Saturation Problems on Graphs

by

Shannon Elizabeth Adele Ogden
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Supervisory Committee

Dr. Natasha Morrison, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. Kieka Mynhardt, Co-Supervisor
(Department of Mathematics and Statistics)

ABSTRACT

In this thesis, we consider two variations on classical saturation problems in extremal graph theory: rainbow saturation and weak saturation.

An edge-coloured graph G is *rainbow* if every edge in G receives a distinct colour. Given a graph H, an edge-coloured graph G is *H-rainbow-saturated* if G does not contain a rainbow copy of H, but the addition of any non-edge to G, in any colour from $\mathbb{N}$, creates a rainbow copy of H. The *rainbow saturation number* of H, denoted by $\mathrm{rsat}(n, H)$, is the minimum number of edges in an H-rainbow saturated graph on n vertices. In Chapter 2, we prove that, like ordinary saturation numbers, the rainbow saturation number of every graph H is linear in n. This result confirms a conjecture of Girão, Lewis, and Popielarz.

In Chapter 3, we consider a specific type of weak saturation known as r-bond bootstrap percolation. In the *r-bond bootstrap percolation* process on a graph G, we start with a set of initially *infected* edges of G, and consider all other edges in G to be *healthy.* At each subsequent step in the process, the infection spreads to a healthy edge if at least one of its endpoints is incident with at least r infected edges. Once an edge is infected, it remains infected indefinitely. If a set of initially infected edges will eventually infect all of $E(G)$, we refer to it as an *r-percolating set* of G. Define $m_e(G, r)$ to be the minimum number of edges in an r-percolating set of G.

Recently, Hambardzumyan, Hatami, and Qian introduced a clever new polynomial method, which they used to provide recursive formulas for $m_e(G, r)$ when G is either a d-dimensional torus or a d-dimensional grid. We push this polynomial method further, in order to determine $m_e(G, r)$ for certain other graphs G. In particular, we provide recursive formulas for $m_e(G, r)$ when G is a Cartesian product of stars or a Cartesian product of joined cycles (cycles with a single chord). We also give upper and lower bounds on $m_e(G, r)$ when G is a Cartesian product of a tree with any graph H, and examine the conditions under which these bounds match.

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I would also like to express my deepest appreciation to Natasha for pushing me out of my comfort zone and into the world of extremal graph theory. Looking back, it does not seem quite so scary after all. This thesis does not mark the end of the adventure, only the end of the beginning; I cannot wait to see all that we will accomplish together during my Ph.D.

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Chapter 1

Introduction

In this thesis, we will consider two saturation problems in extremal graph theory. This chapter provides a brief introduction to saturation problems, including the two variations examined in this thesis: rainbow saturation and weak saturation. Additional history and previous results can be found in the subsequent chapters, in order to place our work in the proper context.

1.1 Saturation

Extremal graph theory is an expansive area of research focused on graphs that are minimal or maximal with respect to some global property. One aspect of extremal graph theory is graph saturation problems, which look to minimize the total number of edges in a graph while maintaining some specified subgraph structures.

A graph G is said to be *H-free* if H is not a subgraph of G. Given graphs G and H, a spanning subgraph F of G is *(G, H)-saturated* if F is H-free, but for every edge $e \in E(G) \setminus E(F)$, the graph $F + e$ contains a copy of H as a subgraph. In this case, we call G the *host graph*. The *saturation number* of H in G, denoted by $\mathrm{sat}(G, H)$, is the minimum number of edges in such a graph F; that is,

$$\mathrm{sat}(G, H) := \min\{|E(F)| : F \text{ is } (G, H)\text{-saturated}\}.$$

When $G = K_n$, we write $\text{sat}(n, H)$ instead of $\text{sat}(K_n, H)$, where as usual K_n denotes the complete n-vertex graph.

The saturation number can be thought of as a dual to the classical Turán extremal number $\text{ex}(n, H)$, which denotes the maximum number of edges in an H-free graph on n vertices. Since a maximal H-free graph is H-saturated, $\text{ex}(n, H)$ equivalently denotes the maximum number of edges in an H-saturated graph on n vertices. The saturation number $\text{sat}(n, H)$, in contrast, denotes the minimum number of edges among such graphs.

Saturation was first studied independently by Zykov [27] and Erdős, Hajnal, and Moon [9], who determined the exact value of $\text{sat}(n, H)$ when H is a complete graph.

Theorem 1.1 [9, 27] *Let* $n \geq m \geq 2$. *Then* $\text{sat}(n, K_m) = (m-2)(n-1) - \binom{m-2}{2}$.

Saturation problems have since become a widely studied area of extremal graph theory. In particular, Kászonyi and Tuza [18] proved that the saturation number of every graph H is linear in n.

Theorem 1.2 [18] *For any graph* H, $\text{sat}(n, H) = O(n)$.

For more information, and many other results related to the saturation number, see the survey of Faudree, Faudree, and Schmitt [10]. In this thesis, we examine two variations on the classic saturation problem: rainbow saturation and weak saturation. Section 1.2 provides background information on rainbow saturation, while Section 1.3 considers weak saturation.

1.2 Rainbow Saturation

For $n \in \mathbb{Z}^+$, we denote by $[n]$ the set of integers $\{1, ..., n\}$. A *t-edge-colouring* of a graph G is a function $c : E(G) \rightarrow [t]$. A t-edge-colouring of a graph G is *proper* if every pair of edges incident with a common vertex receive distinct colours, and *rainbow* if every edge in G is assigned a distinct colour. A *t-edge-coloured graph* is an ordered pair (G, c) where c is a (not necessarily proper) t-edge-colouring of G. When the pallet of possible colours is unlimited (or unimportant), we simply refer to c as an *edge-colouring* of G, and say that (G, c) is an *edge-coloured graph*. For ease of notation, when the edge-colouring c is clear from context, we often refer to an edge-coloured graph (G, c) as simply G.

The generalization of saturation to edge-coloured graphs was first considered by Hanson and Toft [15]. Following this, Barrus, Ferrara, Vandenbussche, and Wenger [4] considered the particular case of t-rainbow saturation, where there are exactly t colours available. Specifically, a t-edge-coloured graph (G, c) is *(H, t)-rainbow saturated* if G does not contain a rainbow copy of H, but the addition of any non-edge e, in any colour from $[t]$, creates a rainbow copy of H in $G+e$. Note that this requires $t \geq |E(H)|$. The *t-rainbow saturation number* of H, denoted by $\operatorname{rsat}_t(n, H)$, is the minimum number of edges in an (H, t)-rainbow-saturated graph on n vertices; that is,

$$\operatorname{rsat}_t(n, H) := \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } (H, t)\text{-rainbow-saturated}\}.$$

The main focus of the study of t-rainbow saturation has been on the asymptotic behaviour of $\operatorname{rsat}_t(n, H)$ as n approaches infinity. Recall that Kászonyi and Tuza [18] proved that the saturation number of any graph H is linear in n (see Theorem 1.2). However, as Barrus, Ferrara, Vandenbussche, and Wenger [4] showed, there exist graphs H for which $\operatorname{rsat}_t(n, H)$ grows much faster (as a function of n) than the ordinary saturation number, $\operatorname{sat}(n, H)$.

Theorem 1.3 [4] *Let $t \geq k \geq 2$ be integers. Then* $\operatorname{rsat}_t(n, K_{1,k}) = \Theta(n^2)$.

In addition, Barrus, Ferrara, Vandenbussche, and Wenger [4] proved the following bounds on the t-rainbow saturation numbers of complete graphs.

Theorem 1.4 [4] *Let $r \geq 3$ and $t \geq \binom{r}{2}$ be integers. There exist positive constants c_1, c_2, n_0 such that, for all $n \geq n_0$,*

$$c_1 \frac{n \log(n)}{\log(\log(n))} \leq \mathrm{rsat}_t(n, K_r) \leq c_2 n \log(n).$$

The lower bound on the t-rainbow saturation numbers of complete graphs was improved independently by Ferrara, Johnston, Loeb, Pfender, Schulte, Smith, Sullivan, Tait, and Tompkins [11], by Girão, Lewis, and Popielarz [13], and by Korándi [19], proving that $\mathrm{rsat}_t(n, K_r)$ behaves asymptotically like $n \log(n)$.

Theorem 1.5 [11, 13, 19] *Let $r \in \mathbb{Z}^+$. If $t \geq \binom{r}{2}$, then $\mathrm{rsat}_t(n, K_r) = \Theta(n \log(n))$.*

Girão, Lewis, and Popielarz [13] also initiated the study of the natural generalization of t-rainbow saturation to the case where the palette of colours available is unlimited, rather than bounded by t. Specifically, an edge-coloured graph (G, c) is *H-rainbow saturated* if G does not contain a rainbow copy of H, but the addition of any non-edge e, in any colour from $\mathbb{N}$, creates a rainbow copy of H in $G + e$. The *rainbow saturation number* of H, denoted by $\mathrm{rsat}(n, H)$, is the minimum number of edges in an H-rainbow saturated graph on n vertices; that is,

$$\mathrm{rsat}(n, H) := \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-rainbow-saturated}\}.$$

In this thesis, we focus on the case where the pallet of possible colours is unlimited.

Girão, Lewis, and Popielarz [13] conjectured that, like ordinary saturation numbers, the rainbow saturation number of any non-empty graph H is linear in n, and proved this for graphs with some particular properties.

Theorem 1.6 [13] *Let H be a graph with a non-pendant edge that is not contained in a triangle. Then $\mathrm{rsat}(n, H) = O(n)$.*

Note that this conjecture is in stark contrast to t-rainbow saturation numbers, which, as shown in Theorems 1.3 and 1.5, can grow considerably faster than n. In Chapter 2, we prove that the rainbow saturation number of every graph H is linear in n, thus confirming the conjecture of Girão, Lewis, and Popielarz [13].

Theorem 1.7 *Every non-empty graph H satisfies* $\mathrm{rsat}(n, H) = O(n)$.

1.3 Weak Saturation

In 1968, motivated by the problem of determining the saturation number of k-uniform hypergraphs, Bollobás [7] introduced the related notion of weak saturation.

Given graphs G and H, a spanning subgraph F of G is *weakly-(G, H)-saturated* if F is H-free, but the edges of $E(G) \setminus E(F)$ can be added one at a time so that each addition creates a new copy of H. The *weak saturation number* of H in G, denoted by $\mathrm{wsat}(G, H)$, is the minimum number of edges in such a graph F; that is,

$$\mathrm{wsat}(G, H) := \min\{|E(F)| : F \text{ is weakly-}(G, H)\text{-saturated}\}.$$

As with saturation, we refer to G as the *host graph*, and write $\mathrm{wsat}(n, H)$ instead of $\mathrm{wsat}(K_n, H)$ when $G = K_n$.

Note that, if F is (G, H)-saturated, then F is also weakly-(G, H)-saturated, where we can take any ordering of the edges in $E(G) \setminus E(F)$. Therefore,

$$\mathrm{wsat}(G, H) \le \mathrm{sat}(G, H).$$

In the paper introducing weak saturation, Bollobás [7] determined the exact value of $\mathrm{wsat}(n, K_m)$ for $3 \le m < 7$, and conjectured that, for any integer m, the weak saturation number $\mathrm{wsat}(n, K_m)$ would be equal to the saturation number $\mathrm{sat}(n, K_m)$. This conjecture was first proved by Lovász [20] in 1977 using a beautiful generalization of the Bollobás Two Families Theorem [6].

Theorem 1.8 [20] *Let* $n \geq m \geq 2$. *Then* $\mathrm{wsat}(n, K_m) = (m-2)(n-1) - \binom{m-2}{2}$.

Theorem 1.8 was later proved independently by Alon [1], Frankl [12], and Kalai [16, 17]. It is worth noting that these proofs all rely on various algebraic techniques. At this time, no purely combinatorial proof of Theorem 1.8 is known.

The concept of weak saturation is intrinsically linked with that of bootstrap percolation. In particular, weak saturation is also commonly referred to as *graph bootstrap percolation*. In this thesis, we restrict our attention to $\mathrm{wsat}(G, S_k)$, where the *star* S_k is the complete bipartite graph $K_{1,k}$, for $k \geq 1$. This specific instance of weak saturation is also referred to as r-bond bootstrap percolation.

In the *r-bond bootstrap percolation* process on a graph G, we start with a set of initially *infected* edges of G, and consider all other edges in G to be *healthy*. At each subsequent step in the process, the infection spreads to a healthy edge if at least one of its endpoints is incident with at least r infected edges. That is, once a vertex has at least r infected incident edges, the infection will spread to all healthy edges incident with this vertex. Once an edge is infected, it remains infected indefinitely. If a set of initially infected edges will eventually infect all of $E(G)$, we say that this set *percolates*, and refer to it as an *r-percolating set* of G.

Define $m_e(G, r)$ to be the minimum number of edges in an r-percolating set of G. Then

$$m_e(G, r) = \mathrm{wsat}(G, S_{r+1}).$$

Let $G \square H$ denote the *Cartesian product* of two graphs G and H, which is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. For ease of notation, given graphs $G_1, ..., G_k$, we often write the Cartesian product $G_1 \square \cdots \square G_k$ as $\prod_{i=1}^{k} G_i$. In what follows, we often refer to a Cartesian product of two graphs as simply a *product*.

Let P_k denote the path on k vertices, and let C_k denote the cycle on k vertices. The product $\prod_{i=1}^{d} P_2$ is often referred to as the *d-dimensional hypercube*, and denoted Q_d. More generally, given integers $a_1, ..., a_d \geq 2$ and $b_1, ..., b_d \geq 3$, the product $\prod_{i=1}^{d} P_{a_i}$ is known as a *d-dimensional grid*, while $\prod_{i=1}^{d} C_{b_i}$ is a *d-dimensional torus*.

In [21], Morrison and Noel determined the value of $m_e(G, r)$ when G is a hypercube. In fact, they proved a more general result that provides a recursive formula for $m_e(G, r)$ when G is a d-dimensional grid.

Recently, Hambardzumyan, Hatami, and Qian [14] introduced a clever polynomial method for establishing lower bounds on $m_e(G, r)$. In their paper, they used this polynomial method to bound $m_e(G \square H, r)$ when H is a path or a cycle. Furthermore, they determined a recursive formula for $m_e(G, r)$ when G is a d-dimensional torus, and provided an alternate and much simpler proof of the result by Morrison and Noel when G is a d-dimensional grid.

In Chapter 3, we will push the power of this polynomial method to determine $m_e(G \square H, r)$ for other graphs H. Section 3.4 deals with the case where H is a star, while Section 3.5 considers graphs H constructed from a cycle by adding a single chord. In Section 3.6, we generalize the bounds on stars found in Section 3.4 to all trees, and investigate the conditions under which these bound match.

Chapter 2

The Rainbow Saturation Number is Linear

The results presented in this chapter are based on work with Natalie Behague, Tom Johnston, Shoham Letzter, and Natasha Morrison, and can also be found in [5].

2.1 Introduction

Recall the following definitions from Chapter 1. A graph G is *H-free* if H is not a subgraph of G. Given graphs G and H, a spanning subgraph F of G is said to be *(G, H)-saturated* if F is H-free, but for every edge $e \in E(G) \setminus E(F)$, the graph $F + e$ contains a copy of H as a subgraph. The *saturation number* of H in G, denoted by $\mathrm{sat}(G, H)$, is the minimum number of edges in such a graph F; that is,

$$\mathrm{sat}(G, H) := \min\{|E(F)| : F \text{ is } (G, H)\text{-saturated}\}.$$

When $G = K_n$, we write $\mathrm{sat}(n, H)$ instead of $\mathrm{sat}(K_n, H)$, where as usual K_n denotes the complete n-vertex graph.

As noted in Section 1.1, Kászonyi and Tuza [18] proved that the saturation number of every graph H is linear in n.

Theorem 1.2 [18] *For any graph H,* $\mathrm{sat}(n, H) = O(n)$.

In this chapter, we provide an analogous result for the rainbow saturation number: a variation on the saturation number for edge-coloured graphs.

For $n \in \mathbb{Z}^+$, we denote by $[n]$ the set of integers $\{1, ..., n\}$. A *t-edge-colouring* of a graph G is a function $c : E(G) \to [t]$. A t-edge-colouring of a graph G is *proper* if every pair of edges incident with a common vertex receive distinct colours, and *rainbow* if every edge in G is assigned a distinct colour. A *t-edge-coloured graph* is an ordered pair (G, c) where c is a (not necessarily proper) t-edge-colouring of G. When the pallet of possible colours is unlimited (or unimportant), we refer to c as an *edge-colouring* of G, and say that (G, c) is an *edge-coloured graph.* For ease of notation, if the edge-colouring c is clear from context, we often refer to an edge-coloured graph (G, c) as simply G.

The generalization of saturation to edge-coloured graphs was first considered by Hanson and Toft [15]. Following this, Barrus, Ferrara, Vandenbussche, and Wenger [4] considered the particular case of t-rainbow saturation, where there are exactly t colours available. A t-edge-coloured graph (G, c) is *(H, t)-rainbow saturated* if G does not contain a rainbow copy of H, but the addition of any non-edge e, in any colour from $[t]$, creates a rainbow copy of H in $G + e$. Note that this requires $t \geq |E(H)|$. The *t-rainbow saturation number* of H, denoted by $\mathrm{rsat}_t(n, H)$, is the minimum number of edges in an (H, t)-rainbow-saturated graph on n vertices; that is,

$$\mathrm{rsat}_t(n, H) := \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } (H, t)\text{-rainbow-saturated}\}.$$

The main focus of the study of t-rainbow saturation has been the asymptotic behaviour of $\mathrm{rsat}_t(n, H)$ as n approaches infinity. As noted in Section 1.2, Barrus, Ferrara, Vandenbussche, and Wenger [4] proved that there exist graphs H for which $\mathrm{rsat}_t(n, H)$ grows much faster (as a function of n) than the ordinary saturation number, $\mathrm{sat}(n, H)$.

Theorem 1.3 [4] *Let $t \geq k \geq 2$ be integers. Then* $\mathrm{rsat}_t(n, K_{1,k}) = \Theta(n^2)$.

In addition, Barrus, Ferrara, Vandenbussche, and Wenger [4] proved the following bounds on $\mathrm{rsat}_t(n, H)$ when H is a complete graph.

Theorem 1.4 [4] *Let $r \geq 3$ and $t \geq \binom{r}{2}$ be integers. There exist positive constants c_1, c_2, n_0 such that, for all $n \geq n_0$,*

$$c_1 \frac{n \log(n)}{\log(\log(n))} \leq \mathrm{rsat}_t(n, K_r) \leq c_2 n \log(n).$$

The lower bound on the t-rainbow saturation number for complete graphs was improved independently by Ferrara, Johnston, Loeb, Pfender, Schulte, Smith, Sullivan, Tait, and Tompkins [11], by Girão, Lewis, and Popielarz [13], and by Korándi [19], proving that $\mathrm{rsat}_t(n, K_r)$ behaves asymptotically like $n \log(n)$.

Theorem 1.5 [11, 13, 19] *Let $r \in \mathbb{Z}^+$. If $t \geq \binom{r}{2}$, then* $\mathrm{rsat}_t(n, K_r) = \Theta(n \log(n))$.

Girão, Lewis, and Popielarz [13] also initiated the study of the natural generalization of t-rainbow saturation to the case where the palette of colours available is unlimited, rather than bounded by t. Specifically, an edge-coloured graph (G, c) is *H-rainbow saturated* if G does not contain a rainbow copy of H, but the addition of any non-edge e, in any colour from $\mathbb{N}$, creates a rainbow copy of H in $G + e$. The *rainbow saturation number* of H, denoted by $\mathrm{rsat}(n, H)$, is the minimum number of edges in an H-rainbow saturated graph on n vertices; that is,

$$\mathrm{rsat}(n, H) := \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-rainbow-saturated}\}.$$

In this thesis, we focus on the case where the pallet of possible colours is unlimited.

Girão, Lewis, and Popielarz [13] conjectured that, like ordinary saturation numbers, the rainbow saturation number of any non-empty graph H is linear in n, and proved this for graphs with certain properties.

Theorem 1.6 [13] *Let H be a graph with a non-pendant edge that is not contained in a triangle. Then* $\text{rsat}(n, H) = O(n)$*.*

In fact, Theorem 1.6 is a consequence of a stronger result, which proves that $\text{rsat}_t(n, H) = O(n)$ for any such H and any $t \geq |E(H)|$.

As noted in Section 1.2, our main result in this chapter proves that the rainbow saturation number of every graph H is linear in n, thus confirming the conjecture of Girão, Lewis, and Popielarz [13]. We restate this result here for convenience.

Theorem 1.7 *Every non-empty graph H satisfies* $\text{rsat}(n, H) = O(n)$*.*

Theorem 1.7 shows that rainbow saturation numbers behave asymptotically like ordinary saturation numbers. This is in stark contrast to t-rainbow saturation numbers, which, as shown in Theorems 1.3 and 1.5, can grow considerably faster than n.

In Section 2.2, we establish some useful basic results regarding rainbow saturation, including Propositions 2.3 and 2.4, which allow us to assume that any counterexample to Theorem 1.7 is connected and has no pendant edges. We describe the construction used to prove Theorem 1.7 in Section 2.3: Given a graph H and an edge $e \in E(H)$, we construct an edge-coloured graph on n vertices with $O(n)$ edges such that the addition of almost any non-edge creates a rainbow copy of H. In Section 2.4, we show that, if e is an edge contained in a triangle, then the graph produced by this construction does not contain a rainbow copy of H. Together with Theorem 1.6, this completes the proof of Theorem 1.7.

2.2 Preliminaries

In this section, we provide preliminary results that deal with some easy classes of graphs. We begin by establishing that the existence of an edge-coloured graph G on n vertices with $O(n)$ edges that is "almost" H-rainbow-saturated (save for at most linearly many problematic non-edges) is enough to prove that $\text{rsat}(n, H) = O(n)$.

Let G and H be graphs. For a given colour c, we say that a non-edge $e \in E(\overline{G})$ is *c-bad* for G if the addition of e in colour c to G does not create a rainbow copy of H. We say that a non-edge e is *bad* if there exists a colour c such that e is c-bad for G.

Proposition 2.1 *Let H be a graph, and let (G, χ) be an n-vertex edge-coloured graph. Suppose that G has m bad non-edges and does not contain a rainbow copy of H. Then $\mathrm{rsat}(n, H) \leq |E(G)| + m$.*

Proof. Let $\{e_1, \ldots, e_m\}$ be the set of bad non-edges of G. We construct an n-vertex graph G_m and colouring χ' of $E(G_m)$ as follows: Set $G_0 := G$, where each edge of G_0 is coloured as in χ. Consider each $i \in [m]$ in turn. If there is a colour c such that e_i is c-bad for G_{i-1}, then we define G_i to be $G_{i-1} \cup \{e_i\}$, set $\chi'(e_i) = c$, and keep the colours of the other edges as in G_{i-1}. Note that there may be multiple such colours c, in which case we pick one arbitrarily. Otherwise, if e_i is not bad for G_{i-1}, set $G_i := G_{i-1}$. Observe that G_m is H-rainbow-saturated, and since we added at most m edges during this process, G_m has at most $|E(G)| + m$ edges. Hence, $\mathrm{rsat}(n, H) \leq |E(G)| + m$. □

In particular, Proposition 2.1 implies the following useful corollary.

Corollary 2.2 *Let H be a graph. Let (G, χ) be an n-vertex edge-coloured graph with $|E(G)| = O(n)$. Suppose that G has $O(n)$ bad non-edges and does not contain a rainbow copy of H. Then $\mathrm{rsat}(n, H) = O(n)$.*

The following consequence of Corollary 2.2 shows that, to prove Theorem 1.7, it suffices to consider connected graphs H.

Proposition 2.3 *Let H be a disconnected graph and let H' be a component of H that has the most vertices, and subject to this, the most edges. If $\mathrm{rsat}(n, H') = O(n)$, then $\mathrm{rsat}(n, H) = O(n)$.*

Proof. Let s denote the number of components of H that are isomorphic to H'. Let $m := 2\left(|V(H)| - |V(H')|\right) - (s-1)$. Then, since $\text{rsat}(n, H') = O(n)$, it follows that $\text{rsat}(n-m, H') = O(n)$. Let (G, χ) be an H'-rainbow-saturated graph on $n-m$ vertices with the minimum number of edges; that is, $|E(G)| = O(n)$.

Define a graph G^* on n vertices as follows: Let H_2 be the graph obtained from H' by gluing two disjoint copies of H' together at an arbitrary vertex. Let G^* be the disjoint union of G with $s-1$ disjoint copies of H_2 and two disjoint copies of each component of H that is not isomorphic to H'. See Figure 2.1 for an example of this construction. Note that

$$\begin{aligned}
|V(G^*)| &= (n-m) + (s-1)(2|V(H')| - 1) + 2(|V(H)| - s|V(H')|) \\
&= n - m + 2(|V(H)| - |V(H')|) - (s-1) \\
&= n.
\end{aligned}$$

In addition,

$$\begin{aligned}
|E(G^*)| &= O(n) + 2(s-1)|E(H')| + 2\left(|E(H)| - s|E(H')|\right) \\
&= O(n).
\end{aligned}$$

Let χ^* be an edge-colouring of $E(G^*)$ where $\chi^*(e) = \chi(e)$ for $e \in E(G)$, and $G^* \setminus V(G)$ is rainbow with colours not used in χ. Observe that G^* contains no rainbow copy of H. Indeed, by our choice of H', there are at most $s-1$ disjoint copies of H' in $G^* \setminus G$, and G contains no rainbow copy of H'. Moreover, since $G^* \setminus V(G)$ contains a rainbow copy of $H \setminus V(H')$ avoiding any given colour, the addition of any non-edge in G in any colour creates a rainbow copy of H in G^*. Since there are at most $m(n-m) + \binom{m}{2} = O(n)$ other non-edges in G^*, by Corollary 2.2 we have $\text{rsat}(n, H) = O(n)$, as required. □

Therefore, in what follows we may (and will) assume that H is a connected graph. Our next result shows that, if H contains a vertex of degree 1, then $\text{rsat}(n, H) = O(n)$.

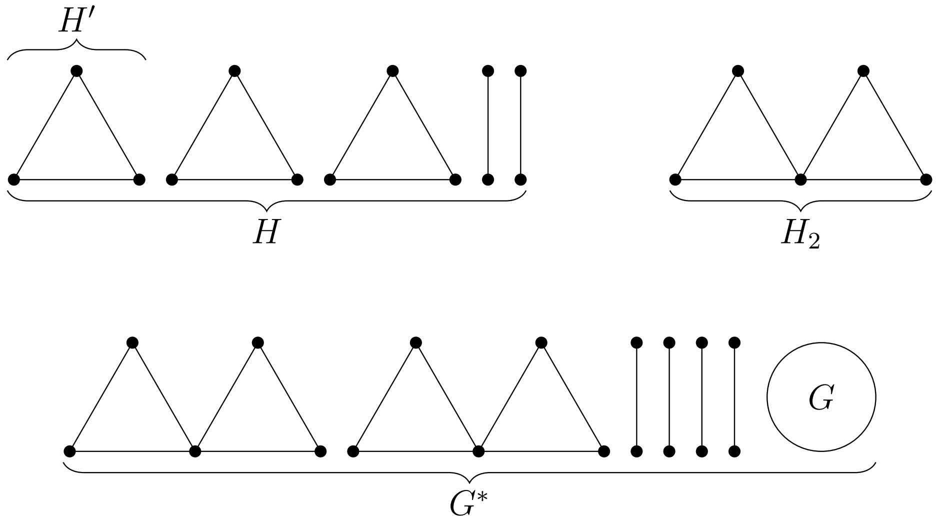

Figure 2.1: The graph G^* created via the construction in Proposition 2.3. Here $s = 3$.

Proposition 2.4 *Let H be a connected graph with $\delta(H) = 1$. Then* $\mathrm{rsat}(n, H) = O(n)$.

Proof. Let $k := |V(H)|$, and write $n = q(k-1) + r$, where $0 \le r < k - 1$. We define an edge-coloured graph (G, χ) on n vertices as follows: Take q disjoint copies of K_{k-1} together with r isolated vertices, and let χ be a rainbow colouring of G. Note that G has $q(k-1)+r = n$ vertices and $q\binom{k-1}{2} = (n-r)\left(\frac{k-2}{2}\right) = O(n)$ edges. By construction, G does not contain a rainbow copy of H. Moreover, since $\delta(H) = 1$, if any non-edge in any colour is added between two distinct copies of K_{k-1} in G, then a rainbow copy of H is created. Therefore, since G has $q(k-1)r + \binom{r}{2} = (n-r)r + \binom{r}{2} = O(n)$ other non-edges, by Corollary 2.2 we have $\mathrm{rsat}(n, H) = O(n)$, as required. □

2.3 The Construction

In this section, we present the construction that lies at the heart of our proof of Theorem 1.7. Given a graph H and an edge $e \in E(H)$, we construct an edge-coloured graph on n vertices with $O(n)$ edges such that the addition of almost any edge creates a rainbow copy of H. The main work in the proof of Theorem 1.7 will be to show that this graph does not contain a rainbow copy of H, which will be done in Section 2.4.

Construction 2.5 Let H be a connected graph of order at least 3. Let $xy \in E(H)$. Define $S := N_H(x) \cap N_H(y)$, and let T be the set of edges in H with one endpoint in S and the other in $\{x, y\}$. Define H' to be the graph obtained from H by contracting the edge xy and replacing any multi-edges by single edges. We label the vertices of H' as in H, with the single vertex obtained from contracting the edge xy labelled by $x|y$. Let T' be the set of edges in H' between $x|y$ and S. Define $H'' := H' \setminus T'$.

For an integer $m \geq 2$, define $F := F(m)$ to be the graph obtained from H' by replacing the vertex $x|y$ with m duplicates of itself, denoted by $v_1, v_2, \ldots, v_m$. Define $M := \{v_1, v_2, \ldots, v_m\}$. We label each vertex in M by $x|y$, and label the remaining vertices of F as in H. Hence F has m vertices labelled $x|y$, and one vertex labelled v for each $v \in V(H) \setminus \{x, y\}$.

Given any integers $m \geq 2$ and $r \geq \max\{|E(H'')| + 1, 2\}$, we define the graph $G := G^H_{xy}(m, r)$ as follows: Start with $|E(H'')|$ disjoint copies of $F(m)$, indexed by the edges in H'' as $F_{e_1}, \ldots, F_{e_{|E(H'')|}}$. To these we add another $r - |E(H'')|$ disjoint copies of $F(m)$, which we index as $F_1, \ldots, F_{r-|E(H'')|}$. Note that this gives a total of r disjoint copies of $F(m)$. Finally, we obtain G by identifying the vertices corresponding to $v_i \in M$ for each i in turn. That is, for each $i \in [m]$, we replace the r vertices $u_1, \ldots, u_r$ corresponding to v_i with one vertex and connect this vertex to the vertices in $\bigcup_{j=1}^{r} N(u_j)$. Note that the graph G has r copies of each vertex in $V(H') \setminus \{x|y\}$ and m copies of the vertex $x|y$. Moreover, this labelling of the vertices naturally defines a labelling of the edges: An edge in G whose vertices are labelled u and v is labelled uv, which, by construction, is an edge in H'.

