DOMINATION AND IRREDUNDANCE IN THE QUEENS' GRAPH

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Abstract

The vertices of the queens' graph Q_n are the squares of an $n \times n$ chessboard and two squares are adjacent if a queen placed on one covers the other. It is shown that the domination number of Q_n is at most $31n/54 + O(1)$, that Q_n possesses minimal dominating sets of cardinality $5n/2 - O(1)$ and that the cardinality of any irredundant set of vertices of Q_n is at most $\lfloor 6n + 6 - 8\sqrt{n + \sqrt{n} + 1} \rfloor$.

1. Introduction. The *open neighbourhood* of a vertex v of a graph $G = (V, E)$ is defined by $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* of v by $N[v] = N(v) \cup \{v\}$. In general, the *closed neighbourhood* of $X \subseteq V$ is defined by $N[X] = \cup_{x \in X} N[x]$. A set $D \subseteq V$ is a *dominating set* of G if $N[D] = V$, while $I \subseteq V$ is an *irredundant set* of G if $PN(x, I) = N[x] - N[I - \{x\}] \neq \emptyset$ for every $x \in I$. $PN(x, I)$ and its elements are respectively called the *private neighbourhood* and the *private neighbours of x relative to I*. The dominating set D is *minimal dominating* if no proper subset of D is dominating – note that D is a minimal dominating set if and only if it is dominating and irredundant. Further, an irredundant set I is *maximal irredundant* if no proper superset of I is irredundant. Finally, $B \subseteq V$ is *independent* if no two vertices of B are adjacent and *maximal independent* if B, but no proper superset of B, is independent. Note that B is maximal independent if and only if B is dominating and independent, and that any independent set is irredundant.

The *lower (upper) domination number* $\gamma(G)$ $(\Gamma(G))$, *independence number* $i(G)$ $(\beta(G))$ and *irredundance number* $ir(G)$ $(IR(G))$ of G are respectively the minimum (maximum) cardinalities among the minimal dominating, maximal independent and maximal irredundant sets of G. These six parameters are well-studied in the literature (see [6]) and satisfy the chain of inequalities

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

In particular, there has been a revival in the study of these parameters for graphs defined from $n \times n$ chessboards ([5]).

The Queens' graph Q_n has the n^2 squares of the $n \times n$ chessboard as its vertex set and two vertices are adjacent if a queen placed on one covers the other, which occurs precisely when the two squares are on the same *line*, *i.e.* row, column or diagonal, of the board. A survey of recent results on the six parameters for Q_n is given in [5]. Further, Weakley [8] and Burger, Cockayne and Mynhardt [2] established new values of $\gamma(Q_n)$, while Cockayne [3] gave an upper bound for $IR(Q_n)$.

Eisenstein, Grinstead, Hahne and Van Stone announced in the abstract [4] that $\gamma(Q_n) \leq i(Q_n) \leq 7n/12 + O(1)$. In this paper we prove a slightly sharper bound for $\gamma(Q_n)$ by showing that $\gamma(Q_n) \leq 31n/54 + O(1)$. In the master's dissertation [1], Burger investigated methods to obtain minimum dominating sets for Q_n. In the process all 5-element dominating set of Q_8 were determined. We briefly explain the simple method used to eliminate isomorphic dominating sets in order to obtain the 638 non-isomorphic 5-element dominating sets of Q_8. A complete list of these sets is given in [1].

Weakley [9] announced that $IR(Q_n) \geq \Gamma(Q_n) \geq 2n - 5$. We improve this result by showing that $IR(Q_n) \geq \Gamma(Q_n) \geq 5n/2 - O(1)$, where the constant depends on the congruency of n modulo 8. We also give an example due to McRae [7], showing that this bound is not exact. Finally, we improve the upper bound for $IR(Q_n)$ found in [3] by proving that $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n + \sqrt{n} + 1} \rfloor$.

2. The upper bound for $\gamma(Q_n)$. The following notation and terminology will be required. Due to the symmetry of the dominating sets described in this and in other sections, we identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes and with the centre of the board at the origin. If n is odd, say $n = 2k+1$, we label the squares according to the coordinates of their centres. Row i, where $i = 0, \pm 1, \ldots, \pm k$ (column j, where $j = 0, \pm 1, \ldots, \pm k$) of the board consists of all squares with y-coordinate i (x-coordinate j). Rows and columns are called *even* or *odd* according to the parity of their labels. Similarly, squares are called *even-even*, *even-odd*, *odd-even* or *odd-odd* according to the parity of their coordinates. The diagonals of squares which rise from left to right are called d-diagonals and are labelled $d = -2k, \ldots, d = 0, \ldots d = 2k$, according to the difference between their y- and x-coordinates (*i.e.*, the intersection of the line through their centres with the y-axis). Similarly, the s-diagonals fall from left to right and are labelled $s = -2k, \cdots, s = 0, \ldots s = 2k$, according to the sum of the coordinates of the squares. If n is even, say $n = 2k$, we label the squares according to the coordinates of their corners furthest away from both axes. Row i, where $i = \pm 1, \ldots, \pm k$, column j, where $j = \pm 1, \ldots, \pm k$, the d-diagonals $d = -2k+1, \ldots, d = 0, \ldots, d = 2k-1$ and the s-diagonals $s = -2k+1, \ldots, s = 0, \ldots, s = 2k-1$ are defined as before (see Figure 1).

