A NOTE ON A CERTAIN INVERSE PAIR OF MULTIPLE SERIES IDENTITIES

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DMS-676-IR

August 1994

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Abstract

We develop a multiple-series generalization of certain series identities considered by T.J. Osler [2] and H.M. Srivastava [4] and derive its inverse. We also apply our pair of multiple series identities to the generalized Kampé de Fériet hypergeometric function in several variables.

1. Introduction

For a suitably bounded multiple sequence $\{C(n_1,\cdots,n_r)\}$, let a general function of r variables be defined by

$$f(z_1,\cdots,z_r)=\sum_{n_1,\cdots,n_r=0}^{\infty} C(n_1,\cdots,n_r)\,z_1^{n_1}\cdots z_r^{n_r} \tag{1.1}$$

$$(|z_j|<R_j;\quad R_j>0;\quad j\in\{1,\cdots,r\}),$$

provided that the multiple series converges absolutely.

Srivastava [4, p. 197] gave the multiple series identity (see also Srivastava and Manocha [6, p. 217, Problem 12]):

$$\sum_{n_1,\cdots,n_r=0}^{\infty} C(n_1,\cdots,n_r)=\sum_{M_1=0}^{N_1-1}\cdots\sum_{M_r=0}^{N_r-1} \cdot\left(\sum_{n_1,\cdots,n_r=0}^{\infty} C(n_1N_1+M_1,\cdots,n_rN_r+M_r)\right) \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 33C65, 33C70; Secondary 33B15.
Key words and phrases. Multiple sequence, multiple series identities, Kampé de Fériet function, inverse series relation, Srivastava's identity, Gauss's multiplication formula.

where N_j $(j = 1, \cdots, r)$ are arbitrary positive integers.

The single-series analogue of (1.2), when $r = 1$, stated in [4, p. 193, Equation (8)] is, in fact, an inverse series relation of a result given earlier by Osler [2, p. 889, Equation (2)]. The general series identity in both of these papers generates precisely the same hypergeometric series identity. The particular case of (1.2), when $r = 2$, was also considered by Sharma [3]. It would thus seem worthwhile to develop a multiple-series generalization of the aforecited series identity and also derive its inverse. The applications of our pair of series identities would be manifold. We, however, illustrate the use by deducing a series identity involving the generalized Kampé de Fériet function.

2. A Theorem on Multiple Series Identities

The inverse pair of multiple series identities which we propose to derive is contained in the following:

Theorem. *For each positive integer N_j, let $M_j \in \{0, 1, 2, \cdots, N_j - 1\}$ $(j \in \{1, \cdots, r\})$, and suppose that there exists a number $w_j = \exp(2\pi i/N_j)$ $(j \in \{1, \cdots, r\})$. Then, corresponding to the multivariable function $f(z_1, \cdots, z_r)$ defined by* (1.1), *there exist the following pair of multiple series identities:*

$$\sum_{m_1, \cdots, m_r = 0}^{\infty} C(m_1 N_1 + M_1, \cdots, m_r N_r + M_r)\, z_1^{m_1 N_1 + M_1} \cdots z_r^{m_r N_r + M_r}$$
$$= \left(\prod_{s=1}^{r} N_s \right)^{-1} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} f(z_1 w_1^{k_1}, \cdots, z_r w_r^{k_r})\, w_1^{-M_1 k_1} \cdots w_r^{-M_r k_r} \tag{2.1}$$

and

$$f(z_1, \cdots, z_r) = \sum_{M_1=0}^{N_1-1} \cdots \sum_{M_r=0}^{N_r-1} \sum_{m_1, \cdots, m_r=0}^{\infty}$$
$$\cdot\, C(m_1 N_1 + M_1, \cdots, m_r N_r + M_r)\, z_1^{m_1 N_1 + M_1} \cdots z_r^{m_r N_r + M_r}, \tag{2.2}$$

provided that each side of (2.1) *and* (2.2) *has a meaning.*

Proof. Denoting the right-hand side of (2.1) by I, we have

$$
\begin{aligned}
I &= \left(\prod_{s=1}^{r} N_s\right)^{-1} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} \left(\sum_{n_1,\cdots,n_r=0}^{\infty} C(n_1,\cdots,n_r) \right. \\
&\qquad \left. \cdot\, z_1^{n_1} \cdots z_r^{n_r} \right) w_1^{(n_1-M_1)k_1} \cdots w_r^{(n_r-M_r)k_r} \\
&= \left(\prod_{s=1}^{r} N_s\right)^{-1} \sum_{n_1,\cdots,n_r=0}^{\infty} \left(\sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} \right. \\
&\qquad \left. \cdot\, w_1^{(n_1-M_1)k_1} \cdots w_r^{(n_r-M_r)k_r} \right) C(n_1,\cdots,n_r)\, z_1^{n_1} \cdots z_r^{n_r}.
\end{aligned}
\tag{2.3}
$$

