2	.8	.002	.002	5
B3	.8	.001	.101	5

[1] $i = 1, \ldots, 99$

Table 5.2: Examples of Dirichlet Priors

Example B1 assumes a high error rate. The small value of K signifies a non-informative prior, causing the subsequent bound to rely heavily on information from the sample. The prior of example B2 represents the belief of a smaller error rate, with the various nonzero taintings being equally likely. The value $K = 5$ suggests the auditor is confident in his prediction of the p_i's. Example B3 has a prior distribution similar to example B2, except here a higher probability is given to 100% overstatement errors.

It is evident from these examples that the Dirichlet distribution provides much flexibility in the model in terms of the actual distribution of the errors and the strength of belief in the prior predictions. More conservative priors can be chosen by placing a higher probability on larger taintings, such as a large value for α_{100}. Thus example B3, which has a more conservative prior, will yield a larger bound than example B2.

To construct a bound for the total error E, the bound for the true mean tainting per dollar μ_t is multiplied by the known constant Y. To this end, we require the posterior distribution of μ_t.

Consider first the distribution of $\boldsymbol{p}$. Since the counts $\boldsymbol{w}$ are generated according to the multinomial distribution, the likelihood can be written

$$\begin{aligned} l(\boldsymbol{w}) &= \frac{n!}{w_0!\ldots w_{100}!}\prod_{i=0}^{100} p_i^{w_i} \\ &= \frac{\Gamma(n+1)}{\Gamma(w_0+1)\ldots\Gamma(w_{100}+1)}\prod_{i=0}^{100} p_i^{w_i} \end{aligned} \tag{5.62}$$

Combining this likelihood with the Dirichlet prior distribution of $\boldsymbol{p}$ yields the posterior distribution of $\boldsymbol{p}$.

$$\begin{aligned} f(\boldsymbol{p}|\boldsymbol{w}) &= cf(\boldsymbol{p})l(\boldsymbol{w}) \\ &= c\,\frac{\Gamma(K)}{\Gamma(K\alpha_0)\ldots\Gamma(K\alpha_{100})}\prod_{i=0}^{100} p_i^{K\alpha_i-1}\frac{\Gamma(n+1)}{\Gamma(w_0+1)\ldots\Gamma(w_{100}+1)}\prod_{i=0}^{100} p_i^{w_i} \end{aligned}$$

$$= \frac{\Gamma(K+n)}{\Gamma(K\alpha_0 + w_0)\ldots\Gamma(K\alpha_{100}+w_{100})} \prod_{i=0}^{100} p_i^{K\alpha_i + w_i - 1} \tag{5.63}$$

where

$$\begin{aligned} c &= \frac{\Gamma(K\alpha_0)\ldots\Gamma(K\alpha_{100})}{\Gamma(K)} \frac{\Gamma(w_0+1)\ldots\Gamma(w_{100}+1)}{\Gamma(n+1)} \\ &\quad \times \frac{\Gamma(K+n)}{\Gamma(K\alpha_0+w_0)\ldots\Gamma(K\alpha_{100}+w_{100})}. \end{aligned} \tag{5.64}$$

$\boldsymbol{p}$ has a Dirichlet posterior distribution with parameters

$$K' = K + n \tag{5.65}$$

and

$$\alpha_i' = \frac{K\alpha_i + w_i}{K+n} \quad \text{for } i = 0, \ldots, 100. \tag{5.66}$$

The posterior distribution of μ_t can be derived as a linear function of the p_i. The exact form of this distribution is very complicated. Percentiles of the posterior distribution which are required to find upper bounds for μ_t must be approximated. Tsui, Matsamura and Tsui [51] explore two methods for approximating the percentiles.

The following relationship between the Dirichlet distribution and the gamma distribution can be used to estimate the percentiles empirically. If Z_i, $i = 1, \ldots, r$, are independent random variables with each $Z_i \sim$ gamma(a_i,1) then the joint distribution of the proportions $p_i = Z_i / (\sum_{j=1}^{r} Z_j)$ is the Dirichlet distribution with parameters $(a_1, \ldots, a_r)$.

For this particular application, 101 random variables Z_i, indexed from 0 to 100, must be employed, each with shape parameter $K'\alpha_i'$. By generating a large number of independent realizations of the Z_i's, the empirical posterior distribution of the linear combination

$$\mu_t = \sum_{i=0}^{100} \frac{i}{100} p_i = \sum_{i=0}^{100} \frac{i}{100} \frac{Z_i}{\sum_{j=0}^{100} Z_j} \tag{5.67}$$

can be obtained. The percentiles of this empirical distribution estimate the percentiles of the true posterior distribution.

Alternatively, the posterior distribution of μ_t can be approximated by the beta distribution. Using the moments of the Dirichlet distribution derived previously, the mean and variance of μ_t can be calculated as follows:

$$\begin{aligned} E(\mu_t) &= E\left(\frac{1}{100}\sum_{i=0}^{100} i p_i\right) \\ &= \frac{1}{100}\sum_{i=0}^{100} i E(p_i) \\ &= \frac{1}{100}\sum_{i=0}^{100} i \alpha_i' \end{aligned} \tag{5.68}$$

$$\begin{aligned} E(\mu_t^2) &= E\left[\left(\frac{1}{100}\sum_{i=0}^{100} i p_i\right)^2\right] \\ &= \frac{1}{100^2} E\left[\left(\sum_{i=0}^{100} i p_i\right)^2\right] \\ &= \frac{1}{100^2} E\left(\sum_{i=0}^{100} i^2 p_i^2 + \sum_{i \neq j} ij p_i p_j\right) \\ &= \frac{1}{100^2}\left[\sum_{i=0}^{100} i^2 E(p_i^2) + \sum_{i \neq j} ij E(p_i p_j)\right] \\ &= \frac{1}{100^2}\left[\sum_{i=0}^{100} i^2 \alpha_i' \frac{K'\alpha_i' + 1}{K' + 1} + \sum_{i \neq j} ij \frac{K'\alpha_i'\alpha_j'}{K' + 1}\right] \\ &= \frac{1}{100^2}\left[\frac{K'}{K'+1}\left(\sum_{i=0}^{100} i^2 \alpha_i'^2 + \sum_{i \neq j} ij \alpha_i'\alpha_j'\right) + \frac{1}{K'+1}\left(\sum_{i=0}^{100} i^2 \alpha_i'\right)\right] \end{aligned}$$

$$= \frac{1}{100^2}\left[\frac{K'}{K'+1}\left(\sum_{i=0}^{100} i\alpha_i'\right)^2 + \frac{1}{K'+1}\left(\sum_{i=0}^{100} i^2\alpha_i'\right)\right] \tag{5.69}$$

$$\begin{aligned}
Var(\mu_t) &= E(\mu_t^2) - E(\mu_t)^2 \\
&= \frac{1}{100^2}\left[\frac{K'}{K'+1}\left(\sum_{i=0}^{100} i\alpha_i'\right)^2 + \frac{1}{K'+1}\left(\sum_{i=0}^{100} i^2\alpha_i'\right)\right] - \left(\frac{1}{100}\sum_{i=0}^{100} i\alpha_i'\right)^2 \\
&= \frac{1}{100^2}\left[\frac{1}{K'+1}\left(\sum_{i=0}^{100} i^2\alpha_i'\right) - \frac{1}{K'+1}\left(\sum_{i=0}^{100} i\alpha_i'\right)^2\right] \\
&= \frac{1}{100^2(K'+1)}\left[\left(\sum_{i=0}^{100} i^2\alpha_i'\right) - \left(\sum_{i=0}^{100} i\alpha_i'\right)^2\right]
\end{aligned} \tag{5.70}$$

The mean and variance of the beta distribution with parameters a and b are

$$\frac{a}{a+b} \tag{5.71}$$

and

$$\frac{ab}{(a+b)^2(a+b+1)} \tag{5.72}$$

respectively. The authors found that matching the mean and variance of the beta distribution with the mean and variance of the posterior distribution of μ_t resulted in a good approximation of the posterior distribution. The parameters a and b giving the appropriate beta distribution are derived below.

Setting the moments equal and solving (5.71) for b.

$$\begin{aligned}
(a+b)E(\mu_t) &= a \\
b &= \frac{a - aE(\mu_t)}{E(\mu_t)} \\
&= \frac{a(1-E(\mu_t))}{E(\mu_t)}
\end{aligned} \tag{5.73}$$

Substitute (5.73) into (5.72) and solve for a.

$$\begin{aligned}
Var(\mu_t) &= \frac{a\frac{a(1-E(\mu_t))}{E(\mu_t)}}{\left(a+\frac{a(1-E(\mu_t))}{E(\mu_t)}\right)^2\left(a+\frac{a(1-E(\mu_t))}{E(\mu_t)}+1\right)} \\
&= \frac{\frac{a^2(1-E(\mu_t))}{E(\mu_t)}}{\left(\frac{aE(\mu_t)+a-aE(\mu_t)}{E(\mu_t)}\right)^2\left(\frac{aE(\mu_t)+a-aE(\mu_t)+E(\mu_t)}{E(\mu_t)}\right)} \\
&= \frac{\frac{a^2(1-E(\mu_t))}{E(\mu_t)}}{\frac{a^2(a+E(\mu_t))}{E(\mu_t)^3}} \\
&= \frac{E(\mu_t)^2(1-E(\mu_t))}{a+E(\mu_t)} \\
a &= E(\mu_t)\left[\frac{E(\mu_t)(1-E(\mu_t))}{Var(\mu_t)}-1\right] \qquad (5.74)
\end{aligned}$$

Substitute (5.74) into (5.73) to find b.

$$\begin{aligned}
b &= E(\mu_t)\left[\frac{E(\mu_t)(1-E(\mu_t))}{Var(\mu_t)}-1\right]\left(\frac{1-E(\mu_t))}{E(\mu_t)}\right) \\
&= [1-E(\mu_t)]\left[\frac{E(\mu_t)(1-E(\mu_t))}{Var(\mu_t)}-1\right] \qquad (5.75)
\end{aligned}$$

The posterior distribution of μ_t is approximated by a beta distribution with parameters a and b, given above in terms of $E(\mu_t)$ and $Var(\mu_t)$. The 95% upper bound for μ_t is taken to be the 95'th percentile of this distribution.

The calculation of the multinomial-Dirichlet bound, with the beta approximation of the posterior distribution, is demonstrated with an example taken from Tsui, Matsamura and Tsui [51]. Consider a dollar-unit sample of size 100, with 6 taints of 1, 1, 5, 11, 27 and 100 cents. For the B3 prior, a 95% upper confidence bound for the total audit error is calculated as follows:

$$K' = 5 + 100 = 105$$

$$\begin{aligned}
\alpha'_0 &= \frac{5(0.8)+94}{5+100} = \frac{98}{105} = .9333 \\
\alpha'_1 &= \frac{5(0.001)+2}{5+100} = \frac{2.005}{105} = .0191 \\
\alpha'_5 &= \frac{5(0.001)+1}{5+100} = \frac{1.005}{105} = .0096 \\
\alpha'_{11} &= \frac{5(0.001)+1}{5+100} = \frac{1.005}{105} = .0096 \\
\alpha'_{27} &= \frac{5(0.001)+1}{5+100} = \frac{1.005}{105} = .0096 \\
\alpha'_{100} &= \frac{5(0.101)+1}{5+100} = \frac{1.505}{105} = .0143 \\
\alpha'_j &= \frac{5(0.001)+0}{5+100} = \frac{0.005}{105} = .00005 \quad j \neq 0, 1, 5, 11, 27, 100
\end{aligned}$$

$$E(\mu_t) = \sum_{i=0}^{100} \frac{i\alpha'_i}{100} = .02098$$

$$Var(\mu_t) = \frac{1}{100^2(K'+1)} \left[\left(\sum_{i=0}^{100} i^2 \alpha'_i \right) - \left(\sum_{i=0}^{100} i\alpha'_i \right)^2 \right] = .0001537$$

$$a = E(\mu_t) \left[\frac{E(\mu_t)(1 - E(\mu_t))}{Var(\mu_t)} - 1 \right] = 2.781708$$

$$b = [1 - E(\mu_t)] \left[\frac{E(\mu_t)(1 - E(\mu_t))}{Var(\mu_t)} - 1 \right] = 129.8309$$

The 95'th percentile of the beta distribution with parameters $a = 2.78$ and $b =$

129.83 is 0.04462. Thus, the 95% Bayesian upper bound for the mean error tainting per dollar is 0.04462 and so for a population of 1 million dollars, the 95% Bayesian upper bound for the total error is $\$1,000,000 \times 0.04462 = \$44,620$. Program dirichlet.bound in Appendix B was designed to perform the above calculations which would be extremely cumbersome by hand.

Tsui, Matsamura and Tsui studied this Bayesian bound for each of the three priors given in the examples. They used data simulated to reflect a variety of possible accounting populations in terms of tainting distribution and error rate. This data included the four distributions suggested by Leitch et al. [28] (J, J-100, unimodal, uniform) and four additional distributions which allowed for a higher percentage of 100% overstatements. The typical tainting patterns of the J, J-100, unimodal and uniform distributions for 6, 10, 15, 20 and 25 percent error rates were used to examine the tightness of the three bounds. Each of the multinomial-Dirichlet bounds was computed twice, once using the beta approximation to the 95'th percentile and once using the empirical 95'th percentile of the posterior distribution of μ_t as generated from 10,000 samples. In each case, the two bounds were practically identical, supporting the use of the beta approximation, which is a simpler method.

Let B1, B2 and B3 signify the multinomial-Dirichlet bounds computed based on the priors in examples B1, B2 and B3 respectively. The priors of B2 and B3 are given equal strength ($K = 5$) but as mentioned earlier B3 is based on a more conservative prior. For this reason, it was expected that B3 would be larger than B2 and this was in fact true for every study population.

The authors also included calculations of the Stringer and modified multinomial bounds, taken from Leitch et al. [28] for comparison purposes. All the bounds were based on a sample size of 100. The multinomial-Dirichlet bounds were tighter than the Stringer bound in all cases, and generally tighter than the multinomial bound. B3 was, in fact, similar in size to the multinomial bound. This is of interest since the multinomial bound is considerably more difficult to compute, requiring numerical methods.

The true proportion of coverage was examined for the multinomial-Dirichlet bounds for all the tainting distributions with 12 different error rates between 1 and 20 percent. The true mean tainting per dollar unit was known for each of the 96 study populations. For each population 500 independent random samples were taken and for each sample the three 95% bounds were calculated. The percentage of times a bound exceeded the true mean tainting out of 500 trials gives a measure of the coverage of the bound. The coverage was examined in this way for sample sizes of 100 and 200. Coverage percentages among the three bounds, B1, B2 and B3, differed more for populations with lower error rates. It was found that, for a fixed error rate, the coverage percentages associated with B1, B2 and B3 were relatively closer for a sample size of 200 than a sample size of 100. One plausible explanation for this is the lessening influence of the prior as more sample information becomes available. Overall, however, increasing the sample size from 100 to 200 did not significantly improve the coverage percentages of the multinomial-Dirichlet bounds.

Whether or not the true coverage level is close to the nominal confidence level depends on the choice of an appropriate prior. For example, the prior associated with B1 is appropriate for a reversed J-shaped tainting distribution. For such a population, the bound B1 results in a significance level close to the nominal level. Similarly, B2 gives correct coverage for a uniform population. The bound B3 was found to result in coverage percentages very close to or greater than the nominal confidence level for all the tainting distributions and error rates considered. As such, the prior associated with B3 may be employed as a conservative specification if the auditor cannot be more certain of the shape of the tainting distribution.

Grimlund and Felix [22] performed a simulation to compare all three Bayesian methods presented in this chapter as well as the modified moment method given in Chapter 4. In the main part of the study the priors chosen for the normal bound were those employed by Menzefricke and Smieliauskas [31], namely $\pi_0 = .1$, $n_0 = 1$, $\mu_0 = .5$, $r_0 = 3$, $\phi_0 = .3$, $\theta_0 = 1$. The Cox and Snell bound was calculated with the "CS10" prior of Godfrey and Neter [20] and the multinomial-Dirichlet bound was calculated with the "B3" prior of Tsui, Matsamura and Tsui [51]. The authors also provide an appendix with results for two alternative bounds: the normal bound with the diffuse prior suggested by Smieliauskas [47] ($n_0 = r_0 = 0$, $\phi_0 = .33$, $\theta_0 = 2$) and Godfrey and Neter's "CS11" bound [20].

To generate study populations Grimlund and Felix followed the approach of

Dworin and Grimlund [14], which was described earlier. They used 36 error combinations and two tainting models for each of accounts receivable and inventory, yielding 144 study populations. For each study population, sample sizes of 60, 120 and 240 and confidence levels of 50, 70, 80, 90 and 95% were employed. In each case 500 random samples were drawn and the bounds were calculated for each sample. Within each sample set the coverage rate, average and advantage of the bounds were calculated. Advantage was defined as the percentage reduction in the average bound compared to the Stringer bound with the LTA adjustment for understatements. The Cox and Snell bound and the multinomial-Dirichlet bounds were not adjusted for understatements in any way.

Not all of the results were included in the paper. Detailed results of coverage and advantage were presented for 36 of the accounts receivable populations at the 95% confidence level, for each sample size. Two important effects were evident from these results. First of all, the coverages of all four bounds, and in particular the modified moment bound, tended to gravitate toward the 95% level as the sample size increased. The authors observed this pattern at other confidence levels as well. The second discovery was the tendency for the coverage to fall below the nominal level when the percentage of 100% overstatements was high (such as 20% or 40% of errors). This problem tended to heighten with larger population error rates and smaller sample sizes. The multinomial-Dirichlet bound exhibited the highest coverage performance for the populations presented, often achieving 100%. This was expected since it is a conservative method.

To judge the advantage of the various bounds over the Stringer bound with the LTA adjustment, the percentage reduction in the average confidence bound was calculated for the same 36 populations; i.e. (Avg. Stringer bound - Avg. bound)/Avg Stringer bound. What is particularly evident from these results is the decline in advantage as either sample size or error rate increases. This same pattern occurred at various confidence levels. Though all of the advantage statistics were positive, the authors report that for the inventory populations, with higher proportions of understatements, this was not always the case.

Finally, it is apparent once again, that rarely do tightness and reliability go hand-in-hand. The multinomial-Dirichlet bound, which exhibited the highest overall reliability tended to provide the smallest advantage. Similarly, in many populations for which the normal and Cox and Snell bounds show the highest advantage, the same bounds exhibit coverage well below the nominal level.

Detailed results for the average bounds were presented for 36 of the inventory populations. Some interesting effects are pointed out by these results. All four bounds tended to decrease substantially as the sample size increased. And all four bounds tended to decrease for a particular sample size, population error rate and 100% overstatement percentage as the percentage of understatements increased. For the Cox and Snell and multinomial-Dirichlet bounds, which ignore understatements, this increase in percentage of understatements is translated as a reduction in error rate, hence the reduction in the bound. For the normal and modified moment bounds, the reduction is essentially due to the "netting" of errors. The difference in the relative reduction of the bounds for these two groups increases with sample size and error occurrence rate i.e., at higher population error rates and higher sample sizes, the normal and modified moment bounds tend to decrease more quickly than the Cox and Snell and multinomial-Dirichlet bounds as the percentage of understatement errors increases.

In order to pick out some general relations from the study, Grimlund and Felix provide aggregate results for coverage and advantage. Prior to this compilation, the populations were divided into two groups: those having no more than 10% of the errors as 100% overstatements and those having 20% to 40% of the errors as 100% overstatements. This division resulted from the conclusion that the level of 100% overstatements had the greatest impact on reliability. The aggregation, within these groups, was performed for accounts receivable and inventory separately; i.e. on 4 sets of 36 populations. The main difference between the accounts receivable and inventory populations is the higher percentage of understatements for inventory.

For those populations having low percentages of 100% overstatements, the modified moment bound had average coverage closest to the nominal level for both accounts receivable and inventory. The multinomial-Dirichlet bound was the most conservative in each case, with the Cox and Snell bound having similar coverage levels for the inventory populations.

The reverse was true in terms of advantage. The multinomial-Dirichlet bound exhibited the smallest advantage for both accounts receivable and inventory and the modified moment bound the highest. The Cox and Snell bound and the normal bound performed well under certain circumstances in this category for accounts receivable and inventory respectively.

The results were not as clear cut for the populations having higher levels of 100% overstatements. The multinomial-Dirichlet bound was again the uniformly most reliable bound, being rather conservative. The performance of the other bounds was not so consistent. The Cox and Snell bound had the highest advantage among the accounts receivable populations and the modified moment among the inventory. These results, however, do pertain to populations which may be viewed as extreme, in terms of the high percentage of 100% overstatements.

The authors conclude that the choice of method depends largely on the auditing situation and the needs of the auditor. For example, if reliability is the main concern, the multinomial-Dirichlet bound should be employed. This reliability comes at the price of efficiency however. Other considerations include confidence level and sample size. The CS10 bound performed well at the 95% confidence level but exhibited reliability problems below this level. For general purpose methods, the authors cite the normal bound, with the priors used in the main part of the paper, and the modified moment bound. These both handle understatements and on average provide coverage levels above the nominal level. These two bounds tend to suffer reliability problems only with populations which occur rarely, for example those with a very high percentage of 100% overstatements.

As mentioned previously, the authors presented similar results for two alternative bounds in an appendix; the Cox and Snell bound and normal bound with alternative priors. Both of these bounds performed well over a range of conditions but neither demonstrated the versatility to be considered a general purpose method.

Chapter 6

Bootstrap Methods

Bootstrapping is a useful, computer-intensive technique which allows the sampling distribution of a statistic to be generated from sample information.

The task of inferential statistics is to make inferences about a population parameter or characteristic θ, based on a sample statistic $\hat{\theta}$. Such inference requires an estimate of the sampling distribution of $\hat{\theta}$. *Traditional parametric inference* gives an analytic estimate using assumptions about the distribution of $\hat{\theta}$ and values for the parameters of this distribution as calculated from the sample. *Bootstrap inference*, on the other hand, gives an empirical estimate of the sampling distribution of the statistic, through repeated resampling, with replacement, from the sample. Thus, bootstrapping relies strictly on the sample information to draw conclusions about the population characteristic rather than on assumptions concerning the population.

The bootstrap approach relies on the sample being representative of the population from which it was drawn. This original sample of size n is treated as the population. A large number of resamples, each of size n, are drawn with replacement from the original sample. Each such resample is called a bootstrap sample. The statistic $\hat{\theta}$, calculated for each bootstrap sample, is denoted $\hat{\theta}^*$. The relative frequency distribution of the $\hat{\theta}^*$'s is taken as an estimate of the sampling distribution of $\hat{\theta}$. This bootstrapped estimate of the sampling distribution can now be used to develop confidence intervals for θ.

6.1 Parametric Bootstrap Bound

The *parametric bootstrap* is another form of the method discussed above. Rather than drawing the bootstrap samples from the original data, the samples are drawn from a parametric estimate of the population density. Tamura and Frost [49] use this technique. They model the tainting distribution using the power function density. From this model Efron's parametric bootstrap method is employed to develop a bound on the population total error. If no errors are found in an audit sample, it is assumed that all potential taints are a maximum of \$1. Then the bound is determined using the attributes method (3.10). If, however, one or more errors are discovered in the sample, the number and value of the taints are used to estimate the parameter of the power function. As with the Stringer bound, utilizing this available information should yield a more precise estimate of the error and hence, a tighter bound.

Consider the mixture distribution, seen earlier (3.1), as a model for tainting

$$t = \begin{cases} z & \text{with probability } \pi \\ 0 & \text{with probability } (1-\pi) \end{cases} \tag{6.1}$$

The parameter π is the population error rate and t is an observation of tainting of a dollar unit. The random variable $z \neq 0$ represents the tainting of a dollar unit. It is assumed that all errors are overstatements with no account error exceeding its book value, i.e. $z \in (0, 1]$. The nonzero taints are modelled by a power density so

$$f(z; \lambda) = \lambda z^{\lambda - 1} \text{ for } 0 < z \leq 1 \text{ and } \lambda > 0 \tag{6.2}$$

and

$$\mu_z = \frac{\lambda}{\lambda + 1}, \tag{6.3}$$

which gives

$$\mu_t = \frac{\pi\lambda}{\lambda + 1}. \tag{6.4}$$

Combining the mixture model (6.1) with the power density (6.2) results in a two parameter model, denoted $f(t; \pi, \lambda)$, for the audit population of dollar-unit taints. This density can be represented as follows:

$$f(t; \pi, \lambda) = (1 - \pi)\delta_0 + \pi\lambda t^{\lambda - 1} \tag{6.5}$$

where

$$\delta_0 = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

Consider a random sample of n dollar units which results in m nonzero taints and $n-m$ zero taints. Based on the model above the joint density of the taints can be written

$$L(\pi,\lambda)=f(t_1,t_2,\ldots,t_n)=\prod_{i=1}^{n}\left[(1-\pi)\delta_0+\pi\lambda t_i^{\lambda-1}\right]. \tag{6.7}$$

With the m nonzero taints indexed separately from the $n-m$ zero taints, the joint distribution can be rewritten as follows:

$$L(\pi,\lambda)=f(t_1,\ldots,t_m,t_{m+1},\ldots,t_n)=(1-\pi)^{n-m}\pi^m\lambda^m\prod_{k=1}^{m}t_k^{\lambda-1} \tag{6.8}$$

where t_k are the nonzero taints in the sample. Taking the natural log of this gives

$$\ln L(\pi,\lambda)=(n-m)\ln(1-\pi)+m\ln(\pi)+m\ln(\lambda)+\sum_{k=1}^{m}(\lambda-1)\ln(t_k) \tag{6.9}$$

and differentiating with respect to π and λ results in the following likelihood equations:

$$\frac{\partial\ln L}{\partial\pi}=\frac{-(n-m)}{1-\pi}+\frac{m}{\pi}=0 \tag{6.10}$$

$$\frac{\partial\ln L}{\partial\lambda}=\frac{m}{\lambda}+\sum_{k=1}^{m}\ln(t_k)=0 \tag{6.11}$$

Solving these equations yields the maximum likelihood estimates

$$\hat{\pi}=\frac{m}{n} \tag{6.12}$$

and

$$\hat{\lambda}=-\frac{m}{\sum\limits_{k=1}^{m}\ln t_k}. \tag{6.13}$$

Invoking the invariance property, maximum likelihood estimates for μ_z and μ_t are given by the following:

$$\hat{\mu}_z = \frac{\hat{\lambda}}{\hat{\lambda}+1} \tag{6.14}$$

and

$$\hat{\mu}_t = \hat{\pi}\hat{\mu}_z. \tag{6.15}$$

In order to construct an upper bound for the population error, one can find an upper bound for the population mean tainting (μ_t) and multiply this value by the total book amount (Y), which is a known constant. Due to the nature of the tainting distribution, it is difficult to derive the exact distribution of the sample mean tainting ($\hat{\mu}_t$). This sampling distribution can be approximated using a parametric bootstrap approach.

A sample of size n is taken from the population and, based on this sample, $\hat{\pi}$ and $\hat{\lambda}$ are calculated using the above formulas. Thus, we can define $\hat{f}(t; \hat{\pi}, \hat{\lambda})$, the parametric bootstrap estimate of the overall tainting distribution. A random sample of size n taken from the population defined by $\hat{f}$ is called a bootstrap sample. The estimate $\hat{\mu}_t^*$ is defined as the value of $\hat{\mu}_t$ computed from a single bootstrap sample. The sampling distribution of $\hat{\mu}_t^*$ under sampling from $\hat{f}(t; \hat{\pi}, \hat{\lambda})$ is the bootstrap distribution of $\hat{\mu}_t$. This distribution can be approximated empirically by sampling B times, where B is sufficiently large (at least 1,000). The 95'th percentile of the bootstrap distribution of $\hat{\mu}_t$ serves as a 95% upper bound for μ_t, assuming that the bootstrap distribution of $\hat{\mu}_t$ approximates the true unknown sampling distribution of $\hat{\mu}_t$. Thus, the parametric bootstrap 95% upper bound for μ_t is $\hat{U}_u$, such that

$$\frac{(\# \text{ of } \hat{\mu}_t^* < \hat{U}_u)}{B} = .95, \tag{6.16}$$

where $\hat{\mu}_t^*$ is calculated from B bootstrap samples.

Tamura and Frost [49] performed a simulation study with data generated from a power function density. The study centered on the 95% upper bound for μ_t. The bootstrap bound was conservative for $\pi \leq .04$; i.e. the probability that the upper bound exceeded μ_t was significantly larger than .95. For $\pi > .04$, the bound was reliable. The bootstrap bound was tighter than the Stringer bound in all cases.

The study did not look at the effects of using this method when the data are generated from the incorrect specification. The authors suggest the need for further research in this area. There is the possibility that alternative models for tainting could prove effective. The performance of the parametric bootstrap bound with proposed models should be examined using real audit data.

6.2 Bootstrap Auxiliary Bounds

Biddle, Bruton and Siegel [6] employed bootstrap techniques to form confidence intervals based on difference and ratio estimation. Difference and ratio estimators of the total population error E, based on line item sampling, were discussed in Chapter 2. Classical confidence intervals based on these estimators assume approximate normality of the population of errors. As previously discussed, this assumption is rarely valid in an audit environment and this leads to unreliability of the confidence coefficient. Bootstrap confidence intervals can be formed without relying on such parametric assumptions.

The reliability of the bootstrap depends on how well the sample represents the population. The authors explore two methods for developing $100(1-\alpha)\%$ bootstrap confidence intervals for E. These differ in the way the sample is assumed to be representative of the population.

Consider $\hat{E}$, the difference or ratio estimator of total error E based on a sample of size n. The sampling distribution of $\hat{E}$ can be estimated via bootstrap samples, from each of which a value $\hat{E}^*$ is obtained. The first method which Biddle et al. [6] term the *reflected bootstrap method* assumes that the sample is representative of the populations in the sense that $(\hat{E}^* - \hat{E})$ and $(\hat{E} - E)$ have approximately the same distribution. The *percentile bootstrap method* assumes that $\hat{E}^*$ and $\hat{E}$ have approximately the same distribution.

For either approach it is necessary to determine the appropriate values, $\hat{E}_L^*$ and $\hat{E}_U^*$, of the bootstrapped sampling distribution so that

$$P^*(\hat{E}_L^* < \hat{E}^* < \hat{E}_U^*) \simeq 1 - \alpha. \tag{6.17}$$

These values may or may not be symmetric about the point estimate $\hat{E}$. Subtracting E from each side of the above inequality yields

$$P^*(\hat{E}_L^* - \hat{E} < \hat{E}^* - \hat{E} < \hat{E}_U^* - \hat{E}) \simeq 1 - \alpha. \tag{6.18}$$

and based on the above assumption for the reflected bootstrap method, $\hat{E}^* - \hat{E}$ is replaced by $\hat{E} - E$ to give

$$\begin{aligned} P^*(\hat{E}_L^* - \hat{E} < \hat{E} - E < \hat{E}_U^* - \hat{E}) &\simeq 1 - \alpha \\ P^*(\hat{E}_L^* - 2\hat{E} < -E < \hat{E}_U^* - 2\hat{E}) &\simeq 1 - \alpha \\ P^*(2\hat{E} - \hat{E}_U^* < E < 2\hat{E} - \hat{E}_L^*) &\simeq 1 - \alpha. \end{aligned} \tag{6.19}$$

Thus, $(2\hat{E} - \hat{E}_U^*, 2\hat{E} - \hat{E}_L^*)$ is a $100(1-\alpha)\%$ confidence interval for E. The authors demonstrate that this interval is simply the reflection of the bootstrap distribution quantiles $\hat{E}_L^*$ and $\hat{E}_U^*$ about the point estimate $\hat{E}$, hence the name.

The percentile method is rather more straightforward. A $100(1-\alpha)\%$ confidence interval for E is given by the $100(\alpha/2)$ and the $100(1-\alpha/2)$ percentiles of the sampling distribution of $\hat{E}$. Since it is assumed that $\hat{E}$ and $\hat{E}^*$ are distributed approximately the same, the appropriate percentiles from the empirical distribution of $\hat{E}^*$, that is $\hat{E}_L^*$ and $\hat{E}_U^*$ give the $100(1-\alpha)\%$ confidence interval for E. Thus, the reflected interval is a reflection of the percentile interval about the point estimate $\hat{E}$.

Biddle, Bruton and Siegel [6] performed a simulation study to compare the performance of the two bootstrap intervals and the classical interval for difference and ratio estimation. They reported results for four of the Neter and Loebbecke [34] populations (1M, 2, 3 and 4), described in Chapter 7. For population 2, an inventory population, the nine error rate, sample size combinations shown in Table 6.1 were used. For the remaining populations, all accounts receivable, the 30% error rate was not used since this was viewed as extreme for accounts receivable.

π		n		
0.3	50,	100,	200,	300
0.1		100,	200,	300
0.05			200,	300

Table 6.1: Error Rates and Sample Sizes for Bootstrap Study

Thus there were 24 combinations of population, error rate and sample size. Consider each combination to be a separate study population. 2000 samples were drawn from each study population and for each sample the classical and bootstrap intervals were calculated for both ratio and difference estimation at the 95% confidence level. The bootstrap intervals were based on 1000 bootstrap samples.

Since the total error amount was known for each study population, the number of bounds which contained the true amount could be counted. The authors reported the achieved coverage level, i.e. the actual percentage of the 2000 bounds which contained the true error E, as well as the percentage of times the true value fell above (below) the upper (lower) confidence limit. To compare the efficiency of the various bounds, the percentage reductions in sample size achieved through use of the bootstrap estimators rather than the corresponding classical estimators were calculated. The results of the study showed that one of the bootstrap intervals outperformed the classical interval in terms of both reliability and efficiency for every study population. In every case, the other bootstrap interval was less reliable and less efficient than the classical interval.

For populations 1M and 2, which exhibited negative skewness and contained both overstatement and understatement errors, the reflected bootstrap exhibited the best performance and the percentile method the worst. The reverse was true for the positively skewed populations 3 and 4 which consisted solely of overstatements. Here the percentile method outperformed the classical interval while the reflective method behaved worse than either of these. The coverage results may provide an explanation for this pattern.

When a two-sided confidence bound falls entirely below the true value E it is considered an upper miss. Similarly a lower miss occurs if the confidence interval is entirely above the true error amount. Biddle et al. [6] observed that for populations 1M and 2, which were negatively skewed, the bounds based on the classical difference and ratio estimators exhibited a higher percentage of lower misses than upper misses. The bounds are interpreted as having a positive bias. The same bounds exhibited a negative bias, i.e. a higher percentage of upper misses, for populations 3 and 4 which were positively skewed.

The reflected and bootstrap intervals are not symmetric about the point estimate $\hat{E}$ for skewed distributions. For a positively skewed distribution, the percentile bootstrap interval exhibits positive asymmetry meaning the difference between the upper bound and the point estimate is greater than the difference between the lower bound and the point estimate. In the same situation, the reflected interval exhibits negative asymmetry; i.e., the difference between the lower limit and the point estimate is greater than the difference between the higher limit and the estimate. For populations which are negatively skewed the opposite holds true; the percentile interval has negative asymmetry and the reflected bound has positive asymmetry. This means that for populations which are positively (negatively) skewed, the percentile (reflected) bootstrap interval is further to the "right" ("left") than the reflected (percentile) bootstrap interval. By employing the bootstrap method with asymmetry in the opposite direction to the bias of the classical estimator, the bias can be reduced.

The authors suggest that if an error population exhibits negative skewness the reflective bootstrap interval, which will have positive asymmetry, should be used. For the case of a positively skewed population, the percentile bootstrap bound will have negative asymmetry and will give the best results. This applies if it can be assumed, as for the populations in this study, that population skewness produces bias in the opposite direction in the classical difference and ratio estimators.

Although these results suggest improvements over classical methods, problems remain. As with the classical intervals of ratio and difference estimation, reliability levels for the bootstrap versions were often considerably less than the nominal confidence level. It appears that bootstrapping alone cannot overcome this difficulty and neither the bootstrap nor the classical versions of these intervals offer a viable solution to finding bounds on audit error.

6.3 Clayton's Combined Bound

Clayton [10] suggests combining another bootstrap confidence interval method, the *bootstrap-t bound*, with the *Hoeffding bound*. His theory is based on dollar-unit sampling and yields an upper bound for the population mean taint. The bootstrap-t and Hoeffding bounds were chosen for several reasons. First of all, both bounds are nonparametric, requiring no distributional assumptions or prior knowledge on the part of the auditor. Secondly, these bounds complement each other in the sense that they perform well under different circumstances. For example, Clayton showed in a preliminary simulation study that as the number of 100% overstatement taints in a sample increases, the efficiency of the Hoeffding bound increases whereas the efficiency of the bootstrap-t bound decreases. Additionally, the bootstrap-t bound can easily handle overstatement and understatement bounds whereas the Hoeffding bound employs an ad hoc adjustment in the presence of understatements. It was hoped that by combining the two bounds in some manner, each would make up for the shortfalls of the other.

To find a bound for the mean taint using the bootstrap-t method, a sample of n taints is drawn and from this initial sample a large number (≥ 1000) of resamples, each of size n, are drawn with replacement. The original sample is considered the bootstrap population and $\hat{\mu}_t$, the mean of this bootstrap population and $\hat{\sigma}$, the estimated standard error of the mean are calculated. The mean, $\hat{\mu}_t^*$, of each resample, called a bootstrap sample, is calculated along with $\hat{\sigma}^*$, the estimated standard error of $\hat{\mu}_t^*$. Thus, for each bootstrap sample, the standardized variable

$$T^* = \frac{\hat{\mu}_t^* - \hat{\mu}_t}{\hat{\sigma}^*} \tag{6.20}$$

can be calculated. The distribution of T^* serves a purpose analogous to Student's t distribution in parametric inference. The 100α percentile of T^*, the value $\hat{t}_\alpha^*$ so that $P^*(T^* < \hat{t}_\alpha^*) = \alpha$, is used to develop an upper confidence bound for μ_t. The 100(1-α)% upper bootstrap-t (BST) bound for μ_t is

$$\hat{\mu}_t - \hat{\sigma}\hat{t}_\alpha^* \tag{6.21}$$

and the 100(1-α)% upper BST bound for E may be obtained by multiplying this quantity by Y, the total book value.

If the initial sample contains no errors, the bootstrap-t approach cannot be employed since each bootstrap sample will contain all zeros leaving the statistic T^* undefined. To handle this case, Clayton proposes the approach adopted by Tamura and Frost [49] for the parametric bootstrap bound. It is assumed that all taints in the population are at the maximum of \$1. Then the conservative attributes bound (3.10) is employed. Thus, the upper bound for μ_t is $\pi_u(0; 1-\alpha)$, the $1-\alpha$ upper confidence bound for the population proportion when no nonzero taints are found in the sample. Based on the binomial distribution this bound has the value $1 - \alpha^{1/n}$.

It can also happen that the initial sample contains nonzero errors but one or more of the bootstrap samples reveals all zero errors, thus failing to provide a value of T^*. To compute the bound for μ_t, only those bootstrap samples with at least one nonzero error are utilized in the percentile computation.

The bootstrap-t bound was found to perform poorly for populations with low error rates. This can be blamed on the low values of $\hat{\sigma}^*$ resulting from the large proportion of zeros in the bootstrap populations. To alleviate this problem, T^* is calculated with $\hat{\sigma}$ in place of $\hat{\sigma}^*$ whenever $\hat{\sigma}^* < \hat{\sigma}/10$.