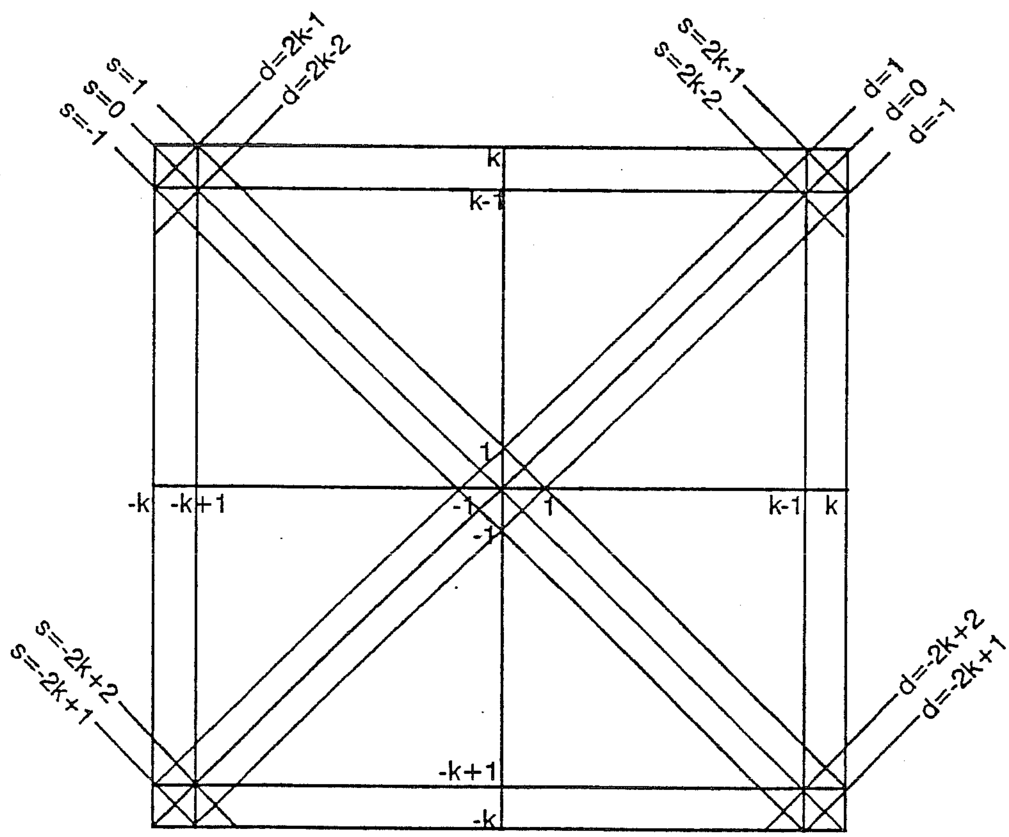

Figure 1

For $n = 108m - 37 \geq 71$, we now give a dominating set of cardinality $62m - 23$ to show that $\gamma(Q_n) \leq (31n - 95)/54$. Note that the set of admissible values for n is an arithmetic progression. For all other values of n, we can create a dominating set by adding queens to a dominating set on a largest admissible board of size less than n. At most one queen is needed for each new row and column. Therefore, the number of added queens is never more than a constant. This will show that $\gamma(Q_n) \leq \frac{31}{54} n + O(1)$.

We first give the positions of the queens and then prove that they form a dominating set. Most of the queens lie on three lines which we denote by Z_1, Z_2 and Z_3 and which form a Z pattern on the board (see Figure 2). The coordinates of the queens on Z_1, Z_2 and Z_3 are:

$$\begin{array}{ll} Z_1: & (-54m+22,\ 54m-20) + i(6,-2),\ \ i = 0,1,\ldots,18m-7 \\ Z_2: & (-54m+24,\ -18m+8) + i(6,2),\ \ i = 0,1,\ldots,18m-8 \\ Z_3: & (-54m+20,\ -18m+6) + i(6,-2),\ \ i = 0,1,\ldots,18m-7. \end{array}$$

We give the coordinates of the remaining queens in the upper half of the board only (on the lines labelled L_1, L_2 and L_3). The positions of the other queens can be obtained by a rotation of 180° of the board.

$$\begin{array}{ll} L_1: & (-22m+5,\ 14m-1) + i(-12,4),\ \ i = 0,1,\ldots,2m-2 \\ L_2: & (-18m+1,\ 18m+3) + i(-4,12),\ \ i = 0,1,\ldots,m-2 \\ L_3: & (-14m+1,\ 38m-13) + i(-4,12),\ \ i = 0,1,\ldots,m-1. \end{array}$$

Let R_m be the set of queens thus constructed.