We now define an edge-colouring χ of G, which we later show does not contain a rainbow copy of H. Let χ_0 be a rainbow edge-colouring of H''. Fix two colours not used in χ_0, say black and $\star$. Extend χ_0 to an edge-colouring of H' by colouring each edge in T' with $\star$. To define the edge-colouring χ, set $\chi(e) = \chi_0(uv)$ for each edge e in G with label uv, recalling that $uv \in E(H')$ by construction. Then, for each edge e in H'', recolour the edge corresponding to e in F_e with the colour black. Finally recolour each edge coloured with $\star$ with unique colours that are not used elsewhere in G.

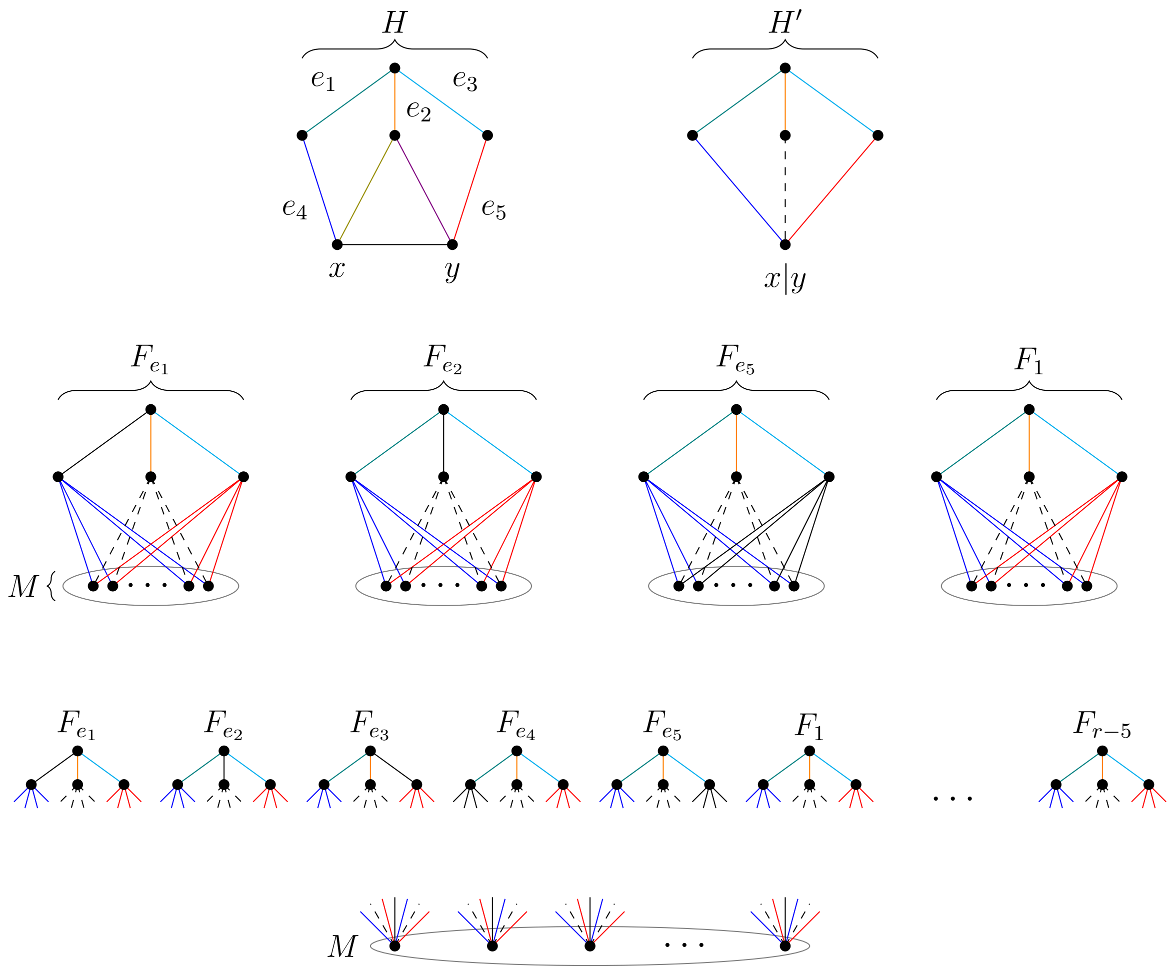

Figure 2.2: An example of the construction of the graph $G^H_{xy}(m, r)$. The dashed edges represent the edges that are originally coloured $\star$ and will be coloured with unique colours by χ. The colour black can be seen replacing other colours according to the index of the copy of F.

Construction 2.5 was inspired by the construction used in [13] to prove Theorem 1.6. In fact, when the edge xy is not contained in a triangle, $N_H(x) \cap N_H(y) = \emptyset$, and the graph described in Construction 2.5 is equivalent to the construction used in [13].

See Figure 2.2 for an example of how Construction 2.5 works. In what follows, we will use $G^H_{xy}(n)$ to refer to the edge-coloured graph $(G^H_{xy}(m, r), \chi)$, where we let $r = \max\{|E(H'')| + 1, 2\}$ and $m = n - r(|V(H)| - 2)$.

Note that $G^H_{xy}(n)$ has $m + r(|V(H)| - 2) = n$ vertices, and, since $r = O(1)$ and $m = n - O(1)$, at most $mr(|V(H)| - 2) + r|E(H'')| = O(n)$ edges. Furthermore, since $r = \max\{|E(H'')|+1, 2\}$, the addition of any non-edge within M, in any colour, creates a rainbow copy of H. Note that $G^H_{xy}(n)$ has at most $m(n-m) + \binom{n-m}{2} = O(n)$ other non-edges, and thus at most $O(n)$ bad non-edges. Therefore, if $G^H_{xy}(n)$ contains no rainbow copy of H, then Corollary 2.2 will yield $\mathrm{rsat}(n, H) = O(n)$, as required.

2.4 Completing the Proof of Theorem 1.7

The goal of this section is to show that $G^H_{xy}(n)$ does not contain a rainbow copy of H. In fact, we prove the following more general result.

Proposition 2.6 *Let H be a non-empty connected graph. For any edge $xy \in E(H)$ that is contained in a triangle, and any integers $m \geq 2$ and $r \geq \max\{|E(H'')| + 1, 2\}$, the graph $G^H_{xy}(m, r)$ does not contain a rainbow copy of H.*

Before proving Proposition 2.6, we show why it implies Theorem 1.7 (restated here for convenience).

Theorem 1.7 *Every non-empty graph H satisfies $\mathrm{rsat}(n, H) = O(n)$.*

Proof of Theorem 1.7. First, note that by Proposition 2.3, it suffices to prove that every connected graph H satisfies $\mathrm{rsat}(n, H) = O(n)$. Let H be a non-empty connected graph. Suppose first that H has a triangle. Let xy be an edge contained in a triangle, and write $G := G^H_{xy}(n)$. Then $|E(G)| = O(n)$. By Proposition 2.6, the graph G does not contain a rainbow copy of H. Moreover, as described at the end of Section 2.3, for all but at most $O(n)$ non-edges e, the addition of e in any colour creates a rainbow copy of H. Therefore, by Corollary 2.2, we have $\mathrm{rsat}(n, H) = O(n)$, as required. Otherwise, if H is triangle-free, then $\mathrm{rsat}(n, H) = O(n)$, using either Proposition 2.4 if H has a pendant edge or Theorem 1.6 if H has a non-pendant edge. □

Proof of Proposition 2.6. Suppose for a contradiction that there exists a connected graph H with an edge $xy \in E(H)$ contained in a triangle such that $G^H_{xy}(m, r)$ contains a rainbow copy of H, for some integers $m \geq 2$ and $r \geq \max\{|E(H'')| + 1, 2\}$. Take H to have the minimum number of vertices with respect to these properties, and fix m and r such that $G := G^H_{xy}(m, r)$ has a rainbow copy $H_{\mathfrak{R}}$ of H. Recall that the vertices of G are labelled by vertices in H' (the graph obtained from H by contracting xy), where M is the set of vertices labelled $x|y$, and that the edges of G are labelled by edges in H'.

First, suppose $H_{\mathfrak{R}}$ does not contain any vertices in M. Then, since $H_{\mathfrak{R}}$ is connected, it must be contained entirely within a single copy of $H \setminus \{x, y\}$. However, this is impossible since $|V(H_{\mathfrak{R}})| = |V(H)|$. Therefore, $H_{\mathfrak{R}}$ contains at least one vertex in M; that is, $|V(H_{\mathfrak{R}}) \cap M| \geq 1$.

Claim 2.7 *For every $v \in V(H) \setminus \{x, y\}$, the rainbow copy $H_{\mathfrak{R}}$ of H uses at least one vertex labelled v.*

Proof of Claim 2.7. Suppose for a contradiction that there exists $v \in V(H) \setminus \{x, y\}$ such that $H_{\mathfrak{R}}$ uses no vertex labelled v. Recall that $S := N_H(x) \cap N_H(y)$, and that T is the set of edges in H with one endpoint in S and the other in $\{x, y\}$. Since xy is contained in a triangle, $|S| \geq 1$.

Suppose first that v is the unique common neighbour of x and y in H; that is, $S = \{v\}$. Hence $|T| = 2$. Recall that T' is the set of edges in H' between $x|y$ and S, that $H'' = H' \setminus T'$, and that χ_0 is a rainbow edge-colouring of H''. Now, since $H_{\mathfrak{R}}$ does not use any vertex labelled v, where $v \in S$, it uses at most $|E(H)| - 2$ distinct colours: at most

$$|E(H'')| = |E(H')| - |T'| = |E(H)| - |T| - 1 \leq |E(H)| - 3$$

colours used by χ_0, and possibly also black. However, since a rainbow copy of H requires $|E(H)|$ colours, this contradicts $H_{\mathfrak{R}}$ being a rainbow copy of H in G. Hence v is not the unique common neighbour of x and y in H.

Now, consider the graph $H^v := H \setminus \{v\}$. Suppose first that H^v is connected. Consider the edge-coloured graph G^* obtained from $G^H_{xy}(m,r)$ by deleting every vertex labelled v. Since $H_{\mathfrak{R}}$ contains no vertex labelled v, it is present in G^*, and, in particular, G^* contains a rainbow copy of H^v. Observe that $G^* = (G^{H^v}_{xy}(m,r), \chi')$, where χ' is the restriction of χ to G^*. Since v is not the unique common neighbour of x and y in H, the edge xy is still in a triangle in H^v. However, since H^v is connected, by minimality of H we know that $G^{H^v}_{xy}(m,r)$ contains no rainbow copy of H^v, a contradiction.

Therefore, for every non-cut vertex u in H, the rainbow copy $H_{\mathfrak{R}}$ contains a vertex labelled u. In particular, v is a cut vertex in H, and thus H^v is disconnected. Note that x and y are in the same component of H^v since $xy \in E(H)$ and $v \in V(H) \setminus \{x,y\}$. Let C be a component of H^v not containing $\{x,y\}$. Take $u \in V(C)$ of maximum distance from v. Note that u is not a cut vertex of H, and thus, by assumption, there is a vertex u' in $H_{\mathfrak{R}}$ labelled u. Since $H_{\mathfrak{R}}$ is connected and contains at least one vertex in M, there is a path in $H_{\mathfrak{R}}$ from M to u'. However, by our choice of C, any such path passes through a vertex labelled v, which is a contradiction. $(\square)$

Note that Claim 2.7 implies that $|V(H_{\mathfrak{R}}) \cap M| \leq 2$. Hence, since $|V(H_{\mathfrak{R}}) \cap M| \geq 1$, it follows that $H_{\mathfrak{R}}$ contains either one or two vertices in M. We will obtain a contradiction by counting the edges in $H_{\mathfrak{R}}$ (via two cases).

Recall that T' is the set of edges in H' between $x|y$ and S, where H' is graph obtained from H by contracting the edge xy and replacing any multi-edges by single edges. Note that $|S| = |T'| = \frac{|T|}{2} \geq 1$ since xy is contained in a triangle.

First, suppose $|H_{\mathfrak{R}} \cap M| = 2$. Then, by Claim 2.7, the rainbow copy $H_{\mathfrak{R}}$ contains exactly one vertex labelled v, for each $v \in V(H) \setminus \{x,y\}$. Hence there are at most $2|T'|$ edges in $H_{\mathfrak{R}}$ with a label in T'. Now, consider an edge $uv \in E(H') \setminus T'$, where $u \neq x|y$ without loss of generality. Since there is exactly one vertex labelled u, all edges labelled uv belong to the same copy of F, and thus have the same colour. Therefore, the edges of $H_{\mathfrak{R}}$ are coloured using at most $|E(H')| - |T'| + 2|T'| \leq |E(H)| - 1$ colours, which contradicts the assumption that $H_{\mathfrak{R}}$ is a rainbow copy of H.

Otherwise, suppose $|H_{\mathfrak{R}} \cap M| = 1$. Then, by Claim 2.7, there is exactly one vertex $v \in V(H) \setminus \{x, y\}$ that labels two vertices in $H_{\mathfrak{R}}$, and every other vertex in $V(H')$ appears once as a label.

Claim 2.8 *For each $e \in E(H') \setminus \{vx|y\}$, all edges labelled e in $H_{\mathfrak{R}}$ are the same colour.*

Proof of Claim 2.8. Suppose first that e is not incident with v. Then, since there is at most one edge labelled e in $H_{\mathfrak{R}}$, the claim clearly holds. Otherwise, suppose $e = uv$ with $u \neq x|y$. Then $e \in E(H') \setminus T'$. Therefore, since all edges labelled e are in the same copy of F, they all receive the same colour, as required. ($\square$)

Now, if $vx|y$ is an edge in H', then there are at most two edges in $H_{\mathfrak{R}}$ with this label. Therefore, by Claim 2.8, the edges of $H_{\mathfrak{R}}$ use at most

$$|E(H')| - 1 + 2 = |E(H')| + 1 = |E(H)| - |T'| \leq |E(H)| - 1$$

colours, which contradicts $H_{\mathfrak{R}}$ being a rainbow copy of H. $\square$

Chapter 3

Weak Saturation Using Polynomial Methods

3.1 Introduction

In this chapter, we use polynomial methods to provide bounds on the weak saturation number (defined in Section 1.3 and below) of products of certain graphs.

Recall the following definitions from Chapter 1. For $n \in \mathbb{Z}^+$, denote by $[n]$ the set of integers $\{1, ..., n\}$. A graph G is H-*free* if H is not a subgraph of G. Given graphs G and H, a spanning subgraph F of G is *weakly-*(G, H)*-saturated* if F is H-free, but there exists an ordering $e_1, ..., e_m$ of $E(G) \setminus E(F)$, such that, for each $i \in [m]$, the addition of the edge e_i to $F \cup \{e_1, ..., e_{i-1}\}$ creates a new copy of H. The *weak saturation number* of H in G, denoted by $\text{wsat}(G, H)$, is the minimum number of edges in such a graph F; that is,

$$\text{wsat}(G, H) = \min\{|E(F)| : F \text{ is weakly-}(G, H)\text{-saturated}\}.$$

The concept of weak saturation is intrinsically linked with that of bootstrap percolation. In particular, weak saturation is also commonly referred to as *graph bootstrap percolation*. In this chapter, we will focus on a specific instance of weak saturation known as r-bond bootstrap percolation.

In the r*-bond bootstrap percolation* process on a graph G, we start with a set of initially *infected* edges of G, and consider all other edges to be *healthy*. At each subsequent step, a healthy edge becomes infected if at least one of its endpoints is incident with at least r infected edges. That is, once a vertex has r infected incident edges, the infection will spread to all healthy edges incident with that vertex. If a set of initially infected edges will eventually infect all of $E(G)$, we say that this set *percolates*, and refer to it as an r*-percolating set* of G. Define $m_e(G, r)$ to be the minimum number of edges in an r-percolating set of G.

Recall that the *star* S_k is the complete bipartite graph $K_{1,k}$, for $k \geq 1$. Therefore, $m_e(G, r) = \text{wsat}(G, S_{r+1})$. In particular, observe that $m_e(G, 0) = 0$. Furthermore, if $r \geq \Delta(G)$, then $m_e(G, r) = |E(G)|$. Therefore, in what follows, we restrict our attention to the case where $0 < r < \Delta(G)$.

Let $G \square H$ denote the *Cartesian product* of two graphs G and H, which is defined to be the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Equivalently, $G \square H$ is the graph obtained from G and H as follows: Take $|V(H)|$ copies of G, indexed by the vertices in $V(H)$. For each $v \in V(H)$, let u_v denote the vertex in G_v corresponding to $u \in V(G)$. For each edge $vw \in E(H)$, add the perfect matching between G_v and G_w that connects the vertices u_v and u_w for each $u \in V(G)$. For ease of notation, given graphs $G_1, ..., G_k$, we often write the Cartesian product $G_1 \square \cdots \square G_k$ as $\prod_{i=1}^{k} G_i$. In what follows, we often refer to a Cartesian product as simply a *product*.

Let P_k denote the path on k vertices, and let C_k denote the cycle on k vertices. The product $\prod_{i=1}^{d} P_2$ is often referred to as the d*-dimensional hypercube*, and denoted Q_d. More generally, given integers $a_1, ..., a_d \geq 2$ and $b_1, ..., b_d \geq 3$, the the product $\prod_{i=1}^{d} P_{a_i}$ is known as a d*-dimensional grid*, while $\prod_{i=1}^{d} C_{b_i}$ is a d*-dimensional torus*.

In [21], Morrison and Noel determined the value of $m_e(G, r)$ when G is a hypercube. Note that Theorem 3.1 is actually a specific case of a more general result by Morrison and Noel [21] that holds for all d-dimensional grids.

Theorem 3.1 [21] *Let $d \geq r \geq 0$ be integers. Then,*

$$m_e(Q_d, r) = r2^{r+1} \sum_{i=1}^{r-1} \binom{d-i-1}{r-i} i2^{i-1}.$$

Note that, to prove $m_e(G, r) \leq k$, it suffices to exhibit an r-percolating set of G containing k edges. However, to show that $m_e(G, r) \geq k$, one must prove that *every* set of $k-1$ edges in G fails to percolate. Therefore, it is generally more difficult to determine lower bounds on $m_e(G, r)$.

The proof of Theorem 3.1 utilized a standard linear algebraic technique to establish the lower bound on $m_e(Q_d, r)$. Recently, Hambardzumyan, Hatami, and Qian [14] introduced a clever polynomial method for establishing lower bounds on $m_e(G, r)$. Their technique is based on the fact that a non-zero polynomial of degree d has at most d roots. See Section 3.2 for more details on this polynomial method.

Hambardzumyan, Hatami, and Qian [14] used their polynomial method to establish lower bounds on $m_e(G \square H, r)$ when H is a cycle or a path. Using these bounds, they found a recursive formula for $m_e(G, r)$ when G is a d-dimensional torus (see Theorem 3.2), and provided an alternate and simpler proof of Morrison and Noel's [21] result for $m_e(G, r)$ when G is a d-dimensional grid (see Theorem 3.3).

Theorem 3.2 [14] *Let $d \in \mathbb{Z}^+$. Let $a_1, ..., a_d \geq 3$ and $r \geq 1$ be integers. Let $G_0 = K_1$, and, for $j \in [d]$, let $G_j = C_{a_1} \square \cdots \square C_{a_j}$. Then*

$$m_e(G_d, r) = m_e(G_{d-1}, r) + (a_d - 2)m_e(G_{d-1}, r-1) + m_e(G_{d-1}, r-2) + p,$$

where

$$p = \begin{cases} 0 & r < 2d-1 \\ \prod_{i=1}^{d-1} a_i & r = 2d-1 \\ \prod_{i=1}^{d} a_i & r \geq 2d. \end{cases}$$

Theorem 3.3 [14][1] *Let $d \in \mathbb{Z}^+$. Let $a_1, ..., a_d \geq 2$ and $r \geq 1$ be integers. Let $G_0 = K_1$, and, for $j \in [d]$, let $G_j = P_{a_1} \square \cdots \square P_{a_j}$. Then*

$$m_e(G_d, r) = m_e(G_{d-1}, r) + (a_d - 1)m_e(G_{d-1}, r-1) + \sum_{\substack{s \subseteq [d-1] \\ |S| = r-d}} 2^{d-1-|S|} \prod_{i \in S} (a_i - 2) + (a_d - 1) \sum_{\substack{s \subseteq [d-1] \\ |S| < r-d}} 2^{d-1-|S|} \prod_{i \in S} (a_i - 2).$$

In this chapter, we will push the power of this polynomial method even further to provide recursive formulas for $m_e(G \square H, r)$ for other graphs G and H. For example, in Section 3.4, we prove the following recursive formula for $m_e(G \square S_k, r)$ when G is a product of stars.

Theorem 3.4 *Let $k, r, a_1, ..., a_k \in \mathbb{Z}^+$. Let $G_i = S_{a_1} \square \cdots \square S_{a_i}$ for each $i \in [k]$, and let $G_0 = K_1$. For $0 \leq i \leq \Delta(G_{k-1})$, let d_i denote the number of vertices in G_{k-1} of degree i. Then*

$$m_e(G_k, r) = m_e(G_{k-1}, r) + a_k m_e(G_{k-1}, r-1) + \sum_{t=1}^{a_k - 1} t d_{r-t} + a_k \sum_{t=a_k}^{r} d_{r-t}.$$

Note that, when $k \geq r - 1$, Theorem 3.4 is actually a specific case of a more general result proved in Section 3.4.3 (see Theorem 3.30). Analogous results are given in Section 3.5 when H is a cycle with a chord, and in Section 3.6 when H is a tree. We refrain from stating these results now, as they require further technical definitions.

However, before proving these results, we first provide more detail regarding the polynomial method in Section 3.2, and prove a useful little lemma in Section 3.3.

[1] The original statement of Theorem 3.3 in [14] contains two small errors, which we have corrected here. First, as $m_e(G, -1)$ is undefined, the result does not hold for $r = 0$. Moreover, the final sum should be multiplied by a factor of $a_d - 1$ rather than a_d, since the path P_{a_d} contains $a_d - 1$ edges.

3.2 A Powerful Polynomial Procedure

In this section, we describe the polynomial method introduced by Hambardzumyan, Hatami, and Qian in [14], which we will use to prove the main results in this chapter.

Recall that an *edge-colouring* of a graph G is a function $c : E(G) \to \mathbb{R}$. An edge colouring c of G is said to be *proper* if $c(e) \neq c(e')$ for all pairs of edges $e, e' \in E(G)$ that share an endpoint. For ease of notation, given an edge-colouring $c : E(G) \to \mathbb{R}$ and an edge $e \in E(G)$, we sometimes use c_e to denote $c(e)$.

Theorem 3.6 provides the essence of the polynomial method of Hambardzumyan, Hatami, and Qian [14]. However, in order to state this result, we first need the following definition. Note that, by convention, the zero polynomial has degree -1.

Definition 3.5 Let $r \geq 0$ be an integer, $G = (V, E)$ a graph, and $c : E \to \mathbb{R}$ a proper edge-colouring of G. For each $v \in V(G)$, let $\gamma_v := \min\{r, \deg(v)\}$. Define $W^r_{G,c}$ to be the vector space consisting of all vectors $\mathbf{p} = (p_v)_{v \in V}$ with entries in $\mathbb{R}[x]$ such that

1. $\deg(p_v) \leq \gamma_v - 1$ for all $v \in V$, and
2. $p_u(c_{uv}) = p_v(c_{uv})$ for every edge $uv \in E$.

Since each polynomial p_v is a vector in $\mathbb{R}^{\gamma_v}$ with basis $\{1, x, ..., x^{\gamma_v - 1}\}$, we can view $W^r_{G,c}$ as a subspace of $\mathbb{R}^m$, where $m = \sum_{v \in V(G)} \gamma_v$. In addition, note that, if $E(G) = \emptyset$ or $r = 0$, then $\dim(W^r_{G,c}) = 0$.

Theorem 3.6 [14] *Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$, and let $r \geq 0$ be an integer. Then $m_e(G, r) \geq \dim\left(W^r_{G,c}\right)$.*

We include a proof of Theorem 3.6 here (inspired by that of Hambardzumyan, Hatami, and Qian [14]) as it provides insight into the specific conditions imposed on the polynomials in Definition 3.5.

Proof of Theorem 3.6. We will show that any r-percolating set of G contains at least $\dim\left(W^r_{G,c}\right)$ edges. Let $F \subseteq E$ be an r-percolating set of G. Let $m := |E \setminus F|$. Since F percolates in G, there exists an ordering $e_1, ..., e_m$ of the edges in $E \setminus F$ so that, for each $i \in [m]$, once the edges in $F \cup \{e_1, ..., e_{i-1}\}$ are infected, e_i will also become infected. For $i \in [m]$, define $F_i := F \cup \{e_1, ..., e_{i-1}\}$.

Claim 3.7 *Let* $(p_v)_{v \in V} \in W^r_{G,c}$. *If* $p_u(c_{uv}) = p_v(c_{uv}) = 0$ *for all* $uv \in F$, *then* $p_v \equiv 0$ *for all* $v \in V$.

Proof of Claim 3.7. We begin by proving that $p_u(c_{uv}) = p_v(c_{uv}) = 0$ for all $uv \in E$. By assumption, this holds for all $uv \in F$. So suppose for a contradiction that there exists $e_i = uv \in E \setminus F$ such that $p_u(c_{uv})$ and $p_v(c_{uv})$ are not both zero. Let i be the smallest index for which this occurs. Since F percolates in G, we may assume without loss of generality that v has at least r incident edges in F_i. By our choice of i, it follows that $p_v(c_e) = 0$ for at least r edges $e \in F_i \cap N_G(v)$. Therefore, since c is a proper edge-colouring, p_v has at least r distinct roots. However, by Condition 1 in Definition 3.5, $\deg(p_v) \leq r - 1$. Therefore, since a non-zero univariate polynomial of degree d has at most d distinct roots, it follows that $p_v \equiv 0$. In particular, $p_v(c_{e_i}) = 0$. But then, by Condition 2 of Definition 3.5, we have $p_u(c_{e_i}) = p_v(c_{e_i}) = 0$, a contradiction. Hence $p_u(c_{uv}) = p_v(c_{uv}) = 0$ for all $uv \in E$.

Now, let $v \in V$. By the above argument, $p_v(c_{uv}) = 0$ for all $u \in N_G(v)$. Therefore, since c is a proper edge-colouring, p_v has at least $\deg_G(v)$ distinct roots. By Condition 1 of Definition 3.5, we have $\deg(p_v) \leq \deg_G(v) - 1$. Therefore, since a non-zero univariate polynomial of degree d has at most d distinct roots, $p_v \equiv 0$, as required. (□)

Now, define X to be the vector space containing all vectors $(p_v)_{v \in V}$ with entries in $\mathbb{R}[x]$ such that $\deg(p_v) \leq \gamma_v - 1$ for each $v \in V$; that is,

$$X := \left\{(p_v)_{v \in V} \in \mathbb{R}[x]^{|V|} : \deg(p_v) \leq \gamma_v - 1 \ \forall v \in V\right\}.$$

Since each polynomial p_v is a vector in $\mathbb{R}^{\gamma_v}$ with basis $\{1, x, ..., x^{\gamma_v - 1}\}$, we can view X as the vector space $\mathbb{R}^m$, where $m = \sum_{v \in V} \gamma_v$.

Define a function $f : F \to V$ where $f(e)$ is an endpoint of e, chosen arbitrarily, for each edge $e \in F$. Define Y to be the subspace of X containing all vectors $(p_v)_{v \in V} \in X$ such that $p_{f(e)}(c_e) = 0$ for all $e \in F$; that is,

$$Y := \left\{ (p_v)_{v \in V} \in X : p_{f(e)}(c_e) = 0 \ \forall e \in F \right\}.$$

Claim 3.8 $\dim(Y) \geq \dim(X) - |F|$.

Proof of Claim 3.8. Let $(p_v)_{v \in V} \in Y$. For each $e \in F$, since $p_{f(e)}(c_e) = 0$, we can write

$$p_{f(e)} = (x - c_e)(b_0 + b_1 x + \cdots + b_{\gamma_{f(e)}-2} x^{\gamma_{f(e)}-2}),$$

where $b_0, ..., b_{\gamma_{f(e)}-2} \in \mathbb{R}$. Hence $\{x - c_e, x^2 - c_e x, ..., x^{\gamma_{f(e)}-1} - c_e x^{\gamma_{f(e)}-2}\}$ is a basis for the vector space spanned by all such polynomials $p_{f(e)}$. Note that the dimension of this space is $\gamma_{f(e)} - 1$. Hence

$$\dim(Y) = \sum_{v \in f(F)} (\gamma_v - 1) + \sum_{v \notin f(F)} \gamma_v = \sum_{v \in V} \gamma_v - |f(F)| = \dim(X) - |f(F)|.$$

Therefore, since $|f(F)| \leq |F|$, it follows that $\dim(Y) \geq \dim(X) - |F|$.$_{(\Box)}$

Now, note that, by Definition 3.5, we have $W^r_{G,c} \subseteq X$. Moreover, $Y \subseteq X$.

Claim 3.9 $W^r_{G,c} \cap Y = \{\overrightarrow{0}\}$.

Proof of Claim 3.9. Clearly $\overrightarrow{0} \in W^r_{G,c} \cap Y$. Let $\mathbf{p} = (p_v)_{v \in V} \in W^r_{G,c} \cap Y$. We will show that $\mathbf{p} = \overrightarrow{0}$. We begin by proving that $p_u(c_{uv}) = p_v(c_{uv}) = 0$ for all $uv \in F$. Let $uv \in F$, and suppose without loss of generality that $f(uv) = v$. Then $p_v(c_{uv}) = 0$ since $\mathbf{p} \in Y$. Therefore, since $\mathbf{p} \in W^r_{G,c}$, Condition 2 of Definition 3.5 implies that $p_u(c_{uv}) = p_v(c_{uv}) = 0$, as required. Hence, by Claim 3.7, $\mathbf{p} = \overrightarrow{0}$. $_{(\Box)}$

Now, by Claim 3.9, $\dim(W^r_{G,c}) + \dim(Y) \leq \dim(X)$. Therefore, by Claim 3.8,

$$\dim(W^r_{G,c}) \leq \dim(X) - \dim(Y) \leq \dim(X) - (\dim(X) - |F|) = |F|.$$

Since this holds for every r-percolating set F of G, we have $\dim(W^r_{G,c}) \leq m_e(G, r)$. $\square$

3.3 A Little Lemma

In this section, we provide a little lemma that will be used in Sections 3.4.2, 3.5.2, and 3.6.2 to determine lower bounds on $m_e(G\square H)$ for various graphs H. However, in order to state Lemma 3.11, we first require the following definition.