Before arriving at this rule, the use of four alternative fractions was investigated, specifically 1/10, 1/25, 1/50 and 1/100. It was found that as the fraction decreased, both the size and reliability of the bound increased. The bound had the best overall performance when the standard error rule was based on the fraction 1/10. This adjusted bound is labeled BST10.

It is evident that the bootstrap-t method presented above requires no modifications for a sample containing understatements. In contrast, to apply the Hoeffding method to an audit population, theory requires that all the errors are overstatements with no error exceeding the corresponding book value, i.e. $0 \leq t \leq 1$.

Since μ_t is the population mean error per dollar unit, $\mu_r = 1 - \mu_t$ is the population mean proportion of a dollar unit that is not in error. The Hoeffding inequality [5] is employed, yielding a lower confidence bound for μ_r and subsequently an upper bound for μ_t.

For a dollar-unit sample of size n, $r_i = 1 - t_i$, $0 \leq r_i \leq 1$, is the proportion of the i'th dollar unit which is not in error. Since this method cannot cope with understatements, any such errors are taken to have a value of zero, yielding an r_i

value of \$1. The mean of this sample is $\overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$.

The Hoeffding inequality can be applied, giving

$$P(\overline{r} - \mu_r \geq c) \leq \left[\left(\frac{\mu_r}{\mu_r + c} \right)^{\mu_r + c} \left(\frac{1 - \mu_r}{1 - (\mu_r + c)} \right)^{1 - (\mu_r + c)} \right]^n \tag{6.22}$$

for some constant c such that $0 < c < 1 - \mu_r$. Replacing $\mu_r + c$ by $\overline{r}$ in the right-hand side of the above inequality, equating this modified expression to α and solving for μ_r yields the approximate $100(1-\alpha)\%$ lower confidence bound for μ_r, denoted L_{μ_r}. Since $\mu_t = 1 - \mu_r$, the approximate $100(1-\alpha)\%$ upper confidence bound for μ_t is $U_{\mu_t} = 1 - L_{\mu_r}$. This approximate bound will be referred to as the Hoeffding bound. The approximate upper confidence bound for the total error in the population results when this bound is multiplied by Y, the total book value of the population.

Though the Hoeffding bound proved reliable in preliminary simulation studies, it was large and thus inefficient. To overcome this, Clayton [10] proposed a modification to the bound based on bootstrap methodology which will be called the *bootstrap Hoeffding bound.* Suppose that many bootstrap samples are drawn from the initial sample. Then $\overline{r}^*$ can be calculated for each bootstrap sample. Applying Hoeffding's inequality,

$$P^*(\overline{r}^* - \overline{r} \geq c) \leq \left[\left(\frac{\overline{r}}{\overline{r} + c} \right)^{\overline{r} + c} \left(\frac{1 - \overline{r}}{1 - (\overline{r} + c)} \right)^{1 - (\overline{r} + c)} \right]^n . \tag{6.23}$$

By setting the left-hand expression to α and solving for c, the $1-\alpha$ lower bootstrap confidence bound for μ_r can be approximated by $\overline{r} - c$. Numerical methods are required for this as no closed form solution for c exists. It is assumed that the empirical bootstrap distribution is representative of the population in the sense that $\overline{r}^* - \overline{r}$ and $\overline{r} - \mu_r$ have approximately the same distribution. Hence $P^*(\overline{r}^* - \overline{r} \geq c)$ approximates $P(\overline{r} - \mu_r \geq c)$. Thus,

$$L^*_{\mu_r} = \overline{r} - c \tag{6.24}$$

serves as the approximate $100(1-\alpha)\%$ lower bootstrap confidence bound for μ_r. The approximate $100(1-\alpha)\%$ upper bootstrap confidence bound for $\mu_t = 1 - \mu_r$ is thus

$$U^*_{\mu_t} = 1 - \overline{r} + c. \tag{6.25}$$

These approximate confidence bounds are the bootstrap Hoeffding bounds. Again the upper confidence bound for E is obtained through multiplication of this bound by Y, the total book value.

Clayton demonstrated that the bootstrap Hoeffding bound (6.25) is actually the Hoeffding bound to the reflected bootstrap bound for μ_t. As discussed in Section 6.2, the first step in determining the reflected bootstrap lower bound for μ_r is to obtain the value $\overline{r}^*_U$ from the empirical bootstrap distribution of $\overline{r}^*$ such that

$$P^*(\overline{r}^* \geq \overline{r}^*_U) = \alpha. \tag{6.26}$$

Then

$$P^*(\overline{r}^* - \overline{r} \geq \overline{r}^*_U - \overline{r}) = \alpha. \tag{6.27}$$

and substituting $\overline{r} - \mu_r$ for $\overline{r}^* - \overline{r}$ leads to

$$P^*(\mu_r \leq 2\overline{r} - \overline{r}^*_U) \simeq \alpha \tag{6.28}$$

and

$$P^*(\mu_r \geq 2\overline{r} - \overline{r}^*_U) \simeq 1 - \alpha. \tag{6.29}$$

Thus,

$$2\overline{r} - \overline{r}^*_U \tag{6.30}$$

is the $1 - \alpha$ reflected bootstrap lower bound for μ_r and

$$1 - 2\overline{r} + \overline{r}^*_U \tag{6.31}$$

is the $1 - \alpha$ reflected bootstrap upper bound for μ_t. We can derive the Hoeffding bound to the reflected bootstrap bound as follows. Suppose that there exists a value c such that

$$P^*(\overline{r}^* - \overline{r} \geq c) \leq \alpha. \tag{6.32}$$

One such value for c is $\overline{r}^*_U - \overline{r}$, according to (6.27). Since $c \geq \overline{r}^*_U - \overline{r}$ from (6.23) and (6.27) and hence $\overline{r}^*_U \leq c + \overline{r}$, where c is calculated from the Hoeffding inequality (6.23) then

$$1 - 2\overline{r} + \overline{r}^*_U \leq 1 - 2\overline{r} + c + \overline{r} = 1 - \overline{r} + c. \tag{6.33}$$

The right-hand expression in (6.33) is the Hoeffding bound to the reflected bootstrap bound for μ_t which is identical to (6.25). The bootstrap Hoeffding bound (6.25) thus combines the Hoeffding bound, which Clayton found to be reliable but conservative with the reflected bootstrap bound which can be tight but unreliable.

As with the bootstrap-t bound, the bootstrap Hoeffding bound cannot be employed for a sample with no nonzero errors. In this case $\overline{r} = 1$ and the right-hand side of the Hoeffding inequality has a value of zero so no solution for c exists. It is impossible to solve for c even if $\bar{r}$ is very close to 1.

For samples which contain no errors, meaning $\bar{r} = 1$, the conservative attributes bound $1 - \alpha^{1/n}$ is used. Thus according to (6.25), $c = U^*_{\mu t} = 1 - \alpha^{1/n}$ when $\bar{r} = 1$. The value of $\bar{r}$ closest to 1 for which a solution for c exists, say $\bar{r}_1$, will vary depending on computer precision. Once the value $\bar{r}_1$ and the corresponding solution c_1 are determined, linear interpolation between $(\bar{r}_1, c_1)$ and $(1, 1 - \alpha^{1/n})$ is employed to estimate c for samples which yield values of $\bar{r}$ close to 1.

Unlike the bootstrap-t bound, the bootstrap Hoeffding bound does not naturally extend to handle understatements. In the case of understatements and overstatements, the net mean error per dollar unit is estimated using the bootstrap Hoeffding bound for overstatements, with the adjustment for understatements suggested by Leslie, Teitlebaum and Anderson [29]. This LTA method was described earlier as an improvement to the Stringer bound (3.23).

After deciding upon the use of the bootstrap Hoeffding bound and the bootstrap-t bound, a rule for their combination had to be determined. Clayton [10] initially investigated the performance of the bootstrap Hoeffding bound and the bootstrap-t bound individually. For this purpose, data was simulated according to the model employed by Dworin and Grimlund [14] and Grimlund and Felix [22], described briefly in Chapter 4 and more extensively in Chapter 7. Clayton used the same 72 error rate combinations as Grimlund and Felix, 36 for accounts receivable and 36 for inventory.

He employed only one taint distribution model for each type of data, thus generating 72 study populations. For each study population, 500 random samples of size 120 were drawn and the bootstrap Hoeffding and bootstrap-t bounds for the mean taint were calculated. The benchmark Stringer bound was also calculated for comparative purposes. The advantage of each bound to the Stringer bound was calculated as (mean Stringer bound - mean bound)/mean Stringer bound, where the means were calculated over the 500 samples. An advantage greater than zero indicates the alternative bound is tighter and, conversely, a negative advantage indicates the Stringer bound is tighter.

By examining the reliability and efficiency of these bounds for the various populations, Clayton [10] determined that error rate, proportion of understatements and proportion of 100% overstatements had the greatest effect on the bounds' performance. Of these variables, error rate proved the most influential. The following univariate combination rule, based only on the number of taints in the sample, reflecting error rate, was proposed. If the total number of taints in the sample m is less than 10% of the sample size n, the bootstrap Hoeffding bound is employed; otherwise the bootstrap-t bound is used. Clayton's investigation was based on samples of size 120, so the corresponding "switchover" point was 12. This particular value was chosen from a broader range as the one which provided the best overall performance in terms of coverage and efficiency.

Combination rules involving two and three variables, for example the total number of taints and the number of understatement taints, were investigated. None of these yielded a uniformly better bound and therefore did not warrant the extra complications. Thus, the initial combination rule involving the total number of taints was used.

The next step was to investigate the performance of the combined bound using the simulated study populations. By examining the coverage and the size of the combined bound, in terms of advantage over the Stringer, Clayton [10] was able to determine certain conditions under which the combined bound is clearly superior. Though not uniformly better than the Stringer bound, the combined bound was deemed preferrable for accounts receivable (inventory) populations with low error rates, low (high) proportions of 100% overstatements and high proportions of understatements.

The performance of the combined bound was then examined for real data. The Neter and Loebbecke [34] populations discussed earlier were used for this purpose. The study populations were each formed from one of the four original populations with an appropriate number of errors being fed in to give the desired error rate (1%, 5%, 10% and 30%). In total, 16 such study populations were examined. The one discrepancy from this pattern is population 2, for which an error rate of 70% replaced the 30% error rate.

Populations 1, 3 and 4 are accounts receivable populations. Population 1, with error rates of 1%, 5% and 10% fit Clayton's description of a favourable environment for the combined bound. As expected, the combined bound performed well for these study populations. The coverage percentage was above the nominal confidence level and the combined bound was tighter than the Stringer bound. None of the other receivables populations fit the ideal conditions, but among these the coverage percentage fell below the nominal confidence level only twice and the Stringer bound was only tighter in one case. Each of these discrepancies occurred for population 3, which had a high proportion of 100% overstatements and no understatements. For this population, the bound was unreliable at higher error rates (10% and 30%) and inefficient at the lowest error rate of 1%.

The inventory population, population 2, has a low proportion of 100% overstatements and and a high proportion of understatements. Although this differs drastically from the recommended condition for use of the combined bound, results were favourable. The coverage percentage of the combined bound did not dip below the nominal confidence level for the four error rates examined (1%, 5%, 10% and 70%) and the combined bound was consistently tighter than the Stringer bound for this population.

Chapter 7

Comparison Study

7.1 Design

This study was designed to provide information on the performance of various methods for finding an upper bound on the mean taint per dollar unit μ_t. The methods included in the study are listed below, in the order in which they have been discussed in this paper. Given in parentheses are the abbreviations used in the discussion of the results.

- Stringer bound (ST)
- Stringer bound with Meikle's adjustment for understatements (ST-meik)
- Stringer bound with the LTA adjustment for understatements (ST-lta)
- modified moment bound (MM)
- Bayesian normal bound (BN)
- Cox and Snell bound (CS)
- multinomial-Dirichlet bound (MD)

- parametric power bound (PP)
- Clayton's combined bootstrap Hoeffding/bootstrap-t bound (CL)

The Stringer bound, which is widely used in practice, has been found to be conservative. Extensive simulation studies by Leitch et al. [28] and Reneau [45] have provided empirical evidence that the achieved confidence level of the Stringer bound exceeds the nominal level and that the Stringer bound is not very tight. The Stringer bound may overestimate the total error amount for an audit population with a small true error amount. This could lead the auditor to falsely conclude that the total book amount contains a material error.

The Stringer bound does not take understatements into account. It is possible, however, to adjust the bound for populations which contain understatements. The most widely used methods for this are Meikle's adjustment and the LTA adjustment. Of 12 large accounting firms surveyed by Grimlund and Schroeder [23], five made no adjustments for understatements, 3 used Meikle's approach and 3 used the LTA approach. One firm employed the modified moment bound which naturally allows for understatements.

The other bounds included in this study have all been suggested as alternatives to the traditional Stringer bound. This simulation was designed to reveal which bounds had acceptable combinations of tightness and good coverage properties, and hence could be employed by auditors.

Several methods discussed in this paper were not included in the simulation analysis. The most notable absence is that of the multinomial bound. The multinomial bound is extremely computer intensive. Following the suggestion by Tsui, Matsamura and Tsui [51] that the multinomial-Dirichlet bound be used as a less expensive and less time-consuming alternative, the multinomial bound was excluded. The other bounds which were not included were all shown in previous studies to be inferior to one or more of the methods in the above list.

The choice of prior parameter values for the three Bayesian methods was based on results from previous simulation studies. In each case, the parameter settings used were believed to yield a fairly general purpose method; i.e. a method which would have good performance for a range of audit populations.

The Bayesian normal bound of Menzefricke and Smieliauskas [31] was found by Grimlund and Felix [22] to provide reliable coverage for the majority of populations in their simulation study. This particular bound sets $\pi_0 = .1$, $n_0 = 1$, $\mu_0 = .5$, $r_0 = 3$, $\phi_0 = .3$ and $\theta_0 = 1$.

For the Cox and Snell method, Godfrey and Neter's [20] CS10 prior was used. The actual values for this model are as follows: $\pi_0 = .1$, $a = 1$, $\mu_0 = .4$ and $b = 6$. Godfrey and Neter, in their study of 21 simulated populations, found the CS10 bound to provide adequate coverage and remain significantly tighter than the benchmark Stringer bound.

Finally, the B3 prior of Tsui, Matsamura and Tsui [51] was employed with the multinomial-Dirichlet bound. Tsui et al. studied the performance of this bound with various priors. The B3 prior had good coverage levels over a range of populations and was deemed the best for those cases where an auditor is uncertain of the shape of a tainting population. The B3 prior sets $K = 5$, $\alpha_0 = .8$, $\alpha_i = .001$ for $i = 1, \ldots, 99$ and $\alpha_{100} = .101$.

The Cox and Snell, multinomial-Dirichlet and parametric power methods do not allow for understatements. For comparative purposes, each of these was also computed with the LTA adjustment for understatements. Thus, a total of 12 bounds were examined.

Each method was applied to numerous study populations, both real and simulated, which are described in the next two sections. Five hundred random samples, each of 100 dollar-unit taints, were drawn from each study population. For each sample the 12 bounds were computed at the 95% confidence level.

7.2 Simulated Data

Data for this study was generated based on the tainting distribution model described by Clayton [10]. The tainting distribution is viewed as a mixture of four separate components; a continuous distribution representing the absolute understatement taints (US), a second continuous distribution, truncated at 1.0, representing the overstatement taints less than 100% (OS) and two point masses, at 0 and 1.0. Neter et al. [36], in their empirical study of dollar-unit taint distributions for accounts receivable and inventory populations, found the existence of a point mass at 1.0, representing 100% overstatements (OS100), to be a common characteristic for such populations. For accounts receivable, a taint of 1.0 would arise, for example, if an account has been paid in full but the payment not recorded.

Let π be the total dollar-unit error rate, then

$$\pi = \pi_{OS} + \pi_{US} + \pi_{OS100} \tag{7.1}$$

where π_{OS} is the OS error rate, π_{US} is the US error rate and π_{OS100} is the OS100 error rate. Each of these probabilities is an overall rate for the population and is thus considered unconditional.

In contrast, we can consider the probabilities, conditional on an error having occurred, of each of the three types of taints. The proportion of nonzero taints that are OS is denoted p_{OS}. Similarly, p_{US} is the proportion of nonzero taints that are US and p_{OS100} gives the proportion of nonzero taints that are OS100. The identity

$$1 = p_{OS} + p_{US} + p_{OS100} \tag{7.2}$$

must hold since every nonzero taint falls into one, and only one, of the three categories. If μ_{OS}, μ_{US} and $\mu_{OS100} = 1$ are the conditional means of the three categories, then the overall mean taint can be calculated as follows:

$$\mu_t = \pi(p_{OS}\mu_{OS} + p_{US}\mu_{US} + p_{OS100}). \tag{7.3}$$

To generate a study population, the distributions to model OS taints and US taints must be chosen and the values of π, p_{OS}, p_{US} and p_{OS100} set. These distributions and parameters can be varied to yield a variety of study populations. For this study the distributions chosen were those used by Dworin and Grimlund [14]. Each of these four models, described in Table 7.1, provides distributions for both US and OS taintings. The χ^2 distributions are scaled by 1/10 in order to yield the means given in the table. According to this scaling, the units of the χ^2 distribution are 10 cents. The means of the OS taints are conditioned on the taints being no greater than 1.

	Model			
	M1	M2	M3	M4
Overstatements distribution	χ_1^2	χ_2^2	χ_3^2	Uniform
μ_{OS}	.0983	.1932	.2827	.5
Understatements distribution	χ_1^2	χ_2^2	χ_3^2	χ_{OS}^2
μ_{US}	-.1	-.2	-.1	-.1
True mean error	$\pi[.0983p_{OS} - .1p_{US} + p_{OS100}]$	$\pi[.1932p_{OS} - .2p_{US} + p_{OS100}]$	$\pi[.2827p_{OS} - .1p_{US} + p_{OS100}]$	$\pi[.5p_{OS} - .1p_{US} + p_{OS100}]$

Table 7.1: Models for OS and US distributions

This choice of distributions is supported by empirical evidence. Neter et al. [36], in their study of accounts receivable and inventory populations, found the distributions of non-100% overstatement and absolute understatement taints to be positively skewed with the distribution concentrated around zero. This study also provided evidence concerning the median taint for those 20 populations having the largest number of errors. For accounts receivable the median nonzero taint was .15 and for inventory, .08. The median values of understatements and overstatements were also computed separately. For accounts receivable, the median overstatement taint was .706 for small book amounts and .348 for large book amounts. The corresponding medians for the inventory populations were .133 and .096. The median understatement taint for receivables was -.057 for small book amounts and -.0044 for large book amounts. Among the inventory populations, the median understatement taints were -.150 and -.075.

Leitch at al. [28] used the χ^2 distribution with 1 and 3 degrees of freedom to model overstatements. They attributed this choice of distributions to empirical evidence provided in studies of real accounting populations by Ramage, Krieger and Spero [44] and Johnson, Leitch and Neter [26]. Though none of the empirical studies mentioned above provided evidence that overstatements follow a uniform distribution, this model was included to simulate an extreme accounting environment.

Values chosen for the various error rates in this study were the same as those used by Grimlund and Felix [22] and Clayton [10]. All 72 possible combinations of the rates, π, p_{US} and p_{OS100}, given in Table 7.2 were used with each of the distribution models in Table 7.1. Since the conditional error percentages (p_{OS}, p_{US}, p_{OS100}) and not the unconditional error rates (π_{OS}, π_{US}, π_{OS100}) are used to generate the study populations, any reference to the amount of the various taints (US, OS, OS100) is assumed to be conditional on the taints being nonzero. For example the percentage of US taints actually refers to the percentage of nonzero taints which are understatements.

		Taintings		
	Error Rate π	% US p_{US}	% OS100 p_{OS100}	% OS $p_{OS} = 1 - p_{US} - p_{OS100}$
Accounts Receivable	3%	0	0	
	6	10	10	[40,100][1]
	10	20	20	
			40	
Inventory	10%	20	0	
	30	50	10	[0,80]
	60	70	20	
			30	

[1] p_{OS} will have a value between 40 and 100 depending on the values of p_{US} and p_{OS100}

Table 7.2: Error Rate Combinations

As with the distribution models, the choice of error rates is supported by empirical evidence. Neter et al. [36] found among the 20 populations in their study which contained the highest number of errors that 40% of the receivable populations had error rates below 2.5% and that approximately 73% of these populations had an error rate below 12%. The error rates were somewhat higher among the inventory populations. Approximately 42% of the inventory populations had error rates below

14% and almost 40% had error rates ranging from 14% to 64.5%.

Among all 55 accounts receivable populations and 26 inventory populations in their study, Neter et al. [36] found proportions of understatement taints ranging primarily from 0-50%, with the higher values being predominant among the inventory populations. The majority of receivable populations (61.9%) contained solely overstatements. Only 10 of the 55 receivable populations examined had more than 20% understatements among the errors. In contrast, less than 8% of the inventory populations examined consisted solely of overstatements. Almost 70% of these populations had understatement percentages falling within the range of understatement rates proposed for this experiment. Among the 20 populations having the largest number of errors, Neter et al. [36] found proportions of 100% overstatements ranging from 0-66% for accounts receivables and from 0-12% for inventory.

Another empirical study by Ham et al. [25], based on annual audits of inventory of 17 companies and annual receivable audits of 20 companies (yielding 49 and 60 data sets respectively), provided results consistent with those of Neter et al. [36].

To generate a sample from a particular study population, a uniform random number generator was used to determine whether a given observation would contain an error. The uniform random number generator was used a second time to classify each error into one of the three categories; US, OS or OS100. The values for the absolute understatement taints and overstatement taints below 1, were generated by an appropriate random number generator, either χ^2 or uniform.

This method of modelling the two components, tainting distributions and error rates, was employed by Dworin and Grimlund [14], Grimlund and Felix [22] and Clayton [10]. Only in the case of Dworin and Grimlund were all four models described in Table 7.1 used. Felix and Grimlund used M1 and M3 to model accounts receivable and M1 and M2 to model inventory. Clayton simply used M1 for accounts receivable and M2 for inventory. All four models were used in this study for both accounts receivable and inventory to provide evidence on the performance of the bounds in extreme audit environments.

The error proportions employed in this study, as in Grimlund and Felix [22] and Clayton [10], are slightly more extreme than those used by Dworin and Grimlund [14] for the OS100 category. The OS100 percentages cover a wider range, particularly for inventory.

Thus, the 144 accounts receivable populations and 144 inventory populations simulated for this study cover a slightly broader range of audit environments than any of the three previous studies which used this model.

7.3 Real Data

The four audit populations from the 1975 Neter and Loebbecke [33] study were chosen to provide evidence concerning the performance of the bounds when applied to real audit data. Neter and Loebbecke caution that these populations cannot be considered representative of all accounting populations, but their use allows easy comparison with previous studies and according to Biddle et al. [6] they are among the most complete data sets available.

These four audit populations, described briefly in Chapter 2, represented the areas of accounts receivable and inventory. Population 1 is accounts receivable of a freight company. Population 2 is inventory of a medium-size manufacturer. Population 3 is accounts receivable of a second medium-size manufacturer and population 4 is accounts receivable of a large manufacturer. Populations 1, 2 and 4 are actually random subsets of the entire populations which were very large. The only other adjustment in the data was the exclusion of very large book amounts from populations 2, 3 and 4. It was felt that these items would be verified by auditors on a 100 percent basis. Tables 7.3 to 7.6, adapted from Neter and Loebbecke [33], give a summary of the book values of the four populations.

Sample information obtained from audits conducted on the populations provided information regarding the error patterns. Table 7.7 gives some general results from these audits, provided by Neter and Loebbecke [33]. The error rates given in this table are actually line item error rates, since the study by Neter and Loebbecke [33] was based on sampling of line items.

The understatement and overstatement errors (line item errors) discovered in the audit of population 1 tended to balance; in fact, the mean error was only \$.22. The mean absolute error, however, was \$3.89. Though this population exhibited a high

Population 1 - accounts receivable, freight company				
Book amount		# of accounts	Summary	
0-	13.50	2,039	Total value	$379,131.00
13.51-	22.50	2,455	Mean	$46.53
22.51-	36.00	1,867	Median	$21.10
36.01-	63.00	852	Standard deviation	$132.61
63.01-	105.00	494	Skewness	22.0
105.01-	195.00	335	Maximum	$6,869.70
195.01-	345.00	136	Minimum	$.50
345.01-	675.00	79		
675.01-	945.00	24		
945.01-	1,545.00	16		
1,545.01-	6,945	12		
Total		8,309		

Table 7.3: Summary of Book Values Population 1 (Source: Neter, J., Loebbecke, J. (1975))

Population 2 - inventory, medium–size manufacturer				
Book amount		# of accounts	Summary	
0-	50.00	1,419	Total value	$3,486,530.00
50.01-	100.00	676	Mean	$636.00
100.01-	200.00	745	Median	$181.00
200.01-	300.00	457	Standard deviation	$1,155.99
300.01-	500.00	538	Skewness	3.5
500.01-	1,000.00	636	Maximum	$9,989.00
1,000.01-	2,000.00	551	Minimum	$1.00
2,000.01-	3,000.00	195		
3,000.01-	5,000.00	175		
5,000.01-	10,000	90		
Total		5,482		

Table 7.4: Summary of Book Values Population 2 (Source: Neter, J., Loebbecke, J. (1975))

Population 3 - accounts receivable, medium-size manufacturer				
Book amount		# of accounts	Summary	
0-	40.00	1,334	Total value	$13,671,500.00
40.01-	136.00	1,438	Mean	$1,945.84
136.01-	400.00	1,475	Median	$236.65
400.01-	800.00	878	Standard deviation	$7,021.61
800.01-	1,400.00	539	Skewness	7.9
1,400.01-	3,000.00	548	Maximum	$98,162.70
3,000.01-	5,000.00	278	Minimum	$.10
5,000.01-	10,000.00	239		
10,000.01-	49,000.00	258		
49,000.01-	100,000	39		
Total		7,026		

Table 7.5: Summary of Book Values Population 3 (Source: Neter, J., Loebbecke, J. (1975))

Population 4 - accounts receivable, large manufacturer				
Book amount		# of accounts	Summary	
0-	90.00	1,070	Total value	$7,502,957.00
90.01-	230.00	715	Mean	$1,860.39
230.01-	400.00	450	Median	$304.80
400.01-	650.00	337	Standard deviation	$3,865.13
650.01-	1,500.00	455	Skewness	3.2
1,500.01-	3,500.00	409	Maximum	$24,928.60
3,500.01-	5,000.00	149	Minimum	$.10
5,000.01-	10,000.00	238		
10,000.01-	25,000.00	210		
Total		4,033		

Table 7.6: Summary of Book Values Population 4 (Source: Neter, J., Loebbecke, J. (1975))

	Population			
	1	2	3	4
# of accounts audited	555	217	123	174
# of errors	159	155	9	10
error rate	28.6%	71%	7.3%	5.7%
direction of errors	OS & US	OS & US	OS only	OS only
mean error	$.22	-$26.00	–[1]	–
mean absolute error	$3.89	$179.00	–	–
maximum understatement	-$35.15	-$2,163	–	–
maximum overstatement	$46.17	$1,450	–	–

[1] This information was not provided in the report.

Table 7.7: General Audit Results (Source: Neter, J., Loebbecke, J. (1975))

error rate, the error amounts were not that large. The maximum understatement error was -$35.15 and the maximum overstatement error was $46.17. The absolute error amount tended to increase with book value. The error rate was not found to differ substantially with book value for this population, or the other three.

The errors discovered in the audit of population 2 were of a much larger magnitude than those of population 1. The largest understatement was -$2,163 and the largest overstatement $1,450. The understatements tended to dominate with a mean error of -$26. The mean absolute error was $179. As with population 1, the size of the absolute error tended to increase with book value.

Due to the small number of errors discovered in the audits of populations 3 and 4, conclusions about the pattern of errors were difficult to make. For population 3 it was found that the larger taints tended to occur for smaller accounts. The audit of population 4 yielded many 100% overstatements, with most of these occuring for accounts with book values below $1,000. Aside from the 100% overstatements, the taints were small, from .005 to .15.

Neter and Loebbecke [33] created several study populations from each of the four populations reflecting a variety of line item error rates. The rates included in their study were .5%, 1%, 5%, 10% and 30%, with the exception of population 2 for which 70% was used in place of 30%. Their study populations were obtained for use in this report. The method Neter and Loebbecke used to create the study populations will be described.

For populations 1 and 2, errors found in the actual audits were divided into 5 pools according to book amount. To create a study population with a particular error rate, the items in the population which were to contain errors were chosen randomly. For each item selected, the error was chosen at random, with replacement, from the appropriate error pool, i.e. the error pool corresponding to the item's book amount. The error amount was then combined with the book amount to yield the audit amount. The number of items chosen to contain an error reflected the desired error rate of the study population.

If the selected error amount led to a negative or zero valued audit amount for population 1, it was replaced by another randomly selected error amount. This was due to the fact that the situation of zero or negative audit amounts never arose in the actual audit of the population. Similarly, for population 2, negative audit amounts were avoided. In terms of taints, this translates to $t_i < 1$ for population 1 and $t_i \leq 1$ for population 2.

Neter and Loebbecke [33] also generated two study populations from a modified version of population 1, with error rates of 1% and 10%. This modification, resulting in population 1M, involved the removal of the largest 27 accounts, having book values over \$950, which represented about 12% of the total book value. It was felt that these accounts would be examined by an auditor on a 100% basis. Therefore, population 1M represents the segment of the total population which would likely be sampled. Table 7.8 gives characteristics of the book amounts for population 1M. It is obvious, upon comparison with Table 7.3, that the skewness of population 1M is considerably less than for population 1. This is to be expected with the exclusion of the line items having the largest book values.

Population 1M	
Summary	
Total value	\$334,212.00
Mean	\$40.35
Median	\$21.05
Standard deviation	\$72.32
Skewness	6.1
Maximum	\$948.28
Minimum	\$.50

Table 7.8: Summary of Book Values Population 1M (Source: Neter, J., Loebbecke, J. (1975))

The process of forming study populations from populations 3 and 4 was slightly different, due to the lack of information obtained from the two audits. Error pools were formed according to book value, but rather than actual errors, the pools contained a probability distribution for the relative error or taint of an overstatement. Thus, for a selected item, an error was chosen as a percentage of the book value according to a particular probability distribution, depending on the book value. The probability distributions ensured that each error fell between zero and the corresponding book value, so the resulting taints were all between zero and one.

Neter and Loebbecke [33] describe the study populations in terms of the audit amounts. Since this study is based on dollar-unit sampling, information concerning the distribution of taintings is of interest. Tables 7.9 to 7.13 give characteristics of the taints in each study populations. The error rates used to describe each study population are actually line item error rates. For these populations, the two rates differ slightly as can be seen in the tables.

Five hundred systematic random samples of 100 dollar units were selected from each study population. To avoid possible patterns the line items were randomized before the selection of dollars took place. For each study population, the sampling interval was determined by dividing the total book value Y by the sample size of 100. One dollar unit was chosen at random from the first $Y/100$ dollar units and every $Y/100$'th dollar unit was then selected until the sample was complete. For each dollar unit selected, the taint of the line item containing that dollar was recorded.

Population 1					
Error %	.5	1	5	10	30
Dollar-unit error rate (π)	.0024	.0096	.0470	.1159	.3184
% OS(p_{OS})	.732	.366	.624	.628	.583
% US (p_{US})	.268	.634	.376	.372	.417
% OS100 (p_{OS100})	.000	.000	.000	.000	.000
OS mean (μ_{OS})	.0829	.0827	.0738	.0524	.0575
US mean (μ_{US})	-.4164	-.1064	-.1156	-.1140	-.0960
Mean taint per dollar-unit (μ_t)	-.00013	-.00035	.00012	-.00110	-.00207

Table 7.9: Summary of Tainting Characteristics for Study Populations from Population 1

Population 1M		
Error %	1	10
Dollar-unit error rate (π)	.0079	.1029
% OS(p_{OS})	.503	.554
% US (p_{US})	.497	.446
% OS100 (p_{OS100})	.000	.000
OS mean (μ_{OS})	.0827	.0742
US mean (μ_{US})	-.1538	-.1185
Mean taint per dollar-unit (μ_t)	-.00028	-.00122

Table 7.10: Summary of Tainting Characteristics for Study Populations from Population 1M

Population 2					
Error %	.5	1	5	10	30
Dollar-unit error rate (π)	.0026	.0084	.0456	.0913	.6881
% OS(p_{OS})	.482	.631	.510	.484	.441
% US (p_{US})	.468	.289	.458	.482	.541
% OS100 (p_{OS100})	.049	.081	.032	.034	.019
OS mean (μ_{OS})	.0520	.0419	.0902	.0899	.0830
US mean (μ_{US})	-.2733	-.2598	-.2292	-.1888	-.1625
Mean taint per dollar-unit (μ_t)	-.00014	.00027	-.00121	-.00127	-.02240

Table 7.11: Summary of Tainting Characteristics for Study Populations from Population 2

Population 3					
Error %	.5	1	5	10	30
Dollar-unit error rate (π)	.0056	.0102	.0496	.0840	.2912
% OS(p_{OS})	.996	.992	.990	.988	.990
% US (p_{US})	.000	.000	.000	.000	.000
% OS100 (p_{OS100})	.004	.008	.010	.012	.010
OS mean (μ_{OS})	.0296	.0301	.0256	.0310	.0307
US mean (μ_{US})	NA	NA	NA	NA	NA
Mean taint per dollar-unit (μ_t)	.00019	.00039	.00173	.00356	.01182

Table 7.12: Summary of Tainting Characteristics for Study Populations from Population 3

Population 4					
Error %	.5	1	5	10	30
Dollar-unit error rate (π)	.0061	.0116	.0444	.0974	.3146
% OS(p_{OS})	.471	.632	.742	.682	.585
% US (p_{US})	.000	.000	.000	.000	.000
% OS100 (p_{OS100})	.529	.368	258	.318	.415
OS mean (μ_{OS})	.0296	.0417	.0591	.0666	.0587
US mean (μ_{US})	NA	NA	NA	NA	NA
Mean taint per dollar-unit (μ_t)	.00331	.00456	.01341	.03541	.14136

Table 7.13: Summary of Tainting Characteristics for Study Populations from Population 4

Chapter 8

Study Results

To facilitate a comparison of the bounds, two performance criteria were examined. For the 500 samples from each study population, the average bound was calculated for each method. The average bounds provide a way of comparing the magnitude or efficiency of the 12 bounds. It is also necessary to examine the reliability of the various methods, since both efficiency and reliability play a role in determining the effectiveness of a particular method. Since the true mean taint per dollar unit was known for each study population, the number of times a particular bound actually covered the mean could be determined. The achieved confidence level, also referred to as the achieved coverage, was obtained by dividing this count by the number of samples from each study population, in this case 500. For a particular population a bound is considered reliable if its achieved confidence level is at least as large as the nominal confidence level which is 95% for this study.

The standard errors of the bounds were also calculated. This is simply the standard deviation of the 500 bounds for each study population. The standard errors provide a measure of the precision of the bounds.

8.1 Results for Simulated Data

8.1.1 General Results

Several factors concerning the performance of the various bounds were common to all the models and data types. All of the bounds increased with error rate and proportion of OS100 taints. In most cases an increase in the proportion of US taints caused the bounds to decrease. For those bounds that incorporate understatements without requiring a special adjustment, this decrease is due to the influence of the US taints. For the bounds which do not handle understatements, a decrease in the percentage of US taints translates to a decrease in the error rate since understatements are essentially treated as nonerrors. Thus, when the LTA adjustment is used, there are two factors decreasing the bound: the "drop" in error rate and the subtraction of the mean expected taint. For populations with a large number of US taints, the LTA adjusted bounds tend to overcompensate, being very tight but unreliable.

The standard errors of all the bounds increased with the error rate and the percentage of 100% overstatements. They also tended to be higher for the inventory populations than the accounts receivable populations; this seemed to be caused by the larger error rates rather than the larger percentage of understatements. No one method exhibited especially high or low standard errors. Thus, the discussion of the results below will pertain only to the coverages and average bounds.

8.1.2 Detailed Results for Accounts Receivable

Results on the performance of the bounds for the simulated accounts receivable populations are given in Tables A.1 to A.12 in Appendix A.

Model M1

The LTA adjusted parametric power (PP-lta) bound had the lowest average bound for 31 of the 36 accounts receivable populations. For each of these populations, the unadjusted parametric power (PP) bound had the second smallest average, if the two were not equal. Four of the 5 populations for which the PP-lta bound was not superior had no OS100 taints.

The percentage of OS100 taints had an interesting effect on the modified moment (MM) bound. As with all the bounds, the MM bound increased as the OS100 percentage increased. Compared to the other bounds, however, the MM bound was small for the populations with no OS100 taints and large for the populations with 40% OS100 taints. This is probably due to the effect of the hypothetical error tainting. As the percentage of OS100 taints increases, so too will the hypothetical tainting and this leads to an increase in the bound. This effect was seen in all the models and is more evident at lower error rates, where the hypothetical tainting has more influence.

In terms of coverage, the Stringer (ST) bound and the two adjusted Stringer bounds (ST-lta and ST-meik) consistently achieved the nominal 95% confidence level and often had 100% coverage. Among the other bounds, Clayton's (CL) bound had the best performance, failing to achieve 95% coverage for only 4 populations, each of which had a 10% error rate. The CL bound employs the bootstrap-t method whenever the percentage of sample errors is 10% or more, otherwise the modified Hoeffding bound is used. For many samples drawn from the populations with a lower error rate (3 or 6%) the CL bound will use the modified Hoeffding method. Clayton found the modified Hoeffding bound to be conservative, hence reliability levels were expected to be higher at lower error rates.

The PP and PP-lta bounds failed to achieve the desired confidence level for those populations with 20 or 40% of taints being OS100. The remaining bounds suffered reliability failures only for populations with 40% OS100 taints. For the Cox and Snell (CS) and LTA adjusted Cox and Snell (CS-lta) bounds, this took place at all error rates. For the MM, normal (BN), multinomial Dirichlet (MD) and adjusted multinomial Dirichlet (MD-lta) bounds, this occurred for error rates of 6 and 10% and for the CL bound this took place only at the 10% error rate. For those cases where the bounds did achieve the nominal 95% confidence level, coverages were often as high as 100%, particularly at lower error rates.

Though the PP bound and the corresponding PP-lta bound had the lowest averages, they had the most reliability problems. The PP-lta bound was unreliable for 12 of the populations in which it achieved the minimum average bound. The ST bounds (ST, ST-meik, ST-lta), which were extremely reliable, were also the most inefficient, consistently having the highest average bounds. This same pattern of mismatched reliability and efficiency was evident throughout the study.

As mentioned above, most of the problems with reliability occurred for populations with 40% of taints being OS100. The smallest reliable bound for the nine populations falling into this category varied with error rate. With a 3% error rate, the BN bound was the smallest bound to achieve reliability for this situation. At the 6% and 10% error rates, the smallest reliable bounds were the CL bound and the ST-lta bound respectively.

The MD bound tended to be smaller than the CL bound and was fairly reliable, failing to achieve the 95% confidence level for only 5 of the 36 populations. The BN bound, which failed to achieve the 95% confidence level for only 6 populations, was generally smaller than the MD bound. The MM bound had similar coverage levels to the BN bound but was slightly larger for most populations.

Model M2

The PP-lta bound achieved the minimum average bound for 17 populations and, again, the PP bound had the same or second smallest average in each case. Most of these populations had error rates of 6 or 10% and at least 10% OS100 taints.