Theorem 1. *For each positive integer m, the set R_m dominates Q_n, where $n = 108m - 37$.*

Proof. We must show that all squares are dominated. The queens on the Z pattern of R_m are situated so that there is a queen on each even row and even column. Thus all even-even, even-odd and odd-even squares are dominated. We therefore only need to check the odd-odd squares, which must be dominated by queens on diagonals with even labels. No two queens on the Z pattern lie on the same diagonal. The positive diagonals through the queens on Z_2 have labels of the form $4j$, and those through the queens on Z_1 and Z_3 have labels of the form $8j+2$ and $8j-2$. Thus all squares between the lines K_1 and K_2 are dominated. (See Figure 2.)

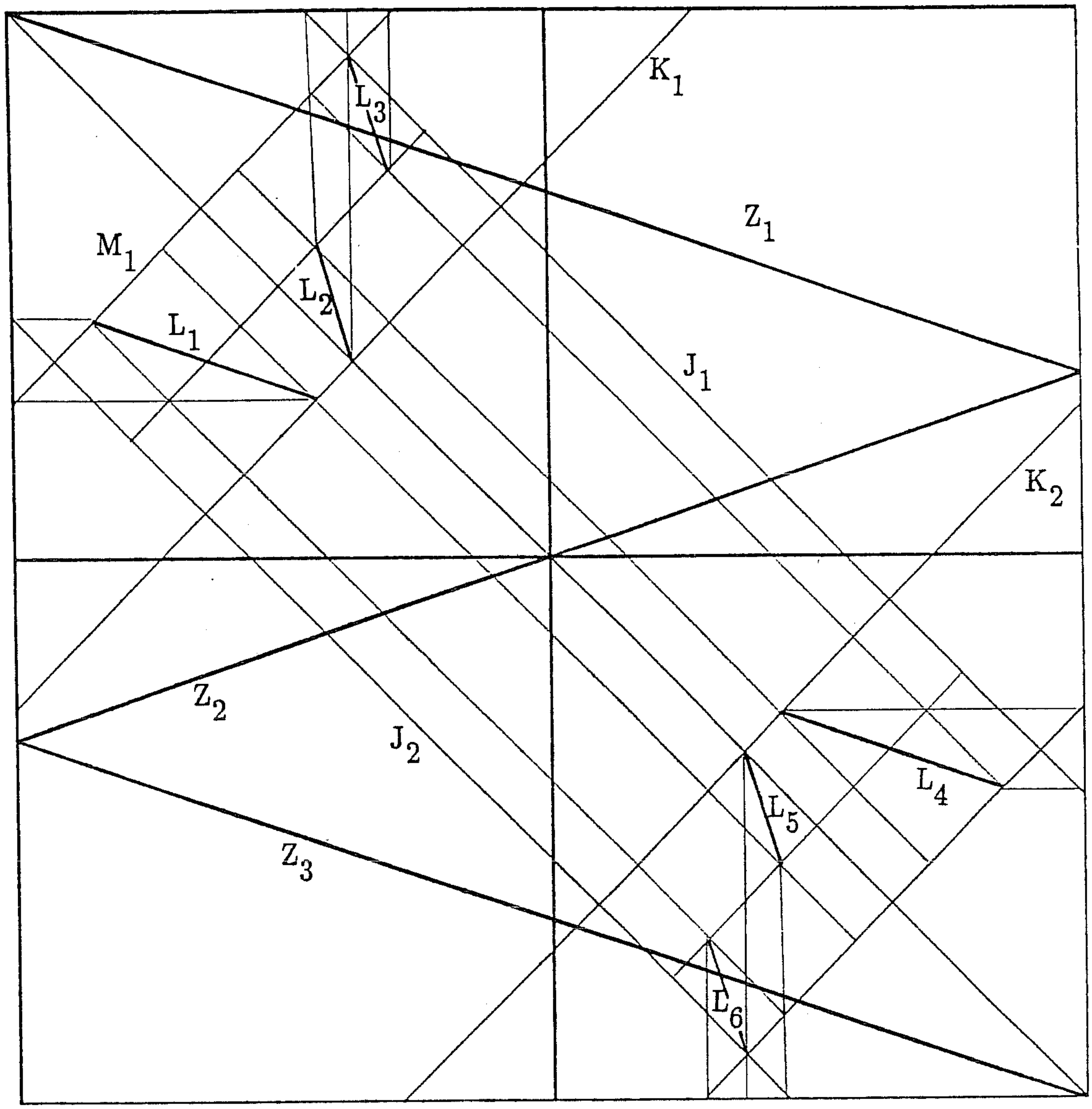

Figure 2

It remains to show that the odd-odd squares above K_1 and below K_2 are dominated. Because of symmetry, we only show this for the squares above K_1. The squares (above K_1) not dominated by the Z pattern are the odd-odd squares which lie on positive as well as negative diagonals which contain no queens from the Z pattern. These squares lie on the intersections of the positive diagonals with labels

$$36m-6,\ 36m+2,\dots,108m-46$$

and the negative diagonals with labels

$$\pm4,\pm12,\dots,\ \pm(72m-36).$$

This gives $9m-4$ positive diagonals above K_1 which must be dominated by the queens not on the Z pattern. There is a total of $4m-2$ queens on the lines L_1, L_2, and L_3. These queens are placed so that they lie on the first (counting from the centre) $4m-2$ diagonals concerned. Thus all squares between M_1 and K_1 are dominated. Also, all the queens on the lines L_1 lie on different negative diagonals. This ensures that the $8m-4$ concerned negative diagonals between the lines J_1 and J_2 are dominated.

The only regions left are two small triangles. The one above M_1 and J_1 is dominated per column by L_2 and L_3, and the one below J_2 and above M_1 is dominated per row by L_1. Thus all squares are dominated. ∎

Corollary 2. *If $n=108m-37\geq 71$, then $\gamma(Q_n)\leq 62m-23$.*

Proof. There are $54m-19$ queens on the Z pattern of R_m and $8m-4$ on the lines L_i, $i=1,\dots,6$. Hence $|R_m|=62m-23$ and the result follows from Theorem 1. ∎

