C*-Algebras from Substitution Tilings: A New Approach

by

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Abstract

C*-algebras from tilings are of particular interest. In 1998 J. Anderson and I. Putnam introduced a C*-algebra obtained from a substitution tiling that is viewed today as a standard invariant for this tilings. In this thesis we introduce another C*-algebra associated to a substitution tiling. We expect this C*-algebra to be in some sense a dual C*-algebra to the one introduced by Anderson and Putnam, but we were not able to make a precise statement. In our effort to characterize this new C*-algebras we prove that they are simple and can be constructed as an inductive limit of recursive subhomogenous algebras. We finish with K-theory computations for a number of examples.

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À MI
To my wife

Introduction

The mathematical study of aperiodical tilings was boosted in 1984 with the the development of the theory of quasicrystals. Geologists had always thought that all crystals had a simple, periodic arrangement, but in 1982 Israeli crystallographer Dan Shechtman discovered quasicrystals: metallic substances which are crystalline but do not follow the usual rules of crystal structure (he actually found some metallic alloys which exhibited a long-range atomic order with five-fold rotational symmetry). It took him two and a half years to get his results published, because the establishment refused to believe that such renegade crystals could exist. Quasicrystalline materials are more than just a geological curiosity - their resistance to wear means they could have many practical applications. One already in use is that they make excellent non-scratch coating for frying pans.

For mathematics the interest part is that the quasicrystal structure can be explained by an aperiodic tiling, see [10] or [30]. This prompted mathematicians and physicists to search for new relations between a quasicrystal and its associated aperiodic tiling. It also made the study of aperiodic tilings much more interesting. Here is where C*-algebras play a role. Anderson and Putnam introduced a C*-algebra, now considered the standard one, that can be obtained from an aperiodic tiling, see [1]. There are other constructions, as introduced by J. Bellissard and J. Kellendonk, see [11]. These C*-algebras are closely related with the physics of quasicrystals, see [11]. The main goal of this thesis is to introduce another C*-algebra for substitution tilings. This C*-algebra should be in some sense a dual C*-algebra to the one introduced by Anderson and Putnam. A general idea of this new C*-algebras is given bellow.

We start by introducing an equivalence relation on R^d, that relates two points, x and y, if the patch defined by y on the tiling matches the patch defined by x translated by y-x. We then associate a C*-algebra to this equivalence relation through the groupoid approach, see [23], and finally use the inflation map on the tiling to get an inductive limit of C*-algebras, which we call $C^*(\cup \mathrm{G}_k)$.

We expect this new C*-algebra to interplay in some dual way with the Anderson-Putnam C*-algebra, because both are associated to a Smale space (see [22] or [29] for details in Smale spaces) formed from the substitution tiling with some mild conditions. Actually $C^*(\cup G_k)$ is the C*-algebra associated to the stable equivalence in the Smale space and the Anderson Putnam C*-algebra is associated to the unstable equivalence (see[1] for details on the last statement).

The thesis is presented in the following manner:

In the first chapter we give a brief introduction to substitution tilings and to the Smith Normal form of a matrix and give some references on K-theory.

The new C*-algebras we are going to define are built from equivalence relations on $\mathbb{R}^d$. So we describe the general method to obtain C*-algebras from equivalence relations in chapter II and also give some important examples that are necessary to understand the C*-algebras defined in the following chapters.

For any tiling (with some mild assumptions, but not necessarily a substitution tiling) we associate a new C*-algebra to it in chapter III. We also make some general K-theory computations for this C*-algebras and illustrate our findings for the Thue-Morse tiling. We finish the chapter with a characterization of the positive cone of K_0 in the 1 dimensional case.

In chapter IV it is essential that we work with substitution tilings. We will use the C*-algebras defined in the previous chapter as steps to introduce a C*-algebra that will encode the dynamics of the substitution rule. We will also characterize these step C*-algebras as recursive subhomogeneous algebras and at the end of the chapter we will make some K-theory computations, including a 1 dimensional example (the Fibonacci tiling).

Finally in chapter V we show, for some particular cases, how to compute the K-theory of the C*-algebras introduced during the thesis.

We finish the work with some final considerations.

Chapter I

Background

I.1 Tilings

In this section we will give a brief definition of a substitution tiling and introduce some conditions we want to be satisfied. Most of the omitted proofs can be found in [1] and other references. We start with the definition of a tiling:

Definition I.1. *A tile is a subset of* $\mathbb{R}^d$ *homeomorphic to a closed ball in* $\mathbb{R}^d$*. A Tiling is a collection of tiles that cover* $\mathbb{R}^d$ *with pairwise disjoint interiors.*

In order to define a substitution tiling we need to make the definition of a tiling a little bit more technical and introduce a few more terms. In particular we also allow labels on the tiles.

Definition I.2. *A partial tiling, or patch, is a collection of tiles in* $\mathbb{R}^d$ *with pairwise disjoint interiors. The support of a partial tiling is the union of all its tiles as a subset of* $\mathbb{R}^d$*.*

Observation I.3. *A tiling is a partial tiling with support the whole* $\mathbb{R}^d$*.*

We may think of a tiling, T, as a multi-valued function, i.e., for $u \in \mathbb{R}^d$ we set

$$\mathrm{T}(u) = \{t \in \mathrm{T} : u \in t\}$$

and for $\mathrm{U} \subseteq \mathbb{R}^d$ we set

$$\mathrm{T}(\mathrm{U}) = \bigcup_{v \in \mathrm{U}} \mathrm{T}(v) = \{t \in \mathrm{T} : t \cap \mathrm{U} \neq \emptyset\}$$

Consider a finite set of tiles, $p_1, p_2, \ldots, p_n$, which we call prototiles and let $\widehat{\Omega}$ be the collection of all partial tilings that contain only translations of these prototiles (so we are assuming none of the prototiles is a translation of another). By the translation of a tile t by a vector $v \in I\!\!R^d$ we mean the set $t+v = \{x+v : x \in t\}$.

Suppose there is an inflation constant $\lambda > 1$ and a substitution rule that associates to each prototile p_i a partial tiling P_i with support p_i and such that $\lambda P_i \in \widehat{\Omega}$. Let $\widehat{\omega}(p_i) := \lambda P_i$. We will extend the definition of $\widehat{\omega}$ by translation for all other tiles in T. If a tile is a translation of a prototile, say $t_i = p_j + u$, we define $\widehat{\omega}(p_j + u) := \widehat{\omega}(p_j) + \lambda u$ for all tiles in T. Finally for any partial tiling $P \in \widehat{\Omega}$ we have that $\widehat{\omega}(P) := \bigcup_{t \in P} \widehat{\omega}(t)$.

The next proposition follows from the definitions introduced above.

Proposition I.4. $\widehat{\omega}(\mathrm{T}+v) = \widehat{\omega}(T) + \lambda v$ *for any* $v \in I\!\!R^d$ *and any* $\mathrm{T} \in \widehat{\Omega}$.

The set $\widehat{\Omega}$ is still too big for our purposes. So we restrict our attention to the set Ω of all tilings, T, in $\widehat{\Omega}$ such that for any partial tiling $P \subseteq \mathrm{T}$, with bounded support, we have $P \subseteq \widehat{\omega}^k(p_i + u)$, for some $k \geq 1$, $1 \leq i \leq n$ and $u \in I\!\!R^d$. The restriction of $\widehat{\omega}$ to Ω is denoted by ω.

Observation I.5. *The set Ω is non-empty, as shown in [1].*

Definition I.6. *A substitution tiling is a tiling that is contained in Ω as above.*

The following properties are standard conditions on a substitution:

1. ω is **primitive** that is, there exists an integer $N_0 > 0$ such that for all pair of prototiles p_i and p_j, the partial tiling $\omega^{N_0}(p_i)$ contains a translation of p_j.
2. Ω satisfies the **finite local complexity** condition that is, for all r in $I\!\!R$, there are only finitely many partial tilings, up to translation, that are subsets of tilings in Ω and whose support has diameter less than r.
3. $\omega : \Omega \to \Omega$ is one to one. This condition is often referred as **recognizability** of ω.

Observation I.7. *The assumption that ω is one to one implies that all tilings in Ω are aperiodic, i.e., if $\mathrm{T} \in \Omega$ and $\mathrm{T} + v = \mathrm{T}$ for some $v \in I\!\!R^d$ then $v = \vec{0}$. Also with all three assumptions above it follows that ω is actually a bijection. We refer the reader to [1] for a proof of these facts.*

We finish this section with a lemma we will need later on.

Lemma I.8. *Let* $p_1, p_2, \ldots, p_n$ *be a finite set of tiles and* Δ *be a collection of tilings such that every tiling in* Δ *contain only translations of the tiles* $p_1, p_2, \ldots, p_n$*. Then there exists* $\delta > 0$ *such that, if* $\mathrm{T} \in \Delta$*,* t *is a tile in* T *and* u *is a vector with* $0 < |u| < \delta$*, then* $t + u$ *is* ***not*** *a tile in* T.

Proof:

For $1 \leq i \leq n$, choose a point x_i in the interior of each prototile p_i and find δ_i such that $B(x_i, \delta_i) \subseteq p_i$. Then $\delta = \min\{\delta_i\}$ has the desired property.

■

I.2 Smith Normal Form of a matrix

In this section we introduce the Smith Normal Form of a matrix in a euclidean ring. Proofs will be omitted. We refer the reader to [26] for details.

Theorem I.9. *Every nonzero* $n \times n$ *matrix* Γ *with entries in a euclidean ring* R *can be written as the product* $\Gamma = P\left[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right]Q$*, where* P *and* Q *are invertible matrices,* $\Sigma = diag(\sigma_1, \ldots, \sigma_q)$*,* $\sigma_1, \sigma_2, \ldots, \sigma_q$ *are nonzero and* σ_i *divides* σ_{i+1} *for all* $1 \leq i < q$ *(the lower blocks of 0's or the right block of 0's may not be present).*

Definition I.10. *The matrix* $\left[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *in the statement of the theorem above is called a Smith normal form of* Γ.

I.3 References on K-theory

It is beyond the scope of this thesis to give the reader enough background in K-theory. We refer the reader to [24], [2] or [31] for a study in K-theory.

Chapter II

Groupoid C*-algebras

Given an equivalence relation G on a locally compact Hausdorff space, under some mild conditions, one can construct a C*-algebra from it. We refer the reader to [23] for the general construction. In the language of [23] we are going to consider the case when G is a principal, r-discrete groupoid with counting measure as a Haar System. In the situation we want to consider these assumptions will translate to properties P1, P2 and P3 below. For us it will be enough to describe how to obtain a C*-algebra from an equivalence relation satisfying P1 - P3 below.

II.1 Étale Equivalence Relations

We start this section with the definition of a groupoid:

Definition II.1. *A groupoid is a set* G *together with a subset* $\mathrm{G}^{(2)}$ *of* $\mathrm{G} \times \mathrm{G}$*, a product map* $\begin{array}{ccc} \mathrm{G}^{(2)} & \rightarrow & \mathrm{G} \\ (x,y) & \mapsto & xy \end{array}$*, and an involution* $\begin{array}{ccc} \mathrm{G} & \rightarrow & \mathrm{G} \\ x & \mapsto & x^{-1} \end{array}$ *that satisfy the following conditions:*

1. *if* (x,y) *and* (y,z) *are in* $\mathrm{G}^{(2)}$*, then* (xy,z) *and* (x,yz) *are in* $\mathrm{G}^{(2)}$ *with* $(xy)z = z(yz)$*;*
2. (x,x^{-1}) *and* (x^{-1},x) *are always in* $\mathrm{G}^{(2)}$ *for all* $x \in \mathrm{G}$ *with* $x^{-1}(xy) = y$ *and* $(zx)x^{-1} = z$ *whenever* (x,y) *and* (z,x) *are in* $\mathrm{G}^{(2)}$*.*

Definition II.2. *A groupoid has an unit space defined by* $\mathrm{G}^0 = \{xx^{-1} : x \in \mathrm{G}\}$*.*

Example II.3. *Observe that any group is a groupoid with* $\mathrm{G}^{(2)} = \mathrm{G} \times \mathrm{G}$*.*

In our work we are interested in C*-algebras obtained from equivalence relations. So we show below that an equivalence relation is a groupoid and hence we can use the approach of [23] to get C*-algebras from an equivalence relation.

Throughout this chapter, X is a locally compact Hausdorff space such that the one-point compactification of X is metrizable.

Let $\mathrm{G} \subseteq X \times X$ be an equivalence relation. Then G with partially defined product given by

$$(x, y)(y', z) = (x, z) \text{ if } y = y'$$

and inverse given by

$$(x, y)^{-1} = (y, x)$$

satisfies the properties of definiton II.1.

Also G can be equipped with two maps, called range and source defined by:

$$\begin{array}{rccc} r: & \mathrm{G} & \rightarrow & \mathrm{X} \\ & (x, y) & \mapsto & x \end{array} \qquad \begin{array}{rccc} s: & \mathrm{G} & \rightarrow & \mathrm{X} \\ & (x, y) & \mapsto & y \end{array}$$

Notice that above we are identifying the unit space of the groupoid, $\mathrm{G}^0 = \Delta_{\mathrm{X}} = \{(x, x) | x \in \mathrm{X}\}$ with the space X itself through the map $x \rightarrow (x, x)$.

We want G to have a topology such that the product and inverse defined above are continuous and have the following properties:

P1. G must be σ-compact, i.e., G is a countable union of compact sets.

P2. $\Delta = \{(x, x) \in \mathrm{G} : x \in \mathrm{X}\}$ is an open subset of G.

P3. For all $(x, y) \in \mathrm{G}$ there exists a neighborhood U of (x, y) in G, such that r restricted to U and s restricted to U are homeomorphisms from U to $r(\mathrm{U})$ and $s(\mathrm{U})$ respectively, and $r(\mathrm{U})$ and $s(\mathrm{U})$ are open subsets of X.

Definition II.4. *If* G *satisfy propertys P1 to P3 above we say* G *is étale.*

Observe that with the above conditions the topology of G is linked to that of X. In particular, since Δ is open, the natural identification of Δ with X is a homeomorphism.

We now can state the first proposition of this chapter:

Proposition II.5. *Let G be an equivalence relation as above.*

1. *G is locally compact and Hausdorff.*

2. *If K is a compact subset of G and $x \in \mathrm{X}$, then $r^{-1}\{x\} \cap \mathrm{K}$ and $s^{-1}\{x\} \cap \mathrm{K}$ are both finite.*

3. *$r^{-1}\{x\}$ and $s^{-1}\{x\}$ and hence the equivalence class of x, $[x]$, are countable for all $x \in \mathrm{X}$.*

4. *If $(x, y) \in \mathrm{G}$ then there are neighborhoods V_x of x and V_y of y in X, and a homeomorphism $\gamma : V_x \to V_y$ such that $\gamma(x) = y$ and*

$$\{(z, \gamma(z)) : z \in V_x\} \subseteq \mathrm{G}$$

   is pre-compact, i.e., its closure is compact.

   Moreover, the above γ is unique.

Observation II.6. *Item 4. above means that G is made up by the graph of "local homeomorphisms" of X.*

Proof:

1. That G is Hausdorff follows from the facts that G is étale and X is Hausdorff.

   To see that G is locally compact, let $(x, y) \in \mathrm{G}$. Since G is étale, there exists a neighborhood of (x, y) such that the range $r : U \to r(U)$ is a homeomorphism.

   Now, $r(x, y) = x$ and since X is locally compact there exists an open set $\mathrm{A} \subset \mathrm{X}$ such that $x \in \mathrm{A}$ and $\overline{\mathrm{A}}$ is compact.

   Then $r(U) \cap \mathrm{A}$ is open and $\overline{r(U) \cap \mathrm{A}}$ is compact and this implies that $r^{-1}(r(U) \cap \mathrm{A})$ is the neighborhood of (x, y) we are looking for.

2. Start by using that G is étale to cover K by open sets around each point where r is a homeomorphism. Since K is compact it has a finite subcover, say $U_1, ..., U_n$ and r is a homeomorphism on each U_i. Let $x \in \mathrm{X}$. If $(x, y) \in \mathrm{K} \cap r^{-1}\{x\}$ for some y, then $(x, y) \in U_j$ for some j and $r(x, y) = x$. Since r is a homeomorphism in U_j, (x, y) is the only point of $r^{-1}\{x\}$ in U_j. So $\mathrm{K} \cap r^{-1}\{x\}$ is bounded by n and in particular is finite.

   An analogous proof work for $s^{-1}\{x\}$.

3. We have that G is σ-compact, so $\mathrm{G} = \bigcup_{i \in \mathbb{N}} \mathrm{K}_i$, where each K_i is compact.

   Then,

$$r^{-1}\{x\} = r^{-1}\{x\} \cap \mathrm{G} = \bigcup_{i \in \mathbb{N}} r^{-1}\{x\} \cap \mathrm{K}_i$$

and by item 2 this is a countable union of finite sets, hence countable.

We proceed in an analogous way for $s^{-1}\{x\}$, and for $[x]$, it is enough to observe that

$$[x] = s\left(r^{-1}\{x\}\right)$$

4. The neighborhoods $\mathrm{V}_x = r(\mathrm{U})$, $\mathrm{V}_y = s(\mathrm{U})$ together with $\gamma = s|_{\mathrm{U}} \circ (r|_{\mathrm{U}})^{-1}$ will have the desired properties.

■

We will now see some examples of groupoids that satisfy properties $P1$, $P2$ and $P3$.

Example II.7. *Let* X *be any countable set with discrete topology and* $\mathrm{G} = \mathrm{X} \times \mathrm{X}$, *the "trivial" equivalence relation with discrete topology. Notice that all points in* X *are equivalent.*

Example II.8. *Let* X *be any locally compact, Hausdorff,* σ*-compact set and* $\mathrm{G} = \Delta_{\mathrm{X}}$ *with the inherited topology from* X, *the "co-trivial" equivalence relation. Notice that a point in* X *is only equivalent to itself.*

Example II.9. *Let* $\mathrm{X} = [0,1] \cup [2,3] \subseteq \mathbb{R}$, $\mathrm{G} = \Delta \cup \{(x, x+2), (x+2, x) : 0 \leq x < 1\}$ *with the relative topology from* $\mathrm{X} \times \mathrm{X}$.

Example II.10. *With* X *as in the previous example,* $\mathrm{G}' = \Delta \cup \{(1,3),(3,1)\}$ *does not satisfy* $P3$.

Proof: Observe that $r(1,3) = 1$ and 1 is closed on X but $(1,3)$ is open in G'.

■

II.2 C*-algebras from G

Given a groupoid as in the previous sections, it is possible to construct a C*-algebra associated with it. In this section we will introduce the reader to this procedure. Proofs will be mostly omitted as they can be found for the general case at [23]. We also refer the reader to [32] and [17] for an approach to groupoid C*-algebras.

The approach we are going to take here is very similar to the approach given in [21]. Examples will be given in the next section.

Begin with a topological equivalence relation, G, as in II.1. Let $C_c(\mathrm{G})$ be the complex linear space of compactly supported, continuous and complex valued functions on G. Define multiplication and involution as follows:

$$(f * g)(x, y) = \sum_{z \in [x]} f(x, z) g(z, y)$$
$$f^*(x, y) = \overline{f(y, x)}$$

where $(x, y) \in \mathrm{G}$ and $f, g \in C_c(\mathrm{G})$

Observation II.11. *The sum defined on the product is actually a finite sum, since* f *has compact support, say* K, *and by II.5 we have that* $\mathrm{K} \cap [x]$ *is finite.*

Observation II.12. *We refer the reader to [23] or [32] for the remaining proofs that* $f * g$ *and* f^* *are well defined.*

In order to obtain a C*-algebra from the equivalence relation G we still need a norm on $C_c(\mathrm{G})$. For this we need to first define a topology in $C_c(\mathrm{G})$.

Definition II.13. *Given a locally compact Hausdorff space* X, *the inductive limit topology is defined in* $\mathrm{C_c(X)}$ *as follows. A sequence* $\{f_n\}$ *converges to* f *in* $\mathrm{C_c(X)}$ *if and only if there exists a compact subset* K *of* X *such that supp*(f) *is in* K, *supp*$(\{f_n\})$ *is eventually in* K, *and for sufficiently large* N, $\{f_n\}_{n \geq \mathrm{N}}$ *converges uniformly to* f *on* K.

We are now able to define the full norm, which will be the supremum over a class of representations of $C_c(\mathrm{G})$.

Definition II.14. *Let* $f \in C_c(\mathrm{G})$. *Define:*

$$\|f\|_{C^*} = \sup\{\|\pi(f)\|_{op}\}$$

where $\pi : C_c(\mathrm{G}) \rightarrow \mathrm{B(H)}$ *is a bounded, non-degenerate* $*-$*homomorphism, that is continuous from the inductive limit topology of* $\mathrm{C_c(G)}$ *to the weak operator topology of* B(H). *By bounded above we mean that* $\|\pi(f)\|_{op} \leq \|f\|_*$ *where*

$$\|f\|_* = \max\left\{\sup_{x \in \mathrm{X}} \sum_{y \in [x]} |f(x, y)|, \sup_{y \in \mathrm{X}} \sum_{x \in [y]} |f(x, y)|\right\}$$

The completion of the $*$*-normed algebra* $(C_c(G), \|\cdot\|_{C^*})$ *(with respect to* $\|\cdot\|_{C^*}$*) is the full C*-algebra of* G, $C^*(\mathrm{G})$.

Observation II.15. *Recall that a representation is non-degenerate if the linear span* $\{\pi(f)\xi : f \in \mathrm{C}_c(\mathrm{G}) \text{ and } \xi \in \mathrm{H}\}$ *is dense in* H.

Observation II.16. *We refer the reader to [32] and [23] for the details that the above norm is well defined, in particular that the norm above is finite.*

To define the reduced C*-algebra we restrict our attention to a smaller class of representations and define a reduced norm, which induces a reduced C*-algebra. For the the cases we are going to be interested both the full and reduced C*-algebras obtained will coincide. To see this we will later on show that the groupoids we are considering are amenable and then use a result from [23].

Definition II.17. *Let* $f \in C_c(\mathrm{G})$. *Let* $x \in \mathrm{X}$. *Consider the Hilbert space* $l^2([x])$ *and define* $\lambda_x : C_c(\mathrm{X}) \to \mathrm{B}(l^2([x]))$ *by*

$$(\lambda_x(f)\xi)(y) = \sum_{z\in[x]} f(y,z)\xi(z)$$

for $\xi \in l^2[x]$, $y \in [x]$.

We can define the reduced norm as:

$$\|f\|_r = \sup_{x\in\mathrm{X}} \|\lambda_x(f)\|_{op}$$

and the completion of $C_c(\mathrm{G})$ *with respect to this norm is a C*-algebra, called the reduced C*-algebra of* G, $C_r^*(\mathrm{G})$.

Observation II.18. *Again we refer the reader to [32] and [23] to the details that the above norm is well defined, in particular that the reduced norm above is dominated by the full norm of definition II.14.*

Observation II.19. *In general* $C_r^*(\mathrm{G})$ *is a quotient of the full C*-algebra of* G.

Observation II.20. *Elements in* $C_r^*(\mathrm{G})$ *can be viewed as functions. Actually* $C_r^*(\mathrm{G})$ *is included in* $C_0(\mathrm{G})$, *[23], (but not as a subalgebra).*

One interesting property of $C_r^*(\mathrm{G})$ is that the commutative C*-algebra $C_0(\mathrm{X})$, with pointwise product and sup norm, can be embedded into $C_r^*(\mathrm{G})$ by means of the $*-$homomorphism

$$\begin{aligned} \alpha : C_c(\mathrm{X}) &\rightarrow C_c(\mathrm{G}) \\ f &\mapsto \alpha(f)(x,y) = \begin{cases} f(x); & \text{if } x = y \\ 0; & \text{otherwise} \end{cases} \end{aligned}$$

One checks that α is isometric, hence extends do the completion of $\mathrm{C}_c(\mathrm{X})$, $\mathrm{C}_0(\mathrm{X})$.

Moreover, in the case X is compact, $C(\mathrm{X})$ is a subalgebra of $C_c(\mathrm{G})$ and one can view $C(\mathrm{X})$ in $C_c(\mathrm{G})$ as an analogue of the diagonal matrices in M_n. Observe that the image under α of the constant function 1 is then a unit for the *-algebra $C_c(\mathrm{G})$.

II.3 Examples

In this section we will see some examples of C*-algebras obtained by from étale equivalence relations. Some of these examples will be used later on.

Example II.21. *Let* $\mathrm{X} = \{1, 2, .., n\}$ *and* $G = \mathrm{X} \times \mathrm{X}$ *as in example II.7. Then*

$$C^*(\mathrm{G}) \cong C_r^*(\mathrm{G}) \cong \mathrm{M}_n(\mathbb{C}).$$

Proof:

Observe that G is the equivalence relation below:

$$\begin{matrix} (1,1) & (1,2) & \dots & (1,n) \\ (2,1) & (2,2) & \dots & (2,n) \\ \vdots & & & \vdots \\ (n,1) & (n,2) & \dots & (n,n) \end{matrix}$$

And this suggests that we have an isomorphism ψ from $C_c(\mathrm{G})$ onto $M_n(\mathbb{C})$ given by:

$$\psi(f) = \begin{pmatrix} f(1,1) & f(1,2) & \dots & f(1,n) \\ f(2,1) & f(2,2) & \dots & f(2,n) \\ \vdots & & & \vdots \\ f(n,1) & f(n,2) & \dots & f(n,n) \end{pmatrix}$$

It is easy to check that ψ is a *-isomorphism. Now since every space of finite dimension is complete we have that

$$C_c(\mathrm{G}) \cong C^*(\mathrm{G}) \cong C_r^*(\mathrm{G}) \cong \mathrm{M}_n(\mathbb{C})$$

In order to illustrate the reduced C*-norm we show bellow that $\|f\|_r$ is equal to the norm of $\psi(f)$ as an operator on $\mathbb{C}^n$.

To compute the reduced C*-norm we need to evaluate the $\sup\limits_{x\in \mathrm{X}} \|\lambda_x(f)\|$. Observe that for any $x \in \mathrm{X}$, $l_2([x]) \cong \mathbb{C}^n$ and for $\xi \in \mathbb{C}^n$ we have that

$$\lambda_x(f)(\xi)(y) = \sum_{z\in[x]} f(y,z)\cdot\xi(z)$$

so for any $x, y \in \mathrm{X}$ we have that $\lambda_x = \lambda_y$ (Notice that $[x] = [y]$). This implies that

$$\|f\|_r = \sup_{x\in \mathrm{X}} \|\lambda_x(f)\| = \|\lambda_1(f)\| = \sup_{\|\xi\|=1} \|\lambda_1(f)\xi\| = \sup_{\|\xi\|=1} \|\psi(f)\xi\| = \|\psi(f)\|$$

■

Example II.22. *Let* $\mathrm{X} = \{1, 2, .., 5\}$ *and* $G = \{1,2,3\}\times\{1,2,3\}\cup\{4,5\}\times\{4,5\}$. *Then*

$$C_r^*(\mathrm{G}) \cong M_3(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Proof:

As in the previous example one can see that $\psi : C_c(\mathrm{G}) \to M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$ given by

$$\psi(f) = \left(\begin{pmatrix} f(1,1) & f(1,2) & f(1,3) \\ f(2,1) & f(2,2) & f(2,3) \\ f(3,1) & f(3,2) & f(3,3) \end{pmatrix}, \begin{pmatrix} f(4,4) & f(4,5) \\ f(5,4) & f(5,5) \end{pmatrix} \right)$$

is an isomorphism and hence $C^*(\mathrm{G}) \cong C_r^*(\mathrm{G}) \cong C_c(\mathrm{G}) \cong M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$

■

Example II.23. *Let* X *be any countable discrete set, say* $\mathrm{X} = \{1, 2, ..\}$ *and* $G = \mathrm{X}\times\mathrm{X}$ *as in example II.7. Then*

$$C_r^*(\mathrm{G}) \cong K(l^2(\mathrm{X})).$$

Proof:

To see that the above isomorphism holds, we first define an isometric $*-$homomorphism between $C_c(\mathrm{G})$ and the finite rank operators, $FR(l^2(\mathrm{X}))$, by:

$$\begin{aligned} \psi : C_c(\mathrm{G}) &\rightarrow K(l^2(\mathrm{X})) \\ f &\mapsto \psi(f)\xi(j) = \sum_{z\in[j]} f(j,z)\xi(z) \end{aligned}$$

where $\xi \in l^2(\mathrm{X})$. Since the support of f is compact it is finite, i.e., there exists $N \in \mathbb{N}$ such that $f(j,z) = 0$ if $j > N$ or $z > N$. So $\psi(f)$ is well defined.

It is not hard to see that ψ is a $*-$homomorphism. In fact ψ is an isometric $*-$homomorphism, since for all $x \in \mathrm{X}$ we have $\lambda_x(f) = \psi(f)$, because all the equivalence classes are the same and so the sum on the definition of ψ is the same as on the definition of λ_x. So we can extend ψ to an isometric $*-$homomorphism, $\tilde{\psi}$ from $C^*(\mathrm{G})$ to the closure of the image of ψ.

We now use that ψ is onto on the block operators of $l_2(\mathrm{X})$ to show that $\tilde{\psi}$ is onto on $\mathrm{K}(l_2(\mathrm{X}))$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $l_2(\mathrm{X})$ and let H_n be the subspace of $l_2(\mathrm{X})$ spanned by $\{e_1, e_2, \ldots, e_n\}$. Let P_n in $\mathrm{B}(l_2(\mathrm{X}))$ be the projection onto H_n. Observe that for $n \geq 1$, ψ is onto on the subalgebra $P_n\mathrm{B}(l_2(\mathrm{X}))P_n$, since operators in $P_n\mathrm{B}(l_2(\mathrm{X}))P_n$ can be represented by a $n \times n$ block matrix. Now it is a standard fact that $\mathrm{K}(l_2(\mathrm{X}))$ is the closure of the union $\cup_{n=1}^{\infty} P_n\mathrm{B}(l_2(\mathrm{X}))P_n$, see [18], and since the image of a C*-algebra under a $*-$homomorphism is always closed we have that

$$K(l^2(\mathrm{X})) = \overline{\mathrm{Im}(\tilde{\psi})} = \mathrm{Im}(\tilde{\psi})$$

■

Example II.24. *Let* X *be any locally compact* σ*-compact set and let* $G = \Delta_{\mathrm{X}}$ *as in II.8. Then*

$$C_r^*(\mathrm{G}) \cong C_0(\mathrm{X}).$$

Example II.25. *Let* $\mathrm{X} = [0,1]\cup[2,3]$ *and* $\mathrm{G} = \Delta\cup\{(x,x+2),(x+2,x) : 0 \leq x < 1\}$ *as in Example II.9.*

If $\mathrm{A} = \{f : [0,1] \rightarrow M_2(\mathbb{C}) : f \textit{is continuous and } f(1) \textit{is diagonal}\}$ *then*

$$C_r^*(\mathrm{G}) \cong \mathrm{A}.$$

Proof:

Define a $*-$homomorphism $\alpha : C_c(\mathrm{G}) \to \mathrm{A}$ by

$$\alpha(f)(t) = \begin{cases} \begin{pmatrix} f(t,t) & f(t,t+2) \\ f(t+2,t) & f(t+2,t+2) \end{pmatrix}; & if \quad 0 \leq t < 1 \\ \begin{pmatrix} f(1,1) & 0 \\ 0 & f(3,3) \end{pmatrix}; & if \quad t = 1 \end{cases}$$

With little trouble, one can check that α is a $*-$homomorphism. To see that α is isometric we will consider a few cases.

First if $x \in [0,1)$. Then the equivalence class of x is $[x] = x, x+2$ and if $\xi = (\xi_x, \xi_{x+2})$ belongs to $l_2[x]$ then

$$\lambda_x(f)(\xi) = \begin{pmatrix} f(x,x) & f(x,x+2) \\ f(x+2,x) & f(x+2,x+2) \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_{x+2} \end{pmatrix}$$

So

$$\|\lambda_x(f)\| = \sup_{\|\xi\|=1} \lambda_x(f)(\xi) = \sup_{\|\eta\|=1} \alpha(f)(x)(\eta) = \|\alpha(f)(x)\|$$

For $x \in [2,3)$ one proceeds analogously as above.

Now notice that $[1] = 1$ and $[3] = 3$ and hence $\|\lambda_1(f)\| = \|f(1,1)\|$ as well as $\|\lambda_3(f)\| = \|f(3,3)\|$. Since the operator norm of the matrix $\begin{pmatrix} f(1,1) & 0 \\ 0 & f(3,3) \end{pmatrix}$ is equal to $\max\{\|f(1,1)\|, \|f(3,3)\|\}$ we have that

$$\|f\|_r = \|\alpha(f)\|_\infty$$

We are left with the task of proving that α has dense range. As noted earlier the range is closed so this will imply that α is onto. So let $f \in \mathrm{A}$. Since f is continuous and $f(1)$ is diagonal, there exists $\delta > 0$ such that $\|f(t)_{12}\| < \frac{\epsilon}{2}$ and $\|f(t)_{21}\| < \frac{\epsilon}{2}$ for all $t \in [1-\delta, 1)$. Define $g \in C_c(\mathrm{G})$ by:

$$\begin{cases} g(t,t) = f(t)_{11} \text{ for } 0 \leq t \leq 1 \\ g(t+2,t+2) = f(t)_{22} \text{ for } 0 \leq t \leq 1 \\ g(t,t+2) = f(t)_{12} \text{ for } 0 \leq t \leq 1-\delta \\ g(t+2,t) = f(t)_{21} \text{ for } 0 \leq t \leq 1-\delta \\ g(1-\delta+t\frac{\delta}{2}, 1-\delta+2+t\frac{\delta}{2}) = -f(t)_{12}t + f(t)_{12} \text{ for } 0 \leq t \leq 1 \\ g(t,t+2) = 0 \text{ for } 1-\frac{\delta}{2} \leq t < 1 \\ g(1-\delta+2+t\frac{\delta}{2}, 1-\delta+t\frac{\delta}{2}) = -f(t)_{21}t + f(t)_{21} \text{ for } 0 \leq t \leq 1 \\ g(t+2,t) = 0 \text{ for } 1-\frac{\delta}{2} \leq t < 1 \end{cases}$$

It follows that $\|\alpha(g)-f\|_\infty = \sup\limits_{t\in[0,1]} \|\alpha(g)(t)-f(t)\| \leq \epsilon$ and hence we have the desired result.

■

Example II.26. *Let* X *and* Y *be countable discrete sets, as in example II.23. Let* $G = (\mathrm{X}\times\mathrm{X})\cup(\mathrm{Y}\times\mathrm{Y})$. *Then*

$$C_r^*(\mathrm{G}) \cong K(l^2(\mathrm{X})) \oplus K(l^2(\mathrm{Y})).$$

Proof:

The proof is basically analogous to the proof of II.23. Just notice that if $\psi : C_c(\mathrm{G}) \to K(l^2(\mathrm{X}))$ and $\psi' : C_c(\mathrm{G}) \to K(l^2(\mathrm{Y}))$ are as in II.23 then

$$\begin{array}{rcl} \psi\oplus\psi' : C_c(\mathrm{G}) & \to & K(l^2(\mathrm{X}))\oplus K(l^2(\mathrm{Y})) \\ f & \mapsto & (\psi(f),\psi'(f)) \end{array}$$

is an isometric $*-$isomorphism from $C_c(\mathrm{G})$ to $FR(l^2(\mathrm{X}))\oplus FR(l^2(\mathrm{Y}))$ since for all $f\in C_c(\mathrm{G})$ we have

$$\begin{aligned} \|f\| = \sup\nolimits_{z\in\mathrm{X}\cup\mathrm{Y}} \|\lambda_z(f)\| &= \max\left\{\sup\nolimits_{x\in\mathrm{X}}\|\lambda_x(f)\|, \sup\nolimits_{y\in\mathrm{Y}}\|\lambda_y(f)\|\right\} \\ &= \max\{\|\psi(f)\|,\|\psi'(f)\|\} = \|\left(\psi\oplus\psi'\right)(f)\| \end{aligned}$$

■

Example II.27. *Let* X *be a locally compact, σ-compact space and* Y *a countable set with the discrete topology. Let* $G = \Delta_x \times (\mathrm{Y}\times\mathrm{Y})$ *on* $\mathrm{X}\times\mathrm{Y}$ *with product topology. So a point* (x_1,y_1) *is equivalent to* (x_2,y_2) *if and only if* $x_1 = x_2$. *Then* G *is étale and*

$$C_r^*(\mathrm{G}) \cong \mathrm{C}_0\left(\mathrm{X}, K(l^2(\mathrm{Y}))\right).$$

Proof:

First we show that G is étale. Let $((x,y_1),(x,y_2)) \in \mathrm{G}$ and let U_x be an open neighborhood of x. Then $\mathrm{U} = \{((x,y_1),(x,y_2)) : x\in\mathrm{U}_x\}$ is an open set of G that contains $((x,y_1),(x,y_2))$ and such that the range map r is a homeomorphism from U onto $r(\mathrm{U}) = \{(x,y_1) : x\in\mathrm{U}_x\}$, which is open in $\mathrm{X}\times\mathrm{Y}$. Analogously we can show that the source map is a local homeomorphism. Also notice that for any $y_j\in\mathrm{Y}$ the set $\{((x,y_j),(x,y_j)) : x\in\mathrm{X}\}$ is open and hence $\Delta = \bigcup\limits_{y_j\in\mathrm{Y}} \{((x,y_j),(x,y_j)) : x\in\mathrm{X}\}$ is open. Finally let K be a compact set in X. Then for any $y_j, y_k \in \mathrm{Y}$ the set $\{((x,y_j),(x,y_k)) : x \in \mathrm{K}\}$ is compact in G and hence $\{((x,y_j),(x,y_k)) : x \in \mathrm{X}\}$ is σ-compact. It follows that G is étale.

We now turn our attention to the isomorphism part of the example. For simplicity we assume $\mathrm{Y} = \{1,2,3,\ldots\}$. We define a isometric $*-$homomorphism ψ from $\mathrm{C}_c(\mathrm{G})$ into $C_0\left(\mathrm{X}, \mathrm{K}\left(l^2(\mathrm{Y})\right)\right)$ by

$$\psi(f)(t)(\xi)(j) = \sum_{z=1,2,\ldots} f\left((t,j),(t,z)\right)\xi(z) = \sum_{(t,z)\in[(t,j)]} f\left((t,j),(t,z)\right)\xi(z)$$

for $t \in \mathrm{X}$ and $\xi \in l_2(\mathrm{Y})$ and $j \in \mathrm{Y}$.