For those populations with a 3% error rate, the BN bound had the lowest average for the populations with no US taints and otherwise the CS-lta bound was smallest on average. For those populations which don't fall into one of the categories just discussed, the average results were mixed, with the MM, BN, CS and MD bounds each possessing the lowest average bound for a few populations.

Neither the ST bounds nor the MD bound had any reliability problems for this model. Each of these 4 bounds repeatedly had achieved confidence levels above 95% and in many cases as high as 100%. As with model M1, reliability problems occurred mostly for populations with a large percentage of OS100 taints. The BN, CS-lta, CS, PP and PP-lta bounds each had failures for populations with 40% OS100 taints at all error rates. The PP-lta bound also failed for two populations with 20% OS100 taints. For those populations with 40% OS100 taints and error rates of 6 or 10%, the MM bound and MD-lta bound also failed to achieve the 95% confidence level and, similarly, the CL bound was unreliable for such populations with a 10% error rate.

For all nine populations with 40% OS100 taints, the smallest bound is unreliable. For these cases the smallest reliable bound seems to depend on the error rate. In the case of a 3% error rate, the CS bound had the lowest reliable average for populations with no US taints, otherwise the BN bound was the lowest reliable bound on average. For populations with a 6% error rate and 40% OS100 taints, the CL bound was the smallest on average and for similar populations with 10% errors the MD bound achieved the minimum reliable average.

For this model, it is difficult to pick out any one bound as having the best overall performance. The BN and CS-lta bounds are fairly reliable, having failed to achieve 95% coverage only 7 and 8 times respectively. Between these, the CS-lta bound tends to be smaller. The one exception to this is populations with no understatements. For this situation, the CS and the CS-lta bound are equivalent and larger than the BN bound. The MM bound is also fairly reliable but is generally higher than either the BN bound or the CS bound.

Model M3

For this model, it was the BN bound which achieved the lowest average for the largest number of populations. All 15 of these populations had a low proportion of understatements (0 or 10%). The CS-lta bound gave the lowest average bound for 8 of the populations, 7 of which had 20% US taints. The PP-lta bound had the lowest average bound for 10 populations, all with at least 20% OS100 taints and error rates of 6 or 10%. In each case where an LTA adjusted bound achieved the minimum average, the corresponding unadjusted bound had the next smallest average.

Again, the ST bounds were consistently reliable, with the smallest achieved confidence level being 97.6%. Like the ST bounds, the MD and CL bounds achieved perfect coverage for more than half of the populations studied and each failed to reach the 95% level on only two occasions. With one exception, all the populations for which one or more bounds failed to achieve the desired confidence level had either

20 or 40% OS100 taints. Again, this affected the BN bound, CS bounds and the PP bounds at all error rates; the MM and MD bounds for populations with 6 and 10% errors and the CL bound for the 10% error rate. The MM bound had only 6 failures, 4 of which were for populations with no US taints.

The lowest average bound was unreliable for 14 of the 36 populations studied. Again, it is difficult to pick out one bound as being superior overall. The MD bound and the CL bound have the best reliability, next to the larger ST bounds. Between these two, the MD is smaller for most populations on average. For the case of a low error rate (3%), the MM bound is consistently smaller than the MD bound and for this error rate the MM bound suffered no reliability problems. At other error rates, the MD bound is smaller and more reliable.

Model M4

The BN bound was very tight for this model. For 30 of the populations, the BN bound had either the lowest average or the second lowest average, after one of the LTA adjusted bounds. For the remaining six populations, all of which had a 10% error rate, the PP-lta bound gave the lowest average bound. The LTA adjusted bounds gave the best performance for populations with a high percentage of US taints.

The ST bounds were extremely reliable for this model, as with the other three. The CL bound was also completely reliable, with its lowest achieved confidence level being 97.6%. The BN, CS and PP bounds, as well as the two corresponding LTA adjusted bounds suffered severe reliability problems, each failing to reach the nominal confidence level for at least 21 populations. The three remaining bounds (MM, MD and MD-lta) were all fairly reliable, failing for no more than 5 populations. Once again, the MM bound reliability problems occurred for populations with few to no US taints and a low error rate (3 or 6%). Each population for which the MD bounds failed to achieve 95% coverage had an error rate of 10% and few to no US taints.

Though the BN bound gave the smallest averages for most populations, it was totally unreliable. Among the more reliable bounds, the MD bound was smaller than the MM bound for the most part and also less than the CL bound. The MD-lta bound was of course smaller than the MD bound but suffered more reliability problems for the 10% error rate. For those populations with a 3% error rate, the PP bounds were fairly reliable, failing to reach the 95% coverage level on only one occasion, and were lower then the MD bound.

8.1.3 Aggregate Results for Accounts Receivable

From the results discussed above, percentage OS100 taints had the greatest impact on reliability of the bounds. For this reason, the populations were divided into two groups, according to the OS100 proportion, before combining results. This separation method is similar to that employed by Grimlund and Felix [22]. Table 8.1 gives the aggregate results for populations with a low percentage of OS100 taints (0 and 10%). Table 8.2 gives the aggregate results for populations with a high percentage of OS100 taints (30 and 40%). Each table gives the average bound over the 18 populations in the category and in parenthesis the average coverage.

These results are presented graphically in Figures 8.1 through 8.4. Figures 8.1 and 8.2 display the average bounds and average coverage for the 18 accounts receivable populations with low percentages of OS100 taints from each of the four models. These plots demonstrate the characteristics of the bounds as well as information regarding the effect of the various models. The same information is displayed for those populations with high percentages of 100% overstatements in Figures 8.3 and 8.4.

Looking first at model M1, the PP bounds were the smallest reliable bounds for populations with a low percentage of OS100 taints (Table 8.1). These bounds also had the lowest average over populations with a high percentage of OS100 taints but were unreliable for this case (Table 8.2). This is supported by the detailed results in Tables A.1 and A.2 which were discussed in the previous section. The aggregate results point to the CS bound as the lowest reliable bound for populations with high levels of OS100 taints. In fact, the CS bound failed to provide desired reliability levels for just as many populations as the CS-lta bound.

For model M2, none of the bounds had average coverage below the nominal 95% level for populations with low levels of OS100 taints. For these populations, the CS-lta bound and the BN bound gave the lowest average bounds. For populations with a high percentage of OS100 taints, the power bounds provided the lowest averages but

Populations with low[1] percentage of OS100				
Bounds	M1	M2	M3	M4
ST	.0435[2]	.0509	.0582	.0758
	(1.000)[3]	(1.000)	(1.000)	(.998)
ST-meik	.0434	.0507	.0581	.0758
	(1.000)	(1.000)	(1.000)	(.998)
ST-lta	.0428	.0497	.0575	.0752
	(1.000)	(1.000)	(1.000)	(.997)
MM	.0324	.0364	.0438	.0650
	(1.000)	(.993)	(.987)	(.969)
BN	.0320	.0362	.0410	.0537
	(1.000)	(.998)	(.990)	(.937)
CS	.0307	.0363	.0420	.0556
	(1.000)	(.998)	(.993)	(.943)
CS-lta	.0301	.0351	.0413	.0549
	(.999)	(.994)	(.990)	(.936)
MD	.0331	.0386	.0443	.0615
	(1.000)	(.999)	(.996)	(.978)
MD-lta	.0324	.0373	.0437	.0609
	(1.000)	(.996)	(.995)	(.974)
PP	.0291	.0377	.0444	.0559
	(.997)	(.996)	(.991)	(.952)
PP-lta	.0284	.0365	.0438	.0553
	(.996)	(.991)	(.987)	(.944)
CL	.0381	.0422	.0484	.0641
	(.988)	(.993)	(.998)	(1.000)

[1] low OS100 percentage = 0 or 10%

[2] average bound over 18 populations

[3] average achieved coverage over 18 populations

Table 8.1: Aggregate Results for Accounts Receivable Populations with Low Percentages of OS100 Taints

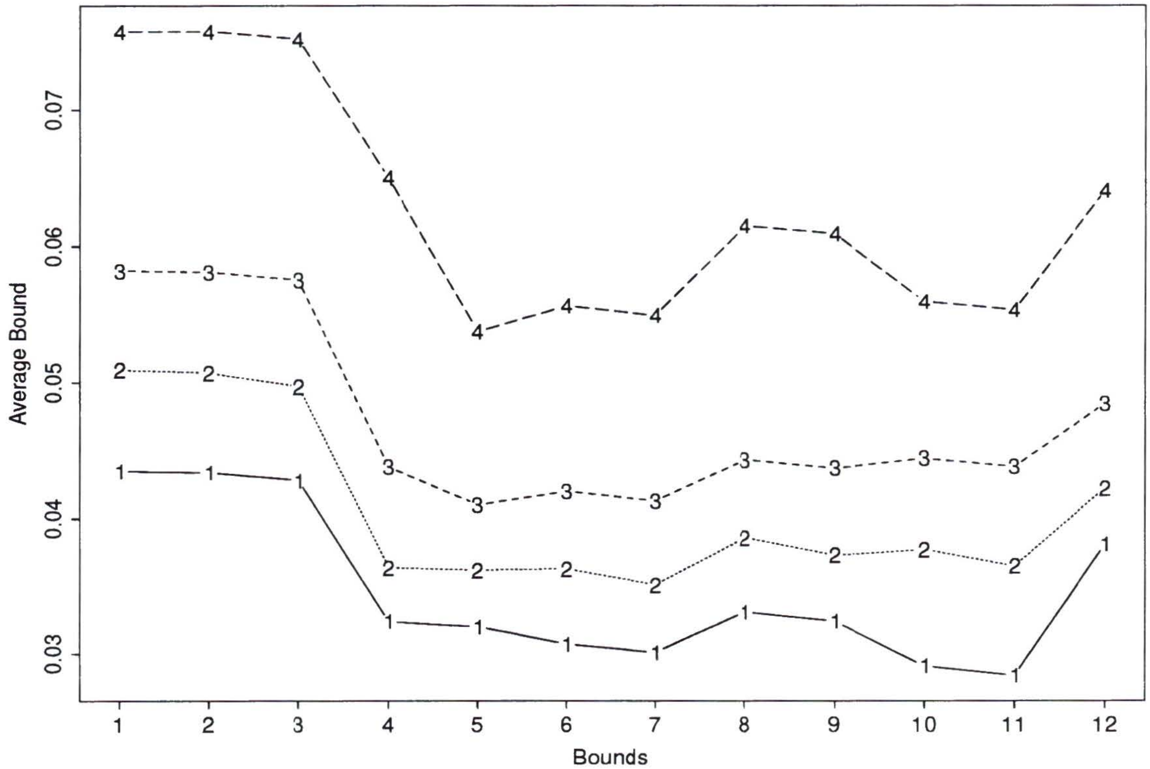
Populations with high[1] percentage of OS100				
Bounds	M1	M2	M3	M4
ST	.0653[2]	.0706	.0749	.0868
	(.998)[3]	(.998)	(.995)	(.993)
ST-meik	.0652	.0704	.0748	.0867
	(.998)	(.998)	(.995)	(.993)
ST-lta	.0647	.0693	.0742	.0862
	(.997)	(.997)	(.994)	(.992)
MM	.0559	.0598	.0648	.0791
	(.969)	(.964)	(.959)	(.961)
BN	.0483	.0511	.0537	.0621
	(.968)	(.961)	(.937)	(.891)
CS	.0468	.0510	.0545	.0640
	(.953)	(.957)	(.938)	(.907)
CS-lta	.0462	.0498	.0538	.0634
	(.946)	(.942)	(.930)	(.901)
MD	.0554	.0592	.0624	.0737
	(.975)	(.981)	(.975)	(.974)
MD-lta	.0547	.0579	.0618	.0731
	(.970)	(.975)	(.973)	(.971)
PP	.0387	.0475	.0537	.0646
	(.890)	(.934)	(.937)	(.921)
PP-lta	.0381	.0463	.0531	.0640
	(.878)	(.918)	(.930)	(.915)
CL	.0580	.0606	.0644	.0749
	(.985)	(.985)	(.984)	(.994)

[1] high OS100 percentage = 20 or 40%

[2] average bound over 18 populations

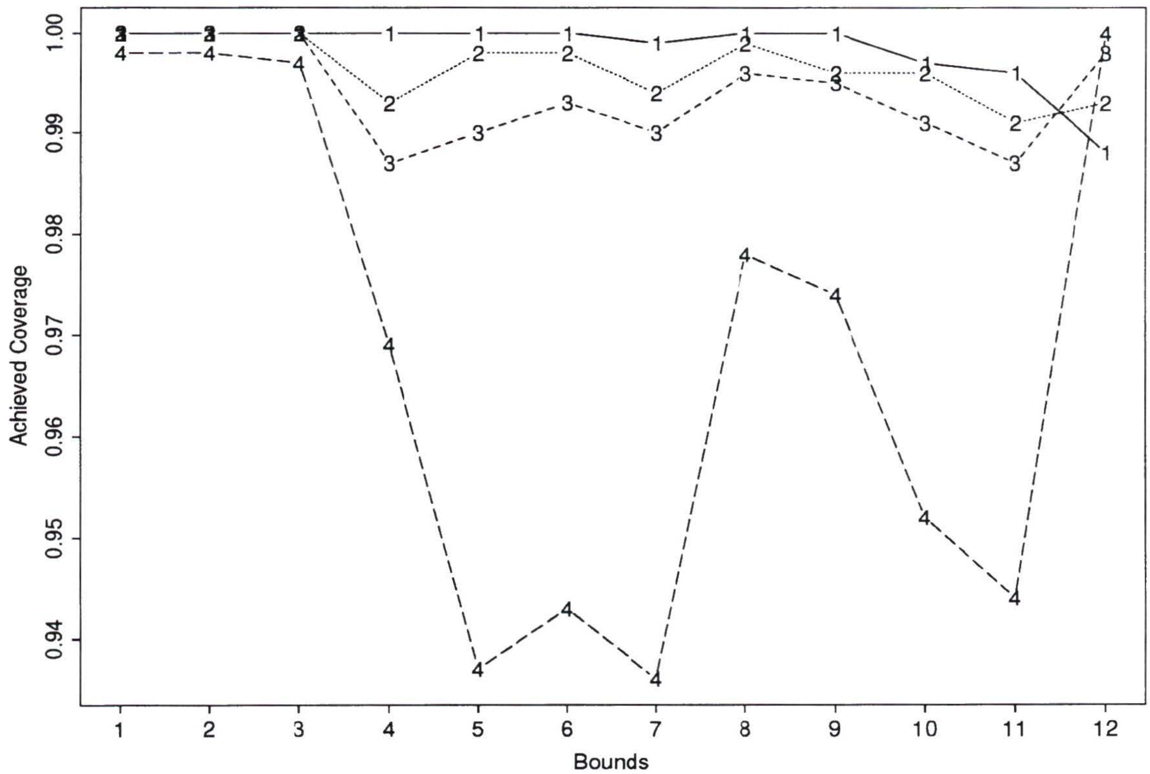
[3] average achieved coverage over 18 populations

Table 8.2: Aggregate Results for Accounts Receivable Populations with High Percentages of OS100 Taints



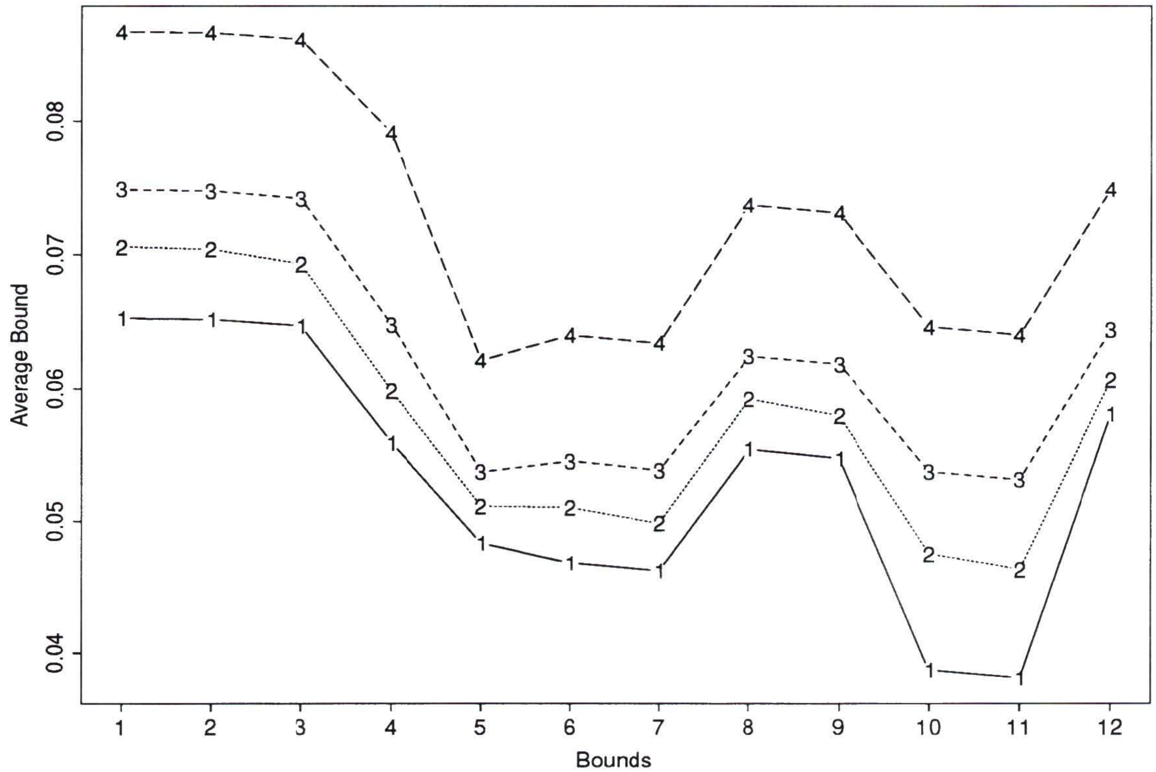
Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.1: Average Bounds for Accounts Receivable Populations With Low Percentages of OS100 Taints



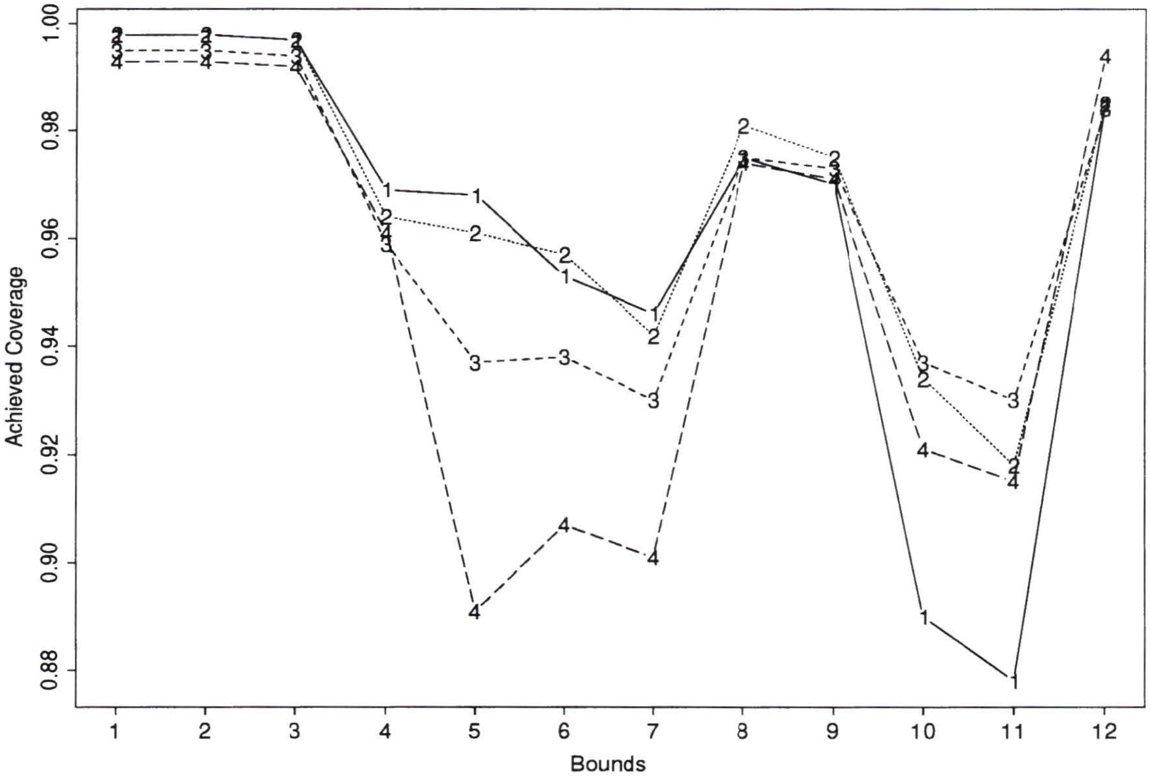
Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.2: Average Coverages for Accounts Receivable Populations With Low Percentages of OS100 Taints



Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.3: Average Bounds for Accounts Receivable Populations With High Percentages of OS100 Taints



Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.4: Average Coverages for Accounts Receivable Populations With High Percentages of OS100 Taints

also showed reliability problems. The BN and CS bounds demonstrated the lowest average reliable bounds over all such populations.

For populations from model M3 with low levels of OS100 taints, there are no reliability problems evident in the aggregate results. The BN bound had the lowest average over all such populations. For populations from this model which had a high percentage of OS100 taints, many of the bounds demonstrated reliability problems. The lowest reliable bound appears to be the MD-lta bound.

Several reliability problems are evident with model M4, even for populations with a low rate of OS100 taints. At the low rate, the BN, CS, CS-lta and PP-lta bounds were unreliable. The PP bound was the lowest reliable bound but this is misleading since the detailed results showed that the PP bound was not reliable for 8 of the 18 populations which fall into this category. For the populations which had a high percentage of OS100 taints, the PP bound was unreliable. The aggregate results for this set show the MD-lta bound to be the lowest reliable bound.

8.1.4 Detailed Results for Inventory

Results on the performance of the bounds for the simulated inventory data are given in Tables A.13 to A.24 in Appendix A.

Model M1

The PP-lta bound was the smallest on average for 23 of the 36 populations studied and the PP bound had the second lowest average for 19 of these populations. The PP and PP-lta bounds tended to be low compared to the other bounds for higher levels of OS100 taints. The one exception to this was populations with a high proportion of US taints (70%). The CL bound was low among populations with no OS100 taints, a situation which also yielded fairly low MM bounds.

The ST bounds were conservative, never failing to achieve the nominal 95% confidence level and often having 100% coverage. Among these, the ST-lta bound achieved coverage closest to the nominal confidence level.

With the exception of the ST-lta bound, bounds with the LTA adjustment were extremely unreliable, failing to have 95% coverage for many more populations than their unadjusted counterparts. For example, the MD bound failed in terms of coverage for only one population, whereas the MD-lta bound failed for 20 populations. This is obviously due to the large proportion of US taints for this set of populations. The MM and CL bounds also suffered major reliability problems, though the MM bound was reliable for populations with no OS100 taints.

The MD bound was very reliable; the BN and CS bounds slightly less so. The MD bound failed to reach the 95% coverage level for one population with 30% OS100 taints. The other two bounds also had problems with higher levels of OS100 taints. The power bound was reliable when the percentage of OS100 taints was low (0 or

10%). For populations with higher error rates and a high percentage of OS100 taints, the PP bound was extremely unreliable, having coverages as low as 39.6%.

For 29 of the 36 populations, the smallest average bound was unreliable. The BN bound and the CS bound were fairly equivalent in terms of reliability. Between these, the BN bound tended to be smaller for high percentages of US taints (50 or 70%) and high error rates (30 or 60%), otherwise the CS bound was usually lower. The MD bound was slightly more reliable than either of these but was also slightly larger.

Model M2

For this model, the PP-lta bound was again the smallest for the largest number of populations; it achieved the smallest average for 20 populations. The PP bound yielded the second smallest average for only 10 of these populations. This demonstrates the effect of the LTA adjustment when the proportion of US taints is large, as with inventory data. The CS-lta and MD-lta bounds combined achieved the smallest average for 13 populations but their unadjusted counterparts had the second smallest average only once. The MM bound was fairly efficient, often having the second lowest average after one of the adjusted bounds.

As is expected, the ST bounds were extremely reliable for this set of populations. The ST-lta bound failed once with an achieved confidence level of 94.4%. Aside from the ST-lta bound, the LTA adjusted bounds were totally unreliable.

The PP bound was much more reliable for model M2 than model M1; failing to achieve the nominal confidence level for only three populations. The BN, CS and MD bounds were also very reliable, with 2, 2 and 0 failures respectively. These failures all occurred for populations with 30% OS100 taints.

There were only 5 populations from this model for which the lowest average bound was reliable. Four of these populations had an error rate of 10%. The CS, BN, MD and PP bounds were the only ones with acceptable reliability levels. The MD bound, which was completely reliable, tended to be the highest on average among these. The PP bound was generally the lowest for populations with a 10% error rate and no more than 50% US taints; otherwise, the BN bound was the most efficient. Though the MM bound did not achieve 95% reliability for many populations, it was reliable whenever a population had no OS100 taints and for such populations, the MM bound was the smallest average bound.

Model M3

For 33 of the 36 populations from this model, one of the three LTA adjusted bounds, excluding the ST-lta bound, yielded the lowest average bound. In 19 of these cases, the corresponding unadjusted bound yielded the second lowest average. The MM bound was fairly efficient, particularly for populations with no OS100 taints.

The ST bounds were all reliable, often yielding coverages of 100% for populations with no OS100 taints. The only other reliable bounds were the BN, CS, MD and PP bounds. The MD bound failed to have 95% coverage for 4 populations, each of which had 20% US taints. The CS and BN bounds failed 5 and 4 times respectively, in each case the percentage of OS100 taints was high (20 or 30%). The PP bound failed to reach the nominal confidence level for 7 populations, all of which had no more than 50% US taints and at least 20% OS100 taints.

Five populations from this model, having error rates of 10 and 30% yielded smallest average bounds which were reliable. The BN bound had the fewest reliability problems and was generally smaller than the MD bound. The CS bound tended to be smaller than the BN bound for populations with a 10% error rate but suffered reliability problems for such populations.

Model M4

This model is not likely to be encountered for inventory data and was included to judge the performance of the bounds in an extreme environment. For every population studied from this model, it was an LTA adjusted bound, though not the ST-lta bound, which gave the lowest average bound. For more than a third of the populations, however, it was not the corresponding unadjusted bound which had the second lowest average.

The power bound was efficient relative to the other bounds for populations which had 20 or 30% OS100 taints, except when this was combined with an US percentage of 70%. For populations with this high percentage of US taints the MM and CS bounds tended to be more efficient.

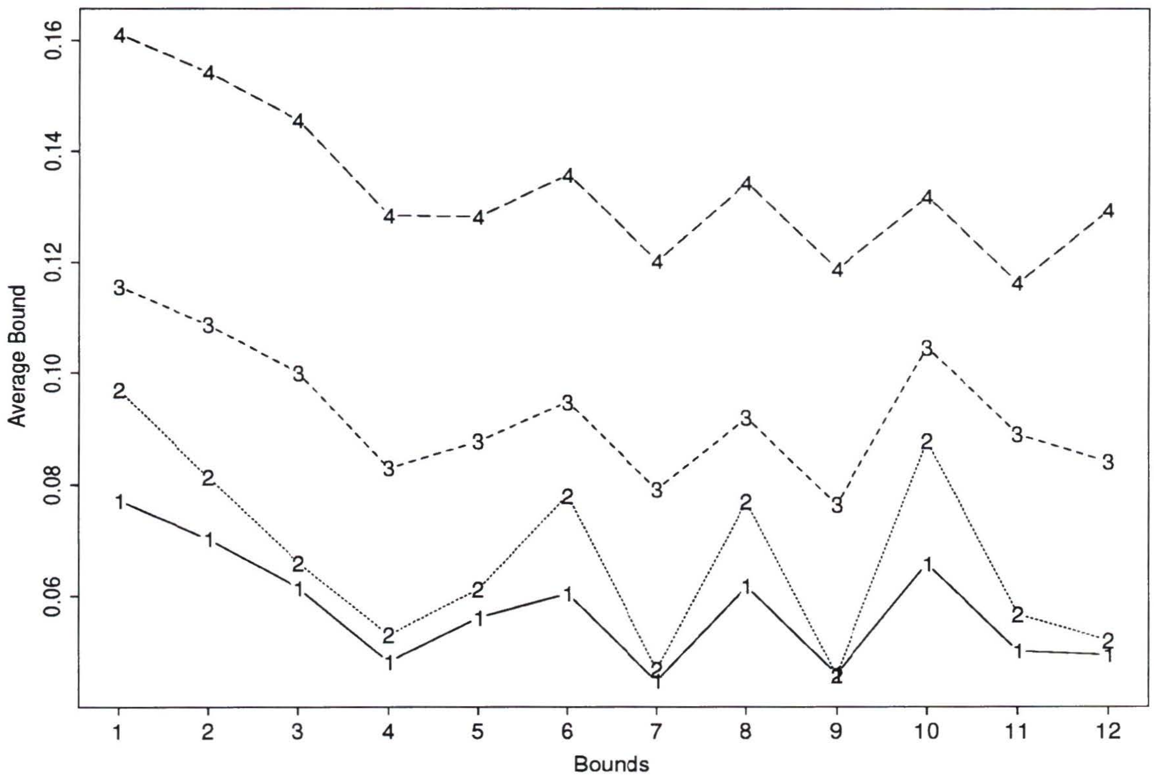
The CL bound was quite large except for populations with no OS taints, i.e. 70% US taints and 30% OS100 taints.

All the bounds had reliability problems except the ST bounds which were completely reliable. The MD bound had the lowest number of failures, only 7, and the BN and CS bounds, which resulted in 10 and 9 failures, were the next most reliable overall. Most of the bounds were reliable in particular situations. For example, the MM bound was reliable for populations with only 20% US taints; the CS bound was reliable for error rates greater than 10%; the BN bound reliable for populations with 50 or 70% US taints; the MD bound was reliable except for populations with error rates of 30 or 60% and 20% US taints.

No one bound was reliable and efficient for a large number of populations. Only the BN, CS and MD bounds were reliable enough to be used over the variety of populations studied for this model. For populations with an error rate higher than 10% the CS bound had the best performance, being relatively small and reliable. For all error rates, the BN and MD bounds were fairly reliable with the BN bound being slightly smaller and slightly less reliable.

8.1.5 Aggregate Results for Inventory

As with the accounts receivable populations, the inventory populations were divided into two groups according to the percentage of OS100 taints before the results were aggregated. Populations with 0 or 10% OS100 taints were classified as having a low



Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.5: Average Bounds for Inventory Populations With Low Percentages of OS100 Taints

level of OS100 taints. The remaining populations were classified as having a high percentage of OS100 taints (20 or 30%). Thus, 18 populations fell into each category. The results are shown numerically in Table 8.3 and Table 8.4 and graphically in Figures 8.5 through 8.8

The aggregate results suggest that for populations from model M1 with a low percentage of OS100 taints, the MM bound was the most efficient reliable bound

Populations with low[1] percentage of OS100				
Bounds	M1	M2	M3	M4
ST	.0772[2]	.0970	.1156	.1611
	(1.000)[3]	(1.000)	(1.000)	(.998)
ST-meik	.0703	.0814	.1087	.1542
	(.999)	(.999)	(.999)	(.996)
ST-lta	.0615	.0660	.1000	.1455
	(.992)	(.986)	(.993)	(.988)
MM	.0484	.0532	.0830	.1284
	(.974)	(.966)	(.962)	(.955)
BN	.0562	.0613	.0878	.1282
	(.993)	(.991)	(.986)	(.972)
CS	.0605	.0781	.0948	.1358
	(.997)	(.999)	(.993)	(.975)
CS-lta	.0449	.0470	.0792	.1202
	(.962)	(.934)	(.948)	(.920)
MD	.0619	.0771	.0921	.1344
	(.997)	(.998)	(.988)	(.978)
MD-lta	.0462	.0460	.0765	.1188
	(.965)	(.928)	(.932)	(.926)
PP	.0659	.0879	.1047	.1319
	(.999)	(.999)	(.996)	(.974)
PP-lta	.0503	.0569	.0891	.1163
	(.946)	(.946)	(.964)	(.907)
CL	.0496	.0522	.0842	.1294
	(.936)	(.940)	(.950)	(.953)

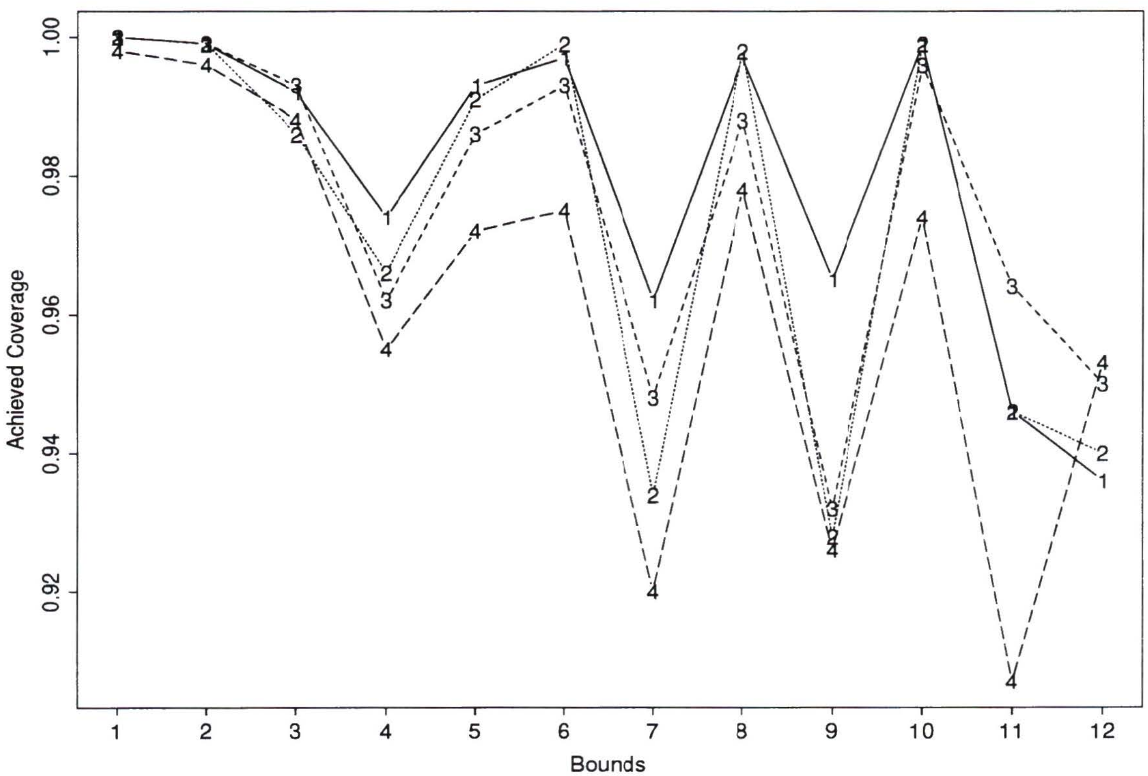
[1] low OS100 percentage = 0 or 10%
[2] average bound over 18 populations
[3] average achieved coverage over 18 populations

Table 8.3: Aggregate Results for Inventory Populations with Low Percentages of OS100 Taints

Populations with high[1] percentage of OS100				
Bounds	M1	M2	M3	M4
ST	.1573[2]	.1683	.1783	.2033
	(.996)[3]	(.999)	(.997)	(.996)
ST-meik	.1504	.1527	.1714	.1964
	(.991)	(.997)	(.994)	(.993)
ST-lta	.1418	.1373	.1628	.1877
	(.976)	(.975)	(.983)	(.981)
MM	.1285	.1250	.1477	.1726
	(.937)	(.942)	(.947)	(.951)
BN	.1287	.1260	.1457	.1675
	(.963)	(.971)	(.963)	(.955)
CS	.1307	.1411	.1506	.1745
	(.961)	(.983)	(.963)	(.956)
CS-lta	.1151	.1101	.1351	.1589
	(.885)	(.877)	(.893)	(.888)
MD	.1398	.1478	.1555	.1780
	(.978)	(.990)	(.975)	(.971)
MD-lta	.1243	.1168	.1400	.1624
	(.933)	(.923)	(.930)	(.927)
PP	.1091	.1301	.1448	.1676
	(.848)	(.975)	(.956)	(.937)
PP-lta	.0936	.0990	.1293	.1520
	(.690)	(.824)	(.873)	(.857)
CL	.1340	.1275	.1504	.1732
	(.959)	(.950)	(.961)	(.956)

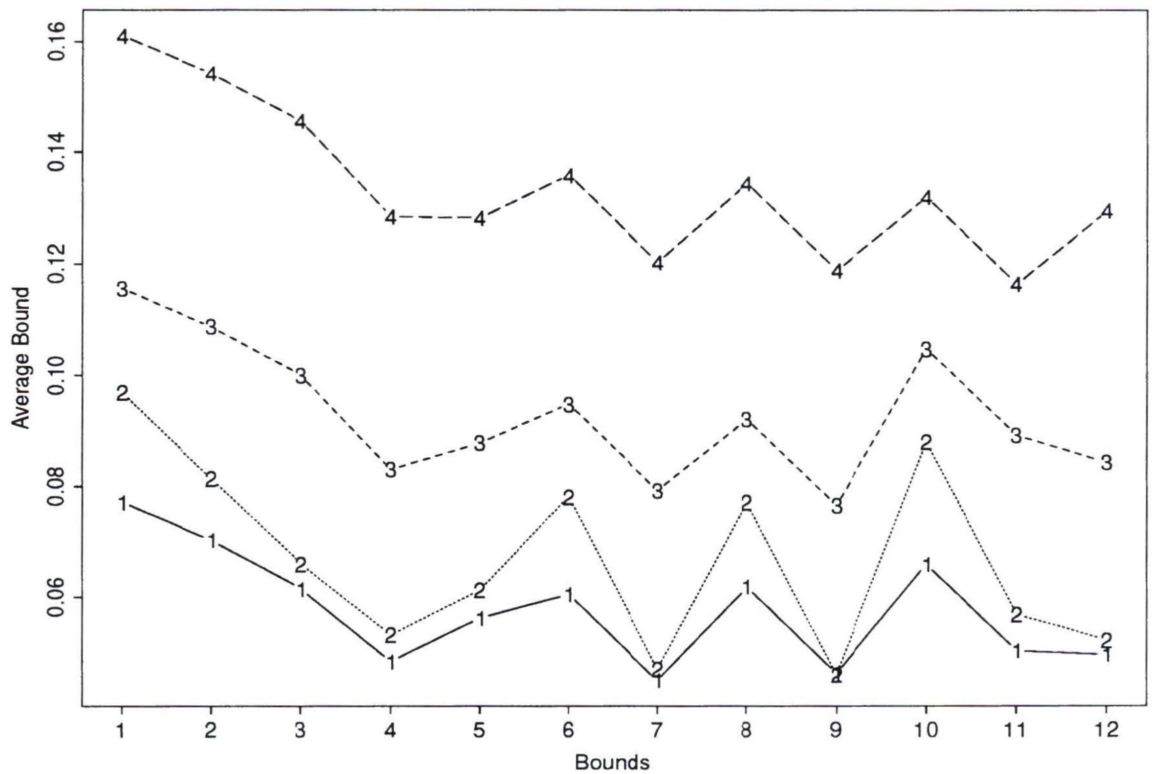
[1] high OS100 percentage = 20 or 30%
[2] average bound over 18 populations
[3] average achieved coverage over 18 populations

Table 8.4: Aggregate Results for Inventory Populations with High Percentages of OS100 Taints



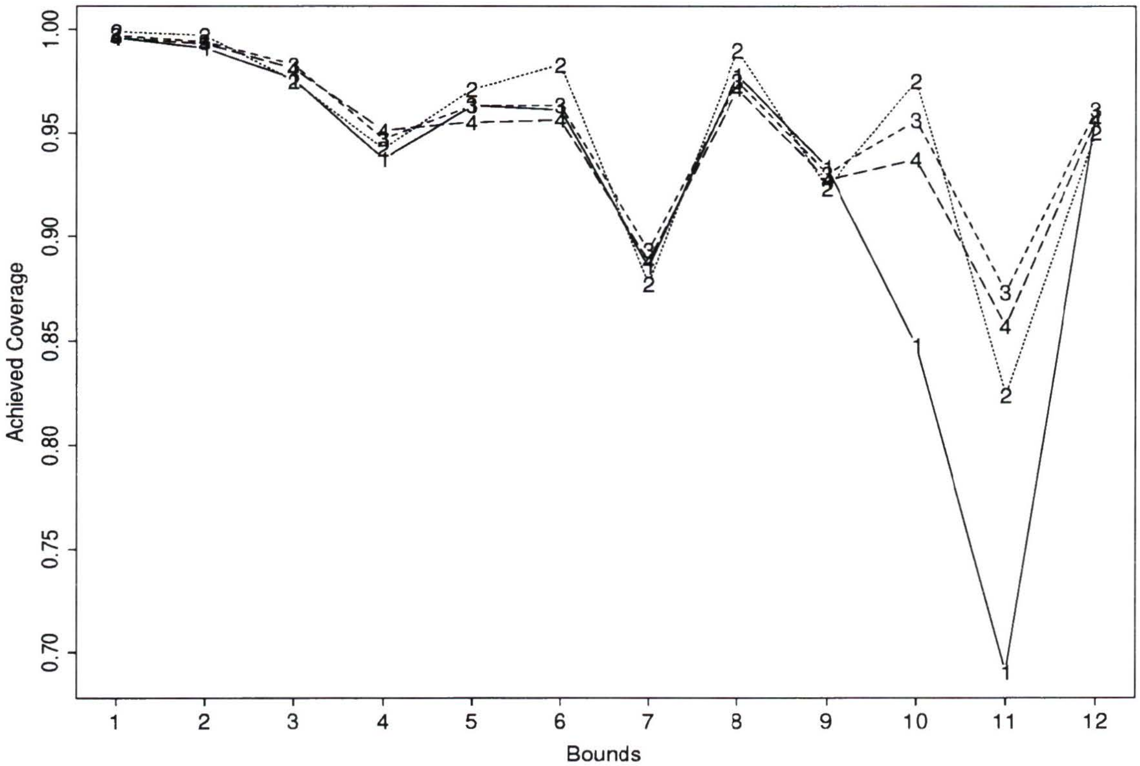
Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.6: Average Coverages for Inventory Populations With Low Percentages of OS100 Taints



Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.7: Average Bounds for Inventory Populations With High Percentages of OS100 Taints



Key for bounds: 1 = ST, 2 = ST-meik, 3 = ST-lta, 4 = MM, 5 = BN, 6 = CS, 7 = CS-lta, 8 = MD, 9 = MD-lta, 10 = PP, 11 = PP-lta, 12 = CL
Key for models: 1 = M1, 2 = M2, 3 = M3, 4 = M4

Figure 8.8: Average Coverages for Inventory Populations With High Percentages of OS100 Taints

(Table 8.3). However, according to the detailed results in Table A.13 which were discussed in the previous section, this bound failed to reach the nominal confidence level for approximately a third of the 18 populations. The aggregate results do accurately demonstrate the unreliability of the PP-lta bound and the CL bound for such populations. The LTA adjusted bounds, except for the ST-lta bound, also yielded low average bounds. For populations from model M1 with a high percentage of OS100 taints, the PP bounds yielded the lowest averages but were unreliable. The MM, CS-lta and MD-lta bounds also had reliability problems for this group of populations. The best bounds, in terms of reliability and efficiency, seemed to be the MD bound and the BN bound.