Definition 3.10 Let $G = (V, E)$ be a graph and $c : E \to \mathbb{R}$ a proper edge-colouring of G. We define Z_c to be the set of vectors $\mathbf{z} = (z_v)_{v\in V} \in \mathbb{R}^{|V|}$ where, for each $v \in V$, the entry z_v is chosen from $\{c(uv) : u \in N_G(v)\}$. Now, let H be a graph, and suppose $\mathbf{p} = (p_u)_{u\in V(G\square H)}$ is a vector where each entry p_u is a univariate polynomial in $\mathbb{R}[x]$. For $\mathbf{z} \in Z_c$, we define $\mathbf{p}(\mathbf{z}) \in \mathbb{R}^{|V(G\square H)|}$ to be the vector where the entry corresponding to a vertex $(u, v) \in V(G\square H)$ is $p_{(u,v)}(z_u)$. Here we say $\mathbf{p}$ is *evaluated* at $\mathbf{z}$.

Lemma 3.11 *Let G and H be graphs, and let $c : E(G) \to \mathbb{R}$ be a proper edge colouring of G. Let $\mathbf{p} = (p_{(u,v)})_{(u,v)\in V(G\square H)}$ be a vector where each entry is a univariate polynomial in $\mathbb{R}[x]$. Suppose there exists $(u, v) \in V(G\square H)$ such that $p_{(u,v)} = q\prod_{\alpha\in A}(x - \alpha)$ for*

- *some non-zero polynomial q with $\deg(q) \leq \deg_G(u) - 1$, and*
- *some set $A \subseteq \mathbb{R}$ with $A \cap c(E(G)) = \emptyset$.*

Then there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$.

Proof. Since $A \cap c(E(G)) = \emptyset$, by definition $p_{(u,v)}$ has at most $\deg(q)$ zeros in $c(E(G))$. Thus, since $\deg(q) \leq \deg_G(u) - 1$, the polynomial $p_{(u,v)}$ cannot evaluate to zero on all $\deg_G(u)$ distinct colours in $\{c_{uw} : w \in N_G(u)\}$. Let $w \in N_G(u)$ such that $p_{(u,v)}(c_w) \neq 0$. Then $\mathbf{p}(\mathbf{z}) \neq 0$ for any vector $\mathbf{z} = (z_x)_{x\in V(G)} \in Z_c$ such that $z_u = c_{uw}$. $\square$

3.4 Products of Stars

Recall that the star S_k is defined to be the complete bipartite graph $K_{1,k}$ for $k \in \mathbb{Z}^+$. Throughout the rest of this chapter, given a graph G, we denote by d_t^G the number of vertices in G of degree t. For ease of notation, if the graph G is clear from context, we will sometimes write d_t^G as d_t.

The main goal of this section is to prove Theorem 3.12, which provides a recursive formula for $m_e(G \square S_k, r)$ for certain graphs G.

Theorem 3.12 *Let $k, r \geq 1$ be integers, and let G be a graph. Suppose there exists a proper edge-colouring c of G such that $m_e(G, i) = \dim(W_{G,c}^i)$ for each $r-1 \leq i \leq r$. Then*

$$m_e(G \square S_k, r) = m_e(G, r) + k m_e(G, r-1) + \sum_{t=1}^{k-1} t d_{r-t}^G + k \sum_{t=k}^{r} d_{r-t}^G.$$

In particular, we prove the following special case of Theorem 3.12 where G is itself a product of stars. Note that this result was initially stated in Section 3.1.

Theorem 3.4 *Let $k, r, a_1, ..., a_k \in \mathbb{Z}^+$. Let $G_i = S_{a_1} \square \cdots \square S_{a_i}$ for each $i \in [k]$, and let $G_0 = K_1$. For $0 \leq i \leq \Delta(G_{k-1})$, let d_i denote the number of vertices in G_{k-1} of degree i. Then*

$$m_e(G_k, r) = m_e(G_{k-1}, r) + a_k m_e(G_{k-1}, r-1) + \sum_{t=1}^{a_k - 1} t d_{r-t} + a_k \sum_{t=a_k}^{r} d_{r-t}.$$

In Sections 3.4.1 and 3.4.2, respectively, we provide upper and lower bounds on $m_e(G \square S_k, r)$. Section 3.4.3 contains proofs of Theorems 3.4 and 3.12.

3.4.1 Upper Bound

In this section, we prove the following upper bound on $m_e(G\square H, r)$ when H is a star.

Proposition 3.13 *Let $r, k \geq 1$ be integers, and let G be a graph. Then*

$$m_e(G\square S_k, r) \leq m_e(G, r) + km_e(G, r-1) + \sum_{t=1}^{k-1} td_{r-t}^G + k\sum_{t=k}^{r} d_{r-t}^G.$$

Proof. Let the vertices of S_k be labelled $0, 1, ..., k$, where 0 is the centre of the star. Let $G_0, G_1, ..., G_k$ denote the $k+1$ copies of G in $G\square S_k$ corresponding to the $k+1$ vertices in S_k. Here, G_0 corresponds to the centre of S_k. For each vertex $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where $0 \leq i \leq k$.

First, we consider the case where $\delta(G) \geq r$. We construct an r-percolating set $F \subseteq E(G\square S_k)$ for $G\square S_k$ as follows: For each $i \in [k]$, pick an optimal $(r-1)$-percolating set F_i on G_i. Finally, let F_0 be an optimal r-percolating set for G_0. Let $F = \bigcup_{i=0}^{k} F_i$. See Figure 3.1 for an illustration of this construction.

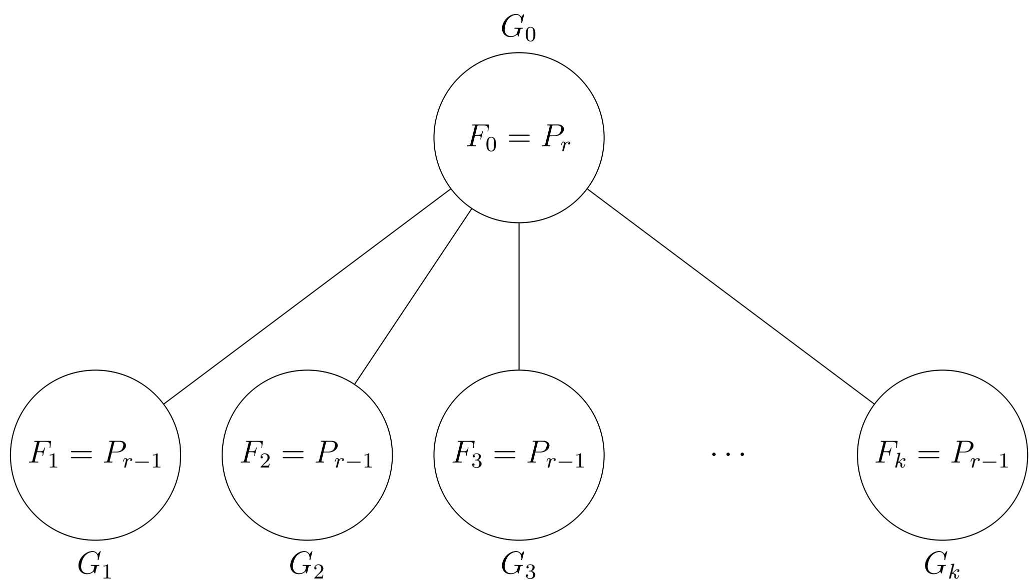

Figure 3.1: The percolating set F chosen for $G\square S_k$ when $\delta(G) \geq r$. Here we denote by P_i an optimal i-percolating set on G.

We claim that F will infect all edges in $G\Box S_k$, starting with those in G_0 and then spreading out to those in $G_1, ..., G_k$. Indeed, by our choice of F_0, after running the r-bond bootstrap percolation process on G_0, all edges in G_0 will be infected. Since each vertex in G has degree at least r, we can then infect all edges between G_0 and G_i, for each $i \in [k]$. Now, for $i \in [k]$, each vertex $v_i \in V(G_i)$ has an infected incident edge coming from G_0. This, together with F_i, will infect all edges in G_i. Therefore, the set F percolates in $G\Box S_k$, as required.

It remains to consider the case where G contains vertices of degree less than r. Let the set F be as define above. We add the following edges to F:

- For each $t \in [k-1]$ and each $v \in V(G)$ with $\deg_G(v) = r - t$, add the edges v_0v_i for $i \in [t]$ to the set F. See Figure 3.2 for an illustration of this construction when $k = 5$ and $t = 2$. Note that adding these edges will guarantee that, once G_0 is fully infected, the infection will spread to the edges v_0v_j where $t+1 \leq j \leq k$, and we can then continue as above. This adds $\sum_{t=1}^{k-1} td_{r-t}^G$ edges to the set F.
- For each $v \in V(G)$ with $\deg_G(v) \leq r - k$, add all edges v_0v_i for $i \in [k]$ to F. This adds $k\sum_{t=k}^{r} d_{r-t}^G$ edges to the set F.

Altogether, we added $\sum_{t=1}^{k-1} td_{r-t}^G + k\sum_{t=k}^{r} d_{r-t}^G$ edges to the $m_e(G,r) + km_e(G,r-1)$ edges initially in F. Therefore, since the resulting set F percolates in G, we have established the desired upper bound on $m_e(G\Box S_k, r)$. $\square$

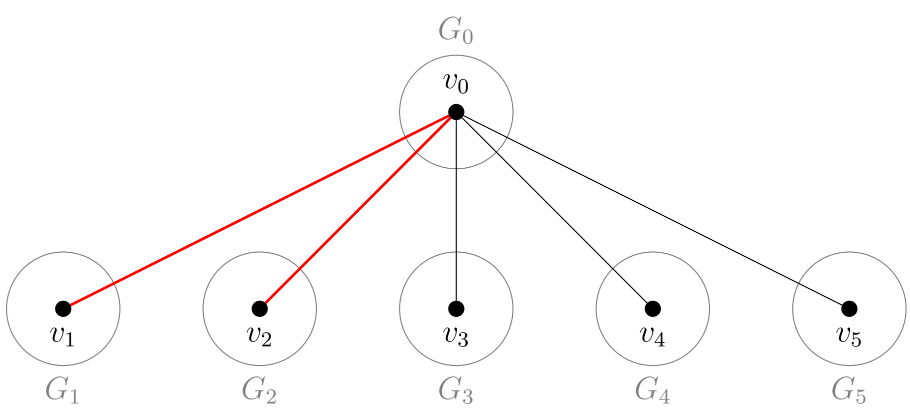

Figure 3.2: The edges added (shown in red) to F for $v \in V(G)$ with $\deg_G(v) = r - 2$.

3.4.2 Lower Bound

The goal of this section is to prove the following lower bound on $m_e(G\Box S_k, r)$.

Proposition 3.14 *Let $r, k \geq 1$ be integers, and let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then*

$$m_e(G\Box S_k, r) \geq \dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right) + \sum_{t=1}^{k-1} t d^G_{r-t} + k\sum_{t=k}^{r} d^G_{r-t}.$$

To this end, we will prove the following result.

Proposition 3.15 *Let $r, k \geq 1$ be integers. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then there exists a proper edge-colouring c' of $G\Box S_k$ for which*

$$\dim\left(W^r_{G\Box S_k,c'}\right) \geq \dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right) + \sum_{t=1}^{k-1} t d^G_{r-t} + k\sum_{t=k}^{r} d^G_{r-t}.$$

Before proving Proposition 3.15, we first demonstrate why it implies the lower bound in Proposition 3.14.

Proof of Proposition 3.14. By Proposition 3.15, there exists a proper edge-colouring c' of $G\Box S_k$ such that

$$\dim\left(W^r_{G\Box S_k,c'}\right) \geq \dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right) + \sum_{t=1}^{k-1} t d^G_{r-t} + k\sum_{t=k}^{r} d^G_{r-t}.$$

By Theorem 3.6, we have $m_e(G\Box S_k, r) \geq \dim\left(W^r_{G\Box S_k,c'}\right)$, and thus

$$m_e(G\Box S_k, r) \geq \dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right) + \sum_{t=1}^{k-1} t d^G_{r-t} + k\sum_{t=k}^{r} d^G_{r-t}.$$

$\Box$

Proof of Proposition 3.15. Let the vertices of S_k be labelled $0, 1, ..., k$, where 0 is the centre of the star. Let $G_0, G_1, ..., G_k$ denote the $k+1$ copies of G in $G\square S_k$ corresponding to the $k+1$ vertices in S_k. Here, G_0 corresponds to the centre of S_k. For each vertex $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where $0 \leq i \leq k$.

Let $\alpha_1, ..., \alpha_k$ be distinct real numbers not in $c(E)$. Define c' to be the proper edge-colouring of $G\square S_k$ that is consistent with c on each of $G_0, G_1, ..., G_k$, and where $c'(v_0 v_i) = \alpha_i$ for all $i \in [k]$ and all $v \in V(G)$.

Let

$$N = \dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right) + \sum_{t=1}^{k-1} t d_{r-t} + k\sum_{t=k}^{r} d_{r-t}.$$

In order to prove Proposition 3.15, we will find a set Y of N linearly independent vectors in $W^r_{G\square S_k,c'}$. We begin by finding a set A of $\dim\left(W^r_{G,c}\right) + k\dim\left(W^{r-1}_{G,c}\right)$ linearly independent vectors in $W^r_{G\square S_k,c'}$.

First, consider a basis $B^{(r)}$ for $W^r_{G,c}$. For each vector $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r)}$, define the vector $\mathbf{p}^0_{\mathbf{q}} = (p_u)_{u\in V(G\square S_k)}$ so that $p_{v_i} = q_v$ for all $0 \leq i \leq k$ and all $v \in V(G)$. See Figure 3.3 for an illustration of this construction.

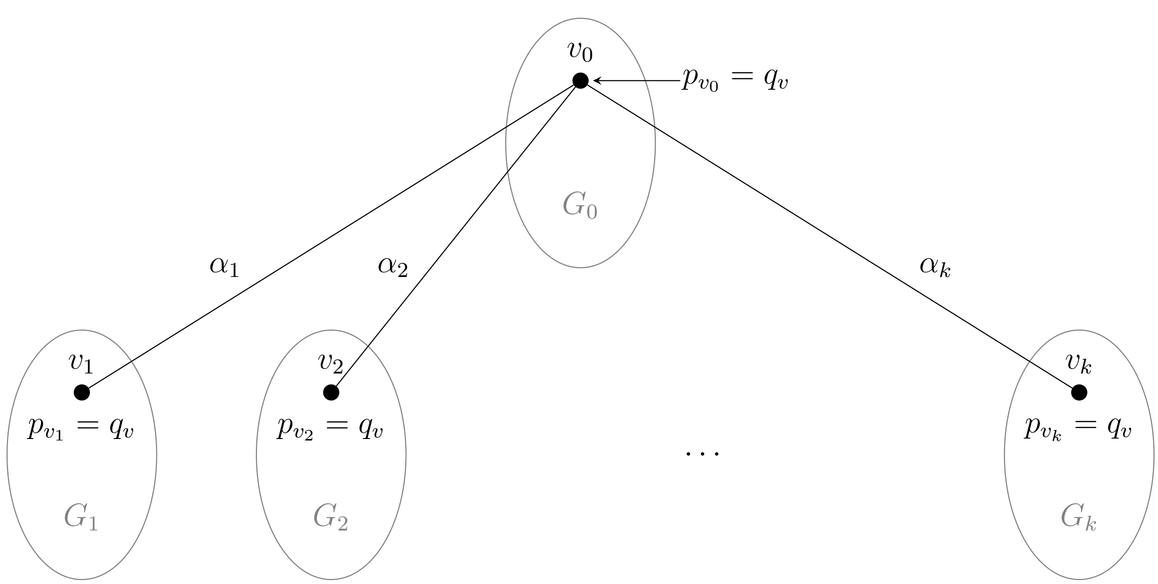

Figure 3.3: The vector $\mathbf{q} \in B^{(r)}$ is used to construct $\mathbf{p}^0_{\mathbf{q}} \in W^r_{G\square S_k,c'}$.

Trivially, the two conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}^0_{\mathbf{q}} \in W^r_{G\Box S_k,c'}$. Now, let $A_0 = \{\mathbf{p}^0_{\mathbf{q}} : q \in B^{(r)}\}$. Note that the restriction of a vector $\mathbf{p}^0_{\mathbf{q}} \in A_0$ to G_0 equals $\mathbf{q}$, and thus the vectors in A_0 are linearly independent since $B^{(r)}$ is a basis.

Now, consider a basis $B^{(r-1)}$ for $W^{r-1}_{G,c}$. Fix $\ell \in [k]$. For each $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r-1)}$, define the vector $\mathbf{p}^\ell_{\mathbf{q}} = (p_u)_{u\in V(G\Box S_k)}$ so that, for each vertex $v \in V(G)$, we have $p_{v_\ell} = q_v(x - \alpha_\ell)$, and $p_{v_i} \equiv 0$ for all other $0 \le i \le k$. See Figure 3.4 for an illustration of this construction.

For $v \in V(G)$, note that $\mathbf{q} \in W^{r-1}_{G,c}$ implies $\deg(q_v) \le \min\{r-1, \deg_G(v)\} - 1$. Thus

$$\begin{aligned}
\deg(p_{v_\ell}) &= \deg(q_v) + 1 \\
&\le \min\{r-1, \deg_G(v)\} \\
&= \min\{r-1, \deg(v_\ell) - 1\} \\
&= \min\{r, \deg(v_\ell)\} - 1.
\end{aligned}$$

In addition, $\mathbf{p}_{v_\ell}(\alpha_\ell) = 0 = \mathbf{p}_{v_0}(\alpha_\ell)$. Therefore, since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G\Box S_k)$, the conditions in Definition 3.5 are satisfied, and thus these vectors $\mathbf{p}^\ell_{\mathbf{q}}$ belong to $W^r_{G\Box S_k,c'}$.

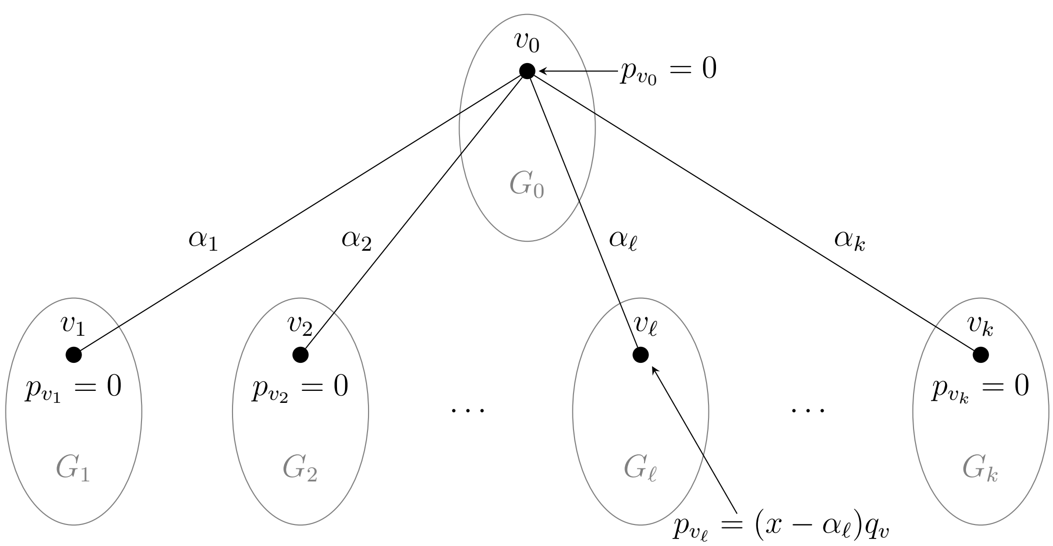

Figure 3.4: The vector $\mathbf{q} \in B^{(r-1)}$ is used to construct $\mathbf{p}^\ell_{\mathbf{q}} \in W^r_{G\Box S_k,c'}$.

For each $\ell \in [k]$, define $A_\ell = \{\mathbf{p}^\ell_\mathbf{q} : \mathbf{q} \in B^{(r-1)}\}$. Note that, for each $\ell \in [k]$, the restriction of $\mathbf{p}^\ell_\mathbf{q}$ to G_ℓ is the vector $(x - \alpha_\ell)\mathbf{q}$. Thus, since $x - \alpha_\ell \not\equiv 0$ and $B^{(r-1)}$ is a basis, the vectors in A_ℓ are linearly independent.

Define $A := \bigcup_{i=0}^{k} A_i$. Note that $|A_0| = \dim\left(W^r_{G,c}\right)$, and $|A_i| = \dim\left(W^{r-1}_{G,c}\right)$ for each $i \in [k]$. Hence $|A| = \dim\left(W^r_{G,c}\right) + k \dim\left(W^{r-1}_{G,c}\right)$.

Claim 3.16 *The vectors in A are linearly independent.*

Proof of Claim 3.16*:* Suppose for a contradiction that there exists a linear combination $S = \sum_{\mathbf{p}\in A} \beta_\mathbf{p}\mathbf{p} = 0$, where $\beta_\mathbf{p} \neq 0$ for some $\mathbf{p} \in A$. Write $S = S_0 + S_1 + \cdots + S_k$ where $S_i = \sum_{\mathbf{p}\in A_i} \beta_\mathbf{p}\mathbf{p}$ for each $0 \leq i \leq k$. Let $0 \leq i \leq k$ be the smallest index such that there exists $\mathbf{p} \in A_i$ with $\beta_\mathbf{p} \neq 0$. Suppose first that $i = 0$. By definition, for each $v \in V(G)$, we have $p_{v_0} \equiv 0$ for each vector $\mathbf{p} \in A_1 \cup \cdots \cup A_k$. Therefore, the restriction of the sum S to the coordinates corresponding to vertices in G_0 is equal to the restriction of S_0 to G_0. However, since the vectors in A_0 are linearly independent, this contradicts our assumption that $\beta_\mathbf{p} \neq 0$ for some $\mathbf{p} \in A_0$. Hence $i \neq 0$, and thus $\beta_\mathbf{p} = 0$ for all $\mathbf{p} \in A_0$.

Otherwise, assume $i \in [k]$. By definition, for each $v \in V(G)$, we have $p_{v_i} \equiv 0$ for each vector $\mathbf{p} \in \bigcup_{j\in[k]\setminus\{i\}} A_j$. Therefore, since $\beta_\mathbf{p} = 0$ for all $\mathbf{p} \in A_0$, the restriction of the sum S to the coordinates corresponding to vertices in G_i is equal to the restriction of S_i to G_i. Again, since the vectors in A_i are linearly independent, this contradicts our assumption that $\beta_\mathbf{p} \neq 0$ for some $\mathbf{p} \in A_i$. ($\Box$)

Now, set $Y = A$. To achieve the desired lower bound on $\dim\left(W^r_{G\Box S_k,c'}\right)$, we must extend the set Y to include an additional $\sum_{t=1}^{k-1} t d^G_{r-t} + k \sum_{t=k}^{r} d^G_{r-t}$ vectors, all of which are linearly independent. To this end, consider the set of vectors $Z_c \in \mathbb{R}^{|V(G)|}$ as defined in Definition 3.10.

Claim 3.17 *For each non-zero $\mathbf{p} \in \operatorname{span}(A)$, there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$.*

Proof of Claim: Let $\mathbf{p} = (p_u)_{u \in V(G \square S_k)}$ be a non-zero vector in span(A). Let j be the smallest index for which there exists some $v_j \in V(G_j)$ with $p_{v_j} \not\equiv 0$. If $j = 0$, then by definition of the vectors in A, we can write $p_{v_0} = q_v$ for some vector $\mathbf{q} = (q_u)_{u \in V(G)}$ in span$(B^{(r)}) = W^r_{G,c}$. Otherwise, if $j \in [k]$, then by definition of the vectors in A, and the minimality of j, we can write $p_{v_j} = q_v(x - \alpha_j)$ for some vector $\mathbf{q} = (q_u)_{u \in V(G)}$ in span$(B^{(r-1)}) = W^{r-1}_{G,c}$. In any case, by Definition 3.5, we have $\deg(q_v) \leq \deg_G(v) - 1$. Therefore, since the colours in $\{\alpha_1, ..., \alpha_k\}$ were chosen to be distinct from those in $c(E)$, by Lemma 3.11 there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$.$_{(\square)}$

Thus, to finish the proof, it suffices to find a set X of $\sum_{t=1}^{k-1} t d^G_{r-t} + k \sum_{t=k}^{r} d^G_{r-t}$ linearly independent vectors in $W^r_{G \square S_k, c'}$ that evaluate to zero on Z_c. This, together with Claim 3.17, will guarantee that these vectors are independent from those in A.

First, for each $v \in V(G)$ with $\deg_G(v) \leq r - k$ and each $\ell \in [k]$, define the vector $\mathbf{p}^\ell_v = (p_u)_{u \in V(G \square S_k)}$ as follows:

- Let p_{v_0} be the unique polynomial of degree $\deg(v_0) - 1$ that is equal to 1 at $c'(v_0 v_\ell)$ and equal to zero on all $\deg(v_0) - 1$ other colours of the edges incident with v_0 in $G \square S_k$. That is, $p_{v_0} = \beta_0 \prod_{u \in N(v_0) \setminus \{v_\ell\}} (x - c'_{v_0 u})$, where β_0 is a non-zero constant chosen so that $p_{v_0}(c'_{v_0 v_\ell}) = 1$.
- Let p_{v_ℓ} be the unique polynomial of degree $\deg(v_\ell) - 1$ that is equal to 1 at $c'(v_0 v_\ell)$ and equal to zero on all $\deg(v_\ell) - 1$ other colours of the edges incident with v_ℓ in $G \square S_k$. That is, $p_{v_\ell} = \beta_\ell \prod_{u \in N(v_\ell) \setminus \{v_0\}} (x - c'_{v_\ell u})$, where β_ℓ is a non-zero constant chosen so that $p_{v_\ell}(c'_{v_0 v_\ell}) = 1$.
- Let $p_u \equiv 0$ for all $u \in V(G \square S_k) \setminus \{v_0, v_\ell\}$.

See Figure 3.5 for an example of this construction for the star S_4 when $\ell = 2$.

Note that, since $\deg_G(v) \leq r - k$, we have

$$\deg(p_{v_0}) = \deg(v_0) - 1 = \deg_G(v) + k - 1 \leq r - 1.$$

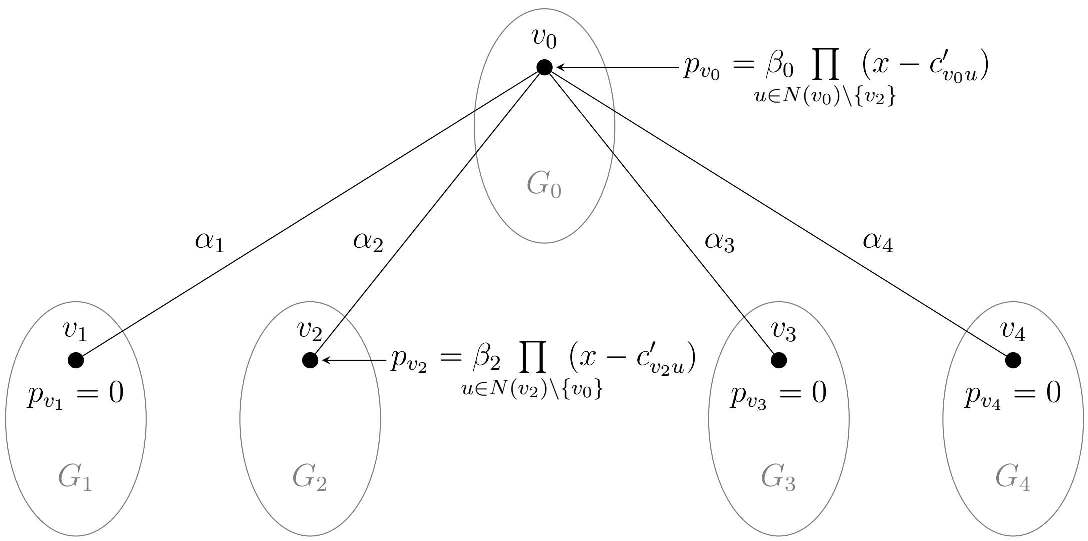

Figure 3.5: The vector $\mathbf{p}_v^2$ constructed for a vertex $v \in V(G)$ with $\deg_G(v) \leq r - 4$.

Moreover, $k \geq 1$ implies that

$$\deg(p_{v_\ell}) = \deg(v_\ell) - 1 = \deg_G(v) \leq r - k \leq r - 1.$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G \square S_k)$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^\ell \in W^r_{G \square S_k, c'}$. Furthermore, $\mathbf{p}_v^\ell$ also evaluates to 0 on all of Z_c. Let $X_0 = \{\mathbf{p}_v^\ell : \deg_G(v) \leq r - k \text{ and } \ell \in [k]\}$. Note that $|X_0| = k \sum_{t=k}^{r} d^G_{r-t}$.

Now, fix $t \in [k-1]$. For each $\ell \in [t]$ and each $v \in V(G)$ with $\deg_G(v) = r - t$, define the vector $\mathbf{p}_v^\ell = (p_u)_{u \in V(G \square S_k)}$ as follows: Let $p_u \equiv 0$ for all $u \notin \{v_0, v_\ell\} \cup \{v_{t+1}, ..., v_k\}$. Let

$$p_{v_0} = \prod_{\substack{i \in [t] \\ i \neq \ell}} (x - \alpha_i) \prod_{u \in N_G(v)} (x - c_{uv}).$$

Finally, for $j \in \{\ell\} \cup \{t+1, ..., k\}$, let

$$p_{v_j} = \gamma_j \prod_{u \in N_G(v)} (x - c_{uv}),$$

where γ_j is a constant chosen so that $p_{v_j}(\alpha_j) = p_{v_0}(\alpha_j)$. Note that $\gamma_j \neq 0$ since $p_{v_0}(\alpha_j) \neq 0$. See Figure 3.6 for an illustration of this construction for $t = 3$ and $\ell = 2$.