We have some work to do. First we need to show that $\psi(f)$ is well defined, i.e., we need to show that $\psi(f)(t)$ is compact for all $t \in \mathrm{X}$ and that $\psi(f)$ is continuous and has limit 0 at infinity. Then we need to show that ψ is a isometric $*-$homomorphism with dense range and hence we can extend it to $\mathrm{C}^*_r(\mathrm{G})$. The details follow below.

- $\psi(f)(t) \in \mathrm{K}(l_2[\mathrm{N}])$ for all $t \in \mathrm{X}$

Notice that for any $i, j \in \mathrm{Y}$ the set $\mathrm{X}\times\{i\}\times\{j\} = \{((x,i),(x,j)) : x \in \mathrm{X}\}$ is open and hence the collection of sets $\{\mathrm{X}\times\{i\}\times\{j\}\}_{i,j\in\mathrm{Y}}$ covers the support of f. Since the support of f is compact there exists a finite subcover and so there exists $\mathrm{N} \in \mathbb{N}$ such that if either $i > \mathrm{N}$ or $j > \mathrm{N}$ then $f\left((t,i)(t,j)\right) = 0$ for all $t \in \mathrm{X}$. But this implies that for any $j > \mathrm{N}$, $\psi(f)(t)(\xi)(j) = 0$ for any $\xi \in l_2(\mathrm{Y})$ and hence $\psi(f)(t)$ is a finite rank operator for all $t \in \mathrm{X}$.

We could also see this by representing $\psi(f)(t)$ in matrix form as below,

$$\psi(f)(t) = \begin{pmatrix} f\left((t,1)(t,1)\right) & f\left((t,1)(t,2)\right) & \cdots & \\ f\left((t,2)(t,1)\right) & f\left((t,2)(t,2)\right) & & \\ \vdots & & & \\ & & f(((t,j)(t,k)) & \cdots \\ & & \vdots & \ddots \end{pmatrix}$$

and from the definition of N we have that the only possible nonzero entries of the matrix above are in the first N × N block, i.e., the matrix looks like:

$$\psi(f)(t) = \begin{array}{cc} & \begin{matrix} & & \mathrm{N} & & & \end{matrix} \\ \begin{matrix} \\ \\ -\mathrm{N}- \\ \\ \\ \end{matrix} & \begin{pmatrix} * & * & * & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

We call operators of the form above the block operators and denote then by $BL(l_2(\mathrm{Y}))$.

- $\psi(f)$ is continuous

Let $t_0 \in \mathrm{X}$ and give $\epsilon > 0$.

Let N as above. Choose an open neighborhood V_{t_0} such that if $t \in \mathrm{V}_{t_0}$ then

$$|f\left((t,j)(t,z)\right) - f\left((t_0,j)(t_0,z)\right)| < \frac{\sqrt{\epsilon}}{\sqrt{\mathrm{N}}}$$

for all $j = 1, 2, \ldots, \mathrm{N}$ and for all $z = 1, 2, \ldots, \mathrm{N}$.

So, if $\xi \in l_2(\mathrm{Y})$ with $\|\xi\| = 1$ and $t \in \mathrm{V}_{t_0}$ then

$$\begin{aligned}
\|\left(\psi(f)(t) - \psi(f)(t_0)\right)\xi\|^2 &= \sum_{j=1,2,3,\ldots} \|\left(\psi(f)(t) - \psi(f)(t_0)\right)\xi(j)\|^2 \\
&= \sum_{j=1..\mathrm{N}} \|\left(\psi(f)(t) - \psi(f)(t_0)\right)\xi(j)\|^2 \\
&= \sum_{j=1..\mathrm{N}} |\sum_{z=1,2,\ldots} f\left((t,j)(t,z)\right)\xi(z) - f\left((t_0,j)(t_0,z)\right)\xi(z)|^2 \\
&= \sum_{j=1..\mathrm{N}} |\sum_{z=1..\mathrm{N}} \left(f\left((t,j)(t,z)\right) - f\left((t_0,j)(t_0,z)\right)\right)\xi(z)|^2 \\
&\leq \sum_{j=1..\mathrm{N}} \sum_{z=1..\mathrm{N}} |f\left((t,j)(t,z)\right) - f\left((t_0,j)(t_0,z)\right)|^2 |\xi(z)|^2 \\
&\leq \sum_{j=1..\mathrm{N}} \sum_{z=1..\mathrm{N}} \frac{\epsilon}{\mathrm{N}}|\xi(z)|^2 \leq \frac{\epsilon}{\mathrm{N}} \sum_{j=1..\mathrm{N}} \sum_{z=1,2,\ldots} |\xi(z)|^2 \\
&= \frac{\epsilon}{\mathrm{N}} \sum_{j=1..\mathrm{N}} \|\xi\|^2 = \frac{\epsilon}{\mathrm{N}} \sum_{j=1..\mathrm{N}} 1 = \epsilon
\end{aligned}$$

and this proves that ψ is continuous at any $t_0 \in \mathrm{X}$.

- $\lim_{t\to\infty} \psi(f)(t) = 0$.

Give $\epsilon > 0$. Denote by π_x the projection of $\mathrm{X}\times\mathrm{Y}$ onto X, i.e., $\pi_x(t, y) = t$. Let $\mathrm{K} = \pi_x(r(\mathrm{supp}f))$ where r is the range map. Notice that K is compact since it is the image by a continuous function of a compact set. Now if $t \notin \mathrm{K}$ then, for any y, z in Y, we have that $((t, y), (t, z)) \notin \mathrm{supp}f$ and hence $\psi(f)(t) = 0$

- ψ is a homomorphism.

First we show that $\psi(f * g) = \psi(f)\psi(g)$.

Let $t \in \mathrm{X}$ and $\xi \in l_2(\mathrm{Y})$. Then

$$\begin{aligned}\psi(f * g)(t)(\xi)(j) &= \sum_{z=1,2,\dots} f * g\left((t,j),(t,z)\right)\xi(z)\\ &= \sum_{z=1,2,\dots}\ \sum_{(t',i)\in[(t,j)]} f\left((t,j),(t',i)\right) g\left((t',i),(t,z)\right)\xi(z)\end{aligned}$$

and $(t', i) \in [(t, j)]$ implies that $t' = t$ and $i = 1, 2, 3, \dots$, so the above sum is equal to

$$\sum_{z=1,2,\dots}\ \sum_{i=1,2,\dots} f\left((t,j),(t,i)\right) g\left((t,i),(t,z)\right)\xi(z) \tag{II.1}$$

On the other hand,

$$\begin{aligned}(\psi(f)\psi(g))(t)(\xi)(j) &= \left[\psi(f)(t)\circ\psi(g)(t)\right](\xi)(j)\\ &= \left[\psi(f)(t)\left(\psi(g)(t)(\xi)\right)\right](j)\\ &= \sum_{z=1,2,\dots} f\left((t,j),(t,z)\right)\left[\left(\psi(g)(t)(\xi)\right)(z)\right]\\ &= \sum_{z=1,2,\dots} f\left((t,j),(t,z)\right)\sum_{i=1,2,\dots} g\left((t,z),(t,i)\right)\xi(i)\\ &= \sum_{z=1,2,\dots}\ \sum_{i=1,2,\dots} f\left((t,j),(t,z)\right) g\left((t,z),(t,i)\right)\xi(i)\end{aligned}$$

and this is equal to II.1 as desired.

- ψ is a $*-$homomorphism.

We only need to show that $\psi(f^*) = \psi(f)^*$ and for this it is enough to show that $\psi(f)^*(t) = \psi(f^*)(t)$ for any $t \in \mathrm{X}$.

Let $t \in \mathrm{X}$ and $\xi, \eta \in l_2(\mathrm{Y})$. Then

$$\psi(f^*)(t)(\xi)(j) = \sum_{z=1,2,\dots} \overline{f\left((t,z),(t,j)\right)}\xi(z)$$

On the other hand $\psi(f)^*(t) = (\psi(f)(t))^*$ and the computation below gives the desired result.

$$\begin{aligned}<\psi(f)(t)(\xi),\eta> &= \sum_{j=1,2,\dots}\left(\sum_{z=1,2,\dots} f\left((t,j),(t,z)\right)\xi(z)\right)\overline{\eta(j)}\\ &= \sum_{j=1,2,\dots}\ \sum_{z=1,2,\dots} f\left((t,j),(t,z)\right)\xi(z)\overline{\eta(j)}\\ &= \sum_{z=1,2,\dots}\ \sum_{j=1,2,\dots} \overline{\overline{f\left((t,j),(t,z)\right)}\eta(j)\overline{\xi(z)}}\\ &= \overline{<\psi(f^*)(t)\eta,\xi>} = <\xi,\psi(f^*)(t)\eta>\end{aligned}$$

Observe that this also follows easily from the matrix representation of $\psi(f)(t)$.

- ψ is *-isometric homomorphism.

Notice that for any $(t,i) \in \mathrm{X}\times\mathrm{Y}$ the associated representation $\lambda_{(t,i)}$ (as in definition II.17) is equal to $\psi(f)(t)$. Hence $\|f\|_r = \sup\limits_{(t,j)\in\mathrm{X}} \|\lambda_{(t,i)}\| = \sup\limits_{t\in\mathrm{X}} \|\psi(f)(t)\| = \|\psi(f)\|$.

- ψ has dense range.

First we observe that for a fixed t in X the set

$$\{\psi(f)(t) : f \in \mathrm{C}_c(\mathrm{G})\}$$

is dense in $\mathrm{K}(l_2(\mathrm{Y}))$. To see this let $\mathrm{T} \in \mathrm{K}(l_2(\mathrm{Y}))$ and let $\epsilon > 0$. From the discussion on example II.23 we can find a function f' on $\mathrm{G}|_{\{((t,i),(t,j)):i,j\in\mathrm{Y}\}}$ such that $\|\psi'(f') - \mathrm{T}\| < \epsilon$, where ψ' is the homomorphism of example II.23. By Tietze extension theorem we can extend f' to a function $f \in \mathrm{C}_c(\mathrm{G})$. It follows that $\|\psi(f)(t) - \mathrm{T}\| < \epsilon$.

Let $f \in \mathrm{C}_c(\mathrm{X}, \mathrm{K}(l_2(\mathrm{Y})))$ and $\epsilon > 0$. Cover the support of f by open sets where f is uniformly compact, that is, for every t in supp(f) let V_t be an open set such that $\|f(z) - f(z')\| < \epsilon$ for all $z, z' \in V_t$ (such sets do exist from continuity of f). Since the support of f is compact, there exists a finite cover, say $\mathrm{V}_{t_1}, \ldots, \mathrm{V}_{t_M}$.

For each t_i, $1 \leq i \leq M$, from the observation above, there exists $f_i \in \mathrm{C}_c(\mathrm{G})$ such that $\|f(t_i) - \psi(f_i)(t_i)\| < \epsilon$. For simplicity we put $k_i := \psi(f_i)(t_i)$.

Let α_i be a partition of the unity with respect to the open sets V_i, $1 \leq i \leq M$, see [28]. Then for all $1 \leq i \leq M$ we have that $\alpha_i \in \mathrm{C}_c(\mathrm{X})$, $\mathrm{supp}(\alpha_i) \subset \mathrm{V}_i$, $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{M} \alpha_i(t) = 1$ for any t in supp(f). Now for any $t \in \mathrm{X}$ we define

$$g(t) = \sum_{i=1..M} \alpha_i(t) k_i$$

This g is a good approximation for f. To see this let $t \in \mathrm{X}$. Notice that if $\alpha_i(t) \neq 0$, $1 \leq i \leq M$, then $t \in \mathrm{supp}(\alpha_i) \subset \mathrm{V}_i$ and hence $\|f(t_i) - f(t)\| < \epsilon$. With this in mind we get that, for any t in X,

$$\begin{aligned}
\|g(t) - f(t)\| &= \|\sum_{i=1..M} \alpha_i(t)k_i - \sum_{i=1..M} \alpha_i(t)f(t)\| \leq \sum_{i=1..M} \|\alpha_i(t)\,[k_i - f(t)]\,\| \\
&\leq \sum_{i=1..M} \alpha_i(t)\,(\|k_i - f(t_i)\| + \|f(t_i) - f(t)\|) \leq \sum_{i=1..M} \alpha_i(t)\,(\epsilon + \epsilon) \\
&\leq 2\epsilon
\end{aligned}$$

so $\|f-g\| = \sup_{t\in \mathrm{X}} \|f(t)-g(t)\| \leq 2\epsilon$.

Finally we have that $g \in \mathrm{C}_c(\mathrm{X}, BL(l_2(\mathrm{N})))$ since sums of block operators and multiplication by scalar yields another block operator. Since $\psi(\mathrm{C}_c(\mathrm{G})) = \mathrm{C}_c(\mathrm{X}, BL(l_2(\mathrm{N})))$ there exists $h \in \mathrm{C}_c(\mathrm{G})$ such that $\psi(h) = g$ and hence ψ has dense range as desired.

■

The next example is of particular importance, since we will later rely on it to understand some ideals of the C*-algebras obtained from tilings that will be introduced in the next chapter.

Example II.28. *Let* $\mathrm{X} = \bigcup_{i\in\mathbb{N}}([0,1],i)$. *Let* Z *be the set of endpoints, i.e.,* $Z = \{(t,i) : i \in \mathbb{N}$ *and* $t = 0$ *or* $t = 1\}$. *Let* $\mathrm{G} = \{((t,i),(t,j)) : i,j \in \mathbb{N}$ *and* $0 < t < 1\} \cup \mathrm{G}' \cup \mathrm{G}''$, *where* G' *is any equivalence relation on the set of left endpoints, i.e., an equivalence on* $\{(0,i) : i \in \mathbb{N}\}$ *and* G'' *is any equivalence relation on the set of right endpoints, i.e. an equivalence on* $\{(1,i) : i \in \mathbb{N}\}$. *A basis for the topology in* G *is given by sets of the form*

$$\mathrm{U} = \{((t,i),(t,j)) : t \in \mathrm{V},\ \textit{where}\ \mathrm{V}\ \textit{is an open set of}\ [0,1]\ \textit{and}\ i,j \in \mathbb{N}\} \cap \mathrm{G}$$

Then $\mathrm{I}_Z = \overline{\{f \in C_c(\mathrm{G}) : f|_{Z\times Z} = 0\}}$ *is isomorphic to* $C_0((0,1), \mathrm{K}(l_2(\mathbb{N})))$.

Proof:

The proof of this example is analogous to the proof of example II.27. We define a isometric $*-$homomorphism ψ from $\mathrm{I}_{\mathrm{Z}_c} = \{f \in C_c(\mathrm{G}) : f|_{\mathrm{Z}\times\mathrm{Z}} = 0\}$ into $C_0((0,1), \mathrm{K}(l_2(\mathbb{N})))$ by

$$\psi(f)(t)(\xi)(j) = \sum_{z=1,2,\ldots} f((t,j),(t,z))\,\xi(z) = \sum_{(t,z)\in[(t,j)]} f((t,j),(t,z))\,\xi(z)$$

for $t \in (0,1)$.

Proceeding as in example II.27 we can show $\psi(f)$ is well defined and that we can extend ψ to I_{Z}.

We just show that $\lim_{t\to \mathrm{Z}} \psi(f)(t) = 0$. Remember N is such that if either $i > \mathrm{N}$ or $j > \mathrm{N}$ then $f((t,i)(t',j)) = 0$ for all $t, t' \in [0,1]$. Give $\epsilon > 0$. Let N as above. Since $f|_{\mathrm{Z}\times\mathrm{Z}} = 0$ and f is continuous there exists a compact set $K \subseteq (0,1)$ such that $|f((t,j)(t,z))| < \frac{\sqrt{\epsilon}}{\sqrt{\mathrm{N}}}$ for all $t \notin K$, $j = 1,2,\ldots,\mathrm{N}$ and $z = 1,2,\ldots,\mathrm{N}$. It now follows by an estimate analogous to the one done to show that f is continuous in example II.27 that $\|\psi(f)(t)\| \leq \epsilon$ for $t \notin K$.

■

Chapter III

C*-algebras from tilings

Given a tiling that satisfies the finite local complexity property and has only a finite number of tiles up to translation, we will define a equivalence relation G "on the tiling". Once we have proved this equivalence relation is étale we can proceed as in chapter II and construct the C*-algebra associated to the tiling, $C^*(\mathrm{G})$. The C*-algebra we get is usually very hard to realize, so we turn our attention to some subalgebras, namely the subalgebras obtained by restricting our equivalence relation to the edges or vertices of the tiling. These are very interesting C*-algebras on their own and they will be very important to the process of computing the K-theory for $C^*(\mathrm{G})$.

We would like to reinforce here that the construction below is not the standard one. For that we refer the reader to [1].

III.1 The C*-algebra of a tiling and ideals

Let T be a tiling on $I\!\!R^d$ with finite local complexity and only a finite number of tiles up to translation.

We say two points x and y of $I\!\!R^d$ are **equivalent** if the patch defined by y on the tiling match the patch defined by x translated by $y - x$. Recall that by the patch defined by x we mean $\mathrm{T}(x) = \{t \in \mathrm{T} : x \in t\}$.

More precisely, we define the equivalence relation by

$$\mathrm{G} = \big\{(x, y) \in I\!\!R^d \times I\!\!R^d : \mathrm{T}(x) - x = T(y) - y\big\} \tag{III.1}$$

and we give G the usual topology of $I\!\!R^d \times I\!\!R^d$

The proof that G is étale, II.1, is straightforward once we proved the two lemmas below.

Lemma III.1. *Given* $x \in I\!R^d$*, there exists* $\delta > 0$ *such that if* y *is in the ball of center* x *and radius* δ*,* $\mathrm{B}(x, \delta)$*, then* $\mathrm{T}(y) \subseteq \mathrm{T}(x)$*.*

Proof:

This follows from the fact that if y is contained in the interior of $\mathrm{T}(x)$ then $\mathrm{T}(y) \subseteq \mathrm{T}(x)$, and x is clearly contained in the interior of $\mathrm{T}(x)$, which is open.

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The next lemma characterizes neighborhoods of a point $(x, y) \in \mathrm{G}$.

Lemma III.2. *Let* $(x, y) \in \mathrm{G}$*. Then there exists* $\delta_0 > 0$ *such that*

$$\mathrm{B}\left((x, y), \delta\right) \cap \mathrm{G} = \left\{(x+v, y+v) : v \in \mathrm{B}\left(0, \frac{\delta}{\sqrt{2}}\right)\right\}$$

for all $\delta \leq \delta_0$*.*

Proof:

For simplicity we assume $d = 2$.

Let δ_0 be small enough so that both lemmas III.1 and I.8 hold.

Let $\delta < \frac{\delta_0}{2}$.

- First we show that if $v \in \mathrm{B}\left(0, \frac{\delta}{\sqrt{2}}\right)$ then $(x+v, y+v)$ is contained in $\mathrm{G} \cap \mathrm{B}\left((x, y), \delta\right)$. It is straightforward to show that $(x+v, y+v) \subseteq \mathrm{B}\left((x, y), \delta\right)$. We now show that $(x+v, y+v) \in \mathrm{G}$, i.e., that

$$T(x+v) - (x+v) = T(y+v) - (y+v).$$

By lemma III.1 and the choice of δ we have that $\mathrm{T}(x+v) + y - x \subseteq \mathrm{T}(x) + y - x = \mathrm{T}(y)$ and hence $\mathrm{T}(x+v) + y - x$ is a patch in T. Since $y + v = (x+v) + y - x$ we have that $\mathrm{T}(y+v) = \mathrm{T}(x+v) + y - x$ as desired.

- To get the other inclusion, suppose $(x', y') \in \mathrm{B}\left((x, y), \delta\right) \cap \mathrm{G}$. Then $x' = x + a$ and $y' = y + b$, where $a, b \in \mathrm{B}(0, \delta)$. Since $(x', y') \in \mathrm{G}$ we have that $\mathrm{T}(y') = \mathrm{T}(x') + y' - x'$, i.e., $\mathrm{T}(y+b) = \mathrm{T}(x+a) + y + b - x - a$, and by the choice of δ and the fact that $(x, y) \in \mathrm{G}$, we have that

$$\mathrm{T}(y+b) \subseteq \mathrm{T}(x) + y - x + b - a = \mathrm{T}(y) + b - a.$$

Also by the choice of δ we have $\mathrm{T}(y+b) \subseteq \mathrm{T}(y)$. We now can use lemma I.8 to prove that $a = b$. Fix a tile $p_i + w$ in $\mathrm{T}(y+b)$. From what is done above we have that $p_i + w \in \mathrm{T}(y)$ and $p_i + w \in \mathrm{T}(y) + b - a$. Let $u = w - b + a$. Then $p_i + u$ and $p_i + u + (b - a)$ belong to $\mathrm{T}(y)$, and since $|b - a| \leq |b| + |a| \leq \delta_0$, by lemma I.8, we must have $a = b$.

We proved that $(x', y') = (x+a, y+a)$ and since $(x', y') \in \mathrm{B}\left((x,y), \delta\right)$ we have that $\sqrt{a^2+a^2} < \delta$ and hence $|a| < \frac{\delta}{\sqrt{2}}$ as desired.

$\blacksquare$

The two lemmas above give us the

Proposition III.3. G *is an étale equivalence relation.*

Lemma III.2 has also another interesting consequence. We can use it to define a trace functional on $\mathrm{C}_c(\mathrm{G})$. But first we need to define G-invariant measures:

Definition III.4. *A measure μ on $\mathbb{R}^d$ is* G*-invariant iff for all $f \in \mathrm{C}_c(\mathrm{G})$ we have*

$$\int_{\mathbb{R}^d} \sum_{z \in [x]} f(x,z) d\mu(x) = \int_{\mathbb{R}^d} \sum_{z \in [x]} f(z,x) d\mu(x).$$

Proposition III.5. *Lebesgue measure is* G*-invariant.*

Proof:

Let $g \in \mathrm{C}_c(\mathrm{G})$. Notice that we can cover the support of g by a finite number of open sets as in lemma III.2. By a partition of unity argument, see [28], we can write g as a finite sum of functions in $\mathrm{C}_c(\mathrm{G})$, where each summand has support contained in an open set as in lemma III.2. So it is enough to show that for any U as in lemma III.2, if $f \in \mathrm{C}_c(\mathrm{U})$ then

$$\int_{\mathbb{R}^d} \sum_{z \in [x]} f(x,z) d\mu(x) = \int_{\mathbb{R}^d} \sum_{z \in [x]} f(z,x) d\mu(x).$$

So suppose $\mathrm{U} = \{(x_0 + v, y_0 + v) : v \in \mathrm{B}\left(0, \epsilon\right)\}$ and $f \in \mathrm{C}_c(\mathrm{U})$. Observe that we only need to show that

$$\int_{\mathrm{B}(x_0,\epsilon)} \sum_{z \in [x]} f(x,z) d\mu(x) = \int_{\mathrm{B}(y_0,\epsilon)} \sum_{z \in [x]} f(z,x) d\mu(x) \tag{III.2}$$

since if $x \notin \mathrm{B}(x_0, \epsilon)$ then $r^{-1}\{x\} \cap \mathrm{U} = \emptyset$ and if if $x \notin \mathrm{B}(y_0, \epsilon)$ then $s^{-1}\{x\} \cap \mathrm{U} = \emptyset$ (remember r and s are the range and source maps).

Now if $x \in \mathrm{B}(x_0, \epsilon)$, say $x = x_0 + v$ for some $v \in \mathbb{R}^d$, then there exists one and only one point (x, y) in $r^{-1}\{x\} \cap \mathrm{U}$. Notice that $y = y_0 - x_0 + x$, so the left hand side in III.2 is equal to $\int_{\mathrm{B}(x_0,\epsilon)} f(x, y_0 - x_0 + x) d\mu(x)$. Analogously if $y \in \mathrm{B}(y_0, \epsilon)$, say $y = y_0 + u$ for some $u \in \mathbb{R}^d$, then there exists one and only one point (t, y) in $s^{-1}\{y\} \cap \mathrm{U}$ and $t = x_0 - y_0 + y$, so the right hand side in III.2 is equal to $\int_{\mathrm{B}(y_0,\epsilon)} f(x_0 - y_0 + y, y) d\mu(y)$. Finally since $\mu(\mathrm{B}(x_0, \epsilon)) = \mu(\mathrm{B}(y_0, \epsilon))$ we have that

$$\int_{\mathrm{B}(x_0,\epsilon)} f(x, y_0 - x_0 + x) d\mu(x) = \int_{\mathrm{B}(y_0,\epsilon)} f(x_0 - y_0 + y, y) d\mu(y)$$

as desired.

■

The G-invariance of Lebesgue measure, denoted by μ, allows us to define a trace functional in $\mathrm{C}_c(\mathrm{G})$ via:

Definition III.6. $\begin{array}{rcl} \tau : \mathrm{C}_c(\mathrm{G}) & \rightarrow & \mathbb{C} \\ f & \mapsto & \int_{\mathbb{R}^d} f(x, x) d\mu(x) \end{array}$.

Observation III.7. *τ cannot be extended to a bounded trace on $\mathrm{C}^*_r(\mathrm{G})$.*

Observation III.8. *It follows from the fact that Lebesgue measure is G-invariant that $\tau(f^* f) = \tau(f f^*)$ for all $f \in \mathrm{C}_c(\mathrm{G})$ and hence τ has the tracial property.*

Later on we will compute traces for a few examples.

We now proceed as in chapter II and consider the full and reduced C*-algebras associated to G, $C^*(\mathrm{G})$ and $C^*_r(\mathrm{G})$ respectively. It is a known fact that if G is amenable then these two C*-algebras are isomorphic.

Observation III.9. *We will show in section IV.1 the G is amenable and as a consequence we will restrict our attention to $C^*_r(\mathrm{G})$ for the rest of our work.*

$C^*_r(\mathrm{G})$ is still usually very hard to realize. To understand it better we will need to study its ideals. Some results are valid in general, when G is any étale equivalence relation (or r-discrete principal groupoid). We refer the reader to [15] for a detailed approach on ideals of $C^*_r(\mathrm{G})$.

From now until the end of this subsection G can be any étale equivalence relation on a locally compact Hausdorff space X.

Definition III.10. *A subset Z of X is G-invariant if, whenever $(x, y) \in \mathrm{G}$ and $x \in \mathrm{Z}$, then $y \in \mathrm{Z}$.*

We are now able to introduce the most important ideal in $C^*_r(\mathrm{G})$ for our work.

Proposition III.11. *If* Z *is a closed,* G*-invariant subset of* X *then*

$$\mathrm{I_Z} = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Z}\times\mathrm{Z}} = 0\}}$$

is an ideal in $C_r^*(\mathrm{G})$.

Proof:

Let $\mathrm{I_{Z_c}} = \{f \in C_c(\mathrm{G}) : f|_{\mathrm{Z}\times\mathrm{Z}} = 0\}$. First we will show this set is an ideal in $C_c(\mathrm{G})$. Let $g \in C_c(\mathrm{G})$ and $f \in \mathrm{I_{Z_c}}$. Suppose $(x, y) \in \mathrm{Z} \times \mathrm{Z}$. Then for any $w \in [x]$ we have that $(x, w) \in \mathrm{G}$ and G-invariance implies that $w \in \mathrm{Z}$. But then,

$$(g * f)(x, y) = \sum_{w\in[x]} g(x, w)f(w, y) = 0.$$

Analogously $f * g|_{\mathrm{Z}\times\mathrm{Z}} = 0$. So $\mathrm{I_{Z_c}}$ is and ideal of $\mathrm{C}_c(\mathrm{G})$.

Now let $b \in \mathrm{I_Z}$ and $a \in \mathrm{C}_r^*(\mathrm{G})$. Notice that $b = \lim b_i$ where each $b_i \in \mathrm{I_{Z_c}}$ and $a = \lim a_j$ where each $a_j \in \mathrm{C}_c(\mathrm{G})$. From the continuity of the product we have that $b \cdot a = \lim b_i \cdot \lim a_i = \lim_i b_i \cdot a_i$. Since each $b_i \cdot a_i \in \mathrm{I_{Z_c}}$ we have that $b \cdot a \in \mathrm{I_Z}$ and hence $\mathrm{I_Z}$ is a closed ideal of $\mathrm{C}_r^*(\mathrm{G})$.

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The ideals above are very important because we have a nice description of the quotient of $C_r^*(\mathrm{G})$ by then; as the proposition below shows.

Proposition III.12. *If* Z *is a closed,* G*-invariant subset of* X *then* $\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z}$ *is étale with unit space* Z*. Moreover, the map that restricts a function in* $\mathrm{C}_c(\mathrm{G})$ *to* $\mathrm{G}|_{\mathrm{Z}\times\mathrm{Z}}$ *is a* $*-$*homomorphism, which extends to a surjection from* $C_r^*(\mathrm{G})$ *onto* $C_r^*(\mathrm{G}|_{\mathrm{Z}\times\mathrm{Z}})$ *with kernel equal to* $\mathrm{I_Z}$.

Proof:

- $\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z}$ is étale.

We will show that the range map r is a local homeomorphism. The proof for the source map s is analogous. Let $(x, y) \in \mathrm{G} \cap \mathrm{Z} \times \mathrm{Z}$. Since G is étale there exists neighborhoods U of (x, y) and V_x of x such that r is a homeomorphism. Now, $x \in \mathrm{Z}$ and G$-$invariance implies that $r_{\mathrm{U}}^{-1}\{\mathrm{V}_x\} \cap \mathrm{Z} \times \mathrm{Z} = r_{\mathrm{U}}^{-1}\{V_x \cap \mathrm{Z}\}$ and hence r is a homeomorphism from $r_{\mathrm{U}}^{-1}\{\mathrm{V}_x\} \cap \mathrm{Z} \times \mathrm{Z}$ onto $V_x \cap \mathrm{Z}$.

- Below we prove the second part of the proposition.

We denote the restriction map by α, i.e., α is defined by

$$\begin{array}{rcl}\alpha : C_c(\mathrm{G}) & \rightarrow & C_c(\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z}) \\ f & \mapsto & f|_{\mathrm{Z}\times\mathrm{Z}}.\end{array}$$

Using G−invariance we can show that α is a $*$−homomorphism. To see it is a homomorphism, let $(x, y) \in \mathrm{Z}$. Then

$$\alpha(f * g)(x, y) = f * g(x, y) = \sum_{z\in[x]_{\mathrm{G}}} f(x, z)g(z, y).$$

Now for any $z \in [x]_{\mathrm{G}}$, since $x \in \mathrm{Z}$ and $(x, z) \in \mathrm{G}$, G-invariance implies that $z \in \mathrm{Z}$ and hence the above sum is equal to:

$$\sum_{z\in[x]_{\mathrm{G}\cap\mathrm{Z}\times\mathrm{Z}}} f(x, z)g(z, y) = (\alpha(f) * \alpha(g))\,(x, y)$$

and hence α is a homomorphism. The proof that $\alpha(f^*) = \alpha(f)^*$ follows from G-invariance in in a similar way.

Our aim is to extend α to $C^*_r(\mathrm{G})$ and in order to do so we need to show that it is continuous. So we will show that $\|\alpha(f)\|_r \leq \|f\|_r$.

Observe again that since Z is G-invariant the G equivalence class of a point $x \in \mathrm{Z}$ is contained in Z, and hence $[x]_{\mathrm{G}} = [x]_{\mathrm{G}\cap\mathrm{Z}\times\mathrm{Z}}$. If we remember the definition of the reduced norm, as introduced in II.17, we conclude that $\|\lambda_x(f)\| = \|\lambda_x(\alpha(f))\|$ for all $x \in \mathrm{Z}$ and λ a representation as in II.17. This implies that

$$\|f\|_r = \sup_{x\in\mathrm{X}} \|\lambda_x(f)\| = \max\left\{\sup_{x\in\mathrm{Z}} \|\lambda_x(f)\|, \sup_{x\notin\mathrm{Z}} \|\lambda_x(f)\|\right\}$$

$$= \max\left\{\sup\nolimits_{x\in\mathrm{Z}} \|\lambda_x(\alpha(f))\|, \sup\nolimits_{x\notin\mathrm{Z}} \|\lambda_x(f)\|\right\}$$

$$= \max\left\{\|\alpha(f)\|_r, \sup\nolimits_{x\notin\mathrm{Z}} \|\lambda_x(f)\|\right\} \geq \|\alpha(f)\|_r.$$

Call the extension of α to $C^*_r(\mathrm{G})$ by $\widetilde{\alpha}$. It is clear that $\frac{C^*_r(\mathrm{G})}{\ker(\widetilde{\alpha})}$ is isomorphic to the image of $\widetilde{\alpha}$.

Notice that $\mathrm{C}_c(\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z})$ is contained in the image of $\widetilde{\alpha}$, since by Tietze extension theorem, see [28], we can extend $f \in \mathrm{C}_c(\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z})$ to $\mathrm{C}_c(\mathrm{G})$. This implies that $\widetilde{\alpha}$ is onto, since the image of a C*-algebra under a $*$−homomorphism is always closed. It remains to be shown that $\ker(\widetilde{\alpha})$ is isomorphic to I_Z.

By [15], page 53, there exists a closed set $\mathrm{Q} \subseteq \mathrm{G}$ such that $\ker(\widetilde{\alpha}) = \mathrm{I}(\mathrm{Q}) = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Q}} = 0\}}$. We will show that $\mathrm{Z} \times \mathrm{Z} = \mathrm{Q}$. It is clear that $\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Z}\times\mathrm{Z}} = 0\} \subseteq \mathrm{I}(\mathrm{Q})$, so that

$$\mathrm{I}_\mathrm{Z} = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Z}\times\mathrm{Z}} = 0\}} \subseteq \mathrm{I}(\mathrm{Q})$$

and hence $\mathrm{Z} \times \mathrm{Z} \supseteq \mathrm{Q}$.

Now suppose there exists a point $p \in \mathrm{Z} \times \mathrm{Z}$ such that $p \notin \mathrm{Q}$. Then $p \in \mathrm{Q}^c$, which is open, and by Urysohn's Lemma, see [28], there exists a function $f \in C_c(\mathrm{G})$ such that $f(p) = 1$ and the support of f is contained in Q^c. This implies that $f \in \ker(\widetilde{\alpha}) = \mathrm{I}(\mathrm{Q})$. But $\widetilde{\alpha}(f) = f|_{\mathrm{Z}\times\mathrm{Z}} \neq 0$ since $f(p) = 1$ and we have a contradiction.

We conclude that $\mathrm{Z} \times \mathrm{Z} = \mathrm{Q}$ and hence $\ker(\widetilde{\alpha}) = \mathrm{I}_{\mathrm{Z}}$ as desired.

∎

Corollary III.13. *It follows from the above proposition that* $\frac{C_r^*(\mathrm{G})}{\mathrm{I}_{\mathrm{Z}}} \cong C_r^*(\mathrm{G} \cap \mathrm{Z} \times \mathrm{Z})$.

The characterization of ideals in $\mathrm{C}_r^*(\mathrm{G})$ used on the proof above will be necessary later. We state it below as a proposition.

Proposition III.14. *Let* I *be an ideal of* $\mathrm{C}_r^*(\mathrm{G})$. *Then there exists a closed set* $\mathrm{Q} \subseteq \mathrm{G}$ *such that* $\mathrm{I} = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Q}} = 0\}}$.

Proof:

See [15], page 53.

∎

III.1.1 The Ideals of $C_r^*(\mathrm{G})$ that we need

In this section we use the results obtained in the general setting of section III.1 to obtain d ideals of $C_r^*(\mathrm{G})$, where G is the equivalence relation associated to a tiling as in III.1, that are very interesting on their own, but that also play a crucial role on the computations of K-theory that will follow on next section.

We need one more assumption about our tilings. From now on we assume every tiling has a cellular structure, for example if $d = 2$ we assume each tiling has vertices, edges and faces. With this assumption we can consider the ideal of all functions that vanish at the edges and the ideal of all functions that vanish at the vertices. The details follow below.

Definition III.15. *For* $0 \leq i < d$ *we denote*

$$\mathrm{X}_i = \left\{x \in I\!\!R^d : \textit{ if } x \in t, \textit{ for some tile } t \in \mathrm{T}, \textit{ then } x \textit{ is in the } i\textit{-skeleton of } t\right\}.$$

That is, X_0 *is the set of vertices in* T, X_1 *is the set of all points of* $I\!\!R^d$ *that are contained in an edge of a tile in* T *and so on.*

Proposition III.16. X_i *is closed and* G*-invariant.*

Proof:

Remember that

$$\mathrm{G} = \left\{(x,y) \in I\!\!R^d \times I\!\!R^d : \mathrm{T}(x) - x = T(y) - y\right\}.$$

Notice that the complement of X_i is open, since

$$(\mathrm{X}_1)^c = \bigcup_{tiles \in \mathrm{T}} \mathrm{int}\ (t)$$

and

$$(\mathrm{X}_0)^c = \bigcup_{tiles \in \mathrm{T}} \mathrm{int}\ (t) \bigcup \bigcup_{edges \in \mathrm{T}} \mathrm{int}\ (e)$$

We now prove G− invariance. Suppose $(x,y) \in \mathrm{G}$ and $x \in \mathrm{X}_i$. By the definition of G we have that $\mathrm{T}(x) - x = \mathrm{T}(y) - y$, i.e., $\mathrm{T}(y) = \mathrm{T}(x) + (y - x)$. This implies that if x is in a face σ of $\mathrm{T}(x)$ then y is in the face $\sigma + (y - x)$ of $\mathrm{T}(y)$ and hence $y \in \mathrm{X}_i$ as desired.

■

The proposition above combined with proposition III.11 gives us ideals in $C_r^*(\mathrm{G})$, namely

$$\mathrm{I}_{\mathrm{X}_i} = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{X}_i \times \mathrm{X}_i} = 0\}}$$

for $0 \leq i < d$.

Observation III.17. *For* $d = 2$, $\mathrm{X}_1 \supseteq \mathrm{X}_0$ *and hence* $\mathrm{I}_{\mathrm{X}_1} \subseteq \mathrm{I}_{\mathrm{X}_0}$.

Our next task is to characterize the ideals above and the quotient of $C_r^*(\mathrm{G})$ by them. From now on we restrict our attention to $d = 2$. Proposition III.11 gives us the following exact sequence:

$$0 \to \mathrm{I}_{\mathrm{X}_1} \to C_r^*(\mathrm{G}) \to C_r^*(\mathrm{G}|_{\mathrm{X}_1}) \to 0 \tag{III.3}$$

where $\mathrm{G}|_{\mathrm{X}_1}$ denotes $\mathrm{G} \cap (\mathrm{X}_1 \times \mathrm{X}_1)$.