The aggregate results for model M2 are similar to those for model M1. At low percentages of OS100 taints, the MM bound was the smallest reliable bound according to the aggregate results. Again, this is misleading. Each of the LTA adjusted bounds, except for the ST-lta bound, were efficient but unreliable, as was the CL bound. The same bounds were unreliable at the higher levels of OS100 taints, as was the MM bound. The smallest reliable bound was the ST-lta bound according to these results.

For model M3, the MM bound is again the smallest reliable bound for populations with low percentages of OS100 taints. As with M1 and M2, however, this is slightly misleading since the MM bound failed to reach the desired confidence level for 6 of the 18 populations. For these populations, the MD-lta and adjusted CS-lta bounds were unreliable. At higher levels of OS100 taints these two bounds, as well as the

MM and MD-lta bounds were unreliable. The PP and BN bounds appear to be the most efficient reliable bounds. The BN bound is slightly larger on average but also slightly more reliable.

The aggregate results for model M4 suggest that the BN bound was the smallest reliable bound for both categories. Among populations with low percentages of OS100 taints, the LTA adjusted bounds, aside from the ST-lta bound, were the smallest but unreliable. These bounds were also small but unreliable for higher levels of OS100 taints. The PP bound was also unreliable at the higher levels.

8.1.6 Comparison of Models

For both the inventory and accounts receivable populations, the bounds increased from model to model, i.e. in terms of average bound M1 < M2 < M3 < M4. This pattern is readily apparent in the plots for the aggregated results, Figures 8.1, 8.3, 8.5 and 8.7. Some exceptions to this occur for bounds which utilize understatements. For example, for inventory populations with a high percentage of US taints, the BN and MM bounds are large for model M1 than model M2. This makes sense because the absolute mean US taint is larger for M2 than M1, and at high US percentages this has a greater effect than the increasing OS mean.

The BN bound had the best reliability levels for those models which were fairly symmetric in terms of overstatements and understatements. For model M4, which a uniform distribution for OS taints, the BN bound suffered severe reliability problems.

This bound assumes the taints are generated by a normal distribution. Model M4 violates that assumption to the greatest degree.

The PP bound was most reliable for populations from models M2 and M3. The CL bound was most reliable for the M4 model, although it failed to achieve the 95% confidence level 11 times for the inventory populations from this model. This does not support the use of this bound since model M4 is viewed as extreme, in that a distribution of this type appears infrequently in auditing.

8.2 Results for Real Data

Results on the performance of the bounds for the Neter and Loebbecke [33] populations are given in Tables A.25 and A.26 in Appendix A. The CS and CS-lta bounds were the smallest bounds to achieve the 95% confidence level for every study population with an error rate of .5 or 1%. For populations with 5 or 10% errors, the PP and PP-lta bounds tended to be the smallest. Population 1 with a 10% error rate is the only exception to this; the CL bound achieved the minimum average for that population. For populations with error rates of 30 and 70%, the method which achieves the minimum reliable average varies. For populations 1 and 1M, the MD-lta bound is the smallest reliable bound for the case of 30% errors. At the highest error rates in populations 2 and 4, the BN bound achieves the minimum reliable average and the CS bound achieves the minimum average for the highest eror rate of population 3.

The greatest number of coverage failures occurred for population 2 with a 70% error rate. Three of the four failures for this population occurred for LTA adjusted bounds (ST-lta, CS-lta and MD-lta). The PP-lta bound is the only adjusted bound which does not fail to reach the 95% confidence level in this case. This is due to the conservatism of the PP bound. In fact, the PP bound is larger than the ST bound for three populations with the highest error rate (30 or 70%). For the same populations the CL bound is extremely low and unreliable. At such high error rates, the CL bound would tend to use the bootstrap-t method rather than the conservative Hoeffding method. This effect does not occur for population 4 with a 30% error rate. This population has a high percentage of OS100 taints (59%). This situation of a large portion of OS100 taints causes the PP bound to decrease and the CL bound to increase with respect to the other bounds.

Chapter 9

Conclusion and Discussion

This report began with a survey of available statistical methods for finding an upper bound for the error in an audit. Those methods which had the greatest potential for use by auditors were included in an extensive simulation study. Prior to this empirical analysis, several shortfalls of the various methods were noted. Below is a summary of these weaknesses.

Modified Moment Bound

- Computation of z^* is ad hoc.
- Conservative for samples with no nonzero taints.

Bayesian Normal Bound

- Certain priors, e.g. that used in this study, do not allow calculation of the bound for samples with no nonzero taints.

- Assumes the distribution of the taints is bell-shaped.

Cox and Snell Bound

- Cannot handle understatements.
- Assumes the probability of a dollar-unit being in error (π) and the mean error per dollar-unit (μ_z) are independent.
- Assumes nonzero taints are generated according to the exponential distribution which allows for large values.

Multinomial-Dirichlet Bound

- Cannot handle understatements.
- Assumes taints are no greater than 1.

Parametric Power Bound

- Cannot handle understatements.
- Assumes taints are no greater than 1.
- Employs the conservative attributes bound for samples with no nonzero taints and samples with only 100% taints.

Clayton's Combined Bound

- Hoeffding bound cannot handle understatements.

- Hoeffding bound assumes taints are no greater than 1.

- Employs the conservative attributes bound for samples with no nonzero taints.

Though methods which cannot handle understatements can be employed with the LTA adjustment for understatements, as in this study, reliability problems could result. The LTA adjustment has no reliability level associated with it but will necessarily decrease the reliability level of any bound it is applied to.

As an alternative to the LTA adjustment, an auditor could compute separate bounds for overstatements and understatements. This involves viewing a sample as two separate samples, one containing only understatements and one containing only overstatements. The method being used is employed with the overstatements and the absolute understatements, to form a two-sided confidence interval for the total error. The auditor bases his conclusions concerning the accuracy of the book total on this two-sided bound.

This approach can also be used to form a one-sided bound by assuming the direction of net error. For example, if it is believed that the net error of a population is positive, an upper bound for the combined error can be found by subtracting the lower bound for absolute understatements from the upper bound for overstatements.

The first of these two approaches provides the greatest amount of information, giving complete consideration to both understatements and overstatements. Though both the total overstatement and understatement error may be immaterial, fairly large bounds may indicate problems with the internal controls of a company. By netting the bounds, this information is lost.

Dollar-unit sampling has become the preferred method for auditors but this sampling technique may not be suitable for populations with understatements. Items with a zero book value are never sampled under dollar-unit sampling and those items with larger audit amounts may have lower chances of selection than desired. Thus, the problem of finding suitable bounds for error in populations with understatements may require new sampling procedures.

The majority of research to date has dealt with the problem of finding upper bounds on audit error. Aside from the classical and CAV methods, only the modified moment bound appears to have been examined using two-sided bounds. Chan and Smieliauskas [9] found the two-sided bound to be unreliable for many populations in which the one-sided upper bound was reliable. This suggests that modifications may be required in this and other methods before suitable two-sided bounds are available.

There is also a notable lack of research dealing with lower one-sided bounds for net error. Lower bounds play an extremely important role in certain auditing situations. For example, if an auditor concludes that a reported balance is significantly overstated, an adjustment is required. One possibility is to adjust the balance downwards by the lower error bound [42]. Hence, reliable methods for finding lower bounds are also required.

This study did not reveal one superior method for finding the upper bound for audit error. Based on the populations included in this study, a tentative guide to choosing the appropriate method was formulated. The choice of method depends on the error rate, the proportion of 100% overstatements and the proportion of understatements. This summary is presented in Table 9.1.

		Error Rate			
		low[1]		high	
		%US		%US	
		low[2]	high	low	high
%OS100	low[3]	BN	PP	MD/CS	BN/ST-lta
	high	ST-lta	BN	ST-lta	BN

[1] low error rate: $\leq 10\%$
[2] low percentage of understatements: $\leq 20\%$
[3] low percentage of 100% overstatements: $\leq 10\%$

Table 9.1: A Guide for Choosing the Appropriate Bound

If an auditor is unsure as to the error rate and distribution of taints, i.e. the proportion of understatements and 100% overstatements, the multinomial-Dirichlet method should be employed. This bound does tend to be larger than those provided by other methods but, aside from the Stringer bounds, it demonstrates the best reliability for a variety of populations.

This study involved both random sampling and systematic sampling of dollar-units. Many auditor's advocate the use of cell sampling. Results on the performance of the bounds may differ according to the type of sampling employed. It would be interesting to extend this study in order to determine if cell sampling could offer further improvements for one or more of the bounds.

In most cases, auditors continue to rely on the conservative Stringer bound with either Meikle's adjustment for understatements or the LTA adjustment for understatements [23]. The firm Arthur Andersen has adopted the modified moment method for finding bounds on total audit error [17]. No evidence could be found of any firm using one of the Bayesian methods in practice [7].

There are still many open problems in the application of statistical methods to auditing. The lack of real audit data available to researchers significantly hinders the process of developing new techniques. If accounting firms hope for improved methods they must ensure that data with which to analyze such methods is available. Additional studies of real data may also serve to provide evidence concerning the occurrence of taints greater than one and any relationship between error rate and error magnitude.

Bibliography

[1] Anderson, R. J., Teitlebaum, A. D. (1973). Dollar-unit sampling. *Canadian Chartered Accountant*, April: 30-39.

[2] Arens, A. A., Loebbecke, J. K., Lemon, W. M. (1984). *Auditing: an integrated approach*. Canada: Prentice-Hall.

[3] Baker, R. L., Copeland, R. M. (1979). Evaluation of the stratified regression estimator for auditing accounting populations. *Journal of Accounting Research*, Vol 17: 606-617.

[4] Beck, P. J. (1980). A critical analysis of the regression estimator in audit sampling. *Journal of Accounting Research*, Vol 18: 16-37.

[5] Bickel, P. J. (1992). Inference and auditing: the Stringer bound. *International Statistical Review*, Vol 60: 197-209.

[6] Biddle, G. C., Bruton, C. M., Siegel, A. F. (1990). Computer-intensive methods in auditing: bootstrap difference and ratio estimation. *Auditing: A Journal of Practice and Theory*, Vol 9, No 3: 92-114.

[7] Byekwaso, S., Flournoy, N., Johnson, J. (1996). Bayesian methods in Auditing. *Bayesian Analysis in Statistics and Econometrics*. New York: John Wiley & Sons.

[8] Carman, L. A. (1933). The efficacy of tests. *American Accountant*, Vol 18: 360-366.

[9] Chan, H., Smieliauskas, W., (1990). Further tests of the modified moment bound in audit sampling of accounting populations. *Auditing: A Journal of Practice and Theory*, Vol 9, No 3: 167-182.

[10] Clayton, H. R. (1994). A combined bound for errors in auditing based on Hoeffding's inequality and the bootstrap. *Journal of Business and Economic Statistics*, Vol 12, No 4: 437-448.

[11] Cochran, W. G. (1963). *Sampling Techniques.* New York: John Wiley & Sons.

[12] Cox, D. R., Snell, E. J. (1979). On sampling and the estimation of rare errors. *Biometrika*, Vol 66, No 1: 125-132.

[13] Dalenius, T., Hodges, J. L. (1959). Minimum variance stratification. *Journal of the American Statistical Association*, Vol 54: 88-101.

[14] Dworin, L., Grimlund, R. A. (1984). Dollar unit sampling for accounts receivable and inventory. *The Accounting Review*, Vol 59, No 2: 218-241.

[15] Dworin, L., Grimlund, R. A. (1986).Dollar-unit sampling: a comparison of the quasi-Bayesian and moment bounds, *The Accounting Review.* Vol 61, No 1: 36-57.

[16] Felix, W. L., Grimlund, R. A. (1977). A sampling model for audit tests of composite accounts. *Journal of Accounting Research*, Vol 15: 23-42.

[17] Felix, W. L., Grimlund, R. A. (1990). Arthur Andersen's new monetary unit sampling approach. *Auditing: A Journal of Practice and Theory*, Vol 9, No 3: 1-16.

[18] Fienberg, S. E., Neter, J., Leitch, R. A. (1977). Estimating the total overstatement error in accounting populations. *Journal of the American Statistical Association*, Vol 72, No 358: 295-302.

[19] Frost, P. A., Tamura, H. (1982). Jackknifed ratio estimation in statistical auditing. *Journal of Accounting Research*, Vol 20: 103-120.

[20] Godfrey, J., Neter, J. (1984). Bayesian bounds for monetary unit sampling in accounting and auditing. *Journal of Accounting Research*, Vol 22, No 2: 497-525.

[21] Goodfellow, J. L., Loebbecke, J. K., Neter, J. (1974). Some perspectives on CAV sampling plans. Part I. *CA Magazine* October: 23-30; Part II. *CA Magazine* November: 46-53.

[22] Grimlund, R. A., Felix, W. L. (1987). Simulation evidence and analysis of alternative methods of evaluating dollar-unit samples. *The Accounting Review*, Vol 62, No 3: 455-479.

[23] Grimlund, R. A., Schroeder, M. S. (1988). On the current use of the Stringer method for MUS: some new directions. *Auditing: A Journal of Practice and Theory*, Vol 8, No 1: 53-62.

[24] Guthrie, D., Birnbaum, Z. W., Dixon, W. J., Fienberg, S. E., Gentleman, J. F., Landwehr, J. M., Mann, N. R., Mielke, P. W., Neter, J., Roberts, D. M., Van Ryzin, J., Tamura, H., Wilburn, A. J., Williams, J. S (1989). Statistical models and analysis in auditing. *Statistical Science*, Vol 4, No 1: 2-33.

[25] Ham, J., Losell, D., Smieliauskas, W. (1985). An empirical study of error characteristics in accounting populations. *The Accounting Review*, Vol 60, No 3: 387-406.

[26] Johnson, J. R., Leitch, R. A., Neter, J. (1981). Characteristics of errors in accounts receivable and inventory audits. *The Accounting Review*, Vol 56, No 2: 270-293.

[27] Kaplan, R. S (1973). Statistical sampling in auditing with auxiliary information estimators. *The Journal of Accounting Research*, Vol 11: 238-258.

[28] Leitch, R. A., Neter, J., Plante, R., Sinha, P. (1982). Modified multinomial bounds for larger numbers of errors in audits. *The Accounting Review*, Vol 57, No 2: 384-400.

[29] Leslie, D. A., Teitlebaum, A. D., Anderson, R. J. (1979). *Dollar-unit sampling: a practical guide for auditors*. Toronto: Copp Clark Pitman.

[30] Meikle, G. R. (1972). *Statistical Sampling in an Audit Context*. Toronto: Canadian Institute of Chartered Accountants.

[31] Menzefricke, U., Smieliauskas, W. (1984). A simulation study of the performance of parametric dollar unit sampling statistical procedures. *Journal of Accounting Research*, Vol 22, No 2: 588-604.

[32] Neter, J., Godfrey, J. (1985). Robust Bayesian bounds for monetary-unit sampling in auditing. *Applied Statistics*, Vol 34, No 2: 157-168.

[33] Neter, J., Loebbecke, J. K. (1975). *Behaviour of major statistical estimators in sampling accounting populations - an empirical study*. New York: American Institute of Certified Public Accountants.

[34] Neter, J., Loebbecke, J. K. (1977). On the behaviour of statistical estimators when sampling accounting populations. *Journal of the American Statistical Association*, Vol 72, No 359: 501-507.

[35] Neter, J., Leitch, R. A., Fienberg, S. E. (1978) Dollar unit sampling: multinomial bounds for overstatement and understatement errors. *The Accounting Review*, Vol 53, No 1: 77-93.

[36] Neter, J., Johnson, J. R., Leitch, R. A. (1985). Characteristics of dollar-unit taints and error rates in accounts receivable and inventory. *The Accounting Review*, Vol 60, No 3: 488-499.

[37] Neter, J. (1986). Boundaries of statistics - sharp or fuzzy? *Journal of the American Statistical Association*, Vol 81, No 393: 1-8.

[38] Odeh, R. E., Owen, D. B., Birnbaum, Z. W., Fisher, L. (1977). *Pocket book of statistical tables*. New York: Marcel Dekker.

[39] Pachares, J. (1960). Tables of confidence limits for the binomial distribution. *Journal of the American Statistical Association*, Vol 55: 521-33.

[40] Pearson, K. (1894). Contribution to the mathematical theory of evolution. *Philosophical Transactions. Royal Society of London. Series A.*, Vol 185: 71-110.

[41] Plante, R., Neter, J., Leitch, R. A. (1985). Comparative performance of multinomial, cell and Stringer bounds. *Auditing: A Journal of Practice and Thoery*. Vol 5, No 1: 40-56.

[42] Plante, R., Neter, J., Leitch, R. A. (1984). A lower multinomial bound for the total overstatement errors in accounting populations. *Management Science*, Vol 30, No 1: 37-50.

[43] Prytherch, R. H. (1942). How much test checking is enough? *The Journal of Accountancy*.

[44] Ramage, J. G., Frieger, A. M., Spero, L. L. (1979). An empirical study of error characteristics in audit populations. *Journal of Accounting Research*, Vol 17 (suppl.): 72-102

[45] Reneau, J. H. (1978). CAV Bounds in Dollar Unit Sampling: Some Simulation Results. *The Accounting Review*, Vol 53, No 3: 669-680.

[46] Robbins, H., Pitman, E. J. (1949). Application of the method of mixtures to quadratic forms in normal variates. *Annals of Mathematical Statistics*, Vol 20: 552-560.

[47] Smieliauskas, W. (1986). A note on comparison of Bayesian with non-Bayesian dollar-unit sampling bounds for overstatement errors of accounting populations. *The Accounting Review*, Vol 61, No 1: 118-128.

[48] Stringer, K. W. (1979). Statistical sampling in auditing - the state of the art. *Annual Accounting Review*, Vol 1: 113-127.

[49] Tamura, H., Frost, P. A. (1986). Tightening CAV (DUS) bounds by using a parametric model. *Journal of Accounting Research*, Vol 24, No 2: 364-371.

[50] Trueblood, R., Cyert, R. (1957). *Sampling techniques in accounting.* New Jersey: Prentice-Hall.

[51] Tsui, K. W., Matsamura, E. M., Tsui, K. L. (1985). Multinomial-Dirichlet bounds for dollar-unit sampling in auditing. *The Accounting Review*, Vol 60, No 1: 76-96.

[52] Vance, L. L. (1950). *Scientific method for auditing.* Berkeley: University of California Press.

[53] Vance, L. L., Neter, J. (1956). *Statistical sampling for auditors and accountants.* New York: John Wiley.

[54] Van Heerden, A. (1961). Steekproeven als middel van accountantscontrole (Statistical sampling as a means of auditing). *Maandblad voor Accountancy en Bedrijfshuishoudkunde*, Vol 11: 453.

[55] Wilburn, A. J. (1984). *Practical statistical sampling for auditors.* New York: Marcel Dekker.

Appendix A

Tables of Results

The tables in this appendix provide results from the simulation study for each of the study populations described in Chapter 7. Tables A.1, A.4, A.7, A.10, A.13, A.16, A.19, A.22 and A.25 contain the averages for each bound. These averages are computed from 500 samples.

Tables A.2, A.5, A.8, A.11, A.14, A.17, A.20, A.23 and A.26 contain the achieved coverage levels for the various bounds. Coverage is simply the percentage of times a particular bound covered the true mean error per dollar unit for the 500 samples.

The standard errors of the bounds are contained in Tables A.3, A.6, A.9, A.12, A.15, A.18, A.21, A.24 and A.27. This is the standard deviation of the bounds over the 500 samples.

The following key explains the abbreviations for the bounds used in the tables.

ST	Stringer
ST-meik	Stringer with Meikle's adjustment
ST-lta	Stringer with LTA adjustment
MM	modified moment
BN	Bayesian normal
CS	Cox and Snell
CS-lta	Cox and Snell with LTA adjustment
MD	multinomial-Dirichlet
MD-lta	multinomial-Dirichlet with LTA adjustment
PP	parametric power
PP-lta	parametric power with LTA adjustment
CL	Clayton's Hoeffding/Bootstrap

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.0350	.0350	.0350	.0237	.0231	.0224	.0224	.0258	.0258	.0202	.0202	.0325
.03	.0	.1	.0394	.0394	.0394	.0290	.0256	.0251	.0251	.0305	.0305	.0216	.0216	.0352
.03	.0	.2	.0441	.0441	.0441	.0343	.0283	.0281	.0281	.0354	.0354	.0246	.0246	.0382
.03	.0	.4	.0511	.0511	.0511	.0425	.0326	.0327	.0327	.0426	.0426	.0298	.0298	.0431
.03	.1	.0	.0345	.0344	.0341	.0239	.0228	.0219	.0216	.0255	.0251	.0191	.0187	.0319
.03	.1	.1	.0392	.0392	.0389	.0287	.0258	.0247	.0243	.0305	.0302	.0220	.0217	.0348
.03	.1	.2	.0438	.0438	.0436	.0336	.0286	.0275	.0273	.0352	.0350	.0251	.0248	.0379
.03	.1	.4	.0525	.0524	.0522	.0440	.0339	.0333	.0330	.0442	.0438	.0320	.0316	.0436
.03	.2	.0	.0340	.0339	.0334	.0237	.0224	.0208	.0203	.0252	.0246	.0191	.0185	.0314
.03	.2	.1	.0385	.0385	.0378	.0282	.0257	.0236	.0228	.0299	.0291	.0217	.0209	.0339
.03	.2	.2	.0433	.0433	.0427	.0334	.0286	.0267	.0261	.0349	.0343	.0249	.0243	.0373
.03	.2	.4	.0515	.0514	.0509	.0423	.0327	.0317	.0311	.0433	.0427	.0324	.0318	.0425
.06	.0	.0	.0394	.0394	.0394	.0276	.0292	.0287	.0287	.0287	.0287	.0282	.0282	.0352
.06	.0	.1	.0483	.0483	.0483	.0362	.0355	.0348	.0348	.0378	.0378	.0309	.0309	.0423
.06	.0	.2	.0570	.0570	.0570	.0464	.0419	.0410	.0410	.0469	.0469	.0340	.0340	.0487
.06	.0	.4	.0719	.0719	.0719	.0639	.0529	.0524	.0524	.0620	.0620	.0406	.0406	.0609
.06	.1	.0	.0385	.0385	.0379	.0277	.0286	.0277	.0271	.0280	.0275	.0267	.0261	.0335
.06	.1	.1	.0480	.0480	.0474	.0367	.0355	.0341	.0335	.0379	.0372	.0292	.0285	.0416
.06	.1	.2	.0553	.0553	.0547	.0445	.0407	.0393	.0387	.0456	.0450	.0321	.0315	.0471
.06	.1	.4	.0724	.0724	.0718	.0641	.0531	.0522	.0516	.0626	.0620	.0422	.0416	.0603
.06	.2	.0	.0373	.0372	.0361	.0267	.0274	.0261	.0248	.0271	.0259	.0241	.0229	.0318
.06	.2	.1	.0455	.0454	.0444	.0346	.0334	.0317	.0306	.0357	.0346	.0273	.0262	.0393
.06	.2	.2	.0541	.0540	.0531	.0433	.0398	.0377	.0366	.0447	.0436	.0313	.0303	.0457
.06	.2	.4	.0728	.0727	.0718	.0643	.0534	.0518	.0507	.0634	.0623	.0431	.0421	.0606
.10	.0	.0	.0453	.0453	.0453	.0331	.0359	.0349	.0349	.0324	.0324	.0405	.0405	.0349
.10	.0	.1	.0589	.0589	.0589	.0471	.0468	.0448	.0448	.0469	.0469	.0422	.0422	.0573
.10	.0	.2	.0750	.0750	.0750	.0646	.0599	.0574	.0574	.0629	.0629	.0482	.0482	.0751
.10	.0	.4	.0967	.0967	.0967	.0898	.0770	.0754	.0754	.0847	.0847	.0570	.0570	.0893
.10	.1	.0	.0440	.0439	.0431	.0329	.0347	.0337	.0327	.0317	.0307	.0378	.0368	.0308
.10	.1	.1	.0580	.0579	.0569	.0459	.0459	.0438	.0427	.0461	.0450	.0410	.0398	.0566
.10	.1	.2	.0725	.0725	.0716	.0615	.0578	.0552	.0543	.0610	.0601	.0459	.0450	.0717
.10	.1	.4	.0948	.0947	.0939	.0868	.0753	.0733	.0724	.0831	.0822	.0556	.0547	.0872
.10	.2	.0	.0423	.0421	.0402	.0323	.0330	.0320	.0299	.0305	.0284	.0341	.0320	.0282
.10	.2	.1	.0566	.0564	.0546	.0450	.0446	.0423	.0403	.0454	.0434	.0380	.0360	.0537
.10	.2	.2	.0702	.0699	.0680	.0586	.0555	.0526	.0505	.0591	.0570	.0424	.0403	.0682
.10	.2	.4	.0963	.0961	.0943	.0875	.0766	.0740	.0720	.0850	.0830	.0562	.0542	.0874

Table A.1: Average Results for Accounts Receivable Model M1

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.0	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.0	.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.988	.988	1.000
.03	.0	.4	1.000	1.000	1.000	1.000	.996	.940	.940	1.000	1.000	.894	.894	1.000
.03	.1	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.1	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.994	1.000
.03	.1	.2	1.000	1.000	1.000	.994	1.000	1.000	.998	1.000	1.000	1.000	.998	1.000
.03	.1	.4	1.000	1.000	1.000	.990	1.000	.930	.924	1.000	1.000	.914	.896	1.000
.03	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.2	.1	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	1.000	1.000	1.000	1.000
.03	.2	.2	1.000	1.000	1.000	.996	1.000	1.000	.998	1.000	1.000	.998	.980	1.000
.03	.2	.4	1.000	1.000	1.000	.986	1.000	.922	.918	1.000	1.000	.928	.914	1.000
.06	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.06	.0	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.988	.988	1.000
.06	.0	.2	1.000	1.000	1.000	.998	.990	.994	.994	1.000	1.000	.970	.970	.994
.06	.0	.4	1.000	1.000	1.000	.910	.884	.912	.912	.920	.920	.784	.784	1.000
.06	.1	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.06	.1	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.994	.988	1.000
.06	.1	.2	1.000	1.000	1.000	.992	.994	.998	.990	1.000	1.000	.948	.940	.996
.06	.1	.4	1.000	1.000	.996	.932	.928	.914	.912	.938	.934	.816	.808	.994
.06	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.990
.06	.2	.1	1.000	1.000	1.000	.998	1.000	1.000	.998	1.000	1.000	.992	.986	.998
.06	.2	.2	1.000	1.000	1.000	1.000	.998	.996	.994	1.000	1.000	.946	.930	.998
.06	.2	.4	1.000	1.000	.998	.948	.926	.908	.900	.948	.944	.850	.828	.998
.10	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.992
.10	.0	.1	1.000	1.000	1.000	.998	1.000	1.000	1.000	1.000	1.000	.996	.996	.980
.10	.0	.2	1.000	1.000	1.000	.964	.978	.974	.974	.960	.960	.954	.954	.984
.10	.0	.4	.982	.982	.982	.918	.904	.884	.884	.926	.926	.698	.698	.932
.10	.1	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.966
.10	.1	.1	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	.994	.994	.992	.952
.10	.1	.2	1.000	1.000	1.000	.978	.988	.978	.958	.974	.952	.938	.920	.984
.10	.1	.4	.986	.986	.982	.928	.920	.910	.904	.946	.940	.734	.718	.942
.10	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.948
.10	.2	.1	1.000	1.000	1.000	.998	1.000	1.000	.996	1.000	.998	.988	.984	.958
.10	.2	.2	1.000	1.000	.998	.986	.988	.982	.950	.992	.952	.938	.904	.980
.10	.2	.4	.988	.988	.988	.922	.926	.904	.882	.954	.924	.726	.680	.930

Table A.2: Coverage Results for Accounts Receivable Model M1

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.00002	.00002	.00002	.00002	.00003	.00003	.00003	.00001	.00001	.00005	.00005	.00001
.03	.0	.1	.00011	.00011	.00011	.00011	.00008	.00007	.00007	.00010	.00010	.00008	.00008	.00004
.03	.0	.2	.00018	.00018	.00018	.00020	.00011	.00011	.00011	.00017	.00017	.00013	.00013	.00007
.03	.0	.4	.00031	.00031	.00031	.00038	.00020	.00021	.00021	.00029	.00029	.00023	.00023	.00017
.03	.1	.0	.00002	.00002	.00002	.00002	.00003	.00002	.00003	.00001	.00001	.00004	.00005	.00001
.03	.1	.1	.00010	.00010	.00011	.00011	.00007	.00007	.00007	.00010	.00010	.00009	.00009	.00004
.03	.1	.2	.00019	.00019	.00019	.00022	.00012	.00012	.00013	.00018	.00018	.00014	.00014	.00009
.03	.1	.4	.00033	.00033	.00033	.00041	.00020	.00022	.00023	.00031	.00032	.00029	.00030	.00017
.03	.2	.0	.00002	.00002	.00002	.00003	.00002	.00003	.00003	.00001	.00001	.00004	.00004	.00001
.03	.2	.1	.00010	.00010	.00011	.00011	.00007	.00007	.00007	.00010	.00010	.00008	.00009	.00004
.03	.2	.2	.00018	.00018	.00018	.00020	.00011	.00012	.00012	.00017	.00017	.00014	.00014	.00010
.03	.2	.4	.00029	.00029	.00029	.00036	.00019	.00020	.00020	.00027	.00028	.00027	.00027	.00015
.06	.0	.0	.00005	.00005	.00005	.00003	.00003	.00004	.00004	.00003	.00003	.00007	.00007	.00004
.06	.0	.1	.00016	.00016	.00016	.00018	.00011	.00010	.00010	.00015	.00015	.00009	.00009	.00019
.06	.0	.2	.00029	.00029	.00029	.00035	.00020	.00019	.00019	.00027	.00027	.00013	.00013	.00030
.06	.0	.4	.00054	.00054	.00054	.00069	.00040	.00040	.00040	.00050	.00050	.00028	.00028	.00053
.06	.1	.0	.00004	.00004	.00004	.00003	.00003	.00003	.00003	.00002	.00003	.00007	.00007	.00004
.06	.1	.1	.00019	.00019	.00020	.00019	.00013	.00012	.00012	.00018	.00018	.00010	.00010	.00020
.06	.1	.2	.00030	.00030	.00029	.00033	.00020	.00019	.00019	.00028	.00028	.00013	.00013	.00030
.06	.1	.4	.00052	.00052	.00052	.00066	.00037	.00038	.00038	.00048	.00048	.00030	.00030	.00048
.06	.2	.0	.00003	.00003	.00004	.00003	.00002	.00003	.00003	.00002	.00002	.00006	.00006	.00004
.06	.2	.1	.00019	.00019	.00019	.00017	.00012	.00012	.00012	.00017	.00018	.00010	.00010	.00017
.06	.2	.2	.00027	.00027	.00027	.00028	.00018	.00018	.00018	.00025	.00025	.00014	.00015	.00032
.06	.2	.4	.00053	.00053	.00053	.00067	.00038	.00040	.00040	.00048	.00049	.00038	.00039	.00051
.10	.0	.0	.00006	.00006	.00006	.00003	.00004	.00004	.00004	.00004	.00004	.00010	.00010	.00014
.10	.0	.1	.00029	.00029	.00029	.00030	.00021	.00019	.00019	.00027	.00027	.00013	.00013	.00069
.10	.0	.2	.00042	.00042	.00042	.00049	.00032	.00030	.00030	.00039	.00039	.00018	.00018	.00088
.10	.0	.4	.00079	.00079	.00079	.00095	.00062	.00063	.00063	.00071	.00071	.00034	.00034	.00098
.10	.1	.0	.00006	.00006	.00006	.00003	.00004	.00004	.00004	.00003	.00004	.00009	.00010	.00017
.10	.1	.1	.00027	.00027	.00027	.00026	.00019	.00017	.00018	.00025	.00026	.00012	.00012	.00096
.10	.1	.2	.00053	.00053	.00053	.00059	.00039	.00038	.00038	.00047	.00048	.00021	.00022	.00093
.10	.1	.4	.00065	.00065	.00065	.00079	.00049	.00052	.00052	.00058	.00058	.00033	.00033	.00082
.10	.2	.0	.00005	.00005	.00006	.00003	.00003	.00003	.00004	.00003	.00004	.00008	.00008	.00021
.10	.2	.1	.00030	.00030	.00030	.00026	.00021	.00019	.00020	.00028	.00028	.00013	.00014	.00078
.10	.2	.2	.00044	.00045	.00046	.00048	.00032	.00031	.00032	.00041	.00042	.00018	.00019	.00087
.10	.2	.4	.00079	.00079	.00079	.00093	.00060	.00064	.00065	.00070	.00070	.00046	.00046	.00098

Table A.3: Standard Error Results for Accounts Receivable Model M1

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.0396	.0396	.0396	.0247	.0246	.0255	.0255	.0293	.0293	.0249	.0249	.0355
.03	.0	.1	.0432	.0432	.0432	.0298	.0268	.0278	.0278	.0330	.0330	.0269	.0269	.0377
.03	.0	.2	.0470	.0470	.047 0	.0349	.0298	.0304	.0304	.0373	.0373	.0288	.0288	.0403
.03	.0	.4	.0547	.0547	.0547	.0451	.0341	.0354	.0354	.0454	.0454	.0350	.0350	.0455
.03	.1	.0	.0387	.0386	.0382	.0245	.0247	.0245	.0240	.0283	.0278	.0249	.0244	.0344
.03	.1	.1	.0422	.0422	.0417	.0290	.0265	.0264	.0259	.0324	.0318	.0259	.0254	.0365
.03	.1	.2	.0450	.0449	.0443	.0324	.0282	.0281	.0274	.0354	.0348	.0282	.0275	.0381
.03	.1	.4	.0527	.0526	.0521	.0424	.0327	.0330	.0324	.0439	.0433	.0348	.0342	.0433
.03	.2	.0	.0372	.0371	.0360	.0232	.0237	.0227	.0214	.0275	.0263	.0229	.0216	.0327
.03	.2	.1	.0413	.0412	.0401	.0281	.0266	.0254	.0242	.0319	.0307	.0258	.0245	.0353
.03	.2	.2	.0449	.0448	.0436	.0325	.0289	.0277	.0265	.0356	.0344	.0287	.0275	.0375
.03	.2	.4	.0536	.0535	.0525	.0433	.0338	.0334	.0322	.0448	.0436	.0366	.0354	.0435
.06	.0	.0	.0483	.0483	.0483	.0329	.0344	.0354	.0354	.0353	.0353	.0378	.0378	.0410
.06	.0	.1	.0550	.0550	.0550	.0413	.0393	.0400	.0400	.0426	.0426	.0397	.0397	.0464
.06	.0	.2	.0641	.0641	.0641	.0528	.0462	.0467	.0467	.0521	.0521	.0442	.0442	.0538
.06	.0	.4	.0781	.0781	.0781	.0704	.0562	.0572	.0572	.0666	.0666	.0512	.0512	.0659
.06	.1	.0	.0458	.0457	.0447	.0309	.0323	.0327	.0317	.0334	.0324	.0343	.0333	.0382
.06	.1	.1	.0531	.0530	.0521	.0395	.0381	.0379	.0369	.0413	.0403	.0370	.0360	.0441
.06	.1	.2	.0624	.0623	.0613	.0503	.0446	.0445	.0435	.0509	.0498	.0413	.0402	.0513
.06	.1	.4	.0773	.0772	.0761	.0684	.0560	.0562	.0549	.0663	.0651	.0509	.0496	.0643
.06	.2	.0	.0443	.0441	.0419	.0298	.0313	.0312	.0288	.0326	.0302	.0322	.0298	.0357
.06	.2	.1	.0525	.0522	.0502	.0384	.0374	.0369	.0345	.0411	.0388	.0355	.0331	.0428
.06	.2	.2	.0604	.0601	.0579	.0475	.0433	.0423	.0399	.0494	.0470	.0392	.0368	.0483
.06	.2	.4	.0747	.0744	.0722	.0644	.0536	.0530	.0506	.0643	.0619	.0496	.0471	.0607
.10	.0	.0	.0590	.0590	.0590	.0434	.0452	.0459	.0459	.0430	.0430	.0540	.0540	.0460
.10	.0	.1	.0708	.0708	.0708	.0574	.0550	.0549	.0549	.0555	.0555	.0577	.0577	.0637
.10	.0	.2	.0858	.0858	.0858	.0751	.0673	.0667	.0667	.0711	.0711	.0632	.0632	.0793
.10	.0	.4	.1064	.1064	.1064	.1000	.0839	.0840	.0840	.0922	.0922	.0730	.0730	.0980
.10	.1	.0	.0565	.0563	.0545	.0405	.0427	.0436	.0415	.0413	.0392	.0501	.0481	.0412
.10	.1	.1	.0677	.0675	.0657	.0531	.0519	.0518	.0498	.0533	.0514	.0529	.0509	.0573
.10	.1	.2	.0800	.0798	.0780	.0671	.0619	.0615	.0594	.0660	.0639	.0582	.0562	.0713
.10	.1	.4	.1030	.1028	.1009	.0940	.0805	.0805	.0784	.0895	.0874	.0697	.0677	.0926
.10	.2	.0	.0541	.0535	.0502	.0381	.0401	.0411	.0372	.0395	.0355	.0459	.0420	.0373
.10	.2	.1	.0668	.0662	.0628	.0513	.0506	.0505	.0464	.0530	.0489	.0504	.0463	.0544
.10	.2	.2	.0777	.0771	.0739	.0637	.0594	.0588	.0550	.0644	.0606	.0542	.0504	.0668
.10	.2	.4	.1023	.1017	.0983	.0914	.0797	.0794	.0754	.0896	.0855	.0688	.0648	.0910