Now noting that (see also Osler [2])

$$
\begin{aligned}
\sum_{k_j=0}^{N_j-1} w_j^{(n_j-M_j)k_j} &= \sum_{k_j=0}^{N_j-1} \exp\bigl(2\pi i(n_j-M_j)\, k_j/N_j\bigr) \\
&= N_j, \qquad \text{if } n_j - M_j = N_j m_j \quad (m_j = 0,1,2,\cdots) \\
&= 0, \qquad \text{otherwise,}
\end{aligned}
\tag{2.4}
$$

(2.3) leads to the desired left-hand side of (2.1), and this proves the multiple series identity (2.1).

To prove the inverse relation (2.2), let us effect the multiple sum over M_j from 0 to $N_j - 1$, for all $j = 1, \cdots, r$, on both the sides of (2.1); then, keeping in mind the relation:

$$
\begin{aligned}
\sum_{M_j=0}^{N_j-1} w_j^{-M_j k_j} &= \sum_{M_j=0}^{N_j-1} \exp\bigl(2\pi i(-M_j k_j)/N_j\bigr) \\
&= N_j, \qquad \text{if } k_j = 0, \\
&= 0, \qquad \text{otherwise,}
\end{aligned}
\tag{2.5}
$$

the multiple series identity (2.2) follows.

Remark 1. The identity (2.2) would also follow at once from Srivastava's identity [4, p. 197, Equation (24)] on replacing

$$C(n_1,\cdots,n_r) \quad \text{by} \quad C(n_1,\cdots,n_r)\, z_1^{n_1} \cdots z_r^{n_r}.$$

3. Applications

We choose the multiple sequence $\{C(n_1, \cdots, n_r)\}$ as follows:

$$C(n_1, \cdots, n_r) = \frac{\prod_{j=1}^{p} (a_j)_{n_1+\cdots+n_r} \prod_{j=1}^{p_1} (b'_j)_{n_1} \cdots \prod_{j=1}^{p_r} (b_j^{(r)})_{n_r}}{\prod_{j=1}^{q} (\alpha_j)_{n_1+\cdots+n_r} \prod_{j=1}^{q_1} (\beta'_j)_{n_1} \cdots \prod_{j=1}^{q_r} (\beta_j^{(r)})_{n_r}} \cdot (n_1! \cdots n_r!)^{-1} \tag{3.1}$$

where, as usual, $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$.

From Gauss's multiplication formula and the simple relations [6, pp. 22-23], we have

$$(\lambda)_{Nm+M} = (\lambda)_M \, N^{Nm} \prod_{j=0}^{N-1} \left(\frac{\lambda + M + j}{N} \right)_m . \tag{3.2}$$

If we denote by $(a_{p_i}^{(i)})$ the array of p_i parameters

$$a_1^{(i)}, \cdots, a_{p_i}^{(i)} \qquad (p_i = 1, 2, 3, \cdots; \ \ i = 1, \cdots, r),$$

and by $[(a_{p_i}^{(i)}) : N]$ the array of Np_i parameters

$$a_j^{(i)}/N, \ \ (a_j^{(i)} + 1)/N, \cdots, (a_j^{(i)} + N - 1)/N \qquad (j = 1, \cdots, p_i; \ i = 1, \cdots, r),$$

then, upon substituting from (3.1) into (2.1), setting $N_i = N$ $(i = 1, \cdots, r)$, and suitably using the relation (3.2), we arrive at the following result (*cf.* [6, p. 65]; see also [5, p. 454]):