We can describe $\mathrm{I}_{\mathrm{X}_1}$ nicely but we are unable to do the same for $C_r^*(\mathrm{G}|_{\mathrm{X}_1})$. All is not lost though as we can apply proposition III.11 for $\mathrm{X}_0 \times \mathrm{X}_0 \subseteq \mathrm{G}|_{\mathrm{X}_1}$ and get the following exact sequence:

$$0 \to C_r^*(\mathrm{G}|_{\mathrm{X}_1 - \mathrm{X}_0}) \to C_r^*(\mathrm{G}|_{\mathrm{X}_1}) \to C_r^*(\mathrm{G}|_{\mathrm{X}_0}) \to 0 \tag{III.4}$$

where $C_r^*(\mathrm{G}|_{\mathrm{X}_1-\mathrm{X}_0}) = \overline{\{f \in C_c(\mathrm{G}|_{\mathrm{X}_1}) : f|_{\mathrm{X}_0\times\mathrm{X}_0} = 0\}}$ and we can realize both end terms of the exact sequence.

In order to characterize the ideals and quotients mentioned above we need a few lemmas:

Lemma III.18. *Let* G *be the equivalence relation given in the beginning of the chapter, III.1, where* T *is a tiling with finite local complexity, only a finite number of tiles up to translation and a cell structure (i.e. vertices, edges and faces). Then the equivalence class of any* $x \in I\!R^d$ *is countable. In particular if* T *is a substitution tiling with finite local complexity, recognizability and primitivity then the equivalence class of any* $x \in I\!R^d$ *is infinitely countable.*

Proof:

If we denote the equivalence class of x by $[x]$ then $y \in [x]$ if and only if $\mathrm{T}(y) = \mathrm{T}(x) + y - x$. So all we need to show is that any patch P in the tiling T appears countably many times on T (In particular $\mathrm{T}(x)$ appears countably many times).

Since the number of tiles in T is countable it is clear that P appears at most a countable number of times in T. To see that it appears infinite many times first notice that from the definition of Ω there exists n, i, u such that $P \subseteq \omega^n(p_i + u)$. For this n, from bijectivety of ω, there exists T'' such that $\omega^n(\mathrm{T}'') = T$. Now p_i appears infinite many times on T'', since if we choose N from primitivity then the fact that ω is a bijection implies that there exists $\mathrm{T}' \in \Omega$ such that $\omega^{\mathrm{N}}(\mathrm{T}') = \mathrm{T}''$. So we have infinite many translations of $\omega^n(p_i)$ in T as desired.

■

Observation III.19. *From now until the rest of this chapter we assume our tilings have finite local complexity, only a finite number of tiles up to translation and a cell structure (i.e. vertices, edges and faces).*

Lemma III.20. *There is only a finite number of vertices equivalence classes in* G*, i.e., there is only a finite number of vertices patterns up to translation.*

Proof:

Choose a vertex x_0 in a tile p_i. Let R be such that if every tile in a partial tiling P contains x_0 then P has diameter less than R (can choose such R since we have only a finite number of tiles up to translation. So each $\mathrm{T}(x_0)$ is contained in $\mathrm{B}(x_0, R)$). By finite local complexity there is only a finite number of such partial tilings P, up to translation, and hence there is only a finite number of possible equivalence relations on x_0.

■

Observation III.21. *We denote the set of all vertex equivalence classes by* $\mathcal{V}$. *So from the lemma above we can write the set of vertices,* X_0, *as a finite disjoint union* $\dot{\bigcup\limits_{[v]\in\mathcal{V}}}[v]$.

We have a similar result for the edge patterns. In order to state it we need to introduce some notation. We say two edges are equivalent if the pattern defined by the interior points of the edges are the same up to translation. We can now state the next lemma:

Lemma III.22. *There is only a finite number of edge equivalence classes in* G, *i.e., there is only a finite number of edge patterns, defined by the interior points of the edges, up to translation.*

Proof: Proof is analogous to lemma III.20.

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Observation III.23. *We denote the set of all edge equivalence classes by* $\mathcal{E}$. *So from the lemma above we can write the set of edges as a finite disjoint union* $\dot{\bigcup\limits_{[e]\in\mathcal{E}}}[e]$.

Proposition III.24. $C_r^*(\mathrm{G}|_{\mathrm{X}_0}) \cong \bigoplus\limits_{[v]\in\mathcal{V}} \mathrm{K}(l^2([v]))$ *where* $[v]$ *are as in observation III.21.*

Proof:

From the lemmas above and observation III.21 we can write the set of vertices, X_0, as a finite disjoint union $\dot{\bigcup\limits_{[v]\in\mathcal{V}}}[v]$ where each v is a representative of a vertex equivalence class. Observe that $\mathrm{G}_{\mathrm{X}_0} = \dot{\bigcup\limits_{[v]\in\mathcal{V}}}\{[v]\}\times\{[v]\}$ and from examples II.23 and II.26 we have the desired result.

■

Proposition III.25. $C_r^*(\mathrm{G}|_{\mathrm{X}_1-\mathrm{X}_0}) \cong \bigoplus\limits_{[e]\in\mathcal{E}} C_0\left(e, \mathrm{K}(l^2([e]))\right)$ *where* $[e]$ *are as in observation III.23.*

Proof:

For each edge equivalence class we have that $[e_i] = e_i + v_0^i, e_i + v_1^i, e_i + v_2^i, e_i + v_3^i, \ldots$ where v_j^i are translation vectors in $I\!R^d$. If we restrict G to only one edge equivalence class $[e_i]$ then we are in the case of example II.28. But this is not exactly what we want. We need to restrict G to all edges of G. Notice that $\mathrm{X}_1 - \mathrm{X}_0$ is a disjoint union of open line segments and $\mathrm{G}|_{\mathrm{X}_1}$ and a point x in the interior of an edge e_j is equivalent to all points of the form $x + u$, where $u \in I\!R^d$ and $e_j + u \in [e_j]$. This is not true for endpoints. Moreover the open sets in $\mathrm{G}|_{\mathrm{X}_1}$ that contain the endpoints are not the same as in example II.28. But since the functions we are looking at vanish at the endpoints the isomorphism ψ of example II.28 still works. We define

$$\begin{aligned} \alpha : \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_1 - \mathrm{X}_0}) &\to \bigoplus_{[e] \in \mathcal{E}} C_0\left(e, \mathrm{K}(l^2([e]))\right) \\ f &\mapsto \{\psi_{[e]}(f)\}_{[e] \in \mathcal{E}} \end{aligned}$$

where $\psi_{[e]}(f) = \psi(f|_{[e]})$, ψ is the isomorphism of example II.28 and the edge e is identified with $[0,1]$. Also $\psi_{[e]}(f)$ is an abuse of notation and rigorously $\psi_{[e]}(f) = \psi(f|_{\{(x,y) \in \mathrm{G} : x \in e; e \in [e]\}})$.

The proof that α extends to an isomorphism is analogous to the proof of example II.28.

■

Finally remember we only have a finite number of tiles up to translation. This means we can also write the set of tiles as a disjoint finite union $\dot{\bigcup_{[p] \in \mathcal{P}}} [p]$, where $[p]$ denote the equivalence class of p, i.e., the set of all tiles that are translations of p and $\mathcal{P}$ denote the set of all tile equivalence classes. We can now state the last proposition of this section.

Proposition III.26. $\mathrm{I}_{\mathrm{X}_1} \cong \bigoplus_{p \in \mathcal{P}} C_0\left(p, \mathrm{K}(l^2([p]))\right)$ *where* $[p]$ *are as described above.*

Proof: This proposition is analogous to the proposition III.25, only in a higher dimension.

■

III.2 Computing K-theory

Our aim in this section is to compute the K_0 and K_1 groups of $C_r^*(G)$. In order to do so we need to equip each edge and tile of T with an orientation. This should be done in a way that if two edges are equivalent in the sense of III.22 then they have the same orientation and analogously if two tiles are equivalent in the setting of proposition III.26 then they have the same orientation. Also each in tile in T is oriented counterclockwise.

With the extra hypotheses above we can use the two exact sequences III.4 and III.3 of the previous section, in conjunction with the six term exact sequence in K-theory, to find the K-groups of $C_r^*(G)$, which we will show that depend only on the vertex, edge and tile patterns of the tiling. The details follows below.

At first we consider the exact sequence III.4 obtained by restricting G to the edges and considering the ideal of functions in $C_r^*(G|_{X_1})$ that vanish at the vertices, i.e.,

$$0 \to C_r^*(G|_{X_1 - X_0}) \xrightarrow{\iota} C_r^*(G|_{X_1}) \xrightarrow{\psi} C_r^*(G|_{X_0}) \to 0$$

where ι is the inclusion map and ψ is the restriction map.

This exact sequence induçes the six term exact sequence in K-theory below

$$\begin{tikzcd}
K_0\left(C_r^*(G|_{X_1-X_0})\right) \arrow[r, "K_0(\iota)"] & K_0\left(C_r^*(G|_{X_1})\right) \arrow[r, "K_0(\psi)"] & K_0\left(C_r^*(G|_{X_0})\right) \arrow[d, "\delta_0"] \\
K_1\left(C_r^*(G|_{X_0})\right) \arrow[u, "\delta_1"] & K_1\left(C_r^*(G|_{X_1})\right) \arrow[l, "K_1(\psi)"'] & K_1\left(C_r^*(G|_{X_1-X_0})\right) \arrow[l, "K_1(\iota)"']
\end{tikzcd} \tag{III.5}$$

where $K_0(\iota)$ and $K_0(\psi)$ are the induced maps in K-theory, δ_0 is the exponential map and δ_1 is the index map.

The computation of the K-groups above will heavily depend on the K-groups of the compact operators, $K_*(K)$, on $K_*(C_0((0,1), K))$ and on $K_*(C_0(U, K))$ where U is an open set of $\mathbb{R}^2$. So we state some facts about this groups below. We refer the reader to [24] and [31] for a proof of the statements below.

We start with K. We have that $K_0(K) \cong \mathbb{Z}$ and the isomorphism takes a projection in K and evaluates its dimension. Also $K_1(K) = 0$.

For $K_*(C_0((0,1), K))$ notice that $C_0((0,1), K))$ is isomorphic to the stabilization of $C_0(0,1)$ (for the reader familiar with tensor products notice that

$C_0((0,1),K)) \cong C_0(0,1) \otimes K)$ and hence the K-groups of $C_0((0,1),K))$ are isomorphic to the K-groups of $C_0(0,1)$, which we know are $K_0(C_0(0,1)) = 0$ and $K_1(C_0(0,1)) \cong \mathbb{Z}$. Furthermore the last isomorphism takes the K_1 class of a function that winds once in $(0,1)$ (for example $f(\lambda) = \exp(2\pi i\lambda)$ for $\lambda \in (0,1)$) to $1 \in \mathbb{Z}$.

Finally $C_0((U),K)$ is isomorphic to $C_0(U) \otimes K)$ and so we only need the K-groups of $C_0(U)$. If we denote the closed disk by $\mathbb{D}$ and the circle by $\mathbb{T}$ then it is clear that $C_0(U) \cong C_0(\mathbb{D} - \mathbb{T})$ and we have that $K_1(C_0(\mathbb{D} - \mathbb{T})) = 0$ and $K_0(C_0(\mathbb{D} - \mathbb{T})) \cong \mathbb{Z}$. Furthermore the last isomorphism takes 1 in $\mathbb{Z}$ to the famous Bott element, given by $\begin{bmatrix} |z|^2 & z(1-|z|^2)^{\frac{1}{2}} \\ \overline{z}(1-|z|^2)^{\frac{1}{2}} & 1-|z|^2 \end{bmatrix}_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$, where z denotes the identity function on the unit disc $\mathbb{D}$ in $\mathbb{C}$. Notice that if r is a homeomorphism from U to $\mathbb{D} - \mathbb{T}$ then 1 in $\mathbb{Z}$ is mapped to the "Bott element" in $K_0(C_0(U))$ given by $\begin{bmatrix} |zor|^2 & zor(1-|zor|^2)^{\frac{1}{2}} \\ \overline{zor}(1-|zor|^2)^{\frac{1}{2}} & 1-|zor|^2 \end{bmatrix}_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$.

Our last remark about the K-groups above is that once we are dealing with $A \otimes K$, we can define $K_*(A)$ directly using the projections and unitaries on $A \otimes K$ instead of the system of all matrix algebras over A. This happens because $A \otimes K$ already contain all matrix algebras over A. We refer the reader to [24] for more details.

We can now go back to the task of computing the K-groups on the six term exact sequence III.5. From propositions III.24 and III.25 we have that

$$C_r^*(G|_{X_0}) \cong \bigoplus_{[v]\in\mathcal{V}} K(l^2([v])) \text{ and } C_r^*(G|_{X_1-X_0}) \cong \bigoplus_{[e]\in\mathcal{E}} C_0\left(e, K(l^2([e]))\right)$$

and this implies that

$$K_0\left(C_r^*(G|_{X_0})\right) \cong \bigoplus_{\mathcal{V}} \mathbb{Z} \quad ; \quad K_1\left(C_r^*(G|_{X_1-X_0})\right) \cong \bigoplus_{\mathcal{E}} \mathbb{Z}$$

and

$$K_1\left(C_r^*(G|_{X_0})\right) = K_0\left(C_r^*(G|_{X_1-X_0})\right) = 0$$

Using the above isomorphisms we can rewrite the six term exact sequence III.5 as below:

$$\begin{tikzcd}
0 \arrow[r] & K_0\left(C_r^*(G|_{X_1})\right) \arrow[r] & \bigoplus_{\mathcal{V}} \mathbb{Z} \arrow[d, "\delta_0"'] \\
0 \arrow[u] & K_1\left(C_r^*(G|_{X_1})\right) \arrow[l] & \bigoplus_{\mathcal{E}} \mathbb{Z} \arrow[l]
\end{tikzcd} \tag{III.6}$$

And from exactness we get that

$$\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \cong \ker \delta_0 \quad \text{and} \quad \mathrm{K}_1\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \cong \frac{\oplus_{\mathcal{E}}\mathbb{Z}}{\mathrm{Im}.(\delta_0)}$$

So all we need to compute the K-theory of $C_r^*(\mathrm{G}|_{\mathrm{X}_1})$ is to find a description for the exponential map δ_0. This is our next proposition. But first we introduce some notation.

Notation III.27. *Given an edge e we write $i(e)$ for its initial point and $t(e)$ for its terminus point. This can be done since every edge has an orientation.*

Proposition III.28. *The exponential map δ_0 of III.6 is a group homomorphism from $\bigoplus_{\mathcal{V}} \mathbb{Z}$ in $\bigoplus_{\mathcal{E}} \mathbb{Z}$ which we can represent in a matrix form, denoted by $[\delta_0]$. The matrix is defined as follows. To find the value at the entry $([e],[v])$ of $[\delta_0]$ we take a representative of the vertex $[v]$, call it v, and look at the edges defining v. If there is a representative of $[e]$, call it e, such that the $i(e) = v$ then $[\delta_0]([e],[v]) = 1$. If $t(e) = v$ then $[\delta_0]([e],[v]) = -1$. If both $i(e) = v$ and $t(e) = v$ then $[\delta_0]([e],[v]) = 0$ and $[\delta_0]([e],[v]) = 0$ otherwise.*

A better description is obtained if we leave behind the bracket notation for equivalence classes and use the same notation for equivalence classes and its representatives. It should be clear from the context whether we are referring to one or the other. With this notation we have:

$$[\delta_0](e,v) = \begin{cases} 1 \text{ if } i(e) = v \\ -1 \text{ if } t(e) = v \\ 0 \text{ otherwise} \end{cases}$$

Proof:

The proof of the proposition is mostly an isomorphism chase, as we know how to evaluate δ_0 from $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_0})\right)$ to $\mathrm{K}_1\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1-\mathrm{X}_0})\right)$ and hence δ_0 from $\bigoplus_{\mathcal{V}} \mathbb{Z}$ to $\bigoplus_{\mathcal{E}} \mathbb{Z}$ is such that the diagram below commutes:

$$\begin{tikzcd}
\bigoplus_{\mathcal{V}} \mathbb{Z} \arrow[r, "\cong"] \arrow[d, dotted, "\delta_0 =?"'] & \bigoplus_{i=1..n} \mathrm{K}_0\left(\mathrm{K}(l^2([v_i]))\right) \arrow[r, "\cong"] & \mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_0})\right) \arrow[d, "\delta_0"'] \\
\bigoplus_{\mathcal{E}} \mathbb{Z} & \bigoplus_{i=1..k} \mathrm{K}_1\left(C_0\left(e_i, \mathrm{K}(l^2([e_i]))\right)\right) \arrow[l, "\cong"'] & \mathrm{K}_1\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1-\mathrm{X}_0})\right) \arrow[l, "\cong"']
\end{tikzcd} \tag{III.7}$$

Since δ_0 is a group homomorphism it is enough to show where the basis element $e_i \in \bigoplus_{\mathcal{V}} \mathbb{Z}$ is mapped by δ_0 (e_i is equal to 1 in the v_i coordinate and 0 in all others).

Observe that e_i is taken to $(0, \dots, 0, \underbrace{[\mathrm{E}]_0}_{v_i}, 0, \dots, 0)$ on $\mathrm{K}_0\left(\mathrm{K}(l_2[v_i])\right)$, where E is a one dimensional projection on $l_2[v_i]$. Let $[v_i] = v_i^0, v_i^1, v_i^2, \dots$ be an enumeration of $[v_i]$ and $\xi \in l_2[v_i]$. For simplicity we denote v_i^0 by v_i. Then

$$\mathrm{E}(\xi)(v_i^j) = \begin{cases} \xi(v_i^0) \ ; \ j = 0 \\ 0 \ \ j \neq 0 \end{cases}$$

Now from proposition III.24 we have that $(0, \dots, 0, \underbrace{[\mathrm{E}]_0}_{v_i}, 0, \dots, 0)$ is taken to $[f]_0 \in \mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G}_{\mathrm{X}_0})\right)$, where the function f is defined by

$$f(x, y) = \begin{cases} 1 \ \text{ if } \ (x, y) = (v_i^0, v_i^0) \\ 0 \ \text{ otherwise} \end{cases}$$

We now apply δ_0 to $[f]_0$. In order to do so we first need a self-adjoint lift for f. So we look at the pattern defined by v_i, i.e., $\mathrm{T}(v_i)$ and consider $\mathrm{T}(v_i) \cap \mathrm{X}_0 = \{v_i, x_0, \dots, x_n\}$. Let r_k be a parametrization of the edge e connecting v_i to x_k that agrees with the orientation of the edge, i.e., $r_k(0) = i(e)$ and $r_k(1) = t(e)$. Define

$$g(x, y) = \begin{cases} 1 - \lambda \ \text{ if } (x, y) = (r_k(\lambda), r_k(\lambda)) \text{ for } \lambda \in [0, 1],\ k \in \{0, .., n\} \text{ and } r_k(0) = v_i \\ \lambda \ \text{ if } (x, y) = (r_k(\lambda), r_k(\lambda)) \text{ for } \lambda \in [0, 1],\ k \in \{0, .., n\} \text{ and } r_k(1) = v_i \\ 0 \ \text{ otherwise} \end{cases}$$

So $g \in \widetilde{\mathrm{C}^*_r(\mathrm{G}_{\mathrm{X}_1})}$ is a self-adjoint lift for f and by the characterization of δ_0 given in proposition 12.2.2. of [24] we have that $\delta_0([f]_0) = -[u]_1$, where $u = \exp 2\pi i g$. We have reached the bottom line of the diagram III.7 and our next step is to follow the isomorphisms to find the image of $-[u]_1$ in $\bigoplus_{\mathcal{E}} \mathbb{Z}$. Before we proceed notice that u is isomorphic to $(u - 1, 1)$ in the unitization of $\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1 - \mathrm{X}_0})$, $\widetilde{\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1 - \mathrm{X}_0})}$ and hence we can use the isomorphism of proposition III.25 to map $(u - 1, 1)_1$ to $\left(\bigoplus_{\mathcal{E}} C_0\left(e_i, \mathrm{K}(l^2([e_i]))\right), \mathbb{C} \right)$.

Suppose all edges in $\mathrm{T}(v_i)$ are not equivalent. Then $(u - 1)(x, y) \neq 0$ on all edges in $\mathrm{T}(v_i)$ and $u - 1$ is mapped to $(u - 1|_{e_1}, u - 1|_{e_2}, \dots, u - 1|_{e_k})$. Observe that if no representative of an edge $[e_l]$ belongs to $\mathrm{T}(v_i)$ then $u-1|_{e_l}=0$. On the other hand if the edge e_l has a representative in $\mathrm{T}(v_i)$ then this representative is parametrized by one of the r_k above, say r_{l_0}, and we have that if $r_{l_0}(0)=v_i$ then

$$\begin{aligned}&(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)) = (\exp(-2\pi i\lambda)-1)\,\xi(r_{l_0}(\lambda))\\ &(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(y)=0 \text{ for all other coordinates } y\end{aligned}$$

where $\xi \in l_2[r_{l_0}(\lambda)]$ and $\lambda \in [0,1]$. Now if $r_{l_0}(1)=v_i$ then

$$\begin{aligned}&(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)) = (\exp(2\pi i\lambda)-1)\,\xi(r_{l_0}(\lambda))\\ &(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(y)=0 \text{ for all other coordinates } y\end{aligned}$$

where $\xi \in l_2[r_{l_0}(\lambda)]$ and $\lambda \in [0,1]$.

Suppose $r_{l_0}(1)=v_i$, i.e., $t(e_l)=v_i$. Observe that $(u-1)|_{e_l}$ belongs to the suspension of the compact operators, $\mathrm{S(K)}$, and so by the the Bott map isomorphism from $\mathrm{K_1(S(K))}$ into $\mathrm{K_0(K)}$ (see the remarks about K-theory before this proposition) we have that $[(u-1)|_{e_l}]_1$ is mapped to $1\in\mathbb{Z}$. This implies that $-[(u-1)|_{e_l}]_1$ is mapped to $-1\in\mathbb{Z}$.

Analogously if $r_{l_0}(0)=v_i$, i.e., $i(e_l)=v_i$ then $-[((u-1)|_{e_l},1)]_1$ is mapped to 1 in $\mathbb{Z}$ and the proposition follows (except for the case below).

Finally we have to consider the case when two edges in $\mathrm{T}(v_i)$ are equivalent. Suppose the edge $[e_l]$ has two representatives in $\mathrm{T}(v_i)$ and call then e_l and e_l+s. Then if $i(e_l)=v_i$ then $t(e_l+s)=v_i$ and analogous if $t(e_l)=v_i$ then $i(e_l+s)=v_i$. We assume without loss of generality that $i(e_l)=v_i$ and $t(e_l+s)=v_i$. Let r_{l_0} be a parametrization of e_l. Then $r_{l_0}+s$ is a parametrization of e_l+s. Now for $\xi\in l_2[r_{l_0}(\lambda)]$ and $\lambda\in[0,1]$ we have that

$$\begin{aligned}&(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)) = (-\exp(2\pi i\lambda)-1)\,\xi(r_{l_0}(\lambda))\\ &(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)+s) = (\exp(2\pi i\lambda)-1)\,\xi(r_{l_0}(\lambda))\\ &(u-1)|_{e_l}(r_{l_0}(\lambda))(\xi)(y)=0 \text{ for all other coordinates } y\end{aligned}$$

We will show that $[(u-1|_{e_l},1)]_1=[(0,1)]_1=[1]_1=0$ in K_1 and for this we only need to show that $(u-1|_{e_l},1)$ is homotopic to $(0,1)$. We use the idea of the Whitehead lemma, see [24], and define

$$h(t)(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)) = \begin{array}{l}\left(-\exp(2\pi i\lambda)\cos^2(t\tfrac{\pi}{2})+\sin^2(t\tfrac{\pi}{2})-1\right)\xi(r_{l_0}(\lambda))+\\ +\left(\sin(t\tfrac{\pi}{2})\cos(t\tfrac{\pi}{2})-\exp(2\pi i\lambda)\sin(t\tfrac{\pi}{2})\cos(t\tfrac{\pi}{2})\right)\xi(r_{l_0}(\lambda)+s)\end{array}$$

$$h(t)(r_{l_0}(\lambda))(\xi)(r_{l_0}(\lambda)+s) = \begin{array}{l}\left(-\sin(t\tfrac{\pi}{2})\cos(t\tfrac{\pi}{2})+\exp(2\pi i\lambda)\sin(t\tfrac{\pi}{2})\cos(t\tfrac{\pi}{2})\right)\xi(r_{l_0}(\lambda))+\\ +\left(\exp(2\pi i\lambda)\cos^2(t\tfrac{\pi}{2})+\sin^2(t\tfrac{\pi}{2})-1\right)\xi(r_{l_0}(\lambda)+s)\end{array}$$

$$h(t)(r_{l_0}(\lambda))(\xi)(y)=0 \text{ for all other coordinates } y$$

for $\xi \in l_2[r_{l_0}(\lambda)]$, $\lambda \in [0,1]$ and $t \in [0,1]$.

Observe that $h(0) = (u-1)|_{e_l}$, $h(1) = 0$ and $h(t)$ is a unitary in $\mathrm{C}_0(e_l, \mathrm{K}(l_2[e_l]))$ and hence $(h,1)$ is the homotopy we are looking for.

■

Now that we have a good description of $\mathrm{K}_*\left(\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1})\right)$ we can look for a description of the K-theory of $\mathrm{C}^*_r(\mathrm{G})$. The idea is very similar to what we did above. We start by considering the exact sequence obtained by including the functions in G that vanish at the edges in $\mathrm{C}^*_r(\mathrm{G})$ and then restricting $\mathrm{C}^*_r(\mathrm{G})$ to the functions on the edges, i.e, we consider the exact sequence III.3,

$$0 \to \mathrm{I}_{\mathrm{X}_1} \to C^*_r(\mathrm{G}) \to C^*_r(\mathrm{G}|_{\mathrm{X}_1}) \to 0$$

This exact sequence induces the six term exact sequence in K-theory below

$$\begin{tikzcd}
\mathrm{K}_0\left(\mathrm{I}_{\mathrm{X}_1}\right) \arrow[r, "\mathrm{K}_0(\iota)"] & \mathrm{K}_0\left(C^*_r(\mathrm{G})\right) \arrow[r, "\mathrm{K}_0(\psi)"] & \mathrm{K}_0\left(C^*_r(\mathrm{G}|_{\mathrm{X}_1})\right) \arrow[d, "\delta_0"] \\
\mathrm{K}_1\left(C^*_r(\mathrm{G}|_{\mathrm{X}_1}))\right) \arrow[u, "\delta_1"] & \mathrm{K}_1\left(C^*_r(\mathrm{G})\right) \arrow[l, "\mathrm{K}_1(\psi)"'] & \mathrm{K}_1\left(\mathrm{I}_{\mathrm{X}_1}\right) \arrow[l, "\mathrm{K}_1(\iota)"']
\end{tikzcd} \tag{III.8}$$

where $\mathrm{K}_0(\iota)$ and $\mathrm{K}_0(\psi)$ are the induced maps in K-theory, δ_0 is the exponential map and δ_1 is the index map.

From proposition III.26 we have that

$$\mathrm{I}_{\mathrm{X}_1} \cong \bigoplus_{[p]\in\mathcal{P}} C_0\left(p, \mathrm{K}(l^2([p]))\right)$$

and this implies that

$$\mathrm{K}_0\left(\mathrm{I}_{\mathrm{X}_1}\right) \cong \bigoplus_{\mathcal{P}} \mathbb{Z} \quad \text{and} \quad K_1\left(\mathrm{I}_{\mathrm{X}_1}\right) = 0$$

Using the above isomorphism and the description of the K-theory groups of $C^*_r(\mathrm{G}|_{\mathrm{X}_1})$ obtained before we can rewrite the six term exact sequence III.8

as below:

$$\begin{tikzcd}
\bigoplus_{\mathcal{P}} \mathbb{Z} \arrow[r, "\mathrm{K}_0(\iota)"] & \mathrm{K}_0\left(C_r^*(\mathrm{G})\right) \arrow[r, "\mathrm{K}_0(\psi)"] & \ker(\delta_0) \arrow[d] \\
\dfrac{\bigoplus_{\mathcal{E}} \mathbb{Z}}{\mathrm{Im}(\delta_0)} \arrow[u, "\delta_1"] & \mathrm{K}_1\left(C_r^*(\mathrm{G})\right) \arrow[l] & 0 \arrow[l]
\end{tikzcd} \qquad (\mathrm{III.9})$$

where δ_0 is the exponential map associated to the exact sequence III.6 as described in proposition III.28.

From exactness we get that $\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \frac{\ker \delta_0}{\mathrm{Im}(\mathrm{K}_0(\iota))}$. Since $\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right)$ is an abelian group and $\ker \delta_0$ is isomorphic to a free abelian group we have that $\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \ker \delta_0 \oplus \mathrm{Im}(\mathrm{K}_0(\iota))$. Still from exactness of the six term sequence above we have that $\mathrm{Im}(\mathrm{K}_0(\iota)) \cong \frac{\oplus_{\mathcal{P}}\mathbb{Z}}{\mathrm{Im}(\delta_1)}$ and we conclude that $\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \ker \delta_0 \oplus \frac{\oplus_{\mathcal{P}}\mathbb{Z}}{\mathrm{Im}(\delta_1)}$. Also $\mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \ker(\delta_1)$.

In order to describe the K-theory of $C_r^*(\mathrm{G})$ any further we need to understand the index map δ_1. That is our next proposition.

Definition III.29. *Let e be an edge and p a tile in* T. *Then* $< e, p >= 1$ *if the counterclockwise orientation of the boundary of p matches the orientation of e and* $< e, p >= -1$ *if the counterclockwise orientation of the boundary of p is contrary to the orientation of e. If none of the equivalent edges to e are contained in p then* $< e, p >= 0$.

Proposition III.30. *The index map* $\delta_1 : \dfrac{\bigoplus_{\mathcal{E}} \mathbb{Z}}{Im(\delta_0)} \to \bigoplus_{\mathcal{P}} \mathbb{Z}$ *of the six term exact sequence III.9 is a group homomorphism, that maps the generator element* $\overline{f_e} = (0, 0, \ldots, 0, \underbrace{1}_{e}, 0, \ldots, 0)$, *i.e., the generator element that is one at the coordinate* $[e]$ *and 0 otherwise, into an element* $(a_i)_{i=1..m}$ *defined as follows. First we fix a representative of the edge equivalence class* $[e]$, *say e, and look at the two tiles that define this edge. These tiles belong to some tile equivalence classes, say* $[p_j]$ *and* $[p_k]$. *If* $[p_j] = [p_k]$ *then* $a_i = 0$ *for all i. Otherwise we define* $a_j =< e, p_j >$, $a_k =< e, p_k >$ *and* $a_i = 0$ *for all other* $i \neq k; i \neq j$.

Proof:

Let $\overline{f_e}$ be one of the generators of $\dfrac{\bigoplus_{\mathcal{E}} \mathbb{Z}}{\mathrm{Im}(\delta_0)}$. By the isomorphism between $\dfrac{\bigoplus_{\mathcal{E}} \mathbb{Z}}{\mathrm{Im}(\delta_0)}$ and $\mathrm{K}_1(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1}))$, given by the six term exact sequence III.6, and the isomorphism chase done in proposition III.28, we know that $\overline{f_e}$ is taken to $[f]_1 \in \mathrm{K}_1(\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1}))$, where f is defined bellow:

$$f(x,y) = \begin{cases} \exp(2\pi i\lambda) \text{ if } (x,y) = (r_e(\lambda), r_e(\lambda)) \text{ , for } \lambda \in [0,1] \\ 1 \text{ if } x = y \text{ and } (x,x) \text{ not as above} \\ 0 \text{ otherwise} \end{cases}$$

where r_e is a parametrization of the edge e such that $r_e(0) = i(e)$ and $r_e(1) = t(e)$.

Let p_j and p_k be two tiles defining the edge e. Without loss of generality assume that the orientation of p_j matches the orientation of e and that p_k has contrary orientation to e (since all tiles are oriented counterclockwise we always have matching and contrary orientations).

We will use the description of the index map given by proposition 9.1.4 of [24]. So we need to find an unitary $v \in \mathrm{M}_2(\mathrm{C}^*_r(\mathrm{G}))$ such that $v|_{\mathrm{G}|\mathrm{X}_1} = \begin{pmatrix} f & 0 \\ 0 & f^* \end{pmatrix}$.

Let r_{p_j} be a map from the tile p_j onto the disk $\mathbb{D}$ such that $r_{p_j}(r_e(0)) = 1$, $r_{p_j}(r_e(\lambda)) = \exp(2\pi i\lambda)$ for $\lambda \in [0,1]$, r_{p_j} is a homeomorphism from the interior of p_j into the interior of $\mathbb{D}$ and $r_{p_j}(x) = 1$ for any other x in an edge different from e of p_j. Similarly let r_{p_k} be a map from the tile p_k onto the disk $\mathbb{D}$ such that $r_{p_k}(r_e(0)) = 1$, $r_{p_k}(r_e(\lambda)) = \exp(-2\pi i\lambda)$ for $\lambda \in [0,1]$, r_{p_k} is a homeomorphism from the interior of p_k into the interior of $\mathbb{D}$ and $r_{p_k}(x) = 1$ for any other x in an edge different from e of p_k. Notice that r_{p_j} and r_{p_k} agree on e.

Now let z denote the identity map on $\mathbb{D}$. We define

$$v(x,y) = \begin{cases} \begin{pmatrix} z \circ r_{p_j}(x) & (1 - |z \circ r_{p_j}(x)|^2)^{\frac{1}{2}} \\ (1 - |z \circ r_{p_j}(x)|^2)^{\frac{1}{2}} & \overline{z} \circ r_{p_j}(x) \end{pmatrix} & \text{if } x = y \text{ and } x \in p_j \\ \begin{pmatrix} \overline{z} \circ r_{p_k}(x) & (1 - |z \circ r_{p_k}(x)|^2)^{\frac{1}{2}} \\ (1 - |z \circ r_{p_k}(x)|^2)^{\frac{1}{2}} & z \circ r_{p_k}(x) \end{pmatrix} & \text{if } x = y \text{ and } x \in p_k \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x = y \text{ and } (x,x) \text{ not in } p_j \text{ or } p_k \\ 0 \text{ otherwise} \end{cases}$$

We then have that if $p = v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^*$ then $\delta_1([f]_1) = [p]_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$ (see proposition 9.1.4 and exercise 9.3 of [24]).

Notice that if $x = y$ and $x \in p_j$ then

$$p(x,y) = \begin{pmatrix} z \circ r_{p_j}(x) & -z \circ r_{p_j}(x)(1 - |z \circ r_{p_j}(x)|^2)^{\frac{1}{2}} \\ -\overline{z} \circ r_{p_j}(x)(1 - |z \circ r_{p_j}(x)|^2)^{\frac{1}{2}} & \overline{z} \circ r_{p_j}(x) \end{pmatrix}$$

analogous if $x = y$ and $x \in p_k$ then

$$p(x,y) = \begin{pmatrix} \overline{z}\circ r_{p_k}(x) & -\overline{z}\circ r_{p_k}(x)(1-|z\circ r_{p_k}(x)|^2)^{\frac{1}{2}} \\ -z\circ r_{p_k}(x)(1-|z\circ r_{p_k}(x)|^2)^{\frac{1}{2}} & z\circ r_{p_k}(x) \end{pmatrix}$$

and $p(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $x = y$ and (x,x) not in p_j or p_k and 0 otherwise.

Observe that p is equal to zero on the off-diagonal points. So we can think of p as a projection on the underlying space $\mathbb{R}^d$, via the embedding of $\mathrm{C}_0(\mathbb{R}^d)$ into $\mathrm{C}_c(\mathrm{G})$ explained right after observation II.20.

Next we need to follow the isomorphism of proposition III.26 to see where $[p]_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$ is mapped. But we can recognize $[p]_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$ as the Bott element on p_j and the conjugate of the Bott element on p_k. This implies that if p_j and p_k are not equivalent then $\delta_1([f]_1) = (a_i) \in \oplus_{\mathcal{P}}\mathbb{Z}$ where $a_{p_j} = 1$, $a_{p_k} = -1$ and $a_{p_i} = 0$ for all other p_i different of p_j and p_k. If p_j and p_k are equivalent then $\delta_1([f]_1) = 0$.

■

Observation III.31. *A matrix representation of δ_1 was not given in the proposition because in general we do not know what are the basis elements of the quotient $\dfrac{\bigoplus_{\mathcal{E}}\mathbb{Z}}{Im(\delta_0)}$. But in the examples, where we know the basis of this quotient, a matrix for δ_1 can be easily derived from the description above.*

Observation III.32. *If one wants to pursue the a matrix representation for δ_1 this can be obtained in the following manner. Let $\tilde{\delta_1} : \bigoplus_{\mathcal{E}}\mathbb{Z} \to \bigoplus_{\mathcal{P}}\mathbb{Z}$ be the matrix defined by*

$$\tilde{\delta_1}([p],[e]) = \begin{cases} 0 \ \textit{if the edge } e \textit{ is defined by two tiles in } [p] \\ < e, p > \quad \textit{otherwise.} \end{cases}$$

Then $Im(\tilde{\delta_1}) = Im(\delta_1)$ and $\mathrm{K}_1(\mathrm{C}^*_r(\mathrm{G})) \cong \frac{\ker(\tilde{\delta_1})}{Im(\delta_0)}$.

As a consequence of this proposition we get the following corollary:

Corollary III.33. *$\frac{\oplus_{\mathcal{P}}\mathbb{Z}}{Im(\delta_1)} \cong \mathbb{Z}$ and hence $\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G})\right) \cong \ker\delta_0 \oplus \mathbb{Z}$.*

Proof:

Let $\{p_i\}_{i=1}^m$ be the set of prototiles and denote the canonical generators of $\bigoplus\limits_{\mathcal{P}}\mathbb{Z}$ by e_{p_i} for $1 \le i \le m$. Then a set of generators for $\frac{\oplus_{\mathcal{P}}\mathbb{Z}}{\mathrm{Im}(\delta_1)}$ is $\{\overline{e_{p_i}}\}_{i=1}^m$.

We will show that for every $1 \leq i, j \leq m$, $\overline{e_{p_i}} = \overline{e_{p_j}}$.

Suppose p_i and p_j share and edge, say e_{ij}, in the tiling. By this we mean that there exists some translations of p_i and p_j that share the edge e_{ij}. Let $\overline{f_{ij}}$ be the element of $\dfrac{\bigoplus_{\mathcal{E}} \mathbb{Z}}{\mathrm{Im}(\delta_0)}$ defined as one at the coordinate $[e_{ij}]$ and 0 otherwise. Then $\delta_1(\overline{f_{ij}}) = \pm \left(e_{p_i} - e_{p_j}\right)$ so that $e_{p_i} - e_{p_j} \in \mathrm{Im}(\delta_1)$ and hence e_{p_i} is equivalent to e_{p_j}.

If p_i and p_j do not share and edge then we can find a path, call it π, of tiles in T, connecting p_i to p_j. Now from transitivity of the equivalence relation on $\frac{\oplus_{\mathcal{P}} \mathbb{Z}}{\mathrm{Im}(\delta_1)}$ and the proved above we have that $e_{p_i} \sim e_{p_j}$ (notice that e_{p_i} is equivalent to e_p, where p is the adjacent tile in π, and so on until we reach e_{p_j}).

