APPLICATIONS OF A CERTAIN LINEAR OPERATOR DEFINED BY A HADAMARD PRODUCT OR CONVOLUTION

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ABSTRACT

Recently, B.C. Carlson and D.B. Shaffer introduced a linear operator defined by means of a Hadamard product (or convolution), and showed how this operator can be fruitfully applied to a systematic study of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions. The object of the present paper is to prove several new characterization theorems involving this general operator and such subclasses of analytic functions as the families of starlike and convex functions of order α $(0 \leqq \alpha < 1)$. Some further consequences of our main results are also indicated.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 \equiv 1) \tag{1.1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z: |z| < 1\}.$$

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be <u>starlike of order</u> α if and only if it satisfies the inequality:

$$(1.2) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{A}$ which are starlike of order α in $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex of order α if and only if it satisfies the inequality:

$$(1.3) \qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. Denoting by $\mathcal{K}(\alpha)$ the class of all functions in $\mathcal{A}$ which are convex of order α in $\mathcal{U}$, it is easily seen that

$$(1.4) \qquad f(z) \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leqq \alpha < 1)$$

and that

$$(1.5) \qquad \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \text{ and } \mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K} \qquad (0 \leqq \alpha < 1).$$

For the functions $f_j(z)$ $(j = 1, 2)$ defined by

$$(1.6) \qquad f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1},$$

we denote (as usual) the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

(1.7) $$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1}.$$

Making use of (1.7), we recall a linear operator $\mathcal{L}(a,c)$ defined by

(1.8) $$\mathcal{L}(a,c)f(z) = \left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right] * f(z)$$

for

$$f(z) \in \mathcal{A} \quad \text{and} \quad c \neq 0, -1, -2, \cdots,$$

where $(\lambda)_n$ is the Pochhammer symbol given by

(1.9) $$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if} \quad n \in \mathcal{N} = \{1,2,3,\ldots\}. \end{cases}$$

Clearly, $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c,a)$ is an inverse of $\mathcal{L}(a,c)$ provided that

$$a \neq 0, -1, -2, \cdots .$$

The operator $\mathcal{L}(a,c)$ was introduced by Carlson and Shaffer [1] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions. The special case of the operator $\mathcal{L}(a,c)$ when $a = 2$ is closely related to some operators of fractional calculus considered in the literature (cf. [4] and [8]). In the present paper we give several new properties and applications of the general operator $\mathcal{L}(a,c)$ involving various subclasses of analytic functions introduced above.

2. STARLIKENESS AND CONVEXITY OF $\mathcal{L}(a,c)f(z)$

We begin by recalling the following results which will be needed in our investigation of starlikeness and convexity of the function $\mathcal{L}(a,c)f(z)$.

LEMMA 1 ([6, Lemma 2.4]). *Let* $h(z)$ *and* $g(z)$ *be analytic in the open unit disk* $\mathcal{U}$ *and satisfy*

$$h(0) = g(0) = 0, \quad h'(0) \neq 0, \quad g'(0) \neq 0.$$

Suppose that, for each σ $(|\sigma| = 1)$ *and* $\rho(|\rho| = 1)$,

$$h(z) * \left[\frac{1 + \rho\sigma z}{1 - \sigma z}\right] g(z) \neq 0 \qquad (z \in \mathcal{U} - \{0\}). \tag{2.1}$$

Then, for each function $F(z)$ *analytic in the open unit disk* $\mathcal{U}$ *and satisfying the inequality*:

$$\mathrm{Re}\{F(z)\} > 0 \qquad (z \in \mathcal{U}),$$

$$\mathrm{Re}\left\{\frac{h * G(z)}{h * g(z)}\right\} > 0 \qquad (z \in \mathcal{U}), \tag{2.2}$$

where $G(z) = F(z)g(z)$.

LEMMA 2 ([5, Theorem 1]; see also [9] for the special case $\alpha = 0$). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*(\alpha)$. *Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leqq \frac{r\ \log\{(1+r)^{2(1-\alpha)}|f(z)|/r\}}{(1-r)\log\{(1+r)/(1-r)\}} + 1 \tag{2.3}$$

for $|z| = r < 1$. *Equality in* (2.3) *holds true for the function* $f(z)$ *defined by*

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{2.4}$$

with $z = r$.

LEMMA 3 ([7, Corollary 1]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*(\alpha)$. *Then*

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \alpha + (1-\alpha)(1-r^2)\left[\frac{|f(z)|}{r}\right]^{1/(1-\alpha)} \tag{2.5}$$

and

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \leqq \frac{1 + (1-2\alpha)r}{1 - r} + \frac{2r\ \log\{(1-r)^{2(1-\alpha)}|f(z)|/r\}}{(1-r^2)\log\{(1+r)/(1-r)\}} \tag{2.6}$$

for $|z| = r < 1$.

By appealing to Lemma 1, we shall prove our first result stated as

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(\alpha)$ and let

$$(2.7) \qquad \mathcal{L}(a,c)\left[\frac{1+\rho\sigma z}{1-\sigma z} f(z)\right] \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for

$$c \neq 0, -1, -2, \cdots .$$

Then $\mathcal{L}(a,c)f(z)$ is also in the class $\mathcal{S}^*(\alpha)$.