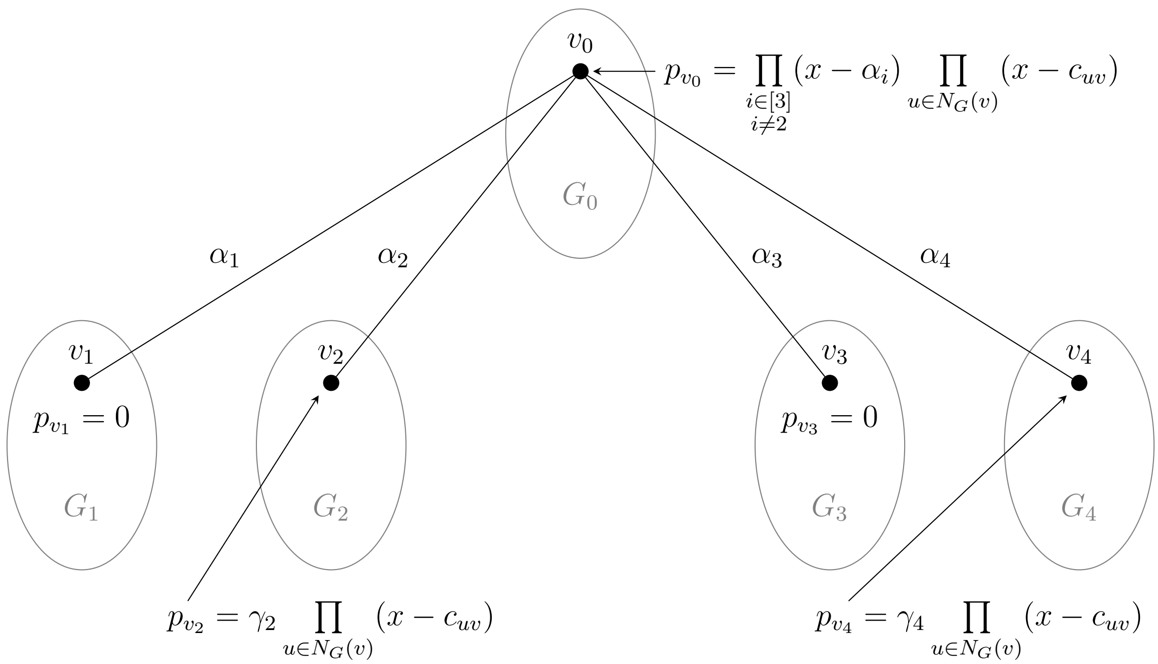

Figure 3.6: The vector $\mathbf{p}_v^2$ constructed for a vertex $v \in V(G)$ with $\deg_G(v) = r - 3$.

Now, $\deg_G(v) = r - t$ implies $\deg(p_{v_0}) = \deg_G(v) + t - 1 = r - 1$. Moreover,

$$\begin{aligned}\deg(p_{v_0}) &= \deg_G(v) + t - 1 \\ &= \deg(v_0) - k + t - 1 \\ &\leq \deg(v_0) - k + (k-1) - 1 \\ &< \deg(v_0) - 1.\end{aligned}$$

Hence $\deg(p_{v_0}) \leq \min\{r, \deg(v_0)\} - 1$. Furthermore, for $j \in \{\ell\} \cup \{t+1, ..., k\}$, since $\deg_G(v) = r - t \leq r - 1$ and $\deg_G(v) = \deg(v_j) - 1$, we have

$$\deg(p_{v_j}) = \deg_G(v) \leq \min\{r-1, \deg(v_j) - 1\} = \min\{r, \deg(v_j)\} - 1.$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G \square S_k)$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^\ell \in W^r_{G\square S_k, c'}$. In addition, $\mathbf{p}_v^\ell$ evaluates to 0 on all of Z_c. For each $t \in [k-1]$, let $X_t = \{\mathbf{p}_v^\ell : \deg_G(v) = r - t \text{ and } \ell \in [t]\}$. Note that $|X_t| = td^G_{r-t}$.

Define $X := \bigcup_{i=0}^{k-1} X_i$. Hence $|X| = \sum_{i=0}^{k-1} |X_i| = k\sum_{t=k}^{r} d_{r-t}^G + \sum_{t=1}^{k-1} td_{r-t}^G$.

Claim 3.18 *The vectors in X are linearly independent.*

Proof of Claim 3.18*:* Suppose for a contradiction that there is a linear combination $S = \sum_{\mathbf{p}\in X} \beta_{\mathbf{p}}\mathbf{p} = 0$, where $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in X$. Note that, for each $v \in V(G)$ with $\deg_G(v) \leq r-k$ and each $\ell \in [k]$, we have $p_{v_\ell} \not\equiv 0$ for the vector $\mathbf{p}_v^\ell \in X_0 \subseteq X$, but $p_{v_\ell} \equiv 0$ for all other vectors $\mathbf{p} \in X$. Similarly, fixing $t \in [k-1]$, for each $v \in V(G)$ with $\deg_G(v) = r-t$ and each $\ell \in [t]$, we have $p_{v_\ell} \not\equiv 0$ for the vector $\mathbf{p}_v^\ell \in X_t \subseteq X$, but $p_{v_\ell} \equiv 0$ for all other vectors $\mathbf{p} \in X$. Therefore, for each vector $\mathbf{p} \in X$, there exists a coordinate p_u which is non-zero in $\mathbf{p}$ but identically zero in all other vectors in X. Thus, restricting the sum S to the entry corresponding to p_u, we see that $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in X$, a contradiction. $(\Box)$

Let $Y = A \cup X$. Therefore, by Claims 3.16, 3.17, and 3.18, we have found

$$|Y| = |A| + |X| = \dim\left(W_{G,c}^r\right) + k\dim\left(W_{G,c}^{r-1}\right) + \sum_{t=1}^{k-1} td_{r-t}^G + k\sum_{t=k}^{r} d_{r-t}^G$$

linearly independent vectors in $W_{G\Box S_k,c'}^r$, as required. $\Box$

3.4.3 Exact Values for Products of Stars

In this section, we provide a recursive formula for $m_e(G,r)$ when G is a product of stars. In fact, we prove the following more general result (initially stated earlier in Section 3.4) that gives a recursive formula for $m_e(G\Box S_k, r)$ for certain graphs G.

Theorem 3.12 *Let $k, r \geq 1$ be integers, and let G be a graph. Suppose there exists a proper edge-colouring c of G such that $m_e(G,i) = \dim(W_{G,c}^i)$ for each $r-1 \leq i \leq r$. Then*

$$m_e(G\Box S_k, r) = m_e(G,r) + km_e(G,r-1) + \sum_{t=1}^{k-1} td_{r-t}^G + k\sum_{t=k}^{r} d_{r-t}^G.$$

Proof. By Proposition 3.13, we have

$$m_e(G\Box S_k, r) \leq m_e(G, r) + km_e(G, r-1) + \sum_{t=1}^{k-1} td^G_{r-t} + k\sum_{t=k}^{r} d^G_{r-t}.$$

The matching lower bound follows from Proposition 3.14, since $m_e(G,i) = \dim(W^i_{G,c})$ for $r-1 \leq i \leq r$. □

Let $d \in \mathbb{Z}^+$. Given integers $a_1, ..., a_d \geq 3$ and $b_1, ..., b_d \geq 2$, let $G_C := \prod_{i=1}^d C_{a_i}$ and $G_P := \prod_{i=1}^d P_{b_i}$. In their proofs of Theorems 3.2 and 3.3, Hambardzumyan, Hatami, and Qian [14] showed that, $m_e(G_C, r) = \dim\left(W^r_{G_C,c}\right)$ and $m_e(G_P, r) = \dim\left(W^r_{G_P,c}\right)$ for all $r \geq 1$. Therefore, Theorems 3.2 and 3.3 can be combined with Theorem 3.12 to determine recursive formulas for $m_e(G\Box S_k)$ when G is a product of cycles or a product of paths. Furthermore, the following result (initially stated in Section 3.1) provides a recursive formula for $m_e(G, r)$ when G is a product of stars.

Theorem 3.4 *Let $k, r, a_1, ..., a_k \in \mathbb{Z}^+$. Let $G_i = S_{a_1}\Box\cdots\Box S_{a_i}$ for each $i \in [k]$, and let $G_0 = K_1$. For $0 \leq i \leq \Delta(G_{k-1})$, let d_i denote the number of vertices in G_{k-1} of degree i. Then*

$$m_e(G_k, r) = m_e(G_{k-1}, r) + a_k m_e(G_{k-1}, r-1) + \sum_{t=1}^{a_k-1} td_{r-t} + a_k\sum_{t=a_k}^{r} d_{r-t}.$$

Proof. Note that $G_k = G_{k-1}\Box S_{a_k}$. Therefore, by Theorem 3.12, it suffices to show that there exists a proper edge-colouring c of G_{k-1} such that $m_e(G_{k-1}, i) = \dim(W^i_{G_{k-1},c})$ for all integers $r-1 \leq i \leq r$. In fact, we prove the following stronger result.

Claim 3.19 *Let $k \geq 0$ be an integer. Then there exists a proper edge-colouring c of G_k such that $m_e(G_k, i) = \dim(W^i_{G_k,c})$ for all $i \geq 0$.*

Proof of Claim 3.19. We prove the result by induction on k. First, suppose $k = 0$. Then $m_e(G_0, i) = \dim(W^i_{G_0,c}) = 0$ for all $i \geq 0$, where c is the empty edge-colouring.

Now, let $k > 0$, and suppose c is a proper edge-colouring of G_{k-1} such that $m_e(G_{k-1}, i) = \dim(W^i_{G_{k-1},c})$ for all $i \geq 0$. By the proof of Proposition 3.15, there exists a proper edge-colouring c' of G_k such that, for any $i \geq 1$,

$$\dim(W^i_{G_k,c'}) \geq \dim(W^i_{G_{k-1},c}) + a_k \dim(W^{i-1}_{G_{k-1},c}) + \sum_{t=1}^{a_k-1} t d_{i-t} + a_k \sum_{t=a_k}^{i} d_{i-t}.$$

Hence, by the induction hypothesis,

$$\dim(W^i_{G_k,c'}) \geq m_e(G_{k-1}, i) + a_k m_e(G_{k-1}, i-1) + \sum_{t=1}^{a_k-1} t d_{i-t} + a_k \sum_{t=a_k}^{i} d_{i-t}.$$

Therefore, by Theorem 3.6 and Proposition 3.13, $m_e(G_k, i) = \dim(W^i_{G_k,c'})$. Since $m_e(G_k, 0) = \dim(W^0_{G_k,c'}) = 0$, Claim 3.19 follows by induction.$_{(\square)}$

Therefore, by Theorem 3.12,

$$m_e(G_k, r) = m_e(G_{k-1}, r) + a_k m_e(G_{k-1}, r-1) + \sum_{t=1}^{a_k-1} t d_{i-t} + a_k \sum_{t=a_k}^{i} d_{i-t}.$$

□

3.5 Products of Joined Cycles

In this section, we establish upper and lower bounds on $m_e(G \square H, r)$ when H is a joined cycle (see below for definition), and show that these bounds match for certain graphs G.

Definition 3.20 Let $k \geq \ell \geq 3$. The *joined cycle* $H_{k,\ell}$ is the graph constructed from a copy of $C_{k+\ell-2}$ by adding an edge between a pair of vertices at distance $\ell - 1$. Hence $H_{k,\ell}$ is the theta graph $\theta_{1,\ell-1,k-1}$ obtained by joining two vertices by three internally vertex-disjoint paths of length 1, $\ell - 1$, and $k - 1$. Note that $H_{k,\ell}$ contains two induced cycles, one of length ℓ and the one of length k.

Throughout Section 3.5, we label the vertices of $H_{k,\ell}$ as follows: Label the ℓ vertices on the induced cycle of length ℓ as $1, ..., \ell$ in order so the two vertices of degree 3 are labelled 1 and 2. Label the vertices on the induced cycle of length k with the symbols $1', ..., k'$ in order, so that $1' = 1$ and $2' = 2$. See Figure 3.7 for an illustration.

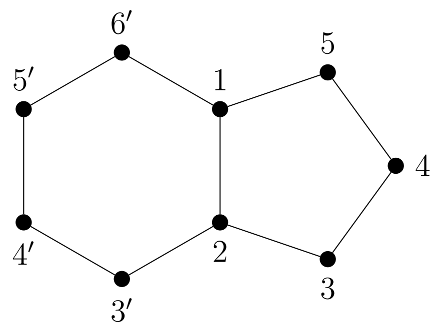

Figure 3.7: The joined cycle $H_{6,5} \simeq \theta_{1,4,5}$

For ease of notation, let $[k'] = \{1', ..., k'\}$. Given integers $k \geq \ell \geq 3$, we define $I = I_{k,\ell}$ to be the set $[\ell] \cup [k'] \setminus \{1', 2'\}$. Hence $V(H_{k,\ell}) = I_{k,\ell}$. We impose an ordering ϕ on $\mathbb{Z} \cup [k']$ where $i < i' < i + 1$ for all $i \in [k]$. Note that $I \subseteq \mathbb{Z} \cup [k']$, and thus ϕ is also an ordering on I.

Recall that, given a graph G, we denote by d_t^G the number of vertices in G of degree t. The main goal of this section is to prove the following recursive formula for $m_e(G \square H_{k,\ell}, r)$.

Theorem 3.21 *Let $k \geq \ell \geq 4$ and $r > 1$ be integers. If there exists a proper edge-colouring c of G such that $m_e(G, i) = \dim(W_{G,c}^i)$ for each $r - 2 \leq i \leq r$, then*

$$m_e(G \square H_{k,\ell}, r) = m_e(G, r) + (k + \ell - 5)m_e(G, r - 1) + 2m_e(G, r - 2)$$
$$+ d_{r-1}^G + (k + \ell - 3)d_{r-2}^G + (k + \ell - 1) \sum_{i=0}^{r-3} d_i^G.$$

In particular, we prove the following special case of Theorem 3.21 where G is itself a product of joined cycles.

Theorem 3.22 *Let $t > 0$ and $r > 1$ be integers. Let $k_1, ..., k_t, \ell_1, ..., \ell_t \in \mathbb{Z}^+$ such that $k_i \geq \ell_i \geq 4$ for all $i \in [t]$. Let $G_i = H_{k_1,\ell_1} \square \cdots \square H_{k_i,\ell_i}$ for $i \in [t]$, and let $G_0 = K_1$. For $0 \leq i \leq \Delta(G_{t-1})$, let d_i denote the number of vertices in G_{t-1} of degree i. Then*

$$m_e(G_t, r) = m_e(G_{t-1}, r) + (k_t + \ell_t - 5)m_e(G_{t-1}, r-1) + 2m_e(G_{t-1}, r-2) + d_{r-1} + (k_t + \ell_t - 3)d_{r-2} + (k_t + \ell_t - 1)\sum_{i=0}^{r-3} d_i.$$

In sections 3.5.1 and 3.5.2, respectively, we provide upper and lower bounds on $m_e(G \square H_{k,\ell})$. Section 3.5.3 contains proofs of Theorems 3.21 and 3.22.

3.5.1 Upper Bound

In this section, we prove the following upper bound on $m_e(G \square H_{k,\ell}, r)$.

Proposition 3.23 *Let $r > 1$ and $k \geq \ell \geq 4$ be integers. For any graph G,*

$$m_e(G \square H_{k,\ell}, r) \leq m_e(G, r) + (k + \ell - 5)m_e(G, r-1) + 2m_e(G, r-2) + d^G_{r-1} + (k + \ell - 3)d^G_{r-2} + (k + \ell - 1)\sum_{t=0}^{r-3} d^G_t.$$

Proof. Label the vertices of $H_{k,\ell}$ as in Definition 3.20. Let $I = [\ell] \cup [k'] \setminus \{1', 2'\}$ so that $V(H_{k,\ell}) = I$, and let ϕ be the ordering of I defined earlier in Section 3.5. Let $\{G_i : i \in I\}$ denote the $k + \ell - 2$ copies of G in $G \square H_{k,\ell}$ corresponding to the $k + \ell - 2$ vertices in $H_{k,\ell}$. For each vertex $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where $i \in I$.

First, we consider the case where $\delta(G) \geq r$. We construct an r-percolating set $F \subseteq E(G \square H_{k,\ell})$ for $G \square H_{k,\ell}$ as follows: Pick an optimal r-percolating set F_1 for G_1. For each $i \in I \setminus \{1, \ell, k'\}$, pick an optimal $(r-1)$-percolating set F_i on G_i. Finally, let F_ℓ and $F_{k'}$ be optimal $(r-2)$-percolating sets on G_ℓ and $G_{k'}$, respectively. Let $F = \bigcup_{i \in I} F_i$. See Figure 3.8 for an illustration of this construction.

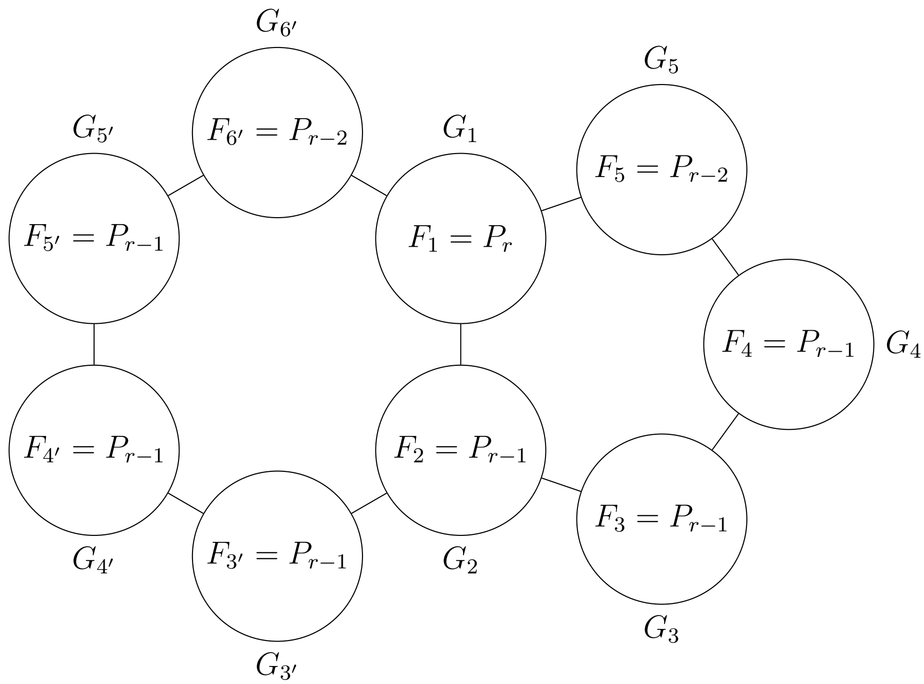

Figure 3.8: The r-percolating set F chosen for $G \square H_{6,5}$ when $\delta(G) \geq r$. Here P_i denotes an optimal i-percolating set on G.

We claim that F percolates in $G \square H_{k,\ell}$. By our choice of F_1, after running the r-bond bootstrap percolation process on G_1, all edges in G_1 will be infected. Since $\delta(G) \geq r$, all edges between G_1 and each of G_2, G_ℓ, and $G_{k'}$ will now become infected. As each vertex $v_2 \in V(G_2)$ has an infected edge coming from G_1, this, together with F_2, will infect the edges in G_2. Again, since $\delta(G) \geq r$, all edges between G_2 and both G_3 and $G_{3'}$ will now become infected. As each vertex $v_3 \in V(G_3)$ has an infected edge coming from G_2, this, together with F_3, will infect the edges in G_3. Continuing in this manner, we can infect $G_3, ..., G_{\ell-1}$, and similarly $G_{3'}, ...G_{(k-1)'}$. Finally, since $\delta(G) \geq r$, all edges between $G_{\ell-1}$ and G_ℓ will become infected. Since each vertex $v_\ell \in V(G_\ell)$ has one infected edge coming from $G_{\ell-1}$ and one from G_1, this, together with F_ℓ, will infect the edges in G_ℓ. A similar argument shows that the edges in $G_{k'}$ will also become infected. Hence F percolates in $G \square H_{k,\ell}$.

It remains to consider the case where $\delta(G) < r$. Let the set F be as defined above. We add the following edges to F (see Figure 3.9 for an illustration of this construction):

- For each $v \in V(G)$ with $\deg_G(v) = r-1$, add the edge v_1v_2 to the set F. Adding this edge will guarantee that, once G_1 is fully infected, the infection will spread to v_1v_ℓ and $v_1v_{k'}$, after which the process continues as described above. Note that this adds an additional d^G_{r-1} edges to the set F.

- For each $v \in V(G)$ with $\deg_G(v) = r-2$, add all edges v_iv_j, except $v_1v_{k'}$ and v_2v_3, to the set F. Note that the edges $v_1v_{k'}$ and v_2v_3 will become infected once the edges in G_1 and G_2 are infected, respectively, allowing the process to continue as described above. This adds an additional $(k+\ell-3)d^G_{r-2}$ edges to the set F.

- For each $v \in V(G)$ with $\deg_G(v) \leq r-3$, add all edges v_iv_j to the set F. This adds an additional $(k+\ell-1)\sum_{t=0}^{r-3} d^G_t$ edges to the set F.

Altogether, we added $d^G_{r-1} + (k+\ell-3)d^G_{r-2} + (k+\ell-1)\sum_{t=0}^{r-3} d^G_t$ edges to the $m_e(G,r)+(k+\ell-5)m_e(G,r-1)+2m_e(G,r-2)$ edges initially in F. Since the resulting set F percolates in G, we have established the desired upper bound on $m_e(G\Box H_{k,\ell}, r)$.

$\square$

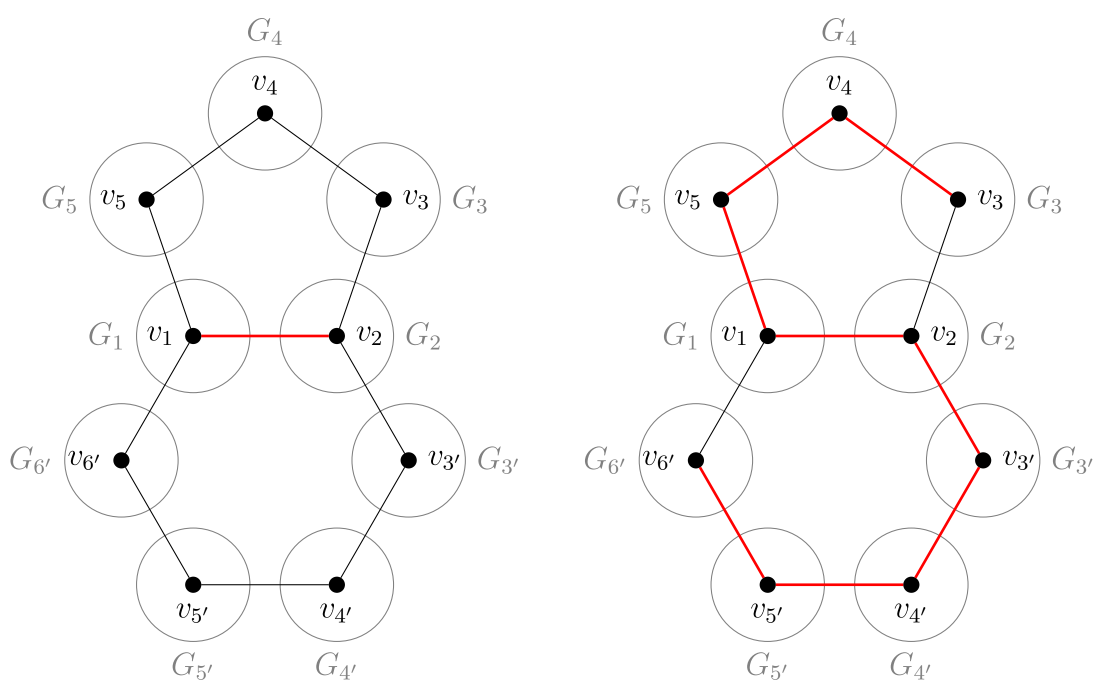

Figure 3.9: The edges added (shown in red) to the set F for a vertex $v \in V(G)$ with $\deg_G(v) = r-1$ (left) and $\deg_G(v) = r-2$ (right).

3.5.2 Lower Bound

The goal of this section is to prove the following lower bound on $m_e(G\Box H_{k,\ell}, r)$.

Proposition 3.24 *Let $r > 1$ and $k \geq \ell \geq 4$ be integers. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then*

$$m_e(G\Box H_{k,\ell}, r) \geq \dim\left(W^r_{G,c}\right) + (\ell + k - 5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right) + d^G_{r-1} + (\ell + k - 3)d^G_{r-2} + (\ell + k - 1)\sum_{t=0}^{r-3} d^G_t.$$

To this end, we will prove the following result.

Proposition 3.25 *Let $r > 1$ and $k \geq \ell \geq 4$ be integers, and let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then there exists a proper edge colouring c' of $G\Box H_{k,\ell}$ for which*

$$\dim\left(W^r_{G\Box H_{k,\ell},c'}\right) \geq \dim\left(W^r_{G,c}\right) + (\ell + k - 5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right) + d^G_{r-1} + (\ell + k - 3)d^G_{r-2} + (\ell + k - 1)\sum_{t=0}^{r-3} d^G_t.$$

Before proving Proposition 3.25, we first demonstrate why it implies the lower bound in Proposition 3.24.

Proof of Proposition 3.24. By Proposition 3.25, there exists a proper edge-colouring c' of $G\Box H_{k,\ell}$ such that

$$\dim\left(W^r_{G\Box H_{k,\ell},c'}\right) \geq \dim\left(W^r_{G,c}\right) + (\ell + k - 5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right) + d^G_{r-1} + (\ell + k - 3)d^G_{r-2} + (\ell + k - 1)\sum_{t=0}^{r-3} d^G_t.$$

By Theorem 3.6, we have $m_e(G\Box H_{k,\ell}, r) \geq \dim\left(W^r_{G\Box H_{k,\ell},c'}\right)$, and thus

$$m_e(G\Box H_{k,\ell}, r) \geq \dim\left(W^r_{G,c}\right) + (\ell+k-5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right) + d^G_{r-1} + (\ell+k-3)d^G_{r-2} + (\ell+k-1)\sum_{t=0}^{r-3} d^G_t.$$

$\square$

Proof of Proposition 3.25. Let the vertices of $H_{k,\ell}$ be labelled as in Definition 3.20. Let $I = [\ell] \cup [k'] \setminus \{1', 2'\}$ so that $V(H_{k,\ell}) = I$, and let ϕ be the ordering of I defined earlier in Section 3.5. Let $\{G_i : i \in I\}$ denote the $k+\ell-2$ copies of G in $G\Box H_{k,\ell}$ corresponding to the $k+\ell-2$ vertices in $H_{k,\ell}$. For convenience, we will sometimes refer to G_i as $G_{i'}$, when $i \in \{1,2\}$. For each vertex $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where we sometimes refer to v_i as $v_{i'}$, for $i \in \{1,2\}$. In addition, we refer to the copy of $G\Box C_\ell$ in $G\Box H_{k,\ell}$ as C, and the copy of $G\Box C_k$ in $G\Box H_{k,\ell}$ as C'. Note that we take the indices in C to be modulo ℓ, and those in C' to be modulo k, where we perform addition on $[k']$ irrespective of the superscripts; for example, $3' + 1 = (3+1)' = 4'$.

Let $\{\alpha_i : i \in I \cup \{2'\}\}$ be distinct real numbers not in $c(E)$. For convenience, we sometimes denote α_1 by $\alpha_{1'}$. Let c' be the proper edge-colouring of $G\Box H_{k,\ell}$ defined as follows: Let c' be consistent with c on each G_i, where $i \in I$. For each $v \in V(G)$ and $i \in [\ell]$, let $c'(v_iv_{i+1}) = \alpha_i$. For each $v \in V(G)$ and $i \in [k'] \setminus \{1'\}$, let $c'(v_iv_{i+1}) = \alpha_i$. See Figure 3.10 for an illustration of this edge-colouring of $G\Box H_{k,\ell}$.

Let

$$N = \dim\left(W^r_{G,c}\right) + (\ell+k-5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right) + d^G_{r-1} + (\ell+k-3)d^G_{r-2} + (\ell+k-1)\sum_{t=0}^{r-3} d^G_t.$$

To prove this result, we will find a set Y of N linearly independent vectors in $W^r_{G\Box H_{k,\ell},c'}$. We begin by finding a set A of $\dim\left(W^r_{G,c}\right) + (\ell+k-5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right)$ linearly independent vectors in $W^r_{G\Box H_{k,\ell},c'}$.

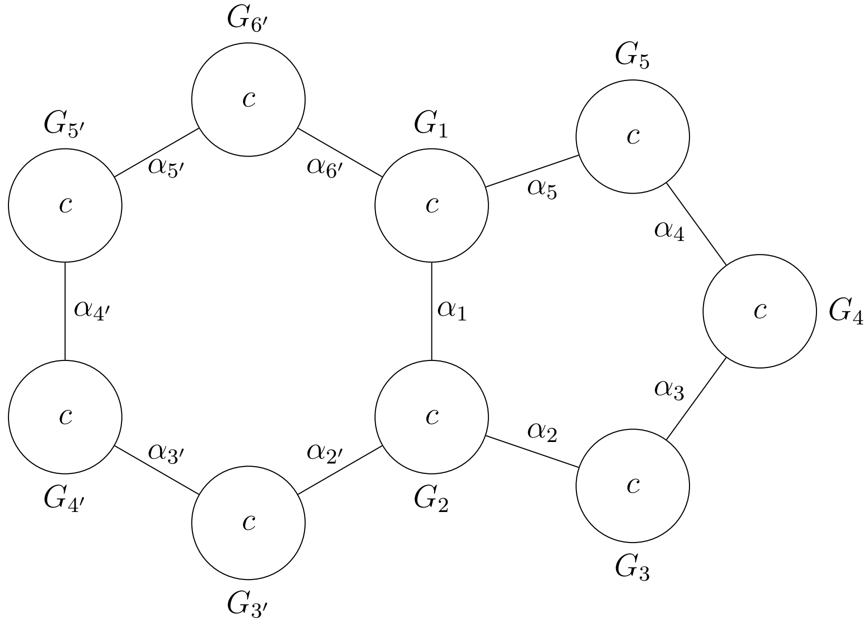

Figure 3.10: The graph $G\Box H_{6,5}$ and its edge colouring c'.

First, consider a basis $B^{(r)}$ for $W^r_{G,c}$. For each vector $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r)}$, define the vector $\mathbf{p}^1_{\mathbf{q}} = (p_u)_{u\in V(G\Box H_{k,\ell})}$ so that $p_{v_i} = q_v$ for all $i \in I$ and $v \in V(G)$. Trivially, the two conditions in Definition 3.5 are satisfied; that is, $\deg(p_v) \leq \min\{r, \deg(v)\} - 1$ for all vertices $v \in V(G\Box H_{k,\ell})$, and $p_u(c_{uv}) = p_v(c_{uv})$ for all edges $uv \in E(G\Box H_{k,\ell})$. Hence $\mathbf{p}^1_{\mathbf{q}} \in W^r_{G\Box H_{k,\ell},c'}$. Let $A_1 = \{\mathbf{p}^1_{\mathbf{q}} : q \in B^{(r)}\}$. Note that the restriction of any vector $\mathbf{p}^1_{\mathbf{q}} \in A_1$ to G_1 equals $\mathbf{q}$, and thus, since $B^{(r)}$ is a basis, the vectors in A_1 are linearly independent.

Now, consider a basis $B^{(r-1)}$ for $W^{r-1}_{G,c}$. First, fix $i \in I \setminus \{1, 2, \ell, k'\}$. For each vector $\mathbf{q} = (q_v)_{v\in V(G)} \in B^{(r-1)}$, define $\mathbf{p}^i_{\mathbf{q}} = (p_u)_{u\in V(G\Box H_{k,\ell})}$ so that, for each $v \in V(G)$,

$$p_{v_i} = \frac{x - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} q_v,$$
$$p_{v_{i+1}} = \frac{x - \alpha_{i+1}}{\alpha_i - \alpha_{i+1}} q_v,$$

and $p_{v_j} \equiv 0$ for all other $j \in I$. See Figure 3.11 for an illustration of this construction when $k = \ell = 5$ and $i = 3$.