Table A.4: Average Results for Accounts Receivable Model M2

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.0	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.998	1.000
.03	.0	.2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.986	.986	1.000
.03	.0	.4	1.000	1.000	1.000	.976	.898	.960	.960	1.000	1.000	.926	.926	1.000
.03	.1	.0	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000	1.000	1.000	1.000	1.000
.03	.1	.1	1.000	1.000	1.000	.994	1.000	1.000	.998	1.000	1.000	1.000	1.000	1.000
.03	.1	.2	1.000	1.000	1.000	.986	1.000	1.000	.986	1.000	.998	1.000	.986	1.000
.03	.1	.4	1.000	1.000	1.000	.978	.964	.940	.932	1.000	.998	.948	.938	1.000
.03	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	1.000	1.000	.998	1.000
.03	.2	.1	1.000	1.000	1.000	.992	1.000	1.000	.994	1.000	.998	1.000	.998	1.000
.03	.2	.2	1.000	1.000	1.000	.982	1.000	1.000	.986	1.000	1.000	1.000	.986	1.000
.03	.2	.4	1.000	1.000	1.000	.964	1.000	.928	.910	1.000	.996	.954	.936	1.000
.06	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.06	.0	.1	1.000	1.000	1.000	.998	.982	.994	.994	1.000	1.000	.982	.982	1.000
.06	.0	.2	1.000	1.000	1.000	.952	.966	.972	.972	1.000	1.000	.960	.960	.988
.06	.0	.4	.996	.996	.996	.942	.918	.922	.922	.950	.950	.856	.856	.986
.06	.1	.0	1.000	1.000	1.000	.994	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000
.06	.1	.1	1.000	1.000	1.000	.992	1.000	.990	.986	1.000	.998	.990	.978	1.000
.06	.1	.2	1.000	1.000	1.000	.978	.986	.982	.968	1.000	.994	.960	.952	1.000
.06	.1	.4	1.000	1.000	.998	.940	.918	.926	.902	.952	.946	.882	.858	.996
.06	.2	.0	1.000	1.000	1.000	.990	1.000	1.000	.994	1.000	.998	1.000	.994	.998
.06	.2	.1	1.000	1.000	1.000	.980	1.000	1.000	.986	1.000	1.000	.996	.980	1.000
.06	.2	.2	1.000	1.000	1.000	.972	.990	.984	.950	1.000	.996	.964	.940	1.000
.06	.2	.4	1.000	1.000	.996	.952	.948	.932	.894	.974	.958	.906	.866	.990
.10	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.998	.996
.10	.0	.1	1.000	1.000	1.000	.962	.986	.986	.986	.982	.982	.986	.986	.984
.10	.0	.2	.996	.996	.996	.958	.970	.972	.972	.966	.966	.966	.966	.988
.10	.0	.4	.996	.996	.996	.956	.934	.930	.930	.952	.952	.860	.860	.948
.10	.1	.0	1.000	1.000	1.000	.998	1.000	1.000	.994	1.000	.996	1.000	.992	.970
.10	.1	.1	1.000	1.000	1.000	.990	.996	.994	.982	1.000	.982	.988	.976	.968
.10	.1	.2	1.000	1.000	.996	.960	.968	.972	.962	.968	.958	.962	.950	.980
.10	.1	.4	.992	.992	.986	.942	.914	.908	.878	.952	.946	.850	.820	.934
.10	.2	.0	1.000	1.000	.998	.994	1.000	1.000	.992	1.000	.990	1.000	.994	.978
.10	.2	.1	1.000	1.000	1.000	.990	.998	.996	.984	1.000	.990	.986	.976	.980
.10	.2	.2	1.000	1.000	.996	.972	.982	.982	.956	.980	.954	.962	.928	.978
.10	.2	.4	.992	.992	.990	.938	.940	.918	.884	.968	.936	.870	.802	.936

Table A.5: Coverage Results for Accounts Receivable Model M2

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.00005	.00005	.00005	.00005	.00005	.00005	.00005	.00004	.00004	.00006	.00006	.00002
.03	.0	.1	.00012	.00012	.00012	.00016	.00009	.00009	.00009	.00010	.00010	.00010	.00010	.00005
.03	.0	.2	.00021	.00021	.00021	.00028	.00014	.00015	.00015	.00019	.00019	.00013	.00013	.00010
.03	.0	.4	.00031	.00031	.00031	.00043	.00021	.00022	.00022	.00029	.00029	.00023	.00023	.00017
.03	.1	.0	.00005	.00005	.00006	.00005	.00004	.00005	.00006	.00003	.00004	.00007	.00007	.00002
.03	.1	.1	.00012	.00012	.00012	.00015	.00008	.00008	.00009	.00010	.00010	.00009	.00009	.00005
.03	.1	.2	.00015	.00015	.00016	.00021	.00010	.00011	.00011	.00014	.00015	.00012	.00012	.00007
.03	.1	.4	.00026	.00026	.00026	.00037	.00017	.00018	.00018	.00025	.00025	.00024	.00025	.00013
.03	.2	.0	.00004	.00004	.00004	.00005	.00003	.00004	.00005	.00002	.00003	.00005	.00006	.00002
.03	.2	.1	.00012	.00012	.00012	.00015	.00008	.00009	.00009	.00010	.00011	.00010	.00010	.00005
.03	.2	.2	.00018	.00018	.00019	.00023	.00011	.00013	.00013	.00017	.00017	.00016	.00017	.00008
.03	.2	.4	.00030	.00030	.00031	.00042	.00019	.00021	.00022	.00029	.00030	.00027	.00028	.00017
.06	.0	.0	.00010	.00010	.00010	.00010	.00007	.00008	.00008	.00007	.00007	.00014	.00014	.00005
.06	.0	.1	.00022	.00022	.00022	.00027	.00016	.00016	.00016	.00019	.00019	.00017	.00017	.00017
.06	.0	.2	.00035	.00035	.00035	.00047	.00025	.00025	.00025	.00033	.00033	.00020	.00020	.00031
.06	.0	.4	.00049	.00049	.00049	.00065	.00038	.00039	.00039	.00045	.00045	.00034	.00034	.00048
.06	.1	.0	.00009	.00009	.00009	.00008	.00006	.00007	.00008	.00006	.00006	.00012	.00013	.00005
.06	.1	.1	.00021	.00021	.00022	.00025	.00014	.00015	.00016	.00019	.00020	.00015	.00016	.00017
.06	.1	.2	.00032	.00032	.00033	.00041	.00023	.00023	.00024	.00030	.00030	.00020	.00020	.00028
.06	.1	.4	.00059	.00059	.00060	.00076	.00044	.00046	.00047	.00053	.00054	.00039	.00041	.00059
.06	.2	.0	.00008	.00008	.00009	.00007	.00005	.00007	.00008	.00006	.00007	.00011	.00012	.00006
.06	.2	.1	.00020	.00020	.00021	.00023	.00014	.00014	.00015	.00018	.00019	.00014	.00015	.00019
.06	.2	.2	.00032	.00032	.00034	.00040	.00022	.00023	.00025	.00030	.00031	.00020	.00021	.00027
.06	.2	.4	.00051	.00051	.00053	.00067	.00038	.00040	.00042	.00047	.00048	.00039	.00040	.00052
.10	.0	.0	.00014	.00014	.00014	.00015	.00010	.00011	.00011	.00010	.00010	.00020	.00020	.00014
.10	.0	.1	.00031	.00031	.00031	.00037	.00023	.00022	.00022	.00027	.00027	.00021	.00021	.00051
.10	.0	.2	.00044	.00044	.00044	.00053	.00035	.00033	.00033	.00040	.00040	.00027	.00027	.00060
.10	.0	.4	.00068	.00068	.00068	.00081	.00055	.00056	.00056	.00060	.00060	.00040	.00040	.00080
.10	.1	.0	.00012	.00012	.00013	.00011	.00008	.00009	.00010	.00009	.00010	.00017	.00018	.00015
.10	.1	.1	.00033	.00033	.00034	.00037	.00024	.00023	.00024	.00029	.00030	.00022	.00023	.00049
.10	.1	.2	.00044	.00044	.00044	.00052	.00033	.00033	.00033	.00040	.00040	.00025	.00025	.00060
.10	.1	.4	.00077	.00077	.00078	.00091	.00061	.00064	.00066	.00066	.00068	.00048	.00049	.00090
.10	.2	.0	.00012	.00012	.00014	.00009	.00008	.00009	.00011	.00009	.00011	.00015	.00017	.00016
.10	.2	.1	.00030	.00030	.00031	.00031	.00021	.00021	.00023	.00026	.00027	.00021	.00023	.00048
.10	.2	.2	.00044	.00044	.00045	.00050	.00032	.00033	.00033	.00040	.00041	.00025	.00026	.00059
.10	.2	.4	.00079	.00079	.00080	.00094	.00061	.00067	.00068	.00070	.00071	.00051	.00052	.00093

Table A.6: Standard Error Results for Accounts Receivable Model M2

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.0442	.0442	.0442	.0293	.0269	.0287	.0287	.0326	.0326	.0294	.0294	.0384
.03	.0	.1	.0480	.0480	.0480	.0344	.0292	.0312	.0312	.0369	.0369	.0318	.0318	.0409
.03	.0	.2	.0485	.0485	.0485	.0359	.0292	.0311	.0311	.0381	.0381	.0316	.0316	.0413
.03	.0	.4	.0568	.0568	.0568	.0474	.0347	.0365	.0365	.0469	.0469	.0388	.0388	.0470
.03	.1	.0	.0428	.0428	.0425	.0280	.0264	.0272	.0268	.0316	.0313	.0279	.0276	.0372
.03	.1	.1	.0446	.0445	.0443	.0310	.0277	.0280	.0277	.0341	.0338	.0295	.0293	.0383
.03	.1	.2	.0489	.0489	.0485	.0366	.0300	.0308	.0305	.0387	.0384	.0322	.0318	.0411
.03	.1	.4	.0555	.0555	.0552	.0458	.0341	.0351	.0348	.0460	.0457	.0389	.0386	.0460
.03	.2	.0	.0409	.0409	.0403	.0267	.0254	.0252	.0247	.0303	.0297	.0264	.0258	.0357
.03	.2	.1	.0458	.0458	.0452	.0323	.0285	.0286	.0280	.0354	.0348	.0300	.0294	.0388
.03	.2	.2	.0472	.0472	.0466	.0350	.0292	.0288	.0282	.0374	.0368	.0312	.0306	.0396
.03	.2	.4	.0558	.0558	.0551	.0461	.0350	.0349	.0342	.0465	.0458	.0403	.0396	.0457
.06	.0	.0	.0560	.0560	.0560	.0411	.0392	.0412	.0412	.0416	.0416	.0444	.0444	.0466
.06	.0	.1	.0619	.0619	.0619	.0489	.0436	.0453	.0453	.0480	.0480	.0466	.0466	.0516
.06	.0	.2	.0702	.0702	.0702	.0593	.0496	.0514	.0514	.0568	.0568	.0507	.0507	.0584
.06	.0	.4	.0809	.0809	.0809	.0733	.0577	.0597	.0597	.0684	.0684	.0569	.0569	.0688
.06	.1	.0	.0539	.0539	.0534	.0385	.0377	.0390	.0385	.0401	.0396	.0415	.0410	.0444
.06	.1	.1	.0590	.0590	.0585	.0452	.0417	.0425	.0419	.0456	.0451	.0435	.0429	.0487
.06	.1	.2	.0670	.0669	.0664	.0553	.0476	.0482	.0475	.0545	.0539	.0474	.0468	.0553
.06	.1	.4	.0800	.0800	.0793	.0717	.0574	.0582	.0575	.0680	.0673	.0563	.0556	.0669
.06	.2	.0	.0513	.0512	.0501	.0358	.0361	.0365	.0353	.0380	.0368	.0386	.0374	.0419
.06	.2	.1	.0578	.0577	.0566	.0439	.0407	.0407	.0395	.0452	.0440	.0408	.0396	.0471
.06	.2	.2	.0649	.0648	.0637	.0529	.0462	.0458	.0446	.0529	.0518	.0447	.0435	.0529
.06	.2	.4	.0767	.0766	.0755	.0679	.0550	.0544	.0532	.0657	.0645	.0539	.0527	.0629
.10	.0	.0	.0716	.0716	.0716	.0569	.0541	.0562	.0562	.0535	.0535	.0641	.0641	.0575
.10	.0	.1	.0822	.0822	.0822	.0698	.0628	.0643	.0643	.0649	.0649	.0678	.0678	.0707
.10	.0	.2	.0931	.0931	.0931	.0829	.0719	.0731	.0731	.0766	.0766	.0727	.0727	.0831
.10	.0	.4	.1123	.1123	.1123	.1063	.0877	.0893	.0893	.0964	.0964	.0829	.0829	.1026
.10	.1	.0	.0678	.0677	.0667	.0518	.0512	.0528	.0517	.0503	.0492	.0597	.0587	.0528
.10	.1	.1	.0794	.0793	.0783	.0657	.0607	.0615	.0605	.0626	.0616	.0642	.0631	.0682
.10	.1	.2	.0875	.0874	.0864	.0759	.0672	.0676	.0665	.0721	.0710	.0662	.0651	.0767
.10	.1	.4	.1086	.1085	.1075	.1012	.0846	.0854	.0842	.0938	.0926	.0793	.0782	.0983
.10	.2	.0	.0638	.0635	.0617	.0475	.0477	.0488	.0467	.0474	.0454	.0543	.0522	.0486
.10	.2	.1	.0757	.0755	.0739	.0615	.0577	.0577	.0558	.0600	.0582	.0589	.0571	.0646
.10	.2	.2	.0874	.0872	.0854	.0751	.0674	.0672	.0652	.0727	.0707	.0656	.0636	.0774
.10	.2	.4	.1064	.1062	.1046	.0982	.0829	.0827	.0808	.0925	.0907	.0775	.0756	.0956

Table A.7: Average Results for Accounts Receivable Model M3

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.03	.0	.1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.996	1.000
.03	.0	.2	1.000	1.000	1.000	1.000	1.000	.962	.962	1.000	1.000	.972	.972	1.000
.03	.0	.4	1.000	1.000	1.000	.952	.898	.932	.932	1.000	1.000	.948	.948	1.000
.03	.1	.0	1.000	1.000	1.000	.996	1.000	1.000	.998	1.000	1.000	1.000	.998	1.000
.03	.1	.1	1.000	1.000	1.000	.990	1.000	1.000	.992	1.000	1.000	.998	.996	1.000
.03	.1	.2	1.000	1.000	1.000	.992	1.000	1.000	.990	1.000	1.000	.998	.986	1.000
.03	.1	.4	1.000	1.000	1.000	.954	.886	.930	.926	1.000	1.000	.950	.946	1.000
.03	.2	.0	1.000	1.000	1.000	.998	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000
.03	.2	.1	1.000	1.000	1.000	.996	1.000	1.000	.998	1.000	1.000	1.000	.996	1.000
.03	.2	.2	1.000	1.000	1.000	.990	1.000	1.000	.986	1.000	1.000	.998	.994	1.000
.03	.2	.4	1.000	1.000	1.000	.972	.950	.918	.906	1.000	1.000	.960	.958	1.000
.06	.0	.0	1.000	1.000	1.000	.998	.992	.992	.992	1.000	1.000	.986	.986	1.000
.06	.0	.1	1.000	1.000	1.000	.948	.958	.970	.970	1.000	1.000	.970	.970	1.000
.06	.0	.2	1.000	1.000	1.000	.936	.942	.954	.954	.982	.982	.956	.956	1.000
.06	.0	.4	.986	.986	.986	.924	.852	.890	.890	.934	.934	.882	.882	.980
.06	.1	.0	1.000	1.000	1.000	.992	.982	1.000	.998	1.000	1.000	.990	.988	1.000
.06	.1	.1	1.000	1.000	1.000	.990	.994	.984	.982	1.000	1.000	.986	.980	1.000
.06	.1	.2	1.000	1.000	1.000	.952	.970	.962	.954	.990	.984	.960	.960	1.000
.06	.1	.4	.994	.994	.994	.952	.912	.908	.898	.958	.958	.896	.890	.990
.06	.2	.0	1.000	1.000	1.000	.994	1.000	.996	.992	1.000	1.000	.998	.984	1.000
.06	.2	.1	1.000	1.000	1.000	.992	.984	.992	.976	1.000	1.000	.976	.968	1.000
.06	.2	.2	1.000	1.000	1.000	.970	.972	.960	.948	1.000	.988	.950	.940	1.000
.06	.2	.4	1.000	1.000	1.000	.956	.922	.910	.884	.968	.958	.900	.884	.996
.10	.0	.0	1.000	1.000	1.000	.968	.982	.992	.992	.992	.992	.998	.998	.998
.10	.0	.1	1.000	1.000	1.000	.976	.980	.992	.992	.984	.984	.988	.988	.996
.10	.0	.2	.996	.996	.996	.948	.932	.936	.936	.950	.950	.948	.948	.974
.10	.0	.4	.980	.980	.980	.962	.916	.916	.916	.956	.956	.886	.886	.938
.10	.1	.0	1.000	1.000	1.000	.990	.992	.994	.990	1.000	.994	.994	.988	1.000
.10	.1	.1	1.000	1.000	1.000	.950	.966	.974	.970	.966	.960	.976	.968	.998
.10	.1	.2	.994	.994	.994	.946	.948	.944	.936	.956	.948	.940	.932	.982
.10	.1	.4	.978	.978	.976	.940	.888	.888	.880	.938	.934	.868	.852	.920
.10	.2	.0	1.000	1.000	1.000	.998	.998	.996	.990	1.000	.998	.994	.988	.984
.10	.2	.1	1.000	1.000	1.000	.982	.988	.984	.984	.990	.980	.980	.980	.992
.10	.2	.2	.996	.996	.986	.952	.962	.960	.938	.968	.960	.956	.942	.972
.10	.2	.4	.986	.984	.980	.958	.922	.914	.906	.954	.954	.894	.862	.956

Table A.8: Coverage Results for Accounts Receivable Model M3

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.00009	.00009	.00009	.00012	.00008	.00008	.00008	.00007	.00007	.00010	.00010	.00004
.03	.0	.1	.00017	.00017	.00017	.00023	.00012	.00013	.00013	.00014	.00014	.00014	.00014	.00008
.03	.0	.2	.00020	.00020	.00020	.00029	.00015	.00015	.00015	.00018	.00018	.00016	.00016	.00010
.03	.0	.4	.00031	.00031	.00031	.00043	.00021	.00023	.00023	.00029	.00029	.00024	.00024	.00018
.03	.1	.0	.00009	.00009	.00010	.00011	.00007	.00008	.00009	.00007	.00007	.00010	.00010	.00004
.03	.1	.1	.00014	.00013	.00014	.00017	.00009	.00011	.00011	.00011	.00011	.00010	.00010	.00006
.03	.1	.2	.00020	.00020	.00020	.00028	.00014	.00014	.00014	.00018	.00019	.00016	.00016	.00010
.03	.1	.4	.00033	.00033	.00033	.00046	.00023	.00025	.00025	.00031	.00030	.00028	.00028	.00019
.03	.2	.0	.00007	.00007	.00007	.00007	.00005	.00007	.00007	.00005	.00005	.00007	.00008	.00003
.03	.2	.1	.00016	.00016	.00016	.00020	.00011	.00012	.00012	.00014	.00014	.00013	.00013	.00007
.03	.2	.2	.00018	.00018	.00018	.00023	.00012	.00013	.00013	.00016	.00016	.00013	.00013	.00008
.03	.2	.4	.00033	.00033	.00033	.00044	.00022	.00025	.00025	.00030	.00030	.00028	.00028	.00019
.06	.0	.0	.00016	.00016	.00016	.00019	.00012	.00013	.00013	.00012	.00012	.00019	.00019	.00009
.06	.0	.1	.00027	.00027	.00027	.00035	.00020	.00020	.00020	.00022	.00022	.00021	.00021	.00019
.06	.0	.2	.00035	.00035	.00035	.00048	.00028	.00027	.00027	.00031	.00031	.00025	.00025	.00031
.06	.0	.4	.00058	.00058	.00058	.00077	.00045	.00046	.00046	.00051	.00051	.00039	.00039	.00058
.06	.1	.0	.00014	.00014	.00014	.00016	.00010	.00011	.00011	.00010	.00010	.00016	.00016	.00007
.06	.1	.1	.00023	.00023	.00023	.00029	.00016	.00017	.00017	.00020	.00020	.00018	.00019	.00015
.06	.1	.2	.00032	.00032	.00033	.00042	.00024	.00025	.00025	.00029	.00029	.00023	.00024	.00028
.06	.1	.4	.00053	.00053	.00053	.00069	.00040	.00043	.00043	.00048	.00048	.00039	.00040	.00055
.06	.2	.0	.00012	.00012	.00013	.00013	.00009	.00010	.00011	.00009	.00009	.00015	.00016	.00007
.06	.2	.1	.00026	.00026	.00026	.00031	.00019	.00020	.00020	.00023	.00023	.00020	.00020	.00018
.06	.2	.2	.00034	.00034	.00035	.00043	.00025	.00026	.00027	.00030	.00031	.00025	.00025	.00029
.06	.2	.4	.00046	.00046	.00047	.00060	.00034	.00037	.00038	.00042	.00043	.00035	.00036	.00046
.10	.0	.0	.00022	.00022	.00022	.00024	.00017	.00017	.00017	.00017	.00017	.00025	.00025	.00017
.10	.0	.1	.00034	.00034	.00034	.00041	.00027	.00026	.00026	.00029	.00029	.00027	.00027	.00039
.10	.0	.2	.00049	.00049	.00049	.00059	.00038	.00038	.00038	.00043	.00043	.00033	.00033	.00057
.10	.0	.4	.00075	.00075	.00075	.00088	.00061	.00064	.00064	.00064	.00064	.00048	.00048	.00083
.10	.1	.0	.00020	.00020	.00021	.00022	.00015	.00016	.00017	.00016	.00016	.00023	.00024	.00017
.10	.1	.1	.00037	.00037	.00037	.00045	.00028	.00029	.00029	.00032	.00032	.00029	.00029	.00043
.10	.1	.2	.00048	.00048	.00049	.00059	.00037	.00038	.00038	.00041	.00042	.00034	.00035	.00057
.10	.1	.4	.00080	.00080	.00080	.00095	.00064	.00068	.00069	.00070	.00071	.00054	.00055	.00090
.10	.2	.0	.00019	.00019	.00020	.00019	.00013	.00015	.00016	.00014	.00015	.00021	.00022	.00019
.10	.2	.1	.00032	.00032	.00032	.00037	.00024	.00025	.00025	.00028	.00028	.00024	.00024	.00040
.10	.2	.2	.00055	.00056	.00056	.00065	.00042	.00044	.00045	.00048	.00049	.00037	.00038	.00065
.10	.2	.4	.00079	.00078	.00079	.00093	.00062	.00068	.00068	.00069	.00069	.00057	.00057	.00088

Table A.9: Standard Error Results for Accounts Receivable Model M3

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.0550	.0550	.0550	.0437	.0335	.0360	.0360	.0432	.0432	.0374	.0374	.0457
.03	.0	.1	.0562	.0562	.0562	.0455	.0339	.0366	.0366	.0450	.0450	.0379	.0379	.0466
.03	.0	.2	.0582	.0582	.0582	.0487	.0352	.0378	.0378	.0477	.0477	.0399	.0399	.0480
.03	.0	.4	.0649	.0649	.0649	.0577	.0397	.0427	.0427	.0548	.0548	.0468	.0468	.0534
.03	.1	.0	.0523	.0522	.0520	.0402	.0319	.0336	.0333	.0412	.0409	.0348	.0345	.0436
.03	.1	.1	.0549	.0548	.0546	.0442	.0332	.0349	.0347	.0442	.0440	.0373	.0371	.0453
.03	.1	.2	.0586	.0586	.0583	.0491	.0358	.0376	.0373	.0481	.0479	.0404	.0401	.0480
.03	.1	.4	.0610	.0610	.0607	.0528	.0369	.0389	.0386	.0514	.0511	.0451	.0448	.0501
.03	.2	.0	.0491	.0490	.0485	.0365	.0297	.0301	.0295	.0385	.0379	.0324	.0319	.0408
.03	.2	.1	.0513	.0513	.0507	.0396	.0316	.0318	.0312	.0409	.0403	.0342	.0336	.0424
.03	.2	.2	.0541	.0540	.0534	.0434	.0332	.0334	.0328	.0442	.0436	.0378	.0372	.0443
.03	.2	.4	.0583	.0583	.0578	.0496	.0356	.0360	.0355	.0490	.0485	.0445	.0439	.0476
.06	.0	.0	.0759	.0759	.0759	.0659	.0530	.0561	.0561	.0612	.0612	.0563	.0563	.0637
.06	.0	.1	.0812	.0812	.0812	.0726	.0571	.0603	.0603	.0666	.0666	.0601	.0601	.0685
.06	.0	.2	.0847	.0847	.0847	.0772	.0597	.0628	.0628	.0708	.0708	.0621	.0621	.0716
.06	.0	.4	.0910	.0910	.0910	.0859	.0641	.0675	.0675	.0779	.0779	.0670	.0670	.0781
.06	.1	.0	.0721	.0720	.0715	.0608	.0505	.0523	.0517	.0580	.0574	.0527	.0521	.0591
.06	.1	.1	.0759	.0758	.0752	.0658	.0533	.0552	.0546	.0621	.0614	.0551	.0544	.0629
.06	.1	.2	.0795	.0795	.0790	.0706	.0560	.0579	.0573	.0663	.0658	.0581	.0575	.0663
.06	.1	.4	.0892	.0891	.0885	.0828	.0638	.0655	.0648	.0767	.0761	.0656	.0650	.0759
.06	.2	.0	.0664	.0663	.0653	.0538	.0463	.0471	.0460	.0529	.0518	.0475	.0463	.0541
.06	.2	.1	.0733	.0732	.0721	.0622	.0520	.0526	.0514	.0598	.0586	.0531	.0519	.0600
.06	.2	.2	.0756	.0755	.0745	.0655	.0535	.0538	.0527	.0633	.0621	.0544	.0533	.0622
.06	.2	.4	.0838	.0837	.0827	.0765	.0596	.0600	.0589	.0724	.0714	.0622	.0612	.0700
.10	.0	.0	.1032	.1032	.1032	.0943	.0789	.0821	.0821	.0839	.0839	.0818	.0818	.0903
.10	.0	.1	.1114	.1114	.1114	.1036	.0858	.0891	.0891	.0925	.0925	.0872	.0872	.0987
.10	.0	.2	.1179	.1179	.1179	.1118	.0911	.0946	.0946	.0997	.0997	.0912	.0912	.1061
.10	.0	.4	.1312	.1312	.1312	.1280	.1022	.1062	.1062	.1137	.1137	.1013	.1013	.1206
.10	.1	.0	.0950	.0949	.0939	.0839	.0724	.0745	.0734	.0773	.0761	.0746	.0735	.0812
.10	.1	.1	.1032	.1031	.1022	.0935	.0792	.0813	.0803	.0859	.0849	.0797	.0788	.0903
.10	.1	.2	.1095	.1094	.1085	.1014	.0843	.0864	.0854	.0926	.0916	.0839	.0829	.0970
.10	.1	.4	.1229	.1228	.1219	.1178	.0955	.0979	.0969	.1071	.1060	.0946	.0936	.1116
.10	.2	.0	.0906	.0903	.0886	.0777	.0689	.0703	.0682	.0731	.0711	.0703	.0683	.0761
.10	.2	.1	.0981	.0978	.0960	.0867	.0752	.0762	.0741	.0814	.0793	.0746	.0725	.0849
.10	.2	.2	.1042	.1040	.1024	.0945	.0804	.0812	.0794	.0880	.0862	.0785	.0766	.0918
.10	.2	.4	.1179	.1176	.1159	.1112	.0918	.0927	.0907	.1034	.1013	.0897	.0877	.1060

Table A.10: Average Results for Accounts Receivable Model M4

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	1.000	1.000	1.000	.992	.912	.950	.950	1.000	1.000	.974	.974	1.000
.03	.0	.1	1.000	1.000	1.000	.978	.864	.952	.952	1.000	1.000	.964	.964	1.000
.03	.0	.2	1.000	1.000	1.000	.966	.880	.932	.932	1.000	1.000	.952	.952	1.000
.03	.0	.4	1.000	1.000	1.000	.962	.864	.916	.916	1.000	1.000	.950	.950	1.000
.03	.1	.0	1.000	1.000	1.000	.996	1.000	.946	.946	1.000	1.000	.976	.970	1.000
.03	.1	.1	1.000	1.000	1.000	.980	.918	.930	.930	1.000	1.000	.978	.976	1.000
.03	.1	.2	1.000	1.000	1.000	.986	.910	.950	.948	1.000	1.000	.982	.980	1.000
.03	.1	.4	1.000	1.000	1.000	.946	.812	.900	.892	1.000	1.000	.940	.938	1.000
.03	.2	.0	1.000	1.000	1.000	.976	1.000	1.000	.962	1.000	1.000	.998	.990	1.000
.03	.2	.1	1.000	1.000	1.000	.988	1.000	.922	.922	1.000	1.000	.982	.978	1.000
.03	.2	.2	1.000	1.000	1.000	.974	.964	.906	.906	1.000	1.000	.980	.978	1.000
.03	.2	.4	1.000	1.000	1.000	.964	.906	.898	.890	1.000	.996	.974	.968	1.000
.06	.0	.0	.996	.996	.996	.952	.900	.942	.942	.972	.972	.922	.922	.996
.06	.0	.1	.998	.998	.998	.964	.898	.928	.928	.968	.968	.924	.924	.996
.06	.0	.2	.994	.994	.994	.968	.892	.930	.930	.972	.972	.932	.932	.990
.06	.0	.4	.984	.984	.984	.946	.828	.876	.876	.950	.950	.880	.880	.958
.06	.1	.0	1.000	1.000	1.000	.966	.940	.954	.946	.980	.974	.952	.946	1.000
.06	.1	.1	1.000	1.000	.996	.938	.906	.916	.906	.972	.964	.930	.916	.996
.06	.1	.2	.996	.996	.994	.948	.886	.908	.906	.962	.958	.912	.902	.990
.06	.1	.4	.994	.994	.994	.950	.880	.866	.862	.972	.962	.890	.882	.976
.06	.2	.0	1.000	1.000	1.000	.968	.952	.938	.932	.990	.976	.952	.940	1.000
.06	.2	.1	1.000	1.000	1.000	.966	.964	.948	.936	.980	.978	.948	.934	1.000
.06	.2	.2	1.000	1.000	.996	.952	.910	.918	.904	.978	.972	.930	.916	.996
.06	.2	.4	.986	.986	.986	.954	.882	.870	.856	.968	.964	.902	.886	.976
.10	.0	.0	.998	.998	.998	.958	.916	.940	.940	.942	.942	.934	.934	.968
.10	.0	.1	.990	.990	.990	.976	.914	.934	.934	.960	.960	.922	.922	.972
.10	.0	.2	.992	.992	.992	.976	.908	.922	.922	.956	.956	.908	.908	.950
.10	.0	.4	.984	.984	.984	.964	.868	.892	.892	.946	.946	.886	.886	.938
.10	.1	.0	.994	.994	.990	.960	.946	.948	.940	.956	.944	.952	.936	.964
.10	.1	.1	.998	.998	.996	.960	.920	.926	.922	.956	.948	.924	.916	.962
.10	.1	.2	.992	.992	.988	.958	.908	.920	.906	.950	.944	.900	.886	.942
.10	.1	.4	.986	.986	.982	.964	.890	.904	.898	.960	.952	.880	.872	.938
.10	.2	.0	.998	.998	.996	.970	.962	.960	.944	.968	.960	.950	.934	.984
.10	.2	.1	.996	.996	.994	.952	.950	.944	.920	.964	.950	.946	.918	.964
.10	.2	.2	.984	.984	.980	.960	.922	.916	.900	.964	.956	.898	.886	.958
.10	.2	.4	.980	.980	.978	.956	.924	.904	.882	.958	.950	.888	.872	.934

Table A.11: Coverage Results for Accounts Receivable Model M4

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.03	.0	.0	.00022	.00022	.00022	.00028	.00016	.00018	.00018	.00018	.00018	.00018	.00018	.00012
.03	.0	.1	.00027	.00027	.00027	.00036	.00020	.00021	.00021	.00023	.00023	.00022	.00022	.00015
.03	.0	.2	.00028	.00028	.00028	.00037	.00019	.00021	.00021	.00025	.00025	.00021	.00021	.00016
.03	.0	.4	.00038	.00038	.00038	.00052	.00028	.00030	.00030	.00035	.00035	.00032	.00032	.00027
.03	.1	.0	.00023	.00023	.00023	.00029	.00017	.00019	.00019	.00019	.00019	.00019	.00019	.00013
.03	.1	.1	.00026	.00026	.00026	.00033	.00018	.00020	.00020	.00023	.00023	.00018	.00019	.00014
.03	.1	.2	.00029	.00029	.00029	.00038	.00021	.00023	.00022	.00025	.00025	.00022	.00022	.00017
.03	.1	.4	.00038	.00038	.00038	.00051	.00027	.00030	.00030	.00034	.00034	.00031	.00031	.00025
.03	.2	.0	.00016	.00016	.00017	.00021	.00011	.00013	.00013	.00014	.00014	.00012	.00012	.00008
.03	.2	.1	.00022	.00022	.00022	.00028	.00015	.00017	.00017	.00019	.00019	.00017	.00017	.00011
.03	.2	.2	.00026	.00026	.00026	.00034	.00017	.00020	.00020	.00022	.00023	.00019	.00019	.00013
.03	.2	.4	.00034	.00034	.00034	.00044	.00023	.00026	.00026	.00030	.00030	.00029	.00029	.00021
.06	.0	.0	.00040	.00040	.00040	.00049	.00031	.00033	.00033	.00032	.00032	.00035	.00035	.00035
.06	.0	.1	.00048	.00048	.00048	.00061	.00038	.00040	.00040	.00041	.00041	.00041	.00041	.00047
.06	.0	.2	.00049	.00049	.00049	.00060	.00040	.00042	.00042	.00040	.00040	.00040	.00040	.00050
.06	.0	.4	.00066	.00066	.00066	.00083	.00055	.00057	.00057	.00058	.00058	.00055	.00055	.00072
.06	.1	.0	.00031	.00031	.00032	.00040	.00025	.00027	.00027	.00025	.00025	.00029	.00029	.00027
.06	.1	.1	.00042	.00042	.00043	.00054	.00032	.00035	.00036	.00035	.00036	.00033	.00034	.00039
.06	.1	.2	.00052	.00052	.00052	.00067	.00040	.00043	.00043	.00046	.00046	.00043	.00043	.00051
.06	.1	.4	.00063	.00064	.00064	.00081	.00050	.00055	.00056	.00055	.00056	.00054	.00055	.00069
.06	.2	.0	.00034	.00034	.00034	.00041	.00026	.00029	.00028	.00028	.00028	.00030	.00030	.00026
.06	.2	.1	.00039	.00039	.00039	.00049	.00029	.00033	.00032	.00034	.00034	.00031	.00031	.00035
.06	.2	.2	.00045	.00045	.00045	.00057	.00034	.00037	.00038	.00039	.00039	.00037	.00037	.00043
.06	.2	.4	.00056	.00056	.00056	.00072	.00042	.00047	.00047	.00049	.00049	.00049	.00049	.00057
.10	.0	.0	.00057	.00057	.00057	.00063	.00046	.00049	.00049	.00046	.00046	.00049	.00049	.00057
.10	.0	.1	.00069	.00069	.00069	.00077	.00057	.00060	.00060	.00058	.00058	.00056	.00056	.00069
.10	.0	.2	.00074	.00074	.00074	.00084	.00062	.00065	.00065	.00062	.00062	.00060	.00060	.00077
.10	.0	.4	.00095	.00095	.00095	.00107	.00080	.00086	.00086	.00080	.00080	.00077	.00077	.00099
.10	.1	.0	.00047	.00047	.00048	.00055	.00036	.00040	.00041	.00039	.00040	.00041	.00042	.00048
.10	.1	.1	.00058	.00058	.00059	.00069	.00046	.00050	.00050	.00051	.00052	.00048	.00049	.00062
.10	.1	.2	.00075	.00075	.00076	.00088	.00060	.00066	.00066	.00065	.00065	.00062	.00063	.00082
.10	.1	.4	.00084	.00084	.00084	.00098	.00069	.00076	.00076	.00070	.00071	.00069	.00070	.00092
.10	.2	.0	.00049	.00049	.00049	.00056	.00038	.00042	.00042	.00042	.00042	.00042	.00042	.00048
.10	.2	.1	.00057	.00058	.00059	.00068	.00044	.00049	.00051	.00049	.00050	.00045	.00047	.00062
.10	.2	.2	.00073	.00073	.00074	.00086	.00057	.00063	.00064	.00061	.00061	.00060	.00061	.00080
.10	.2	.4	.00087	.00087	.00088	.00103	.00069	.00079	.00080	.00075	.00076	.00074	.00075	.00097