$$F^{pN:p_1N+1;\cdots;p_rN+1}_{qN:q_1N+N;\cdots;q_rN+N} \left(\begin{matrix} [(a_p) + M_1 + \cdots + M_r; \, N] & : & [(b'_{p_1}) + M_1; \, N], 1; \cdots; \\ [(\alpha_q) + M_1 + \cdots + M_r; \, N] & : & [(\beta'_{q_1+1}) + M_1; \, N]; \cdots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b_{p_r}^{(r)}) + M_r; \, N], 1; \\ [(\beta_{q_r+1}^{(r)}) + M_r; \, N]; \end{matrix} \ \ (z_1 \, N^{\lambda_1})^N, \cdots, (z_r \, N^{\lambda_r})^N \right) \tag{3.3}$$

$$= N^{-r} \, \Delta(M_1, \cdots, M_r) \prod_{i=1}^{r} \left\{ z_i^{-M_i} \sum_{k_i=0}^{N-1} w^{-M_i k_i} \right\}$$

$$\cdot \, F^{p:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r} \left(z_1 w^{k_1}, \cdots, z_r w^{k_r} \right),$$

where

(i) $\beta'_{q_1+1} = \cdots = \beta^{(r)}_{q_r+1} = 1$,

(ii) $1 + q + q_j - p - p_j \geq 0 \quad (j \in \{1, \cdots, r\})$; the equality holds when $|z_i|$ are suitably restricted;

$$\lambda_i = p + p_i - q - q_i - 1 \qquad (i \in \{1, \cdots, r\}), \tag{3.4}$$

and

$$\Delta(M_1, \cdots, M_r) = \frac{\prod_{i=1}^{q} (\alpha_i)_{M_1+\cdots+M_r} \prod_{i=1}^{q_1+1} (\beta'_i)_{M_1} \cdots \prod_{i=1}^{q_r+1} (\beta^{(r)}_i)_{M_r}}{\prod_{i=1}^{p} (a_i)_{M_1+\cdots+M_r} \prod_{i=1}^{p_1+1} (b'_i)_{M_1} \cdots \prod_{i=1}^{p_r+1} (b^{(r)}_i)_{M_r}}. \tag{3.5}$$

The inverse of the identity (3.3) can be deduced in a similar manner from (2.2). The result thus obtained is

$$F^{p:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r}(z_1, \cdots, z_r) = \sum_{M_1=0}^{N-1} \cdots \sum_{M_r=0}^{N-1} [\Delta(M_1, \cdots, M_r)]^{-1}$$

$$\cdot z_1^{M_1} \cdots z_r^{M_r} \; F^{pN:p_1N+1;\cdots;p_rN+1}_{qN:q_1N+N;\cdots;q_rN+N} \left(\begin{matrix} [(a_p) + M_1 + \cdots + M_r; N] : \\ [(\alpha_q) + M_1 + \cdots + M_r; N] : \end{matrix} \right. \tag{3.6}$$

$$\left. \begin{matrix} [(b'_{p_1}) + M_1; N], 1; \cdots; [(b^{(r)}_{p_r}) + M_r; N], 1; \\ [(\beta'_{q_1+1}); N]; \cdots; [(\beta_{q_r+1}) + M_r; N]; \end{matrix} (z_1 N^{\lambda_1})^N, \cdots, (z_r N^{\lambda_r})^N \right),$$

provided that Conditions (i) and (ii) stated with (3.3) are satisfied, $\Delta(M_1, \cdots, M_r)$ and λ_i $(i = 1, \cdots, r)$ being defined by (3.4) and (3.5) above.

Remark 2. It should be noted that any one of the numerator parameters 1 can get cancelled by one of the denominator parameters

$$(\beta'_{q_1+1} + M_1)/N, \; (\beta'_{q_1+1} + M_1 + 1)/N, \cdots, (\beta'_{q_1+1} + M_1 + N - 1)/N,$$

and so on, which is due to the fact that

$$\beta'_{q_1+1} = \cdots = \beta^{(r)}_{q_r+1} = 1.$$

Remark 3. For $r = 1$, the pair of identities (3.3) and (3.6) would correspond essentially to the identities given in [2]. Furthermore, the pair of identities stated in [3] would correspond to (3.3) and (3.6), when $r = 2$, and when z_i is replaced by $z_i N^{-\lambda_i}$ $(i = 1, \cdots, r)$, λ_i being defined by (3.4).

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353. The first-named author wishes to thank the University Grants Commission of India for providing financial support for this project.

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