For each $v \in V(G)$, note that $\mathbf{q} \in W^{r-1}_{G,c}$ implies $\deg(q_v) \leq \min\{r-1, \deg_G(v)\} - 1$. Hence

$$\begin{aligned}\deg(p_{v_i}) &\leq \deg(q_v) + 1 \\ &\leq \min\{r-1, \deg_G(v)\} \\ &= \min\{r-1, \deg(v_i) - 2\} \\ &\leq \min\{r, \deg(v_i)\} - 1.\end{aligned}$$

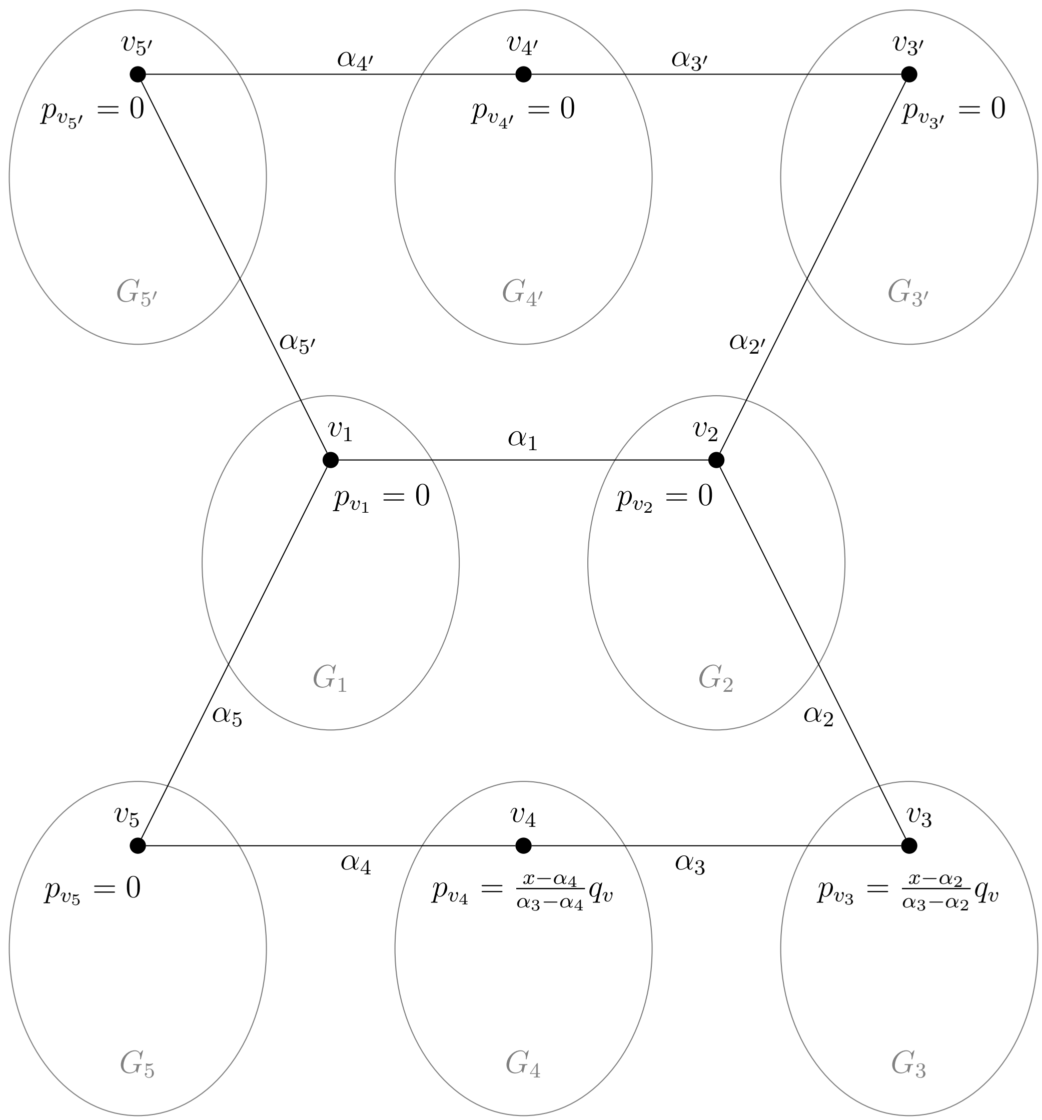

Figure 3.11: The vector $\mathbf{q} \in B^{(r-1)}$ is used to construct $\mathbf{p}^3_{\mathbf{q}} \in W^r_{G\Box H_{5,5},c'}$.

Similarly, $\deg(p_{v_{i+1}}) \leq \min\{r, \deg(v_{i+1})\} - 1$. Moreover, note that

$$\begin{aligned} p_{v_{i-1}}(\alpha_{i-1}) &= 0 = p_{v_i}(\alpha_{i-1}), \\ p_{v_i}(\alpha_i) &= q_v(\alpha_i) = p_{v_{i+1}}(\alpha_i), \text{ and} \\ p_{v_{i+1}}(\alpha_{i+1}) &= 0 = p_{v_{i+2}}(\alpha_{i+1}). \end{aligned}$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G \square H_{k,\ell})$, the two conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}^i_{\mathbf{q}} \in W^r_{G\square H_{k,\ell},c'}$. For each $i \in I \setminus \{1, 2, \ell, k'\}$, define $A_i := \{\mathbf{p}^i_{\mathbf{q}} : \mathbf{q} \in B^{(r-1)}\}$. Note that the restriction of any vector $\mathbf{p}^i_{\mathbf{q}} \in A_i$ to G_i is $\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\mathbf{q}$. Therefore, since $x - \alpha_{i-1} \not\equiv 0$ and $B^{(r-1)}$ is a basis, the vectors in A_i are linearly independent.

Now, for each $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r-1)}$, define the vector $\mathbf{p}^2_{\mathbf{q}} = (p_u)_{u\in V(G\square H_{k,\ell})}$ so that, for each $v \in V(G)$,

$$\begin{aligned} p_{v_3} &= \frac{x - \alpha_3}{(\alpha_2 - \alpha_3)(\alpha_{2'} - \alpha_1)} q_v, \\ p_{v_2} &= \frac{x - \alpha_1}{(\alpha_2 - \alpha_1)(\alpha_{2'} - \alpha_1)} q_v, \\ p_{v_{3'}} &= \frac{x - \alpha_{3'}}{(\alpha_2 - \alpha_1)(\alpha_{2'} - \alpha_{3'})} q_v, \end{aligned}$$

and $p_{v_i} \equiv 0$ for all other $i \in I$. See Figure 3.12 for an illustration of this construction.

For each $v \in V(G)$, note that $\mathbf{q} \in W^{r-1}_{G,c}$ implies $\deg(q_v) \leq \min\{r-1, \deg_G(v)\} - 1$. Hence

$$\begin{aligned} \deg(p_{v_2}) &= \deg(q_v) + 1 \\ &\leq \min\{r - 1, \deg_G(v)\} \\ &= \min\{r - 1, \deg(v_2) - 3\} \\ &\leq \min\{r, \deg(v_2)\} - 1. \end{aligned}$$

In addition,

$$\begin{aligned}\deg(p_{v_3}) &= \deg(q_v) + 1 \\ &\le \min\{r-1, \deg_G(v)\} \\ &= \min\{r-1, \deg(v_3) - 2\} \\ &\le \min\{r, \deg(v_3)\} - 1.\end{aligned}$$

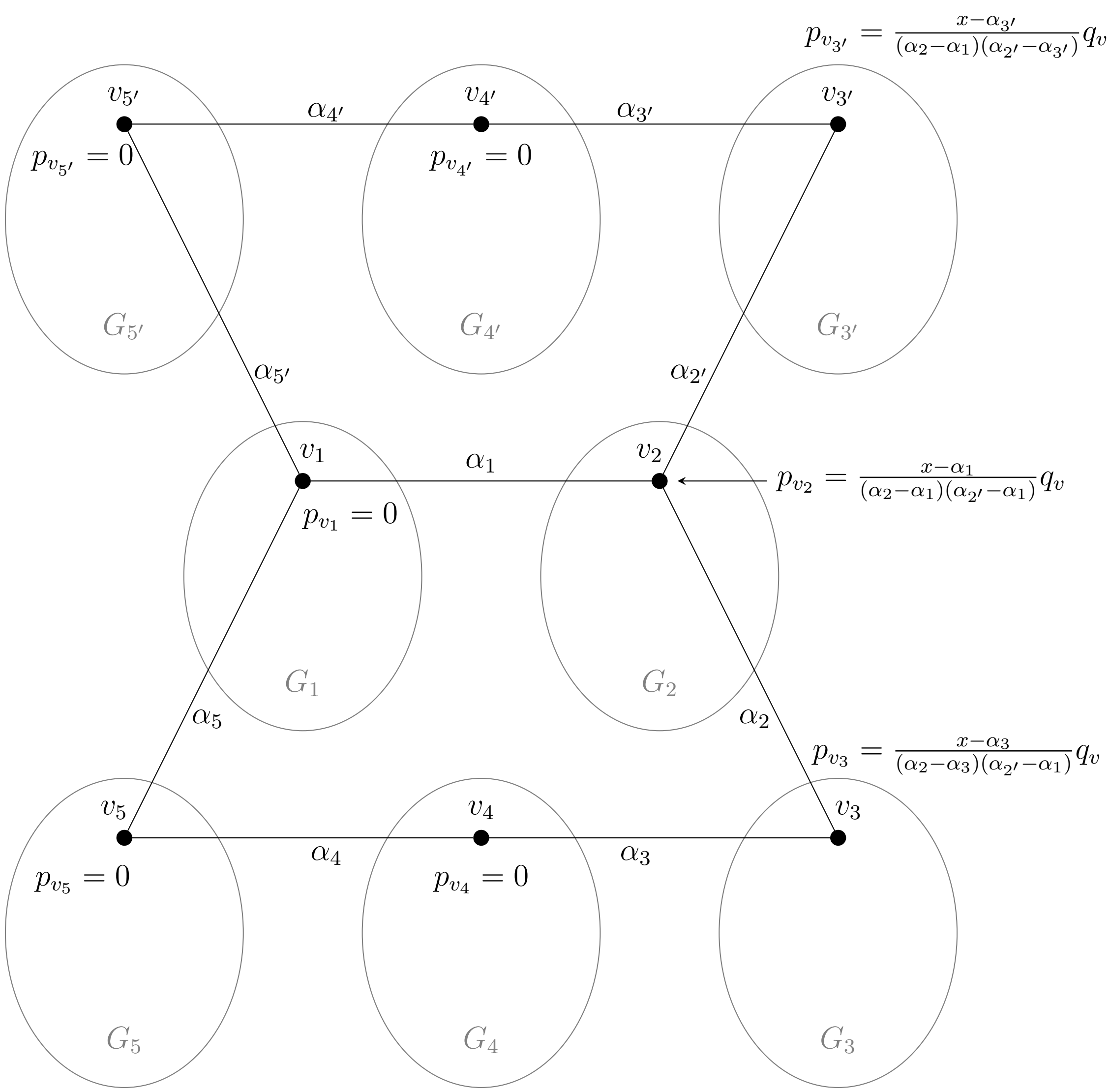

Figure 3.12: The vector $\mathbf{q} \in B^{(r-1)}$ is used to construct $\mathbf{p}_{\mathbf{q}}^2 \in W^r_{G\Box H_{5,5,c'}}$.

Similarly, $\deg(p_{v_{3'}}) \leq \min\{r, \deg(v_{3'})\} - 1$. Now, note that

$$
\begin{aligned}
p_{v_1}(\alpha_1) &= 0 = p_{v_2}(\alpha_1), \\
p_{v_2}(\alpha_2) &= \frac{1}{\alpha_{2'} - \alpha_1} q_v(\alpha_2) = p_{v_3}(\alpha_2), \\
p_{v_3}(\alpha_3) &= 0 = p_{v_4}(\alpha_3), \\
p_{v_{3'}}(\alpha_{3'}) &= 0 = p_{v_{4'}}(\alpha_{3'}), \text{ and} \\
p_{v_2}(\alpha_{2'}) &= \frac{1}{\alpha_2 - \alpha_1} q_v(\alpha_{2'}) = p_{v_{3'}}(\alpha_{2'}).
\end{aligned}
$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G \square H_{k,\ell})$, the two conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_{\mathbf{q}}^2 \in W^r_{G\square H_{k,\ell},c'}$. Define $A_2 := \{\mathbf{p}_{\mathbf{q}}^2 : \mathbf{q} \in B^{(r-1)}\}$. Note that the restriction of any vector $\mathbf{p}_{\mathbf{q}}^2 \in A_2$ to G_2 is $\frac{x-\alpha_1}{(\alpha_2-\alpha_1)(\alpha_{2'}-\alpha_1)}\mathbf{q}$. Therefore, since $x - \alpha_1 \not\equiv 0$ and $B^{(r-1)}$ is a basis, the vectors in A_2 are linearly independent.

Finally, consider a basis $B^{(r-2)}$ for $W^{r-2}_{G,c}$. Fix $i \in \{\ell, k'\}$. For each $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r-2)}$, define the vector $\mathbf{p}_{\mathbf{q}}^i = (p_u)_{u\in V(G\square H_{k,\ell})}$ such that $p_{v_i} = (x - \alpha_{i-1})(x - \alpha_i)q_v$ for all v in $V(G)$, and $p_u \equiv 0$ for all other $u \in V(G\square H_{k,\ell})$. See Figure 3.13 for an illustration of this construction.

For $v \in V(G)$, note that $\mathbf{q} \in W^{r-2}_{G,c}$ implies $\deg(q_v) \leq \min\{r-2, \deg_G(v)\}-1$. Thus

$$
\begin{aligned}
\deg(p_{v_i}) &= \deg(q_v) + 2 \\
&\leq \min\{r-2, \deg_G(v)\} + 1 \\
&= \min\{r-2, \deg(v_i) - 2\} + 1 \\
&= \min\{r, \deg(v_i)\} - 1.
\end{aligned}
$$

In addition, $p_{v_{i-1}}(\alpha_{i-1}) = 0 = p_{v_i}(\alpha_{i-1})$ and $p_{v_i}(\alpha_i) = 0 = p_{v_1}(\alpha_i)$. Therefore, since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G\square H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_{\mathbf{q}}^i \in W^r_{G\square H_{k,\ell},c'}$. For $i \in \{\ell, k'\}$, let $A_i := \{\mathbf{p}_{\mathbf{q}}^i : \mathbf{q} \in B^{(r-2)}\}$. Note that the restriction of any vector $\mathbf{p}_{\mathbf{q}}^i \in A_i$ to G_i is $(x-\alpha_{i-1})(x-\alpha_i)\mathbf{q}$. Thus, since $(x - \alpha_{i-1})(x - \alpha_i) \not\equiv 0$ and $B^{(r-2)}$ is a basis, the vectors in A_i are linearly independent.

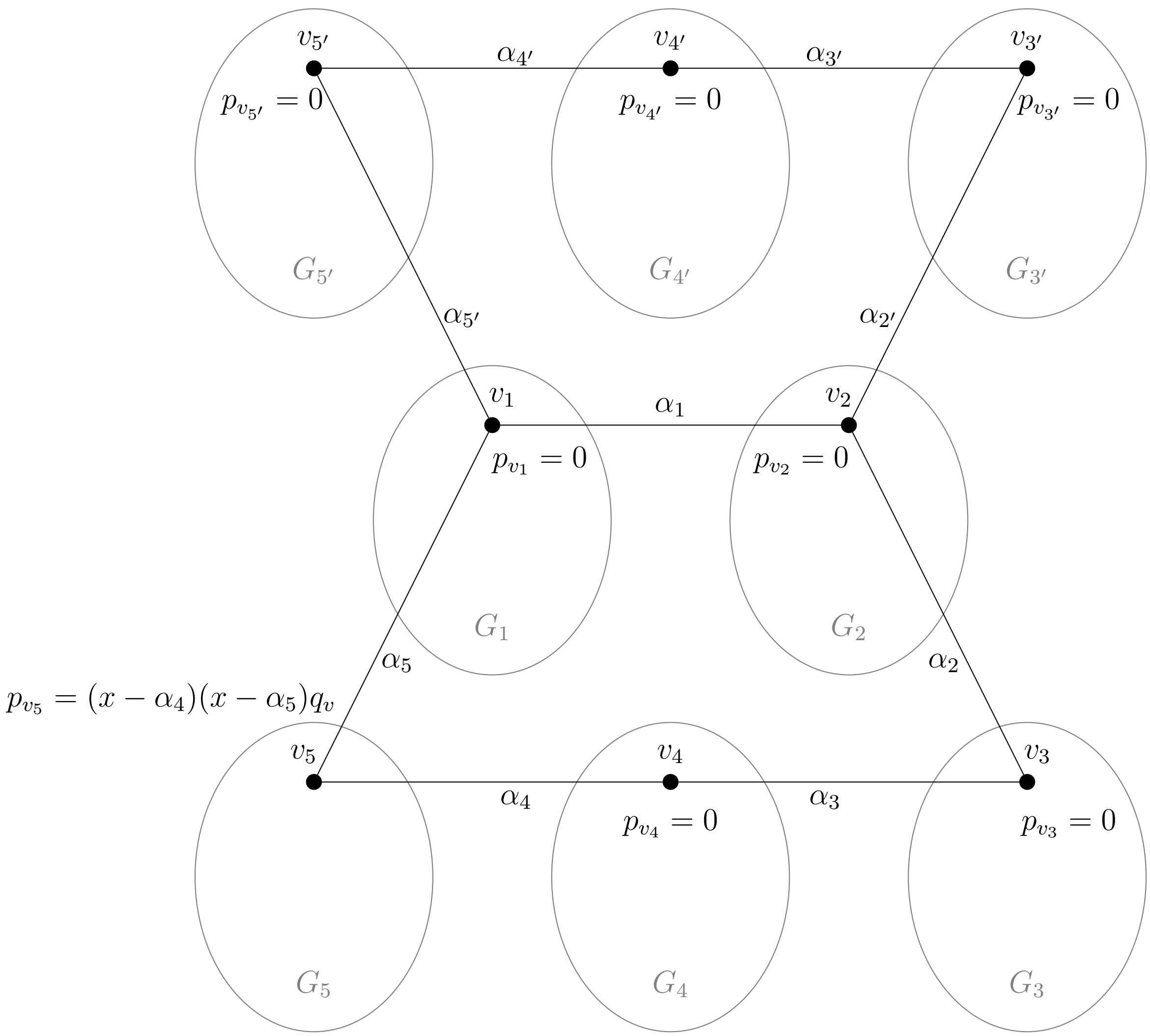

Figure 3.13: The vector $\mathbf{q} \in B^{(r-1)}$ is used to construct $\mathbf{p}^5_{\mathbf{q}} \in W^r_{G\square H_{5,5,c'}}$.

Define $A := \bigcup_{i \in I} A_i$. Note that

$$|A_1| = \dim(W^r_{G,c}),$$

$$|A_i| = \dim(W^{r-1}_{G,c}) \text{ for } i \in I \setminus \{1, \ell, k'\}, \text{ and}$$

$$|A_i| = \dim(W^{r-2}_{G,c}) \text{ for } i \in \{\ell, k'\}.$$

Hence $|A| = \dim\left(W^r_{G,c}\right) + (\ell + k - 5)\dim\left(W^{r-1}_{G,c}\right) + 2\dim\left(W^{r-2}_{G,c}\right)$.

Claim 3.26 *The vectors in A are linearly independent.*

Proof of Claim 3.26. Suppose for a contradiction that there exists a linear combination $S = \sum_{\mathbf{p} \in A} \beta_{\mathbf{p}} \mathbf{p} = 0$, where $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in A$. Write $S = \sum_{i \in I} S_i$ where $S_i = \sum_{\mathbf{p} \in A_i} \beta_{\mathbf{p}} \mathbf{p}$ for each $i \in I$. Let $j \in I$ be the smallest index, relative to the ordering ϕ, such that there exists $\mathbf{p} \in A_j$ with $\beta_{\mathbf{p}} \neq 0$.

Suppose first that $j = 1$. For each $v \in V(G)$, by definition, $p_{v_1} \equiv 0$ for each vector $\mathbf{p} \in \bigcup_{i \in I \setminus \{1\}} A_i$. Hence the restriction of the sum S to the coordinates corresponding to vertices in G_1 is equal to the restriction of S_1 to G_1. However, since the vectors in A_1 are linearly independent, this contradicts our assumption that $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in A_1$. Hence $j \neq 1$, and thus $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in A_1$.

Next, assume $j = 2$. Similarly to the previous case, for each $v \in V(G)$, we have $p_{v_2} \equiv 0$ for each vector $\mathbf{p} \in \bigcup_{i \in I \setminus \{1,2\}} A_i$. Since $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in A_1$, it follows that the restriction of the sum S to the coordinates corresponding to vertices in G_2 is equal to the restriction of S_2 to G_2. However, since the vectors in A_2 are linearly independent, this contradicts our assumption that $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in A_2$. Hence $j \neq 2$, and thus $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in A_2$.

Continuing in this manner, we find that $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in A$, a contradiction. Therefore, the vectors in A are linearly independent. $(\Box)$

Now, set $Y = A$. To achieve the desired lower bound on $\dim\left(W^r_{G \Box H_{k\ell},c'}\right)$, we must extend Y to include an additional $d^G_{r-1} + (\ell+k-3)d^G_{r-2} + (\ell+k-1)\sum_{t=0}^{r-3} d^G_t$ vectors, all of which are linearly independent. To this end, consider the set of vectors $Z_c \in \mathbb{R}^{|V(G)|}$ as defined in Definition 3.10.

Claim 3.27 *For each non-zero* $\mathbf{p} \in \operatorname{span}(A)$*, there exists* $\mathbf{z} \in Z_c$ *such that* $\mathbf{p}(\mathbf{z}) \neq 0$.

Proof of Claim: Let $\mathbf{p} = (p_u)_{u \in V(G \Box H_{k,\ell})}$ be a non-zero vector in $\operatorname{span}(A)$. Let j be the smallest index, relative to the ordering ϕ, for which there exists $v_j \in V(G_j)$ with $p_{v_j} \not\equiv 0$. By definition of A, if $j = 1$, we can write $p_{v_1} = q_v$ for some $\mathbf{q} \in (q_u)_{u \in V(G)}$ in

$\mathrm{span}(B^{(r)}) = W^r_{G,c}$. Similarly, if $j \in I \setminus \{1, \ell, k'\}$, then we can write $p_{v_j} = q_v(x - \alpha_{j-1})$ for some $\mathbf{q} \in (q_u)_{u \in V(G)}$ in $\mathrm{span}(B^{(r-1)}) = W^{r-1}_{G,c}$. Finally, if $j \in \{\ell, k'\}$, then we can write $p_{v_j} = q_v(x - \alpha_{j-1})(x - \alpha_j)$ for some $\mathbf{q} \in (q_u)_{u \in V(G)}$ in $\mathrm{span}(B^{(r-2)}) = W^{r-2}_{G,c}$. In any case, by Definition 3.5, we have $\deg(q_v) \leq \deg_G(v) - 1$. Therefore, since the colours α_i were chosen to be distinct from those in $c(E)$, by Lemma 3.11, there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$. $_{(\Box)}$

Therefore, to finish the proof of Proposition 3.25, it suffices to find a set X of $d^G_{r-1} + (\ell + k - 3)d^G_{r-2} + (\ell + k - 1)\sum_{t=0}^{r-3} d^G_t$ linearly independent vectors in $W^r_{G\Box H_{k,\ell},c'}$, each of which evaluates to zero on all of Z_c. This, together with Claim 3.27, will guarantee that these vectors are independent from those in A.

First, for each $v \in V(G)$ with $\deg_G(v) = r-1$, define the vector $\mathbf{p}_v = (p_u)_{u \in V(G\Box H_{k,\ell})}$ such that

$$p_{v_i} = \prod_{u \in N_G(v)} (x - c_{uv}) \text{ for all } i \in I,$$

and $p_u \equiv 0$ for all other $u \in V(G\Box H_{k,\ell})$. Note that $\deg_G(v) \leq \deg(v_i) - 2$ for each $i \in I$. Therefore, since $\deg_G(v) = r - 1$, for each $i \in I$,

$$\deg(p_{v_i}) = \deg_G(v) \leq \min\{r - 1, \deg(v_i) - 2\} \leq \min\{r, \deg(v_i)\} - 1.$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G\Box H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v \in W^r_{G\Box H_{k,\ell},c'}$. Moreover, $\mathbf{p}_v$ also evaluates to 0 on all of Z_c. Define $X_v := \{\mathbf{p}_v\}$. Let $X_1 := \bigcup X_v$, where the union is over all $v \in V(G)$ such that $\deg_G(v) = r - 1$. Note that $|X_1| = d^G_{r-1}$.

Now, let $v \in V(G)$ with $\deg_G(v) = r - 2$. First, for each $i \in I \setminus \{1, 2, \ell, k'\}$, define the vector $\mathbf{p}^i_v = (p_u)_{u \in V(G\Box H_{k,\ell})}$ as follows:

- Let p_{v_i} be the unique polynomial of degree $\deg(v_i) - 1$ that evaluates to 1 at α_i and evaluates to zero at all $\deg(v_i) - 1$ other colours of the edges incident with v_i in $G\Box H_{k,\ell}$. That is, $p_{v_i} = \beta_i \prod_{u \in N(v_i) \setminus \{v_{i+1}\}}(x - c'_{v_i u})$, where β_i is a non-zero constant chosen so that $p_{v_i}(\alpha_i) = 1$.

- Let $p_{v_{i+1}}$ be the unique polynomial of degree $\deg(v_{i+1}) - 1$ that evaluates to 1 at α_i and evaluates to zero at all $\deg(v_{i+1}) - 1$ other colours of the edges incident with v_{i+1} in $G \square H_{k,\ell}$. That is, $p_{v_{i+1}} = \beta_{i+1} \prod_{u \in N(v_{i+1}) \setminus \{v_i\}} (x - c'_{v_{i+1}u})$, where β_{i+1} is a non-zero constant chosen so that $p_{v_{i+1}}(\alpha_i) = 1$.

- Let $p_u \equiv 0$ for all $u \notin \{v_i, v_{i+1}\}$.

See Figure 3.14 for an illustration of this construction for $H_{5,5}$ when $i = 3$.

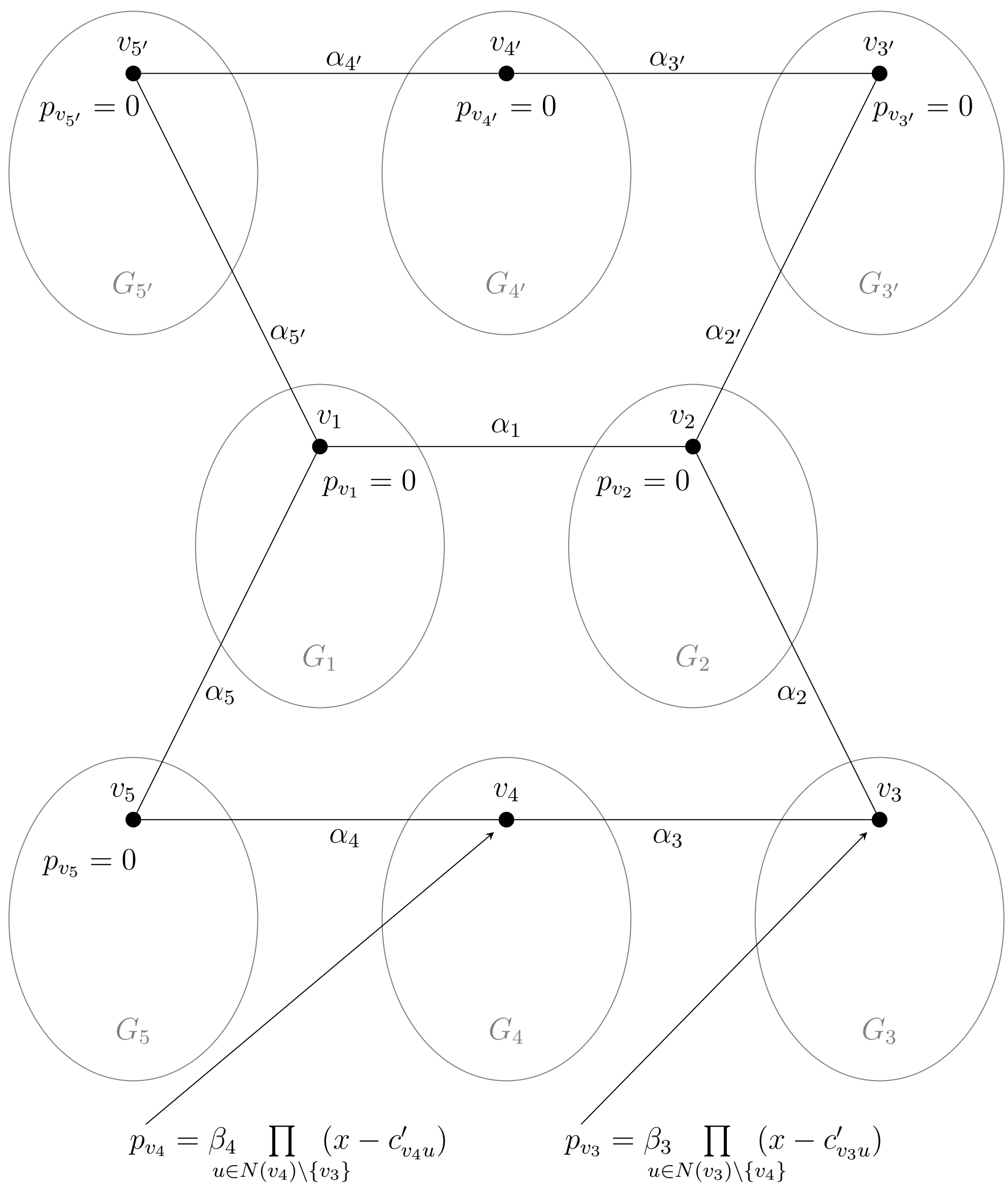

Figure 3.14: The vector $\mathbf{p}_v^3 \in W^r_{G \square H_{5,5},c'}$ defined for $v \in V(G)$ with $\deg_G(v) = r - 2$.

Note that

$$\deg(p_{v_i}) = \deg(v_i) - 1 = \deg_G(v) + 1 = r - 1.$$

Similarly, $\deg(p_{v_{i+1}}) = \deg(v_{i+1}) - 1 = r - 1$. Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G\Box H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus the vector $\mathbf{p}_v^i$ belongs to $W^r_{G\Box H_{k,\ell},c'}$. Furthermore, $\mathbf{p}_v^i$ evaluates to 0 on all of Z_c.