Table A.12: Standard Error Results for Accounts Receivable Model M4

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.0428	.0426	.0408	.0260	.0334	.0323	.0303	.0308	.0288	.0344	.0324	.0294
.1	.2	.1	.0566	.0563	.0546	.0407	.0445	.0422	.0402	.0453	.0433	.0382	.0362	.0520
.1	.2	.2	.0729	.0727	.0709	.0588	.0579	.0550	.0530	.0619	.0598	.0442	.0422	.0712
.1	.2	.3	.0842	.0840	.0823	.0718	.0666	.0638	.0618	.0731	.0712	.0486	.0466	.0775
.1	.5	.0	.0376	.0365	.0326	.0204	.0286	.0265	.0216	.0274	.0224	.0251	.0202	.0202
.1	.5	.1	.0535	.0524	.0485	.0363	.0414	.0377	.0326	.0437	.0386	.0309	.0258	.0473
.1	.5	.2	.0664	.0653	.0613	.0499	.0518	.0472	.0421	.0568	.0518	.0373	.0323	.0607
.1	.5	.3	.0805	.0794	.0755	.0656	.0628	.0578	.0528	.0709	.0659	.0502	.0452	.0713
.1	.7	.0	.0350	.0331	.0277	.0152	.0259	.0225	.0153	.0258	.0186	.0200	.0128	.0146
.1	.7	.1	.0497	.0478	.0426	.0311	.0378	.0318	.0248	.0411	.0341	.0294	.0224	.0369
.1	.7	.2	.0644	.0627	.0578	.0472	.0498	.0421	.0355	.0559	.0493	.0454	.0388	.0524
.1	.7	.3	.0788	.0770	.0720	.0624	.0611	.0527	.0459	.0701	.0633	.0769	.0700	.0614
.3	.2	.0	.0644	.0630	.0585	.0414	.0508	.0516	.0457	.0463	.0405	.0772	.0714	.0367
.3	.2	.1	.1045	.1031	.0985	.0826	.0872	.0838	.0778	.0870	.0810	.0858	.0798	.0936
.3	.2	.2	.1395	.1381	.1336	.1191	.1190	.1146	.1087	.1212	.1153	.0954	.0895	.1260
.3	.2	.3	.1730	.1715	.1668	.1537	.1491	.1455	.1394	.1532	.1470	.1083	.1021	.1572
.3	.5	.0	.0523	.0464	.0372	.0259	.0350	.0413	.0263	.0374	.0223	.0542	.0392	.0154
.3	.5	.1	.0919	.0860	.0764	.0644	.0704	.0726	.0571	.0779	.0624	.0637	.0482	.0732
.3	.5	.2	.1288	.1232	.1142	.1018	.1044	.1051	.0905	.1139	.0993	.0778	.0632	.1096
.3	.5	.3	.1650	.1592	.1497	.1384	.1365	.1386	.1234	.1484	.1332	.1008	.0856	.1430
.3	.7	.0	.0441	.0348	.0228	.0123	.0246	.0337	.0124	.0318	.0105	.0375	.0162	.0020
.3	.7	.1	.0843	.0754	.0639	.0546	.0611	.0645	.0441	.0729	.0525	.0498	.0295	.0616
.3	.7	.2	.1206	.1114	.0996	.0894	.0933	.0957	.0748	.1080	.0871	.0763	.0554	.0963
.3	.7	.3	.1596	.1505	.1389	.1289	.1284	.1314	.1107	.1452	.1245	.1541	.1334	.1331
.6	.2	.0	.0948	.0907	.0828	.0631	.0729	.0771	.0651	.0688	.0568	.1377	.1257	.0586
.6	.2	.1	.1699	.1656	.1573	.1370	.1426	.1394	.1269	.1428	.1303	.1519	.1394	.1428
.6	.2	.2	.2325	.2284	.2206	.1994	.2012	.1967	.1848	.2026	.1906	.1686	.1566	.2043
.6	.2	.3	.3010	.2970	.2895	.2671	.2645	.2619	.2504	.2666	.2551	.1925	.1810	.2695
.6	.5	.0	.0721	.0567	.0412	.0306	.0397	.0582	.0272	.0519	.0210	.0934	.0624	.0206
.6	.5	.1	.1469	.1320	.1169	.1029	.1095	.1207	.0907	.1260	.0960	.1110	.0810	.1074
.6	.5	.2	.2172	.2024	.1877	.1715	.1753	.1863	.1568	.1932	.1637	.1353	.1059	.1767
.6	.5	.3	.2799	.2647	.2495	.2318	.2324	.2472	.2167	.2514	.2210	.1722	.1418	.2357
.6	.7	.0	.0572	.0347	.0161	.0072	.0194	.0456	.0045	.0409	-.0002	.0634	.0223	-.0025
.6	.7	.1	.1317	.1085	.0894	.0804	.0872	.1079	.0656	.1157	.0734	.0830	.0407	.0835
.6	.7	.2	.2033	.1797	.1606	.1483	.1534	.1753	.1325	.1836	.1408	.1278	.0850	.1533
.6	.7	.3	.2642	.2408	.2218	.2076	.2098	.2350	.1926	.2404	.1979	.2525	.2100	.2119

Table A.13: Average Results for Inventory Model M1

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.956
.1	.2	.1	1.000	1.000	1.000	.992	1.000	1.000	.996	1.000	1.000	.986	.972	.934
.1	.2	.2	1.000	1.000	.994	.914	.990	.978	.948	.978	.946	.926	.888	.974
.1	.2	.3	.990	.988	.984	.954	.960	.954	.934	.956	.956	.858	.814	.974
.1	.5	.0	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	1.000	1.000	.992	.934
.1	.5	.1	1.000	1.000	1.000	.954	1.000	1.000	.970	1.000	.992	.998	.930	.908
.1	.5	.2	1.000	1.000	.994	.914	1.000	.976	.898	1.000	.950	.910	.802	.952
.1	.5	.3	1.000	1.000	.986	.950	.958	.930	.884	.958	.948	.852	.762	.970
.1	.7	.0	1.000	1.000	1.000	1.000	1.000	1.000	.994	1.000	1.000	1.000	.982	.954
.1	.7	.1	1.000	1.000	1.000	.942	1.000	1.000	.952	1.000	.990	1.000	.906	.884
.1	.7	.2	1.000	1.000	.996	.926	1.000	.964	.878	1.000	.974	.954	.850	.946
.1	.7	.3	1.000	1.000	.982	.952	.968	.952	.822	.952	.952	1.000	.978	.952
.3	.2	.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000	1.000	.936
.3	.2	.1	1.000	1.000	.998	.942	.990	.984	.958	.980	.942	.998	.990	.954
.3	.2	.2	.986	.982	.974	.930	.946	.942	.910	.952	.926	.910	.820	.976
.3	.2	.3	.986	.984	.968	.934	.940	.916	.882	.948	.926	.646	.530	.960
.3	.5	.0	1.000	1.000	1.000	.992	1.000	1.000	.992	1.000	.986	1.000	.996	.908
.3	.5	.1	1.000	.998	.988	.978	.990	.996	.942	.992	.964	1.000	.926	.952
.3	.5	.2	.992	.984	.976	.954	.972	.972	.914	.982	.948	.900	.668	.974
.3	.5	.3	.992	.990	.966	.928	.946	.946	.888	.986	.924	.728	.526	.966
.3	.7	.0	1.000	1.000	1.000	.998	1.000	1.000	.992	1.000	.990	1.000	.992	.938
.3	.7	.1	1.000	1.000	.978	.948	.994	1.000	.916	1.000	.942	.998	.776	.932
.3	.7	.2	.996	.990	.956	.944	.962	.978	.852	.994	.934	.926	.664	.954
.3	.7	.3	.998	.984	.966	.936	.960	.950	.850	.998	.938	.998	.952	.948
.6	.2	.0	1.000	1.000	1.000	.994	1.000	1.000	.990	1.000	.954	1.000	1.000	.934
.6	.2	.1	.998	.996	.980	.942	.968	.968	.914	.970	.922	1.000	1.000	.960
.6	.2	.2	.992	.990	.984	.942	.956	.952	.884	.960	.912	.904	.750	.960
.6	.2	.3	.996	.994	.984	.954	.958	.944	.882	.968	.910	.396	.250	.966
.6	.5	.0	1.000	1.000	.990	.978	1.000	1.000	.968	1.000	.930	1.000	1.000	.920
.6	.5	.1	1.000	.996	.972	.940	.966	.994	.878	.996	.906	1.000	.884	.946
.6	.5	.2	.998	.994	.970	.936	.954	.988	.892	.998	.918	.876	.454	.950
.6	.5	.3	.996	.982	.968	.930	.950	.970	.880	.986	.908	.572	.292	.950
.6	.7	.0	1.000	1.000	.996	.992	1.000	1.000	.974	1.000	.952	1.000	.990	.956
.6	.7	.1	1.000	.998	.960	.940	.960	1.000	.886	1.000	.912	1.000	.686	.940
.6	.7	.2	1.000	.996	.958	.934	.962	.998	.850	1.000	.906	.912	.482	.942
.6	.7	.3	1.000	.984	.958	.938	.950	.988	.880	.996	.910	1.000	.944	.946

Table A.14: Coverage Results for Inventory Model M1

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.00006	.00006	.00007	.00004	.00004	.00005	.00005	.00004	.00005	.00010	.00011	.00019
.1	.2	.1	.00029	.00029	.00029	.00031	.00020	.00018	.00019	.00027	.00027	.00013	.00014	.00076
.1	.2	.2	.00053	.00053	.00054	.00061	.00038	.00037	.00038	.00049	.00049	.00020	.00021	.00100
.1	.2	.3	.00054	.00054	.00054	.00064	.00039	.00041	.00041	.00049	.00049	.00025	.00025	.00081
.1	.5	.0	.00003	.00003	.00005	.00005	.00002	.00003	.00004	.00002	.00003	.00006	.00008	.00016
.1	.5	.1	.00030	.00030	.00032	.00033	.00021	.00020	.00022	.00028	.00031	.00014	.00016	.00099
.1	.5	.2	.00048	.00049	.00051	.00055	.00033	.00034	.00036	.00045	.00048	.00027	.00029	.00101
.1	.5	.3	.00063	.00063	.00064	.00072	.00045	.00051	.00052	.00057	.00058	.00053	.00054	.00086
.1	.7	.0	.00002	.00003	.00005	.00006	.00002	.00003	.00005	.00001	.00004	.00005	.00007	.00019
.1	.7	.1	.00027	.00027	.00030	.00032	.00018	.00018	.00021	.00025	.00028	.00023	.00026	.00064
.1	.7	.2	.00045	.00046	.00048	.00052	.00031	.00034	.00037	.00042	.00045	.00053	.00053	.00082
.1	.7	.3	.00063	.00063	.00064	.00072	.00044	.00054	.00055	.00056	.00057	.00058	.00059	.00081
.3	.2	.0	.00012	.00012	.00013	.00009	.00009	.00008	.00010	.00009	.00011	.00015	.00017	.00021
.3	.2	.1	.00062	.00063	.00065	.00064	.00052	.00046	.00049	.00055	.00058	.00023	.00025	.00064
.3	.2	.2	.00100	.00101	.00103	.00105	.00083	.00081	.00084	.00087	.00090	.00034	.00036	.00087
.3	.2	.3	.00131	.00133	.00138	.00140	.00111	.00112	.00119	.00111	.00118	.00048	.00052	.00120
.3	.5	.0	.00009	.00010	.00013	.00010	.00007	.00006	.00010	.00006	.00011	.00012	.00017	.00018
.3	.5	.1	.00054	.00055	.00058	.00050	.00044	.00040	.00044	.00048	.00052	.00020	.00025	.00070
.3	.5	.2	.00093	.00095	.00098	.00096	.00078	.00077	.00083	.00081	.00086	.00036	.00041	.00089
.3	.5	.3	.00138	.00142	.00150	.00149	.00118	.00125	.00136	.00117	.00128	.00087	.00097	.00130
.3	.7	.0	.00006	.00007	.00012	.00011	.00006	.00004	.00010	.00004	.00009	.00009	.00016	.00011
.3	.7	.1	.00060	.00062	.00068	.00059	.00051	.00045	.00053	.00054	.00061	.00025	.00031	.00089
.3	.7	.2	.00093	.00097	.00104	.00096	.00080	.00082	.00092	.00081	.00092	.00070	.00080	.00100
.3	.7	.3	.00144	.00150	.00159	.00155	.00124	.00142	.00157	.00122	.00136	.00130	.00145	.00151
.6	.2	.0	.00019	.00021	.00025	.00017	.00017	.00014	.00019	.00014	.00020	.00022	.00029	.00026
.6	.2	.1	.00091	.00092	.00095	.00088	.00080	.00071	.00075	.00079	.00083	.00027	.00033	.00089
.6	.2	.2	.00139	.00141	.00146	.00136	.00121	.00119	.00126	.00119	.00126	.00039	.00045	.00131
.6	.2	.3	.00216	.00219	.00224	.00204	.00184	.00194	.00201	.00175	.00183	.00071	.00078	.00194
.6	.5	.0	.00015	.00020	.00026	.00020	.00016	.00010	.00021	.00010	.00021	.00019	.00032	.00025
.6	.5	.1	.00105	.00113	.00122	.00104	.00099	.00084	.00100	.00090	.00106	.00033	.00049	.00116
.6	.5	.2	.00168	.00182	.00195	.00176	.00159	.00148	.00173	.00140	.00165	.00060	.00078	.00174
.6	.5	.3	.00223	.00232	.00242	.00220	.00198	.00205	.00223	.00182	.00199	.00129	.00144	.00213
.6	.7	.0	.00009	.00016	.00024	.00020	.00014	.00006	.00020	.00006	.00020	.00014	.00031	.00020
.6	.7	.1	.00097	.00107	.00116	.00098	.00095	.00079	.00098	.00085	.00103	.00034	.00050	.00115
.6	.7	.2	.00172	.00190	.00202	.00177	.00166	.00158	.00188	.00143	.00171	.00114	.00137	.00181
.6	.7	.3	.00216	.00233	.00244	.00220	.00199	.00210	.00238	.00176	.00202	.00188	.00215	.00214

Table A.15: Standard Error Results for Inventory Model M1

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.0542	.0535	.0501	.0339	.0402	.0412	.0372	.0393	.0353	.0466	.0426	.0367
.1	.2	.1	.0654	.0648	.0614	.0471	.0494	.0493	.0452	.0521	.0481	.0489	.0448	.0532
.1	.2	.2	.0763	.0757	.0722	.0594	.0583	.0576	.0535	.0633	.0592	.0532	.0491	.0665
.1	.2	.3	.0900	.0894	.0862	.0755	.0696	.0689	.0651	.0771	.0734	.0608	.0570	.0800
.1	.5	.0	.0455	.0428	.0353	.0223	.0316	.0323	.0221	.0333	.0231	.0335	.0232	.0219
.1	.5	.1	.0595	.0569	.0496	.0373	.0430	.0425	.0326	.0480	.0381	.0401	.0302	.0412
.1	.5	.2	.0708	.0682	.0611	.0501	.0523	.0503	.0406	.0600	.0503	.0462	.0365	.0540
.1	.5	.3	.0838	.0812	.0741	.0645	.0626	.0604	.0507	.0733	.0636	.0586	.0489	.0665
.1	.7	.0	.0394	.0350	.0256	.0128	.0258	.0256	.0118	.0290	.0152	.0253	.0115	.0134
.1	.7	.1	.0512	.0467	.0370	.0261	.0352	.0329	.0187	.0416	.0274	.0332	.0190	.0279
.1	.7	.2	.0647	.0603	.0507	.0411	.0461	.0421	.0281	.0559	.0419	.0480	.0340	.0420
.1	.7	.3	.0749	.0705	.0608	.0522	.0543	.0493	.0351	.0664	.0523	.0731	.0590	.0506
.3	.2	.0	.0961	.0925	.0841	.0652	.0728	.0795	.0674	.0713	.0593	.1077	.0956	.0622
.3	.2	.1	.1283	.1248	.1165	.0991	.1021	.1058	.0940	.1044	.0926	.1156	.1038	.1024
.3	.2	.2	.1624	.1586	.1497	.1344	.1327	.1362	.1235	.1387	.1260	.1268	.1141	.1368
.3	.2	.3	.1931	.1895	.1809	.1669	.1609	.1648	.1526	.1684	.1562	.1422	.1300	.1679
.3	.5	.0	.0717	.0579	.0410	.0307	.0404	.0577	.0270	.0525	.0218	.0728	.0421	.0223
.3	.5	.1	.1067	.0933	.0767	.0655	.0719	.0860	.0560	.0889	.0589	.0837	.0536	.0662
.3	.5	.2	.1413	.1278	.1112	.1000	.1033	.1167	.0867	.1232	.0932	.1000	.0699	.1033
.3	.5	.3	.1708	.1571	.1403	.1297	.1299	.1442	.1137	.1523	.1218	.1204	.0899	.1327
.3	.7	.0	.0562	.0351	.0146	.0058	.0201	.0434	.0018	.0408	-.0009	.0501	.0085	-.0017
.3	.7	.1	.0931	.0713	.0500	.0434	.0519	.0722	.0290	.0794	.0363	.0639	.0207	.0424
.3	.7	.2	.1251	.1045	.0840	.0768	.0822	.0998	.0587	.1116	.0705	.0897	.0486	.0790
.3	.7	.3	.1582	.1370	.1163	.1090	.1116	.1299	.0881	.1439	.1021	.1527	.1109	.1118
.6	.2	.0	.1518	.1421	.1282	.1053	.1138	.1286	.1049	.1145	.0908	.1891	.1654	.1026
.6	.2	.1	.2172	.2077	.1939	.1702	.1743	.1842	.1610	.1784	.1552	.2052	.1820	.1722
.6	.2	.2	.2746	.2643	.2498	.2261	.2267	.2375	.2126	.2344	.2095	.2245	.1997	.2284
.6	.2	.3	.3330	.3232	.3093	.2848	.2815	.2933	.2696	.2896	.2659	.2493	.2256	.2859
.6	.5	.0	.1090	.0749	.0486	.0397	.0490	.0908	.0304	.0813	.0209	.1269	.0666	.0317
.6	.5	.1	.1739	.1398	.1136	.1014	.1084	.1464	.0861	.1461	.0858	.1466	.0864	.1017
.6	.5	.2	.2360	.2026	.1767	.1623	.1673	.2046	.1453	.2063	.1470	.1731	.1139	.1651
.6	.5	.3	.3003	.2671	.2411	.2253	.2271	.2672	.2080	.2675	.2083	.2165	.1574	.2278
.6	.7	.0	.0802	.0293	-.0029	-.0079	.0063	.0653	-.0178	.0592	-.0238	.0855	.0025	-.0151
.6	.7	.1	.1473	.0962	.0640	.0594	.0674	.1226	.0392	.1271	.0438	.1082	.0249	.0581
.6	.7	.2	.2082	.1565	.1241	.1176	.1238	.1800	.0959	.1868	.1026	.1512	.0671	.1194
.6	.7	.3	.2667	.2150	.1826	.1739	.1784	.2375	.1534	.2426	.1585	.2548	.1707	.1770

Table A.16: Average Results for Inventory Model M2

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	1.000	1.000	1.000	.992	1.000	1.000	.994	1.000	.996	1.000	.994	.972
.1	.2	.1	1.000	1.000	.996	.964	.998	.996	.958	1.000	.966	.988	.954	.948
.1	.2	.2	1.000	1.000	1.000	.926	.978	.970	.926	.974	.954	.964	.914	.980
.1	.2	.3	.988	.988	.984	.938	.952	.940	.918	.962	.948	.934	.882	.958
.1	.5	.0	1.000	1.000	1.000	.990	1.000	1.000	.980	1.000	.990	1.000	.972	.956
.1	.5	.1	1.000	1.000	1.000	.938	1.000	1.000	.948	1.000	.976	1.000	.914	.934
.1	.5	.2	1.000	1.000	.986	.940	.998	.996	.912	1.000	.956	.984	.888	.958
.1	.5	.3	1.000	1.000	.988	.964	.988	.986	.882	.996	.964	.964	.846	.960
.1	.7	.0	1.000	1.000	1.000	1.000	1.000	1.000	.966	1.000	.992	1.000	.966	.966
.1	.7	.1	1.000	1.000	.996	.944	1.000	1.000	.902	1.000	.966	1.000	.900	.898
.1	.7	.2	1.000	1.000	.986	.896	1.000	1.000	.832	1.000	.948	1.000	.894	.902
.1	.7	.3	1.000	.998	.958	.914	.984	.928	.778	1.000	.924	1.000	.954	.918
.3	.2	.0	1.000	1.000	.998	.988	.998	1.000	.982	1.000	.952	1.000	.998	.946
.3	.2	.1	.998	.996	.982	.944	.970	.992	.938	.976	.918	1.000	.988	.940
.3	.2	.2	1.000	1.000	.992	.944	.966	.976	.906	.978	.916	.980	.914	.968
.3	.2	.3	.996	.994	.984	.946	.946	.960	.902	.962	.914	.896	.794	.956
.3	.5	.0	1.000	1.000	.996	.974	1.000	1.000	.956	1.000	.924	1.000	.988	.932
.3	.5	.1	1.000	.998	.974	.958	.986	1.000	.898	1.000	.912	1.000	.914	.936
.3	.5	.2	1.000	1.000	.984	.966	.986	1.000	.904	1.000	.940	.998	.786	.966
.3	.5	.3	.998	.994	.964	.926	.964	.990	.862	.996	.902	.958	.712	.934
.3	.7	.0	1.000	1.000	.980	.982	1.000	1.000	.944	1.000	.936	1.000	.964	.934
.3	.7	.1	1.000	1.000	.968	.946	.998	1.000	.876	1.000	.912	1.000	.806	.914
.3	.7	.2	1.000	1.000	.964	.960	.986	1.000	.828	1.000	.922	1.000	.790	.946
.3	.7	.3	1.000	.998	.970	.970	.974	.998	.842	1.000	.938	1.000	.966	.962
.6	.2	.0	1.000	1.000	.996	.974	.996	1.000	.958	1.000	.866	1.000	1.000	.938
.6	.2	.1	1.000	1.000	.990	.952	.964	.994	.926	.980	.886	1.000	.996	.954
.6	.2	.2	1.000	.996	.982	.942	.960	.984	.886	.978	.870	.998	.876	.952
.6	.2	.3	1.000	.996	.986	.934	.932	.976	.898	.974	.894	.882	.652	.944
.6	.5	.0	1.000	1.000	.976	.960	.994	1.000	.918	1.000	.840	1.000	.996	.936
.6	.5	.1	1.000	.998	.960	.950	.964	1.000	.890	1.000	.884	1.000	.914	.934
.6	.5	.2	1.000	.996	.954	.936	.952	1.000	.870	1.000	.882	.996	.724	.930
.6	.5	.3	1.000	.996	.978	.964	.974	.998	.916	.998	.920	.990	.610	.970
.6	.7	.0	1.000	1.000	.982	.982	.998	1.000	.926	1.000	.894	1.000	.976	.942
.6	.7	.1	1.000	.996	.956	.946	.980	1.000	.860	1.000	.888	1.000	.792	.934
.6	.7	.2	1.000	.996	.944	.946	.964	1.000	.844	1.000	.902	1.000	.680	.942
.6	.7	.3	1.000	.994	.954	.950	.972	1.000	.876	1.000	.918	1.000	.942	.954

Table A.17: Coverage Results for Inventory Model M2

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.00012	.00013	.00015	.00011	.00009	.00010	.00012	.00009	.00011	.00017	.00020	.00017
.1	.2	.1	.00034	.00034	.00037	.00040	.00024	.00024	.00026	.00031	.00034	.00020	.00022	.00057
.1	.2	.2	.00046	.00046	.00047	.00054	.00033	.00034	.00036	.00042	.00043	.00027	.00028	.00066
.1	.2	.3	.00063	.00063	.00064	.00076	.00047	.00050	.00052	.00056	.00057	.00035	.00037	.00081
.1	.5	.0	.00008	.00009	.00012	.00011	.00005	.00007	.00011	.00005	.00009	.00012	.00016	.00017
.1	.5	.1	.00029	.00031	.00035	.00035	.00020	.00021	.00026	.00027	.00032	.00019	.00024	.00053
.1	.5	.2	.00043	.00044	.00047	.00049	.00030	.00033	.00037	.00039	.00043	.00028	.00031	.00064
.1	.5	.3	.00059	.00059	.00063	.00069	.00042	.00050	.00053	.00053	.00057	.00050	.00054	.00075
.1	.7	.0	.00005	.00006	.00010	.00010	.00003	.00005	.00010	.00003	.00008	.00007	.00011	.00017
.1	.7	.1	.00028	.00029	.00034	.00037	.00018	.00020	.00026	.00027	.00032	.00021	.00027	.00046
.1	.7	.2	.00042	.00044	.00049	.00051	.00028	.00033	.00039	.00039	.00045	.00042	.00048	.00068
.1	.7	.3	.00066	.00067	.00074	.00078	.00045	.00055	.00063	.00059	.00067	.00061	.00069	.00084
.3	.2	.0	.00026	.00028	.00033	.00026	.00022	.00020	.00026	.00020	.00027	.00029	.00035	.00033
.3	.2	.1	.00070	.00071	.00076	.00074	.00059	.00054	.00061	.00060	.00066	.00036	.00043	.00079
.3	.2	.2	.00104	.00105	.00109	.00110	.00086	.00087	.00092	.00090	.00094	.00051	.00057	.00099
.3	.2	.3	.00146	.00148	.00155	.00155	.00124	.00128	.00136	.00123	.00131	.00075	.00083	.00138
.3	.5	.0	.00017	.00022	.00030	.00024	.00017	.00012	.00026	.00012	.00025	.00021	.00035	.00027
.3	.5	.1	.00064	.00070	.00079	.00066	.00057	.00050	.00064	.00056	.00070	.00035	.00049	.00086
.3	.5	.2	.00093	.00096	.00103	.00094	.00079	.00079	.00090	.00081	.00091	.00052	.00064	.00098
.3	.5	.3	.00134	.00144	.00157	.00150	.00120	.00122	.00144	.00114	.00136	.00090	.00111	.00147
.3	.7	.0	.00013	.00019	.00029	.00024	.00016	.00010	.00026	.00009	.00025	.00016	.00032	.00024
.3	.7	.1	.00061	.00068	.00078	.00071	.00055	.00049	.00065	.00054	.00070	.00035	.00051	.00087
.3	.7	.2	.00100	.00111	.00124	.00108	.00091	.00091	.00114	.00087	.00109	.00079	.00097	.00124
.3	.7	.3	.00130	.00142	.00153	.00140	.00117	.00128	.00152	.00110	.00132	.00117	.00140	.00145
.6	.2	.0	.00038	.00043	.00052	.00040	.00036	.00029	.00043	.00028	.00041	.00036	.00052	.00046
.6	.2	.1	.00098	.00105	.00116	.00103	.00092	.00080	.00097	.00083	.00100	.00050	.00066	.00107
.6	.2	.2	.00157	.00167	.00179	.00161	.00143	.00136	.00157	.00130	.00150	.00071	.00093	.00157
.6	.2	.3	.00197	.00207	.00218	.00196	.00176	.00177	.00196	.00162	.00182	.00089	.00107	.00189
.6	.5	.0	.00027	.00045	.00059	.00045	.00039	.00021	.00052	.00022	.00053	.00032	.00069	.00050
.6	.5	.1	.00107	.00129	.00146	.00118	.00114	.00088	.00125	.00092	.00129	.00055	.00095	.00137
.6	.5	.2	.00167	.00196	.00216	.00187	.00173	.00148	.00195	.00139	.00184	.00086	.00127	.00197
.6	.5	.3	.00194	.00224	.00244	.00217	.00194	.00178	.00226	.00157	.00202	.00129	.00171	.00211
.6	.7	.0	.00023	.00040	.00053	.00044	.00035	.00017	.00047	.00018	.00046	.00026	.00060	.00044
.6	.7	.1	.00116	.00146	.00167	.00145	.00130	.00098	.00148	.00100	.00149	.00064	.00111	.00160
.6	.7	.2	.00172	.00207	.00226	.00190	.00182	.00159	.00212	.00143	.00194	.00133	.00179	.00208
.6	.7	.3	.00191	.00226	.00247	.00213	.00198	.00186	.00241	.00155	.00208	.00166	.00220	.00220

Table A.18: Standard Error Results for Inventory Model M2

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.0638	.0635	.0617	.0446	.0477	.0487	.0466	.0473	.0452	.0543	.0522	.0487
.1	.2	.1	.0745	.0742	.0723	.0573	.0566	.0568	.0546	.0589	.0567	.0588	.0566	.0626
.1	.2	.2	.0854	.0851	.0834	.0707	.0657	.0652	.0632	.0706	.0687	.0635	.0615	.0747
.1	.2	.3	.0972	.0970	.0952	.0847	.0755	.0751	.0731	.0825	.0805	.0708	.0688	.0865
.1	.5	.0	.0525	.0514	.0477	.0327	.0388	.0376	.0328	.0388	.0340	.0395	.0346	.0351
.1	.5	.1	.0622	.0611	.0571	.0434	.0468	.0440	.0389	.0499	.0448	.0437	.0386	.0500
.1	.5	.2	.0744	.0733	.0695	.0573	.0571	.0533	.0484	.0627	.0578	.0523	.0474	.0636
.1	.5	.3	.0844	.0834	.0796	.0695	.0651	.0607	.0560	.0734	.0687	.0617	.0570	.0719
.1	.7	.0	.0440	.0422	.0372	.0242	.0321	.0285	.0217	.0327	.0259	.0297	.0229	.0255
.1	.7	.1	.0554	.0535	.0484	.0364	.0415	.0358	.0288	.0451	.0381	.0377	.0306	.0409
.1	.7	.2	.0655	.0636	.0584	.0476	.0500	.0426	.0356	.0563	.0493	.0503	.0433	.0505
.1	.7	.3	.0764	.0746	.0696	.0601	.0592	.0506	.0437	.0679	.0610	.0746	.0677	.0588
.3	.2	.0	.1231	.1217	.1171	.0962	.0994	.1037	.0977	.0941	.0881	.1288	.1227	.0949
.3	.2	.1	.1536	.1521	.1473	.1286	.1270	.1293	.1230	.1253	.1190	.1387	.1324	.1301
.3	.2	.2	.1810	.1795	.1748	.1581	.1523	.1541	.1479	.1527	.1466	.1503	.1441	.1588
.3	.2	.3	.2077	.2063	.2018	.1871	.1770	.1791	.1731	.1793	.1734	.1641	.1581	.1861
.3	.5	.0	.0896	.0838	.0744	.0577	.0653	.0731	.0579	.0675	.0523	.0867	.0715	.0552
.3	.5	.1	.1207	.1148	.1054	.0895	.0937	.0988	.0835	.0997	.0844	.0994	.0841	.0938
.3	.5	.2	.1503	.1444	.1349	.1208	.1211	.1252	.1098	.1301	.1147	.1143	.0989	.1252
.3	.5	.3	.1767	.1711	.1622	.1496	.1461	.1498	.1353	.1568	.1424	.1337	.1193	.1528
.3	.7	.0	.0681	.0588	.0469	.0351	.0436	.0530	.0317	.0509	.0296	.0601	.0388	.0285
.3	.7	.1	.0989	.0894	.0776	.0659	.0719	.0770	.0558	.0838	.0626	.0726	.0514	.0705
.3	.7	.2	.1295	.1202	.1084	.0973	.1004	.1038	.0826	.1150	.0939	.0976	.0765	.1029
.3	.7	.3	.1556	.1465	.1349	.1249	.1249	.1274	.1067	.1415	.1208	.1503	.1296	.1295
.6	.2	.0	.2075	.2035	.1960	.1667	.1709	.1799	.1684	.1596	.1481	.2310	.2195	.1657
.6	.2	.1	.2621	.2576	.2493	.2204	.2207	.2274	.2147	.2139	.2012	.2472	.2345	.2214
.6	.2	.2	.3152	.3112	.3036	.2749	.2713	.2770	.2655	.2667	.2551	.2682	.2566	.2755
.6	.2	.3	.3646	.3605	.3526	.3239	.3168	.3246	.3126	.3141	.3021	.2933	.2813	.3228
.6	.5	.0	.1438	.1286	.1133	.0934	.1017	.1224	.0920	.1105	.0800	.1542	.1237	.0913
.6	.5	.1	.2007	.1857	.1707	.1500	.1552	.1720	.1420	.1674	.1375	.1740	.1441	.1531
.6	.5	.2	.2584	.2438	.2291	.2083	.2102	.2265	.1972	.2241	.1948	.2037	.1744	.2118
.6	.5	.3	.3065	.2919	.2772	.2566	.2554	.2736	.2443	.2717	.2424	.2372	.2079	.2595
.6	.7	.0	.1026	.0798	.0612	.0492	.0575	.0850	.0436	.0776	.0362	.1031	.0617	.0432
.6	.7	.1	.1582	.1353	.1166	.1032	.1098	.1329	.0913	.1351	.0936	.1247	.0831	.1059
.6	.7	.2	.2130	.1892	.1699	.1560	.1609	.1847	.1416	.1902	.1471	.1651	.1220	.1607
.6	.7	.3	.2675	.2441	.2252	.2109	.2129	.2382	.1960	.2433	.2011	.2555	.2133	.2153

Table A.19: Average Results for Inventory Model M3

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	1.000	1.000	1.000	.980	.996	.994	.986	1.000	.992	.994	.988	.994
.1	.2	.1	1.000	1.000	1.000	.918	.980	.976	.958	.982	.944	.978	.962	.986
.1	.2	.2	1.000	1.000	.998	.938	.956	.956	.936	.964	.946	.934	.928	.978
.1	.2	.3	.998	.998	.998	.958	.958	.954	.934	.980	.972	.938	.918	.980
.1	.5	.0	1.000	1.000	1.000	.990	1.000	1.000	.994	1.000	1.000	1.000	.990	.960
.1	.5	.1	1.000	1.000	.996	.968	1.000	.994	.954	1.000	.978	.982	.944	.944
.1	.5	.2	1.000	1.000	.988	.942	.990	.952	.908	.986	.948	.954	.900	.966
.1	.5	.3	.996	.994	.988	.962	.974	.934	.880	.972	.954	.936	.874	.966
.1	.7	.0	1.000	1.000	1.000	.994	1.000	1.000	.988	1.000	1.000	1.000	.986	.942
.1	.7	.1	1.000	1.000	1.000	.936	1.000	1.000	.912	1.000	.992	1.000	.942	.932
.1	.7	.2	1.000	1.000	.990	.918	1.000	.936	.834	1.000	.952	.976	.902	.942
.1	.7	.3	1.000	1.000	.984	.946	.972	.950	.774	.950	.950	1.000	.980	.948
.3	.2	.0	1.000	1.000	.998	.960	.982	.992	.972	.974	.924	1.000	1.000	.956
.3	.2	.1	.996	.994	.984	.944	.962	.970	.940	.958	.924	.992	.976	.952
.3	.2	.2	.992	.992	.988	.954	.958	.962	.934	.948	.926	.982	.956	.968
.3	.2	.3	.992	.992	.986	.950	.934	.932	.908	.938	.912	.900	.850	.954
.3	.5	.0	1.000	1.000	.994	.976	.996	1.000	.960	1.000	.926	1.000	.986	.918
.3	.5	.1	1.000	.998	.986	.948	.978	.990	.920	.986	.918	.996	.930	.946
.3	.5	.2	.996	.994	.982	.942	.964	.982	.904	.982	.926	.970	.844	.966
.3	.5	.3	.992	.986	.972	.928	.930	.948	.894	.970	.918	.910	.796	.948
.3	.7	.0	1.000	1.000	.998	.990	1.000	1.000	.976	1.000	.972	1.000	.988	.932
.3	.7	.1	1.000	1.000	.984	.956	.984	.994	.896	.998	.928	.996	.878	.934
.3	.7	.2	.998	.996	.974	.950	.970	.978	.874	.996	.938	.968	.822	.960
.3	.7	.3	.996	.984	.962	.936	.954	.948	.838	.996	.932	.996	.956	.958
.6	.2	.0	1.000	1.000	1.000	.972	.986	.998	.980	.962	.860	1.000	1.000	.964
.6	.2	.1	1.000	.998	.996	.948	.954	.978	.930	.938	.848	.998	.998	.948
.6	.2	.2	.998	.994	.988	.958	.958	.970	.936	.956	.892	.976	.934	.962
.6	.2	.3	.998	.996	.996	.956	.946	.960	.916	.934	.878	.884	.794	.956
.6	.5	.0	1.000	1.000	.992	.970	.986	1.000	.948	1.000	.864	1.000	1.000	.948
.6	.5	.1	.998	.996	.984	.940	.970	.996	.898	.992	.892	1.000	.942	.950
.6	.5	.2	1.000	.996	.984	.956	.972	.994	.916	.994	.922	.980	.832	.966
.6	.5	.3	.998	.992	.974	.956	.964	.986	.916	.988	.922	.928	.710	.962
.6	.7	.0	1.000	1.000	.994	.982	.998	1.000	.962	1.000	.914	1.000	.992	.954
.6	.7	.1	1.000	.992	.968	.952	.968	.996	.884	.998	.898	1.000	.850	.944
.6	.7	.2	1.000	.992	.970	.944	.966	.996	.884	.996	.914	.986	.754	.960
.6	.7	.3	.998	.994	.976	.960	.966	.992	.894	.998	.934	.998	.966	.964