So we proved that $\frac{\oplus_{\mathcal{P}} \mathbb{Z}}{\mathrm{Im}(\delta_1)}$ is isomorphic to the subgroup generated by $\overline{e_{p_1}}$. We still need to show that $\overline{e_{p_1}} \neq 0$, i.e., that $e_{p_1} \notin \mathrm{Im}(\delta_1)$. We prove by contradiction. Suppose that $e_{p_1} \in \mathrm{Im}(\delta_1)$. Then δ_1 is onto, what implies that $\mathrm{K}_0(\iota)$ is the zero map and hence $\mathrm{K}_0(\psi)$ is 1-1. We now define an element in $\ker(\mathrm{K}_0(\psi))$.

Let r_{p_1} be a homeomorphism from the tile p_1 into the disc $\mathbb{D}$ such that the border of p_1 is taken to the circle. Now for $(x, y) \in \mathrm{G}_0$ define

$$p(x,y) = \begin{pmatrix} z \circ r_{p_1}(x) & -z \circ r_{p_1}(x)(1 - |z \circ r_{p_1}(x)|^2)^{\frac{1}{2}} \\ -\overline{z} \circ r_{p_1}(x)(1 - |z \circ r_{p_1}(x)|^2)^{\frac{1}{2}} & \overline{z} \circ r_{p_1}(x) \end{pmatrix}$$

if $x = y$ and $x \in p_1$; $p(x, y) = 1$ if $x = y$ and $x \notin p_1$ and $p(x, y) = 0$ otherwise. Then $[p]_0 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_0$ is on the kernel of $\mathrm{K}_0(\psi)$ and it is not the zero element (notice the Bott element on the tile p_1).

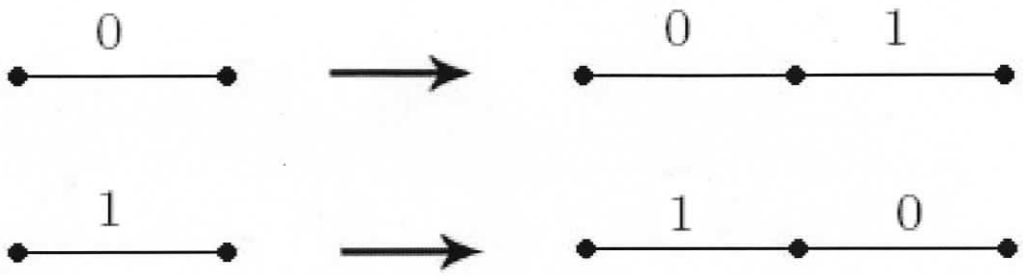
Finally observe that $\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G})\right)$ is an abelian group and $\ker \delta_0$ is a free abelian group on a finite set of generators. So $\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G})\right) \cong \ker \delta_0 \oplus \ker(\mathrm{K}_0(\psi))$ and since $\ker(\mathrm{K}_0(\psi)) \cong \mathrm{Im}(\mathrm{K}_0(\iota)) \cong \frac{\oplus_{\mathcal{P}} \mathbb{Z}}{\mathrm{Im}(\delta_1)}$ we have that

$$\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G})\right) \cong \ker \delta_0 \oplus \mathbb{Z}$$

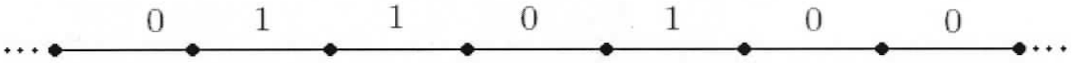
■

III.3 The Thue Morse Tiling example

The Morse tiling is given by the substitution represented below, where the segments have length 1 and are inflated by 2. Observe that since both segments have the same length, labels are needed to distinguish between then.

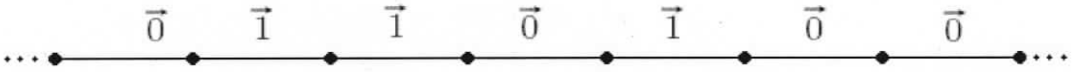


And we can tile the line as show below



Observe that in the one dimensional case $\mathrm{C}_r^*(\mathrm{G}) = \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$ and we only need to use the six term exact sequence III.6 to compute the K-theory groups.

Remember that in order to use the results of section III.2 we need to give an orientation to all edges in the tiling. We do so by giving all edges the same orientation, to the right, as shown below.



We can easily see that there are four equivalence classes of vertices, namely $v_1 = \vec{0} \bullet \vec{0}$, $v_2 = \vec{0} \bullet \vec{1}$, $v_3 = \vec{1} \bullet \vec{0}$ and $v_4 = \vec{1} \bullet \vec{1}$. Also there are two edge equivalence classes, namely $e_1 = \vec{0}$ and $e_2 = \vec{1}$. So the six term exact sequence III.6 becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \arrow[r] & \mathbb{Z}^4 \arrow[d, "\delta_0"'] \\
0 \arrow[u] & \mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \arrow[l] & \mathbb{Z}^2 \arrow[l]
\end{tikzcd}$$

where δ_0 is the matrix given by

$$[\delta_0] = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

as described in proposition III.28.

The vectors $(1,0,0,0)$, $(0,1,1,0)$ and $(0,0,0,1)$ generate the kernel of δ_0 and $(1,-1)$ generates the image of δ_0. So we can conclude that

$$\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \ker(\delta_0) \cong \mathbb{Z}^3 \quad \text{and} \quad \mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)} \cong \mathbb{Z}$$

III.4 K_0 as an ordered group for the 1 dimensional case

In this section we will completely characterize the positives elements in $\mathrm{C}_r^*(\mathrm{G})$ when T is a tiling covering $I\!\!R$, i.e., when the equivalence relation restricted to the edges is in fact the whole equivalence relation. We remind the reader that in this case we only need (actually only have) one six-term exact sequence, III.4, to compute the K-groups. Our motivation to do this follows from the fact that K_0 as an ordered abelian group is a much better invariant for C*-algebras.

We start by reminding the reader of the definition of the positive cone of $\mathrm{K}_0(\mathrm{A})$.

Definition III.34. *Let* A *be a C*-algebra. The positive cone of* $\mathrm{K}_0(\mathrm{A})$ *is the set* $\{[p]_0 : p \in \mathrm{P}_\infty(\mathrm{A})\}$. *We denote this set by* $\mathrm{K}_0(\mathrm{A})^+$.

To make the discussion more clear we repeat below the exact sequence III.6.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \arrow[r, "\mathrm{K}_0(\psi)"] & \bigoplus_{\mathcal{V}} \mathbb{Z} \arrow[d, "\delta_0"'] \\
0 \arrow[u] & \mathrm{K}_1\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right) \arrow[l] & \bigoplus_{\mathcal{E}} \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Since $\mathrm{K}_0(\psi)\left([p]_0)\right) = [\psi(p)]_0$ we have that $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+$ is "contained" in $(\ker\delta_0)^+$. Furthermore since $\mathrm{K}_0(\psi)$ is injective this is actually an inclusion. We will prove that $\mathrm{K}_0(\psi)$ is actually an order isomorphism, and for that we only need to show that a positive element in $(\ker\delta_0)$ can be lifted to a positive in

$\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)$ (i.e., $\mathrm{K}_0(\psi)$ is surjective). Once we have this it is enough to characterize the positive elements in $(\ker\delta_0)^+$ to get a complete picture of $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+$. We do the work below.

Proposition III.35. *Under the isomorphism of* $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)$ *with* $(\ker\delta_0)$, *the positive cone,* $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+$ *is identified with* $(\ker\delta_0)^+$ *which is defined as:*

$$(\ker\delta_0)^+ = \ker\delta_0 \cap \left(\bigoplus_{\mathcal{V}}\mathbb{Z}\right)^+.$$

Proof:

We have to lift a positive element in $\ker\delta_0$ to a positive element in $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+$. So let $(b_1, b_2, \ldots, b_n) = a_1u_1 + \ldots + a_ku_k$ be an element in $(\ker\delta_0)^+$. Observe that $b_i \geq 0$ for all $i = 1, ..n$.

Our first step is to find where $(b_1, b_2, \ldots, b_n)$ is mapped under the isomorphism from $\ker\delta_0$ to $\mathrm{K}_0(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}))$. For this we will choose a rather special representative of the equivalence class in K_0 that is isomorphic to $(b_1, b_2, \ldots, b_n)$. Details follows.

For each vertex equivalence class v_i choose b_i representatives of v_i, so that they do not intersect (so there is a gap between then). Name this representatives by $v_i + s_1^i, v_i + s_2^i, \ldots, v_i + s_{b_i}^i$ and define

$$f(x,y) = \begin{cases} 1; & \text{if } (x,y) = (v_i + s_j^i, v_i + s_j^i), \text{ for } i = 1,..,n \text{ and } j = 1..b_i \\ 0; & \text{otherwise} \end{cases}$$

It is clear that f is a projection and that $[f]_0$ is taken to $(b_1, b_2, \ldots, b_n)$. We now have an element in $\mathrm{K}_0(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}))$!!

Our next task is to lift $[f]_0$ to $\mathrm{K}_0\left(C_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)$. For this we need the lemma below.

By an edge e in the representation of f we mean an edge such that either $t(e)$ or $i(e)$ is equal to some vertex $v_i + s_j^i$ for some $i = 1,..,n$, $j = 1,..,b_i$. By an edge equivalence class in the representation of f we mean an equivalence class that contain an edge in the representation of f.

Lemma III.36. *Let* $\left\{v_i + s_j^i\right\}_{i=1..n}^{j=1..b_i}$ *as above. Then for any edge equivalence class* $[e]$, *the number of times* $i(e) \in \left\{v_i + s_j^i\right\}_{i=1..n}^{j=1..b_i}$, *for some edge* e *in the representation of* f *and* $e \in [e]$, *is equal to the number of times* $t(e) \in \left\{v_i + s_j^i\right\}_{i=1..n}^{j=1..b_i}$ *for some edge* e *in the representation of* f *and* $e \in [e]$.

Proof:

Very shortly the lemma follows from the information on the lines of the matrix for δ_0 and the fact that $(b_1, \ldots, b_n) \in (\ker \delta_0)^+$. We elaborate more below.

We denote the number of times $i(e) \in \left\{v_i + s_j^i\right\}_{i=1..n}^{j=1..b_i}$ by $Ni(e)$, and the number of times $t(e) \in \left\{v_i + s_j^i\right\}_{i=1..n}^{j=1..b_i}$ by $Nt(e)$.

Since $(b_1, \ldots, b_n) \in (\ker \delta_0)^+$ we have that each $b_i \geq 0$ and

$$\sum_{i=1..n} \delta_0(e, v_i) b_i = 0$$

for any fixed edge equivalence class $[e]$.

We can rewrite the above equation as

$$\sum_{i:\delta_0(e,v_i)>0} \delta_0(e, v_i) b_i + \sum_{i:\delta_0(e,v_i)<0} \delta_0(e, v_i) b_i = 0$$

and hence

$$\sum_{i:\delta_0(e,v_i)>0} \delta_0(e, v_i) b_i = - \sum_{i:\delta_0(e,v_i)<0} \delta_0(e, v_i) b_i \tag{III.10}$$

Finally observe that $\delta_0(e, v) = 0$ iff both $i(e)$ and $t(e)$ are different from v_{i_0} or $i(e) = t(e) = v_{i_0}$. So for each v_{i_0} such that $\delta_0(e, v_{i_0}) = 0$ and $i(e) = t(e) = v_{i_0}$ we add b_{i_0} to both sides of equation III.10 and get $Ni(e)$ and $Nt(e)$ respectively.

■

From the lemma the number of edges equivalent to e in the representation of f is even, and we can write the edges as $e + w_1^e, e + w_2^e, \ldots, e + w_{2m_e}^e$ where

$$\begin{aligned} t(e + w_l^e) &= v_i + s_j^i \quad \text{for } l \text{ even} \\ i(e + w_l^e) &= v_i + s_j^i \quad \text{for } l \text{ odd} \end{aligned}$$

for some $i = 1, .., n$, $j = 1, .., b_i$.

From the choice of vertex equivalence classes representatives it is clear that we can write the edges in the representation of f as a finite disjoint union.

Summarizing for each edge equivalence class $[e]$ in the representation of f we have chosen a representative e and written the equivalent edges in the representation of f in an appropriate manner, namely $e + w_1^e, e + w_2^e, \ldots, e + w_{2m_e}^e$.

For each representative e as in the paragraph above let r_e be a parametrization of e such that $r_e(0) = i(e)$ and $r_e(1) = t(e)$. We want to define functions

$g_e : e \longrightarrow M_{2m_e}(\mathbb{C})$. To do this let $\lambda \in [0,1]$ and define $g_e\left(r_e(\lambda)\right)$ by the matrix below

$$\begin{array}{r|ccccc}
 & e+w_1^e & e+w_2^e & \cdots & & e+w_{2m_e}^e \\
\hline
-e+w_1^e- & 1-\lambda & \sqrt{\lambda(1-\lambda)} & & & \\
-e+w_2^e- & \sqrt{\lambda(1-\lambda)} & \lambda & & & \\
\vdots & & & \ddots & & \\
 & & & & 1-\lambda & \sqrt{\lambda(1-\lambda)} \\
-e+w_{2m_e}^e- & & & & \sqrt{\lambda(1-\lambda)} & \lambda
\end{array}$$

Observe that $g_e\left(r_e(\lambda)\right)$ is a direct sum of blocks of 2 by 2 matrices and hence the only non-zero entries of the matrix are on the diagonal and secondary diagonals.

We can now finally define a function $\tilde{f} \in \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$ such that $[\tilde{f}]_0$ lifts $[f]_0$ as desired. Let $\tilde{f}(x,y) = g_e\left(r_e(\lambda)\right)[k,l]$ if $(x,y) = (r_e(\lambda) + w_k^e, r_e(\lambda) + w_l^e)$, for some $k, l \in 1, .., m_e$ and some edge e as above and let $\tilde{f}(x,y) = 0$ otherwise. Since $\tilde{f}$ is a projection we have that $[\tilde{f}]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+$ and it follows that

$$\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})\right)^+ \cong (\ker\delta_0)^+$$

■

III.5 $\mathrm{C}_r^*(\mathrm{G})$ as a Recursive Subhomogeneous C*-algebra

The notion of a recursive subhomogeneous C*-algebra, was introduced by N. C. Phillips in [19]. In this section we will describe $\mathrm{C}_r^*(\mathrm{G})$ as a recursive subhomogeneous algebra. But we will need a slight generalization of the notion introduced in [19]. Basically we will have to replace continuous functions taking values on the nxn matrices by continuous functions taking values on the compact operators on some l_2 space. This is a rather natural generalization, but doing this we get C*-algebras that are non-unital. This means we will not be able to apply the results about recursive subhomogeneous C*-algebras of [19] for $\mathrm{C}_r^*(\mathrm{G})$. But we expect that this new description of $\mathrm{C}_r^*(\mathrm{G})$ will help with the classification of these C*-algebras (see [25], [13] and [19] for references on Elliott's classification program). Also the ideas used in this section were very usefull in the computation of traces done in the next chapter and hopefully the results in [19] will be generalized in the future. Below we introduce the definition of a recursive subhomogeneous C*-algebra, RSA, already modified to fit our needs.

Definition III.37. *Let* X *be a compact Hausdorff space. Then,*

1. $\mathrm{C}\left(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])\right)$ *is an RSA.*
2. *If* A *is RSA;* $\mathrm{X}^{(0)} \subseteq \mathrm{X}$*;* $\mathrm{X}^{(0)}$ *is closed;* $\varphi : \mathrm{A} \to \mathrm{C}\left(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}])\right)$ *is any homomorphism, and* $\rho : \mathrm{C}\left(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])\right) \to \mathrm{C}\left(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}])\right)$ *is the restriction homomorphism, then the pull back*

   $$A \oplus_{\mathrm{C}\left(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}])\right)} \mathrm{C}\left(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])\right) = \left\{(a, f) \in \mathrm{A} \oplus \mathrm{C}\left(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])\right) : \varphi(a) = \rho(f)\right\}$$

   is a RSA.

Observation III.38. $\mathrm{X}^{(0)} = \emptyset$ *is allowed in which case the pull back is the ordinary direct sum.*

Example III.39. *Any direct sum of RSA is an RSA.*

We will show that $\mathrm{C}_r^*\left(\mathrm{G}\right)$ is an RSA. We start by showing that $\mathrm{C}_r^*\left(\mathrm{G}|_{\mathrm{X}_1}\right)$ is isomorphic to an RSA which we construct below.

From lemma III.22 we know that there is only a finite number of edge equivalence classes. So we can write the set of edges, E, as

$$E = \left\{e_j + u_l^j : j = 1..k; l = 0, 1, 2, \ldots\right\}$$

where each u_l^j is a vector in $\mathbb{R}^n$, $u_0^j = 0$ for every j and each e_j is a representative of one edge equivalence class.

Consider the set $[0,1] \times E$ and identify each point $(t, e_j + u_l^j)$ in this set with the point $r_{e_j}(t) + u_l^j$, where r_{e_j} is a parametrization of the edge e_j. On $[0,1] \times E$ introduce the equivalence relation $\Delta\times \sim$, where $\sim$ is the equivalence relation on the edges (so $e_j + u_l^j$ is equivalent to any other translation $e_j + u_h^j$). Observe that $(0, e_j + u_l^j)$ and $(1, e_j + u_l^j)$ are the endpoints of the edge $e_j + u_l^j$ and we also consider the set $\{0,1\} \times E$ with the equivalence relation $\Delta\times \sim$. Also notice that $\mathrm{C}_r^*([0,1] \times E, \Delta\times \sim)$ is isomorphic to $\bigoplus_{j=1..k} \mathrm{C}\left([0,1], \mathrm{K}(l_2([e_j]))\right)$ and $\mathrm{C}_r^*(\{0,1\} \times E, \Delta\times \sim)$ is isomorphic to $\bigoplus_{j=1..k} \mathrm{C}\left(\{0,1\}, \mathrm{K}(l_2([e_j]))\right)$ and hence the restriction map ρ from $\mathrm{C}_r^*([0,1] \times E, \Delta\times \sim)$ onto $\mathrm{C}_r^*(\{0,1\} \times E, \Delta\times \sim)$ is defined as required by definition III.37.

Now observe that $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0})$ is an RSA, since from proposition III.24 $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}) \cong \bigoplus_{v \in \mathcal{V}} \mathrm{K}(l^2([v]))$, and let $\varphi : \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_0}) \to \mathrm{C}_c(\{0,1\} \times E, \Delta\times \sim)$ be the homomorphism defined by

$$\varphi(f)\left((i, e_j + u_l^j), (i, e_j + u_h^j)\right) = f\left(r_{e_j}(i) + u_l^j, r_{e_j}(i) + u_h^j\right)$$

if $\left(r_{e_j}(i) + u_l^j, r_{e_j}(i) + u_h^j\right) \in \mathrm{G}$, $i \in \{0,1\}$ and to be equal to 0 otherwise.

The homomorphism φ above can be extended to a homomorphism from $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0})$ onto $\mathrm{C}_r^*(\{0,1\} \times E, \Delta\times \sim)$. We will still denote the extension by φ as this should not bring any confusion. Also a proof that φ can be extend (is isometric) is ommited here, as this proof is analogous to the proof that the homomorphism α_2 defined bellow is isometric (with $t \in \{0,1\}$).

With this setting we can consider the pull back C*-algebra,

$$\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_r^*(\{0,1\}\times E, \Delta\times\sim)} \mathrm{C}_r^*([0,1] \times E, \Delta\times \sim)$$

which by definition III.37 is an RSA.

We will show that the pull back C*-algebra above is isomorphic to $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$. For this we need a lemma:

Lemma III.40. $\ker\rho \cap \mathrm{C}_c([0,1] \times E, \Delta\times \sim)$ *is dense in* $\ker\rho$.

Proof:

From proposition III.14 we know that there exists a closed set $\mathrm{Q} \subseteq ([0,1] \times E, \Delta\times \sim)$ such that $\ker\rho = \overline{\{f \in C_c(([0,1] \times E, \Delta\times \sim)) : f|_{\mathrm{Q}} = 0\}}$. With this characterization of $\ker\rho$ it is clear that $\ker\rho \cap \mathrm{C}_c([0,1] \times E, \Delta\times \sim)$ contains $\{f \in C_c(([0,1] \times E, \Delta\times \sim)) : f|_{\mathrm{Q}} = 0\}$ and hence the lemma follows.

■

We are now able to show the proposition below.

Proposition III.41. $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1}) \cong \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_r^*(\{0,1\}\times E,\Delta\times\sim)} \mathrm{C}_r^*([0,1] \times E, \Delta\times \sim)$

Proof:

Let α be the homomorphism defined by

$$\begin{array}{rcl} \alpha : \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_1}) & \rightarrow & \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_r^*(\{0,1\}\times E,\Delta\times\sim)} \mathrm{C}_r^*([0,1] \times E, \Delta\times \sim) \\ f & \mapsto & (\alpha_1(f), \alpha_2(f)) \end{array}$$

where $\alpha_1(f)$ is the restriction to the vertices map ($\alpha_1(f) = f|_{\mathrm{X}_0\times\mathrm{X}_0}$)) and

$$\alpha_2(f)\left((t, e_j + u_l^j), (t, e_j + u_h^j)\right) = f\left(r_{e_j}(t) + u_l^j, r_{e_j}(t) + u_h^j\right)$$

if $\left(r_{e_j}(t) + u_l^j, r_{e_j}(t) + u_h^j\right) \in \mathrm{G}$ and to be equal to 0 otherwise.

It follows from the definitions above that $\varphi(\alpha_1(f)) = \rho(\alpha_2(f))$ and hence $\alpha(f)$ is well defined. One can also check that α is a $*-$homomorphism. below we show that α is isometric and has dense range.

- α is isometric.

Since $\|\alpha(f)\| = \max\{\|\alpha_1(f)\|, \|\alpha_2(f)\|\}$ it is enough to show that $\|f\| = \|\alpha_2(f)\|$. From definition II.17 we have that

$$\|\alpha_2(f)\| = \sup_{(t,e_j+u_l^j)\in[0,1]\times E} \|\lambda_{(t,e_j+u_l^j)}(\alpha_2(f))\|$$

where $\lambda_{(t,e_j+u_l^j)}$ was defined in II.17.

So if $t \in (0,1)$ then $l_2[(t, e_j + u_l^j)] = l_2([r_{e_j}(t) + u_l^j])$ and hence

$$\lambda_{(t,e_j+u_l^j)}(\alpha_2(f)) = \lambda_{r_{e_j}(t)+u_l^j}(f)$$

Also if $t \in \{0,1\}$ then $[(t, e_j + u_l^j)]_{\Delta\times\sim}$ is equal to the disjoint union $\bigcup_{h=1}^{h_0}[(t, e_j + u_h^j)]_0$, where each $r_{e_j}(t) + u_h^j$ is a representative of a different vertex equivalence class in G and $[(t, e_j + u_h^j)]_0$ is the set of all points (t, e) in $[0,1]\times E$ such that $r_{e_j}(t) + u_h^j$ is equivalent to $r_e(t)$ in G. This implies (the proof is analogous to proposition IV.9) that

$$\lambda_{(t,e_j+u_l^j)}(\alpha_2(f)) = \bigoplus_{h=1}^{h_0} \lambda_{r_{e_j}(t)+u_h^j}(f)$$

and hence $\|\lambda_{(t,e_j+u_l^j)}(\lambda_2(f))\| = \max_{h=1..h_0} \|\lambda_{r_{e_j}(t)+u_h^j}(f)\|$ which implies that α is isometric.

- α has dense range.

Let $(a_2, a_1) \in \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_r^*(\{0,1\}\times E,\Delta\times\sim)} \mathrm{C}_r^*([0,1]\times E, \Delta\times\sim)$. Notice that $\varphi(a_2) = \rho(a_1)$.

Let $\epsilon > 0$. Using the continuity of φ find $a_2' \in \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_0})$ such that $\|a_2' - a_2\| < \epsilon$ and

$$\|\varphi(a_2') - \varphi(a_2)\| < \epsilon$$

Since ρ is onto there exists $a_1' \in \mathrm{C}_c([0,1]\times E, \Delta\times\sim)$ such that $\rho(a_1') = \varphi(a_2')$ (This is actually the Tietze extension theorem (see [28]), i.e., since $\varphi(a_2')$ belongs to $\mathrm{C}_c(\{0,1\}\times E, \Delta\times\sim)$ we can extend it to a function in $\mathrm{C}_c([0,1]\times E, \Delta\times\sim)$. It follows that

$$\|\rho(a_1' - a_1)\| = \|\varphi(a_2' - a_2)\| < \epsilon$$

Next observe that $\|\rho(a_1' - a_1)\|$ is a quotient norm, that is, since ρ is onto and $\frac{\mathrm{C}_r^*([0,1]\times E,\Delta\times\sim)}{\ker\rho} \cong \mathrm{Im}\rho$ via $f + \ker\rho \to \rho(f)$, we have that $\|\rho(a_1' - a_1)\| =$

$\|(a_1' - a_1) + \ker \rho\| = \inf\{\|a_1' - a_1 + c\| : c \in \ker \rho\}$. Hence there exists $c \in \ker \rho$ such that $\|a_1' + c - a_1\| < 2\epsilon$. Furthermore from lemma III.40 there exists $c' \in \ker \rho$, $c' \in \mathrm{C}_c([0,1] \times E, \Delta\times \sim)$, such that

$$\|a_1' + c' - a_1\| < 3\epsilon$$

Now notice that $\rho(a_1' + c') = \rho(a_1') = \varphi(a_2')$ and hence $(a_2', a_1' + c')$ belongs to the pull back $\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}^*_r(\{0,1\}\times E, \Delta\times\sim)} \mathrm{C}^*_r([0,1] \times E, \Delta\times \sim)$. Also

$$\|(a_2, a_1) - (a_2', a_1' + c')\| < 3\epsilon$$

Finally since $(a_2', a_1'+c')$ belongs to the pull back and both a_2' and $a_1'+c'$ have compact support, we can find $f \in \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_1})$ such that $\alpha(f) = (a_2', a_1'+c')$, which implies that α has dense range.

■

Corollary III.42. $\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1})$ *is an RSA.*

Our next task is to show that $\mathrm{C}^*_r(\mathrm{G})$ is an RSA C*-algebra.

Theorem III.43. $\mathrm{C}^*_r(\mathrm{G})$ *is an RSA.*

Proof:

Since there are only a finite number of prototiles we can write the set of tiles, τ, as

$$\tau = \left\{t_j + u_l^j : j = 1..m; l = 0, 1, 2, \ldots\right\}$$

where each u_l^j is a vector in $\mathbb{R}^n$, $u_0^j = 0$ for every j and each t_j is a prototile. Notice that the vectors u_l^j are not the same as on the previous proposition.

Let D denote the disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and S denote the unitary circle. Consider the set $D \times \tau$ and identify each point $(s, t_j + u_l^j)$ in this set with the point $r_{t_j}(s) + u_l^j$, where r_{t_j} is a homeomorphism from the tile t_j into the disk D. On $D \times \tau$ introduce the equivalence relation $\Delta\times \sim$, where $\sim$ is the equivalence relation on the tiles (so $t_j + u_l^j$ is equivalent to any other translation $t_j + u_h^j$).

To construct an RSA, let $\varphi : \mathrm{C}_c(\mathrm{G}|_{\mathrm{X}_1}) \to \mathrm{C}_c(S \times \tau, \Delta\times \sim)$ be the homomorphism defined by

$$\varphi(f)\left((s, t_j + u_l^j), (i, t_j + u_h^j)\right) = f\left(r_{t_j}(s) + u_l^j, r_{t_j}(s) + u_h^j\right)$$

if $\left(r_{t_j}(s) + u_l^j, r_{t_j}(s) + u_h^j\right) \in \mathrm{G}$ and to be equal to 0 otherwise. As in the previous proposition this homomorphism can be extended to $\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1})$.

Remember we just proved above that $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$ is an RSA and hence we can consider the pull back C*-algebra,

$$\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1}) \oplus_{\mathrm{C}_r^*(S\times\tau,\Delta\times\sim)} \mathrm{C}_r^*(D\times\tau,\Delta\times\sim)$$

which by definition III.37 is an RSA.

Now let α be the homomorphism defined by

$$\begin{array}{rcl} \alpha : \mathrm{C}_c(\mathrm{G}) & \rightarrow & \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1}) \oplus_{\mathrm{C}_r^*(S\times\tau,\Delta\times\sim)} \mathrm{C}_r^*(D\times\tau,\Delta\times\sim) \\ f & \mapsto & (\alpha_1(f), \alpha_2(f)) \end{array}$$

where $\alpha_1(f)$ is the restriction to the edges map $(\alpha_1(f) = f|_{\mathrm{X}_1\times\mathrm{X}_1}))$ and

$$\alpha_2(f)\left((s, t_j + u_l^j), (s, t_j + u_h^j)\right) = f\left(r_{t_j}(s) + u_l^j, r_{t_j}(s) + u_h^j\right)$$

if $\left(r_{t_j}(s) + u_l^j, r_{t_j}(s) + u_h^j\right) \in \mathrm{G}$ and to be equal to 0 otherwise.

It follows analogous to what we did for $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$ that α is an isomorphism and hence $\mathrm{C}_r^*(\mathrm{G}) \cong \mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1}) \oplus_{\mathrm{C}_r^*(S\times\tau,\Delta\times\sim)} \mathrm{C}_r^*(D\times\tau,\Delta\times\sim)$ as desired.

■

Chapter IV

C*-algebras that Encode the Dynamics on the Tiling

The C*-algebras from tilings defined in the previous chapter could be constructed not only for substitution tilings, but for any tiling with finite local complexity and a finite number of tiles up to translation. This is not the case in this chapter and for what follows we need a substitution tiling with primitivity, finite local complexity and recognizability.

We now want a way to capture the dynamics of the inflation rule on the equivalence relation induced by the tilings $\omega^k(\mathrm{T})$, $k = 1, 2, 3, \ldots$. For this we define a equivalence relation on each tiling $\omega^k(\mathrm{T})$ and show that these relations are comparable by inclusion. This gives rise to a inductive limit of C*-algebras that we intend to study.

IV.1 The C*-algebra Induced by the Inflation Map

From now on call G_0 what we were calling G and define:

$$G_k = \{(x, y) \in I\!R^2 \times I\!R^2 : \mathrm{T}_k(\lambda^k x) - \lambda^k x = \mathrm{T}_k(\lambda^k y) - \lambda^k y\}$$

where $\mathrm{T}_k = \omega^k(\mathrm{T})$; $k = 1, 2, 3, \ldots$

Proposition IV.1. *Each* G_k *is contained in* G_{k+1}, *i.e.,* $\mathrm{G}_0 \subseteq \mathrm{G}_1 \subseteq \mathrm{G}_2 \subseteq \ldots$.

Proof:

We will show that $G_0 \subseteq G_1$. The others inclusions are proved analogously. Let $(x, y) \in \mathrm{G}_0$. We need to show that $(x, y) \in \mathrm{G}_1$, i.e.,

$$\omega(\mathrm{T})(\lambda x) - \lambda x = \omega(\mathrm{T})(\lambda y) - \lambda y.$$

From the definition of ω, $\omega(\mathrm{T})(\lambda x) \subseteq \omega(\mathrm{T}x)$. Also $(x, y) \in G_0$ implies that $\omega(\mathrm{T}y) = \omega(\mathrm{T}x) + \lambda(y - x)$. So we have

$$\omega(\mathrm{T})(\lambda x) + \lambda(y - x) \subseteq \omega(\mathrm{T}x) + \lambda(y - x) = \omega(\mathrm{T}y)$$

and hence $\omega(\mathrm{T})(\lambda x) + \lambda(y - x)$ is a patch in $\omega(\mathrm{T})$. Moreover, since $\lambda y = \lambda x + \lambda(y - x)$ this patch is exactly $\omega(\mathrm{T})(\lambda y)$ and it follows that $(x, y) \in \mathrm{G}_1$.

■

The next step is to show that each G_k is open in the next. In order to do so we need two lemmas that are generalizations of lemmas III.1 and III.2.

Lemma IV.2. *Fix* $\mathrm{N} \in \mathbb{N}$. *Given* $x \in I\!R^d$, *there exists* $\delta > 0$ *such that if* y *belongs to the ball of center* x *and radius* δ, $\mathrm{B}(x, \delta)$, *then* $\mathrm{T}_k\left(\lambda^k(y)\right) \subseteq \mathrm{T}_k(x)\left(\lambda^k(x)\right)$ *for all* $0 \leq k \leq \mathrm{N}$.

Proof:

For any $0 \leq k \leq \mathrm{N}$ the set $\mathrm{U} = \frac{1}{\lambda^k}\left(\mathrm{int}\left(\mathrm{T}_k(\lambda^k x)\right)\right)$ is open and $x \in \mathrm{U}$. So there exists δ_k such that $\mathrm{B}(x, \delta_k) \subseteq \mathrm{U}$. Now if $y \in \mathrm{B}(x, \delta_k)$ then $\lambda^k y \in \mathrm{int}\left(T_k(\lambda^k x)\right)$ and hence $\mathrm{T}_k(\lambda^k y) \subseteq \mathrm{T}_k(\lambda^k x)$. Finally choose δ as the minimum of all δ_k.

■

The next lemma characterizes neighborhoods of a point $(x, y) \in \mathrm{G}_k$.

Lemma IV.3. *Let* $(x, y) \in \mathrm{G}_k$ *with* $k \in \mathbb{Z}$, $k \geq 0$. *Then there exists* $\delta_k > 0$ *such that*

$$\mathrm{B}\left((x, y), \delta\right) \cap \mathrm{G}_k = \left\{(x + v, y + v) : v \in \mathrm{B}\left(0, \frac{\delta}{\sqrt{2}}\right)\right\}$$

for all $\delta \leq \delta_k$.

Proof:

Let δ be small enough so that both lemmas IV.2, applied for x, and I.8 hold. Let $\delta_k < \frac{\delta}{2\lambda^k}$. The rest of the proof is a straightforward adaptation of the proof of lemma III.2, just replacing T with T_k and multiplying by λ^k where necessary.

■

Observation IV.4. *It is clear from the lemma above that each* G_k *is étale.*

Proceeding as in chapter III we can show that Lebesgue measure is G_k- invariant and hence we have a trace functional in $\mathrm{C}_c(\mathrm{G}_k)$. We state this below.

Proposition IV.5. *For any $k \in \mathbb{N}$, Lebesgue measure is G_k invariant and*

$$\begin{array}{rcl}\tau : \mathrm{C}_c(\mathrm{G}_k) & \rightarrow & \mathbb{C} \\ f & \mapsto & \int_{I\!R^d} f(x,x)d\mu(x)\end{array}$$

is a trace functional.

Proof: Analogous to proposition III.5.

■

Latet we will need a better description for the trace formula to use it on examples. We give it below.

Lemma IV.6. *For $x \in I\!R^d$ let π_x be the representation on $l_2([x])$ as in the definition of the reduced norm. Then*

$$\tau(f) = \sum_{[p]\in\mathcal{P}} \int_{x\in p} Tr(\pi_x(f))dx.$$

If f is a projection then

$$\tau(f) = \sum_{[p]\in\mathcal{P}} Tr(\pi_x(f))\mu(p)$$

where for each $[p] \in \mathcal{P}$, x is a point in the interior of p and μ is Lebesgue measure.

Proof:

Remember that if $\{\delta_y\}_{y\in[x]}$ is the orthonormal basis of $l_2([x])$ given by $\delta_y(\xi) = 1$ if $\xi = y$ and $\delta_y(\xi) = 0$ if $\xi \neq y$ then $\mathrm{Tr}(\pi_x(f)) = \sum_{y\in[x]} \langle \pi_x(f)\delta_y, \delta_y\rangle = \sum_{y\in[x]} f(y,y)$.

Also we already know that for each $[p] \in \mathcal{P}$ we can write $[p] = \{p + w_i^p\}_{i=1}^{\infty}$, where $w_i^p \in I\!R^d$. Notice that if x is a point in the interior of p then $[x] = \{x + w_i^p\}_{i=1}^{\infty}$. Denote the interior of p by $\mathrm{int}(p)$. We now have

$$\begin{aligned}\int_{\mathbb{R}^d} f(x,x)d\mu(x) &= \sum_{t\in\{\text{tiles}\}}\int_t f(x,x)dx = \sum_{[p]\in\mathcal{P}}\sum_{t\in[p]}\int_t f(x,x)dx \\ &= \sum_{[p]\in\mathcal{P}}\sum_{i=1}^{\infty}\int_{p+w_i^p} f(x,x)dx = \sum_{[p]\in\mathcal{P}}\sum_{i=1}^{\infty}\int_p f(x+w_i^p, x+w_i^p)dx \\ &= \sum_{[p]\in\mathcal{P}}\int_{\text{int}(p)}\sum_{i=1}^{\infty} f(x+w_i^p, x+w_i^p)dx = \sum_{[p]\in\mathcal{P}}\int_{\text{int}(p)}\sum_{y\in[x]} f(y,y)dx \\ &= \sum_{[p]\in\mathcal{P}}\int_{\text{int}(p)}\text{Tr}(\pi_x(f))dx = \sum_{[p]\in\mathcal{P}}\int_p \text{Tr}(\pi_x(f))dx\end{aligned}$$

and the first part of the proposition is proved.

Now suppose f is a projection. Then $\text{Tr}(\pi_x(f)) \in \mathbb{Z}$ and for any tile t the map $x \mapsto \text{Tr}(\pi_x(f))$ is a continuous function from the interior of t into the integers, and hence a constant function. This implies the second part of the proposition.