Next, define $\mathbf{p}_v^2 = (p_u)_{u\in V(G\Box H_{k,\ell})}$ so that $p_u \equiv 0$ for all $u \notin \{v_2, v_3, v_{3'}\}$,

$$p_{v_2} = (x - \alpha_1) \prod_{u\in N_G(v)} (x - c_{uv}),$$
$$p_{v_3} = \beta(x - \alpha_3) \prod_{u\in N_G(v)} (x - c_{uv}),$$

where β is a non-zero constant chosen so that $p_{v_3}(\alpha_2) = p_{v_2}(\alpha_2)$, and

$$p_{v_{3'}} = \beta'(x - \alpha_{3'}) \prod_{u\in N_G(v)} (x - c_{uv}),$$

where β' is a non-zero constant chosen so that $p_{v_{3'}}(\alpha_{2'}) = p_{v_2}(\alpha_{2'})$. See Figure 3.15 for an illustration of this construction.

Note that

$$\deg(p_{v_2}) = \deg(p_{v_3}) = \deg(p_{v_{3'}}) = \deg_G(v) + 1 = r - 1.$$

Therefore, since $\deg(v_2) = \deg_G(v) + 3$, it follows that $\deg(p_{v_2}) \le \min\{r, \deg(v_2)\} - 1$. Similarly, $\deg(v_3) = \deg(v_{3'}) = \deg_G(v)+2$ implies that $\deg(p_{v_3}) \le \min\{r, \deg(v_3)\} - 1$ and $\deg(p_{v_{3'}}) \le \min\{r, \deg(v_{3'})\} - 1$. Now, by construction, $p_{v_3}(\alpha_2) = p_{v_2}(\alpha_2)$ and $p_{v_{3'}}(\alpha_{2'}) = p_{v_2}(\alpha_{2'})$. Furthermore,

$$p_{v_1}(\alpha_1) = 0 = p_{v_2}(\alpha_1),$$
$$p_{v_4}(\alpha_3) = 0 = p_{v_3}(\alpha_3), \text{ and}$$
$$p_{v_{4'}}(\alpha_{3'}) = 0 = p_{v_{3'}}(\alpha_{3'}).$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G\square H_{k,\ell})$, the two conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^2 \in W^r_{G\square H_{k,\ell},c'}$. Furthermore, $\mathbf{p}_v^2$ evaluates to 0 on all of Z_c.

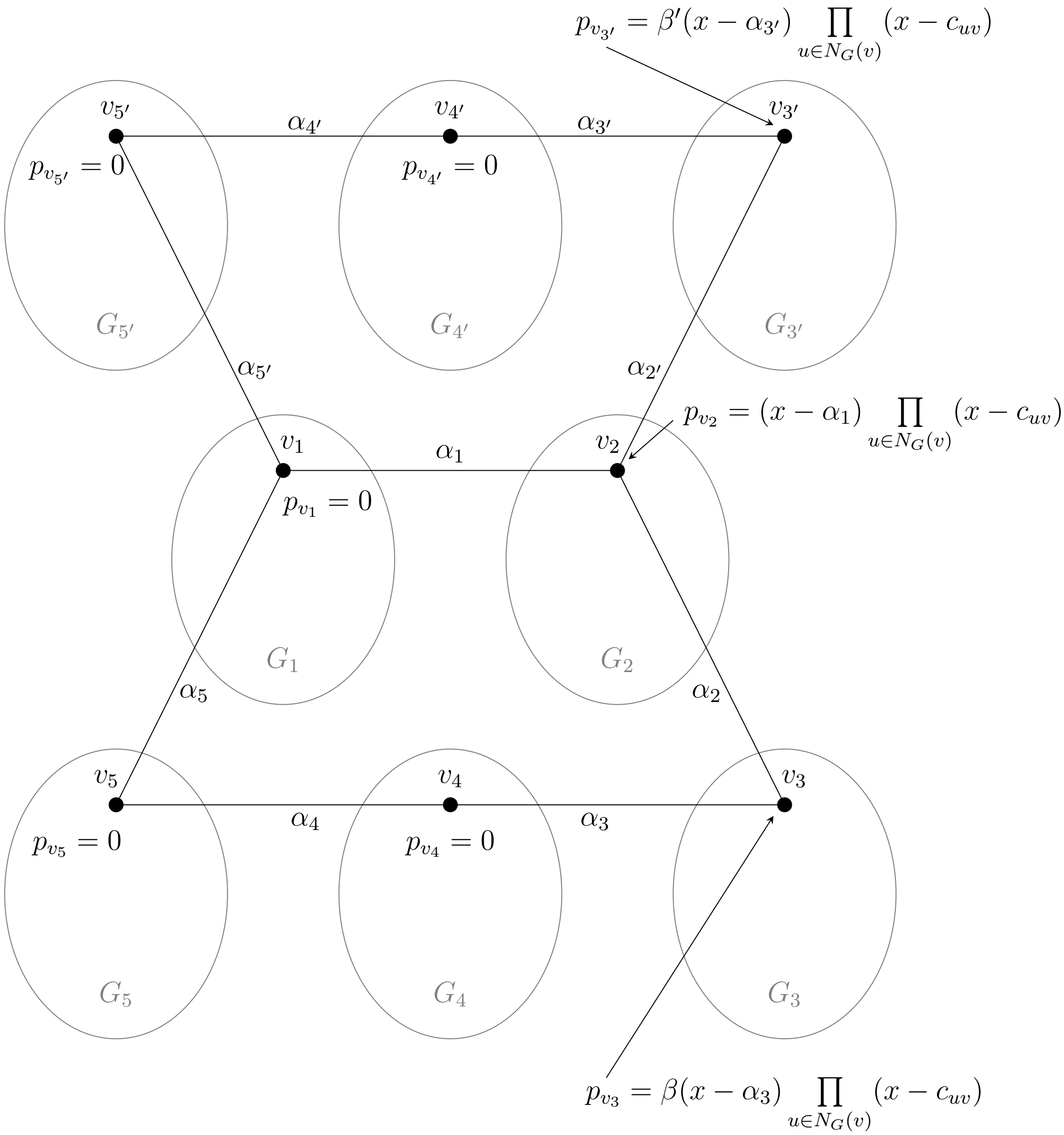

Figure 3.15: The vector $\mathbf{p}_v^2 \in W^r_{G\square H_{5,5},c'}$ defined for $v \in V(G)$ with $\deg_G(v) = r-2$.

Similarly, define $\mathbf{p}_v^1 = (p_u)_{u \in V(G \square H_{k,\ell})}$ so that $p_u \equiv 0$ for all $u \notin \{v_1, v_\ell, v_{k'}\}$,

$$p_{v_1} = (x - \alpha_1) \prod_{u \in N_G(v)} (x - c_{uv}),$$
$$p_{v_\ell} = \beta(x - \alpha_{\ell-1}) \prod_{u \in N_G(v)} (x - c_{uv}),$$

where β is a non-zero constant chosen so that $p_{v_\ell}(\alpha_\ell) = p_{v_1}(\alpha_\ell)$, and

$$p_{v_{k'}} = \beta'(x - \alpha_{(k-1)'}) \prod_{u \in N_G(v)} (x - c_{uv}),$$

where β' is a non-zero constant chosen so that $p_{v_{k'}}(\alpha_{k'}) = p_{v_1}(\alpha_{k'})$.

By an argument analogous to the one for $\mathbf{p}_v^2$, we have $\deg(p_{v_i}) \leq \min\{r, \deg(v_i)\} - 1$ for $i \in \{1, \ell, k'\}$. Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G \square H_{k,\ell})$, the conditions of Definition 3.5 are satisfied, and thus $\mathbf{p}_v^1 \in W^r_{G \square H_{k,\ell}, c'}$. Furthermore, $\mathbf{p}_v^1$ evaluates to 0 on all of Z_c.

Finally, define $\mathbf{p}_v^0 = (p_u)_{u \in V(G \square H_{k,\ell})}$ so that $p_u \equiv 0$ for all $u \notin \{v_1, v_2, v_3, v_{k'}\}$,

$$p_{v_1} = \frac{x - \alpha_\ell}{\alpha_1 - \alpha_\ell} \prod_{u \in N_G(v)} (x - c_{uv}),$$
$$p_{v_2} = \frac{x - \alpha_{2'}}{\alpha_1 - \alpha_{2'}} \prod_{u \in N_G(v)} (x - c_{uv}),$$
$$p_{v_3} = \beta(x - \alpha_3) \prod_{u \in N_G(v)} (x - c_{uv}),$$

where β is a non-zero constant chosen so that $p_{v_3}(\alpha_2) = p_{v_2}(\alpha_2)$, and

$$p_{v_{k'}} = \beta'(x - \alpha_{(k-1)'}) \prod_{u \in N_G(v)} (x - c_{uv}),$$

where β' is a non-zero constant chosen so that $p_{v_{k'}}(\alpha_{k'}) = p_{v_1}(\alpha_{k'})$. See Figure 3.16 for an illustration of this construction for $H_{5,5}$.

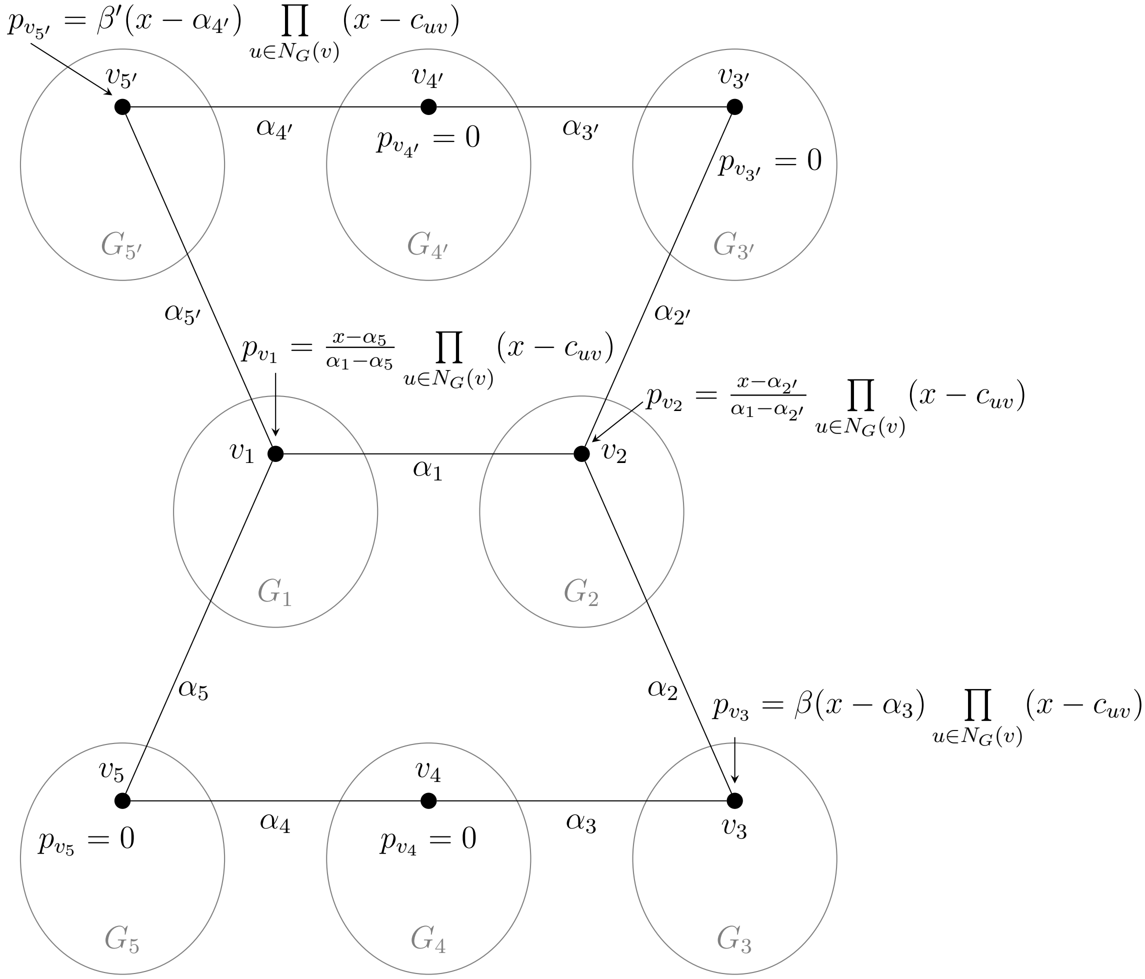

Figure 3.16: The vector $\mathbf{p}_v^2 \in W^r_{G\Box H_{5,5,c'}}$ defined for $v \in V(G)$ with $\deg_G(v) = r-2$.

For $i \in \{1, 2, 3, k'\}$, since $\deg(p_{v_i}) = \deg_G(v)+1 = r-1$ and $\deg(v_i) \geq \deg_G(v)+2$, it follows that $\deg(p_{v_i}) \leq \min\{r, \deg(v_i)\} - 1$. By assumption, $p_{v_3}(\alpha_2) = p_{v_2}(\alpha_2)$ and $p_{v_{k'}}(\alpha_{k'}) = p_{v_1}(\alpha_{k'})$. Furthermore,

$$
\begin{aligned}
&p_{v_1}(\alpha_\ell) = 0 = p_{v_\ell}(\alpha_\ell),\\
&p_{v_2}(\alpha_{2'}) = 0 = p_{v_{3'}}(\alpha_{2'}),\\
&p_{v_3}(\alpha_3) = 0 = p_4(\alpha_3),\\
&p_{v_{k'}}(\alpha_{(k-1)'}) = 0 = p_{v_{(k-1)'}}(\alpha_{(k-1)'}), \text{ and}\\
&p_{v_1}(\alpha_1) = \prod_{u\in N_G(v)} (\alpha_1 - c_{uv}) = p_{v_2}(\alpha_1).
\end{aligned}
$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other $uv \in E(G\Box H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^0 \in W^r_{G\Box H_{k,\ell},c'}$. Note that $\mathbf{p}_v^0$ also evaluates to 0 on all of Z_c.

For each $v \in V(G)$ with $\deg_G(v) = r-2$, let $X_v := \{\mathbf{p}_v^i : i \in \{0\} \cup I \setminus \{\ell, k'\}\}$. Let $X_2 := \bigcup X_v$, where the union is taken over all $v \in V(G)$ with $\deg_G(v) = r-2$. Note that, since $|I \setminus \{\ell, k'\}| = k + \ell - 4$, we have $|X_2| = (\ell + k - 3)d^G_{r-2}$.

Finally, suppose $v \in V(G)$ with $\deg_G(v) \leq r-3$. First, for each $i \in [\ell]$, define the vector $\mathbf{p}_v^i = (p_u)_{u \in V(G\Box H_{k,\ell})}$ as follows:

- Let p_{v_i} be the unique polynomial of degree $\deg(v_i) - 1$ that evaluates to 1 at α_i and evaluates to zero on all $\deg(v_i) - 1$ other colours of the edges incident with v_i in $G\Box H_{k,\ell}$. That is, $p_{v_i} = \beta_i \prod_{u \in N(v_i)\setminus\{v_{i+1}\}}(x - c'_{v_i u})$, where β_i is a non-zero constant chosen so that $p_{v_i}(\alpha_i) = 1$.
- Let $p_{v_{i+1}}$ be the unique polynomial of degree $\deg(v_{i+1}) - 1$ that evaluates to 1 at α_i and evaluates to zero on all $\deg(v_{i+1}) - 1$ other colours of the edges incident with v_{i+1} in $G\Box H_{k,\ell}$. That is, $p_{v_{i+1}} = \beta_{i+1} \prod_{u \in N(v_{i+1})\setminus\{v_i\}}(x - c'_{v_{i+1} u})$, where β_{i+1} is a non-zero constant chosen so that $p_{v_{i+1}}(\alpha_i) = 1$.
- Let $p_u \equiv 0$ for all $u \notin \{v_i, v_{i+1}\}$.

See Figure 3.17 for an illustration of this construction for $H_{5,5}$ when $i = 3$.

Now, for $j \in \{i, i+1\}$, note that $\deg(v_j) \leq \deg_G(v) + 3 \leq r$. Hence

$$\deg(p_{v_j}) = \deg(v_j) - 1 \leq \min\{r, \deg(v_j)\} - 1.$$

By definition, $p_{v_i}(\alpha_i) = 1 = p_{v_{i+1}}(\alpha_i)$. Furthermore,

$$p_{v_i}(\alpha_{i-1}) = 0 = p_{v_{i-1}}(\alpha_{i-1}), \text{ and}$$
$$p_{v_{i+1}}(\alpha_{i+1}) = 0 = p_{v_{i+2}}(\alpha_{i+1}).$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other $uv \in E(G\Box H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^i \in W^r_{G\Box H_{k,\ell},c'}$. Note that $\mathbf{p}_v^i$ also evaluates to 0 on all of Z_c.

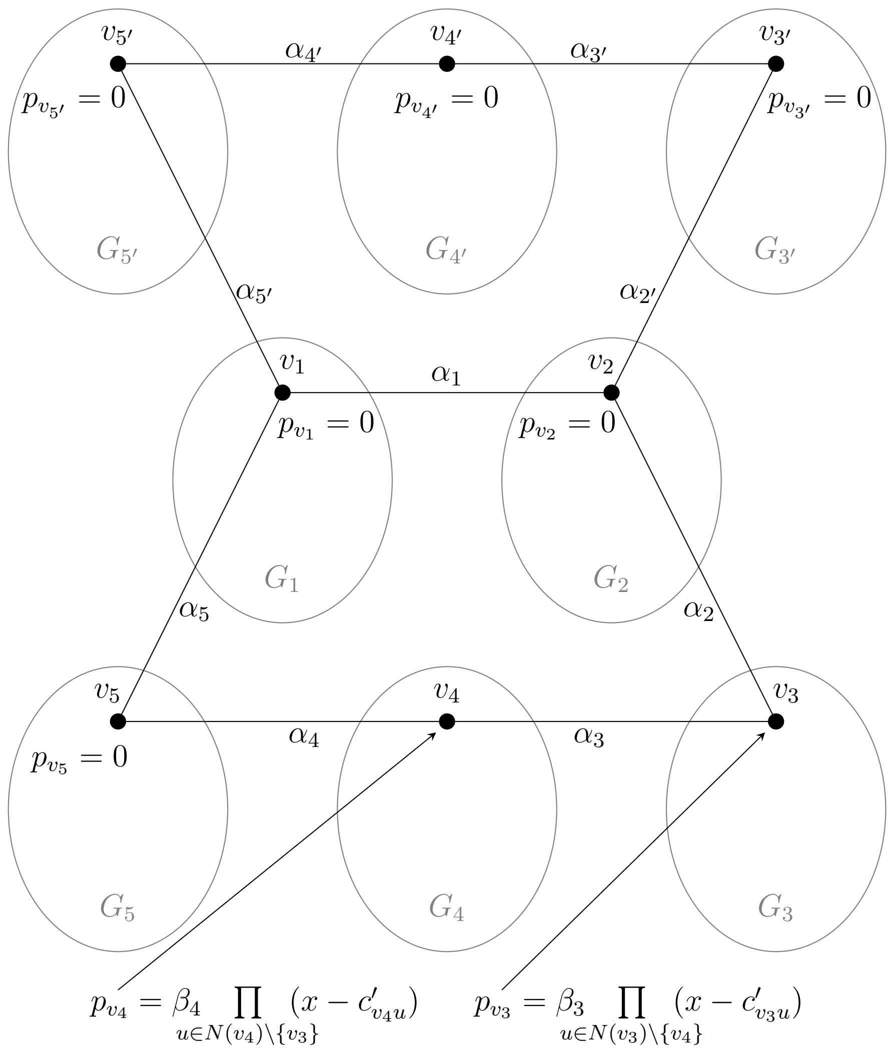

Figure 3.17: The vector $\mathbf{p}_v^3 \in W^r_{G\Box H_{5,5,c'}}$ defined for $v \in V(G)$ with $\deg_G(v) \le r - 3$.

Similarly, for $i \in [k'] \setminus \{1'\}$, define the vector $\mathbf{p}_v^i = (p_u)_{u \in V(G\Box H_{k,\ell})}$ as follows:

- Let p_{v_i} be the unique polynomial of degree $\deg(v_i) - 1$ that evaluates to 1 at α_i and evaluates to zero on all $\deg(v_i) - 1$ other colours of the edges incident with v_i in $G\Box H_{k,\ell}$.
- Let $p_{v_{i+1}}$ be the unique polynomial of degree $\deg(v_{i+1}) - 1$ that evaluates to 1 at α_i and evaluates to zero on all $\deg(v_{i+1}) - 1$ other colours of the edges incident with v_{i+1} in $G\Box H_{k,\ell}$.
- Let $p_u \equiv 0$ for all $u \notin \{v_i, v_{i+1}\}$.

By an argument analogous to the one above, we have $\deg(p_{v_j}) \leq \min\{r, \deg(v_j)\} - 1$ for $j \in \{i, i+1\}$. Therefore, since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G\Box H_{k,\ell})$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^i \in W^r_{G\Box H_{k,\ell},c'}$. Furthermore, $\mathbf{p}_v^i$ evaluates to 0 on all of Z_c.

For each $v \in V(G)$ with $\deg_G(v) \leq r-3$, define $X_v := \{\mathbf{p}_v^i : i \in I \cup \{2'\}\}$. Let $X_3 := \bigcup X_v$, where the union is taken over all $v \in V(G)$ such that $\deg_G(v) \leq r-3$. Note that, since $|I \cup \{2'\}| = \ell + k - 1$, we have $|X_3| = (\ell+k-1)\sum_{t=0}^{r-3} d_t^G$.

Define $X := X_1 \cup X_2 \cup X_3$. Note that

$$|X| = |X_1| + |X_2| + |X_3| = d_{r-1}^G + (\ell+k-3)d_{r-2}^G + (\ell+k-1)\sum_{t=0}^{r-3} d_t^G.$$

Claim 3.28 *The vectors in X are linearly independent.*

Proof of Claim 3.28. Since each vector in X is only non-zero in the entries corresponding to a unique vertex $v \in V(G)$, it suffices to show that the vectors in each set X_v are linearly independent, where $\deg_G(v) \leq r-1$. When $\deg_G(v) = r-1$, we have $|X_v| = 1$, and thus X_v is clearly linearly independent. So assume $\deg_G(v) < r-1$.

First, consider the case where $\deg_G(v) = r-2$. Suppose for a contradiction that $S = \sum_{\mathbf{p}\in X_v} \gamma_{\mathbf{p}}\mathbf{p} = 0$ is a non-trivial linear combination. For ease of notation, we write γ_i for $\gamma_{\mathbf{p}_v^i}$. First, suppose $\gamma_i \neq 0$ for some $i \in I\setminus\{1,2,3,3',\ell,k'\}$. Note that $\mathbf{p}_v^i$ and $\mathbf{p}_v^{i-1}$ are the only two vectors in X_v for which $p_{v_i} \not\equiv 0$. Hence $\gamma_{i-1} \neq 0$. Moreover, we have $\gamma_{i-1}\,\mathbf{p}_v^{i-1}|_{v_i} + \gamma_i\,\mathbf{p}_v^i|_{v_i} \equiv 0$. But $\mathbf{p}_v^{i-1}|_{v_i}$ evaluated at α_i is 0, while $\mathbf{p}_v^i|_{v_i}$ evaluated at α_i is 1, a contradiction. Hence $\gamma_i = 0$ for all $i \in I\setminus\{1,2,3,3',\ell,k'\}$, and thus $\gamma_i \neq 0$ for some $i \in \{0,1,2,3,3'\}$. Now, since $p_{v_4} \not\equiv 0$ in $\mathbf{p}_v^3$, but $p_{v_4} \equiv 0$ in $\mathbf{p}_v^i$ for all $i \in \{0,1,2,3'\}$, we have $\gamma_3 = 0$. Similarly, $\gamma_{3'} = 0$. Thus $S = \gamma_0\mathbf{p}_v^0 + \gamma_1\mathbf{p}_v^1 + \gamma_2\mathbf{p}_v^2$. Now, since $p_{v_{3'}} \not\equiv 0$ in $\mathbf{p}_v^2$, but $p_{v_{3'}} \equiv 0$ in $\mathbf{p}_v^i$ for all $I \in \{0,1\}$, we have $\gamma_2 = 0$. Similarly, $\gamma_1 = 0$. But then $S = \gamma_0\mathbf{p}_v^0 = 0$ with $\mathbf{p}_v^0 \neq 0$, and thus $\gamma_0 = 0$, which contradicts our assumption that S was a nontrivial linear combination. Hence X_v is linearly independent, as required.

Finally, let $\deg_G(v) \leq r-3$. Suppose for a contradiction that $S = \sum_{\mathbf{p}\in X_v} \gamma_{\mathbf{p}}\mathbf{p} = 0$ is a non-trivial linear combination. As before, we write γ_i for $\gamma_{\mathbf{p}_v^i}$. First, suppose $\gamma_i \neq 0$ for some $i \in I \setminus \{1,2\}$. Note that $\mathbf{p}_v^i$ and $\mathbf{p}_v^{i-1}$ are the only two vectors in X_v for which $p_{v_i} \not\equiv 0$. Hence $\gamma_{i-1} \neq 0$. Moreover, we have $\gamma_{i-1}\, \mathbf{p}_v^{i-1}|_{v_i} + \gamma_i\, \mathbf{p}_v^i|_{v_i} \equiv 0$. But $\mathbf{p}_v^{i-1}|_{v_i}$ evaluated at α_i is 0, while $\mathbf{p}_v^i|_{v_i}$ evaluated at α_i is 1, a contradiction. Hence $\gamma_i = 0$ for all $i \in I \setminus \{1,2\}$, and thus $S = \gamma_1\mathbf{p}_v^1 + \gamma_2\mathbf{p}_v^2 + \gamma_{2'}\mathbf{p}_v^{2'}$. Now, since $p_{v_3} \not\equiv 0$ in $\mathbf{p}_v^2$, but $p_{v_3} \equiv 0$ in $\mathbf{p}_v^i$ for all $i \in \{1, 2'\}$, we have $\gamma_2 = 0$. Similarly, $\gamma_{2'} = 0$. But then $S = \gamma_1\mathbf{p}_v^1 = 0$ with $\mathbf{p}_v^1 \neq 0$, and thus $\gamma_1 = 0$, which contradicts our assumption that S was a nontrivial linear combination. Hence X_v is linearly independent, as required. $(\square)$

Let $Y = A \cup X$. Therefore, by Claims 3.26, 3.27, and 3.28, we have found

$$\begin{aligned}|Y| &= |A| + |X| \\ &= \dim\left(W_{G,c}^r\right) + (\ell+k-5)\dim\left(W_{G,c}^{r-1}\right) + 2\dim\left(W_{G,c}^{r-2}\right) \\ &\qquad + d_{r-1}^G + (\ell+k-3)d_{r-2}^G + (\ell+k-1)\sum_{t=0}^{r-3} d_t^G\end{aligned}$$

linearly independent vectors in $W^r_{G\square H_{k,\ell},c'}$, as required. $\square$

3.5.3 Exact Values for Products of Joined Cycles

In this section, we determine a recursive formula for $m_e(G,r)$ when G is a product of joined cycles. In fact, we prove the following more general result (initially stated earlier in Section 3.5) that gives a recursive formula for $m_e(G\square H_{k,\ell}, r)$ for certain graphs G.

Theorem 3.21 *Let $k \geq \ell \geq 4$ and $r > 1$ be integers. If there exists a proper edge-colouring c of G such that $m_e(G,i) = \dim(W_{G,c}^i)$ for each $r-2 \leq i \leq r$, then*

$$\begin{aligned}m_e(G\square H_{k,\ell}, r) &= m_e(G,r) + (k+\ell-5)m_e(G,r-1) + 2m_e(G,r-2) \\ &\qquad + d_{r-1}^G + (k+\ell-3)d_{r-2}^G + (k+\ell-1)\sum_{i=0}^{r-3} d_i^G.\end{aligned}$$

Proof. By Proposition 3.23,

$$m_e(G\square S_k, r) \leq m_e(G, r) + (k + \ell - 5)m_e(G, r - 1) + 2m_e(G, r - 2) + d^G_{r-1} + (k + \ell - 3)d^G_{r-2} + (k + \ell - 1)\sum_{i=0}^{r-3} d^G_i.$$

The matching lower bound follows from Proposition 3.24, since $m_e(G, i) = \dim(W^i_{G,c})$ for $r - 2 \leq i \leq r$. $\square$

Let $d \in \mathbb{Z}^+$. Given integers $a_1, ..., a_d \geq 3$ and $b_1, ..., b_d \geq 2$, let $G_C := \prod_{i=1}^d C_{a_i}$ and $G_P := \prod_{i=1}^d P_{b_i}$. As noted in Section 3.4.3, in their proofs of Theorems 3.2 and 3.3, Hambardzumyan, Hatami, and Qian [14] showed that, $m_e(G_C, r) = \dim\left(W^r_{G_C,c}\right)$ and $m_e(G_P, r) = \dim\left(W^r_{G_P,c}\right)$ for all $r \geq 1$. Hence Theorems 3.2 and 3.3 can be combined with Theorem 3.21 to determine recursive formulas for $m_e(G\square H_{k,\ell})$ when G is a product of cycles or a product of paths. Moreover, the following consequence of Theorem 3.21 (initially stated earlier in Section 3.5) gives a recursive formula for $m_e(G, r)$ when G is a product of joined cycles.

Theorem 3.22 *Let $t > 0$ and $r > 1$ be integers. Let $k_1, ..., k_t, \ell_1, ..., \ell_t \in \mathbb{Z}^+$ such that $k_i \geq \ell_i \geq 4$ for all $i \in [t]$. Let $G_i = H_{k_1,\ell_1}\square\cdots\square H_{k_i,\ell_i}$ for $i \in [t]$, and let $G_0 = K_1$. For $0 \leq i \leq \Delta(G_{t-1})$, let d_i denote the number of vertices in G_{t-1} of degree i. Then*

$$m_e(G_t, r) = m_e(G_{t-1}, r) + (k_t + \ell_t - 5)m_e(G_{t-1}, r - 1) + 2m_e(G_{t-1}, r - 2) + d_{r-1} + (k_t + \ell_t - 3)d_{r-2} + (k_t + \ell_t - 1)\sum_{i=0}^{r-3} d_i.$$

Proof. Note that $G_t = G_{t-1}\square H_{k_t,\ell_t}$. Therefore, by Theorem 3.21, it suffices to show that there exists a proper edge-colouring c of G_{t-1} such that $m_e(G_{t-1}, i) = \dim(W^i_{G_{t-1},c})$ for all integers $r - 2 \leq i \leq r$. In fact, we prove the following stronger result.

