AN APPLICATION OF THE THEORY OF THE DOUBLE GAMMA FUNCTION

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ABSTRACT. The main purpose of this paper is first to show how one can apply the theory of the double Gamma function, which has recently been revived in the study of the determinants of Laplacians, to evaluate some classes of series involving the Zeta function. The determinants of Laplacians on the n-sphere S^n $(n = 1, 2, 3)$ are then computed by using our evaluations of series involving the Zeta function. Relevant connections of the results presented here with those given in earlier works are also pointed out.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

In recent years the problem of evaluating the determinants of Laplacians on Riemann manifolds has received considerable attention. D'Hoker and Phong ([8] and [9]), Sarnak [12], and Voros [17] computed the determinants of Laplacians on compact Riemann surfaces of constant curvature in terms of special values of the Selberg Zeta function. The theory of multiple Gamma functions also plays an important role in computations of the determinants of Laplacians on manifolds of constant curvature ([5],[11], and [16]). These functions were introduced by Barnes ([1] to [4]) and others in about 1900.

Although recent interest in the determinants of Laplacians is mainly for Riemann surfaces, it is also interesting to compute these determinants for classical Riemannian manifolds of higher dimensions, such as spheres. We are particularly interested here in the functional determinant for the n-sphere S^n with the standard metric. We recall in this connection that Choi [5] evaluated the determinants of Laplacians on S^n $(n = 1, 2, 3)$ by factorizing the analogous Weierstrass canonical product forms of a shifted sequence of eigenvalues on S^n into multiple Gamma functions. The object of the present paper is first to give (in Section 2) some formulas for series involving the Zeta function and then to compute (in Section 3) the determinants of Laplacians on S^n $(n = 1, 2, 3)$ using these summation formulas.

It should be remarked in passing that, while some of the formulas (presented in Section 2) for series involving the Zeta function were derived earlier by other means or in other contexts, our main objective in this paper is to apply these summation formulas with a view to presenting (in Section 3) a novel and natural approach to the familiar problem of computation of the determinants of Laplacians on the

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n-sphere S^n $(n = 1, 2, 3)$. We do indeed point out the relevant connections of the various results considered here with those given by earlier workers on the subject.

We begin by introducing the double Gamma and related functions as follows: Barnes [1] defined the double Gamma function $\Gamma_2 = 1/G$ satisfying each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z) \quad (z \in \mathbf{C})$;

(b) $G(1) = 1$;

(c) As $n \to \infty$,

$$\log G(z+n+2) = \frac{n+1+z}{2}\log(2\pi) + \left[\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right]\log n - \frac{3z^2}{4} - n - nz - \log A + \frac{1}{12} + O(\frac{1}{n}),$$

where Γ is the well-known Gamma function given by

$$\{\Gamma(z+1)\}^{-1} = e^{\gamma z}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)e^{-z/k}\right\}, \tag{1.1}$$

γ denotes the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577\,215\,664\ \ldots, \tag{1.2}$$

and A is Glaisher's (or Kinkelin's) constant defined by

$$\log A = \lim_{N\to\infty}\left\{\log(1^1 \cdot 2^2 \cdots N^N) - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12}\right)\log N + \frac{N^2}{4}\right\}, \tag{1.3}$$

the approximate numerical value of A being $1.282427130\cdots$. Voros [17] expressed $\log A$ in terms of the Riemann Zeta function (cf., e.g., [15]):

$$\log A = -\zeta'(-1) + \frac{1}{12}, \tag{1.4}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \quad (\mathrm{Re}(s) > 1)$$

is the Riemann Zeta function which can be continued analytically to the whole complex s-plane with only a simple pole at $s = 1$ having its residue 1 [18, p. 265].