■

At first sight, for two dimensional examples, the formula for the trace above do not appear to be helpful, since we usually do not know the values of a projection on the interior of a tile. But we usually know the values at the vertices and we use this to get a better description of the trace:

Lemma IV.7. *For each $[p] \in \mathcal{P}$ we choose one and only one vertex on the tile p, which we call v_p. Using the language of lemma IV.6 we have that $[p] = \{p + w_i^p\}_{i=1}^{\infty}$, where $w_i^p \in \mathbb{R}^d$. Let $\mathcal{V}_{v_p} = \{[v] \in \mathcal{V} : \exists\ w_i^p$ such that $v_p + w_i^p \in int(\mathrm{T}(v))\}$. Then for any projection $f \in \mathrm{C}_c(\mathrm{G}_k)$ we have*

$$\tau(f) = \sum_{[p]\in\mathcal{P}}\sum_{[v]\in\mathcal{V}_{v_p}} Tr(\pi_v(f))\mu(p)$$

Proof:

Observe that for $x \in \text{int}(p)$ we have $\text{Tr}(\pi_x(f)) = \sum_{i=1}^{\infty} f(x+w_i^p, x+w_i^p)$ and for each v_p we have

$$\sum_{i=1}^{\infty} f(v_p + w_i^p, v_p + w_i^p) = \sum_{[v]\in\mathcal{V}_{v_p}}\sum_{z\in[v]} f(z,z) = \sum_{[v]\in\mathcal{V}_{v_p}} \text{Tr}(\pi_v(f)).$$

Let $[p] \in \mathcal{P}$. Notice that $x \mapsto \mathrm{Tr}(\pi_x(f))$ is not necessarily continuous on the border of p. Actually we have that

$$\lim_{\substack{x \to v_p \\ x \in \mathrm{int}(p)}} \mathrm{Tr}(\pi_x(f)) = \sum_{[v] \in \mathcal{V}_{v_p}} \mathrm{Tr}(\pi_v(f)).$$

Now since $x \mapsto \mathrm{Tr}(\pi_x(f))$ is constant in the interior of p we have that for any $x \in \mathrm{int}(p)$, $\mathrm{Tr}(\pi_x(f)) = \sum_{[v] \in \mathcal{V}_{v_p}} \mathrm{Tr}(\pi_v(f))$ and the lemma follows.

■

Proposition IV.8. *Each* G_k *is open in* G_{k+1}*;* $k = 0, 1, 2, \ldots$

Proof: We will show that G_0 is open on G_1. The proof for k is the same, one just need to change indices.

Let $(x, y) \in \mathrm{G}_0$. Notice that (x, y) is also in G_1. Let $\delta = \min\{\delta_0, \delta_1\}$, where δ_0 and δ_1 are obtained from lemma IV.3 applied for G_0 and G_1, respectively. Then

$$B\left((x, y), \delta\right) \cap \mathrm{G}_1 = \left\{(x + v, y + v) : v \in B\left(0, \frac{\delta}{\sqrt{2}}\right)\right\} = B\left((x, y), \delta\right) \cap \mathrm{G}_0$$

and hence $B\left((x, y), \delta\right) \cap \mathrm{G}_1$ is a neighborhood of (x, y) in G_1 completely contained in G_0.

■

With the two propositions above we are now able to show that $C_r^*(\mathrm{G}_k)$ is a sub-algebra of $C_r^*(\mathrm{G}_{k+1})$.

Proposition IV.9. $C_r^*(\mathrm{G}_k)$ *is a subalgebra of* $C_r^*(\mathrm{G}_{k+1})$*, for all* $k = 0, 1, 2, \ldots$.

Proof:

It is enough to show the proposition for $k = 0$.

The idea is to include a function $f \in C_c(\mathrm{G}_0)$ into $C_c(\mathrm{G}_1)$ by extending it to 0 on $\mathrm{G}_1 - \mathrm{G}_0$. We show that this is an isometric $*-$homomorphism and hence it can be extended to an isometric $*-$homomorphism from $C_r^*(\mathrm{G}_0)$ into $C_r^*(\mathrm{G}_1)$.

More explicitly, we define $\iota : C_c(\mathrm{G}_0) \to C_c(\mathrm{G}_1)$ by

$$\iota(f)(x, y) = \tilde{f}(x, y) = \begin{cases} f(x, y); & (x, y) \in \mathrm{G}_0 \\ 0; & (x, y) \in \mathrm{G}_1 - \mathrm{G}_0 \end{cases}$$

Observe that $\tilde{f} \in C_c(\mathrm{G}_1)$ since from proposition IV.8 G_0 is an open subset of the locally compact space G_1.

We now show that ι is an isometric $*-$homomorphism. We denote the equivalence class of a point x in G_k by $[x]_k$.

- ι is a $*-$homomorphism.

Notice that

$$\widetilde{f * g}(x,y) \begin{cases} f * g(x,y) = \sum_{z\in[x]_0} f(x,z)g(z,y); & \text{if} \quad (x,y) \in \mathrm{G}_0 \\ 0; \quad \text{if} \quad (x,y) \in \mathrm{G}_1 - \mathrm{G}_0 \end{cases}$$

On the other hand, $\tilde{f} * \tilde{g}(x,y) = \sum_{z\in[x]_1} \tilde{f}(x,z)\tilde{g}(z,y)$ and we have two possibilities. If $(x,y) \in \mathrm{G}_1 - \mathrm{G}_0$, then for all $z \in [x]_1$ either $(x,z) \in \mathrm{G}_1 - \mathrm{G}_0$ or $(z,y) \in \mathrm{G}_1 - \mathrm{G}_0$ otherwise we would have $(x,y) \in \mathrm{G}_0$. So for all $z \in [x]_1$ either $\tilde{f}(x,z) = 0$ or $\tilde{g}(x,z) = 0$ and hence $\tilde{f} * \tilde{g}(x,y) = 0$ if $(x,y) \in \mathrm{G}_1 - \mathrm{G}_0$.

If $(x,y) \in \mathrm{G}_0$. Let $[x]_0$ be the set $\{z \in [x]_1 : (x,z) \in \mathrm{G}_0\}$ and $[x]_1 - [x]_0 = \{z \in [x]_1 : (x,z) \in \mathrm{G}_1 - \mathrm{G}_0\}$. Observe that if $z \in [x]_0$ then $(z,y) \in \mathrm{G}_0$ and if $z \in [x]_1 - [x]_0$ then $(z,y) \in \mathrm{G}_1 - \mathrm{G}_0$ and hence $\tilde{g}(z,y) = 0$. This implies that

$$\tilde{f}*\tilde{g}(x,y) = \sum_{z\in[x]_0} \tilde{f}(x,z)\tilde{g}(z,y) + \sum_{z\in[x]_1-[x]_0} \tilde{f}(x,z)\tilde{g}(z,y) = \sum_{z\in[x]_0} f(x,z)g(z,y) = \widetilde{f * g}(x,y)$$

and ι is a homomorphism.

The *-condition follows promptly and hence ι is a $*-$homomorphism.

- ι is an isometric $*-$homomorphism.

Remember that $\|\tilde{f}\| = \sup_{x\in\mathrm{X}} \|\lambda_x(\tilde{f})\|$ where

$$\lambda_x(\tilde{f})(\xi)(y) = \sum_{z\in[x]_1} \tilde{f}(y,z)\xi(z)$$

for $\xi \in l_2([x]_1)$.

We have that $[x]_1$ is equal to the disjoint union $[x_0]_0 \cup [x_1]_0 \cup [x_2]_0 \cup \ldots$ and hence $l_2([x]_1) = l_2([x_0]_0) \oplus l_2([x_1]_0) \oplus l_2([x_2]_0) \oplus \ldots$.

Let $\lambda^1_x(\tilde{f}) := \lambda_x(\tilde{f})$ and $\lambda^0_{x_i}(f) = \lambda_{x_i}(f)$. Observe that $\lambda^1_x(\tilde{f})$ acts on $l^2([x]_1)$ and $\lambda^0_{x_i}(f)$ acts on $l_2([x_i]_0)$.

Once we show that $\lambda^1_x(\tilde{f}) = \oplus\lambda^0_{x_i}(f)$, we have that

$$\|\tilde{f}\| = \sup_{x\in\mathrm{X}} \|\lambda^1_x(\tilde{f})\| = \sup_{x\in\mathrm{X}} \| \oplus \lambda^0_{x_i}(f)\|$$

It is a fact from functional analysis that if a Hilbert space $\mathrm{H} = \oplus \mathrm{H}_i$, $\mathrm{T} = \oplus \mathrm{T}_i$ is a bounded operator in H and $\|\mathrm{T}_i\|$ is uniformly bounded than $\|\mathrm{T}\| = \sup \|\mathrm{T}_i\|$. Using this fact we have that

$$\sup_{x\in \mathrm{X}} \| \oplus \lambda^0_{x_i}(f)\| = \sup_{x\in \mathrm{X}} \left\{ \sup_{x_i:[x]_1=\cup[x_i]_0} \|\lambda^0_{x_i}(f)\| \right\} = \sup_{x\in \mathrm{X}} \|\lambda^0_x(f)\| = \|f\|$$

and hence ι is isometric.

It remains to be proved that $\lambda^1_x(\tilde{f}) = \oplus\lambda^0_{x_i}(f)$.

Notice that $l_2([x_i]_0) \subseteq l_2([x]_1)$ is invariant under $\lambda^1_x(\tilde{f})$ for any $i \in \mathbb{N}$. To see this let let $i \in \mathbb{N}$, and $\xi \in l_2([x]_1)$ be supported in $l_2([x_i]_0)$. Now suppose $y \notin [x_i]_0$. We want to show that $\lambda^1_x(\tilde{f})(\xi)(y) = \sum_{z\in[x]_1} f(y,z)\xi(z) = 0$. But this follows promptly once we notice that if $(y,z) \notin \mathrm{G}_0$ then $\tilde{f}(y,z) = 0$ and if $(y,z) \in \mathrm{G}_0$ then $z \notin [x_i]_0$ and hence $\xi(z) = 0$.

Next we show that $\lambda^1_x(\tilde{f})|_{l_2([x_i]_0)} = \lambda^0_{x_i}(f)$ and hence the desired results follows since $l_2([x]_1) = l_2([x_0]_0) \oplus l_2([x_1]_0) \oplus l_2([x_2]_0) \oplus \ldots$. So take ξ supported on $l_2([x_i]_0)$ and let $y \in [x_i]_0$. Also let $\mathrm{Z}_1 = \{z \in [x]_1 : (y,z) \in \mathrm{G}_0 \Leftrightarrow z \in [x_i]_0\}$ and $\mathrm{Z}_2 = \{z \in [x]_1 : (y,z) \notin \mathrm{G}_0\}$. Observe that $\tilde{f}(y,z) = 0$ in Z_2. Then

$$\begin{aligned}
\lambda^1_x(\tilde{f})(\xi)(y) &= \sum_{z\in[x]_1} \tilde{f}(y,z)\xi(z) = \sum_{z\in \mathrm{Z}_1} \tilde{f}(y,z)\xi(z) + \sum_{z\in \mathrm{Z}_2} \tilde{f}(y,z)\xi(z) \\
&= \sum_{z\in[x_i]_0} f(y,z)\xi(z) = \lambda^0_{x_i}(f)(\xi|_{[x_i]_0})(y)
\end{aligned}$$

as desired.

■

As promised in section III.1 we now prove that G_k is amenable.

Proposition IV.10. G_k *is amenable, for every* $k \in \mathbb{N}$.

Proof:

The definition of amenability in [15] requires us to show that there exists a sequence of functions $\{f_i\}_{i\in\mathbb{N}} \in \mathrm{C}_c(\mathrm{G}_k)$ such that:

1. $x \mapsto \sum_{z\in[x]} |f_i(x,z)|^2$ is uniformly bounded in the sup norm.

2. $(x,y) \mapsto \sum_{z\in[x]} f_i(x,z)\overline{f_i(y,z)}$ converges to 1 uniformly on compact subsets of G_k

First we show that G_0 is amenable. For this, let $r : [0,\infty) \to [0,1]$ be any function such that

- $r = 1$ on a neighborhood of 0
- r is continuous and the support of r is contained in $[0,1)$

For any $(x,y) \in \mathrm{G}_0$, define

$$\delta(x,y) = \sup\left\{\delta \le 1 : \mathrm{T}(x+z) - x - z = \mathrm{T}(y+z) - y - z \ \ \forall z, |z| \le \delta\right\}. \quad \text{(IV.1)}$$

From lemma IV.3 one can see that $\delta(x,y) > 0$ for any $(x,y) \in \mathrm{G}_0$, since there exists $\epsilon > 0$ such that $\left\{(x+v, y+v) : v \in \mathrm{B}\left(0, \frac{\delta}{\sqrt{2}}\right)\right\}$ is a neighborhood of (x,y).

We will show that δ is a continuous function. We prove that $\delta^{-1}((a,b))$, $0 \le a < b \le 1$, is an open set in G_0. The proof that $\delta^{-1}((a,1])$ is open is analogous. So let $(x,y) \in \delta^{-1}((a,b))$. Then $\delta(x,y) = c$; $a < a_1 < c < b_1 < b$. Let ϵ be small enough so that $c - \epsilon > a_1$, $b_1 - \epsilon > c$ and $\mathrm{B}\left((x,y),\epsilon\right) = \left\{(x+v, y+v) : v \in \mathrm{B}\left(0, \frac{\epsilon}{\sqrt{2}}\right)\right\}$ (use lemma IV.3). Then $\mathrm{B}\left((x,y),\epsilon\right)$ is a neighborhood of (x,y) contained in $\delta^{-1}((a,b))$. To see this suppose $(x',y') \in \mathrm{B}\left((x,y),\epsilon\right)$. Then $(x',y') = (x+v, y+v)$ for some $|v| < \epsilon$. Now if $|z| \le a_1$ then $|v+z| < \epsilon + a_1 < c - a_1 + a_1 = c$ and hence $\mathrm{T}(x+(v+z)) - x - (v+z) = \mathrm{T}(y+(v+z)) - y - (v+z)$ for all z with $|z| \le a_1$ so that $\delta(x',y') \ge a_1 > a$. On the other hand if $|z| \ge b_1$ then $|v+z| \ge |z| - |v| > b_1 - \epsilon > c$ and hence $\mathrm{T}(x+(v+z)) - x - (v+z) = \mathrm{T}(y+(v+z)) - y - (v+z)$ do NOT hold for all z with $|z| \le b_1$ so that $\delta(x',y') \le b_1 < b$. We conclude that $\delta(x',y') \in (a,b)$ as desired.

Back to the construction of a net of functions that will satisfy the amenability conditions, for each $n \in \mathbb{N}$, define

$$g_n(x,y) = r(|x|n^{-1})r(|y|n^{-1})(1 - r(n\delta(x,y)))$$

One can easily check that each $g_n \in \mathrm{C}_c(\mathrm{G}_0)$. Finally we define, for each $n \in \mathbb{N}$,

$$f_n(x,y) = \left(1 + \sum_{(x,z)\in \mathrm{G}_0} g_n^2(x,z)\right)^{-\frac{1}{2}} g_n(x,y)$$

We need to check conditions 1 and 2 of the amenability definition. Observe that if $|x| \ge n$ then $g_n(x,y) = 0$ so that $f_n(x) = 0$. Also if $|z| \ge n$ then

$g_n(x,z)=0$ and the sum in the definition of f is actually finite. To see that condition 1 holds, let $x \in \mathrm{X}$. Then

$$\left|\sum_{w\in[x]} |f_n(x,w)|^2\right| = \sum_{w\in[x]} \left| \frac{g_n(x,w)}{\left(1+\sum\limits_{(x,z)\in \mathrm{G}_0} g_n^2(x,z)\right)^{\frac{1}{2}}} \right|^2 = \sum_{w\in[x]} \frac{g_n^2(x,w)}{1+\sum\limits_{(x,z)\in \mathrm{G}_0} g_n^2(x,z)}$$

$$= \frac{\sum\limits_{w\in[x]} g_n^2(x,w)}{1+\sum\limits_{(x,z)\in \mathrm{G}_0} g_n^2(x,z)} < 1$$

We still need to prove condition 2. So let K be a compact in G_0.

Let $\delta_0 = \min\limits_{(x,y)\in \mathrm{K}} \delta(x,y)$. Observe that since δ is continuous and K is compact, δ has a minimum which is attained in K. This implies that $\delta_0 > 0$.

Choose $N \in \mathbb{N}$ such that $r(|x|N^{-1}) = r(|y|N^{-1}) = 1$ for all $(x,y)\in \mathrm{K}$ and such that $N\delta_0 \geq 1$.

Let $(x,y)\in\mathrm{K}$. Let $z\in[x]_0$ such that $|z|<N$ (remember that if $|z|\geq N$ then $g_N(x,z)=0$). Then $\delta(x,z)=\delta(y,z)$ or both $\delta(x,z),\delta(y,z)\geq\delta$. To see this observe that if $w\in \mathbb{R}$ and $|w|<\delta<\delta(x,y)$ then $\mathrm{T}(x+w)-x-w=\mathrm{T}(y+w)-y-w$ and hence for such w we have that $\mathrm{T}(z+w)-z-w=\mathrm{T}(x+w)-x-w \Leftrightarrow \mathrm{T}(z+w)-z-w=\mathrm{T}(y+w)-y-w$.

Remember that we need to prove that $(x,y)\mapsto \sum\limits_{w\in[x]} f_n(x,w)\overline{f_n(y,w)}$ converges to 1 uniformly on compact subsets of G_0. Observe that

$$\sum_{w\in[x]} f_N(x,w)\overline{f_N(y,w)} = \frac{\sum\limits_{w\in[x]} g_N(x,w)g_N(y,w)}{\left(1+\sum\limits_{z\in[x]} g_N^2(x,z)\right)^{\frac{1}{2}}\left(1+\sum\limits_{z\in[x]} g_N^2(z,y)\right)^{\frac{1}{2}}}$$

and uniform convergence follows once we prove that

$$\sum_{w\in[x]} g_N(x,w)g_N(y,w) = \sum_{z\in[x]} g_N^2(x,z) = \sum_{z\in[x]} g_N^2(z,y) \tag{IV.2}$$

To do this we define $\mathrm{Z}_1=\{w\in[x]:\delta(x,w)\geq\delta; |w|<N\}$ and $\mathrm{Z}_2=\{w\in[x]:\delta(x,w)<\delta; |w|<N\}$. Observe that $g_N(x,w)=g_N(y,w)$ in Z_2. Also if

$w \in \mathrm{Z}_1$ then $g_N(x,w)g_N(y,w) = r^2(|w|N^{-1}) = g_N^2(x,w) = g_N^2(y,w)$. So if we separate the above sums as the sum in Z_1 plus the sum in Z_2 it is clear that equality holds in equation IV.2 above.

For G_k the proof is analogous. We only need to use the tiling $w^k(\mathrm{T}) = \mathrm{T}_k$ instead of T. For example the function $\delta(x,y)$ of equation IV.1, is defined by

$$\delta_k(x,y) = \sup\left\{\delta \le 1 : \mathrm{T}_k(\lambda^k(x+z) - \lambda(x+z)) = \mathrm{T}_k(\lambda(y+z) - \lambda(y+z)) \;\; \forall z, |z| \le \delta\right\}$$

for any $(x,y) \in \mathrm{G}_k$, and the rest of the proof follows.

■

It is natural for us now to consider the inductive limit of the C*-algebras $\mathrm{C}_r^*(\mathrm{G}_k)$, with inclusion as connecting map. This is the C*-algebra encoding the inflation map that we were looking for. Next proposition characterizes this inductive limit.

Proposition IV.11. *The inductive limit* $\mathrm{C}_r^*(\mathrm{G}_k)_{\rightarrow}$ *is isomorphic to* $\mathrm{C}_r^*(\bigcup \mathrm{G}_k)$ *where a basis for the topology of* $\cup\mathrm{G}_k$ *is given by sets* U_k *such that* U_k *is open in* G_k *(So a set* U *is open in* $(\cup\mathrm{G}_k)$ *if* $\mathrm{U} \cap \mathrm{G}_k$ *is open in* G_k *for all* k*).*

Proof:

In order to prove this proposition we first need to define a family of inclusions of $\mathrm{C}_r^*(\mathrm{G}_k)$ into $\mathrm{C}_r^*(\bigcup \mathrm{G}_k)$. We define this inclusions in the same way we included $\mathrm{C}_r^*(\mathrm{G}_0)$ into $\mathrm{C}_r^*(\mathrm{G}_1)$ on proposition IV.9. So for $n = 0, 1, 2, \dots$ we define $\lambda_n : \mathrm{C}_c(\mathrm{G}_n) \to \mathrm{C}_c(\bigcup \mathrm{G}_k)$ by

$$\lambda_n(f)(x,y) = \begin{cases} f(x,y); & (x,y) \in \mathrm{G}_n \\ 0; & (x,y) \in (\bigcup \mathrm{G}_k) - \mathrm{G}_n \end{cases}$$

Each λ_n is an isometric $*-$homomorphism and hence it can be extended to an isometric $*-$homomorphism from $C_r^*(\mathrm{G}_n)$ into $\mathrm{C}_r^*(\bigcup \mathrm{G}_k)$ (the proof of this statements is analogous to proposition IV.9).

Now denote by ι_k the inclusion of $C_r^*(\mathrm{G}_k)$ into $C_r^*(\mathrm{G}_{k+1})$. Then $\lambda_n = \lambda_{n+1} \circ \iota_n$ for all $n = 0, 1, 2, \dots$ and by definition 6.2.2ii) in [24] it follows that $\mathrm{C}_r^*(\bigcup \mathrm{G}_k)$ is isomorphic to the inductive limit $\mathrm{C}_r^*(\mathrm{G}_k)_{\rightarrow}$ as desired.

■

We finish the section with a brief note about traces.

Proposition IV.12. *Lebesgue measure is* $\cup\mathrm{G}_k-$*invariant and hence the trace functional of proposition IV.5 is well defined.*

IV.2 $C_r^*\left(\bigcup \mathrm{G}_k\right)$ is simple.

One of the very interesting properties of $C_r^*\left(\bigcup \mathrm{G}_k\right)$ is its simplicity. This gives us hope of using Elliott's classification program to characterize $C_r^*\left(\bigcup \mathrm{G}_k\right)$ completely in terms of its K-theory.

In order to prove simplicity we need the following characterization of ideals in $C_r^*(\mathrm{G})$ given by [15].

Proposition IV.13. *Let* G *be principal, r-discrete groupoid with counting measure as a Haar system and* I *an ideal of* $C_r^*(\mathrm{G})$. *Then* $\mathrm{I} = \mathrm{I(Q)} = \overline{\{f \in C_c(\mathrm{G}) : f|_{\mathrm{Q}} = 0\}}$, *where* Q *is a closed subset of* G *such that* $\mathrm{G} \circ \mathrm{Q} \circ \mathrm{G} \subseteq \mathrm{Q}$, *i.e., if* (x, y) *and* $(z, w) \in \mathrm{G}$ *and* $(y, z) \in \mathrm{Q}$ *then* $(x, w) = (x, y)(y, z)(z, w) \in \mathrm{Q}$.

Observation IV.14. *If* Q *is a set as above and* $(x, y) \in \mathrm{Q}$ *then* $(y, x) \in \mathrm{Q}$ *since* $(y, x) \in \mathrm{G}$ *and* $(y, x) = (y, x)(x, y)(y, x)$.

Observation IV.15. *Furthermore, if* Q *is a set as above and* $(x, y) \in \mathrm{Q}$ *then* $[x] \times [x] \subseteq \mathrm{Q}$, *where* $[x]$ *denotes the equivalence class of* x.

We now return to our problem at hand of showing that $C_r^*(\bigcup \mathrm{G}_k)$ is simple. We still need two more lemmas. From now on we denote the union $\bigcup \mathrm{G}_k$ by G and the equivalence class of a point $x \in I\!R^d$ with respect to G by $[x]$.

Lemma IV.16. *Let* p_i *be a prototile. Then there exists* $M > 0$ *such that* $\omega^M(p_i)$ *contains at least one translation of all prototiles, vertex patterns and edge patterns.*

Proof:

Take N from primitivity (as defined right below definition I.6). Then $\omega^{\mathrm{N}}(p_i)$ contain a translation of any prototile. Notice that if $k \in \mathbb{N}$ then $\omega^{k+\mathrm{N}}(p_i)$ also contains a translation of all prototiles, since $\omega^{k+\mathrm{N}}(p_i) = \omega^{\mathrm{N}}(\omega^k(p_i))$ (use the definition of primitivity once again).

Now let P be a translation of one of the prototiles or a translation of a vertex pattern or a translation of an edge pattern. Then there exists $n_P \in \mathbb{N}$ such that $\omega^{n_P}(p_i)$ contains a translation of P (to see this notice that from definition I.6 there exists a prototile p_j and $m \in \mathbb{N}$ such that $\omega^m(p_j)$ contains a translation of P. So take $n_P = m + \mathrm{N}$).

Let MX be the maximum of all n_P as above. Observe that this is really a maximum since we only have a finite number of prototiles, vertex and edge patterns.

We will show that $MX + \mathrm{N}$ satisfies the lemma. So let P as before, that is, P is a translation of one of the prototiles or a translation of a vertex pattern or a translation of an edge pattern. Then $\omega^l(p_i) \supseteq P$ for some $l \in \mathbb{N}$. Now observe that $\omega^{MX+\mathrm{N}}(p_i) = \omega^l(\omega^{(MX-l)+\mathrm{N}}(p_i))$, and from the first paragraph of the this proof we have that $\omega^{(MX-l)+\mathrm{N}}(p_i)$ contains a translation of p_i. We conclude that $\omega^{MX+\mathrm{N}}(p_i)$ contains a translation of P as desired.

■

Lemma IV.17. *For any $x \in \mathbb{R}^d$, $[x]$ is dense in $\mathbb{R}^d$.*

Proof:

Given a ball $\mathrm{B}(y, \epsilon)$ in $\mathbb{R}^d$ we want to find a point in the equivalence class of x within the ball.

Since the ball has a fixed radius we can find a $n \in \mathbb{N}$ such that one tile of $\lambda^{-n}\omega^n(\mathrm{T})$ is completely contained in $\mathrm{B}(y, \epsilon)$. Choose M as in lemma IV.16. Let $\mathrm{N} = M + n$. Then a translation of any prototile, vertex and edge patterns of $\lambda^{-\mathrm{N}}\omega^{\mathrm{N}}(\mathrm{T})$ appear at least once inside the ball $\mathrm{B}(y, \epsilon)$. In particular a translation of $\lambda^{-\mathrm{N}}\left(\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}x)\right)$ is contained in $\mathrm{B}(y, \epsilon)$ and hence there exists a point $y \in \mathrm{B}(y, \epsilon)$ such that $\left(\lambda^{-\mathrm{N}}\omega^{\mathrm{N}}(\mathrm{T})\right)(y) = \lambda^{-\mathrm{N}}\left(\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}x)\right) + y - x$. This last equality implies that $\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}y) = \omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}(x)) + \lambda^{\mathrm{N}}(y - x)$ and hence $(y, x) \in \mathrm{G}_{\mathrm{N}}$. So y is equivalent to x in G and the proof is complete.

■

Corollary IV.18. *For any $x \in \mathbb{R}^d$, $[x] \times [x]$ is dense in $\mathbb{R}^d \times \mathbb{R}^d$.*

We are now able to prove the theorem of this section. Remember a basis for the topology of G is given by the sets of the form $\mathrm{U}_k \cap \mathrm{G}$, where U_k is any open set of G_k.

Theorem IV.19. *$C_r^*\left(\bigcup \mathrm{G}_k\right)$ is simple.*

Proof:

Let I be an ideal of $C_r^*(\mathrm{G}) = C_r^*\left(\bigcup \mathrm{G}_k\right)$. By proposition IV.13 we know that $\mathrm{I} = \mathrm{I(Q)}$ for some closed set Q satisfying all the conditions of proposition IV.13.

We will show that either Q is empty, which implies that $\mathrm{I} = C_r^*(\mathrm{G})$, or Q is dense in G, which implies that $\mathrm{I} = \{0\}$ since Q is closed.

Suppose Q is not empty.

Let $\mathrm{U} \neq \emptyset$ be an open subset of G. Then there exists $k \in \mathbb{N}$ such that $\mathrm{U} \cap G_k \neq \emptyset$. So let $(x, y) \in \mathrm{U} \cap \mathrm{G}_k$. From lemma IV.3 and the fact that $\mathrm{U} \cap \mathrm{G}_k$ is an open set of G_k we have that there exists $\delta > 0$ such that $\mathrm{U} \cap \mathrm{G}_k \supseteq \mathrm{B}((x,y),\delta) \cap \mathrm{G}_k = \left\{(x+v, y+v) : v \in \mathrm{B}\left(\vec{0}, \frac{\delta}{\sqrt{2}}\right)\right\}$

Now by corollary IV.18 there exists $(z,w) \in \mathrm{Q}$ such that $(z,w) \in B\left(\vec{0}, \frac{\delta}{4\sqrt{2}}\right)$. So $z = x + u_1$ and $w = y + u_2$ where $|u_1|, |u_2| \leq \frac{\delta}{4\sqrt{2}} < \frac{\delta}{\sqrt{2}}$.

From the choice of δ, $(x+u_1, y+u_1) \in \mathrm{B}((x,y),\delta) \cap \mathrm{G}_k \subseteq G$ and hence $(y+u_1, x+u_1) \in \mathrm{G}$. This implies that $(y+u_1, y+u_2) \in \mathrm{Q}$ since

$$(y+u_1, y+u_2) = (y+u_1, x+u_1)(x+u_1, y+u_2)(y+u_2, y+u_2)$$

and from observation IV.14 we have that $(y+u_2, y+u_1) \in \mathrm{Q}$. But this implies that $(x+u_1, y+u_1) \in \mathrm{Q}$ since

$$(x+u_1, y+u_1) = (x+u_1, y+u_2)(y+u_2, y+u_1)(y+u_1, y+u_1)$$

It is clear that $(x+u_1, y+u_1) \in \mathrm{B}((x,y),\delta) \cap \mathrm{G}_k \subseteq \mathrm{U} \cap \mathrm{G}_k$ and hence Q is dense in G as desired.

■

In our effort to further characterize $\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)$ in the next section we turn our attention to computing its K-theory.

IV.2.1 K-theory of $\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)$

Unfortunately we can not find a general description for the K-theory of the inductive limit C*-algebra $\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)$ as we did for $\mathrm{C}^*_r(\mathrm{G}_0)$. Nevertheless we will show in this section what can be done without restricting our attention to specific examples. We refer the reader to [24], [31] or [2] for the results in K-theory used below.

Remember that $\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)$ is isomorphic to the inductive limit C-* algebra, $\mathrm{C}^*_r(\mathrm{G}_0) \hookrightarrow \mathrm{C}^*_r(\mathrm{G}_1) \hookrightarrow \mathrm{C}^*_r(\mathrm{G}_2) \hookrightarrow \cdots$, as showed in prop. IV.11. This direct limit of C-* algebras induces a direct limit in K-theory, namely

$$\mathrm{K}_*\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \overset{\mathrm{K}_*(\iota)}{\to} K_*\left(\mathrm{C}^*_r(\mathrm{G}_1)\right) \overset{\mathrm{K}_*(\iota)}{\to} \mathrm{K}_*\left(\mathrm{C}^*_r(\mathrm{G}_2)\right) \overset{\mathrm{K}_*(\iota)}{\to} \cdots \qquad \text{(IV.3)}$$

and from continuity of K-theory, see [24] section 6.2, we have that the K-groups of $\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)$ are isomorphic to the direct limits IV.3 above.

In order to find the direct limit IV.3 we need to compute the K-groups of $\mathrm{C}^*_r(\mathrm{G}_k)$ for each k and to find the connecting maps $\mathrm{K}_*(\iota)$. A closer look in the terms of the direct limits tell us that the K_*-groups are all isomorphic and that is enough to compute the connecting map from $K_*(C_r^*(G_0))$ into $K_*(C_r^*(G_1))$, since all others are equal to this one. This is our next proposition. But first we need to introduce some formal definitions.

Definition IV.20. *A vertex in* G_k *is a point* (x, x) *such that* $\lambda^k x$ *is a vertex in* T_k. *A point* (x, x) *belongs to an edge of* G_k *if* $\lambda^k x$ *belongs to an edge of* T_k. *An edge e in* G_k *is the collection of all points of the form* (x, x) *such that* $\lambda^k x$ *belong to the same edge in* T_k.

Proposition IV.21. *For any* $k \in \mathbb{N}$, $K_*(C_r^*(G_k))$ *is isomorphic to* $K_*(C_r^*(G_0))$. *With these identifications the connecting map between the groups* $K_*(C_r^*(G_k))$ *and* $K_*(C_r^*(G_{k+1}))$ *is equal to the connecting map between* $K_*(C_r^*(G_0))$ *and* $K_*(C_r^*(G_1))$.

Proof:

Just note that the K-groups of $C_r^*(G_1)$ depend on the same vertex, edge and tile patterns used to compute the K-groups of $C_r^*(G_0)$

■

In light of the above proposition we only need to compute the connecting map between $K_*(C_r^*(G_0))$ and $K_*(C_r^*(G_1))$. Unfortunately we do not have a way of doing this in general, but rather we have a method that can be used in the specific examples. To illustrate the method suppose

$$K_0(C_r^*(G_0)) \cong \mathbb{Z}^2 \cong K_0(C_r^*(G_1))$$

Then the connecting map c, from $\mathbb{Z}^2$ to $\mathbb{Z}^2$, is such that the diagram below is commutative.

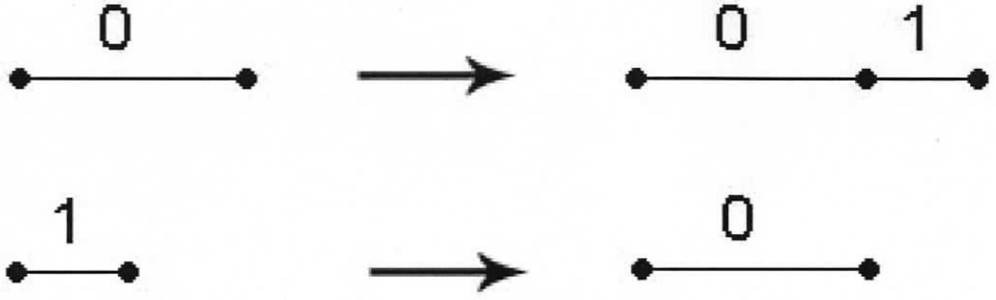
$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "\cong"] \arrow[d, dotted, "c"'] & K_0(C_r^*(G_0)) \arrow[d, hook, "K_0(\iota)"'] \\
\mathbb{Z}^2 & K_0(C_r^*(G_1)) \arrow[l, "\cong"']
\end{tikzcd}$$

where ι is the inclusion map from $C_r^*(G_0)$ into $C_r^*(G_1)$ as described in proposition IV.9.

The rest of our work is dedicated to computing K-theory for a number of examples.

IV.3 The Fibonacci Tiling

The Fibonacci tiling is associated with the substitution matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and has inflation constant γ, the golden mean. The substitution can be represented as below, where the segment labeled 0 has length one and the segment labeled 1 has length $1/\gamma$.



And we can tile the line as shown below

Since this tiling satisfies the finite pattern condition, primitivity and recognizability we can build $C_r^*(G_0)$ and $C_r^*(\bigcup G_k)$. Our task here is to compute the K-theory of these C*-algebras. The K-theory of $C_r^*(\bigcup G_k)$ will be computed using the method described in the previous section. But first we need to find the K-theory of $C_r^*(G_0)$ and to this end we proceed analogously to section III.3.

Remember that in the one dimensional case $C_r^*(G) = C_r^*(G|_{X_1})$ and we need to give an orientation to all edges in the tiling. We do so by giving all edges the same orientation, to the right, as shown below.

We can easily see that there are three equivalence classes of vertices, namely $v_1 = \vec{0} \bullet \vec{1}$, $v_2 = \vec{1} \bullet \vec{0}$ and $v_3 = \vec{0} \bullet \vec{0}$. Also there are two equivalence classes of edges, namely $e_1 = \vec{0}$ and $e_2 = \vec{1}$. So the six term exact sequence III.6 of section III.2 becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1})\right) \arrow[r] & \mathbb{Z}^3 \arrow[d, "\delta_0"'] \\
0 \arrow[u] & \mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1})\right) \arrow[l] & \mathbb{Z}^2 \arrow[l]
\end{tikzcd}$$

where δ_0 is the matrix given by

$$[\delta_0] = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

as described in proposition III.28.

The vectors $(0,0,1)$ and $(1,1,0)$ generate the kernel of δ_0 and $(1,-1)$ generates the image of δ_0. So we can conclude that

$$\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \cong \ker(\delta_0) \cong \mathbb{Z}^2 \quad \text{and} \quad \mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathrm{Im}(\delta_0)} \cong \mathbb{Z}$$

We now proceed to compute

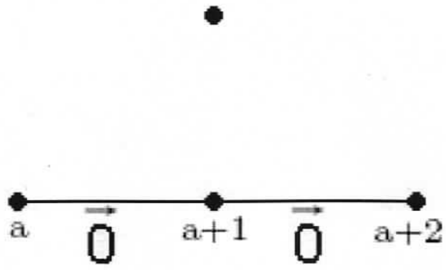
- $\mathrm{K}_0\left(\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)\right)$

From what is done above and proposition IV.21 we conclude that $\mathrm{K}_0\left(\mathrm{C}^*_r\left(\mathrm{G}_k\right)\right)$ is isomorphic to $\mathbb{Z}^2$ for all $k \in \mathbb{Z}^+$ and the connecting maps from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ are all the same.

Our next step is to compute the connecting map. In order to do so we need to specify the isomorphism between $\ker(\delta_0)$ and $\mathbb{Z}^2$. As expected we will use the isomorphism that maps $(0,0,1)$ to $(0,1)$ and $(1,1,0)$ to $(1,0)$.

Since the connecting map is a group homomorphism it is enough to find where the basis elements $(0,1)$ and $(1,0)$ are mapped.

We start with $(0,1)$. By our choice of isomorphism $(0,1)$ is mapped to the vector $(0,0,1)$ in the kernel of δ_0. Proceeding analogously to the first part of the proof of Proposition III.28 we conclude that $(0,0,1)$ is mapped to $[f]_0 \in \mathrm{K}_0\left(\mathrm{C}^*_r\left(\mathrm{G}_0|_{\mathrm{X}_0}\right)\right)$, where the function $f \in C_c(\mathrm{G}_0|_{\mathrm{X}_0})$ is defined to be 1 at one representative of the vertex v_3 and 0 otherwise. Letting $(a+1, a+1)$, for some $a \in \mathbb{R}$, be the representative of v_3 we have the following very illustrative picture of f.



and we can write f as $f(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) = (a+1, a+1) \\ 0 & \text{otherwise} \end{cases}$

Unfortunately we can not include this function in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1|_{\mathrm{X}_0}\right)\right)$ directly as we do not have this map. But we know that $[f]_0$ lifts to an element in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$. Moreover $[f]_0$ lifts to $[\tilde{f}]_0$ where $\tilde{f} \in \mathrm{C}_c(\mathrm{G}_0)$ is the projection defined below.

