EXPECTED WAITING TIMES

(Mathematics Magazine Proposal 1268)

By

BRUCE R. JOHNSON

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Bruce R. Johnson
Department of Mathematics
University of Victoria
Victoria, B.C.
CANADA V8W 2Y2

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1268. *Proposed by Lawrence Stout, Illinois Wesleyan University, Bloomington.*

An experiment with probability p of success is repeated. Each time a failure occurs, \$1 is put into a kitty. Each time a success occurs, the contents of the kitty are paid out as a jackpot.

a. What is the expected waiting time until the kitty reaches \$n for the first time starting from an empty kitty?

b. What is the expected waiting time until a jackpot of exactly \$n is won?

Solution by Bruce R. Johnson, University of Victoria, Canada.

Our solution uses the powerful tool of conditioning not only to solve parts (a) and (b), but also to find the expected waiting time until a jackpot of at least \$n is won. We begin by defining three random variables:

X = number of trials until the kitty reaches \$n for the first time,

Y = number of trials until a jackpot of at least \$n is won for the first time,

Z = number of trials until a jackpot of exactly \$n is won for the first time.

Our intuition tells us that $E(X) < E(Y) < E(Z)$. More precisely, we will achieve our goals by showing

$$E(X) = (1-q^n)/q^n p, \ E(Y) = 1/q^n p, \ E(Z) = 1/q^n p^2,$$

where $q = 1 - p$. To compute each of these three expectations, we will condition on the random variable W = number of trials until the first success occurs. We know that W has a geometric distribution with parameter p. Since the process restarts with an empty kitty following the first success, we find

$$E(X|W=w) = \begin{cases} w + E(X) & \text{for } w = 1, 2, \ldots, n \\ n & \text{for } w = n+1, n+2, \ldots \end{cases} .$$

Therefore,

$$E(X) = \sum_{w=1}^{\infty} E(X|W=w)P(W=w) = \sum_{w=1}^{n} (w+E(X))P(W=w) + n\ P(W \geq n+1)$$

$$= \sum_{w=1}^{n} w\ q^{w-1}\ p + E(X)P(W \leq n) + n\ P(W \geq n+1).$$

Solving for $E(X)$,

$$E(X) = \frac{p \sum_{w=1}^{n} wq^{n-1}}{P(W \geq n+1)} + n = \frac{p\{(nq^{n+1}-(n+1)q^{n} + 1)/p^{2}\}}{q^{n}} + n = \frac{1 - q^{n}}{q^{n}\ p}\ ,$$

where the formula $\sum_{w=1}^{n} wq^{w-1} = (nq^{n+1}-(n+1)q^{n}+1)/p^{2}$ is obtained by differentiating both sides of the equation

$$1 + q + q^{2} + \cdots + q^{n} = (1-q^{n+1})/(1-q)$$

with respect to q. Similarly,

$$E(Y) = \sum_{w=1}^{\infty} E(Y|W=w)P(W=w) = \sum_{w=1}^{n} (w+E(Y))P(W=w) + \sum_{w=n+1}^{\infty} w\ P(W=w)$$

$$= E(W) + E(Y)P(W \leq n),$$

so we find

$$E(Y) = \frac{E(W)}{P(W \geq n+1)} = \frac{1/p}{q^{n}} = \frac{1}{q^{n}\ p}\ .$$

Finally,

$$E(Z) = \sum_{w=1}^{\infty} E(Z|W=w)P(W=w) = \sum_{w \neq n+1} (w+E(Z))P(W=w) + (n+1)P(W=n+1)$$

$$= E(W) + E(Z)P(W \neq n+1),$$

from which we obtain

$$E(Z) = \frac{E(W)}{P(W=n+1)} = \frac{1/p}{q^{n}\ p} = \frac{1}{q^{n}\ p^{2}}\ .$$