PROOF. It is sufficient to show that

$$(2.8) \qquad \mathrm{Re}\left\{\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right\} > \alpha$$

for $z \in \mathcal{U}$. Since

$$(2.9) \qquad \mathrm{Re}\left\{\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right\} = \mathrm{Re}\left\{\frac{\mathcal{L}(a,c)\{zf'(z)\}}{\mathcal{L}(a,c)f(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right] * \{zf'(z)\}}{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right] * f(z)}\right\},$$

setting

$$h(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad F(z) = \frac{zf'(z)}{f(z)} - \alpha, \quad \text{and} \quad g(z) = f(z)$$

in Lemma 1, we readily observe that

$$\text{(2.10)} \qquad \mathrm{Re}\left\{\frac{h * G(z)}{h * g(z)}\right\} = \mathrm{Re}\left\{\frac{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right] * \{zf'(z) - \alpha f(z)\}}{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right] * f(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right\} - \alpha$$

$$> 0,$$

which evidently completes the proof of Theorem 1.

COROLLARY 1. <u>Under the hypotheses of Theorem</u> 1,

$$\text{(2.11)} \qquad \left|\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right| \leqq \frac{r \log\{(1+r)^{2(1-\alpha)}|\mathcal{L}(a,c)f(z)|/r\}}{(1-r)\log\{(1+r)/(1-r)\}} + 1$$

<u>for</u> $|z| = r < 1$. <u>Equality in</u> (2.11) <u>holds true for the function</u> $f(z)$ <u>given by</u>

$$\text{(2.12)} \qquad f(z) = \mathcal{L}(c,a)\left[\frac{z}{(1-z)^{2(1-\alpha)}}\right] \qquad (a \neq 0, -1, -2, \cdots).$$

provided that the function on the right-hand side is in the class $\mathcal{S}^*(\alpha)$, that is, that

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(a)_n} z^{n+1} \in \mathcal{K} \supseteq \mathcal{K}(\alpha) \qquad (0 \leqq \alpha < 1).$$

PROOF. Corollary 1 follows immediately from Theorem 1 if we apply Lemma 2.

Next, by applying Lemma 3 to Theorem 1, we deduce

COROLLARY 2. Under the hypotheses of Theorem 1,

$$\text{(2.13)} \qquad \operatorname{Re}\left\{\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right\} \geqq \alpha + (1-\alpha)(1-r^2)\left[\frac{|\mathcal{L}(a,c)f(z)|}{r}\right]^{1/(1-\alpha)}$$

and

$$\text{(2.14)} \qquad \operatorname{Re}\left\{\frac{z\{\mathcal{L}(a,c)f(z)\}'}{\mathcal{L}(a,c)f(z)}\right\}$$

$$\leqq \frac{1+(1-2\alpha)r}{1-r} + \frac{2r\,\log\{(1-r)^{2(1-\alpha)}|\mathcal{L}(a,c)f(z)|/r\}}{(1-r^2)\log\{(1+r)/(1-r)\}}$$

for $|z| = r < 1$.

Our main result involving the convexity of the function $\mathcal{L}(a,c)f(z)$ is contained in

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}(\alpha)$ and let

$$\mathcal{L}(2,1)\mathcal{L}(a,c)\left[\frac{1+\rho\sigma z}{1-\sigma z}\, f(z)\right] \neq 0 \qquad (z \in \mathcal{U} - \{0\}) \tag{2.15}$$

for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$, and for

$$c \neq 0, -1, -2, \cdots .$$

Then $\mathcal{L}(a,c)f(z)$ is also in the class $\mathcal{K}(\alpha)$.

PROOF. By using (1.4) and Theorem 1, we observe that

$$\begin{aligned} f(z) \in \mathcal{K}(\alpha) &\Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \\ &\Rightarrow \mathcal{L}(a,c)\{zf'(z)\} \in \mathcal{S}^*(\alpha) \\ &\Leftrightarrow z\{\mathcal{L}(a,c)f(z)\}' \in \mathcal{S}^*(\alpha) \\ &\Leftrightarrow \mathcal{L}(a,c)f(z) \in \mathcal{K}(\alpha), \end{aligned}$$

which completes the proof of Theorem 2.

3. FURTHER APPLICATIONS OF THE OPERATOR $\mathcal{L}(a,c)$

With a view to deriving some further results characterizing the operator $\mathcal{L}(a,c)$, we recall here the following lemmas.

LEMMA 4 ([6, Lemma 2.7]; see also [2]). Let $h(z) \in \mathcal{K}$ and $g(z) \in \mathcal{S}^*$. Then, for each function $F(z)$ analytic in $\mathcal{U}$ and satisfying

$$\text{Re}\{F(z)\} > 0 \qquad (z \in \mathcal{U}),$$

$$\text{Re}\left\{\frac{h * G(z)}{h * g(z)}\right\} > 0 \qquad (z \in \mathcal{U}), \tag{3.1}$$

where $G(z) = F(z)g(z)$.