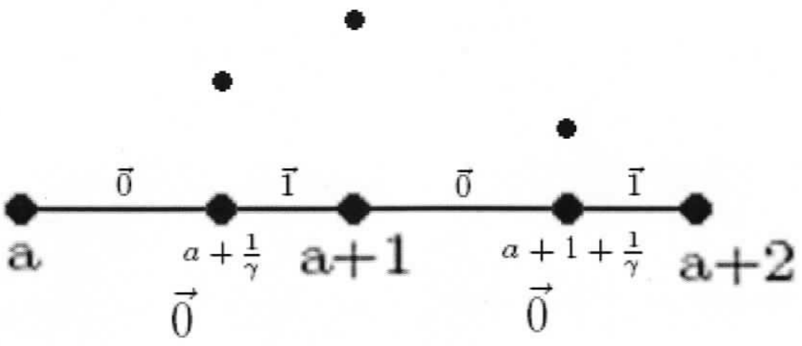
$$\tilde{f}(x,y) = \begin{cases} t \quad \text{if} \quad (x,y) = (a+t, a+t) \quad \text{for } 0 \leq t \leq 1 \\ 1-t \quad \text{if} \quad (x,y) = (a+1+t, a+1+t) \quad \text{for } 0 \leq t \leq 1 \\ \sqrt{t-t^2} \quad \text{if} \quad (x,y) = (a+t, a+1+t) \quad \text{for } 0 < t < 1 \\ \sqrt{t-t^2} \quad \text{if} \quad (x,y) = (a+1+t, a+t) \quad \text{for } 0 < t < 1 \\ 0 \quad \text{otherwise} \end{cases}$$

So we have that $(0,1)$ is "lifted" to $[\tilde{f}]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$, which we can include in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ via $\mathrm{K}_0(\iota)$, where ι is the inclusion map as in proposition IV.9.

Our next task is to follow the isomorphism from $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ to $\mathbb{Z}^2$ to find out where $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ is mapped. The first step in this direction is to restrict $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ to the vertices in G_1, i.e., we need to compute $\mathrm{K}_0(\psi)\left(\mathrm{K}_0(\iota)([\tilde{f}]_0)\right)$ where ψ is the restriction map. So we need to recognize the points in $\mathrm{G}_1|_{\mathrm{X}_0}$ where the function $\tilde{f}$ is non zero. Following the definition IV.20 of a vertex in G_1, we have that $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ is mapped to $[g]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1|_{\mathrm{X}_0}\right)\right) \cong \ker \delta_0$ where g is the projection below:

$$g(x,y) = \begin{cases} \frac{1}{\gamma} \quad \text{if} \quad (x,y) = (a+\frac{1}{\gamma}, a+\frac{1}{\gamma}) \\ 1 \quad \text{if} \quad (x,y) = (a+1, a+1) \\ 1-\frac{1}{\gamma} \quad \text{if} \quad (x,y) = (a+1+\frac{1}{\gamma}, a+1+\frac{1}{\gamma}) \\ \sqrt{\frac{1}{\gamma}-\frac{1}{\gamma^2}} \quad \text{if} \quad (x,y) = (a+\frac{1}{\gamma}, a+1+\frac{1}{\gamma}) \\ \sqrt{\frac{1}{\gamma}-\frac{1}{\gamma^2}} \quad \text{if} \quad (x,y) = (a+1+\frac{1}{\gamma}, a+\frac{1}{\gamma}) \\ 0 \quad \text{otherwise} \end{cases}$$

Observe that $(a+1, a+1)$ is a representative of the vertex v_2 in G_1, as $(a+\frac{1}{\gamma}, a+\frac{1}{\gamma})$ and $(a+1+\frac{1}{\gamma}, a+1+\frac{1}{\gamma})$ are representatives of the vertex v_1 in G_1. The picture of g below makes this more clear.



Following the isomorphism of proposition III.24 from $K_0(C_r^*(G_1|_{X_0}))$ to $\bigoplus_{i=1..3} K_0(K(l_2[v_i]))$ we have that $[g]_0$ is mapped into $([T_1]_0, [T_2]_0, [T_3]_0)$, where $T_3 = 0$ in $l_2[v_3]$ and T_1 and T_2 are both rank 1 projections on $l_2[v_1]$ and $l_2[v_2]$ respectively.

Since the isomorphism between $\bigoplus_{i=1..3} K_0(K(l_2[v_i]))$ and $\mathbb{Z}^3$ is given by the dimension of the operators we have that $([T_1]_0, [T_2]_0, [T_3]_0)$ is mapped to $(1,1,0)$ and from our choice of isomorphism from $\ker \delta_0$ into $\mathbb{Z}^2$ we have that $(1,1,0)$ is mapped to $(1,0)$.

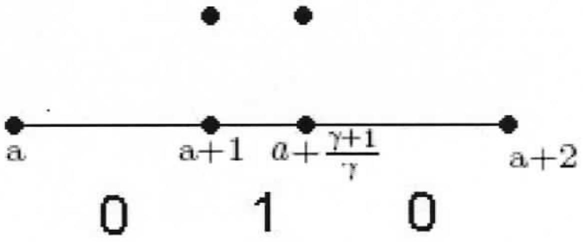
So we conclude that the connecting map c in K_0 takes the vector $(0,1)$ to the vector $(1,0)$. We believe that all the steps of this isomorphism chase were well explained but maybe it is not so clear why T_1 is a one dimensional projection. We explain this in further detail now.

Let ξ be a vector in $l_2[v_1]$. We have that

$$\begin{aligned}
&T_1(\xi)(a+\tfrac{1}{\gamma}) = \tfrac{1}{\gamma}\xi(a+\tfrac{1}{\gamma}) + \sqrt{\tfrac{1}{\gamma} - \tfrac{1}{\gamma^2}}\ \xi(a+1+\tfrac{1}{\gamma})\\
&T_1(\xi)(a+1+\tfrac{1}{\gamma}) = \sqrt{\tfrac{1}{\gamma} - \tfrac{1}{\gamma^2}}\ \xi(a+\tfrac{1}{\gamma}) + \left(1-\tfrac{1}{\gamma}\right)\xi(a+1+\tfrac{1}{\gamma})\\
&T_1(\xi)(y) = 0 \quad \text{for all other } y \in [v_1]
\end{aligned}$$

Using the fact that $\gamma^2 = \gamma + 1$ it is not hard to see that T_1 is a projection. Furthermore if a basis for the range of T_1 is given by the vector $\left(\frac{1}{\gamma}, \sqrt{\frac{1}{\gamma} - \frac{1}{\gamma^2}}, 0, 0, 0, \dots\right)$ and hence T_1 is a one dimensional compact operator.

We still need to find out where the basis element $(1,0)$ is mapped by the connecting map. We proceed analogously as above. Remember that by our choice of isomorphism between $\mathbb{Z}^2$ and $\ker \delta_0$, $(1,0)$ is mapped to $(1,1,0)$. Looking once more at the first part of the proof of proposition III.28 we conclude that $(1,1,0)$ is mapped to $[f]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0|_{\mathrm{X}_0}\right)\right)$, where the function f is defined to be 1 at one representative of the vertex v_1, to be 1 at one representative of the vertex v_2 and 0 otherwise. Letting $(a+1, a+1)$, for some $a \in I\!R$, be the representative of v_1 and $(a+1+\frac{1}{\gamma}, a+1+\frac{1}{\gamma})$ be the representative of v_2 we have the following very illustrative picture of f.



and we can write f as $f(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) = (a+1, a+1) \\ 1 & \text{if} \quad (x,y) = (a+1+\frac{1}{\gamma}, a+1+\frac{1}{\gamma}) \\ 0 & \text{otherwise} \end{cases}$

We know that $[f]_0$ lifts to an element in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$. Moreover $[f]_0$ lifts to $[\tilde{f}]_0$ where $\tilde{f}$ is the projection defined below.

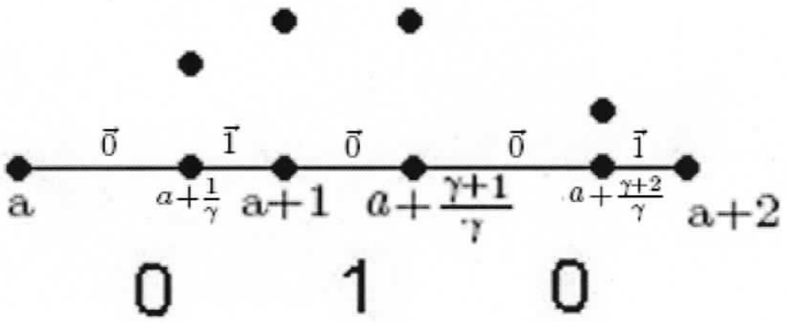
$$\tilde{f}(x,y) = \begin{cases} t \ \ \text{if} \quad (x,y) = (a+t, a+t) \quad \text{for } 0 \le t \le 1 \\ 1 \ \ \text{if} \quad (x,y) = (a+1+t, a+1+t) \quad \text{for } 0 \le t \le \frac{1}{\gamma} \\ 1-t \ \ \text{if} \quad (x,y) = (a+1+\frac{1}{\gamma}+t, a+1+\frac{1}{\gamma}+t) \quad \text{for } 0 \le t \le 1 \\ \sqrt{t-t^2} \ \ \text{if} \quad (x,y) = (a+t, a+1+\frac{1}{\gamma}+t) \quad \text{for } 0 < t < 1 \\ \sqrt{t-t^2} \ \ \text{if} \quad (x,y) = (a+1+\frac{1}{\gamma}+t, a+t) \quad \text{for } 0 < t < 1 \\ 0 \ \ \text{otherwise} \end{cases}$$

So we have that $(0,1)$ is "lifted" to $[\tilde{f}]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$, which we can include in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ via $\mathrm{K}_0(\iota)$, where ι is the inclusion map as in proposition IV.9. Next we restrict $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ to the vertices in G_1, i.e., we need to compute $\mathrm{K}_0(\psi)\left(\mathrm{K}_0(\iota)([\tilde{f}]_0)\right)$ where ψ is the restriction map. So we need to recognize the points in $\mathrm{G}_1|_{\mathrm{X}_0}$ where the function $\tilde{f}$ is non zero. Following the definition IV.20 of a vertex in G_1, we have that $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ is mapped to $[g]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1|_{\mathrm{X}_0}\right)\right) \cong \ker \delta_0$

where g is the projection below:

$$g(x,y)=\begin{cases}\frac{1}{\gamma} \text{ if } (x,y)=(a+\frac{1}{\gamma},a+\frac{1}{\gamma})\\ 1-\frac{1}{\gamma} \text{ if } (x,y)=(a+1+\frac{2}{\gamma},a+1+\frac{2}{\gamma})\\ 1 \text{ if } (x,y)=(a+1,a+1)\\ 1 \text{ if } (x,y)=(a+1+\frac{1}{\gamma},a+1+\frac{1}{\gamma})\\ \sqrt{\frac{1}{\gamma}-\frac{1}{\gamma^2}} \text{ if } (x,y)=(a+\frac{1}{\gamma},a+1+\frac{2}{\gamma})\\ \sqrt{\frac{1}{\gamma}-\frac{1}{\gamma^2}} \text{ if } (x,y)=(a+1+\frac{2}{\gamma},a+\frac{1}{\gamma})\\ 0 \text{ otherwise}\end{cases}$$

Observe that $(a+1,a+1)$ is a representative of the vertex v_2 in G_1, $(a+1+\frac{1}{\gamma},a+1+\frac{1}{\gamma})$ is a representative of the vertex v_3 in G_1 as well as $(a+\frac{1}{\gamma},a+\frac{1}{\gamma})$ and $(a+1+\frac{2}{\gamma},a+1+\frac{2}{\gamma})$ are representatives of the vertex v_1 in G_1. The picture of g below makes this more clear.



Following the isomorphism of proposition III.24 from $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_1|_{\mathrm{X}_0}))$ to $\bigoplus_{i=1..3} \mathrm{K}_0(\mathrm{K}(l_2[v_i]))$ we have that $[g]_0$ is mapped into $([T_1]_0,[\mathrm{T}_2]_0,[\mathrm{T}_3]_0)$, where T_1, T_2 and T_3 are 1-dimensional compact operators in $l_2[v_1]$, $l_2[v_2]$ and $l_2[v_3]$ respectively.

Since the isomorphism between $\bigoplus_{i=1..3} \mathrm{K}_0(\mathrm{K}(l_2[v_i]))$ and $\mathbb{Z}^3$ is given by the dimension of the operators we have that $([T_1]_0,[\mathrm{T}_2]_0,[\mathrm{T}_3]_0)$ is mapped to $(1,1,1)$ and from our choice of isomorphism from $\ker\delta_0$ into $\mathbb{Z}^2$ we have that $(1,1,1)$ is mapped to $(1,1)$.

So we conclude that the connecting map c in K_0 takes the vector $(1,0)$ to the vector $(1,1)$ and the connecting map in $\mathbb{Z}^2$ is given by the matrix

$$c=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

The matrix c above has determinant -1 and hence it is an isomorphism from $\mathbb{Z}^2$ into $\mathbb{Z}^2$. This implies that the inductive limit

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}} \cdots$$

is isomorphic to $\mathbb{Z}^2$ and hence

$$\mathrm{K}_0\left(\mathrm{C}_r^*\left(\cup \mathrm{G}_k\right)\right) \cong \mathbb{Z}^2$$

Before proceeding to K_1 we compute the trace of $\mathrm{K}_0(\mathrm{C}_r^*(\cup \mathrm{G}_k))$ using the formula of lemma IV.6.

Proposition IV.22. *$\mathrm{K}_0(\tau)(\mathrm{K}_0(\mathrm{C}_r^*(\cup \mathrm{G}_k))) = \mathbb{Z}[\frac{1}{\gamma}] \oplus \gamma\mathbb{Z}$ where $\mathrm{K}_0(\tau)$ is the map in K-theory induced by τ, and γ is the golden mean.*

Remember that for any $k \in \mathbb{N}$, $\mathrm{K}_0(\mathrm{C}_r^*(\mathrm{G}_k)) = \langle [f_1^k]_0, [f_2^k]_0 \rangle$, where $[f_1^k]_0 \cong (1,1,0)$ and $[f_2^k]_0 \cong (0,0,1)$. We have explicitly defined f_1^0 and f_2^0 and the definition of f_1^k, f_2^k are analogous. Once we have this we can just use the trace formula to compute $\tau(f_1^k)$ and $\tau(f_2^k)$ (even the definition works!). Just observe that edges in G_k are scaled down by γ^{-1}. For example if $(a+\frac{1}{\gamma^k}, (a+\frac{1}{\gamma^k})$ is a representative of v_3 in G_k then

$$f_2^k(x,y) = \begin{cases} t \quad \text{if} \quad (x,y) = (a+\frac{t}{\gamma^k}, a+\frac{t}{\gamma^k}) \quad 0 \le t \le 1 \\ 1-t \quad \text{if} \quad (x,y) = (a+\frac{1+t}{\gamma^k}, a+\frac{1+t}{\gamma^k}) \quad 0 \le t \le 1 \\ 0 \quad \text{if } x = y \text{ not as above} \\ * \quad \text{if} \quad x \neq y \end{cases}$$

Notice that the off diagonal values of f_2^k are irrelevant for the trace. With this description we get $\mathrm{K}_0(\tau)([f_2^k]_0) = \tau(f_2^k) = \frac{1}{\gamma^k}$.

Analogously we get that $\mathrm{K}_0(\tau)([f_1^k]_0) = \frac{1}{\gamma^k} + \frac{1}{\gamma^{k+1}} = \frac{1}{\gamma^{k-1}}$. Observe that $\mathrm{K}_0(\tau)([f_1^0]_0) = \gamma$.

Finally for all $k \in \mathbb{N}$, let $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau)$. Then $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau_{k+1}) \circ \mathrm{K}_0(\iota)$ and hence the collection of maps $\{\mathrm{K}_0(\tau_k)\}_{i=0}^{\infty}$ gives a well defined map τ_u, from the direct limit $\mathrm{K}_0(\mathrm{C}_r^*(\cup \mathrm{G}_k))$ onto $\mathbb{Z}[\frac{1}{\gamma}] \oplus \gamma\mathbb{Z}$ as desired.

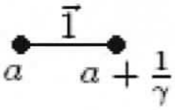
■

- $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\bigcup \mathrm{G}_k\right)\right)$

We know that $\mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G}_0)\right) \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)} \cong \mathbb{Z}$. To compute $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\bigcup \mathrm{G}_k\right)\right)$ we need to perform an isomorphism chase very similar to the one we did to compute $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\bigcup \mathrm{G}_k\right)\right)$. Similarly we know that $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_k\right)\right)$ is isomorphic to $\mathbb{Z}$ for all $k \in \mathbb{Z}^+$ and the connecting maps from $\mathbb{Z}$ to $\mathbb{Z}$ are all the same.

Since the connecting map, c, in K_1 is an group homorphism it is enough to find where 1 is mapped by c. Following the isomorphism from $\mathbb{Z}$ to $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)}$ we have that 1 is mapped to $\overline{(0,1)}$ which in turn is mapped to $(0,1)$ in $\bigoplus_{\mathcal{E}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$. (We could also choose to take 1 to $\overline{(1,0)}$).

We now choose a representative of the edge $\vec{1}$, say $(a+\lambda, a+\lambda)$ for some $a \in I\!R$ and $0 \leq \lambda \leq \frac{1}{\gamma}$ as in the picture below



Proceeding analogously to the second part of the proof of proposition III.28 we have that $(1,0)$ is mapped to $[g]_1 \in \mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_0|_{\mathrm{X}_1-\mathrm{X}_0}\right)\right)$, where the function g in $\mathrm{C}_r^*\left(\widetilde{\mathrm{G}_0|_{\mathrm{X}_1-\mathrm{X}_0}}\right)$ is defined as follows

$$g(x,y) = \begin{cases} \exp(2\pi i\lambda) & \text{if} \quad (x,y) = (a+\lambda\frac{1}{\gamma}, a+\lambda\frac{1}{\gamma}) \quad \text{for} \quad \lambda \in [0,1] \\ 1 \quad \text{if } x = y \\ 0 \quad \text{otherwise} \end{cases}$$

and we can include $[g]_1$ in $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$ and then in $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ via $\mathrm{K}_1(\iota)$. Our final step is to follow back the isomorphism from $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ to $\mathbb{Z}$.

Observe that our edge representative is still an edge in G_1. Actually $(a+\lambda, a+\lambda)$ with $0 \leq \lambda \leq \frac{1}{\gamma}$ is a representative of the edge $\vec{0}$ in G_1. And we have that g winds once in this edge (g here means the inclusion of g in $\mathrm{C}_r^*\left(\mathrm{G}_1\right)$). So proceeding again analogously to the second part of the proof of proposition III.28 we have that $[g]_1$ is mapped to $\overline{(0,1)}$ in $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)}$ which is mapped to 1 in Z.

So the connecting map c in K_1 is just the identity map and hence the inductive limit $\mathbb{Z} \overset{\mathrm{id.}}{\to} \mathbb{Z} \overset{\mathrm{id.}}{\to} \mathbb{Z} \overset{\mathrm{id.}}{\to} \cdots$ is isomorphic to $\mathbb{Z}$ what implies that

$$\mathrm{K}_1\left(\mathrm{C}_r^*\left(\cup \mathrm{G}_k\right)\right) \cong \mathbb{Z}$$

Chapter V

Examples

V.1 The Thue Morse tiling revisited

The Thue Morse tiling was introduced in section III.3. It was shown that

$$\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \ker(\delta_0) \cong \mathbb{Z}^3 \quad \text{and} \quad \mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G})\right) \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)} \cong \mathbb{Z}$$

where the vertex equivalence classes are $v_1 = \vec{0}\bullet\vec{0}$, $v_2 = \vec{0}\bullet\vec{1}$, $v_3 = \vec{1}\bullet\vec{0}$ and $v_4 = \vec{1}\bullet\vec{1}$ and the edge equivalence classes are $e_1 = \vec{0}$ and $e_2 = \vec{1}$. Also the vectors $(1,0,0,0)$, $(0,1,1,0)$ and $(0,0,0,1)$ generate the kernel of δ_0 and $(1,-1)$ generates the image of δ_0.

We now compute the K-theory of $\mathrm{C}_r^*(\cup \mathrm{G}_k)$ following the same ideas used in section IV.3.

- $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\bigcup \mathrm{G}_k\right)\right)$

We have that $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_k\right)\right)$ is isomorphic to $\mathbb{Z}^3$ for all $k \in \mathbb{Z}^+$ and the connecting maps from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ are all the same (using proposition IV.21). In order to compute the connecting map we need to specify the isomorphism between $\ker(\delta_0)$ and $\mathbb{Z}^3$. We will use the isomorphism that maps $(1,0,0,0)$ to $(1,0,0)$, $(0,1,1,0)$ to $(0,1,0)$ and $(0,0,0,1)$ to $(0,0,1)$. Since the connecting map is a group homomorphism it is enough to find where the basis elements $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are mapped.

We start with $(1,0,0)$. By our choice of isomorphism $(1,0,0)$ is mapped to the vector $(1,0,0,0)$ in the kernel of δ_0. Proceeding analogously to the first part of the proof of proposition III.28 we conclude that $(1,0,0,0)$ is mapped to $[f]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0|\mathrm{x}_0\right)\right)$, where the function f is defined to be 1 at one representative of the vertice v_1 and 0 otherwise. Letting $(a+1, a+1)$, for some $a \in \mathbb{R}$, be the representative of v_1 we we can write f as

$$f(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) = (a+1, a+1) \\ 0 & \text{otherwise} \end{cases}$$

We know that $[f]_0$ lifts to an element in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$. Actually, $[f]_0$ lifts to $[\tilde{f}]_0$ where $\tilde{f}$ is the projection defined below.

$$\tilde{f}(x,y) = \begin{cases} t \quad \text{if} \quad (x,y) = (a+t, a+t) \quad \text{for } 0 \leq t \leq 1 \\ 1-t \quad \text{if} \quad (x,y) = (a+1+t, a+1+t) \quad \text{for } 0 \leq t \leq 1 \\ \sqrt{t-t^2} \quad \text{if} \quad (x,y) = (a+t, a+1+t) \quad \text{for } 0 < t < 1 \\ \sqrt{t-t^2} \quad \text{if} \quad (x,y) = (a+1+t, a+t) \quad \text{for } 0 < t < 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

So we have that $(1,0,0)$ is "lifted" to $[\tilde{f}]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$, which we can include in $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ via $\mathrm{K}_0(\iota)$, where ι is the inclusion map as in proposition IV.9. Notice that we could also have "lifted" $(1,0,0)$ to $[\hat{f}]_0$ where $[\hat{f}]_0 = [\tilde{f}]_0$ and $\hat{f}$ is either 1 or 0 vertices of G_1. We could argue that it would be easier to find where this projection is mapped under the isomorphism of $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ to $\mathbb{Z}^3$. We define

$$\hat{f}(x,y) = \begin{cases} 2t \quad \text{if} \quad (x,y) = (a+t, a+t) \quad \text{for } 0 \leq t \leq \frac{1}{2} \\ 1-2t \quad \text{if} \quad (x,y) = (a+1+t, a+1+t) \quad \text{for } 0 \leq t \leq \frac{1}{2} \\ \sqrt{2t-4t^2} \quad \text{if} \quad (x,y) = (a+t, a+1+t) \quad \text{for } 0 < t < \frac{1}{2} \\ \sqrt{2t-4t^2} \quad \text{if} \quad (x,y) = (a+1+t, a+t) \quad \text{for } 0 < t < \frac{1}{2} \\ 1 \quad \text{if} \quad (x,y) = (a+t, a+t) \quad \text{for } \frac{1}{2} \leq t \leq 1 \\ 1 \quad \text{if} \quad (x,y) = (a+1+t, a+1+t) \quad \text{for } \frac{1}{2} \leq t \leq 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Our next task is to follow the isomorphism from $\mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ to $\mathbb{Z}^3$ to find out where $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ is mapped. We proceed by restricting $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ to the vertices in G_1. Following the definition IV.20 of a vertice in G_1, we have that $\mathrm{K}_0(\iota)([\tilde{f}]_0)$ is mapped to $[g]_0 \in \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1|_{\mathrm{X}_0}\right)\right) \cong \ker \delta_0$ where g is the projection below:

$$g(x,y) = \begin{cases} \frac{1}{2} \quad \text{if} \quad (x,y) = (a+\frac{1}{2}, a+\frac{1}{2}) \\ 1 \quad \text{if} \quad (x,y) = (a+1, a+1) \\ \frac{1}{2} \quad \text{if} \quad (x,y) = (a+1+\frac{1}{2}, a+1+\frac{1}{2}) \\ \sqrt{\frac{1}{2}-\frac{1}{4}} = \frac{1}{2} \quad \text{if} \quad (x,y) = (a+\frac{1}{2}, a+1+\frac{1}{2}) \\ \frac{1}{2} \quad \text{if} \quad (x,y) = (a+1+\frac{1}{2}, a+\frac{1}{2}) \\ 0 \quad \text{otherwise} \end{cases}$$

Observe that $(a+1, a+1)$ is a representative of the vertice v_3 in G_1 as $(a+\frac{1}{2}, a+\frac{1}{2})$ and $(a+1+\frac{1}{2}, a+1+\frac{1}{2})$ are representatives of the vertice v_2 in G_1.

Following the isomorphism of proposition III.24 from $\mathrm{K}_0\left(\mathrm{C}^*_r\left(\mathrm{G}_1|_{\mathrm{X}_0}\right)\right)$ to $\bigoplus_{i=1..4} \mathrm{K}_0\left(\mathrm{K}(l_2[v_i])\right)$ we have that $[g]_0$ is mapped into $([T_1]_0, [\mathrm{T}_2]_0, [\mathrm{T}_3]_0, [\mathrm{T}_4]_0)$, where $\mathrm{T}_1 = 0$, $\mathrm{T}_4 = 0$ and T_2 and T_3 are both rank 1 projections in $l_2[v_2]$ and $l_2[v_3]$ respectively.

Since the isomorphism between $\bigoplus_{i=1..4} \mathrm{K}_0\left(\mathrm{K}(l_2[v_i])\right)$ and $\mathbb{Z}^4$ is given by the dimension of the operators we have that $([T_1]_0, [\mathrm{T}_2]_0, [\mathrm{T}_3]_0, [\mathrm{T}_4]_0)$ is mapped to $(0,1,1,0)$ and from our choice of isomorphism from $\ker \delta_0$ into $\mathbb{Z}^3$ we have that $(0,1,1,0)$ is mapped to $(0,1,0)$.

We conclude that the connecting map C in K_0 takes the vector $(1,0,0)$ to the vector $(0,1,0)$. Analogously one can see that $(0,0,1)$ is mapped to $(0,1,0)$ and that $(0,1,0)$ is mapped to $(1,1,1)$. So the connecting map C in $\mathbb{Z}^3$ is given by the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\mathrm{K}_0\left(\mathrm{C}^*_r\left(\cup \mathrm{G}_k\right)\right)$ is isomorphic to the direct limit induced by this matrix.

Next we give a partial description of the inductive limit above. We will define a homomorphism from the direct limit in $\mathbb{Z}^3$, denoted by $\mathbb{Z}^3_{\rightarrow}$, onto $\mathbb{Z}[\frac{1}{2}]$, with kernel $\mathbb{Z}$. Notice that 2 is a nonzero eigenvalue of $C^T = C$ with associate eigenvector $w = (1,2,1)$. Now let $\varphi_i : \mathbb{Z}^3 \to \mathbb{Z}[\frac{1}{2}]$ be defined by $\varphi_i(v) =< v, w > 2^{-i}$ where $v \in \mathbb{Z}^3$. Observe that for all $i \in \mathbb{N}$, $\varphi_i(v) = \varphi_{i+1} \circ C(v)$ and hence the collection of maps $\{\varphi_i\}_{i=0}^{\infty}$ gives a well defined map φ, from the direct limit $\mathbb{Z}^3_{\rightarrow}$ onto $\mathbb{Z}[\frac{1}{2}]$. Notice that each φ_i is a map from $\mathbb{Z}^3$ at step i of $\mathbb{Z}^{10}_{\rightarrow}$. It is clear that φ_0 is onto in $\mathbb{Z}$ and this implies that φ is onto. To find the kernel of φ we first notice that the definition of φ_i implies that $\ker \varphi_i = \ker \varphi_0$ for all i. One can check that a basis for the null space of φ_0 is given by the vectors $n_1 = (1,0,-1)$ and $n_2 = (1,-1,1)$. Now we observe that $C \cdot n_1 = 0$ so this generator vanishes at the inductive limit. Also $C(n_2) = -n_2$ and this implies that any vector in the $\ker \varphi_i$, for any i, is equivalent to a vector in $\ker \varphi_0$ in the inductive limit $\mathbb{Z}^3_{\rightarrow}$. This implies that $\ker \varphi \cong < \overline{n_2} > \cong \mathbb{Z}$, where $\overline{n_2}$ is the inclusion of n_2 in the direct limit $\mathbb{Z}^3_{\rightarrow}$. We can now write the following exact sequence:

$$0 \to \mathbb{Z} \to \mathrm{K}_0\left(\mathrm{C}^*_r\left(\cup \mathrm{G}_k\right)\right) \to \mathbb{Z}[\frac{1}{2}] \to 0$$

Unfortunately we can not say whether this exact sequence splits and hence this is the best we can describe K_0 at the moment.

As we did for the Fibonacci tiling, before proceeding to K_1 we compute the trace of $\mathrm{K}_0(\mathrm{C}^*_r(\cup \mathrm{G}_k))$.

Proposition V.1. $\mathrm{K}_0(\tau)(\mathrm{K}_0(\mathrm{C}^*_r(\cup \mathrm{G}_k))) = \mathbb{Z}[\frac{1}{2}]$ *where* $\mathrm{K}_0(\tau)$ *is the map in* K*-theory induced by* τ*.*

Proof:

The proof is analogous to the proof of proposition IV.22. We elaborate a little below.

Remember that for any $k \in \mathbb{N}$, $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_k)) = \langle [f_1^k]_0, [f_2^k]_0, [f_3^k]_0 \rangle$, where $[f_1^k]_0 \cong (1,0,0,0)$, $[f_2^k]_0 \cong (0,1,1,0)$ and $[f_3^k]_0 \cong (0,0,0,1)$. For example if $(a + \frac{1}{2^k}, (a + \frac{1}{2^k})$ is a representative of v_1 in G_k then

$$f_1^k(x,y) = \begin{cases} t \quad \text{if} \quad (x,y) = (a + \frac{t}{2^k}, a + \frac{t}{2^k}) \quad 0 \le t \le 1 \\ 1-t \quad \text{if} \quad (x,y) = (a + \frac{1+t}{2^k}, a + \frac{1+t}{2^k}) \quad 0 \le t \le 1 \\ 0 \quad \text{if } x = y \text{ not as above} \\ * \quad \text{if} \quad x \neq y \end{cases}$$

Notice that the off diagonal values of f_1^k are irrelevant for the trace. With this description we get $\mathrm{K}_0(\tau)([f_1^k]_0) = \tau(f_1^k) = \frac{1}{2^k}$.

Analogously we get that $\mathrm{K}_0(\tau)([f_2^k]_0) = \frac{1}{2^{k-1}}$ and $\mathrm{K}_0(\tau)([f_3^k]_0) = \frac{1}{2^k}$.

Finally for all $k \in \mathbb{N}$, let $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau)$. Then $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau_{k+1}) \circ \mathrm{K}_0(\iota)$ and hence the collection of maps $\{\mathrm{K}_0(\tau_k)\}_{i=0}^{\infty}$ gives a well defined map τ_u, from the direct limit $\mathrm{K}_0(\mathrm{C}^*_r(\cup \mathrm{G}_k))$ onto $\mathbb{Z}[\frac{1}{2}]$ as desired.

■

- $\mathrm{K}_1\left(\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)\right)$

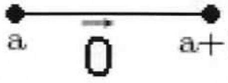
To compute $\mathrm{K}_1\left(\mathrm{C}^*_r\left(\bigcup \mathrm{G}_k\right)\right)$ we need to perform an isomorphism chase very similar to the one in section IV.3. A new idea will be introduced though, as we need to be careful with our choice of representative of the class in K_1 isomorphic to 1.

We know that $\mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)} \cong \mathbb{Z}$ and hence $\mathrm{K}_1\left(\mathrm{C}^*_r\left(\mathrm{G}_k\right)\right)$ is isomorphic to $\mathbb{Z}$ for all $k \in \mathbb{Z}^+$. Also the connecting maps from $\mathbb{Z}$ to $\mathbb{Z}$ are all the same.

Again it is enough to find where 1 is mapped by the connecting map c. Following the isomorphism from $\mathbb{Z}$ to $\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathrm{Im}(\delta_0)}$ we have that 1 is mapped to $\overline{(1,0)}$

which in turn is mapped to $(1,0)$ in $\bigoplus\limits_{\mathcal{E}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$. (We could also choose to take 1 to $\overline{(0,1)}$).

We now choose a representative of the edge $\vec{0}$, say $(a+\lambda, a+\lambda)$ for some $a \in I\!R$ and $0 \leq \lambda \leq 1$ as in the picture below



If we proceeded in an analogous manner to section IV.3 we would have $(0,1)$ mapped to $[g']_1 \in \mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_0|_{\mathrm{X}_1-\mathrm{X}_0}\right)\right)$, where the function g' in $\mathrm{C}_r^*\left(\widetilde{\mathrm{G}_0|_{\mathrm{X}_1-\mathrm{X}_0}}\right)$ is defined as follows

$$g'(x,y) = \begin{cases} \exp(2\pi i\lambda) & \text{if} \quad (x,y) = (a+\lambda, a+\lambda) \quad \text{for} \quad \lambda \in [0,1] \\ 1 \quad \text{if } x = y \\ 0 \quad \text{otherwise} \end{cases}$$

The next step would be to include $[g']_1$ in $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ and then follow the isomorphism back to $\mathbb{Z}$. Here is where the problem would arise as g' does not wind in only one edge of G_1 and hence we would not know how to follow the isomorphism back to $\mathbb{Z}$.

We solve this by choosing another representative for $[g']_1$, i.e., we will find a function g that is homotopic to g' and such that we know where $[g]_1$ is mapped. The function g is defined by

$$g(x,y) = \begin{cases} \exp(2\pi i 2\lambda) & \text{if} \quad (x,y) = (a+\lambda, a+\lambda) \quad \text{for} \quad \lambda \in [0,\frac{1}{2}] \\ 1 \quad \text{if } x = y \\ 0 \quad \text{otherwise} \end{cases}$$

and we include $[g]_1$ in $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right)$ and then in $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ via $\mathrm{K}_1(\iota)$. Our final step is to follow back the isomorphism from $\mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right)$ to $\mathbb{Z}$.

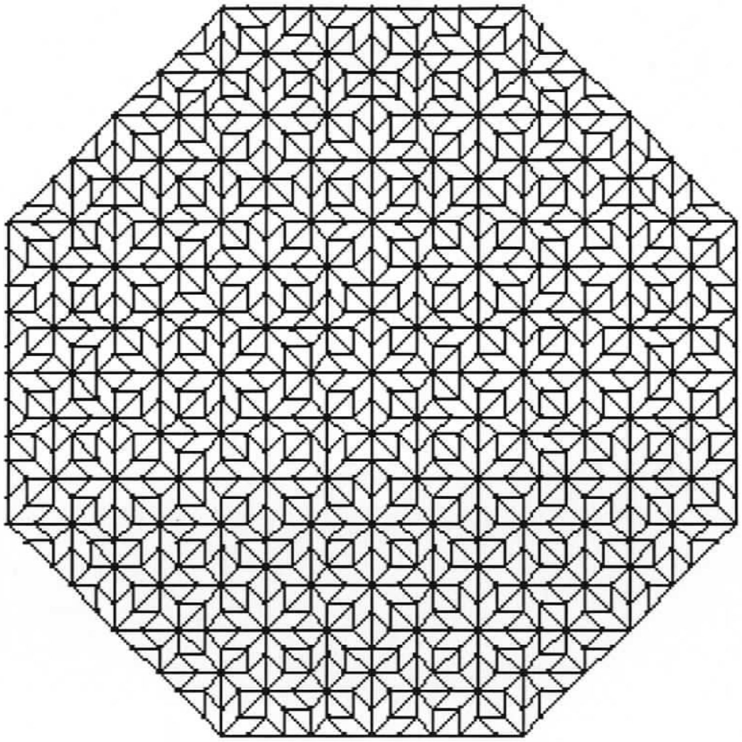
Observe that $(a+\lambda, a+\lambda)$ with $0 \leq \lambda \leq \frac{1}{2}$ is a representative of the edge $\vec{0}$ in G_1 and g winds once in this edge. Also $(a+\lambda, a+\lambda)$ with $\frac{1}{2} \leq \lambda \leq 1$ is a representative of the edge $\vec{1}$ in G_1 and g is equal to one in this edge. g here means the inclusion of g in $\mathrm{C}_r^*\left(\mathrm{G}_1\right)$). So proceeding again analogously to the second part of the proof of proposition III.28 we have that $[g]_1$ is mapped to $\overline{(1,0)}$ in $\frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathrm{Im}(\delta_0)}$ which is mapped to 1 in $\mathbb{Z}$.

So the connecting map c in K_1 is just the identity map and hence the inductive limit $\mathbb{Z} \xrightarrow{\text{id.}} \mathbb{Z} \xrightarrow{\text{id.}} \mathbb{Z} \xrightarrow{\text{id.}} \cdots$ is isomorphic to $\mathbb{Z}$ what implies that

$$\mathrm{K}_1\left(\mathrm{C}_r^*\left(\cup \mathrm{G}_k\right)\right) \cong \mathbb{Z}.$$

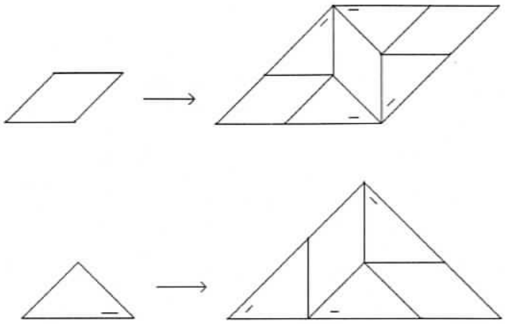
V.2 Examples in two dimensions

V.2.1 The Octagonal Tiling



A patch of the octagonal tiling

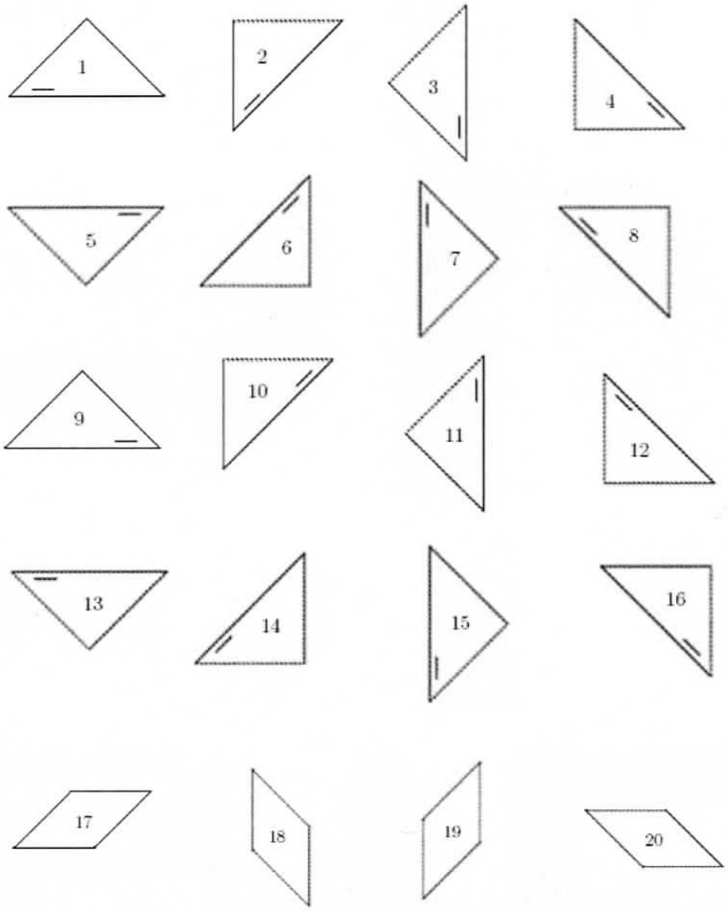
In what follows we will compute the K-theory groups for the C*-algebras introduced in the previous chapter in the case of the octagonal tiling as defined in [11]. This is actually called the triangular version of the octagonal tiling and it is obtained by the substitution given below:



So the tiles in the octagonal tiling are triangles and rhombi. The prototiles consist of the two tiles on the left hand side above and all its rotates around $\frac{n\pi}{4}$ and reflections along the boundaries of the tiles. The substitution is extended to all prototiles by symmetry. Thus the octagonal tiling has twenty prototiles; four of them congruent to the rhombus and the remaining sixteen congruent to the triangle. We refer the reader to [11] to check that the substitution is primitive and recognizable. Local finite complexity follows readily.