From this definition, Barnes [1] deduced the following explicit *Weierstrass canonical product* expression for the double Gamma function:

$$\begin{aligned}\{\Gamma_2(z+1)\}^{-1} &= G(z+1)\\ &= (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}[(1+\gamma)z^2+z]}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right\}.\end{aligned} \tag{1.5}$$

We summarize below various known formulas for Γ_2 [1] for later use:

$$\int_0^z \log \Gamma(1+t)dt = z \log \Gamma(z+1) - \frac{z(z+1)}{2} + \frac{z}{2}\log(2\pi) - \log G(z+1), \tag{1.6}$$

which is known as Alexeiewsky's theorem. Setting $z = 1$ in (1.6) and noting that $G(2) = 1$, we obtain

$$\int_0^1 \log \Gamma(1+t)dt = \frac{1}{2}\log(2\pi) - 1. \tag{1.7}$$

We can also readily evaluate the following identity directly:

$$\int_0^1 \log G(1+t)dt = \frac{1}{12} + \frac{1}{4}\log(2\pi) - 2\log A. \tag{1.8}$$

Considering the functional relations:

$$G(t+2) = \Gamma(t+1)G(t+1) \quad \text{and} \quad \Gamma(t+2) = (t+1)\Gamma(t+1)$$

in conjunction with (1.7) and (1.8), we have

$$\begin{aligned}\int_0^{-1} \log G(1+t)dt &= \int_0^1 \log \Gamma(1+t)dt - \int_0^1 \log G(1+t)dt + 1 \\ &= -\frac{1}{12} + \frac{1}{4}\log(2\pi) + 2\log A.\end{aligned} \tag{1.9}$$

It is also known [1] that

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} \quad \text{and} \quad \Gamma_2\left(\frac{1}{2}\right) = 2^{-\frac{1}{24}} \cdot \pi^{\frac{1}{4}} \cdot e^{-\frac{1}{8}} \cdot A^{\frac{3}{2}}. \tag{1.10}$$

Furthermore, we recall from [18, p. 267, Example 2] that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = \left(1 - 2^{-s}\right)\zeta(s) \qquad (\mathrm{Re}(s) > 1) \tag{1.11}$$

and that the Riemann's functional equation for $\zeta(s)$ has the form [18, p. 269]:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s). \tag{1.12}$$

Some special values of the ζ-function are given below (see [18, p. 271]):

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log(2\pi), \tag{1.13}$$

and

$$\lim_{s \to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma. \tag{1.14}$$

2. Series involving the Zeta function

Srivastava ([13] and [14]) and others have investigated this subject rather systematically. Choi *et al.* ([6] and [7]) computed sums of certain series of the Riemann Zeta function. Many of our evaluations here turn out to be closely related to various known results. However, we evaluate some series associated with the Zeta function, using the theory of the Gamma function and the double Gamma function, to show the usefulness of the double Gamma function and to give an easy reference for Section 3 of this paper.

Recalling Equation (5.2) in [13, p. 13] and considering (1.6), we obtain

(2.1)
$$\sum_{k=2}^{\infty}(-1)^k\zeta(k)\frac{z^{k+1}}{k+1}=\frac{1}{2}[1-\log(2\pi)]z+\frac{1+\gamma}{2}z^2+\log G(z+1),\quad |z|<1;$$

(2.2)
$$\sum_{k=2}^{\infty}\zeta(k)\frac{z^{k+1}}{k+1}=\frac{1}{2}[1-\log(2\pi)]z-\frac{1+\gamma}{2}z^2-\log G(1-z),\quad |z|<1.$$

Adding (2.1) to (2.2), and subtracting (2.2) from (2.1), we have

(2.3)
$$\sum_{k=1}^{\infty}\zeta(2k)\frac{z^{2k+1}}{2k+1}=\frac{1}{2}[1-\log(2\pi)]z+\frac{1}{2}\log G(1+z)-\frac{1}{2}\log G(1-z),\quad |z|<1;$$

(2.4)
$$\sum_{k=2}^{\infty}\zeta(2k-1)\frac{z^{2k}}{k}=-(1+\gamma)z^2-\log G(1+z)-\log G(1-z),\quad |z|<1.$$

Letting $z=\frac{1}{2}$ in (2.3) and (2.4), and using the identity:

$$G(z+1)=\Gamma(z)G(z)$$

and (1.10), we obtain the following formulas for series involving the Zeta function:

(2.5)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)2^{2k+1}}=\frac{1}{4}(1-\log 2),$$

which is precisely the same as the known result recorded by Srivastava [13, p. 15, Eq. (5.14)];

(2.6)
$$\sum_{k=2}^{\infty}\frac{\zeta(2k-1)}{k\cdot 2^{2k}}=-\frac{1}{2}-\frac{\gamma}{4}-\frac{1}{12}\log 2+3\log A,$$

which is presumably new.