Claim 3.29 *Let $t \geq 0$ be an integer. Then there exists a proper edge-colouring c of G_t such that $m_e(G_t, i) = \dim(W^i_{G_t,c})$ for all $i \geq 0$.*

Proof of Claim 3.19. We prove the result by induction on t. First, suppose $t = 0$. Then $m_e(G_0, i) = \dim(W^i_{G_0,c}) = 0$ for all $i \geq 0$, where c is the empty edge-colouring. Now, let $t > 0$, and suppose c is a proper edge-colouring of G_{t-1} such that, for all $i \geq 0$, we have $m_e(G_{t-1}, i) = \dim(W^i_{G_{t-1},c})$. By the proof of Proposition 3.25, there exists a proper edge-colouring c' of G_t such that, for any $i \geq 1$,

$$\dim\left(W^i_{G_t,c'}\right) \geq \dim\left(W^i_{G_{t-1},c}\right) + (\ell_t + k_t - 5)\dim\left(W^{i-1}_{G_{t-1},c}\right) + 2\dim\left(W^{i-2}_{G_{t-1},c}\right)$$
$$+ d_{i-1} + (\ell_t + k_t - 3)d_{i-2} + (\ell_t + k_t - 1)\sum_{j=0}^{i-3} d_j.$$

Hence, by induction hypothesis,

$$\dim(W^i_{G_k,c'}) \geq m_e(G_{t-1}, i) + (\ell_t + k_t - 5)m_e(G_{t-1}, i-1) + 2m_e(G_{t-1}, i-2)$$
$$+ d_{i-1} + (\ell_t + k_t - 3)d_{i-2} + (\ell_t + k_t - 1)\sum_{j=0}^{i-3} d_j.$$

Therefore, by Theorem 3.6 and Proposition 3.23, $m_e(G_t, i) = \dim(W^i_{G_t,c'})$. Since $m_e(G_t, 0) = \dim(W^0_{G_t,c'}) = 0$, Claim 3.29 follows by induction.(□)

Therefore, by Theorem 3.21,

$$m_e(G_t, r) = m_e(G_{t-1}, r) + (k_t + \ell_t - 5)m_e(G_{t-1}, r-1) + 2m_e(G_{t-1}, r-2)$$
$$+ d_{r-1} + (k_t + \ell_t - 3)d_{r-2} + (k_t + \ell_t - 1)\sum_{i=0}^{r-3} d_i.$$

□

3.6 Products of Trees

Recall that, given a graph G, we denote by d^G_t the number of vertices in G of degree t. The main goal of this section is to prove the following theorem, which provides a recursive formula for $m_e(G \square T, r)$ when the minimum degree of G is sufficiently large (compared to r).

Theorem 3.30 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let G be a graph with $\delta(G) \geq r-2$. Suppose there exists a proper edge-colouring c of G such that $m_e(G,i) = \dim(W^i_{G,c})$ for each $r-1 \leq i \leq r$. Then*

$$m_e(G\Box T, r) = m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + d^G_{r-2}\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

In Sections 3.6.1 and 3.6.2, respectively, we provide upper and lower bounds on $m_e(G\Box T, r)$ when T is a tree. Section 3.6.3 explores the conditions under which these bounds match, and provides a proof of Theorem 3.30.

3.6.1 Upper Bound

In this section, we establish the following upper bound on $m_e(G\Box T, r)$ for trees T of order $n \geq 2$.

Proposition 3.31 *Let T be a tree of order $n \geq 2$. Let $r \in \mathbb{Z}^+$, and let G be a graph. Then*

$$m_e(G\Box T, r) \leq m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + \sum_{t=1}^{r-1} d^G_{r-1-t}\left(1 + t\sum_{i=t+1}^{\Delta(T)} d^T_i + \sum_{i=2}^{t}(i-1)d^T_i\right).$$

Proof. Consider T to be rooted at a leaf, which we label 0. Let the levels of T with respect to this root be labelled $L_0, L_1, ..., L_k$ for some $k \geq 1$, where L_0 contains the root of T, and $v \in L_d$ for each $v \in V(T)$ such that $d_T(0,v) = d$. Let $l(v)$ denote the level of $v \in V(T)$; that is, $l(v) = d$ if and only if $v \in L_d$. Label the remaining vertices of T with the integers in $[n-1]$ such that $l(i) \leq l(j)$ for all $0 \leq i < j \leq n-1$. See Figure 3.18 for an illustration of this labelling when T is the spider $S_3(2,2,2)$, which is the tree constructed from the star S_3 by subdividing each edge exactly once.

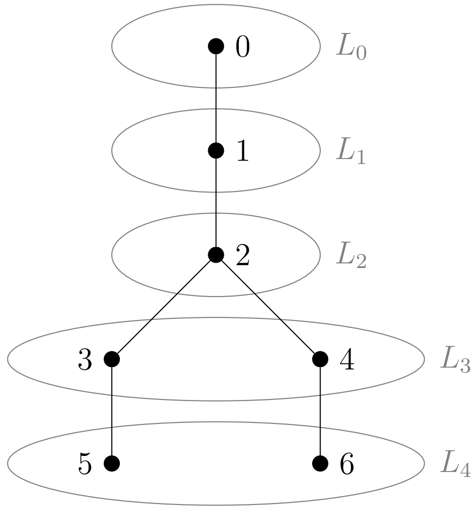

Figure 3.18: The spider $S_3(2,2,2)$ drawn as a rooted tree.

Let $G_0, G_1, ..., G_{n-1}$ denote the n copies of G in $G\square T$ corresponding to the n vertices in T, where G_i corresponds to the vertex $i \in V(T)$. Define $l(G_i) := l(i)$ for each $i \in V(T)$. For ease of notation, we write $G_i \in L_d$ when $l(G_i) = d$. For each vertex $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where $0 \le i \le n-1$.

First, we consider the case where $\delta(G) \ge r$. We construct an r-percolating set $F \subseteq E(G\square T)$ for $G\square T$ as follows: Let F_0 be an optimal r-percolating set for G_0. Finally, for each $i \in [n-1]$, pick an optimal $(r-1)$-percolating set F_i on G_i. Let $F = \bigcup_{i=0}^{n-1} F_i$. See Figure 3.19 for an illustration of this construction for $T = S_3(2,2,2)$.

We claim that F percolates in G, starting with the edges in G_0 and then spreading through those in $L_1, ..., L_k$, in that order. Indeed, by our choice of F_0, after running the r-bond bootstrap percolation process on G_0, all edges in G_0 will be infected. Since $\delta(G) \ge r$, all edges between G_0 and G_1 will then become infected. Now, since each vertex in G_1 has an infected edge coming from G_0, this, together with F_1, will infect the edges of G_1. Again, since $\delta(G) \ge r$, we can then infect all edges between G_1 and each $G_i \in L_2$. Now, for each $i \in L_2$, every vertex $v_i \in V(G_i)$ has an infected edge coming from G_1, which, together with F_i, will infect all edges in G_i. Hence we can infect the edges in each $G_i \in L_2$. Continuing in this manner, we can infect $L_1, ..., L_k$ in order. Hence F percolates in $G\square T$, as required.

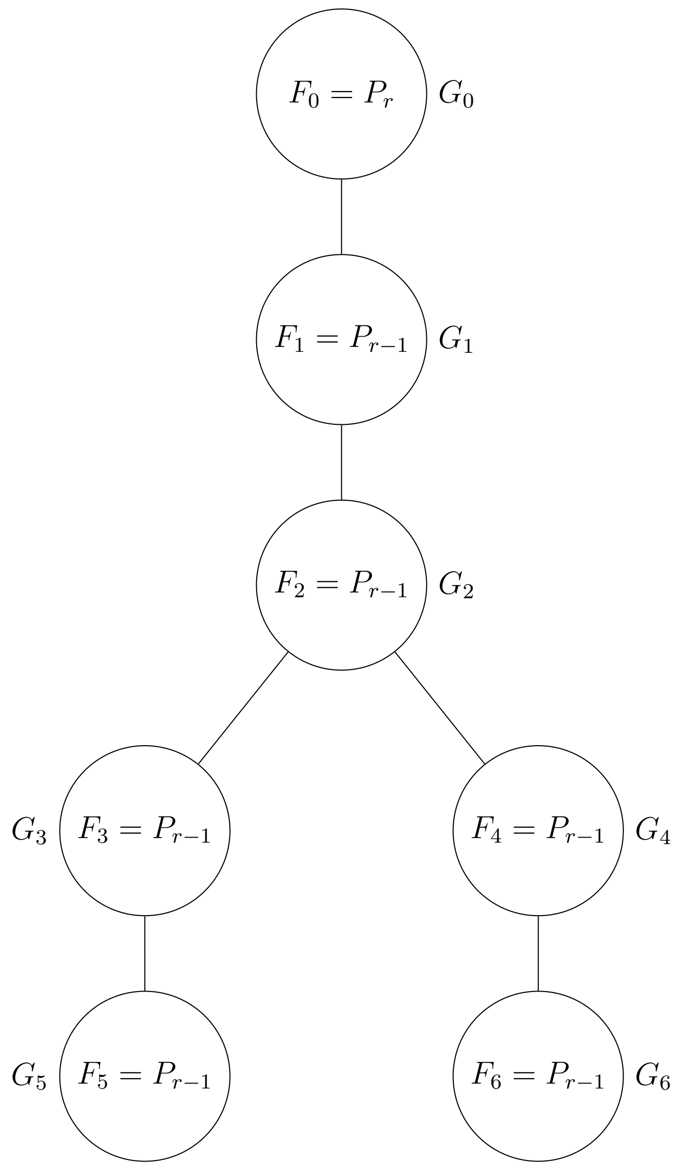

Figure 3.19: The percolating set F chosen for $G\Box S_3(2,2,2)$ when $\delta(G) \geq r$. Here we denote by P_i an optimal i-percolating set on G.

It remains to consider the case where $\delta(G) < r$. Let the set F be defined as above. We add the following edges to F:

- For each $v \in V(G)$ with $\deg_G(v) = r - 1$, add the edge v_0v_1 to the set F. Note that adding this edge will guarantee that, once the edges in G_0 are infected, the infection will spread to those in G_1, and we can then continue as above. This adds an additional d_{r-1}^G edges to the set F.

- Fix $t \in [r-1]$. For each $v \in V(G)$ such that $\deg_G(v) = r-1-t$, add the following edges to the set F (see Figure 3.20 for an illustration of this construction):
  - Add the edge $v_0 v_1$.
  - For each vertex $a \in V(T)$ with $\deg_T(a) > t$, add t edges from v_a down to its neighbours v_b where $l(b) = l(a) + 1$. This adds an additional $t \sum_{i=t+1}^{\Delta(T)} d_i^T$ edges to F, for this specific vertex v with $\deg_G(v) = r-1-t$.
  - For each $i \in \{2, ..., t\}$, and each vertex $a \in V(T)$ with $\deg_T(a) = i$, add all $i-1$ edges from v_a down to its neighbours v_b such that $l(b) = l(a) + 1$. This adds an additional $\sum_{i=2}^{t}(i-1)d_i^T$ edges to F, for this specific vertex v with $\deg_G(v) = r-1-t$.

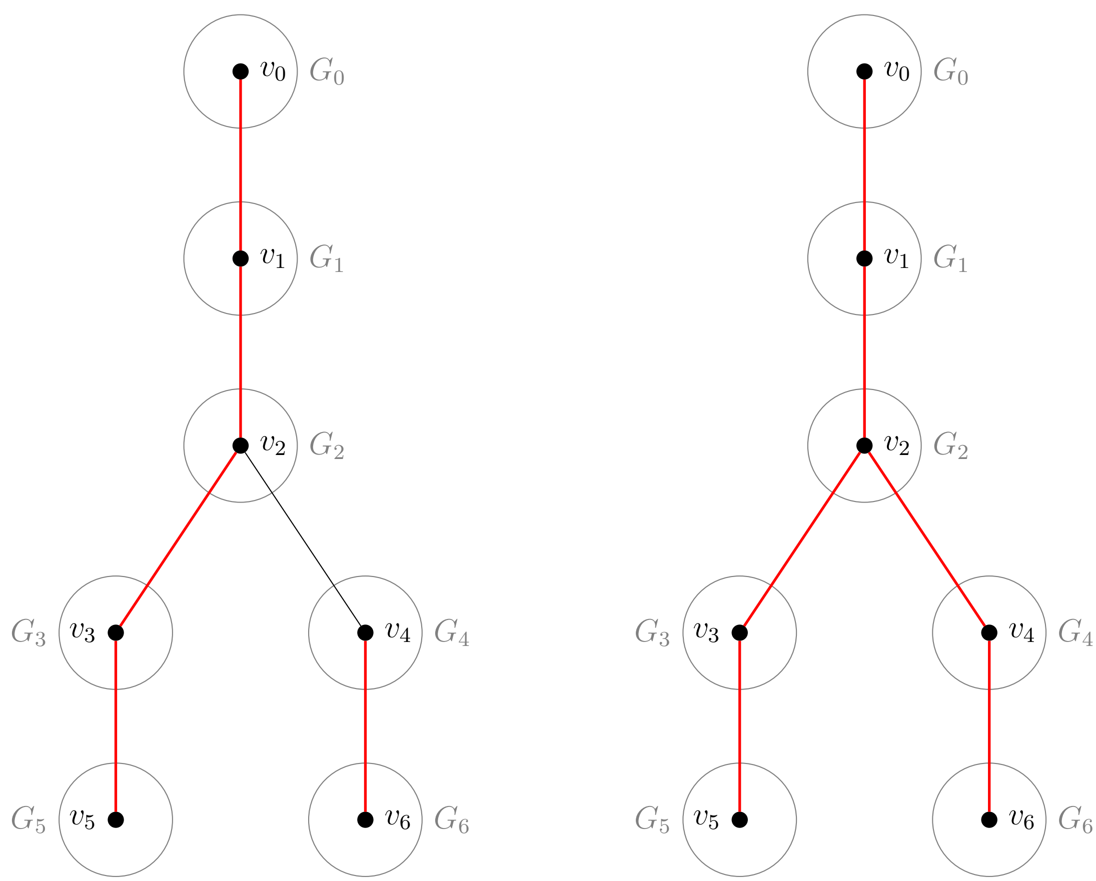

Figure 3.20: The edges added (shown in red) to the set F for a vertex $v \in V(G)$ with $\deg_G(v) = r-2$ (left) and $\deg_G(v) = r-3$ (right).

Note that adding these edges will guarantee that, for each $0 \leq a \leq n-1$, once the edges in G_a are infected, the infection will spread to the remaining edges $v_a v_b$ where $l(b) = l(a) + 1$.

Now, in total, we added

$$d^G_{r-1} + \sum_{t=1}^{r-1} d^G_{r-1-t} \left(1 + t \sum_{i=t+1}^{\Delta(T)} d^T_i + \sum_{i=2}^{t} (i-1) d^T_i \right)$$

edges to the $m_e(G, r) + (n-1)m_e(G, r-1)$ edges initially in F. Therefore, since the resulting set F percolates in G, we have established the desired upper bound on $m_e(G \square T, r)$. □

3.6.2 Lower Bound

The goal of this section is to prove the following lower bound on $m_e(G \square T, r)$ for trees T of order at least 2.

Proposition 3.32 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then*

$$m_e(G \square T, r) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

To this end, we will prove the following result.

Proposition 3.33 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then there exists a proper edge colouring c' of $G \square T$ such that*

$$\dim\left(W^r_{G \square T, c'}\right) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

Before proving Proposition 3.33, we first demonstrate why it implies the desired lower bound in Proposition 3.32.

Proof of Proposition 3.32. By Proposition 3.33, there exists a proper edge-colouring c' of $G\Box T$ such that

$$\dim\left(W^r_{G\Box T,c'}\right) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

By Theorem 3.6, we have $m_e(G\Box T, r) \geq \dim(W^r_{G\Box T,c'})$, and thus

$$m_e(G\Box T, r) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

□

Proof of Proposition 3.33 . Consider T to be rooted at a leaf, which we label 0. Let the levels of T with respect to this root be labelled $L_0, L_1, ..., L_k$ for some $k \geq 1$, where L_0 contains the root of T, and $v \in L_d$ for each $v \in V(T)$ such that $d_T(0, v) = d$. Let $l(v)$ denote the level of $v \in V(T)$; that is, $l(v) = d$ if and only if $v \in L_d$. Label the remaining vertices of T with the integers in $[n-1]$ such that $l(i) \leq l(j)$ for all $0 \leq i < j \leq n-1$. For each edge $ij \in E(T)$ where $l(i) = l(j) + 1$, let T_i denote the component of $T \setminus \{ij\}$ that contains the vertex i; we consider T_i to be rooted at i. See Figure 3.21 for an illustration of this labelling when T is the spider $S_3(2, 2, 2)$.

Let $G_0, G_1, ..., G_{n-1}$ denote the n copies of G in $G\Box T$ corresponding to the n vertices in T, where G_i corresponds to the vertex $i \in V(T)$. Define $l(G_i) := l(i)$ for $i \in V(T)$. For each $v \in V(G)$, denote its corresponding vertex in G_i by v_i, where $0 \leq i \leq n-1$.

Now, let $\{\alpha_i : i \in [n-1]\}$ be distinct real numbers that do not belong to $c(E)$. Let c' be the proper edge-colouring of $G\Box T$ defined as follows: Let c' be consistent with c on each of $G_0, G_1, ..., G_{n-1}$. For each $v \in V(G)$, let $c'(v_iv_j) = \alpha_i$ for all $i, j \in V(T)$ where j is the parent of i in T; that is, $j \in N_T(i)$ such that $l(j) = l(i) - 1$.

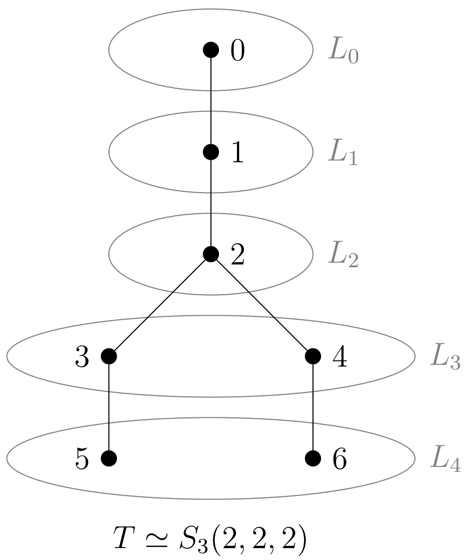

Figure 3.21: The spider $T \simeq S_3(2,2,2)$ drawn as a rooted tree, and it's subtree T_2.

Let

$$N = \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

In order to prove Proposition 3.33, we will find a set Y of N linearly independent vectors in $W^r_{G\square T,c'}$. We begin by finding a set A of $\dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right)$ linearly independent vectors in $W^r_{G\square T,c'}$.

Consider a basis $B^{(r)}$ for $W^r_{G,c}$. For each vector $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r)}$, define the vector $\mathbf{p}^0_{\mathbf{q}} = (p_u)_{u\in V(G\square T)}$ so that $p_{v_i} = q_v$ for all $0 \le i \le n-1$ and $v \in V(G)$. Trivially, the two conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}^0_{\mathbf{q}} \in W^r_{G\square T,c'}$. Let $A_0 := \{\mathbf{p}^0_{\mathbf{q}} : q \in B^{(r)}\}$. Note that the restriction of any vector $\mathbf{p}^0_{\mathbf{q}} \in A_0$ to G_0 equals $\mathbf{q}$. Therefore, since $B^{(r)}$ is a basis, the vectors in A_0 are linearly independent.

Now, consider a basis $B^{(r-1)}$ for $W^{r-1}_{G,c}$. Fix $\ell \in [n-1]$. For each $\mathbf{q} = (q_v)_{v\in V(G)}$ in $B^{(r-1)}$, define the vector $\mathbf{p}^\ell_{\mathbf{q}} = (p_u)_{u\in V(G\square T)}$ such that, for each $v \in V(G)$, we have $p_{v_i} = (x-\alpha_\ell)q_v$ for all $i \in V(T_\ell)$, and $p_{v_i} \equiv 0$ for all other $i \in V(T)$. See Figure 3.22 for an illustration of the vector $\mathbf{p}^4_{\mathbf{q}}$ defined when $T = S_3(2,2,2)$.

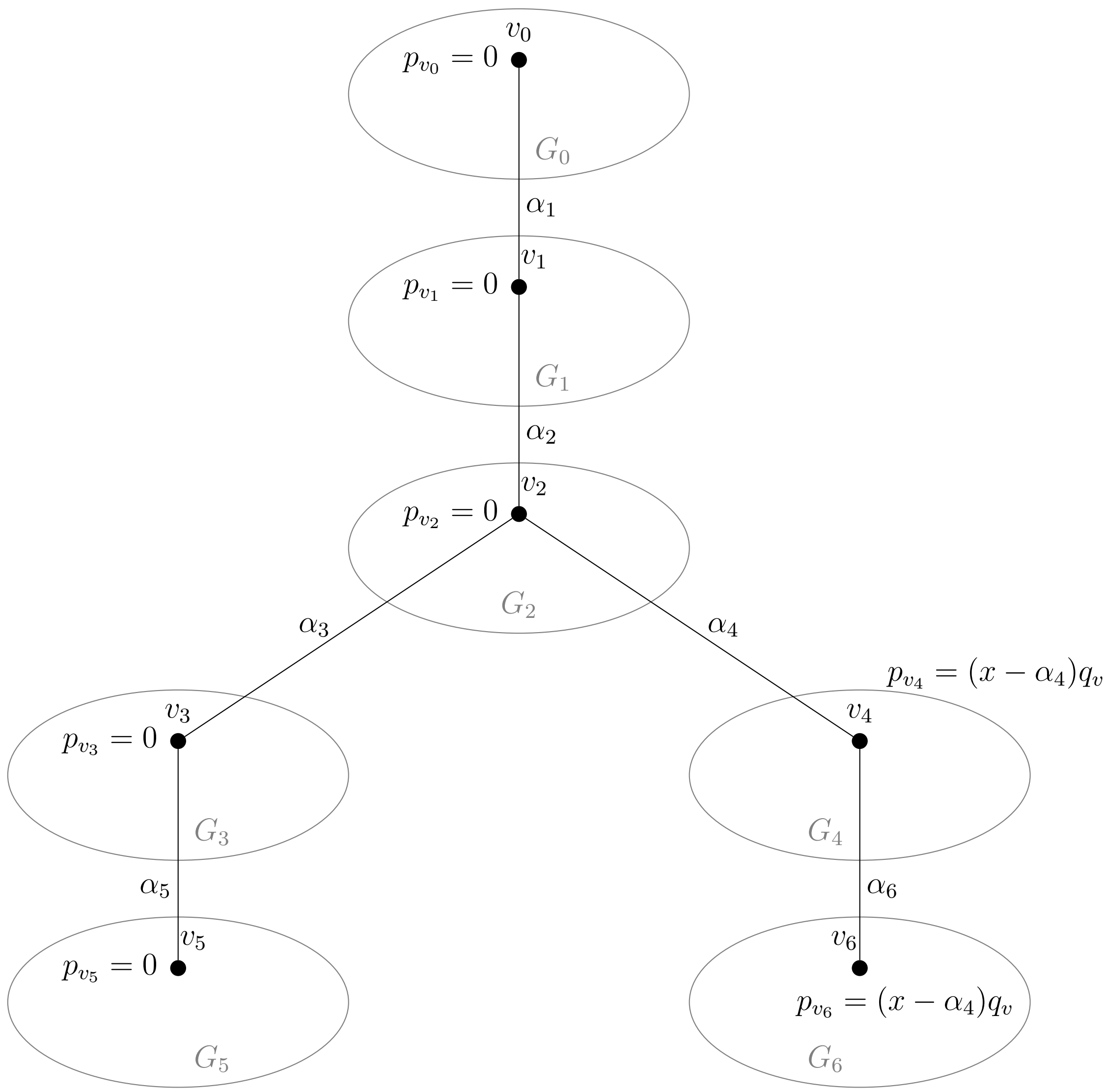

Figure 3.22: The vector $\mathbf{q} \in B^{(r-1)}$ is used to construct $\mathbf{p}^4_{\mathbf{q}} \in W^r_{G\square T,c'}$ for $T \simeq S_3(2,2,2)$.

For each $v \in V(G)$, note that $\mathbf{q} \in W^{r-1}_{G,c}$ implies $\deg(q_v) \leq \min\{r-1, \deg_G(v)\} - 1$. Therefore, for each $i \in V(T_\ell)$,

$$\begin{aligned}
\deg(p_{v_i}) &= \deg(q_v) + 1 \\
&\leq \min\{r-1, \deg_G(v)\} \\
&\leq \min\{r-1, \deg(v_i) - 1\} \\
&= \min\{r, \deg(v_i)\} - 1.
\end{aligned}$$

Moreover, $\mathbf{p}_{v_\ell}(\alpha_\ell) = 0 = \mathbf{p}_{v_j}(\alpha_\ell)$, where j is the parent of ℓ in T. Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other edges $uv \in E(G\square T)$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}^\ell_{\mathbf{q}} \in W^r_{G\square T,c'}$.

For each $\ell \in [n-1]$, define $A_\ell := \{\mathbf{p}^\ell_{\mathbf{q}} : \mathbf{q} \in B^{(r-1)}\}$. Note that, for each $\ell \in [n-1]$, the restriction of any vector $\mathbf{p}^\ell_{\mathbf{q}} \in A_\ell$ to G_ℓ equals $(x-\alpha_\ell)\mathbf{q}$. Therefore, since $x-\alpha_\ell \not\equiv 0$ and $B^{(r-1)}$ is a basis, the vectors in A_ℓ are linearly independent.

Define $A := \bigcup_{i=0}^{n-1} A_i$. Note that $|A_0| = \dim(W^r_{G,c})$, and $|A_\ell| = \dim(W^{r-1}_{G,c})$ for each $\ell \in [n-1]$. Therefore, $|A| = \dim(W^r_{G,c}) + (n-1)\dim(W^{r-1}_{G,c})$.

Claim 3.34 *The vectors in A are linearly independent.*

Proof of Claim 3.34. Suppose for a contradiction that there exists a linear combination $S = \sum_{\mathbf{p}\in A} \beta_{\mathbf{p}}\mathbf{p} = 0$, where $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in A$. Write $S = \sum_{j=0}^{n-1} S_j$ where $S_j = \sum_{\mathbf{p}\in A_j} \beta_{\mathbf{p}}\mathbf{p}$ for each $j \in V(T)$. Let i be the smallest index for which there exists some $\mathbf{p}_i \in A_i$ such that $\beta_{\mathbf{p}_i} \neq 0$. By definition, for each $v \in V(G)$, we have $p_{v_i} \equiv 0$ for all $\mathbf{p} \in \bigcup_{j=i+1}^{n-1} A_j$. Thus, since $\beta_{\mathbf{p}} = 0$ for all $\mathbf{p} \in \bigcup_{j=0}^{i-1} A_j$ by our choice of i, the restriction of the sum S to the coordinates corresponding to vertices in G_i is equal to the restriction of S_i to G_i. However, since the vectors in A_i are linearly independent, this contradicts our assumption that $\beta_{\mathbf{p}} \neq 0$ for some $\mathbf{p} \in A_i$.$_{(\square)}$

Now, set $Y = A$. To achieve the desired lower bound on $\dim\left(W^r_{G\square T,c'}\right)$, we must extend Y to include an additional $d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right)$ vectors, all of which are linearly independent. To this end, consider the set of vectors $Z_c \in \mathbb{R}^{|V(G)|}$ defined in Definition 3.10.

Claim 3.35 *For each non-zero $\mathbf{p} \in \operatorname{span}(A)$, there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$.*

Proof of Claim: Let $\mathbf{p} = (p_u)_{u\in V(G\square T)}$ be a non-zero vector in span(A). Let j be the smallest index for which there exists some $v_j \in V(G_j)$ with $p_{v_j} \not\equiv 0$. If $j = 0$, then by definition of the vectors in A, we can write $p_{v_0} = q_v$ for some vector $\mathbf{q} = (q_u)_{u\in V(G)}$ in

$\mathrm{span}(B^{(r)}) = W^r_{G,c}$. Otherwise, if $j \in [n-1]$, then we can write $p_{v_j} = q_v(x - \alpha_j)$ for some vector $\mathbf{q} = (q_u)_{u \in V(G)}$ in $\mathrm{span}(B^{(r-1)}) = W^{r-1}_{G,c}$. In either case, by Definition 3.5, $\deg(q_v) \leq \deg_G(v) - 1$. Thus, since the colours in $\{\alpha_i : i \in [n-1]\}$ were chosen to be distinct from those in $c(E)$, by Lemma 3.11 there exists $\mathbf{z} \in Z_c$ such that $\mathbf{p}(\mathbf{z}) \neq 0$. $(\Box)$

Therefore, in order to finish the proof of Proposition 3.33, it suffices to find a set X containing $d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right)$ linearly independent vectors in $W^r_{G\Box T,c'}$, each of which evaluates to zero on all of Z_c. This, together with Claim 3.35, will ensure that these vectors are independent from A.

First, for each $v \in V(G)$ with $\deg_G(v) \leq r-1$, define the vector $\mathbf{p}^0_v = (p_u)_{u \in V(G\Box T)}$ so that $p_{v_i} = \prod_{u \in N_G(v)}(x - c_{uv})$ for all $i \in V(T)$, and $p_u \equiv 0$ for all other $u \in V(G\Box T)$. Note that $p_u(c'_{uv}) = p_v(c'_{uv})$ for all edges $uv \in E(G\Box T)$. Furthermore, for all $i \in V(T)$, since $\deg_G(v) \leq r-1$ and $\deg_G(v) \leq \deg(v_i) - 1$, we have

$$\deg(p_{v_i}) = \deg_G(v) \leq \min\{r, \deg(v_i)\} - 1.$$

Hence the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}^0_v \in W^r_{G\Box T,c'}$. Moreover, $\mathbf{p}^0_v$ evaluates to 0 on all of Z_c. Let $X_v := \{\mathbf{p}^0_v\}$ for each $v \in V(G)$ with $\deg_G(v) \leq r-1$.

Now, let $v \in V(G)$ with $\deg_G(v) \leq r-2$. For each vertex $i \in V(T)$ with $\deg_T(i) \geq 2$, define the vector $\mathbf{p}^i_v = (p_u)_{u \in V(G\Box T)}$ as follows: Let

$$p_{v_i} = (x - \alpha_i) \prod_{u \in N_G(v)} (x - c_{uv}).$$

For each child j of i in T, and for each $\ell \in V(T_j)$, let

$$p_{v_\ell} = (\alpha_j - \alpha_i) \prod_{u \in N_G(v)} (x - c_{uv}).$$

Finally, let $p_u \equiv 0$ for all other $u \in V(G\Box T)$. See Figure 3.23 for an illustration of this construction for $T = S_3(2,2,2)$ when $i = 2$.