In order to compute the K-theory groups it will be very useful to label all prototiles, vertex patterns and edge patterns of the octagonal tiling. It is also necessary to give orientation to the edges and tiles. We assume all tiles are oriented counterclockwise and the edges are oriented as shown in the labeling below:

First we label the prototiles: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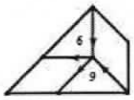
The vertex patterns are labeled below:



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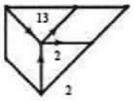

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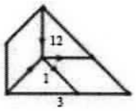

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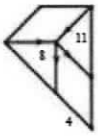

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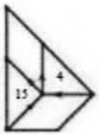

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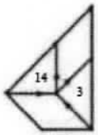

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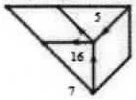

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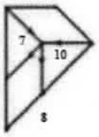

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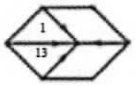

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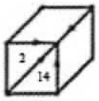

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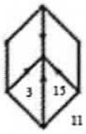

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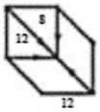

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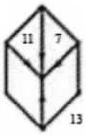

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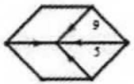

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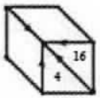

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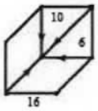

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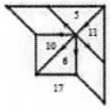

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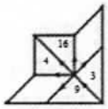

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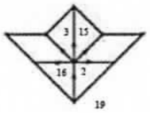

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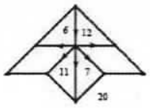

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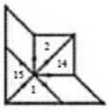

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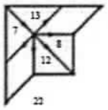

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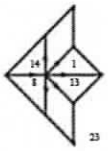

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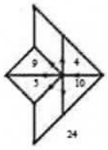

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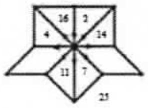

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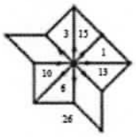

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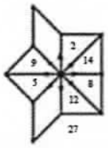

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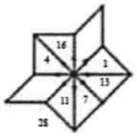

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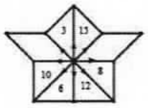

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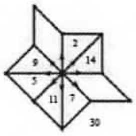

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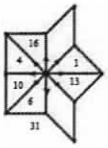

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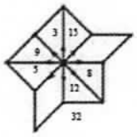

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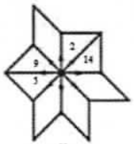

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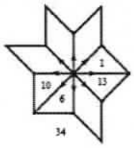

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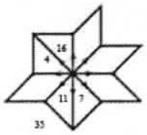

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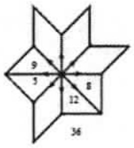

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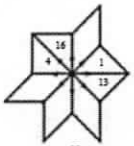

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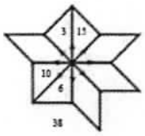

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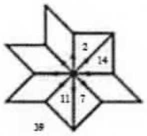

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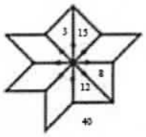

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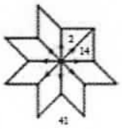

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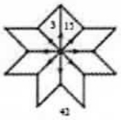

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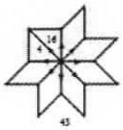

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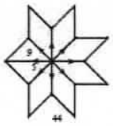

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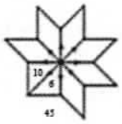

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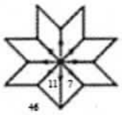

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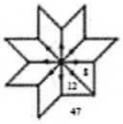

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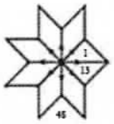

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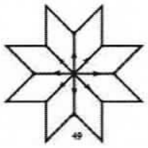

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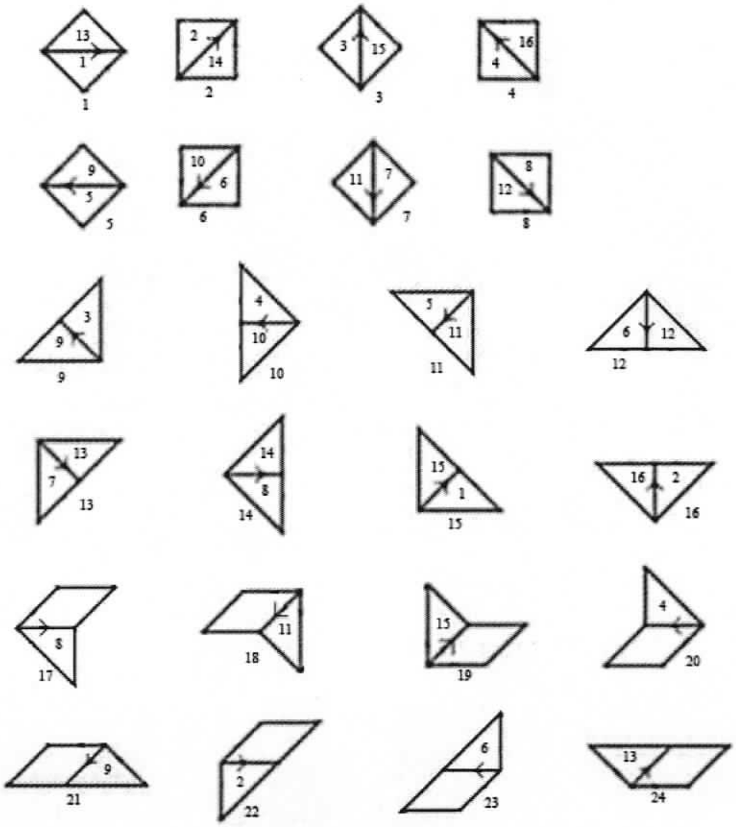
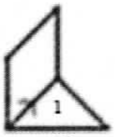
Finally the edge patterns are labeled as shown:





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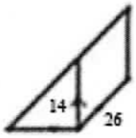

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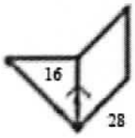

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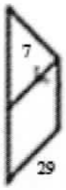

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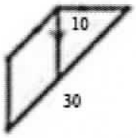

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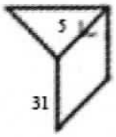

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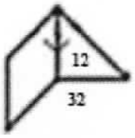

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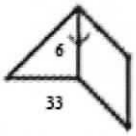

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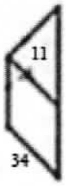

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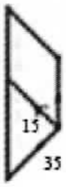

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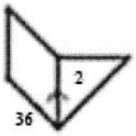

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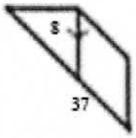

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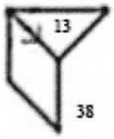

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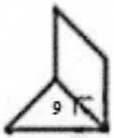

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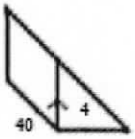

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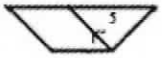

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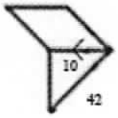

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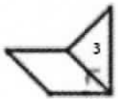

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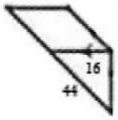

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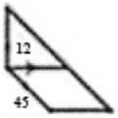

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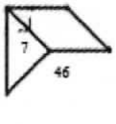

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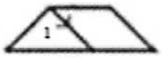

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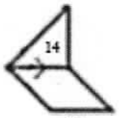

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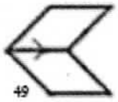
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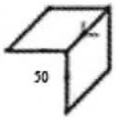
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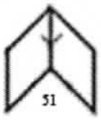
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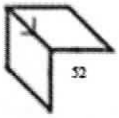
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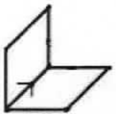
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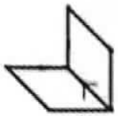
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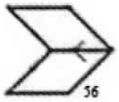
56

This vertex and edge patterns were obtained by applying the substitution enough times to the rhombus prototile (p_{17}) so that the vertex and edge patterns in $w^n(p_{17})$ are the same as in $w^{n+1}(p_{17})$ (in this case $n = 5$).

We now proceed to compute the K-theory groups of $C_r^*(G_0)$. We will use both exact sequences III.6 and III.9. We start by restricting our equivalence relation to the edges, G_{X_1}. Using proposition III.28 we get the following 56x49 matrix for δ_0.

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&1&1&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&1&1&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&1&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1
\end{bmatrix}$$

Using the software gap, [9], we have that the kernel of δ_0 is generated by the 17 vectors below

$$\ker\delta_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 2 & 0 & -1 & 2 & 1 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 2 & -1 & 0 & 2 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 2 & 1 & 0 & 0 & 1 & 0 & -2 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 2 & 0 & -1 & -2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & 0 & -2 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 1 & 0 & -1 & -2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 & -1 & 2 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 2 & 1 & 0 & 1 & 1 & -1 & -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & -7 & -8 & 1 & 1 & -8 & 0 & 8 & 8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \tag{V.1}$$

and hence $\mathrm{K}_0\left(\mathrm{C}_r^*(\mathrm{G}_0|_{\mathrm{X}_1}\right)$ is isomorphic to $\mathbb{Z}^{17}$.

Also the integer image of δ_0 is given by the matrix below:

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
-1&0&1&0&-1&0&1&0&0&1&0&-1&0&1&0&-1&1&-1&-1&1&1&-1&-1&1&1&1&-1&-1&-1&1&1&0\\
0&0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
1&0&-1&0&1&0&-1&1&0&-1&0&0&0&-1&0&1&-1&1&1&-1&-1&1&1&-1&-1&-1&1&1&1&-1&-1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&1&-1&0&0&-1&0&0&0&-1&-1&0&0&0&1&1&1&-1&1&-1&-1&1&1&-1&0&-1&1&1&0&0&-1&1\\
1&1&-1&0&1&-1&-1&0&0&-1&-1&1&0&-1&1&1&0&0&1&-1&-2&1&2&-1&0&-2&2&1&1&-1&-1&1\\
0&1&-1&1&0&0&-1&0&0&-1&0&0&1&-1&1&0&0&0&1&-1&-1&0&1&0&0&-1&1&0&1&-1&0&1\\
0&0&0&0&-1&0&1&0&0&1&0&0&0&0&0&-1&1&-1&-1&1&0&0&0&0&0&0&0&-1&-1&1&1&1\\
1&0&-1&0&-1&0&1&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&1&0&-1&0&-1&1&-1&0&0&1&1\\
1&1&-2&0&-1&0&0&1&1&-1&0&0&0&-1&1&0&0&0&1&-1&0&1&0&-1&0&-2&2&0&0&0&0&1\\
1&1&-1&0&-1&-1&1&0&0&0&-1&0&0&0&1&0&1&-1&1&-1&0&0&0&0&0&-1&1&0&-1&1&0&1
\end{bmatrix}$$

and we have that $\mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G}_0|_{\mathrm{X}_1}\right)$ is isomorphic to $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0}$.

We now encounter a problem as we can not promptly identify this quotient. We will solve this using the Smith Normal Form of δ_0, introduced in section I.2. Again using the software gap we have that the Smith Normal Form of δ_0 is given by the diagonal matrix $\delta = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ with 32 1's and 24 0's.

So it is clear that $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta}$ is isomorphic to $\mathbb{Z}^{24}$ and hence $\mathrm{K}_1\left(\mathrm{C}_r^*(\mathrm{G}_0|_{\mathrm{X}_1}\right) \cong \frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0} \cong \mathbb{Z}^{24}$.

We have computed the K-groups for $\mathrm{C}_r^*(\mathrm{G}|_{\mathrm{X}_1})$ and can now use the six term exact sequence III.9 to compute the K-groups of $C_r^*(G_0)$. Corollary III.33 implies that $K_0(C_r^*(G_0))$ is isomorphic to $\ker \delta_0 \oplus \mathbb{Z} \cong \mathbb{Z}^{18}$. In order to compute K_1 we need first to compute the index map.

To get a matrix for the index map we need a basis for $K_1\left(C_r^*(G_0|_{X_1})\right)$. From the matrix description of the image of δ_0 given before it follows promptly that $\{\overline{e_{32}}, \overline{e_{33}}, \ldots, \overline{e_{56}}\}$ is a generating set for $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0}$. Also from the last column of the matrix for the image of δ_0, we conclude that $\overline{e_{56}}$ is equivalent to $-\overline{e_{55}} - \overline{e_{54}} - \overline{e_{53}} - \overline{e_{52}} - \overline{e_{51}} - \overline{e_{50}} - \overline{e_{49}}$ and hence $\{\overline{e_{32}}, \overline{e_{33}}, \ldots, \overline{e_{55}}\}$ is a basis for $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0} \cong \mathbb{Z}^{24}$. Now using the description of the index map given in proposition III.30 we get the following matrix for δ_1:

$$\delta_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

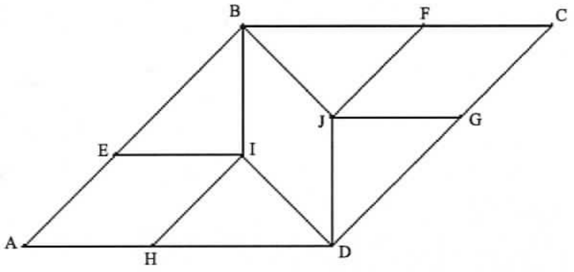
And the following vectors are a basis for the kernel of δ_1:

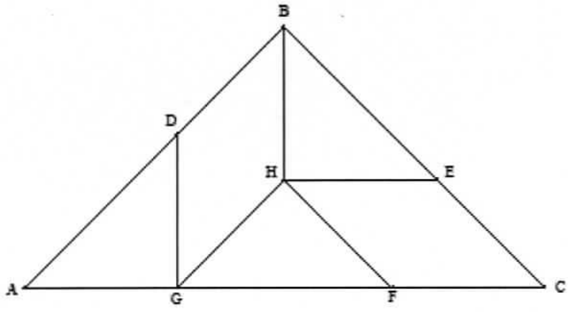
$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 \\
-1 & 1 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & -1
\end{bmatrix} \tag{V.2}$$

and hence $K_1\left(C_r^*(G_0)\right) \cong \mathbb{Z}^5$.

K-theory of the inductive limit C*-algebra

We have shown that $K_0(C_r^*(G_0)) \cong \mathbb{Z}^{18}$ and $K_1(C_r^*(G_0)) \cong \mathbb{Z}^5$. We now proceed to compute the K-theory of $C_r^*(\cup G_k)$ as introduced in section IV.2.1. But in the two dimensional case it is not so easy to follow all isomorphisms. We will need to make some "smart" choices for representatives and isomorphisms and this will depend mostly on the map F defined below. Roughly F is a map that collapses the new vertices and edges obtained by applying the inflation map to a tile into the "old" vertices and edges of this tile. We start by defining F on a rectangle prototile and a triangle prototile. We then extend F to $\mathbb{R}^2$ by symmetry (using the fact that the octagonal tiling covers the plane). To make it easier to define F we name the vertices of this prototiles and their inflation as show in the pictures below:





Now we define F as a map that takes the rhombus vertices A, E, H, I to the vertex A, the vertices J, F, G, C to the vertex C and leaves the vertices B and D fixed. Moreover we require F to be a homeomorphism from the interior of the rhombus defined by BIDJ into the interior of the rhombus ABCD. Also F should take the triangle vertices A, D, G to the vertex A, the vertices F, C to C and the vertices H, E, B to the vertex B. Moreover F should be a homeomorphism from the interior of the triangle GHF into the interior of the triangle ABC. Finally we require F to be a continuous map. Since F agrees on the borders of the matching tiles of the octagonal tiling we can extend it to $\mathbb{R}^2$ by symmetry. Observe that vertices in $\omega(\mathrm{T})$ are mapped to vertices in T under F and edges in $\omega(\mathrm{T})$ are mapped to either vertices or edges in T. Furthermore notice that the deformation of the tiles induced by F can be done in a continuous way, so that there exists a path of continuous maps F_t such that F_0 is the identity and $F_1 = F$. For example we can define $F_t(x) = tF(x) + (1-t)(x)$, for $x \in I\!R^d$, $t \in [0,1]$. Now for $f \in \mathrm{C}_c(\mathrm{G}_0)$ we define $\alpha_t : \mathrm{C}_c(\mathrm{G}_0) \to \mathrm{C}_c(\mathrm{G}_0)$ by

$$\alpha_t(f)(x,y) = \begin{cases} f\left(F_t(x), F_t(y)\right) & \text{if} \quad (F_t(x), F_t(y)) \in \mathrm{G}_0 \\ 0 \quad \text{otherwise} \end{cases}$$

and this implies that $\alpha(f) := \alpha_1(f)$ is homotopic to f. We still need to show that each α_t is well defined, i.e., $\alpha_t(f)$ is continuous and has compact support. We will do the proof for $\alpha(f)$ and for $\alpha_t(f)$ one proceed analogously.

- $\alpha(f)$ *is continuous.*

Let (x_n, y_n) be a sequence in G_0 such that (x_n, y_n) converges to $(x,y) \in \mathrm{G}_0$. We have to consider two cases.

First suppose $(F(x), F(y)) \in \mathrm{G}_0$. From continuity of F we have that there exists N such that $(F(x_n), F(y_n)) \in \mathrm{T}(F(x)) \times \mathrm{T}(F(y))$ for all $n > \mathrm{N}$. Furthermore we can choose N such that $(\mathrm{T}(F(x_n)), \mathrm{T}(F(y_n))) \subseteq \mathrm{T}(F(x)) \times \mathrm{T}(F(y))$ for all $n > \mathrm{N}$. This implies that $(F(x_n), F(y_n)) \in \mathrm{G}_0$ for all $n > \mathrm{N}$ and from continuity of F we have that $(F(x_n), F(y_n))$ converges to $(F(x), F(y))$. Now it follows from the continuity of f that $\alpha(f)(x_n, y_n)$ converge to $\alpha(f)(x,y)$.

Now suppose $(F(x), F(y)) \notin \mathrm{G}_0$. We have that $\alpha(f)(x,y) = 0$. Let K be the support of f. Observe that $(F(x), F(y)) \notin \mathrm{K}$ since $(F(x), F(y)) \notin \mathrm{G}_0$. We will show that there exists $\mathrm{N} > 0$ such that $(F(x_n), F(y_n)) \notin \mathrm{K}$ for all $n > \mathrm{N}$. Suppose not. Then there exists a sequence $(F(x_{n_k}), F(y_{n_k}))$ in K and since K is compact this sequence has a converging subsequence, say $(F(x_{n_{k_i}}), F(y_{n_{k_i}}))$. Since $(F(x_n), F(y_n))$ converges to $(F(x), F(y))$ we have that $(F(x_{n_{k_i}}), F(y_{n_{k_i}}))$ also converges to $(F(x), F(y))$ and since K is compact this implies that $(F(x), F(y)) \in \mathrm{K}$, which is a contradiction. We conclude that $\alpha(f)(x_n, y_n) = 0$ for $n > \mathrm{N}$ and hence α is continuous.

- $\alpha(f)$ *has compact support.*

Let K be the support of f and K_α be the support of $\alpha(f)$. Also let D be the maximum distance between two points inside the same prototile, i.e., $D = \max_{t \in \mathcal{P}} \max_{x,y \in t} d(x,y)$ and let M be such that $\mathrm{K} \subseteq \mathrm{B}(0, M)$. Then $\mathrm{K}_\alpha \subseteq \overline{B(0, M+2D)}$ since f vanishes on $(\mathrm{T}(x) \times \mathrm{T}(y)) \cap \mathrm{G}_0$ for any $(x,y) \notin B(0, M+2D)$ and $F \times F(x,y) \subseteq \mathrm{T}(x) \times \mathrm{T}(y)$. We conclude that K_α is totally bounded since $\overline{B(0, M+2D)} \cap \mathrm{G}_0$ also is. Next we show that K_α is complete.

So let (x_n, y_n) be a Cauchy sequence in K_α. Then this sequence converges to a point $(x, y) \in I\!R^2 \times I\!R^2$ and it is enough to show that $(x, y) \in \mathrm{G}_0$. To do so first we show that $(F(x_n), F(y_n))$ is a sequence is K. Observe that if $\alpha(f)(x_n, y_n) \neq 0$ then $(F(x_n), F(y_n))$ is clearly in K. If $\alpha(f)(x_n, y_n) = 0$ then, since $(x_n, y_n) \in \mathrm{K}_\alpha$, there exists a sequence $(x_n^k, y_n^k) \in \mathrm{K}_\alpha$ such that $\alpha(f)(x_n^k, y_n^k) \neq 0$ and (x_n^k, y_n^k) converges to (x_n, y_n). But this implies that $(F(x_n^k), F(y_n^k))$ is a sequence in K that converges to $(F(x_n), F(y_n))$ and since K is compact $(F(x_n), F(y_n)) \in \mathrm{K}$. Now this implies that $(F(x), F(y)) = \lim(F(x_n), F(y_n))$ belongs to K and hence $(F(x), F(y)) \in \mathrm{G}_0$. Finally since F collapses points on the border of the tiles to vertex points we have that $(F(x), F(y)) \in \mathrm{G}_0$ implies that $(x, y) \in \mathrm{G}_0$. (Observe that if $(x, y) \notin \mathrm{G}_0$ then (x, y) has to belong to the 1-skeleton of a tile).

■

We are now able to compute some inductive limits.

$\mathrm{K}_0(C_r^*(\cup \mathrm{G}_k))$

From what was done previously we have that $\mathrm{K}_0(C_r^*(\mathrm{G}_k)) \cong \mathbb{Z}^{18}$. We want to compute the connecting map such that the diagram below is commutative

$$\begin{tikzcd}
\mathbb{Z}^{18} \arrow[r, "\cong"] \arrow[d, dotted, "c"'] & \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right) \arrow[d, hook, "\mathrm{K}_0(\iota)"'] \\
\mathbb{Z}^{18} & \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right) \arrow[l, "\cong"']
\end{tikzcd}$$

In order to make the isomorphisms chases more clear we repeat here the six term exact sequence III.9

$$\begin{tikzcd}
\mathbb{Z}^{20} \cong \mathrm{K}_0(\mathrm{I}_{\mathrm{X}_1}) \arrow[r, "\mathrm{K}_0(\iota)"] & \mathrm{K}_0\left(C_r^*(\mathrm{G}_0)\right) \arrow[r, "\mathrm{K}_0(\psi)"] & \mathbb{Z}^{17} \arrow[d] \\
\mathbb{Z}^{24} \cong \dfrac{\bigoplus_{\text{edge}} \mathbb{Z}}{\mathrm{Im}(\delta_0)} \arrow[u, "\delta_1"] & \mathrm{K}_1\left(C_r^*(\mathrm{G}_0)\right) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Here we need to make a choice of isomorphism. Since $\mathrm{K}_0(C_r^*(\mathrm{G}_0)) \cong \ker \delta_0 \oplus \mathbb{Z}$ we will say that, for $i = 1..17$, the canonical basis vectors e_i in $\mathbb{Z}^{18}$ is isomorphic to the column vector c_i in the kernel of δ_0 as described in the matrix V.1. Also e_{18} is isomorphic to an element $[g]_0 \in \mathrm{K}_0(\mathrm{I}_{\mathrm{X}_1})$ included in $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_0))$ that is NOT on $\ker(\mathrm{K}_0(i)) = \mathrm{Im}\delta_1$.

We start with e_1. We deal with the vectors e_2 to e_{17} in a similar way. From the choice of isomorphism above e_1 is mapped to the first column vector of the matrix for $\ker \delta_0$, c_1. This vector c_1 is isomorphic to $[f]_0$, where f is a function in $\ker \delta_0$. We then lift $[f]_0$ to $\mathrm{K}_0(C^*_r(\mathrm{G}_0))$ and call this lift $[\tilde{f}]_0$, where $\tilde{f} \in \mathrm{C}_c(\mathrm{G}_0)$. From what we did above we have that $[\tilde{f}]_0 = [\alpha(\tilde{f})]_0$. We then include $[\alpha(\tilde{f})]_0$ in $\mathrm{K}_0(C^*_r(\mathrm{G}_1))$ and restrict it to the vertices of G_1 in order to follow the isomorphism from $\mathrm{K}_0(C^*_r(\mathrm{G}_1))$ into $\ker \delta_0$. So far things do not look much simpler, but since the definition of $\alpha(\tilde{f})$ depends only on the values of f on the vertices, we can find a matrix for ”F”, where for example the function δ_{v_1} that is defined as 1 on the vertex v_1 and 0 otherwise is taken to the function $\delta_{v_2} + \delta_{v_3} + \delta_{v_6} + \delta_{v_9} + \delta_{v_{10}} + \delta_{v_{40}}$. Actually our matrix, which we call $\mathcal{F}$, is a composition of homomorphisms. It takes a vector in $\ker \delta_0$ in G_0 to a vector in $\ker \delta_0$ in G_1. So once we compute $\mathcal{F}(c_1)$ we only need to write this vector as a linear combination of the basis of $\ker \delta_0$, i.e., as a linear combination of c_1 to c_{17}. The coefficients of this linear combination is the vector in $\mathbb{Z}^{17}$ we are looking for. The diagram below illustrates the definition of $\mathcal{F}$.

$$\begin{tikzcd}
\mathbb{Z}^{17} \arrow[r] & \ker \delta_0 \subseteq \mathrm{K}_0\left(C^*_r(\mathrm{G}_0|_{\mathrm{X}_0})\right) \arrow[r] & \mathrm{K}_0\left(C^*_r(\mathrm{G}_0)\right) \arrow[r] & \mathrm{K}_0\left(C^*_r(\mathrm{G}_0)\right) \\
e_1 \arrow[r] & c_1 \to [f]_0 \arrow[r] & [\tilde{f}]_0 \arrow[r] & [\alpha(\tilde{f})]_0 \arrow[d] \\
 & \mathcal{F}(c_1) & & [\alpha(\tilde{f})]_0 \arrow[ll] \\
\mathbb{Z}^{17} & \ker \delta_0 \subseteq \mathrm{K}_0\left(C^*_r(\mathrm{G}_1|_{\mathrm{X}_0})\right) \arrow[l] & & \mathrm{K}_0\left(C^*_r(\mathrm{G}_1)\right) \arrow[ll]
\end{tikzcd}$$

Proceeding as explained above we get the following matrix for $\mathcal{F}$:

$$
\begin{bmatrix}
0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&1&0&0&1&0&0&1&1&0&1&0&1&0&1&1&1&1&1&1&0&1\\
1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&1&0&1&0&1&1&1&1&1&0&0&1&1&1&1\\
1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&1&0&1&0&0&1&0&0&0&0&1&0&0&0&1&1&1&1&1&0&0&1&1&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&1&0&1&0&0&1&0&1&0&0&1&1&1&1&1&1\\
0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0&0&1&0&0&0&0&1&0&1&1&0&0&1&1&0&0&1&1&1&1&1&0&0&1&1\\
1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&1&0&0&1&1&0&0&1&1&1&1&1&0&0&1&1&1\\
0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&0&0&0&1&0&0&1&0&1&0&0&1&1&0&1&1&1&1&1&1&0&0&1\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&1&1&1&0&0&0&0&0&1&1&1&1&1&1&1\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}
$$

So proceeding as described above we can find where e_1 to e_{17} are mapped under the connecting map. We still need to find where e_{18} is mapped. If we take $[g']_0$ to be the Bott element in the tile p_{17} and 1 everywhere else then $[g']_0$ do not belong to the $\ker(\mathrm{K}_0(i)) = \mathrm{Im}\delta_1$ and hence e_{18} is isomorphic to $[g']_0$. But this is not a great choice of representative. Instead we take $[g]_0$ to be equal to the Bott element on one of the copies of p_{17} obtained by applying the inflation map to p_{17} and 1 everywhere else. One can see that g and g' are homotopic and hence e_{18} is isomorphic to $[g]_0$. Now once we include $[g]_0$ in $\mathrm{K}_0\left(\mathrm{C}^*_r(\mathrm{G}_1)\right)$ we have that $[g]_0$ is a Bott element in exactly one tile of G_1 (the copy of p_{17}) and 1 otherwise and hence $[g]_0$ in $\mathrm{K}_0\left(C^*_r(\mathrm{G}_1)\right)$ is isomorphic to e_{18} in $\mathbb{Z}^{18}$. With all this in mind we get the following connecting map in K_0:

$$C = \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & -1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 2 & 2 & -3 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -2 & 1 & 2 & -2 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & 1 & 2 & -1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 3 & 2 & -3 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 3 & 2 & -3 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 1 & -2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -3 & 2 & 2 & -3 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -4 & 3 & 3 & -4 & -1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and hence $\mathrm{K}_0(C_r^*(\cup \mathrm{G}_k))$ is isomorphic to the direct limit in $\mathbb{Z}^{18}$ induced by C.

Unfortunately we can not describe this limit any further. We do have a better description for K_1 though:

$\mathrm{K}_1(C_r^*(\cup \mathrm{G}_k))$

From what was done previously we have that $\mathrm{K}_1(C_r^*(\mathrm{G}_k)) \cong \mathbb{Z}^5$. We want to compute the connecting map such that the diagram below is commutative

$$\begin{tikzcd}
\mathbb{Z}^5 \arrow[r, "\cong"] \arrow[d, dotted, "c"'] & \mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right) \arrow[d, hook, "\mathrm{K}_1(\iota)"'] \\
\mathbb{Z}^5 & \mathrm{K}_1\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right) \arrow[l, "\cong"']
\end{tikzcd}$$

To find c we will use the fact proved above that $\alpha(f)$ is homotopic to f for any $f \in \mathrm{C}_c(\mathrm{G}_0)$ and then read the values of $\alpha(f)$ on the edges of G_1 from the values of f on the edges in G_0. But first we need to make some explicit choices of isomorphisms.

Since $\mathrm{K}_1(C_r^*(\mathrm{G}_1)) \cong \ker \delta_1$ we will say that, for $i = 1..5$, the canonical basis vectors e_i in $\mathbb{Z}^5$ is isomorphic to the column vector c_i in the kernel of δ_1 as described in the matrix V.2. Now each c_i is a vector in $\mathbb{Z}^{24}$ which is isomorphic to $\frac{\bigoplus_{edges} \mathbb{Z}}{\mathrm{Im}\delta_0} = \frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0}$ via the map $e_i \rightarrow \overline{e_{31+i}}$ where e_i are the canonical basis elements of $\mathbb{Z}^{24}$ and $\overline{e_i}$ are the basis elements of $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0}$.

We may now proceed. We start with the vector $e_5 \in \mathbb{Z}^5$. All other basis elements of $\mathbb{Z}^5$ are dealt in a similar way. By the choice of isomorphisms above e_5 is mapped to $\overline{e_{52}} - \overline{e_{55}}$ in $\frac{\mathbb{Z}^{56}}{\mathrm{Im}\delta_0}$. This element on its turn is mapped to $[f]_1$ in

$\mathrm{K}_1(C^*_r(\mathrm{G}_0|_{\mathrm{X}_1}))$ where f is defined as

$$f(x,y) = \begin{cases} \exp(2\pi i\lambda) & \text{if } \quad (x,y) = (r_{52}(\lambda), r_{52}(\lambda)) \\ \exp(-2\pi i\lambda) & \text{if } \quad (x,y) = (r_{55}(\lambda), r_{55}(\lambda)) \\ 1 \quad \text{if } x = y \\ 0 \quad \text{otherwise} \end{cases}$$

where r_{52} and r_{55} are parametrizations of the edges e_{52} and e_{55} respectively.

Next we need to lift $[f]_1$ to $\mathrm{K}_1(C^*_r(\mathrm{G}_0))$. Observe that the exact sequence III.9 guarantees that such a lift exists. We call the lift $[\tilde{f}]_1$ where $\tilde{f}$ is an extension of f to $C^*_r(\mathrm{G}_0)$. From what we did above we have that $[\tilde{f}]_1 = [\alpha(\tilde{f})]_1$. So we include $[\alpha(\tilde{f})]_1$ in $\mathrm{K}_1(C^*_r(\mathrm{G}_1))$ and restrict it to the edges of G_1 in order to follow the isomorphism from $\mathrm{K}_1(C^*_r(\mathrm{G}_1))$ into $\mathrm{K}_1(C^*_r(\mathrm{G}_1|_{\mathrm{X}_1}))$. The diagram below illustrates the chase so far:

$$\begin{tikzcd}
\mathbb{Z}^5 \arrow[r] & \ker\delta_1 \subseteq \mathrm{K}_1\left(C^*_r(\mathrm{G}_0|_{\mathrm{X}_1})\right) \arrow[r] & \mathrm{K}_1\left(C^*_r(\mathrm{G}_1)\right) \arrow[r] & \mathrm{K}_1\left(C^*_r(\mathrm{G}_1)\right) \\
e_5 \arrow[r] & \overline{e_{52}} - \overline{e_{55}} \to [f]_1 \arrow[r] & {[\tilde{f}]_1} \arrow[r] & {[\alpha(\tilde{f})]_1} \arrow[d] \\
 & & {[\alpha(\tilde{f})|_{\mathrm{X}_1}]_1} & {[\alpha(\tilde{f})]_1} \arrow[l] \\
\mathbb{Z}^5 & & \mathrm{K}_1\left(C^*_r(\mathrm{G}_1|_{\mathrm{X}_1})\right) \arrow[ll] & \mathrm{K}_1\left(C^*_r(\mathrm{G}_1)\right) \arrow[l]
\end{tikzcd}$$

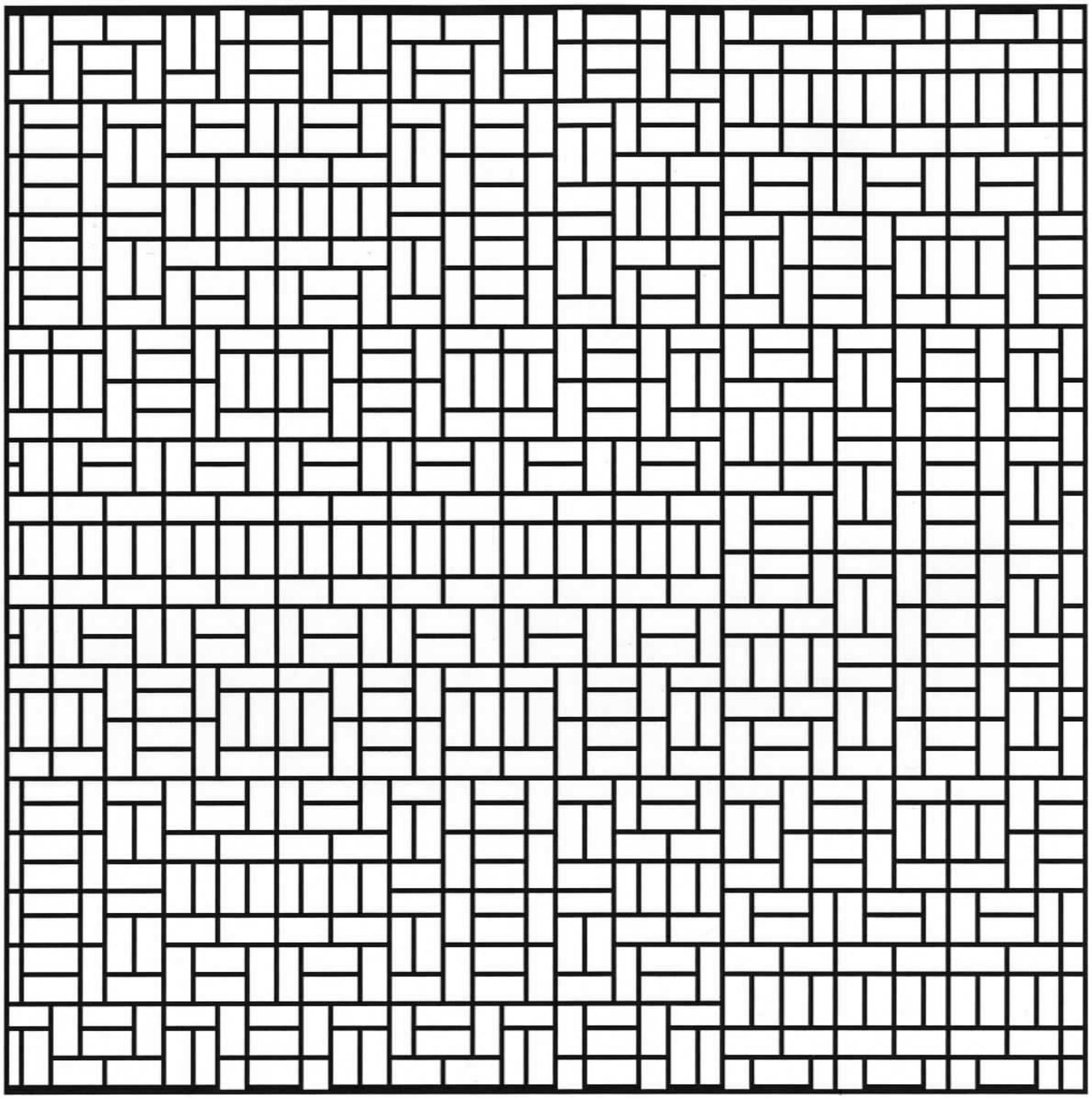
Now from the definition of α (and so the definition of F), we have that the values of $\alpha(\tilde{f})$ on the edges of G_1 can be read from the values of f on the edges of G_0. This implies that $\alpha(\tilde{f})|_{\mathrm{X}_1}$ winds once on the edges e_{32}, e_8 and e_{17} and winds once with contrary orientation (one could say -1 times) on the edges e_{20}, e_4 and e_{28}. We conclude that $[\alpha(\tilde{f})|_{\mathrm{X}_1}]_1$ is mapped to the vector $\overline{e_{32}} + \overline{e_8} + \overline{e_{17}} - \overline{e_{20}} - \overline{e_4} - \overline{e_{28}}$ in $\frac{\mathbb{Z}^{56}}{\mathrm{Im}(\delta_0)}$. Finally we write this vector as a linear combination of the basis vectors of $\frac{\mathbb{Z}^{56}}{\mathrm{Im}(\delta_0)}$ and follow the isomorphisms described above between $\frac{\mathbb{Z}^{56}}{\mathrm{Im}(\delta_0)}$ and $\mathbb{Z}^5$ to get that $\overline{e_{32}} + \overline{e_8} + \overline{e_{17}} - \overline{e_{20}} - \overline{e_4} - \overline{e_{28}}$ is mapped to $(0,2,1,2,2)$.