Similarly, using Equation (5.3) in [13, p. 13] and considering (1.6), we obtain

$$\sum_{k=1}^{\infty}\{\zeta(2k)-1\}\frac{z^{2k+1}}{2k+1}=\frac{1}{2}\{[3-\log(2\pi)]z-\log(z+1)+\log G(1+z) + \log(1-z)-\log G(1-z)\}, \quad |z|<1; \tag{2.7}$$

$$\sum_{k=2}^{\infty}\{\zeta(2k-1)-1\}\frac{z^{2k}}{k}=-\gamma z^2+\log(z+1) - \log G(z+1)+\log(1-z)-\log G(1-z), \quad |z|<1. \tag{2.8}$$

Letting $z\to 1$ in (2.7) and (2.8), and noting that

$$\lim_{z\to 1}\{\log(1-z)-\log G(1-z)\}=0, \tag{2.9}$$

we obtain

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{2k+1}=\frac{1}{2}[3-\log(4\pi)] \tag{2.10}$$

and

$$\sum_{k=2}^{\infty}\frac{\zeta(2k-1)-1}{k}=-\gamma+\log 2, \tag{2.11}$$

which are known results recorded by Srivastava [13, p. 14, Eqs. (5.8) and (5.9)].

Setting $z=\frac{1}{2}$ in (2.7) and (2.8), we obtain

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{(2k+1)2^{2k-1}}=3-\log 18, \tag{2.12}$$

which is precisely the same as the known result recorded by Srivastava [13, p. 15, Eq. (5.15)];

$$\sum_{k=2}^{\infty}\frac{\zeta(2k-1)-1}{k\cdot 2^{2k}}=-\frac{1}{4}-\frac{\gamma}{4}-\frac{25}{12}\log 2+\log 3+3\log A, \tag{2.13}$$

which is presumably new.

Differentiating both sides of Equation (4.2) in [13, p. 11] with respect to t and multiplying the resulting equations by t^2, and integrating these equations from $t=0$ to $t=z$, we have

$$\sum_{k=1}^{\infty}\{\zeta(2k)-1\}\frac{z^{2k+2}}{k+1}=\int_0^z t^2\psi(2+t)dt+\int_0^{-z} t^2\psi(2+t)dt, \quad |z|<2, \tag{2.14}$$

where $\psi(t) = \Gamma'(t)/\Gamma(t)$ is the Psi (or Digamma) function.

Similarly, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\{\zeta(2k+1)-1\}\frac{z^{2k+3}}{2k+3} &= \frac{1}{2}\left(\int_0^{-z} t^2\psi(2+t)dt - \int_0^{z} t^2\psi(2+t)dt\right) \\ &\quad + \frac{1}{3}(\gamma-1)z^3, \quad |z|<2.\end{aligned} \tag{2.15}$$

Recalling that

$$\psi(2+t) = \frac{1}{t+1} + \psi(1+t)$$

and integrating by parts successively with the aid of (1.6), we have

$$\begin{aligned}\int_0^{z} t^2\psi(2+t)dt &= \frac{z^3}{3} + \frac{1}{4}[3-\log(2\pi)]z^2 - z \\ &\quad + \log(z+1) + z\log G(z+1) - \int_0^{z}\log G(t+1)dt.\end{aligned} \tag{2.16}$$

Combining (2.14), (2.15), and (2.16), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\{\zeta(2k)-1\}\frac{z^{2k+2}}{k+1} &= \frac{1}{2}[3-\log(2\pi)]z^2 + \log(z+1) \\ &\quad + z\log G(z+1) + \log(1-z) - z\log G(1-z) \\ &\quad - \int_0^{z}\log G(t+1)dt - \int_0^{-z}\log G(t+1)dt, \quad |z|<1;\end{aligned} \tag{2.17}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\{\zeta(2k+1)-1\}\frac{z^{2k+3}}{2k+3} &= \frac{\gamma-2}{3}z^3 + z \\ &\quad + \frac{1}{2}\{\log(1-z) - z\log G(1-z) - \log(1+z) - z\log G(1+z)\} \\ &\quad + \frac{1}{2}\left\{\int_0^{z}\log G(t+1)dt - \int_0^{-z}\log G(t+1)dt\right\}, \quad |z|<1.\end{aligned} \tag{2.18}$$