LEMMA 5 ([3, Theorem 1]). *Given* μ, *with* $-\infty < \mu < \infty$, *let*

$$(3.2) \qquad f_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\mu} z^{n+1}$$

for $z \in \mathcal{U}$. *Then* $f_\mu(z)$ *is in the class* $\mathcal{K}$ *whenever* $\mu \geqq 0$.

We now state

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{A}$ *and let*

$$(3.3) \qquad \mathcal{L}(a,c)\left[\frac{1+\rho\sigma z}{1-\sigma z}\{f_\mu * f(z)\}\right] \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

for each σ $(|\sigma| = 1)$ *and* ρ $(|\rho| = 1)$, *where* $f_\mu(z)$ *is given by* (3.2), *and*

$$\mu \geqq 0; \quad c \neq 0, -1, -2, \cdots .$$

Then $\mathcal{L}(a,c)\{f_\mu * f(z)\}$ *is in the class* $\mathcal{S}^* \equiv \mathcal{S}^*(0)$.

PROOF. Since

$$(3.4)\qquad \operatorname{Re}\left\{\frac{z[\mathcal{L}(a,c)\{f_\mu * f(z)\}]'}{\mathcal{L}(a,c)\{f_\mu * f(z)\}}\right\}$$

$$= \operatorname{Re}\left\{\frac{\mathcal{L}(a,c)\{f * zf_\mu'(z)\}}{\mathcal{L}(a,c)\{f * f_\mu(z)\}}\right\}$$

$$= \operatorname{Re}\left\{\frac{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}\right] * \{zf_\mu'(z)\}}{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}\right] * f_\mu(z)}\right\},$$

we find from Lemma 5 that

$$f_\mu(z) \in \mathcal{K} \quad \mathcal{S}^* \quad \text{for} \quad \mu \geqq 0.$$

Setting

$$(3.5)\qquad h(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}, \quad F(z) = \frac{zf_\mu'(z)}{f_\mu(z)}, \quad \text{and} \quad g(z) = f_\mu(z)$$

in Lemma 1, we conclude that

$$\mathcal{L}(a,c)\{f_\mu * f(z)\} \in \mathcal{S}^*,$$

which precisely is the assertion of Theorem 3.

The following immediate consequence of Theorem 3 is worthy of mention.

COROLLARY 3. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.1) <u>be in the class</u> $\mathcal{S}^*$ <u>and satisfy the condition</u> (3.2) <u>for each</u> σ ($|\sigma| = 1$) <u>and</u> ρ ($|\rho| = 1$). <u>Then</u> $\mathcal{L}(a,c)\{f_\mu * f(z)\}$ <u>is also in the class</u> $\mathcal{S}^*$.

Finally, we prove

THEOREM 4. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}(\alpha)$ and let

$$\mathcal{L}(2,1)\mathcal{L}(a,c)\left[\frac{1+\rho\sigma z}{1-\sigma z}\{f_{\mu}*f(z)\}\right] \neq 0 \qquad (z \in \mathcal{U} - \{0\}) \tag{3.6}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), where $f_{\mu}(z)$ is given by (3.2), and

$$\mu \geqq 0; \quad c \neq 0, -1, -2, \cdots .$$

Then $\mathcal{L}(a,c)\{f_{\mu}*f(z)\}$ is also in the class $\mathcal{K}(\alpha)$.

PROOF. Putting

$$F(z) = \frac{z\{zf'(z)\}'}{zf'(z)} - \alpha, \quad g(z) = zf'(z), \text{ and } \quad h(z) = f_{\mu}(z) \tag{3.7}$$

in Lemma 4, we have

$$\begin{aligned} \operatorname{Re}\left\{\frac{h * G(z)}{h * g(z)}\right\} &= \operatorname{Re}\left\{\frac{f_{\mu} * [z\{zf'(z)\}' - \alpha zf'(z)]}{f_{\mu} * \{zf'(z)\}}\right\} \\ &= \operatorname{Re}\left\{\frac{z[f_{\mu}*\{zf'(z)\}]'}{f_{\mu} * \{zf'(z)\}}\right\} - \alpha \\ &> 0, \end{aligned} \tag{3.8}$$

which shows that $f_{\mu} * \{zf'(z)\} \in \mathcal{S}^{*}(\alpha)$. Therefore, by appealing to Theorem 1, we find that

$$(3.9) \qquad f(z) \in \mathcal{K}(\alpha) \Rightarrow f_\mu * \{zf'(z)\} \in \mathcal{S}^*(\alpha)$$
$$\Rightarrow \mathcal{L}(a,c)[f_\mu*\{zf'(z)\}] \in \mathcal{S}^*(\alpha)$$
$$\Leftrightarrow z[\mathcal{L}(a,c)\{f_\mu*f(z)\}]' \in \mathcal{S}^*(\alpha)$$
$$\Leftrightarrow \mathcal{L}(a,c)\{f_\mu*f(z)\} \in \mathcal{K}(\alpha),$$

which indeed is asserted by Theorem 4.

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