We conclude that $(0,0,0,0,1)$ is mapped to $(0,2,1,2,2)$ and proceeding analogous as above to the other four basis vectors e_1, e_2, e_3 and e_4 we get that the connecting map in K_1 is given by the matrix below

$$c = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 2 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & -1 & 0 & 1 & 2 \\ 2 & 0 & 1 & 1 & 2 \end{bmatrix}$$

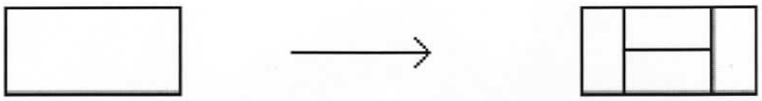
which is an isomorphism and hence $\mathrm{K}_1(C_r^*(\cup \mathrm{G}_k))$ is isomorphic to $\mathbb{Z}^5$.

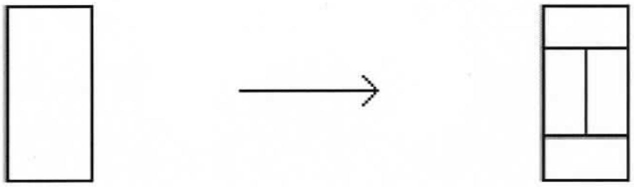
V.2.2 The Table tiling



A patch of the table tiling

We now focus our attention on the table tiling. We will compute the K-theory groups for the C*-algebras introduced in the previous chapter for the table tiling given by the substitution below:





And the inflation constant in this case is $\lambda = 2$

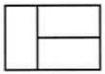
The table tiling has the interesting property that tiles do not necessarily match edge by edge. This will induce two natural choices for a cell complex on the tiling. One being just the vertices and edges on the tiling and the other being the previous one with added vertices on the middle of some edges so that the tiles meet edge to edge. Notice that the choice of cell complex do not change the equivalence relation in $\mathbb{R}^d \times \mathbb{R}^d$ and so do not change the C*-algebra of the tiling at all. This implies we can use either of the two choices to compute the K-theory groups. Actually it has proven easier to use the cell complex with added vertices to compute K_0 and the cell complex induced by the tiling to compute K_1.

We will start with the cell complex induced by the tiling, compute the K groups for $C^*_r(G_0)$ and then $K_1\left(C^*_r(\cup G_k)\right)$. Then we introduce the new vertices in our cell complex and compute $K_0\left(C^*_r(G_k)\right)$.

K-groups of $C^*_r(G_0)$

Again we assume that all tiles are oriented counterclockwise. We also assume that horizontal edges are oriented to the left (pointing left) and vertical edges are oriented upwards (pointing up). We label the edges and vertices as follows (notice we only have two tiles so do not need to label them):

Vertex patterns are labeled below: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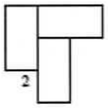
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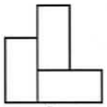
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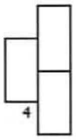
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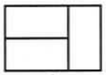
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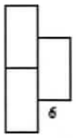
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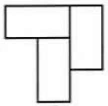
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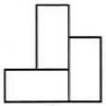
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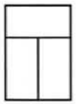
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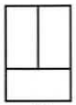
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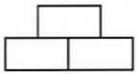
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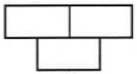
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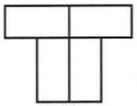
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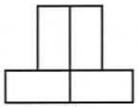
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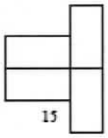
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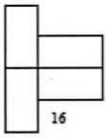
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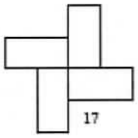
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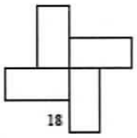
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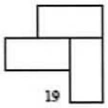
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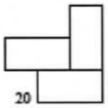
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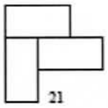
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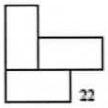
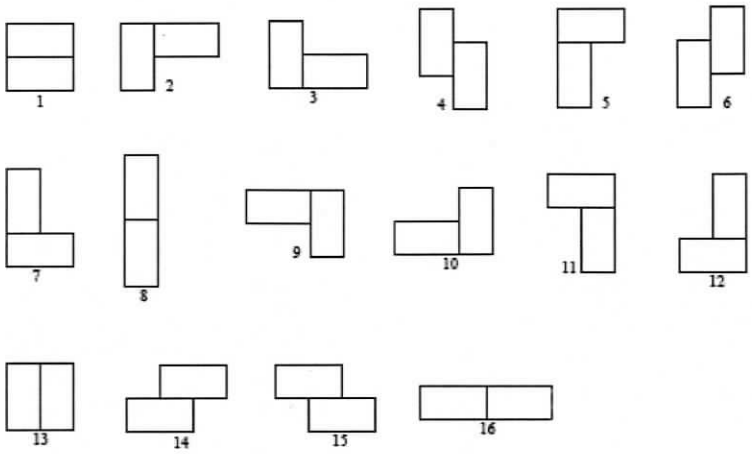

22

And edge patterns are labeled as shown:



We now proceed to compute the K-theory groups of $C^*_r(G_0)$. We will use both exact sequences III.6 and III.9. We start by restricting our equivalence relation to the edges, G_{X_1}. Using proposition III.28 we get the following 16x22 matrix for δ_0.

$$\delta_0 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Using the software gap, [9], we have that the kernel of δ_0 is generated by the 9 vectors below:

$$\ker \delta_0 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{V.3}$$

So we have that $\mathrm{K}_0\,(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1})$ is isomorphic to $\mathbb{Z}^9$.

We also know that $\mathrm{K}_1\,(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1})$ is isomorphic to $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$. To identify this quotient we will need to use again the Smith Normal Form of δ_0, introduced in section I.2.

Using the software gap we have that the Smith Normal Form of δ_0 is given by the diagonal matrix $\delta := \mathrm{diag}(1, \dots, 1, 2, 0, \dots, 0)$ with twelve 1's and nine 0's. Notice that the thirteenth diagonal entry of δ is 2. So it is clear that $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}^3$ and hence $\mathrm{K}_1\,(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1}) \cong \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0} \cong \mathbb{Z}_2 \oplus \mathbb{Z}^3$.

We have computed the K-groups for $\mathrm{C}^*_r(\mathrm{G}|_{\mathrm{X}_1})$ and can now use then on the six term exact sequence III.9 to compute the K-groups of $\mathrm{C}^*_r(\mathrm{G}_0)$. Corollary III.33 implies that $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_0))$ is isomorphic to $\ker \delta_0 \oplus \mathbb{Z} \cong \mathbb{Z}^{10}$. In order to compute K_1 we need first to compute the index map.

Observe that we used above that $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0} \cong \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta}$. Since $\mathrm{Im}\delta_0 = P^{-1}(\mathrm{Im}\delta)$, where P is such that $\delta = P\delta_0 Q$, the isomorphism above maps a basis element $\overline{e_i} \in \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta}$ to the element $\overline{P^{-1}(e_i)} \in \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$. We will need this isomorphism description to compute the index map. The matrix for P^{-1} is given below:

$$P^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Now using the description of the index map given in proposition III.30 we get the following matrix for $\tilde{\delta_1}$:

$$\tilde{\delta_1} = \left[\begin{smallmatrix} 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]$$

notice that this matrix take an element in $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$ to an element in $\mathbb{Z}^2$. But we want a 2x4 matrix from $\mathbb{Z}_2 \oplus \mathbb{Z}^3$ into $\mathbb{Z}^2$. So we do the necessary isomorphisms chases to get the matrix for δ_1. For example the vector $(1,0,0,0) \in \mathbb{Z}_2 \oplus \mathbb{Z}^3$ is mapped to $\overline{e_{13}} \in \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta}$ then mapped to $\overline{P^{-1}(e_{13})} \in \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$ and finally mapped to $(0,0) \in \mathbb{Z}^2$ by $\tilde{\delta_1}$. Proceeding analogous to the other three generators we get the following matrix for δ_1:

$$\delta_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and it is clear that the kernel of δ_1 is equal to $\langle (1,0,0,0), (0,1,0,0), (0,0,0,1) \rangle$ and hence $\mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2$.

K-theory of the inductive limit C*-algebra

$\mathrm{K}_1(C^*_r(\cup \mathrm{G}_k))$

From what was done previously we have that $\mathrm{K}_1(C^*_r(\mathrm{G}_k)) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$. We want to compute the connecting map such that the diagram below is commutative

$$\begin{tikzcd}
\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\cong"] \arrow[d, dotted, "c"'] & \mathrm{K}_1\left(\mathrm{C}^*_r\left(\mathrm{G}_0\right)\right) \arrow[d, hook, "\mathrm{K}_1(\iota)"'] \\
\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} & \mathrm{K}_1\left(\mathrm{C}^*_r\left(\mathrm{G}_1\right)\right) \arrow[l, "\cong"']
\end{tikzcd}$$

We have also shown that the generators of $\mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right) \cong \frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$ are $n_1 = P^{-1}(e_{13})$, $n_2 = P^{-1}(e_{14})$ and $n_3 = P^{-1}(e_{15})$, which are associated with $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. In order to find the connecting map c we need to find where each of this generators is mapped. But we are not able to just follow the isomorphisms and find to which functions in $\mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right)$ this generators are mapped. This is due basically to our inability to extend unitary functions in the edges to the whole tiling. We go around this problem by finding different representations of our generators (or linear combination of then) such that we know how to follow the isomorphisms. In particular we know how to extend to $\mathrm{K}_1\left(\mathrm{C}^*_r(\mathrm{G}_0)\right)$ a function that winds the same number of times on the pairs of edges (5,7), (11,12), (3,10), (2,9), (14,15) and (4,6).

Using a program in Mathematica, [14], we obtain that $e_{14} + e_{15} \sim 2n_2$ (meaning that $e_{14} + e_{15} - 2P^{-1}(e_{14}) \in \mathrm{Im}(\delta_0)$), $e_4 + e_6 \sim 2n_2 - 2n_3$, $e_5 + e_7 \sim 3n_2$,

$e_{11}+e_{12} \sim 3n_2$, $e_2+e_9 \sim n_1+3n_2-3n_3$, $e_3+e_{10} \sim n_1+3n_2-3n_3$ and $e_2+e_4+e_{10} \sim n_1+4n_2-4n_3$. Now a little bit of arithmetic yields that

$$\begin{aligned}
&2n_3 \sim e_{14}+e_{15}-e_4-e_6\\
&n_2 \sim e_{11}+e_{12}-e_{14}-e_{15}\\
&n_1-3n_3 \sim e_2+e_9-e_5-e_7\\
&n_1+4n_2-4n_3 \sim e_2+e_4+e_{10}
\end{aligned}$$

Once we find where each of the above elements is mapped by the connecting map we will be able to find where the generators n_1, n_2 and n_3 are mapped and hence we will be able to compute the map c. This requires that we keep the isomorphisms chase.

We start with $2n_3$. By above $2n_3 \sim e_{14}+e_{15}-e_4-e_6$ and hence $2n_3$ is mapped to $[f]_1 \in \mathrm{K}_1(\mathrm{C}^*_r(\mathrm{G}_0|_{\mathrm{X}_1}))$, where f is a function that winds once on the edges e_{14} and e_{15} and winds once with contrary orientation on the edges e_4 and e_6 (we say it winds -1 on this edges). Since this function f will be eventually included in $\mathrm{C}^*_r(\mathrm{G}_1)$ we need to choose a representative of $[f]_1$ so that we can recognize $[f]_1$ in $\mathrm{K}_1(\mathrm{C}^*_r(\mathrm{G}_1|_{\mathrm{X}_1}))$. So we choose f to wind once on the right half of the edges e_{14} and e_{15} and to wind -1 on the top half of the edges e_4 and e_6. We then need to extend this function to $\mathrm{C}^*_r(\mathrm{G}_0)$. Notice that once we rotate e_{14} and e_{15} by $\frac{\pi}{2}$ on the counterclockwise direction we get the edges e_4 and e_6 respectively but with contrary orientation. So the right half in e_{14} becomes the top half in e_4 and hence the extension of f to the tiles defining e_4 and e_6 can be easily derived from the extension of f to the tiles defining e_{14} and e_{15}.

In order to extend f we need to make a few precise statements. So let r_{14} be a parametrization of e_{14} such that $r_{14}(0) = i(e_{14})$, $r_{14}(1) = t(e_{14})$ and $r_{14}(\frac{1}{2}) = \frac{i(e_{14})+t(e_{14})}{2}$. Let r_{15} be a similar parametrization of e_{15}. We define f on this edges by

$$f((r_{14}(\lambda), r_{14}(\lambda))) = \begin{cases} \exp(2\pi i 2\lambda) & \text{for} \quad \lambda \in [0, \frac{1}{2}] \\ 1 \quad \text{for} \quad \lambda \in [\frac{1}{2}, 1] \end{cases}$$

$$f((r_{15}(\lambda), r_{15}(\lambda))) = \begin{cases} \exp(2\pi i 2\lambda) & \text{for} \quad \lambda \in [0, \frac{1}{2}] \\ 1 \quad \text{for} \quad \lambda \in [\frac{1}{2}, 1] \end{cases}$$

and f is defined accordingly in e_4 and e_6, $f(x, y) = 1$ if $y = x$ and $f(x, y) = 0$ in any other point of $\mathrm{G}|_{\mathrm{X}_1}$ not mentioned above.

Next we need to lift $[f]_1$ to $\mathrm{K}_1(C^*_r(\mathrm{G}_0))$. Observe that the exact sequence III.9 guarantees that such a lift exists. We call the lift $[\tilde{f}]_1$ where $\tilde{f}$ is an extension of f to $C_c(\mathrm{G}_0)$. To define $\tilde{f}$ let t_1 be the bottom tile defining e_{14} and t_2 be the top tile defining e_{14}. Similarly let t_3 be the bottom tile defining e_{15} and t_4 be the top tile defining e_{15}. We will use the fact that $t_4 = t_1 + u$ and $t_3 = t_2 + v$ for some vectors $u, v \in \mathbb{R}^2$ to define $\tilde{f}$ in the inside of the tiles t_1, t_2, t_3 and t_4 in a way that $\tilde{f}$ is equal to 1 on the border of the tiles (except where f winds). Once we do this we can define $\tilde{f}$ on the tiles defining e_4 and e_6 by the rotation symmetry and on all other points to be 1.

First we define $\tilde{f}$ on t_2 and t_3. In order to simplify notation we identify the point (λ, β) with $(r_{14}(\lambda) + (0, \beta), r_{14}(\lambda) + (0, \beta))$ in t_2 for $\lambda \in [0, 1/2]$, $\beta \in [0, 1]$.Now for any $(\lambda, \beta) \in [0, 1/2] \times [0, 1/2]$ we define $\tilde{f}$ via the matrix

$$\begin{pmatrix} \exp(2\pi i 2\lambda) & 0 \\ 0 & 1 \end{pmatrix}$$

where the entry 11 of the matrix above defines $\tilde{f}\,((\lambda, \beta), (\lambda, \beta))$, entry 12 gives $\tilde{f}\,((\lambda, \beta), (\lambda, \beta) + v)$, entry 21 gives $\tilde{f}\,((\lambda, \beta) + v, (\lambda, \beta))$ and finally entry 22 gives $\tilde{f}\,((\lambda, \beta) + v, (\lambda, \beta) + v)$. For $\lambda \in [0, 1/2]$, $\beta \in [\frac{1}{2}, 1]$ we define $\tilde{f}$ by

$$\begin{pmatrix} \cos \pi(\beta - \frac{1}{2}) & -\sin \pi(\beta - \frac{1}{2}) \\ \sin \pi(\beta - \frac{1}{2}) & \cos \pi(\beta - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} \exp(2\pi i 2\lambda) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi(\beta - \frac{1}{2}) & -\sin \pi(\beta - \frac{1}{2}) \\ \sin \pi(\beta - \frac{1}{2}) & \cos \pi(\beta - \frac{1}{2}) \end{pmatrix}^{-1}$$

where the entries of the matrix define $\tilde{f}$ in the same way as above.

To define $\tilde{f}$ on t_2 and t_3 we identify the point (λ, β) with the point $(r_{14}(\lambda) - (0, \beta), r_{14}(\lambda) - (0, \beta))$ in t_1 for $\lambda \in [0, 1/2]$, $\beta \in [0, 1]$. Now for any $\lambda \in [0, 1/2]$, $\beta \in [0, 1]$ we define $\tilde{f}$ by

$$\begin{pmatrix} \cos \beta\frac{\pi}{2} & -\sin \beta\frac{\pi}{2} \\ \sin \beta\frac{\pi}{2} & \cos \beta\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} \exp(2\pi i 2\lambda) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta\frac{\pi}{2} & -\sin \beta\frac{\pi}{2} \\ \sin \beta\frac{\pi}{2} & \cos \beta\frac{\pi}{2} \end{pmatrix}^{-1}$$

where the entries of the matrix define $\tilde{f}$ in an analogous way (for example the entry 12 of the matrix defines $\tilde{f}\,((\lambda, \beta), (\lambda, \beta) - v))$.

Next we include $\tilde{f}$ in $\mathrm{K}_1(C^*_r(\mathrm{G}_1))$ and follow the isomorphism back to $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$. We notice that $\tilde{f}$ restricted to the edges of G_1 winds once on the edges 1, 5 and 7 and winds -1 on the edges 3, 10, 13 and hence $\tilde{f}$ is taken to $\overline{e_1 + e_5 + e_7 - e_3 - e_{10} - e_{13}}$ in $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta_0}$, which in turn is taken to the vector

$$\overline{(-2, 2, 3, 2, -5, 6, 7, 5, -4, 4, -2, -4, -6, 0, 4, 0)}$$

in $\frac{\mathbb{Z}^{16}}{\mathrm{Im}\delta}$ via the matrix P of the smith normal form of δ_0.

Finally notice that $(-2, 2, 3, 2, -5, 6, 7, 5, -4, 4, -2, -4, -6, 0, 4, 0)$ is equivalent to $-6e_{13} + 4e_{15}$, since $(-2, 2, 3, 2, -5, 6, 7, 5, -4, 4, -2, -4, -6, 0, 4, 0) +$

$6e_{13} - 4e_{15}$ belongs to the image of δ, and hence $2n_3$ is taken to $(-6, 0, 4) = (0, 0, 4)$ in $Z_2 \oplus Z^2$. We conclude that $n_3 = (0, 0, 1)$ is taken to $(*, 0, 2)$ (Observe we can not say what $*$ is yet).

Following in an analogous way as above we get that $n_2 = (0, 1, 0)$ is taken to $(0, 2, 0)$, $n_1 - 3n_3$ is taken to $(0, 0, -6)$, $n_1 = (1, 0, 0)$ is taken to $(*, 0, 0)$ and $n_1 + 4n_2 - 4n_3$ is taken to $(1, 8, -8)$. It follows that $n_3 = (n_1 - 3n_3) + 4n_2(-n_1 - 4n_2 + 4n_3)$ is taken to $(1, 0, 2)$ and this implies that n_1 is taken to $(1, 0, 0)$. So the connecting map in K_1 is given by

$$c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The final step is to identify the inductive limit in $\mathbb{Z}_2 \oplus \mathbb{Z}^2$ induced by the matrix c above. To do this we will show that the map c is conjugate to a map c' that induces a limit easily identified.

Observe that we can rewrite the map c as

$$c : \begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & \to & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \\ (a, b, c) & \mapsto & (2a, 2b, a + c) \end{array}$$

Now let

$$c' : \begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & \to & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \\ (a, b, c) & \mapsto & (2a, 2b, c) \end{array}$$

To see that c and c' are conjugate we define

$$\phi : \begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & \to & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \\ (a, b, c) & \mapsto & (a, b, a + c) \end{array}$$

which has inverse defined by $\phi^{-1}(a, b, c) = (a, b, a + c)$, and notice that $\phi^{-1} \circ c' \circ \phi = c$.

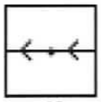
Finally we notice that the inductive limit induced by c' is isomorphic to $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}_2$ and hence $\mathrm{K}_1(C_r^*(\cup \mathrm{G}_k)) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}_2$.

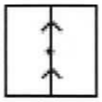
■

$\mathrm{K}_0(C_r^*(\cup \mathrm{G}_k))$

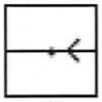
As we mention earlier it is easier to compute K_0 if we add vertices in the middle of some edges on the tiling so that tiles meet edge to edge. Basically we add one vertex on the middle of the previous edges 1 and 13. With this complex we replace our old edges 1 and 13 by the "new" edges shown below. We also add edges 17 and 18 as show below. Notice that we also get two new vertex patterns, 23 and 24, as described below (we keep all the same orientation on tiles and edges).

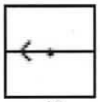
New vertex patterns 23 and 24:

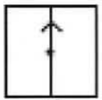

23

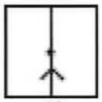

24

New edge patterns 1, 13, 17 and 18:


1


17


13


18

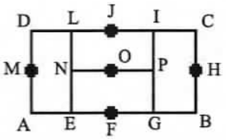
The computations change slightly but for K_0 we only need the matrix for δ_0, which we show below:

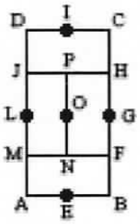
$$\delta_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

As expected this matrix has kernel of dimension 9. We show a basis below:

$$\ker(\delta_0) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}$$

So, as before, corollary III.33 implies that $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_0))$ is isomorphic to $\ker \delta_0 \oplus \mathbb{Z} \cong \mathbb{Z}^{10}$. In order to computer K_0 for the inductive limit C*-algebra, we will have to proceed analogous to what we did in the octagonal tiling and define a F map that collapses the new vertices obtained by applying the inflation map to a tile, into the "old" vertices of this tile. We start by defining F on the two prototiles and then extend F to $\mathbb{R}^2$ by translation. To make it easier to define F we name the vertices of this prototiles and their inflation as show in the pictures below:





Now we define F on the horizontal prototile as a map that takes the vertices M, N, E to the vertex A, the vertices O, P, G to the vertex F, the vertex H to the vertex B, the vertex I to the vertex J, the vertex L to the vertex D and leaves all other vertices fixed. Moreover we require F to be a homeomorphism from the interior of the rectangle defined by NOJL into the interior of the rectangle AFJD and a homeomorphism from the interior of the rectangle defined by PHCI into the interior of the rectangle FBCJ. Also on the vertical prototile we define F as a map that takes the vertices J, P, O to the vertex L, the vertices M, N, E to the vertex A, the vertex H to the vertex G, the vertex I to the vertex D, the vertex F to the vertex B and leaves all other vertices fixed. Moreover we require F to be a homeomorphism from the interior of the rectangle defined by ICHC into the interior of the rectangle DLGC and a homeomorphism from the interior of the rectangle defined by ONFG into the interior of the rectangle LABG. Finally we require F to be a continuous map.

Since F agrees on the borders of the matching tiles of the table tiling we can extend it to $\mathbb{R}^2$ by translation. Observe that vertices in $\omega(\mathrm{T})$ are mapped to vertices in T under F but this is not true for the edges in $\omega(\mathrm{T})$ (that is why we can not use this method for K_1). Following the same ideas already introduced for the octagonal tiling we have that $\alpha(f)$ is homotopic to f, where $f \in \mathrm{C}_c(\mathrm{G}_0)$ and $\alpha : \mathrm{C}_c(\mathrm{G}_0) \to \mathrm{C}_c(\mathrm{G}_0)$ is defined by

$$\alpha(f)(x,y) = \begin{cases} f\left(F(x), F(y)\right) & \text{if} \quad (F(x), F(y)) \in \mathrm{G}_0 \\ 0 \quad \text{otherwise} \end{cases}$$

Notice that the proof that α is well defined is analogous to the one we presented for the octagonal tiling.

We are now able to compute K_0.

From what was done previously we have that $\mathrm{K}_0(C_r^*(\mathrm{G}_k)) \cong \mathbb{Z}^{10}$. We want to compute the connecting map such that the diagram below is commutative

$$\begin{tikzcd}
\mathbb{Z}^{10} \arrow[r, "\cong"] \arrow[d, dotted, "c"'] & \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_0\right)\right) \arrow[d, hook, "\mathrm{K}_0(\iota)"'] \\
\mathbb{Z}^{10} & \mathrm{K}_0\left(\mathrm{C}_r^*\left(\mathrm{G}_1\right)\right) \arrow[l, "\cong"']
\end{tikzcd}$$

Remember that $\mathrm{K}_0(C_r^*(\mathrm{G}_0)) \cong \ker\delta_0 \oplus \mathbb{Z}$ and we say that, for $i = 1..9$, the canonical basis vectors e_i in $\mathbb{Z}^{10}$ is isomorphic to the column vector c_i in the kernel of δ_0. Also e_{10} is isomorphic to an element $[g]_0 \in \mathrm{K}_0(\mathrm{I}_{\mathrm{X}_1})$ included in $\mathrm{K}_0(\mathrm{C}_r^*(\mathrm{G}_0))$ that is NOT on $\ker(\mathrm{K}_0(i)) = \mathrm{Im}\delta_1$. Proceeding analogous to the octagonal case we have that e_{10} is mapped to itself under the connecting map. We need to worry about the other vectors. For this we proceed again as in the octagonal case and find the matrix $\mathcal{F}$ for the table tiling. Remember $\mathcal{F}$ is a composition of homomorphisms. It takes a vector in $\ker\delta_0$ in G_0 to a vector in $\ker\delta_0$ in G_1. The diagram below illustrates the definition of $\mathcal{F}$.

$$\begin{tikzcd}
\mathbb{Z}^9 \arrow[r] & \ker\delta_0 \subseteq \mathrm{K}_0\left(C_r^*(\mathrm{G}_0|_{\mathrm{X}_0})\right) \arrow[r] & \mathrm{K}_0\left(C_r^*(\mathrm{G}_0)\right) \arrow[r] & \mathrm{K}_0\left(C_r^*(\mathrm{G}_0)\right) \\
e_1 \arrow[r] & c_1 \to [f]_0 \arrow[r] & [\tilde{f}]_0 \arrow[r] & [\alpha(\tilde{f})]_0 \arrow[d] \\
 & \mathcal{F}(c_1) & & [\alpha(\tilde{f})]_0 \arrow[ll] \\
\mathbb{Z}^9 & \ker\delta_0 \subseteq \mathrm{K}_0\left(C_r^*(\mathrm{G}_1|_{\mathrm{X}_0})\right) \arrow[l] & & \mathrm{K}_0\left(C_r^*(\mathrm{G}_1)\right) \arrow[ll]
\end{tikzcd}$$

Proceeding as explained for the octagonal tiling we get the following matrix for $\mathcal{F}$:

$$\begin{bmatrix}
1&1&0&1&0&0&0&0&0&0&0&1&1&0&0&1&0&1&0&0&0&1&0&0\\
0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0\\
0&0&0&0&1&1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&2&0\\
0&0&0&0&1&1&1&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&2
\end{bmatrix}$$

And so to find where the vector e_j is mapped under the connecting map we only need to write the vector $\mathcal{F}\ (c_j)$ as a linear combination of the basis of $\ker \delta_0$ and take the coefficients of this linear combination. Doing this we get the following connecting map in K_0

$$C = \begin{bmatrix}
2&1&1&1&0&1&1&0&0&0\\
1&0&2&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&2&0&0&0\\
1&0&0&0&0&0&0&2&0&0\\
2&1&1&1&1&1&-1&-2&-2&0\\
0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

and hence $\mathrm{K}_0(C_r^*(\cup \mathrm{G}_k))$ is isomorphic to the direct limit in $\mathbb{Z}^{10}$ induced by C.

What is the inductive limit above? Even though we can not give a complete answer to this question we can elaborate more than on the octagonal case. Details follows below.

We will work with C^2 in order to have only nonnegative eigenvalues. We show the matrix below:

$$C^2 = \begin{bmatrix}
7&4&4&2&2&2&4&0&0&0\\
4&5&1&1&0&1&1&0&0&0\\
4&1&5&1&0&1&1&0&0&0\\
2&1&1&1&0&1&1&0&0&0\\
6&4&4&2&3&2&4&0&0&0\\
1&1&1&1&1&1&1&0&0&0\\
1&1&1&1&3&1&5&0&0&0\\
4&1&1&1&0&1&1&4&0&0\\
2&3&3&1&-1&1&3&0&4&0\\
0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

We will define an onto homomorphism from the direct limit in $\mathbb{Z}^{10}$, denoted by $\mathbb{Z}^{10}_{\rightarrow}$, onto $\mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z}[\frac{1}{4}]$. In order to do this we need to find all nonzero eigenvalues and associated eigenvectors of $(C^2)^T$. Using the software Mathematica we get that the eigenvalues are 1, 1, 1, 1, 4, 4, 4, 4, 16, which we name $\lambda_1, \lambda_2, \ldots, \lambda_9$

respectively, and the associated eigenvectors are

$$\begin{aligned} &w_1 = (0,0,0,0,0,0,0,0,0,1), \quad w_2 = (-3,2,2,0,0,0,2,0,0,0), \\ &w_3 = (-1,0,0,2,0,2,0,0,0,0), \quad w_4 = (-1,0,0,0,1,0,0,0,0,0), \\ &w_5 = (0,1,0,0,-1,0,0,0,1,0), \quad w_6 = (0,-1,0,0,0,0,0,1,0,0), \\ &w_7 = (-1,-1,0,0,1,0,1,0,0,0), \quad w_8 = (0,-1,1,0,0,0,0,0,0,0), \\ &w_9 = (3,2,2,1,1,1,2,0,0,0), \end{aligned}$$

where w_i is associated with λ_i.

Now let $\varphi_i : \mathbb{Z}^{10} \to \mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z}[\frac{1}{4}]$ be defined by

$$\varphi_i(v) = \left(< v, w_1 > \lambda_1^{-i}, < v, w_2 - w_3 - w_4 > \lambda_2^{-i}, < v, w_5 > \lambda_5^{-i}, \ldots, < v, w_9 > \lambda_9^{-i}\right)$$

where $v \in \mathbb{Z}^{10}$ and $i = 0, 1, 2, \ldots$. Observe that for all i, $\varphi_i(v) = \varphi_{i+1} \circ C^2(v)$ (remember that $\lambda_2 = \lambda_3 = \lambda_4 = 1$) and hence the collection of maps $\{\varphi_i\}_{i=0}^{\infty}$ gives a well defined map φ, from the direct limit $\mathbb{Z}^{10}_{\rightarrow}$ into $\mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z}[\frac{1}{4}]$. Notice that each φ_i is a map from $\mathbb{Z}^{10}$ at step i of $\mathbb{Z}^{10}_{\rightarrow}$.

We proceed to show that φ is onto. To do this we notice that φ_0 can be represented by the matrix

$$\varphi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & -2 & -1 & -2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

and use the software GAP to find its integer column space. It happens that φ_0 is onto in $\mathbb{Z}^7$ and this implies that φ is onto.

To find the kernel of φ we first notice that the definition of φ_i implies that $\ker \varphi_i = \ker \varphi_0$ for all i. Using the software GAP we get that a basis for the null space of φ_0 is given by the vectors $n_1 = (1,3,3,0,13,-10,-9,3,10,0)$, $n_2 = (0,5,5,0,18,-12,-13,5,13,0)$ and $n_3 = (0,0,0,1,0,-1,0,0,0,0)$. Now we observe that $C^2 \cdot n_3 = 0$ so this generator vanishes at the inductive limit. Also $C^2\left(C^2(n_1)\right) = C^2(n_1)$ and $C^2\left(C^2(n_2)\right) = C^2(n_2)$ what implies that any vector in the $\ker \varphi_i$, for any i, is equivalent to a vector in $\ker \varphi_0$ in the inductive limit $\mathbb{Z}^{10}_{\rightarrow}$. This implies that $\ker \varphi \cong < \overline{n_1}, \overline{n_2} > \cong \mathbb{Z}^2$, where $\overline{n_j}$ is the inclusion of n_j in the direct limit $\mathbb{Z}^{10}_{\rightarrow}$.

The above discussion allow us to write the following exact sequence:

$$0 \to \mathbb{Z}^2 \to \mathrm{K}_0(C_r^*(\cup \mathrm{G}_k)) \to \mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]^4 \oplus \mathbb{Z}[\frac{1}{4}] \to 0.$$

Unfortunately we can not say whether this exact sequence splits and hence this is the best we can describe $\mathrm{K}_0(C_r^*(\cup \mathrm{G}_k))$ at the moment.

We finish the chapter with a trace calculation.

Proposition V.2. $\mathrm{K}_0(\tau)(\mathrm{K}_0(\mathrm{C}^*_r(\cup \mathrm{G}_k)))$ *maps onto* $\mathbb{Z}[\frac{1}{2}]$ *where* $\mathrm{K}_0(\tau)$ *is the map in* K*-theory induced by* τ.

Proof:

For simplicity we will use the cell complex used to compute K_1, that is, the cell complex without added vertices. Remember that $\mathrm{K}_0(\mathrm{C}^*_r(\mathrm{G}_k))$ is generated by the projections arising from the kernel of δ_0 and one projection from $\mathrm{I}_{\mathrm{X}_1}$. We focus on the projections arising from δ_0 as they have finite trace. Remember that $\ker \delta_0$ was given in V.3. Denote the column i of the matrix in V.3 by c_i. Each c_i is isomorphic to the class of a projection f_i^k in $\mathrm{C}^*_r(\mathrm{G}_k)$. We will use lemma IV.7 to show how to compute the trace for f_1^0. The others are analogous.

Notice that f_1^0 is equal to 1 on the vertices v_3, v_7, v_{19} and v_{22}. Also f_1^0 is 0 on all other vertices (by v_i we mean the point (v_i, v_i)). Call the prototile with length greater than height p_1 (horizontal prototile) and the other one p_2. Notice that both prototiles have area 2. By lemma IV.6 we have that

$$\tau(f_1^0) = \mathrm{Tr}(\pi_x(f_1^0)) \cdot 2 + \mathrm{Tr}(\pi_y(f_1^0)) \cdot 2$$

where x is is the interior of p_1 and y is in the interior of p_2. Now take x_1 to be the bottom right corner point on p_1 and y_1 to be the bottom right corner point on p_2. Then by lemma IV.7 we have that if $\mathcal{V}_{x_1} = \{[v_5], [v_7], [v_{12}], [v_{13}], [v_{15}, [v_{17}], [v_{20}]\}$ and $\mathcal{V}_{y_1} = \{[v_6], [v_8], [v_{10}], [v_{14}], [v_{16}, [v_{18}], [v_{22}]\}$ then

$$\tau(f_1^0) = \sum_{\substack{z \in [v] \\ [v] \in \mathcal{V}_{x_1}}} f(z,z) \cdot 2 + \sum_{\substack{z \in [v] \\ [v] \in \mathcal{V}_{y_1}}} f(z,z) \cdot 2 = 2 + 2 = 4.$$

Now observe that tiles in G_k are scaled down by 2^{-k} and hence $\tau(f_1^k) = 2^{k-2}$. Also notice that for any $1 \leq i \leq 9$, $\tau(f_i^k) = a \cdot 2^{k-j}$ for some $a, j \in \mathbb{Z}$. This implies that $\tau(f_i^k)$ is contained in $\mathbb{Z}[\frac{1}{2}]$ for every k and every $1 \leq i \leq 9$.

Finally for all $k \in \mathbb{N}$, let $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau)$. Then $\mathrm{K}_0(\tau_k) = \mathrm{K}_0(\tau_{k+1}) \circ \mathrm{K}_0(\iota)$ and hence the collection of maps $\{\mathrm{K}_0(\tau_k)\}_{i=0}^{\infty}$ gives a well defined map τ_u, from the direct limit $\mathrm{K}_0(\mathrm{C}^*_r(\cup \mathrm{G}_k))$ onto $\mathbb{Z}[\frac{1}{2}]$ as desired.

■

Conclusion

C*-algebras associated to tilings have been object of study for a number of years now and one of the milestones on the subject was the introduction by Anderson and Putnam of a C*-algebra associated with a substitution tiling. The Anderson-Putnam C*-algebra arises from an unstable equivalence relation on a Smale space. In this thesis we studied the C*-algebras arising from the stable equivalence relation in the same Smale space. The first step was to introduce an equivalence relation on $\mathbb{R}^d$ associated to a given tiling. We then constructed a C*-algebra from this equivalence relation through the groupoid approach. The study of these C*-algebras led us to a characterization of its ideals which was used to compute the K-theory of these C*-algebras. We computed the K-groups for a few examples and in the one dimensional case we were able to completely describe the order structure of K_0. We also constructed these C*-algebras as recursive subhomogeneous algebras. Up to this point the assumption of a substitution tiling was not necessary, but the inductive limit C*-algebras defined in chapter IV only exist for a substitution tiling. These C*-algebras were constructed using the C*-algebras introduced in chapter III. We proved that the inductive limit C*-algebras are simple and that each of the step C*-algebras used to build the inductive limit is a recursive subhomogeneous algebra. We have also computed traces for the Fibonacci, Thue Morse and table tilings. Finally on the last chapter we compute the K-theory of the octagonal and table tiling.

The inductive limit C*-algebra of chapter IV is the C*-algebra associated to the stable equivalence relation mentioned before. So we expect it to interplay in some sense with the Anderson-Putnam C*-algebra. We note that for the table tiling example we obtained torsion on K_1 whereas for the Anderson-Putnam C*-algebra this torsion appears on K_0. Also for the octagonal tiling we obtained $\mathbb{Z}^5$ for K_1 which agrees with K_1 of the AP C*-algebra. We were not able to find a precise relation between the K-groups of these two C*-algebras and this is one of many other problems that we plan to study in the future.

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