Letting $z \to 1$ in (2.16) and (2.17), and making use of (1.9), (1.10), and (2.9), we have

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k+1} = \frac{3}{2} - \log\pi, \tag{2.19}$$

which was considered by Choi *et al.* [7, p. 388, Eq. (2.17)];

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+3} = \frac{5}{12} + \frac{\gamma}{3} - \frac{1}{2}\log 2 - 2\log A, \tag{2.20}$$

which is a known result given (with correction) by Choi *et al.* [7, p. 389, Eq. (2.18)].

3. Determinants of Laplacians on Spheres

Choi [5] computed the determinants of Laplacians on the n-sphere S^n ($n = 1, 2, 3$) by factorizing the analogous Weierstrass canonical product form of a shifted sequence of eigenvalues of the Laplacians on S^n into multiple Gamma functions. Using the formula (3.2) below, appearing in Voros [17], and some results given in Section 2, we propose to develop here a novel method to evaluate the determinants of Laplacians on the n-sphere S^n ($n = 1, 2, 3$) with standard metric.

Let $\{\lambda_n\}$ be a sequence such that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots ; \; \lambda_n \uparrow \infty \;\; (n \to \infty) \tag{3.1}$$

and henceforth consider only such nonnegative increasing sequences. Then we can show that

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

converges absolutely in the half-plane $\mathrm{Re}(s) > \sigma$ for some real number σ.

Definition 3.1 (cf. Osgood *et al.* [10]). The determinant of the Laplacian Δ on the compact manifold M is defined to be

$$\det{}'\Delta = \prod_{\lambda_j \neq 0} \lambda_j,$$

where $\{\lambda_n\}$ is the sequence (3.1) of eigenvalues of the Laplacian Δ on M. But this is always divergent; so, to make sense of this expression, some sort of regularization procedure must be used. It is easily seen that, formally, $e^{-Z'(0)}$ is the product of nonzero eigenvalues of Δ. This product does not converge, but $Z(s)$ can be continued analytically to a neighborhood of $s = 0$, and we define

$$\det{}'\Delta = e^{-Z'(0)}$$

to be the *Functional Determinant of the Laplacian* Δ on M.

Definition 3.2. Let

$$\mu = \inf \left\{ \alpha > 0 \;\middle|\; \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\alpha}} < \infty \right\}.$$

Then we call μ the *order* of the sequence $\{\lambda_k\}$. We also let

$$Z(s, a) := \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + a)^s}$$

and the analogous Weierstrass canonical product

$$E(\lambda) = \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k} + \frac{\lambda^2}{2\lambda_k^2} + \cdots + \frac{\lambda^{[\mu]}}{[\mu]\lambda_k^{[\mu]}} \right) \right\},$$

where $[\mu]$ denotes the integer part of the order μ of the sequence $\{\lambda_n\}$. Let

$$D(\lambda) = \exp\left(-Z'(0,-\lambda)\right).$$

Formally, indeed,

$$Z'(0,-\lambda) = -\sum_{k=1}^{\infty} \log(\lambda_k - \lambda); \quad \text{hence} \quad D(\lambda) = \prod_{k=1}^{\infty} (\lambda_k - \lambda).$$

Also, Voros [17] gave the formula:

$$\begin{aligned} D(\lambda) &= \exp(-Z'(0)) \exp\left(- \sum_{m=1}^{[\mu]} \mathrm{FP} Z(m) \frac{\lambda^m}{m} \right) \\ &\quad \cdot \exp\left(- \sum_{m=2}^{[\mu]} C_{-m} \left(1 + \cdots + \frac{1}{m-1}\right) \frac{\lambda^m}{m!} \right) E(\lambda), \end{aligned} \tag{3.2}$$

where the *finite part* prescription is applied, as usual, as follows (cf. [17, p. 446]):

$$\mathrm{FP} f(s) = \begin{cases} f(s), & \text{if } s \text{ is not a pole,} \\ \lim_{\epsilon \to 0} \left(f(s+\epsilon) - \frac{\text{Residue}}{\epsilon} \right), & \text{if } s \text{ is a simple pole;} \end{cases}$$

and $Z(m) = (-1)^m m! C_{-m}$.