Table A.20: Coverage Results for Inventory Model M3

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.00020	.00020	.00021	.00021	.00014	.00016	.00017	.00014	.00015	.00025	.00026	.00017
.1	.2	.1	.00039	.00039	.00040	.00046	.00029	.00030	.00031	.00034	.00035	.00031	.00031	.00047
.1	.2	.2	.00052	.00052	.00052	.00062	.00039	.00041	.00041	.00046	.00046	.00035	.00035	.00060
.1	.2	.3	.00058	.00058	.00059	.00070	.00045	.00048	.00049	.00051	.00051	.00039	.00040	.00070
.1	.5	.0	.00014	.00014	.00016	.00012	.00009	.00012	.00013	.00010	.00012	.00017	.00019	.00021
.1	.5	.1	.00028	.00028	.00030	.00030	.00019	.00021	.00023	.00025	.00027	.00020	.00023	.00050
.1	.5	.2	.00052	.00052	.00054	.00061	.00038	.00041	.00043	.00047	.00049	.00038	.00039	.00071
.1	.5	.3	.00056	.00056	.00058	.00066	.00041	.00047	.00049	.00050	.00052	.00047	.00049	.00074
.1	.7	.0	.00009	.00010	.00011	.00009	.00005	.00008	.00010	.00007	.00008	.00010	.00011	.00019
.1	.7	.1	.00030	.00030	.00033	.00033	.00020	.00022	.00025	.00027	.00030	.00023	.00026	.00053
.1	.7	.2	.00051	.00051	.00053	.00056	.00035	.00040	.00043	.00046	.00048	.00046	.00049	.00070
.1	.7	.3	.00064	.00064	.00067	.00074	.00045	.00054	.00057	.00057	.00060	.00059	.00062	.00090
.3	.2	.0	.00040	.00040	.00042	.00039	.00031	.00032	.00034	.00029	.00030	.00041	.00044	.00039
.3	.2	.1	.00082	.00082	.00084	.00084	.00067	.00067	.00069	.00066	.00068	.00055	.00058	.00079
.3	.2	.2	.00101	.00102	.00104	.00106	.00084	.00086	.00089	.00086	.00088	.00060	.00063	.00095
.3	.2	.3	.00143	.00143	.00144	.00143	.00118	.00126	.00127	.00117	.00118	.00090	.00091	.00126
.3	.5	.0	.00032	.00034	.00039	.00030	.00026	.00025	.00032	.00025	.00031	.00032	.00040	.00041
.3	.5	.1	.00072	.00075	.00079	.00074	.00061	.00059	.00065	.00063	.00069	.00048	.00054	.00079
.3	.5	.2	.00100	.00103	.00108	.00108	.00085	.00087	.00096	.00086	.00094	.00066	.00074	.00098
.3	.5	.3	.00138	.00142	.00148	.00149	.00117	.00127	.00137	.00117	.00126	.00104	.00113	.00132
.3	.7	.0	.00019	.00021	.00025	.00019	.00015	.00015	.00021	.00015	.00020	.00022	.00029	.00029
.3	.7	.1	.00064	.00068	.00073	.00063	.00055	.00053	.00061	.00056	.00065	.00043	.00051	.00081
.3	.7	.2	.00106	.00109	.00115	.00107	.00089	.00098	.00106	.00091	.00099	.00088	.00097	.00107
.3	.7	.3	.00131	.00135	.00142	.00138	.00111	.00129	.00141	.00111	.00122	.00118	.00129	.00132
.6	.2	.0	.00052	.00054	.00059	.00050	.00044	.00043	.00049	.00042	.00047	.00047	.00055	.00051
.6	.2	.1	.00122	.00125	.00131	.00118	.00105	.00103	.00111	.00103	.00111	.00067	.00075	.00118
.6	.2	.2	.00157	.00162	.00169	.00152	.00137	.00138	.00150	.00129	.00139	.00086	.00098	.00147
.6	.2	.3	.00200	.00205	.00213	.00190	.00172	.00180	.00192	.00164	.00174	.00109	.00119	.00184
.6	.5	.0	.00048	.00055	.00064	.00049	.00047	.00039	.00054	.00036	.00050	.00048	.00067	.00058
.6	.5	.1	.00115	.00125	.00136	.00120	.00109	.00097	.00117	.00096	.00116	.00067	.00087	.00125
.6	.5	.2	.00158	.00167	.00175	.00159	.00142	.00141	.00158	.00130	.00146	.00097	.00112	.00155
.6	.5	.3	.00205	.00216	.00228	.00206	.00184	.00189	.00211	.00166	.00188	.00153	.00172	.00199
.6	.7	.0	.00034	.00042	.00049	.00037	.00035	.00027	.00042	.00027	.00042	.00037	.00053	.00047
.6	.7	.1	.00110	.00121	.00131	.00110	.00106	.00095	.00115	.00094	.00114	.00071	.00091	.00124
.6	.7	.2	.00155	.00165	.00173	.00155	.00143	.00144	.00163	.00130	.00148	.00126	.00143	.00156
.6	.7	.3	.00199	.00217	.00230	.00207	.00186	.00193	.00224	.00162	.00190	.00173	.00202	.00202

Table A.21: Standard Error Results for Inventory Model M3

Error Rate	Taints		Average Bound											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.0896	.0893	.0875	.0746	.0681	.0694	.0673	.0726	.0705	.0695	.0675	.0752
.1	.2	.1	.0976	.0973	.0956	.0844	.0747	.0756	.0736	.0813	.0793	.0743	.0723	.0840
.1	.2	.2	.1018	.1015	.0997	.0898	.0782	.0789	.0768	.0862	.0842	.0765	.0745	.0890
.1	.2	.3	.1108	.1105	.1087	.1004	.0858	.0866	.0846	.0956	.0936	.0838	.0817	.0983
.1	.5	.0	.0689	.0678	.0639	.0500	.0515	.0495	.0445	.0551	.0502	.0498	.0449	.0535
.1	.5	.1	.0738	.0727	.0688	.0563	.0558	.0525	.0476	.0609	.0559	.0525	.0476	.0603
.1	.5	.2	.0830	.0819	.0778	.0667	.0636	.0598	.0546	.0710	.0658	.0610	.0558	.0695
.1	.5	.3	.0904	.0893	.0853	.0757	.0699	.0655	.0604	.0792	.0741	.0699	.0648	.0774
.1	.7	.0	.0539	.0521	.0471	.0340	.0399	.0348	.0280	.0427	.0359	.0361	.0293	.0371
.1	.7	.1	.0629	.0610	.0559	.0442	.0475	.0411	.0341	.0526	.0457	.0449	.0379	.0470
.1	.7	.2	.0696	.0676	.0621	.0514	.0533	.0457	.0382	.0603	.0528	.0557	.0482	.0535
.1	.7	.3	.0782	.0765	.0715	.0619	.0605	.0522	.0455	.0695	.0628	.0763	.0695	.0614
.3	.2	.0	.1914	.1900	.1855	.1678	.1600	.1654	.1594	.1583	.1524	.1646	.1586	.1646
.3	.2	.1	.2085	.2070	.2024	.1859	.1756	.1807	.1746	.1756	.1695	.1725	.1664	.1824
.3	.2	.2	.2304	.2289	.2243	.2094	.1958	.2013	.1952	.1979	.1918	.1872	.1811	.2049
.3	.2	.3	.2495	.2481	.2435	.2302	.2135	.2196	.2136	.2172	.2112	.1999	.1939	.2248
.3	.5	.0	.1346	.1288	.1193	.1026	.1053	.1121	.0969	.1105	.0953	.1121	.0968	.1043
.3	.5	.1	.1550	.1494	.1403	.1248	.1245	.1300	.1153	.1317	.1170	.1228	.1081	.1270
.3	.5	.2	.1726	.1669	.1576	.1438	.1409	.1462	.1311	.1504	.1354	.1363	.1212	.1461
.3	.5	.3	.1948	.1891	.1799	.1677	.1617	.1671	.1522	.1731	.1582	.1572	.1423	.1687
.3	.7	.0	.0985	.0891	.0772	.0640	.0704	.0778	.0565	.0802	.0588	.0776	.0563	.0646
.3	.7	.1	.1202	.1109	.0990	.0864	.0909	.0959	.0747	.1037	.0824	.0917	.0705	.0908
.3	.7	.2	.1388	.1297	.1180	.1066	.1088	.1122	.0914	.1234	.1027	.1109	.0901	.1121
.3	.7	.3	.1563	.1472	.1357	.1256	.1255	.1280	.1075	.1422	.1216	.1509	.1304	.1302
.6	.2	.0	.3312	.3269	.3187	.2871	.2823	.2951	.2825	.2728	.2603	.2922	.2797	.2845
.6	.2	.1	.3702	.3660	.3582	.3262	.3186	.3313	.3193	.3106	.2986	.3115	.2995	.3232
.6	.2	.2	.4070	.4029	.3950	.3629	.3528	.3667	.3547	.3475	.3355	.3332	.3212	.3593
.6	.2	.3	.4400	.4358	.4281	.3963	.3836	.3991	.3872	.3804	.3685	.3566	.3447	.3917
.6	.5	.0	.2277	.2130	.1981	.1751	.1791	.1992	.1696	.1879	.1583	.1981	.1685	.1762
.6	.5	.1	.2664	.2516	.2368	.2140	.2154	.2351	.2054	.2272	.1976	.2189	.1893	.2159
.6	.5	.2	.3016	.2867	.2717	.2496	.2487	.2691	.2392	.2631	.2333	.2436	.2138	.2514
.6	.5	.3	.3370	.3223	.3073	.2858	.2824	.3040	.2744	.2980	.2684	.2749	.2452	.2869
.6	.7	.0	.1528	.1298	.1109	.0959	.1033	.1293	.0874	.1260	.0841	.1287	.0868	.0960
.6	.7	.1	.1961	.1728	.1539	.1384	.1441	.1688	.1265	.1697	.1275	.1564	.1142	.1419
.6	.7	.2	.2325	.2092	.1901	.1751	.1791	.2037	.1613	.2073	.1649	.1903	.1479	.1793
.6	.7	.3	.2650	.2414	.2223	.2081	.2103	.2358	.1931	.2411	.1984	.2532	.2105	.2123

Table A.22: Average Results for Inventory Model M4

Error Rate	Taints		Achieved Coverage											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.996	.996	.994	.952	.962	.956	.938	.968	.954	.954	.932	.984
.1	.2	.1	1.000	1.000	.994	.964	.956	.944	.928	.976	.966	.940	.916	.970
.1	.2	.2	1.000	1.000	.996	.952	.932	.918	.900	.952	.946	.906	.880	.956
.1	.2	.3	.988	.988	.988	.952	.918	.906	.884	.962	.952	.898	.872	.936
.1	.5	.0	1.000	1.000	1.000	.978	.998	.978	.944	1.000	.978	.982	.948	.970
.1	.5	.1	1.000	1.000	.988	.938	.980	.934	.876	.974	.936	.940	.874	.958
.1	.5	.2	.992	.992	.984	.930	.958	.908	.858	.966	.942	.920	.870	.954
.1	.5	.3	.990	.988	.974	.924	.954	.906	.854	.970	.946	.916	.880	.940
.1	.7	.0	1.000	1.000	1.000	.948	1.000	1.000	.934	1.000	.988	1.000	.958	.920
.1	.7	.1	1.000	1.000	.998	.944	1.000	.954	.904	1.000	.982	.992	.936	.932
.1	.7	.2	1.000	1.000	.984	.936	1.000	.934	.822	1.000	.942	.982	.918	.942
.1	.7	.3	1.000	1.000	.974	.942	.954	.948	.804	.948	.944	1.000	.964	.940
.3	.2	.0	.990	.988	.984	.958	.948	.960	.936	.944	.918	.964	.940	.956
.3	.2	.1	.992	.992	.990	.956	.940	.960	.924	.946	.908	.934	.884	.952
.3	.2	.2	.988	.986	.986	.976	.954	.970	.944	.956	.930	.924	.872	.974
.3	.2	.3	.988	.988	.986	.964	.944	.952	.926	.952	.930	.890	.828	.962
.3	.5	.0	.996	.992	.988	.946	.984	.984	.916	.984	.928	.990	.920	.958
.3	.5	.1	.996	.992	.976	.926	.956	.966	.882	.986	.906	.946	.854	.942
.3	.5	.2	.996	.978	.956	.928	.936	.946	.884	.954	.912	.924	.810	.940
.3	.5	.3	.996	.992	.982	.956	.962	.964	.898	.986	.942	.938	.846	.970
.3	.7	.0	1.000	1.000	.994	.970	.998	1.000	.924	1.000	.930	1.000	.920	.932
.3	.7	.1	.998	.996	.976	.968	.978	.990	.898	.996	.954	.988	.880	.954
.3	.7	.2	1.000	1.000	.986	.966	.988	.986	.884	1.000	.958	.980	.870	.972
.3	.7	.3	.992	.988	.968	.950	.966	.962	.860	.992	.950	.992	.964	.962
.6	.2	.0	.998	.996	.990	.960	.956	.976	.944	.924	.842	.980	.950	.954
.6	.2	.1	1.000	1.000	.990	.952	.940	.966	.938	.924	.872	.946	.880	.948
.6	.2	.2	.998	.996	.996	.964	.938	.964	.932	.926	.850	.876	.772	.962
.6	.2	.3	1.000	1.000	.998	.970	.944	.976	.938	.932	.886	.834	.746	.966
.6	.5	.0	1.000	.996	.982	.948	.964	.994	.918	.990	.882	.996	.930	.954
.6	.5	.1	1.000	.998	.988	.958	.972	.996	.932	.990	.904	.990	.874	.970
.6	.5	.2	1.000	.998	.982	.950	.960	.990	.918	.992	.914	.962	.796	.958
.6	.5	.3	1.000	.992	.980	.962	.964	.990	.910	.988	.904	.938	.792	.964
.6	.7	.0	1.000	.996	.982	.958	.984	1.000	.918	1.000	.908	1.000	.900	.944
.6	.7	.1	1.000	.992	.976	.958	.972	1.000	.904	.998	.914	.998	.830	.964
.6	.7	.2	1.000	.998	.976	.950	.972	.996	.872	.998	.910	.994	.792	.962
.6	.7	.3	.998	.990	.962	.946	.948	.994	.890	.998	.932	.998	.948	.950

Table A.23: Coverage Results for Inventory Model M4

Error Rate	Taints		Standard Error											
	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
.1	.2	.0	.00048	.00048	.00049	.00055	.00037	.00041	.00042	.00039	.00040	.00041	.00042	.00048
.1	.2	.1	.00057	.00058	.00059	.00068	.00044	.00050	.00051	.00048	.00049	.00047	.00048	.00063
.1	.2	.2	.00060	.00060	.00060	.00069	.00047	.00052	.00053	.00052	.00052	.00051	.00051	.00068
.1	.2	.3	.00085	.00085	.00086	.00100	.00068	.00076	.00077	.00074	.00075	.00069	.00070	.00095
.1	.5	.0	.00033	.00033	.00035	.00036	.00023	.00027	.00029	.00028	.00030	.00029	.00031	.00038
.1	.5	.1	.00046	.00046	.00048	.00051	.00031	.00037	.00040	.00039	.00041	.00036	.00038	.00058
.1	.5	.2	.00063	.00063	.00064	.00073	.00045	.00053	.00054	.00055	.00056	.00054	.00055	.00076
.1	.5	.3	.00064	.00065	.00067	.00077	.00047	.00056	.00059	.00056	.00059	.00057	.00059	.00077
.1	.7	.0	.00024	.00024	.00027	.00025	.00015	.00019	.00022	.00020	.00022	.00019	.00021	.00035
.1	.7	.1	.00037	.00038	.00039	.00040	.00025	.00029	.00031	.00033	.00035	.00030	.00032	.00055
.1	.7	.2	.00050	.00050	.00052	.00055	.00035	.00041	.00042	.00046	.00047	.00046	.00048	.00072
.1	.7	.3	.00066	.00066	.00069	.00077	.00046	.00056	.00059	.00059	.00062	.00061	.00064	.00089
.3	.2	.0	.00104	.00105	.00108	.00103	.00085	.00091	.00094	.00085	.00088	.00085	.00088	.00094
.3	.2	.1	.00125	.00126	.00129	.00126	.00103	.00110	.00115	.00102	.00106	.00097	.00102	.00113
.3	.2	.2	.00139	.00140	.00144	.00140	.00116	.00125	.00130	.00115	.00119	.00107	.00112	.00125
.3	.2	.3	.00152	.00154	.00159	.00155	.00128	.00139	.00145	.00126	.00133	.00124	.00130	.00138
.3	.5	.0	.00077	.00078	.00081	.00076	.00061	.00067	.00071	.00060	.00063	.00066	.00070	.00073
.3	.5	.1	.00118	.00120	.00125	.00122	.00097	.00104	.00111	.00095	.00102	.00090	.00098	.00112
.3	.5	.2	.00127	.00130	.00134	.00135	.00105	.00115	.00123	.00107	.00114	.00104	.00111	.00121
.3	.5	.3	.00152	.00155	.00160	.00159	.00127	.00143	.00151	.00124	.00131	.00131	.00138	.00142
.3	.7	.0	.00058	.00061	.00067	.00055	.00048	.00050	.00059	.00047	.00056	.00049	.00058	.00069
.3	.7	.1	.00081	.00084	.00090	.00081	.00068	.00072	.00081	.00069	.00079	.00068	.00077	.00088
.3	.7	.2	.00100	.00104	.00111	.00105	.00086	.00094	.00105	.00085	.00095	.00097	.00107	.00103
.3	.7	.3	.00132	.00137	.00143	.00140	.00112	.00130	.00141	.00112	.00122	.00119	.00130	.00134
.6	.2	.0	.00132	.00135	.00140	.00120	.00109	.00116	.00124	.00103	.00112	.00101	.00109	.00119
.6	.2	.1	.00164	.00168	.00174	.00152	.00138	.00146	.00156	.00131	.00140	.00112	.00121	.00146
.6	.2	.2	.00203	.00210	.00218	.00188	.00172	.00183	.00197	.00159	.00171	.00149	.00161	.00181
.6	.2	.3	.00200	.00205	.00213	.00181	.00168	.00182	.00194	.00159	.00171	.00152	.00162	.00175
.6	.5	.0	.00115	.00121	.00128	.00112	.00101	.00100	.00113	.00092	.00103	.00090	.00104	.00113
.6	.5	.1	.00146	.00156	.00164	.00147	.00131	.00131	.00148	.00118	.00135	.00110	.00127	.00142
.6	.5	.2	.00169	.00180	.00190	.00172	.00153	.00155	.00175	.00135	.00154	.00136	.00154	.00164
.6	.5	.3	.00213	.00225	.00235	.00211	.00188	.00197	.00219	.00169	.00190	.00183	.00202	.00201
.6	.7	.0	.00084	.00092	.00101	.00080	.00078	.00073	.00089	.00066	.00082	.00070	.00085	.00093
.6	.7	.1	.00135	.00151	.00165	.00142	.00132	.00122	.00151	.00112	.00140	.00108	.00134	.00151
.6	.7	.2	.00179	.00195	.00206	.00184	.00168	.00169	.00195	.00147	.00172	.00161	.00184	.00184
.6	.7	.3	.00185	.00201	.00212	.00191	.00172	.00180	.00206	.00150	.00176	.00161	.00187	.00187

Table A.24: Standard Error Results for Inventory Model M4

Pop.	Error Rate	Taints		Average Bound											
		US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
1	.005	.268	.000	.0302	.0302	.0300	.0280	.0265	.0127	.0125	.0228	.0226	.0265	.0263	.0295
1	.010	.634	.000	.0306	.0305	.0299	.0240	.0220	.0137	.0130	.0230	.0223	.0244	.0238	.0292
1	.050	.376	.000	.0334	.0332	.0315	.0245	.0245	.0214	.0195	.0247	.0228	.0200	.0181	.0294
1	.100	.372	.000	.0359	.0351	.0310	.0300	.0290	.0271	.0222	.0261	.0212	.0283	.0234	.0140
1	.300	.417	.000	.0461	.0422	.0331	.0353	.0336	.0365	.0235	.0327	.0197	.0602	.0472	.0119
1M	.010	.497	.000	.0306	.0305	.0300	.0245	.0227	.0137	.0131	.0230	.0224	.0239	.0233	.0293
1M	.100	.446	.000	.0370	.0360	.0320	.0295	.0289	.0270	.0220	.0267	.0218	.0283	.0233	.0187
2	.005	.468	.049	.0304	.0303	.0301	.0281	.0268	.0126	.0124	.0230	.0228	.0276	.0274	.0295
2	.010	.289	.081	.0312	.0312	.0308	.0254	.0232	.0147	.0142	.0238	.0233	.0228	.0223	.0298
2	.050	.458	.032	.0364	.0359	.0317	.0198	.0263	.0224	.0177	.0280	.0233	.0197	.0150	.0286
2	.100	.482	.034	.0420	.0407	.0333	.0237	.0331	.0295	.0208	.0327	.0240	.0260	.0173	.0266
2	.700	.541	.019	.0871	.0610	.0263	.0289	.0411	.0690	.0082	.0688	.0079	.1038	.0429	.0195
3	.005	.000	.004	.0302	.0302	.0302	.0291	.0231	.0142	.0142	.0229	.0229	.0196	.0196	.0297
3	.010	.000	.008	.0307	.0307	.0307	.0291	.0215	.0160	.0160	.0234	.0234	.0164	.0164	.0299
3	.050	.000	.010	.0329	.0329	.0329	.0306	.0254	.0234	.0234	.0251	.0251	.0164	.0164	.0311
3	.100	.000	.012	.0358	.0358	.0358	.0339	.0301	.0271	.0271	.0275	.0275	.0229	.0229	.0271
3	.300	.000	.010	.0488	.0488	.0488	.0465	.0446	.0371	.0371	.0385	.0385	.0569	.0569	.0392
4	.005	.000	.529	.0358	.0358	.0358	.0324	.0243	.0171	.0171	.0285	.0285	.0297	.0297	.0330
4	.010	.000	.368	.0370	.0370	.0370	.0310	.0232	.0194	.0194	.0295	.0295	.0254	.0254	.0337
4	.050	.000	.368	.0512	.0512	.0512	.0412	.0363	.0354	.0354	.0424	.0424	.0307	.0307	.0432
4	.100	.000	.258	.0825	.0825	.0825	.0737	.0662	.0634	.0634	.0717	.0717	.0531	.0531	.0794
4	.300	.000	.415	.2215	.2215	.2215	.2113	.1975	.1898	.1898	.1984	.1984	.1483	.1483	.2085

Table A.25: Average Results for Real Data

Population	Error Rate	Taints		Achieved Coverage											
		US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
1	.005	.268	.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000	1.000	1.000	1.000	1.000
1	.010	.634	.000	1.000	1.000	1.000	1.000	1.000	1.000	.992	1.000	1.000	1.000	.998	1.000
1	.050	.376	.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000	.998	1.000	.994	1.000
1	.100	.372	.000	1.000	1.000	1.000	1.000	1.000	1.000	.988	1.000	.986	1.000	.986	.978
1	.300	.417	.000	1.000	1.000	.976	.998	1.000	1.000	.966	1.000	.954	1.000	.998	.930
1M	.010	.497	.000	1.000	1.000	1.000	1.000	1.000	1.000	.992	1.000	1.000	1.000	.996	1.000
1M	.100	.446	.000	1.000	1.000	1.000	1.000	1.000	1.000	.990	1.000	.990	1.000	.988	.972
2	.005	.468	.049	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000	1.000	1.000	.998	1.000
2	.010	.289	.081	1.000	1.000	1.000	1.000	1.000	1.000	.998	1.000	1.000	1.000	.996	1.000
2	.050	.458	.032	1.000	1.000	.996	1.000	1.000	1.000	.970	1.000	.986	1.000	.962	.996
2	.100	.482	.034	1.000	1.000	.976	1.000	1.000	1.000	.954	1.000	.962	1.000	.938	.958
2	.700	.541	.019	1.000	1.000	.928	.984	.996	1.000	.860	1.000	.862	1.000	.966	.944
3	.005	.000	.004	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	.010	.000	.008	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	.050	.000	.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996
3	.100	.000	.012	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.902
3	.300	.000	.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.740
4	.005	.000	.529	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4	.010	.000	.368	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4	.050	.000	.258	1.000	1.000	1.000	1.000	1.000	.986	.986	1.000	1.000	.994	.994	1.000
4	.100	.000	.318	.988	.988	.988	.956	.954	.942	.942	.956	.956	.912	.912	.968
4	.300	.000	.415	.990	.990	.990	.980	.960	.912	.912	.954	.954	.614	.614	.988

Table A.26: Coverage Results for Real Data

Pop	Error	Taints		Standard Error											
	Rate	US	OS100	ST	ST-meik	ST-lta	MM	BN	CS	CS-lta	MD	MD-lta	PP	PP-lta	CL
1	.050	.268	.000	.00000	.00000	.00000	.00002	.00003	.00000	.00001	.00000	.00000	.00005	.00005	.00000
1	.010	.634	.000	.00000	.00000	.00001	.00004	.00004	.00001	.00001	.00000	.00001	.00006	.00007	.00001
1	.050	.376	.000	.00001	.00001	.00002	.00004	.00001	.00002	.00003	.00001	.00002	.00004	.00005	.00002
1	.100	.372	.000	.00002	.00002	.00006	.00006	.00002	.00002	.00005	.00001	.00005	.00005	.00008	.00019
1	.300	.417	.000	.00005	.00006	.00014	.00009	.00004	.00003	.00012	.00003	.00012	.00011	.00022	.00014
1M	.010	.497	.000	.00000	.00000	.00001	.00004	.00004	.00001	.00001	.00000	.00001	.00007	.00007	.00001
1M	.100	.446	.000	.00002	.00003	.00005	.00006	.00001	.00002	.00004	.00001	.00004	.00007	.00009	.00022
2	.005	.468	.049	.00001	.00001	.00001	.00002	.00003	.00001	.00001	.00001	.00001	.00004	.00004	.00000
2	.010	.289	.081	.00002	.00002	.00002	.00005	.00005	.00002	.00002	.00002	.00002	.00008	.00009	.00001
2	.050	.458	.032	.00007	.00007	.00011	.00012	.00005	.00005	.00008	.00007	.00011	.00005	.00009	.00006
2	.100	.482	.034	.00011	.00012	.00023	.00014	.00008	.00007	.00018	.00011	.00022	.00005	.00017	.00041
2	.700	.541	.019	.00040	.00058	.00129	.00065	.00055	.00026	.00112	.00035	.00121	.00023	.00113	.00078
3	.005	.000	.004	.00000	.00000	.00000	.00000	.00005	.00001	.00001	.00000	.00000	.00012	.00012	.00000
3	.010	.000	.008	.00001	.00001	.00001	.00000	.00004	.00001	.00001	.00001	.00001	.00010	.00010	.00000
3	.050	.000	.010	.00003	.00003	.00003	.00001	.00002	.00002	.00002	.00003	.00003	.00002	.00002	.00002
3	.100	.000	.012	.00006	.00006	.00006	.00002	.00004	.00003	.00003	.00005	.00005	.00003	.00003	.00019
3	.300	.000	.010	.00019	.00019	.00019	.00007	.00014	.00011	.00011	.00017	.00017	.00006	.00006	.00106
4	.005	.000	.529	.00009	.00009	.00009	.00005	.00004	.00005	.00005	.00009	.00009	.00017	.00017	.00003
4	.010	.000	.368	.00010	.00010	.00010	.00008	.00006	.00006	.00006	.00010	.00010	.00016	.00016	.00004
4	.050	.000	.258	.00029	.00029	.00029	.00036	.00019	.00018	.00018	.00029	.00029	.00010	.00010	.00015
4	.100	.000	.318	.00051	.00051	.00051	.00062	.00039	.00038	.00038	.00047	.00047	.00018	.00018	.00103
4	.300	.000	.415	.00118	.00118	.00118	.00116	.00098	.00103	.00103	.00098	.00098	.00041	.00041	.00104

Table A.27: Standard Error Results for Real Data

Appendix B

Programs for Computing Bounds

Each section in this appendix provides the programs necessary for the computation of a particular bound. The main program appears first, followed by every program called in the main and subsequent programs, except in those cases where a program was given in a previous section.

B.1 Stringer Bound

B.1.1 stringer.pois.bound()

```
function(taints, confidence, maxit)
{
# This function returns the Stringer bound for the mean error per
# dollar unit as described by Goodfellow et al. (1974), with the
# upper bound for the error rate based on the Poisson distribution.
# For a (Bxn) matrix of taints, a (Bx1) vector of bounds for mut is
# returned.  n represents the  sample size and B the number of
# samples.  maxit is the maximum nunber of iterations to be
# employed in finding the error rate upper confidence bound based
# on the Newton-Rhapson method.  The function spb is called if at
# least one sample has one or more nonzero taints.  For samples
# with no errors, the Stringer bound is simply the attributes bound
# for no sample errors.  This bound handles overstatements only.
# Find the sample size, number of samples and number of
# errors per sample.
    n <- ncol(taints)
    B <- nrow(taints)
```

```
    m <- apply((taints > 0), 1, sum)
# If at least one sample has one or more nonzero taints,
# call spb. Otherwise, a vector of attribute bounds is
# formed.
    bound.mut <- switch((max(m) > 0) + 1,
        rep( - log(1 - confidence)/n, B),
        spb(taints, confidence, maxit))
    list(bound.mut = bound.mut, m.over = m)
}
```

B.1.2 spb()

```
function(taints, confidence, maxit)
{
# This function returns the Stringer bound based on the Poisson
# distribution.  For a (Bxn) matric of taints, a (Bx1) vector of
# bounds for mut is returned. n represents the sample size and B
# the number of samples.  This function will work only if at least
# one sample contains one or more nonzero taints.  The upper bound
# for the error rate is calculated using the Newton-Rhapson method
# via the function pois.up.bound.newton.  maxit gives the maximum
# number of iterations to be used for this process.  This bound
# handles overstatements only.
# Find the sample size and number of samples.
    n <- ncol(taints)
    B <- nrow(taints)
# Disregard understatements.
    taints.over <- as.numeric(taints > 0) * taints
# Find the number of overstatement errors per sample.
    m <- apply((taints.over > 0), 1, sum)
# Arrange the taints in descending order by sample and truncate
# the columns containing all zeros.
    taints.order <- t(apply(apply(taints.over, 1, sort),
       2, rev))
    taints.trunc <- matrix(taints.order[, 1:max(m)], B,
       max(m))
# Form a vector of m values for which upper bounds on the
# population error rate must be calculated.
```

```
    factor.m <- c(1:max(m))
# Find the upper bound on the population error rate for the case
# of 0 sample errors.
    prop.ci.zero <-  - log(1 - confidence)/n
# Set the starting values and call pois.up.bound.newton to find
# the upper bounds on the error rate.
    last.lambda <- rep(1000, length(factor.m))
    lambda <- factor.m
    factor1 <- pois.up.bound.newton(lambda, last.lambda,
    factor.m, confidence, maxit)/n
    factor2 <- c(prop.ci.zero, factor1[1:max(m) - 1])
# Determine the precision factors.
    precision.factor <- factor1 - factor2
# Sum the product of the precision factors by the ordered taints.
    adjust <- t(precision.factor * t(taints.trunc))
    adjust <- apply(adjust, 1, sum)
# Determine the Stringer bound.
    bound.mut <- prop.ci.zero + adjust
    return(bound.mut)
}
```

B.1.3 pois.up.bound.newton()

```
function(lambda, last.lambda, m, confidence, maxit)
{
# This function returns a vector of upper confidence bounds
# for the Poisson parameter lambda, given a vector of sample
# number of errors, m.  To find the corresponding bound for the
# population error rate, simply divide by the sample size.  This
# calculation is based on the Newton-Rhapson method.  lambda is a
# vector of starting values, last.lambda is a vector of values to
# start the loop and maxit gives the maximum number of iterations
# to be employed.  If maxit is exceeded, the function stops,
# returning an error.  h1.up is a function giving the equation to
# be solved and h2.up gives the first derivative of h1.up.
    it <- 0
    while(max(abs(lambda - last.lambda)) > 1e-08 & maxit > it)
    {
```

```
        last.lambda <- lambda
        lambda <- lambda - h1.up(lambda, m, confidence)/
            h2.up(lambda, m)
        it <- it + 1
    }
    if(maxit <= it)
        stop("Too many iterations.")
    return(lambda)
}
```

B.1.4 h1.up()

```
function(lambda, m, confidence)
{
# This function gives the equation to be solved in order to find
# the upper Poisson confidence limit for lambda.
    r <- c(0:max(m))
    r1 <- rep(r, length(m))
    rep.temp <- rep(length(r), length(m))
    lambda1 <- rep(lambda, rep.temp)
    Lambda <- (exp( - lambda1) * lambda1^r1)/gamma(r1 + 1)
    Lambda <- matrix(Lambda, nrow = length(m), ncol =
        length(r), byrow = T)
    m1 <- rep(m, rep.temp)
    M <- matrix(m1, nrow = length(m), ncol = length(r),
        byrow = T)
    R <- matrix(r1, nrow = length(m), ncol = length(r),
        byrow = T)
    Lambda[R > M] <- 0
    h1 <- apply(Lambda, 1, sum) - (1 - confidence)
    return(h1)
}
```

B.1.5 h2.up()

```
function(lambda, m)
{
# This function gives the first derivative of the equation to be
```

```
# solved in order to find the upper Poisson confidence limit for
# lambda.
    r <- c(0:max(m))
    r1 <- rep(r, length(m))
    rep.temp <- rep(length(r), length(m))
    lambda1 <- rep(lambda, rep.temp)
    Lambda <- (r1 * lambda1^(r1 - 1) * exp( - lambda1) -
       lambda1^r1 * exp(-lambda1))/gamma(r1 + 1)
    Lambda <- matrix(Lambda, nrow = length(m), ncol =
       length(r), byrow = T)
    m1 <- rep(m, rep.temp)
    M <- matrix(m1, nrow = length(m), ncol = length(r), byrow = T)
    R <- matrix(r1, nrow = length(m), ncol = length(r), byrow = T)
    Lambda[R > M] <- 0
    h2 <- apply(Lambda, 1, sum)
    return(h2)
}
```

B.2 Meikle's Adjustment to the Stringer Bound

B.2.1 meikle.pois.bound()

```
function(taints, confidence, maxit)
{
# This function returns the Stringer bound for the mean error per
# dollar unit based on the Poisson distribution with Meikle's
# adjustment for understatements as described by Goodfellow et al.
# (1974).  For a (Bxn) matrix of taints, a (Bx1) vector of bounds
# is returned.  n represents the sample size and B the number of
# samples.  maxit represents the maximum number of iterations to
# be employed for finding confidence intervals for lambda based on
# the Newton-Rhapson method.  This bound handles overstatements
# and understatements.
# Find the sample size.
    n <- ncol(taints)
# Separate the overstatement taints and find the Stringer bound
# for the overstatements.
    taints.over <- as.numeric(taints > 0) * taints
```

```
    over <- stringer.pois.bound(taints.over, confidence, maxit)
    bound.over.mut <- over$bound.mut
    m.over <- over$m.over
# Separate the understatement taints and find the number of
# understatements per sample.
    taints.under <- abs(as.numeric(taints < 0) * taints)
    m.under <- apply((taints.under != 0), 1, sum)
# If at least one of the samples has more than one understatement,
# call mpb.  Otherwise, return a vector of lower bounds for the
# mean understatement based on 0 or 1 sample errors.
    bound.under.mut <- switch(((max(m.under) > 1) + 1),
       pois.low.bound.newton(1, 1000, 1, confidence, maxit)/n
        * apply(taints.under, 1, max),
       mpb(taints, confidence, maxit))
# Subtract the Meikle adjustment from the Stringer bound.
    bound.mut <- bound.over.mut - bound.under.mut
    list(bound.mut = bound.mut, m.under = m.under, m.over = m.over)
}
```

B.2.2 pois.low.bound.newton()

```
function(lambda, last.lambda, m, confidence, maxit)
{
# This function returns a vector of lower confidence
# bounds for the Poisson parameter lambda, given a vector of
# sample number of errors, m. To find the corresponding bound for
# the population error rate, simply divide by the sample size.
# The calculation is based on the Newton-Rhapson method.  lambda
# is a vector of starting values, last.lambda is a vector of
# values to start the loop and maxit gives the maximum number of
# iterations to be employed.  If maxit is exceeded, the function
# stops, returning an error.  h1.up is a function giving the
# equation to be solved and h2.up gives the first derivative of
# h1.up.
    it <- 0
    while(max(abs(lambda - last.lambda)) > 1e-08 & maxit > it)
    {
       last.lambda <- lambda
```

```
      lambda <- lambda - h1.low(lambda, m, confidence)/
         h2.low(lambda, m)
      it <- it + 1
   }
   if(maxit <= it)
      stop("Too many iterations.")
   return(lambda)
}
```

B.2.3 h1.low()

```
function(lambda, m, confidence)
{
# This function gives the equation to be solved in order to find
# the lower Poisson confidence limit for lambda.
   r <- c(0:(max(m) - 1))
   r1 <- rep(r, length(m))
   rep.temp <- rep(length(r), length(m))
   lambda1 <- rep(lambda, rep.temp)
   Lambda <- (exp( - lambda1) * lambda1^r1)/gamma(r1 + 1)
   Lambda <- matrix(Lambda, nrow = length(m), ncol =
      length(r), byrow = T)
   m1 <- rep(m, rep.temp)
   M <- matrix(m1, nrow = length(m), ncol = length(r), byrow = T)
   R <- matrix(r1, nrow = length(m), ncol = length(r), byrow = T)
   Lambda[R >= M] <- 0
   h1 <- apply(Lambda, 1, sum) - confidence
   return(h1)
}
```

B.2.4 h2.low()

```
function(lambda, m)
{
# This function gives the first derivative of the equation to be
# solved in order to find the lower Poisson confidence limit for
# lambda.
   r <- c(0:(max(m) - 1))
```

```
    r1 <- rep(r, length(m))
    rep.temp <- rep(length(r), length(m))
    lambda1 <- rep(lambda, rep.temp)
    Lambda <- (r1 * lambda1^(r1 - 1) * exp( - lambda1) -
    lambda1^r1 * exp( - lambda1))/gamma(r1 + 1)
    Lambda <- matrix(Lambda, nrow = length(m), ncol =
       length(r), byrow = T)
    m1 <- rep(m, rep.temp)
    M <- matrix(m1, nrow = length(m), ncol = length(r), byrow = T)
    R <- matrix(r1, nrow = length(m), ncol = length(r), byrow = T)
    Lambda[R >= M] <- 0
    h2 <- apply(Lambda, 1, sum)
    return(h2)
}
```

B.2.5 mpb()

```
function(taints, confidence, maxit)
{
# This function returns Meikle's understatement adjustment to the
# Stringer bound based on the Poisson distribution.  For a (Bxn)
# matrix of taints, a (Bx1) vector of bounds for mut is returned.
# This function will work only if at least one sample contains
# more than one understatement taint.  The lower bound for the
# understatement error rate is determined using the Newton-Rhapson
# method via the function pois.low.bound.newton.  maxit gives the
# maximum number of iterations to be used for this process.  This
# function deals only with understatements.
# Find the sample size and the number of samples.
    n <- ncol(taints)
    B <- nrow(taints)
# Remove the overstatements and take the absolute value of the
# understatements.
    taints.under <- abs(as.numeric(taints < 0) * taints)
# Find the number of understatement errors per sample.
    m.under <- apply((taints.under != 0), 1, sum)
# Arrange the taints in descending order by sample and truncate
# the columns containing all zeros.
```

```
    taints.under.order <- t(apply(apply(taints.under, 1,
       sort), 2, rev))
    taints.under.trunc <- matrix(taints.under.order[,
       1:max(m.under)], B, max(m.under))
# Form a vector of m values for which lower bounds on the
# population understatement error rate must be calculated.
    factor.m.under <- c(1:max(m.under))
# Set the starting values and call pois.low.bound.newton to find
# the lower error rate bounds.
    last.lambda <- rep(1000, length(factor.m.under))
    lambda <- factor.m.under
    lambda[lambda == 2] <- 1
    factor <- pois.low.bound.newton(lambda, last.lambda,
       factor.m.under, confidence, maxit)/n
    factor1 <- factor[2:max(m.under)]
    factor2 <- factor[1:max(m.under) - 1]
# Determine the precision factors.
    precision.factor <- factor1 - factor2
# Sum the product of the precision factors by the ordered taints.
    temp <- matrix(taints.under.trunc[, 2:max(m.under)], nrow = B)
    adjust <- t(precision.factor * t(temp))
    if((max(m.under) - 1) * B != 1)
       adjust <- apply(adjust, 1, sum)
# Determine the understatement adjustment.
    bound.under.mut <- factor[1] * taints.under.trunc[, 1] + adjust
    return(bound.under.mut)
}
```

B.3 LTA Adjustment to the Stringer Bound

B.3.1 lta.pois.bound()

```
function(taints, confidence, maxit)
{
# This function returns the Stringer bound for the mean error per
# dollar unit based on the Poisson distribution with the LTA
# adjustment for understatements (Leslie, Teitlebaum and Anderson,
# 1980). This adjustment is calculated via the function lta.  For
```

```
# a (Bxn) matrix of taints a (Bx1) vector of bounds is returned.
# n represents the sample size and B the number of samples.  maxit
# gives the maximum number of iterations to be used in finding the
# upper bound for lambda using the Newton-Rhapson method for the
# Stringer bound calculation.  This bound handles overstatements
# and understatements.
# Find the sample size and the number of samples.
    n <- ncol(taints)
    B <- nrow(taints)
# Separate the overstatement taints and find the Stringer
# bound based on these overstatements.
    taints.over <- as.numeric(taints > 0) * taints
    over <- stringer.pois.bound(taints.over, confidence, maxit)
    bound.over.mut <- over$bound.mut
# Find the number of overstatements per sample.
    m.over <- over$m.over
# Call the function lta to determine the necessary adjustment for
# understatements and the number of understatements.
    under <- lta(taints)
    m.under <- under$m.under
# Compute the Stringer bound with the LTA adjustment for
# understatements.
    bound.mut <- bound.over.mut - under$adjust
    list(bound.mut = bound.mut, m.under = m.under, m.over = m.over)
}
```