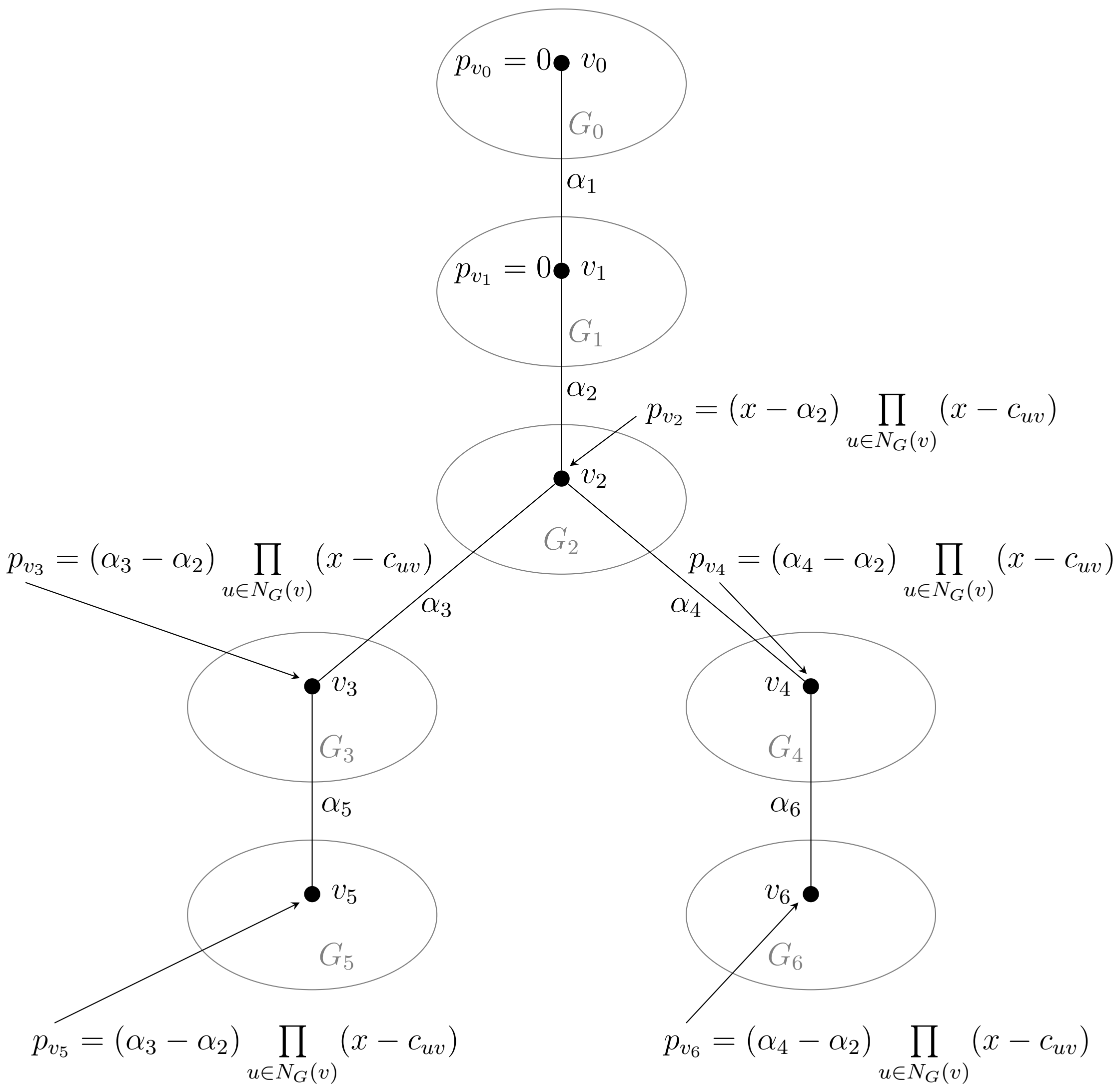

Figure 3.23: The vector $\mathbf{p}_v^2$ defined for $v \in V(G)$ with $\deg_G(v) \le r - 2$.

Now, $\deg_T(i) \ge 2$ implies $\deg_G(v) \le \deg(v_i) - 2$. Therefore, since $\deg_G(v) \le r - 2$,

$$\deg(p_{v_i}) = \deg_G(v) + 1 \le \min\{r, \deg(v_i)\} - 1.$$

In addition, for each child j of i in T, and each $\ell \in V(T_j)$, since $\deg_G(v) \le \deg(v_\ell) - 1$ and $\deg_G(v) \le r - 2$,

$$\deg(p_{v_\ell}) = \deg_G(v) \le \min\{r, \deg(v_\ell)\} - 1.$$

Now, $p_{v_i}(\alpha_i) = 0 = p_{v_a}(\alpha_i)$, where a is the parent of i in T. Moreover, for every child j of i in T,

$$p_{v_i}(\alpha_j) = (\alpha_j - \alpha_i) \prod_{u \in N_G(v)} (\alpha_j - c_{uv}) = p_{v_j}(\alpha_j).$$

Since $p_u(c'_{uv}) = p_v(c'_{uv})$ for all other $uv \in E(G \square T)$, the conditions in Definition 3.5 are satisfied, and thus $\mathbf{p}_v^i \in W^r_{G\square T, c'}$. Note that $\mathbf{p}_v^i$ also evaluates to zero on all of Z_c. For each $v \in V(G)$ with $\deg_G(v) \leq r-2$, and each $2 \leq i \leq \Delta(T)$, add $\mathbf{p}_v^i$ to the set X_v.

Let $X := \bigcup X_v$, where the union is taken over all sets X_v where $\deg_G(v) \leq r-1$. Then

$$|X| = \sum_{t=0}^{r-1} d_t^G + \sum_{t=0}^{r-2} d_t^G \sum_{i=2}^{\Delta(T)} d_i^T = d_{r-1}^G + \sum_{t=0}^{r-2} d_t^G \left(1 + \sum_{i=2}^{\Delta(T)} d_i^T\right).$$

Claim 3.36 *The vectors in X are linearly independent.*

Proof of Claim 3.36. Since each vector in X is only non-zero in the entries corresponding to a unique vertex $v \in V(G)$, it suffices to show that the vectors in each X_v are linearly independent. If $\deg_G(v) = r-1$, then $|X_v| = 1$, and thus X_v is clearly linearly independent. So we may assume $v \in V(G)$ with $\deg_G(v) = t$, where $t \leq r-2$.

Suppose for a contradiction that $S = \sum_{\mathbf{p} \in X_v} \gamma_{\mathbf{p}} \mathbf{p} = 0$ is a non-trivial linear combination. For ease of notation, we write γ_i for $\gamma_{\mathbf{p}_v^i}$. Let i be the smallest index such that $\gamma_i \neq 0$. By definition, $p_{v_i} \not\equiv 0$ in $\mathbf{p}_v^i$, but $p_{v_i} \equiv 0$ for all $\mathbf{p}_v^j$ where $j > i$. Since $\gamma_j = 0$ for all $j < i$, restricting the sum S to the coordinate corresponding to p_{v_i} gives $\gamma_i \mathbf{p}_v^i|_{v_i} \equiv 0$. Therefore, since $p_{v_i} \not\equiv 0$ in $\mathbf{p}_v^i$, it follows that $\gamma_i = 0$, a contradiction. Hence the vectors in X_v are linearly independent. (□)

Let $Y = A \cup X$. Therefore, by Claims 3.34, 3.35, and 3.36, we have found

$$|Y| = |A| + |X| = \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d_{r-1}^G + \sum_{t=0}^{r-2} d_t^G \left(1 + \sum_{i=2}^{\Delta(T)} d_i^T\right)$$

linearly independent vectors in $W^r_{G\square T, c'}$, as required. □

3.6.3 Some Exact Values for Trees

In this section, we investigate the conditions under which the upper and lower bounds on $m_e(G\Box T)$ given in Sections 3.6.1 and 3.6.2, respectively, match.

Lemma 3.37 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$, and suppose $m_e(G, i) = \dim(W^i_{G,c})$ for $r - 1 \leq i \leq r$. If the upper and lower bounds on $m_e(G\Box T)$ given in Propositions* 3.31 *and* 3.32 *match, then either $\delta(G) \geq r - 2$ or T is a path.*

Proof. By Propositions 3.31 and 3.32, since $m_e(G, i) = \dim(W^i_{G,c})$ for $r - 1 \leq i \leq r$,

$$d^G_{r-1} + \sum_{t=1}^{r-1} d^G_{r-1-t}\left(1 + \sum_{i=t+1}^{\Delta(T)} t d^T_i + \sum_{i=2}^{t}(i-1)d^T_i\right) = d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

Cancelling like terms and reindexing the sums on the left hand side, it follows that

$$\sum_{t=0}^{r-2} d^G_t\left(1 + \sum_{i=r-t}^{\Delta(T)}(r-1-t)d^T_i + \sum_{i=2}^{r-1-t}(i-1)d^T_i\right) = \sum_{t=0}^{r-2} d^G_t\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

Therefore, for each $t \leq r - 2$ such that $d^G_t > 0$,

$$\sum_{i=r-t}^{\Delta(T)}(r-1-t)d^T_i + \sum_{i=2}^{r-1-t}(i-1)d^T_i = \sum_{i=2}^{\Delta(T)} d^T_i. \tag{3.1}$$

Fix $t \leq r - 2$ such that $d^G_t > 0$. From the first sum on the left side of Equation 3.1, if there exists $r - t \leq i \leq \Delta(T)$ such that $d^T_i > 0$, then $r - 1 - t = 1$. Therefore, since $d^T_{\Delta(T)} > 0$, either

(a) $t = r - 2$, or

(b) $\Delta(T) < r - t$.

From the second sum on the left side of Equation 3.1, if there exists $2 \leq i \leq r - 1 - t$ such that $d^T_i > 0$, then $r - 1 - t \leq 2$. Therefore, either

(c) $t \geq r-3$, or

(d) T has no vertices of degree $2 \leq i \leq r-1-t$.

Now, suppose for a contradiction that $\delta(G) < r-2$ and T is not a path. Since $\delta(G) < r-2$, there exists some $t \leq r-3$ such that $d_t^G > 0$. Note that condition (a) fails for this value of t, and thus condition (b) holds; that is, $\Delta(T) < r-t$. Now, since T is not a path, $\Delta(T) \geq 3$. Hence $3 < r-t$. Therefore, condition (c) fails for this value of t, and thus condition (d) holds; that is, T has no vertices of degree $2 \leq i \leq r-1-t$. However, T contains at least one vertex of degree $\Delta(T)$, contradicting the fact that $3 \leq \Delta(T) \leq r-1-t$. Therefore, if the upper and lower bounds on $m_e(G\Box T)$ given in Propositions 3.31 and 3.32 match, then either $\delta(G) \geq r-2$ or T is a path. $\Box$

Therefore, in this section, we prove two results that give recursive formulas for $m_e(G\Box T, r)$ for certain trees T and graphs G. Theorem 3.30 (originally stated earlier in Section 3.6) considers the case where the minimum degree of G is at least $r-2$, while Theorem 3.38 deals with the case where T is a path.

Theorem 3.30 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let G be a graph with $\delta(G) \geq r-2$. Suppose there exists a proper edge-colouring c of G such that $m_e(G, i) = \dim(W_{G,c}^i)$ for each $r-1 \leq i \leq r$. Then*

$$m_e(G\Box T, r) = m_e(G, r) + (n-1)m_e(G, r-1) + d_{r-1}^G + d_{r-2}^G \left(1 + \sum_{i=2}^{\Delta(T)} d_i^T\right).$$

Proof. Since $\delta(G) \geq r-2$, by Proposition 3.31,

$$\begin{aligned} m_e(G\Box T, r) &\leq m_e(G, r) + (n-1)m_e(G, r-1) + d_{r-1}^G \\ &\qquad + \sum_{t=1}^{r-1} d_{r-1-t}^G \left(1 + \sum_{i=t+1}^{\Delta(T)} t d_i^T + \sum_{i=2}^{t} (i-1) d_i^T\right) \\ &= m_e(G, r) + (n-1)m_e(G, r-1) + d_{r-1}^G + d_{r-2}^G \left(1 + \sum_{i=2}^{\Delta(T)} d_i^T\right). \end{aligned}$$

Moreover, by Proposition 3.32,

$$m_e(G\Box T, r) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right)$$
$$= \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + d^G_{r-2}\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

Therefore, since $m_e(G,i) = \dim(W^i_{G,c})$ for all $r-1 \leq i \leq r$,

$$m_e(G\Box T, r) = m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + d^G_{r-2}\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

□

Note that Theorem 3.30 agrees with Theorem 3.12 when T is the star S_{n-1}.

Theorem 3.38 *Let P_n be a path on $n \geq 2$ vertices, and let $r \geq 1$ be an integer. If $m_e(G,i) = \dim(W^i_{G,c})$ for all $r-1 \leq i \leq r$, then*

$$m_e(G\Box P_n, r) = m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + (n-1)\sum_{t=0}^{r-2} d^G_t.$$

Proof. Since P_n is a path of order $n \geq 2$, it follows that $d^{P_n}_2 = n-2$. Therefore, by Proposition 3.31,

$$m_e(G\Box P_n, r) \leq m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1}$$
$$+ \sum_{t=1}^{r-1} d^G_{r-1-t}\left(1 + \sum_{i=t+1}^{\Delta(P_n)} t d^{P_n}_i + \sum_{i=2}^{t}(i-1)d^{P_n}_i\right)$$
$$= m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + \sum_{t=1}^{r-1} d^G_{r-1-t}\left(1 + d^{P_n}_2\right)$$
$$= m_e(G,r) + (n-1)m_e(G,r-1) + d^G_{r-1} + (n-1)\sum_{t=0}^{r-2} d^G_t.$$

Moreover, by Proposition 3.32,

$$\begin{aligned} m_e(G\Box P_n, r) &\geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t \left(1 + \sum_{i=2}^{\Delta(P_n)} d^{P_n}_i\right) \\ &= \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t (1+n-2) \\ &= \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + (n-1)\sum_{t=0}^{r-2} d^G_t. \end{aligned}$$

Therefore, since $m_e(G, i) = \dim(W^i_{G,c})$ for all $r-1 \leq i \leq r$,

$$m_e(G\Box P_n, r) = m_e(G, r) + (n-1)m_e(G, r-1) + d^G_{r-1} + (n-1)\sum_{t=0}^{r-2} d^G_t.$$

□

As noted in Sections 3.4.3 and 3.5.3, in the proof of Theorem 3.3, Hambardzumyan, Hatami, and Qian [14] showed that, for any $d, r \in \mathbb{Z}^+$, if $G_P = \prod_{i=1}^d P_{b_i}$ for some $b_1, ..., b_d \geq 2$, then $m_e(G_P, r) = \dim\left(W^r_{G_P,c}\right)$. Hence Theorem 3.38 is a generalization of Theorem 3.3.

Chapter 4

Conclusion and Open Problems

In this chapter, we present some open problems and areas for future study. Section 4.1 considers future work on rainbow saturation, while Section 4.2 focuses on polynomial methods for weak saturation.

4.1 Future Work on Rainbow Saturation

Recall that, in Chapter 2, we showed that the rainbow saturation number of a graph H is linear in n. We restate this result here for convenience.

Theorem 1.7 *Every non-empty graph H satisfies* $\mathrm{rsat}(n, H) = O(n)$.

Therefore, it is natural to ask the following question.

Question 4.1 *For every graph H, does there exist a constant $c = c(H)$ such that*

$$\mathrm{rsat}(n, H) = (c(H) + o(1))n?$$

Analogous questions have been considered for both saturation and weak saturation. For example, the limit $\lim_{n\to\infty} \frac{\mathrm{sat}(n,H)}{n}$ was conjectured to exist by Tuza [25, 26]. While some progress has been made towards Tuza's conjecture, (see [22, 24]), it remains open.

4.1.1 Weak Rainbow Saturation

Another natural direction would be to generalize the notion of weak saturation to edge-coloured graphs. Recall that, given graphs F and H, a spanning subgraph G of F is *weakly* (F, H)*-saturated* if the edges of $E(F) \setminus E(G)$ can be added, one at a time, so that the addition of each edge creates a new copy of H. The *weak saturation number* of H in F, denoted by $\text{wsat}(F, H)$, is the minimum number of edges in a weakly (F, H)-saturated graph. When $F = K_n$, we write $\text{wsat}(n, H)$ instead of $\text{wsat}(K_n, H)$.

We say an edge-coloured subgraph (G, χ) of F is *weakly* H*-rainbow saturated* if there exists an ordering $e_1, ..., e_m$ of $E(F) \setminus E(G)$ such that, for any list $c_1, ..., c_m$ of distinct colours from $\mathbb{N}$, the non-edges e_i in colour c_i can be added to G, one at a time, so that the addition of e_i to $G \cup \{e_1, ..., e_{i-1}\}$ creates a new rainbow copy of H. The *weak rainbow saturation number* of H in F, denoted by $\text{rwsat}(F, H)$, is the minimum number of edges in a weakly H-rainbow saturated graph. When $F = K_n$, we write $\text{rwsat}(n, H)$ instead of $\text{rwsat}(K_n, H)$.

Note that, in the above definition, we require the collection of added edges to receive distinct colours. In particular, we wish to exclude the possibility that all added edges receive the same colour, in which case the previously added edges do not contribute to making new rainbow copies of H, and the problem reduces to the standard rainbow saturation number.

In 1985, Alon [2] proved that $\text{wsat}(n, H) = (c(H) + o(1))n$, for all graphs H. The natural generalization of this to hypergraphs was recently proved by Shapira and Tyomkin [23]. An analogous question can be considered regarding the weak rainbow saturation number.

Question 4.2 *For every graph H, does there exist a constant $c = c(H)$ such that*

$$\text{rwsat}(n, H) = (c(H) + o(1))n?$$

4.1.2 Proper rainbow saturation

Note that the definition of rainbow saturation does not require the graph (G, c) to be properly edge-coloured. As a result, we introduce the following definition, which restricts our attention to properly edge-coloured graphs.

Definition 4.3 A properly edge-coloured graph (G, c) is *properly H-rainbow saturated* if (G, c) does not contain a rainbow copy of H, but the addition of any non-edge, in any colour from $\mathbb{N}$ which preserves the proper edge-colouring, creates a rainbow copy of H. The *proper rainbow saturation number* of H, denoted by $\mathrm{prsat}(n, H)$, is the minimum number of edges in a properly H-rainbow saturated edge-coloured graph on n vertices.

One natural problem would be to determine how the proper rainbow saturation number of a graph H compares to its saturation number.

Question 4.4 *Is* $\mathrm{prsat}(n, H) \leq \mathrm{rsat}(n, H)$ *for all graphs H?*

It is worth noting that the phrase 'rainbow saturation' has also appeared in the literature in a different context. Recently, Bushaw, Johnston, and Rombach [8] defined a different form of rainbow saturation which also requires the edge-colouring to be proper. We will refer to this concept as *BJR proper rainbow saturation* to distinguish it from the type of proper rainbow saturation defined in Definition 4.3.

Definition 4.5 A graph G is *BJR properly H-rainbow saturated* if there exists a proper edge-colouring of G that does not contain a rainbow copy of H, but, if any non-edge e is added to G, then any proper edge-colouring of the resulting graph $G + e$ contains a rainbow copy of H. The *BJR proper rainbow saturation number* of H, denoted by $\mathrm{prsat}'(n, H)$, is the minimum number of edges in a BJR properly H-rainbow saturated graph on n vertices.

Note that Definitions 4.3 and 4.5 are subtly different. Any graph G that is BJR properly H-rainbow saturated gives rise to a properly H-rainbow saturated edge-coloured graph (G, c) by taking c to be the proper edge-colouring of G that does not contain a rainbow copy of H. As a result, we obtain the following observation.

Observation 4.6 $\text{prsat}(n, H) \leq \text{prsat}'(n, H)$ *for every graph* H*.*

However, the converse of Observation 4.6 may not hold: If an edge-coloured graph (G, c) is properly H-rainbow saturated, there is no guarantee that, for every non-edge e of G, every recolouring of $G + e$ contains a rainbow copy of H.

In [8], Bushaw, Johnston, and Rombach proved that, for any graph H that does not contain an induced even cycle, the BJR proper rainbow saturation number of H is linear in n. Regarding graphs that do contain induced even cycles, they also showed that $\text{prsat}'(n, C_4) = O(n)$. Bushaw, Johnston and Rombach [8] conjectured that, analogously to ordinary saturation numbers, the BJR proper rainbow saturation number of every graph H is linear in n.

Conjecture 4.7 [8] *Every graph* H *satisfies* $\text{prsat}'(n, H) = O(n)$.

Private correspondence with Bushaw, Johnston, and Rombach, and independently with Barnabás Janzer, informs us that Conjecture 4.7 can be proved via a straightforward application of a result of Kazsonyi and Tuza [18]. By Observation 4.6, it follows that $\text{prsat}(n, H)$ must also be linear in n for all H. It would be interesting to know if there are any graphs H for which $\text{prsat}(n, H)$ and $\text{prsat}'(n, H)$ differ considerably.

4.2 Future Work on Polynomial Methods for Weak Saturation

In this section, we present some open problems related to the study of weak saturation via polynomial methods.

4.2.1 The Power of the Polynomial Procedure

Recall that, in Chapter 3, we provided recursive definitions of the value of $m_e(G, r)$ for various graphs G. In particular, we considered the case where G is a Cartesian product of stars (see Section 3.4), or a Cartesian product of joined cycles (see Section 3.5). We restate these results here for convenience.

Theorem 3.4 *Let $k, r, a_1, ..., a_k \in \mathbb{Z}^+$. Let $G_i = S_{a_1} \square \cdots \square S_{a_i}$ for each $i \in [k]$, and let $G_0 = K_1$. For $0 \le i \le \Delta(G_{k-1})$, let d_i denote the number of vertices in G_{k-1} of degree i. Then*

$$m_e(G_k, r) = m_e(G_{k-1}, r) + a_k m_e(G_{k-1}, r-1) + \sum_{t=1}^{a_k-1} t d_{r-t} + a_k \sum_{t=a_k}^{r} d_{r-t}.$$

Theorem 3.22 *Let $t > 0$ and $r > 1$ be integers. Let $k_1, ..., k_t, \ell_1, ..., \ell_t \in \mathbb{Z}^+$ such that $k_i \ge \ell_i \ge 4$ for all $i \in [t]$. Let $G_i = H_{k_1,\ell_1} \square \cdots \square H_{k_i,\ell_i}$ for $i \in [t]$, and let $G_0 = K_1$. For $0 \le i \le \Delta(G_{t-1})$, let d_i denote the number of vertices in G_{t-1} of degree i. Then*

$$\begin{aligned} m_e(G_t, r) = m_e(G_{t-1}, r) + (k_t + \ell_t - 5) m_e(G_{t-1}, r-1) + 2m_e(G_{t-1}, r-2) \\ + d_{r-1} + (k_t + \ell_t - 3) d_{r-2} + (k_t + \ell_t - 1) \sum_{i=0}^{r-3} d_i. \end{aligned}$$

Hambardzumyan, Hatami, and Qian [14] had previously provided recursive formulas for $m_e(G, r)$ when G is a Cartesian product of cycles or a Cartesian product of paths (see Theorems 3.2 and 3.3, respectively). Note that the lower bounds on $m_e(G, r)$ in these proofs all relied on the polynomial method of Hambardzumyan, Hatami, and Qian [14], which we restate here for convenience.

Theorem 3.6 [14] *Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$, and let $r \ge 0$ be an integer. Then $m_e(G, r) \ge \dim\left(W^r_{G,c}\right)$.*

Therefore, a natural line of inquiry would be to utilize Theorem 3.6 to determine the value of $m_e(G, r)$ for other graphs G. In addition to continuing this investigation for Cartesian products of graphs, one could also consider other graph products, such as a tensor product or a strong product.

Now, recall that, in Section 3.6, we provided upper and lower bounds on $m_e(G\Box T, r)$ when T is a tree. We restate these results here for convenience.

Proposition 3.31 *Let T be a tree of order $n \geq 2$. Let $r \in \mathbb{Z}^+$, and let G be a graph. Then*

$$m_e(G\Box T, r) \leq m_e(G, r) + (n-1)m_e(G, r-1) + d^G_{r-1} + \sum_{t=1}^{r-1} d^G_{r-1-t}\left(1 + t\sum_{i=t+1}^{\Delta(T)} d^T_i + \sum_{i=2}^{t}(i-1)d^T_i\right).$$

Proposition 3.32 *Let T be a tree of order $n \geq 2$, and let $r \in \mathbb{Z}^+$. Let $c : E \to \mathbb{R}$ be a proper edge-colouring of a graph $G = (V, E)$. Then*

$$m_e(G\Box T, r) \geq \dim\left(W^r_{G,c}\right) + (n-1)\dim\left(W^{r-1}_{G,c}\right) + d^G_{r-1} + \sum_{t=0}^{r-2} d^G_t\left(1 + \sum_{i=2}^{\Delta(T)} d^T_i\right).$$

In addition, we showed that the bounds in Propositions 3.31 and 3.32 match in certain cases, namely when either $\delta(G) \geq r-2$ or T is a path. Therefore, it is natural to ask the following question.

Question 4.8 *Is the exact value of $m_e(G\Box T, r)$ nearer to the lower bound presented in Proposition 3.32 or to the upper bound presented in Proposition 3.31?*

It is our belief that the upper bound presented in Proposition 3.31 is closer to the exact value of $m_e(G\Box T, r)$. It is possible that a different choice of polynomials in the proof of Proposition 3.33 could have improved the lower bound in Proposition 3.32.

Question 4.9 *Is it possible to improve the lower bound on $m_e(G\Box T, r)$ presented in Proposition* 3.32 *so it matches the upper bound presented in Proposition* 3.31 *for all trees T?*

Finally, note that the proofs in Chapter 3, and those in [14], which give recursive formulas for $m_e(G, r)$ when G is a Cartesian product of certain graphs, all showed that $m_e(G, r) = \dim(W^r_{G,c})$ for these graphs G. Therefore, it would be interesting to classify the graphs for which this property holds.

Question 4.10 *Under what conditions is $m_e(G, r) = \dim(W^r_{G,c})$ for a graph G and a proper edge-colouring c of G?*

4.2.2 A Connection to Linear Algebra

As noted in Section 3.1, the standard method for determining lower bounds on weak saturation numbers, and thus on $m_e(G, r)$, is to utilize techniques from linear algebra. In particular, the following lovely linear algebraic lemma of Balogh, Bollobás, Morris, and Riordan [3] has been used to prove many results in weak saturation, including the result of Morrison and Noel [21] that determined $m_e(Q_d, r)$ (see Theorem 3.1).

Lemma 4.11 [3] *Let F be a graph, let H be a subgraph of F, and let W be a vector space. Suppose there exists a set $S = \{f_e : e \in E(F)\} \subseteq W$ of vectors in W such that, for every copy H' of H in F, there exists a set of non-zero scalars $\{c_e : e \in E(H')\}$ such that $\sum_{e\in E(H')} c_e f_e = 0$. Then $\mathrm{wsat}(F, H) \geq \dim(S)$.*

It would be interesting to compare this linear algebraic lemma with the polynomial method of Hambardzumyan, Hatami, and Qian [14]. Note that Morrison and Noel [21] determined $m_e(G, r)$ for d-dimensional grids using Lemma 4.11, and Hambardzumyan, Hatami, and Qian [14] provided an alternate proof of this result using Theorem 3.6. Therefore, it is natural to ask whether the existence of a proof using one of these methods implies the existence of a proof using the other.

Question 4.12 *Let G be a graph, and let $r \geq 0$ be an integer. Does a proof that $m_e(G, r) \geq k$ exist using Lemma* 4.11 *if and only if one exists using Theorem* 3.6*?*

Bibliography

[1] N. Alon. An extremal problem for sets with applications to graph theory. *J. Combin. Theory Ser. A*, 40(1):82–89, 1985.

[2] N. Alon. An extremal problem for sets with applications to graph theory. *J. Combin. Theory Ser. A*, 40(1):82–89, 1985.

[3] J. Balogh, B. Bollobás, R. Morris, and O. Riordan. Linear algebra and bootstrap percolation. *J. Combin. Theory Ser. A*, 119(6):1328–1335, 2012.

[4] M. D. Barrus, M. Ferrara, J. Vandenbussche, and P. S. Wenger. Colored saturation parameters for rainbow subgraphs. *J. Graph Theory*, 86(4):375–386, 2017.

[5] N. Behague, T. Johnston, S. Letzter, N. Morrison, and S. Ogden. The rainbow saturation number is linear. *arXiv*, 2022. arXiv:2211.08589.

[6] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, 16:447–452, 1965.

[7] B. Bollobás. Weakly k-saturated graphs. *Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967)*, pages 25–31, 1968.

[8] N. Bushaw, D. Johnston, and P. Rombach. Rainbow Saturation. *Graphs Combin.*, 38(5):Paper No. 166, 2022.

[9] P. Erdős, A. Hajnal, and J. W. Moon. A problem in graph theory. *Amer. Math. Monthly*, 71:1107–1110, 1964.

[10] J. R. Faudree, R. J. Faudree, and J. R. Schmitt. A survey of minimum saturated graphs. *Electron. J. Combin.*, DS19(Dynamic Surveys):36, 2011.

[11] M. Ferrara, D. Johnston, S. Loeb, F. Pfender, A. Schulte, H. C. Smith, E. Sullivan, M. Tait, and C. Tompkins. On edge-colored saturation problems. *J. Comb.*, 11(4):639–655, 2020.

[12] P. Frankl. An extremal problem for two families of sets. *European J. Combin.*, 3(2):125–127, 1982.

[13] A. Girão, D. Lewis, and K. Popielarz. Rainbow saturation of graphs. *J. Graph Theory*, 94(3):421–444, 2020.

[14] L. Hambardzumyan, H. Hatami, and Y. Qian. Lower bounds for graph bootstrap percolation via properties of polynomials. *arXiv*, 2021. arXiv:1708.04640v2.

[15] D. Hanson and B. Toft. Edge-colored saturated graphs. *J. Graph Theory*, 11(2):191–196, 1987.

[16] G. Kalai. Weakly saturated graphs are rigid. *Convexity and graph theory (Jerusalem, 1981), North-Holland Math. Stud.*, 87:189–190, 1984.

[17] G. Kalai. Hyperconnectivity of graphs. *Graphs Combin.*, 1(1):65–79, 1985.

[18] L. Kászonyi and Z. Tuza. Saturated graphs with minimal number of edges. *J. Graph Theory*, 10(2):203–210, 1986.

[19] D. Korándi. Rainbow saturation and graph capacities. *SIAM J. Discrete Math.*, 32(2):1261–1264, 2018.

[20] L. Lovász. Flats in matroids and geometric graphs. *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages 45–86, 1977.

[21] N. Morrison and J. Noel. Extremal bounds for bootstrap percolation in the hypercube. *J. Combin. Theory Ser. A*, 156:61–84, 2018.

[22] O. Pikhurko. Results and open problems on minimum saturated hypergraphs. *Ars Combin.*, 72:111–127, 2004.

[23] A. Shapira and M. Tyomkyn. Weakly saturated hypergraphs and a conjecture of Tuza. *arXiv*, 2021. arXiv:2111.02373v1.

[24] M. Truszczyński and Z. Tuza. Asymptotic results on saturated graphs. *Discrete Math.*, 87(3):309–314, 1991.

[25] Z. Tuza. A generalization of saturated graphs for finite languages. In *Proceedings of the 4th International Meeting of Young Computer Scientists, IMYCS '86 (Smolenice Castle, 1986)*, number 185, pages 287–293, 1986.

[26] Z. Tuza. Extremal problems on saturated graphs and hypergraphs. *Ars Combin.*, 25(B):105–113, 1988. Eleventh British Combinatorial Conference (London, 1987).

[27] A. A. Zykov. On some properties of linear complexes. *Mat. Sbornik N.S.*, 24(66):163–188, 1949. [In Russian].