Now consider the sequence of eigenvalues on the standard Laplacian on S^n. Let Δ_n be the standard Laplacian on S^n , where S^n is the unit n-sphere. It is known [15] that the standard Laplacian Δ_n $(n \geq 1)$ has eigenvalues $k(k+n-1)$ with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n}.$$

Let us take the fundamental sequence as the spectrum shifted by $\left(\frac{n-1}{2}\right)^2$. Then we get a sequence $\{\lambda_k\}$:

$$\lambda_k = k(k+n-1) + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2 \tag{3.3}$$

with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n}, \quad k = 0,\ 1,\ 2, \ldots .$$

We will exclude the zero mode, *i.e.*, start the sequence at $k = 1$ for later use.

To emphasize n on S^n, we use the notations $Z_n(s)$, $Z_n(s,a)$, $E_n(\lambda)$ and $D_n(\lambda)$ instead of $Z(s)$, $Z(s,a)$, $E(\lambda)$ and $D(\lambda)$, respectively. We are particularly interested in the cases when $n = 1, 2, 3$.

We readily observe from (3.2) that

$$D_n\left(\left(\frac{n-1}{2}\right)^2\right) = \det \Delta_n, \tag{3.4}$$

where $\det \Delta_n$ are the determinants of the Laplacians on S^n $(n = 1, 2, 3, \dots)$.

Using (1.11) and the definition of the ζ-function, we can readily obtain

$$\begin{aligned} Z_1(s) &= \sum_{k=1}^{\infty} \frac{2}{k^{2s}} = 2\zeta(2s), \\ Z_2(s) &= \sum_{k=1}^{\infty} \frac{2k+1}{(k+1/2)^{2s}} = \left(2^{2s}-2\right)\zeta(2s-1) - 4^s, \\ Z_3(s) &= \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k+1)^{2s}} = \zeta(2s-2) - 1. \end{aligned} \tag{3.5}$$

We see that the original sequence and the shifted sequence of eigenvalues on S^1 are the same. So, using the last formula of (1.13), we have (*cf.*, *e.g.*, [5, p. 162])

$$\det \Delta_1 = \exp\left(-Z_1'(0)\right) = \exp\left[-4\zeta'(0)\right] = 4\pi^2. \tag{3.6}$$

To compute $\det \Delta_2$, we find from (3.2) and (3.4) that

$$\det \Delta_2 = D_2\left(\frac{1}{4}\right) = \exp\left[-Z_2'(0) - \frac{\mathrm{FP} Z_2(1)}{4}\right] E_2\left(\frac{1}{4}\right). \tag{3.7}$$

From (1.13) and (3.5) we have

$$Z_2'(0) = -\frac{13\log 2}{6} - 2\zeta'(-1). \tag{3.8}$$

Since $Z_2(s)$ has a unique simple pole at $s = 1$ with the residue 1, we have

$$\begin{aligned} \mathrm{FP} Z_2(1) &= \lim_{\epsilon\to 0}\left[\left(2^{2(1+\epsilon)}-2\right)\zeta(\,2(1+\epsilon)-1\,) - 4^{1+\epsilon} - \frac{1}{\epsilon}\right] \\ &= \lim_{\epsilon\to 0}\left[2\left(2^{2\epsilon+1}-1\right)\left\{\zeta(\,2\epsilon+1\,) - \frac{1}{2\epsilon}\right\} + \frac{2\left(2^{2\epsilon}-1\right)}{\epsilon}\right] - 4 \\ &= 2\gamma + 4\log 2 - 4, \end{aligned} \tag{3.9}$$

where we have used (1.14) for the last equality.