B.3.2 lta()

```
function(taints)
{
# This function computes the LTA adjustment for understatements
# (Leslie, Teitlebaum and Anderson, 1980) for finding bounds on
# the mean error per dollar unit.  For a (Bxn) matrix of taints,
# a (Bx1) vector of adjustments is returned.  n is the sample size
# and B the number of samples.
    taints.under <- abs(as.numeric(taints < 0) * taints)
    adjust <- apply(taints.under, 1, mean)
    m.under <- apply((taints.under != 0), 1, sum)
```

```
    list(adjust = adjust, m.under = m.under)
}
```

B.4 Modified Moment Bound

B.4.1 mod.moment.a.bound()

```
function(taints, confidence)
{
# This function calculates the modified moment bound for the mean
# error per dollar unit for accounts receivable data, as developed
# by Dworin and Grimlund (1986).  For a (Bxn) matrix of taints, a
# (Bx1) vector of bounds is returned.  The moment bound allows for
# both overstatements and understatements.
    n <- ncol(taints)
    m.over <- apply((taints > 0), 1, sum)
    m.under <- apply((taints < 0), 1, sum)
    m <- m.over + m.under
    zsum <- apply(taints, 1, sum)
    zmean <- div.zero(zsum, m)
# Calculate the hypothetical taint.
    zstar <- 0.81 * (1 - 0.667 * tanh(10 * abs(zmean))) *
       (1 + 0.667 * tanh(m/10))
# Calculate the error tainting noncentral moments.
    z.non.1 <- (zstar + zsum)/(m + 1)
    z.non.2 <- (zstar^2 + apply(taints^2, 1, sum))/(m + 1)
    z.non.3 <- (zstar^3 + apply(taints^3, 1, sum))/(m + 1)
# Calculate the error rate noncentral moments.
    pi.non.1 <- (m + 1)/(n + 2)
    pi.non.2 <- (m + 2)/(n + 3) * pi.non.1
    pi.non.3 <- (m + 3)/(n + 4) * pi.non.2
# Calculate the mean error noncentral moments.
    mu.non.1 <- pi.non.1 * z.non.1
    mu.non.2 <- (pi.non.1 * z.non.2 + (n - 1) * pi.non.2 *
       z.non.1^2)/n
    mu.non.3 <- (pi.non.1 * z.non.3 + 3 * (n - 1) * pi.non.2 *
       z.non.1 * z.non.2 + (n - 1) * (n - 2) * pi.non.3 *
       z.non.1^3)/(n^2)
```

```
# Calculate the mean error central moments.
   mu.cen.2 <- mu.non.2 - mu.non.1^2
   mu.cen.3 <- mu.non.3 - 3 * mu.non.1 * mu.non.2 + 2 * mu.non.1^3
# Calculate the gamma distribution parameters.
   gamma.a <- (4 * mu.cen.2^3)/(mu.cen.3^2)
   gamma.b <- (mu.cen.3)/(2 * mu.cen.2)
   gamma.g <- mu.non.1 - (2 * mu.cen.2^2)/(mu.cen.3)
# Calculate the upper confidence bound for the mean error.
   percentile <- confidence * (gamma.b > 0) + (1 - confidence)
      * (gamma.b < 0)
   bound.mut <- gamma.g + gamma.a * gamma.b * (1 +
      qnorm(percentile)/sqrt(9 * gamma.a) - 1/(9 * gamma.a))^3
   list(bound.mut = bound.mut, m.over = m.over, m.under = m.under)
}
```

B.4.2 mod.moment.i.bound()

```
function(taints, confidence)
{
# This function calculates the modified moment bound for the mean
# error per dollar unit for inventory data, as developed by Dworin
# and Grimlund (1986).  For a (Bxn) matrix of taints, a (Bx1)
# vector of bounds is returned.  The moment bound allows for both
# overstatements and understatements.
   n <- ncol(taints)
   m.over <- apply((taints > 0), 1, sum)
   m.under <- apply((taints < 0), 1, sum)
   m <- m.over + m.under
   zsum <- apply(taints, 1, sum)
   zmean <- div.zero(zsum, m)
# Calculate the hypothetical taint.
   zstar <- 0.81 * (1 - 0.667 * tanh(10 * abs(zmean)))
# Calculate the error tainting noncentral moments.
   z.non.1 <- (zstar + zsum)/(m + 1)
   z.non.2 <- (zstar^2 + apply(taints^2, 1, sum))/(m + 1)
   z.non.3 <- (zstar^3 + apply(taints^3, 1, sum))/(m + 1)
# Calculate the error rate noncentral moments.
   pi.non.1 <- (m + 1)/(n + 2)
```

```
    pi.non.2 <- (m + 2)/(n + 3) * pi.non.1
    pi.non.3 <- (m + 3)/(n + 4) * pi.non.2
# Calculate the mean error noncentral moments.
    mu.non.1 <- pi.non.1 * z.non.1
    mu.non.2 <- (pi.non.1 * z.non.2 + (n - 1) * pi.non.2 *
       z.non.1^2)/n
    mu.non.3 <- (pi.non.1 * z.non.3 + 3 * (n - 1) * pi.non.2 *
       z.non.1 * z.non.2 + (n - 1) * (n - 2) * pi.non.3 *
       z.non.1^3)/(n^2)
# Calculate the mean error central moments.
    mu.cen.2 <- mu.non.2 - mu.non.1^2
    mu.cen.3 <- mu.non.3 - 3 * mu.non.1 * mu.non.2 + 2 *
    mu.non.1^3
# Calculate the gamma distribution parameters.
    gamma.a <- (4 * mu.cen.2^3)/(mu.cen.3^2)
    gamma.b <- (mu.cen.3)/(2 * mu.cen.2)
    gamma.g <- mu.non.1 - (2 * mu.cen.2^2)/(mu.cen.3)
# Calculate the upper confidence bound for the mean error.
    percentile <- confidence * (gamma.b > 0) + (1 - confidence)
       * (gamma.b < 0)
    bound.mut <- gamma.g + gamma.a * gamma.b * (1 +
       qnorm(percentile)/sqrt(9 * gamma.a) - 1/(9 * gamma.a))^3
    list(bound.mut = bound.mut, m.over = m.over, m.under = m.under)
}
```

B.5 Bayesian Normal Bound

B.5.1 normal.bound

```
function(taints, confidence, prior.pi, prior.n, prior.mu, prior.r,
 prior.phi, prior.theta)
{
# This function returns the normal bound for the mean error per
# dollar unit as described by Smieliauskas (1986, Table 1).  For a
# (Bxn) matrix of taints, a (Bx1) vector of bounds is returned.
# n represents the sample size and B the number of samples.  For
# samples containing no nonzero errors, the bound is set equal to
# the attributes bound for no errors.  The function
```

```
# normal.non.bound is called to compute the bound for those
# samples with at least one nonzero error.  The bound handles
# overstatements and understatements.  Six prior parameter values
# are required.
# Calculate the sample size and the number of samples.
   n <- ncol(taints)
   B <- nrow(taints)
# Calculate the number of nonzero errors per sample.
   m <- apply((taints != 0), 1, sum)
# Sort the matrix of taints by number of errors.
   ord <- order(m)
   taints <- matrix(taints[ord,  ], nrow = B, ncol = n)
# Calculate the number of overstatements and understatements.
   m.over <- apply((taints > 0), 1, sum)
   m.under <- apply((taints < 0), 1, sum)
   m <- m[ord]
# Determine the number of samples with no nonzero errors.
   B.zero <- sum(m == 0)
# If at least one sample has no nonzero errors, return a vector of
# attribute bounds.
   bound.mut.zero <- switch((B.zero > 0) + 1,numeric(0),
      rep(1 - (1 - confidence)^(1/n), B.zero))
# Determine the number of samples with at least one nonzero error.
   B.non <- sum(m != 0)
# If at least one sample has one or more nonzero errors, form a
# matrix of taints for these samples and call normal.non.bound.
   if(B.non > 0)
      taints.non <- matrix(taints[m != 0,  ], ncol = n)
   bound.mut.non <- switch((B.non > 0) + 1, numeric(0),
      normal.non.bound(taints.non, confidence, prior.pi, prior.n,
      prior.mu, prior.r, prior.phi, prior.theta))
# Combine the two sets of bounds.
   bound.mut <- c(bound.mut.zero, bound.mut.non)
   list(bound.mut = bound.mut, m.over = m.over, m.under = m.under)
}
```

B.5.2 normal.non.bound()

```
function(taints.non, confidence, prior.pi, prior.n, prior.mu,
        prior.r, prior.phi, prior.theta)
{
# This function returns the normal bound for the mean error per
# dollar unit for samples containing at least one nonzero error.
# Calculate the sample size and the number of samples.
    n <- ncol(taints.non)
    B <- nrow(taints.non)
    m.non <- apply((taints.non != 0), 1, sum)
# Calculate the mean and variance for the nonzero errors.
    zmean <- apply(taints.non, 1, sum)/m.non
    zvar <- div.zero(apply((as.numeric(taints.non != 0) *
       (taints.non - zmean))^2, 1, sum), (m.non - 1))
# Calculate the posterior parameters.
    post.m <- prior.n * prior.pi + m.non
    post.n <- prior.n + n
    post.r <- prior.r + m.non
    post.mu <- (prior.r * prior.mu + m.non * zmean)/(post.r)
    post.theta <- prior.theta + dirac.inverse(prior.r) + m.non - 1
    post.phi <- (1/post.theta) * (prior.theta * prior.phi
       + (m.non - 1) * zvar + (prior.r * m.non *
       (prior.mu - zmean)^2)/(post.r))
# Find the upper bound for mut.
    bound.mut <- (post.mu * post.m)/post.n + qt(confidence,
       post.theta) * sqrt((post.phi/post.r * (post.m + 1) *
       post.m)/(post.n * (post.n + 1)) + ((post.mu^2 * post.m *
       (post.n - post.m))/(post.n^2 * (post.n + 1)) * (post.theta
       - 2))/post.theta)
    return(bound.mut)
}
```

B.6 Cox and Snell Bound

B.6.1 cs.bound()

```
function(taints, confidence, prior.pi, prior.a, prior.mu, prior.b,
```

```
        LTA)
{
# This function returns the Cox and Snell (1979) bound for the mean
# error per dollar unit and is based on theory presented in Godfrey
# and Neter (1984, equation 12).  For a (Bxn) matrix of taints, a
# (Bx1) vector of bounds is returned.  n represents the sample size
# and B the number of samples.  If LTA = F, understatements are
# ignored.  If LTA = T, the LTA adjustment for understatements
# (Leslie, Teitlebaum and Anderson, 1980) is employed, through the
# function lta.  Four prior parameter values are required.
# prior.a and prior.b must be multiples of 1/2.
# Calculate the sample size and the number of overstatements.
    n <- ncol(taints)
    m.over <- apply((taints > 0), 1, sum)
# If the LTA adjustment is being used, compute the number
# of understatements.
    if(LTA == T) m.under <- apply((taints < 0), 1, sum)
# Find the mean overstatement taint.
    zsum <- apply(as.numeric(taints > 0) * taints, 1, sum)
    zmean <- div.zero(zsum, m.over)
# Calculate the bound for mut.
    bound.mut <- ((m.over * zmean + (prior.b - 1) * prior.mu)/
       (prior.a/prior.pi + n) * (m.over + prior.a))/(m.over +
       prior.b) * qf(confidence, 2 * (m.over + prior.a), 2 *
       (m.over + prior.b))
# If LTA = T, call lta to adjust the bound for understatements.
    if(LTA == T)
       bound.mut <- bound.mut - lta(taints)$adjust
    if(LTA == T)
       list(bound.mut = bound.mut, m.over = m.over,
          m.under = m.under)
    else list(bound.mut = bound.mut, m.over = m.over)
}
```

B.7 Multinomial-Dirichlet Bound

B.7.1 dirichlet.bound()

```
function(taints, confidence, prior.K, prior.alpha, LTA)
{
# This function returns the multinomial-Dirichlet bound for the
# mean error per dollar unit.  It is based on the theory of Tsui,
# Matsamura and Tsui (1985, p.79-83).  For a (Bxn) matrix of taints
# a (Bx1) vector of bounds is returned.  n refers to the sample
# size and B the number of samples.  prior.alpha is a (101x1)
# vector giving the prior class probabilities which must sum to 1.
# If LTA=F, understatements are disregarded.  If LTA=T, the LTA
# adjustment for understatements (Leslie, Teitlebaum and Anderson,
# 1980) is employed through the function lta.
# Calculate the sample size and number of samples.
    n <- ncol(taints)
    B <- nrow(taints)
# Calculate the number of overstatements per sample.
    m.over <- apply((taints > 0), 1, sum)
# If the LTA adjustment is being used, compute the number of
# understatements per sample.
    if(LTA == T) m.under <- apply((taints < 0), 1, sum)
# Round the taints for classification, ignoring understatements.
    taints.round <- round(as.numeric(taints > 0) * taints, 2)
# Form a matrix of class counts.
    w <- matrix(0, nrow = B, ncol = 101)
    for(i in 1:101) {
       w[, i] <- apply(((100 * taints.round) == i - 1), 1, sum)
    }
# Calculate the posterior values of K and alpha.
    post.K <- prior.K + n
    post.alpha <- matrix(0, nrow = B, ncol = 101)
    for(i in 1:B) {
       post.alpha[i, ] <- (prior.K * prior.alpha + w[i, ])/
          (post.K)
    }
# Calculate the mean and variance of mut.
```

```
    i <- matrix(0:100, nrow = B, ncol = 101, byrow = T)
    mut.mean <- apply((i * post.alpha)/100, 1, sum)
    mut.var <- ((apply((i^2 * post.alpha), 1, sum)) -
       (apply((i * post.alpha), 1, sum))^2)/(10000 * (post.K + 1))
# Calculate the parameters of the beta distribution.
    beta.a <- mut.mean * ((mut.mean * (1 - mut.mean))/mut.var - 1)
    beta.b <- (1 - mut.mean) * ((mut.mean * (1 - mut.mean))/
       mut.var - 1)
# Calculate the bound for mut.
    bound.mut <- qbeta(confidence, beta.a, beta.b)
# If LTA=T call lta to adjust the bound for understatements.
    if(LTA == T)
       bound.mut <- bound.mut - lta(taints)$adjust
    if(LTA == T)
       list(bound.mut = bound.mut, m.over = m.over,
          m.under = m.under)
    else list(bound.mut = bound.mut, m.over = m.over)
}
```

B.8 Parametric Power Bound

B.8.1 power.bound()

```
function(taints, confidence, B.star, maxit, LTA)
{
# This function returns the parametric bootstrap bound based on
# the power function for the mean error per dollar unit.  This
# bound was developed by Tamura and Frost (1986, p.366-367).  For
# a (Bxn) matrix of taints, a (Bx1) vector of bounds is returned.
# n represents the sample size and B the number of samples.  B.star
# gives the number of bootstrap replications used to estimate each
# bound.  For samples containing no overstatements other than 100%
# overstatements, the bound is set equal to the attributes bound.
# The function power.non.bound is called to compute the bound for
# those samples with at least one non-100% overstatement.  If
# LTA=F, understatements are disregarded.  If LTA=T, the LTA
# adjustment for understatements (Leslie, Teitlebaum and Anderson,
# 1980) is employed via the function lta.
```

```
# Calculate the sample size and the number of samples.
    n <- ncol(taints)
    B <- nrow(taints)
# Calculate the number of overstatements and 100% overstatements
# in each sample.
    m.over <- apply((taints > 0), 1, sum)
    m.ones <- apply((taints == 1), 1, sum)
# Sort the matrix of taints by number of overstatements and 100%
# overstatements.
    ord <- order(m.over, m.ones)
    taints <- matrix(taints[ord,  ], nrow = B, ncol = n)
    m.over <- m.over[ord]
    m.ones <- m.ones[ord]
# If the LTA adjustment is being used, calculate the number of
# understatements per sample.
    if(LTA == T) m.under <- apply((taints < 0), 1, sum)
# Determine the number of samples with no overstatement errors.
    B.zero <- sum(m.over == 0)
# If at least one sample has no overstatements, return a vector of
# attribute bounds for no errors.
    bound.mut.zero <- switch((B.zero > 0) + 1, numeric(0),
       rep(1 - (1 - confidence)^(1/n), B.zero))
# Determine the number of samples with only 100% overstatements.
    B.ones <- sum(m.over != 0 & m.over == m.ones)
# If at least one sample has only 100% overstatements, determine
# the number of 100% overstatements for these samples and call the
# function bino.up.bound.newton to compute the appropriate
# attribute bounds.
    if(B.ones > 0)
       m.att <- m.ones[m.ones != 0 & m.ones == m.over]
    bound.mut.ones <- switch((B.ones > 0) + 1, numeric(0),
       bino.up.bound.newton(m.att/n, rep(10, length(m.att)), m.att,
       n, confidence, maxit))
# Determine the number of samples with at least one non-100%
# overstatement.
    B.non <- sum(m.over != 0 & m.over > m.ones)
# If at least one sample has one or more non-100% overstatements,
# form a matrix of overstatement taints for these samples and call
```

```
# power.non.bound.
    if(B.non > 0)
       taints.non <- matrix((as.numeric(taints > 0) * taints)
          [m.over != 0 & m.over > m.ones,  ], ncol = n)
       bound.mut.non <- switch((B.non > 0) + 1,numeric(0),
          power.non.bound(taints.non, confidence, B.star))
# Combine the three sets of bounds.
    bound.mut <- c(bound.mut.zero, bound.mut.ones, bound.mut.non)
# If LTA=T call lta to adjust the bound for understatements.
    if(LTA == T)
       bound.mut <- bound.mut - lta(taints)$adjust
    if(LTA == T)
       list(bound.mut = bound.mut, m.over = m.over,
          m.under = m.under)
    else list(bound.mut = bound.mut, m.over = m.over)
}
```

B.8.2 bino.up.bound.newton()

```
function(pp, lastp, m, n, confidence, maxit)
{
# This function returns a vector of binomial upper confidence
# bounds for the error rate given a vector of sample error counts,
# m, in a sample of size n.  The function works for m>0 and m not
# too near n.  This calculation is based on the Newton-Rhapson
# method.  pp is a vector of starting values, lastp is a vector of
# values to start the loop and maxit gives the maximum number of
# iterations to be employed.  If maxit is exceeded, the function
# stops, returning an error.  g1.up is a function giving the
# equation to be solved and g2.up gives the first derivative of
# g1.up.
    it <- 0
    while(max(abs(pp - lastp)) > 1e-08 & maxit > it)
    {
       lastp <- pp
       pp <- pp - g1.up(pp, m, n, confidence)/g2.up(pp, m, n)
       it <- it + 1
    }
```

```
    if(maxit <= it)
       stop("Too many iterations.")
    return(pp)
}
```

B.8.3 g1.up()

```
function(pp, m, n, confidence)
{
# This function gives the equation to be solved in order to find the
# upper binomial confidence limit for a population proportion.
    r <- c(0:max(m))
    r1 <- rep(r, length(m))
    rep.temp <- rep(length(r), length(m))
    pp1 <- rep(pp, rep.temp)
    comb <- rep(0, length(r1))
    for(i in 1:length(r1))
    comb[i] <- choose(n, r1[i])
    PP <- comb * pp1^r1 * (1 - pp1)^(n - r1)
    PP <- matrix(PP, nrow = length(m), ncol = length(r), T)
    m1 <- rep(m, rep.temp)
    M <- matrix(m1, nrow = length(m), ncol = length(r), T)
    R <- matrix(r1, nrow = length(m), ncol = length(r), T)
    PP[R > M] <- 0
    g1 <- apply(PP, 1, sum) - (1 - confidence)
    return(g1)
}
```

B.8.4 g2.up()

```
function(pp, m, n)
{
# This function returns the first derivative of the equation to
# be solved in order to find the upper binomial confidence limit
# for a population proportion.
    r <- c(0:max(m))
    r1 <- rep(r, length(m))
    rep.temp <- rep(length(r), length(m))
```

```
    pp1 <- rep(pp, rep.temp)
    comb <- rep(0, length(r1))
    for(i in 1:length(r1))
       comb[i] <- choose(n, r1[i])
    PP <- comb * (r1 * pp1^(r1 - 1) * (1 - pp1)^(n - r1)
       - pp1^r1 * (n - r) * (1 - pp1)^(n - r1 - 1))
    PP <- matrix(PP, nrow = length(m), ncol = length(r), T)
    m1 <- rep(m, rep.temp)
    M <- matrix(m1, nrow = length(m), ncol = length(r), T)
    R <- matrix(r1, nrow = length(m), ncol = length(r), T)
    PP[R > M] <- 0
    g2 <- apply(PP, 1, sum)
    return(g2)
}
```

B.8.5 power.non.bound()

```
function(taints.non, confidence, B.star)
{
# This function returns the parametric bootstrap bound based
# on the power function for the mean error per dollar unit for
# samples containing at least one overstatement taint.
# Calculate the sample size and the number of samples.
    n <- ncol(taints.non)
    B.non <- nrow(taints.non)
# Initialize the vector bound and set iseed to the Splus random
# seed.
    bound <- rep(0, B.non)
    iseed <- .Random.seed
# Call the Fortran function pwernon to compute the bound.
    non <- .Fortran("pwernon",
            as.double(taints.non),
            as.integer(B.non),
            as.integer(n),
            as.double(confidence),
            as.integer(B.star),
            bound = as.double(bound),
            as.integer(iseed))
```

```
    bound.mut.non <- non$bound
    return(bound.mut.non)
}
```

B.8.6 pwernon.f()

```
subroutine pwernon(taints,nonB,n,conf,nbstar,bound,iseed)

c This subroutine determines the parametric bootstrap bound
c based on the power function for the mean error
c per dollar unit.  This function will work only for
c samples containing at least one non-100 percent
c overstatement, since rlamht would involve division
c by zero for the case of no overstatements or only
c 100 percent overstatements.

c taints is the matrix of taints for which the bounds are
c to be determined.  n is the sample size and nonB the
c number of samples. conf is the confidence level, between
c 0 and 1, at which the bounds are to be determined.
c nbstar gives the number of bootstrap samples to be
c employed.  bound gives the vector of bounds and iseed
c provides a start point for the random number generator.

    implicit real*8 (a-h,o-z)
    parameter (nbstmx=1000)
    dimension taints(nonB,n),rmuhts(nbstmx),bound(nonB)
    external g05cbf, g05caf, m01caf

c go5cbf sets the seed for the generator in g05caf.

    call  g05cbf(iseed)

    do 100 i=1,nonB

       mnon = 0
       rlamht = 0.0d0
```

```
c Determine the number of overstatements and estimates of
c pi and lambda for sample i.

        do 200 j=1,n
          if(taints(i,j).gt.0.0d0) then
             mnon = mnon + 1
             rlamht = rlamht + dlog(taints(i,j))
          end if
 200    continue

        pihat = dfloat(mnon)/dfloat(n)
        rlamht = -dfloat(mnon)/rlamht

c Construct bootstrap samples.  Taints are generated from the
c power distribution with parameter rlamht with probability
c pihat.  The remainder of the sample is zeros. Calculate
c the number of overstatements in each bootstrap sample.

        istar=0
           do 300 ib = 1,nbstar
              mstar = 0
              rlamhs = 0.0d0

c g05caf generates uniform(0,1) numbers

                 do 400 jb = 1,n
                    data1 = g05caf(x)
                    data = g05caf(x)
                    if (data1.le.pihat) then
                       mstar = mstar + 1
                       rlamhs = rlamhs + dlog(data**(1.0d0/rlamht))
                    end if
 400             continue

c For those bootstrap samples with at least one non-100
c percent overstatement, calculate the bootstrap
c estimates of pi, lambda and mut.
```

```
         if (mstar.eq.0.or.rlamhs.eq.0) then
            go to 300
         else
            istar=istar+1
         endif

         pihats = dfloat(mstar)/dfloat(n)
         rlamhs = -dfloat(mstar)/dfloat(rlamhs)
         rmuhts(istar) = (pihats*rlamhs)/(rlamhs + 1.0d0)
300     continue

c If less than ten of the bootstrap samples have at least
c one non-100 percent overstatement, bound is set equal to
c the attributes bound for the case of no errors.  Otherwise,
c bound is set equal to the conf'th percentile of the
c bootstrap estimates of mut.

c m01caf sorts the vector muhats in ascending order

        if (istar.lt.10) then
           bound(i)=1.0d0-(1.0d0-conf)**(1.0d0/dfloat(n))
        else
           call m01caf(rmuhts,1,istar,'a',ifail)

           rindex = dfloat(istar)*conf
           bound(i) = rmuhts(min(int(rindex)+1,istar))
           if(mod(rindex,1.0d0).eq.0.0d0)  then
              bound(i) = (bound(i) + rmuhts(int(rindex)))*.5d0
           endif
        endif
100   continue

      return
      end
```

B.9 Clayton's Combined Bound

B.9.1 clayton.bound()

```
function(taints, confidence, B.star, maxit)
{
# This function returns the combined bound based on Hoeffding's
# inequality and the bootstrap for the mean error per dolar unit.
# This bound was developed by Clayton (1994).  For a (Bxn) matrix
# of taints a (Bx1) vector of bounds  is returned.  n represents
# the sample size and B the number of samples. The function
# hoeff.bound is called if there is at least one sample with less
# than 10% nonzero taints. The function boot.t.bound is called if
# there is at least one sample with 10% or more nonzero taints.
# maxit gives the maximum number of iterations for finding the
# Hoeffding bound and B.star gives the number of bootstrap samples
# for the bootstrap-t method.  This bound handles overstatements
# and understatements.  WARNING: The function hoeff.bound is only
# valid for n=100, confidence=.95 (see hoeff.bound comments for
# details).
# Find the sample size and the number of errors per sample.
    n <- ncol(taints)
    m <- apply((taints != 0), 1, sum)
# Determine the number of samples with less than 10% errors.
    B.hoeff <- sum(m < (0.1 * n))
# If there is at least one sample with less than 10% nonzero
# taints, form a matrix of taints for these samples and call
# hoeff.bound.
    if(B.hoeff > 0)
       taints.hoeff <- matrix(taints[m < (0.1 * n),  ], ncol = n)
    h <- switch((B.hoeff > 0) + 1,
       list(bound.mut = numeric(0), m.over = numeric(0), m.under =
          numeric(0)), hoeff.bound(taints.hoeff, confidence,
          maxit))
# Determine the number of samples with at least 10% nonzero errors.
    B.boot <- sum(m >= (0.1 * n))
# If there is at least one sample with 10% or more nonzero taints,
# form a matrix of taints for these samples and call boot.t.bound.
```

```
    if(B.boot > 0)
       taints.boot <- matrix(taints[m >= (0.1 * n),  ], ncol = n)
    bt <- switch((B.boot > 0) + 1, list(bound.mut = numeric(0),
       m.over = numeric(0), m.under = numeric(0)),
    boot.t.bound(taints.boot, confidence, B.star))
# Combine the overstatement and understatement counts, and the
# bounds from the two cases above.
    m.over <- c(h$m.over, bt$m.over)
    m.under <- c(h$m.under, bt$m.under)
    bound.mut <- c(h$bound.mut, bt$bound.mut)
    list(bound.mut = bound.mut, m.over = m.over, m.under = m.under)
}
```

B.9.2 hoeff.bound()

```
function(taints, confidence, maxit)
{
# This function returns the modified Hoeffding bound as described
# by Clayton (1994, Section 5.2).  For a (Bxn) matrix of taints, a
# (Bx1) vector of bounds is returned.  n represents the sample
# size and B the number of samples. In order to determine the
# bound, a solution to the Hoeffding inequality must be found.
# The Newton-Rhapson method is used for this purpose. For rbar
# values greater than .97, a solution is impossible and so linear
# interpolation is employed through the function h.interp.  Note
# that the value .97 was determined for n=100 and confidence=.95,
# restricting the program to use in this case.  For rbar values
# less than or equal to .97, the function h.bound is used to find
# the bound.  This bound employs the LTA adjustment (Leslie,
# Teitlebaum and Anderson, 1980) for understatements, through the
# function lta.  maxit is the maximum number of iterations to be
# used for finding the solution to the Hoeffding inequality via
# the Newton-Rhapson method.
# Calculate the mean taint per sample and order the sample
# by tbar.
    tbar <- apply(as.numeric(taints >= 0) * taints, 1, mean)
    ord <- order(tbar)
    taints <- matrix(taints[ord,  ], ncol = ncol(taints))
```

```
# Calculate the number of overstatements and understatements
# per sample.
    m.over <- apply((taints > 0), 1, sum)
    m.under <- apply((taints < 0), 1, sum)
    # Compute rbar.
    tbar <- tbar[ord]
    rbar <- 1 - tbar
# Separate those samples having rbar less than or equal to .97.
    rbar.low <- rbar[rbar <= 0.97]
# If at least one sample has rbar less than or equal to .97, call
# h.bound to return the bound for mut for those samples.
    bound.mut.low <- switch((length(rbar.low) > 0) + 1,
       numeric(0), h.bound(rbar.low, maxit))
# Separate those samples having rbar greater than .97.
    rbar.high <- rbar[rbar > 0.97]
# If at least one sample has rbar greater than .97, call h.interp
# to return the bound for mut for those samples.
    bound.mut.high <- switch((length(rbar.high) > 0) + 1,
       numeric(0), h.interp(rbar.high, maxit))
# Combine the two sets of bounds.
    bound.mut <- c(bound.mut.high, bound.mut.low)
# Call the function lta to adjust for understatements.
    under <- lta(taints)
    bound.mut <- bound.mut - under$adjust
    list(bound.mut = bound.mut, m.over = m.over, m.under = m.under)
}
```

B.9.3 h.bound

```
function(rbar.low, maxit)
{
# This function returns the Hoeffding bound for mut for rbar
# values less than or equal to .97.  The starting value is
# determined and then hoeff.newton, a function based on the
# Newton-Rhapson method, is called to give the solution to the
# Hoeffding inequality.
    lastc <- c(rep(10, length(rbar.low)))
    cc <- 1 - rbar.low - 0.0001
```

```
# When calling hoeff.newton, the values of n and confidence are
# set to 100 and .95 respectively, since the value .97 was
# determined for that case.
    bound.c <- hoeff.newton(cc, lastc, rbar.low, 100, 0.95, maxit)
    bound.mut <- 1 - rbar.low + bound.c
    return(bound.mut)
}
```

B.9.4 hoeff.newton

```
function(cc, lastc, rbar, n, confidence, maxit)
{
# This function returns the solution to the Hoeffding inequality,
# for a vector of rbar values, using the Newton-Rhapson method.
# cc is a vector of starting values, lastc is a vector of values
# to begin the loop and maxit gives the maximum number of
# iterations to be employed.  f1 is a function giving the equation
# to be solved, and f2 gives the first derivative of this function.
# This function stops, returning an error if maxit is exceeded.
    it <- 0
    while(max(abs(cc - lastc)) > 1e-08 & maxit > it)
    {
       lastc <- cc
       cc <- cc - f1(cc, rbar, n, confidence)/f2(cc, rbar, n)
       it <- it + 1
     }
    if(maxit <= it)
       stop("Too many iterations.")
    return(cc)
}
```

B.9.5 f1()

```
function(cc, rbar, n, confidence)
{
# This function returns the Hoeffding inequality equation.
    n * (rbar + cc) * (log(rbar) - log(rbar + cc)) + n *
       (1 - rbar - cc) * (log(1 - rbar) - log(1 - rbar - cc))
```

```
        - log(1 - confidence)
}
```

B.9.6 f2()

```
function(cc, rbar, n)
{
# This function returns the first derivative of the Hoeffding
# inequality equation.
    n * (log(rbar) - log(rbar + cc) - log(1 - rbar) +
       log(1 - rbar - cc))
}
```

B.9.7 h.interp()

```
function(rbar.high, maxit)
{
# This function returns the Hoeffding bound for mut when rbar is
# greater than .97, n=100 and confidence=.95.  For rbar=1, the
# solution to the Hoeffding inequality is assumed to be equal to
# the attributes bound, based on no non-zero errors.
    bound.att <- 1 - (0.05)^(1/100)
# For n=100 and confidence=.95, the largest value of rbar for which
# a solution to the Hoeffding inequality could be found was
# determined to be .97.
    limit.rbar <- 0.97
# hoeff.newton is called to return the solution for such a value
# of rbar.
    limit.c <- hoeff.newton(1 - limit.rbar - 0.0001, 10,
       limit.rbar, 100, 0.95, maxit)
# Solutions for rbar values between .97 and 1 are found using
# linear interpolation.
    interpolate.slope <- (bound.att - limit.c)/(1 - limit.rbar)
    interpolate.constant <- bound.att - interpolate.slope
    interpolate.c <- interpolate.slope * rbar.high +
       interpolate.constant
    bound.mut <- 1 - rbar.high + interpolate.c
    return(bound.mut)
```

```
}
```

B.9.8 boot.t.bound()

```
function(taints, confidence, B.star)
{
# This function returns the modified bootstrap-t bound as
# described by Clayton (1994, Section 4), for the mean error per
# dollar unit.  For a (Bxn) matrix of taints a (Bx1) vector of
# bounds is returned.  n represents the sample size and B the
# number of samples.  B.star gives the  number of bootstrap
# replications used to estimate each bound.  For samples
# containing no nonzero errors, the bound is set equal to the
# attributes bound for no errors.  The function boot.t.non is
# called to compute the bound for those samples with at least
# one nonzero taint.  This bound handles overstatements and
# understatements.
# Calculate the sample size and the number of errors per sample.
    n <- ncol(taints)
    m <- apply((taints != 0), 1, sum)
# Sort the matrix of taints by number of errors.
    ord <- order(m)
    taints <- matrix(taints[ord,  ], ncol = n)
# Calculate the number of overstatements and understatements.
    m.over <- apply((taints > 0), 1, sum)
    m.under <- apply((taints < 0), 1, sum)
    m <- m[ord]
# Find the number of samples with no errors.
    B.zero <- sum(m == 0)
# If at least one sample has no errors, return a vector of
# attribute bounds.
    bound.mut.zero <- switch((B.zero > 0) + 1, numeric(0),
       rep(1 - (1 - confidence)^(1/n), B.zero))
# Find the number of samples with at least one error.
    B.non <- sum(m != 0)
# If at least one sample contains one or more nonzero taints, form
# a matrix of taints for these samples and call boot.t.non.
    if(B.non > 0)
```

```
        taints.non <- matrix(taints[m != 0,  ], ncol = n)
    bound.mut.non <- switch((B.non > 0) + 1, numeric(0),
        boot.t.non(taints.non, confidence, B.star))
# Combine the two sets of bounds.
    bound.mut <- c(bound.mut.zero, bound.mut.non)
    list(bound.mut = bound.mut, m.under = m.under, m.over = m.over)
}
```

B.9.9 boot.t.non()

```
function(taints.non, confidence, B.star)
{
# This function returns the bootstrap-t bound for mut for samples
# containing at least one nonzero taint.
# Calculate the sample size and the number of samples.
    n <- ncol(taints.non)
    B.non <- nrow(taints.non)
# Initialize the vector bound and set iseed to the Splus
# random seed.
    bound <- rep(0, B.non)
    iseed <- .Random.seed
# Call the Fortran function bootnon to compute the bound.
    non <- .Fortran("bootnon",
        as.double(taints.non),
        as.integer(B.non),
        as.integer(n),
        as.double(confidence),
        as.integer(B.star),
        bound = as.double(bound),
        as.integer(iseed))
    bound.mut.non <- non$bound
    return(bound.mut.non)
}
```

B.9.10 bootnon.f()

```
subroutine bootnon(taints,nonB,n,conf,nbstar,bound,iseed)
```

```
c This subroutine determines the bootstrap-t bound for the mean
c error per dollar unit.  This function will only work for
c samples containing at least one nonzero taint.
c taint is the matrix of taints for which the bounds are to be
c determined.  n is the sample size and nonB the number of samples.
c conf is the confidence level, between 0 and 1, at which the
c bounds are to be determined.  nbstar gives the number of
c bootstrap samples to be employed.  bound is the vector of
c bounds and iseed provides a start point for the random
c number generator.

      implicit real*8 (a-h, o-z)
      parameter(nbstmx=1000)
      dimension taints(nonB,n),tstar(nbstmx),talpha(nbstmx)
      dimension bound(nonB)
      integer g05dyf
      external g05cbf, g05dyf, m01caf

c g05cbf sets the seed for the generator in g05duf.

      call g05cbf(iseed)
      do 100 i=1,nonB

c Determine the mean and standard deviation for sample i

         mnon=0
         sum=0.0d0
         sum2=0.0d0

         do 200 j=1,n
            sum=sum+taints(i,j)
            sum2=sum2+taints(i,j)**2
 200     continue

         rmean=sum/dfloat(n)
         var=(sum2-(sum**2)/dfloat(n))/(dfloat(n-1))
         sig=dsqrt(var)/dsqrt(n)
```

```
c Construct bootstrap samples by taking nbstar samples of size
c n from the original sample.  For each bootstrap sample,
c calculate the number of nonzero taints.

       istar=0

       do 300 ib=1,nbstar

          mstar=0
          sumb=0.0d0
          sumb2=0.0d0

          do 400 jb=1,n

c g05dyf generates a random integer

             inum=g05dyf(1,n)

             data=taints(i,inum)
             if (data.gt.0.0d0) then
                mstar=mstar + 1
             endif
             sumb=sumb+data
             sumb2=sumb2+data**2

400       continue

c For those bootstrap samples with at least one nonzero taint,
c calculate the mean, standard deviation and value of tstar.
c If the standard deviation of the bootstrap sample is less
c than 1/10 the standard deviation of the original sample,
c the bootstrap standard deviation is replaced by the
c original standard deviation in the computation of tstar.
          if (mstar.eq.0) then
             go to 300
          else
             istar=istar+1
          endif
```

```
          rmeanb=sumb/dfloat(n)
          varb=(sumb2-(sumb**2)/dfloat(n))/(dfloat(n-1))
          sigb=dsqrt(varb)/dsqrt(n)

          if(sigb.lt.(sig/dfloat(10))) then
             tstar(istar)=(rmeanb-rmean)/sig
          else
             tstar(istar)=(rmeanb-rmean)/sigb
          endif

300    continue

c If less than ten of the bootstrap samples have at least
c one nonzero taint, the bound is set equal to the
c attributes bound for the case of no errors.  Otherwise
c the bound is computed using the (1-conf)'th percentile
c of the empirical distribution of tstar.

       if (istar.lt.10) then
          bound(i)=1.0d0-(1.0d0-conf)**(1.0d0/dfloat(n))
       else
          call m01caf(tstar,1,istar,'a',ifail)
          rindex=dfloat(istar)*(1.0d0-conf)
          talpha(i)=tstar(min(int(rindex)+1,istar))
          if(mod(rindex,1.0d0).eq.0.0d0) then
             talpha(i)=(talpha(i)+tstar(int(rindex)))*.5d0
          endif
       endif

100   continue

      return
      end
```

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