Now we note that the shifted sequence of eigenvalues of Δ_2 on S^2 is of order $\mu = 1$. So we have

$$E_2\left(\frac{1}{4}\right) = \prod_{k=1}^{\infty}\left(1 - \frac{1}{(2k+1)^2}\right)^{2k+1} \exp\left(\frac{1}{2k+1}\right). \tag{3.10}$$

Taking logarithms on both sides of (3.10) and considering the Maclaurin series of $\log(1+x)$, we obtain

$$\begin{aligned} \log E_2\left(\frac{1}{4}\right) &= -\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{n=2}^{\infty}\frac{1}{(2k-1)^{2n-1}}-1\right) \\ &= -\sum_{n=2}^{\infty}\frac{\zeta(2n-1)-1}{n}+\sum_{n=2}^{\infty}\frac{\zeta(2n-1)}{n\cdot 2^{2n-1}} \\ &= -1+\frac{\gamma}{2}-\frac{7}{6}\log 2+6\log A, \end{aligned} \tag{3.11}$$

where we have used (2.6) and (2.11) for the last equality.

Finally, it follows from (3.7), (3.8), (3.9), and (3.11) that, using (1.4) for the second equality,

$$\det \Delta_2 = \exp[2\zeta'(-1)+6\log A] = A^4 e^{\frac{1}{6}}, \tag{3.12}$$

which was considered (among others) by Choi [5, p. 162].

Finally, we shall evaluate $\det \Delta_3$. Letting $n=3$ in (3.3), we see that the Laplacian Δ_3 on S^3 has the shifted sequence by 1 of eigenvalues, $\{(k+1)^2\}$ with multiplicity $(k+1)^2$, $k=0,1,2,\ldots$. This sequence has order $\mu=\frac{3}{2}$. Consequently, the analogous Weierstrass canonical product is

$$E_3(\lambda) = \prod_{k=1}^{\infty}\left[\left(1-\frac{\lambda}{(1+k)^2}\right)\exp\left(\frac{\lambda}{(1+k)^2}\right)\right]^{(k+1)^2}. \tag{3.13}$$

It follows from (3.2) that

$$D_3(\lambda) = \exp\left[-Z_3'(0)-\mathrm{FP}Z_3(1)\lambda\right]E_3(\lambda). \tag{3.14}$$

By the definition of $D_3(\lambda)$ and the shifted part $\left(\frac{3-1}{2}\right)=1$ by letting $n=3$ in (3.3), we see that

$$\det \Delta_3 = D_3(1) = \exp\left[-Z_3'(0)-\mathrm{FP}Z_3(1)\right]E_3(1). \tag{3.15}$$

We note that $Z_3(s)$ has a unique simple pole at $s=\frac{3}{2}$ with residue $\frac{1}{2}$. Then, by observing the fact that

$$\mathrm{FP}Z_3(1) = Z_3(1) = \zeta(0,2) = -\frac{3}{2},$$

we obtain

$$\det \Delta_3 = D_3(1) = \exp\left(-2\zeta'(-2,2)+\frac{3}{2}\right)E_3(1). \tag{3.16}$$

Now we recall that

$$\zeta(s,2) = \zeta(s)-1$$

and so

$$\zeta'(s,2) = \zeta'(s).$$

Differentiating the Riemann functional equation (1.12) with respect to s, we have

$$\zeta'(-2) = \zeta'(-2,2) = -\frac{\zeta(3)}{4\pi^2}. \tag{3.17}$$

Letting $\lambda = 1$ in (3.13) and taking logarithms of the resulting equation with the Maclaurin series of $\log(1+x)$, by using (2.26), we obtain

$$\log E_3(1) = -\sum_{k=2}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)k^{2n}}\right) = -\sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n+1} = \log\pi - \frac{3}{2}. \tag{3.18}$$

Finally, it follows from (3.16), (3.17), and (3.18) that

$$\det\Delta_3 = \pi\exp\left(\frac{\zeta(3)}{2\pi^2}\right), \tag{3.19}$$

which was considered (for example) by Choi [5, p. 162].

4. Concluding Remarks and Observations

We conclude this paper by reiterating the fact that, even though each of the determinantal expressions (3.6), (3.12), and (3.19) has been considered by other workers on this subject, our novel approach to the familiar problem of computation of the determinants of Laplacians on the n-sphere S^n $(n = 1, 2, 3)$ is relatively more natural than the earlier approaches to these and analogous problems. We have first shown how the theory of the double Gamma function can be applied in order to evaluate the sums of various classes of series involving the Zeta function and then made use of these summation formulas, in a rather natural manner, with a view to deriving the determinantal expressions (3.6), (3.12), and (3.19).

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