3. Minimum dominating sets for Q_8. In order to investigate patterns which may occur in dominating sets of Q_n of smallest cardinality, all such dominating sets of Q_n, for $5\leq n\leq 11$, are generated in [1]. Where the number of solutions obtained is small, it is easy to give a list of non-isomorphic solutions only. This is not the case for $n=8$, however; hence the following procedure is applied. To ensure that no reflections or rotations of dominating sets are repeated, dominating sets are oriented so that:

(1) most (3 or more if $n=8$) of the queens are in the lower half of the board, *i.e.*, below the X-axis;

(2) most of the queens are in the left half of the board;

(3) most of the queens are on or below the positive main diagonal, *i.e.*, the diagonal $d = 0$. Note that (1) ensures that reflections about the X-axis and 180° rotations are not repeated and (2) ensures that reflections about the Y-axis are not repeated. Also, (1) combined with (2) ensures that 90° and 270° rotations as well as reflections about the diagonal $s = 0$ are not repeated. Note that all these symmetries are eliminated, because n is even and the number of queens is odd. The only remaining symmetry is the reflection about the diagonal $d = 0$. Most of the dominating sets which are isomorphic under this symmetry are eliminated by (3). The reflections which (3) does not eliminate are the dominating sets with an odd number of queens on the diagonal $d = 0$, with the same number of queens below and above the concerned diagonal. In this case the first dominating set according to the lexicographic ordering of ordered pairs is listed.

Combining this method with a computer search, all non-isomorphic 5-element dominating sets of Q_8 are found – there are 638 of them! A complete list of coordinates of these dominating sets is given in [1].

We digress briefly to remark that it was mentioned in [5,8] that $\gamma(Q_{12}) = 6$ while $i(Q_{12}) \in \{6, 7\}$. However, a computer search for dominating sets of Q_{12} of cardinality 6, with no two queens on the same row or column, shows that the dominating set given in [8] is the only such set. Since this set is not independent, it follows that $i(Q_{12}) = 7$.

4. Lower bounds for $\Gamma(Q_n)$ and $IR(Q_n)$. We begin by describing a minimal dominating set of Q_n of cardinality $2n-5$ for $n \geq 5$ odd and of cardinality $2n-6$ for $n \geq 6$ even. These sets are later used to construct minimal dominating sets of Q_n of cardinality $5n/2 - O(1)$.

We consider the cases n even and n odd separately and first describe the placement of queens on the board to form a minimal dominating set D_n of Q_n for $n \geq 5$ odd. Since D_n is symmetric with respect to column 0, we only describe the queens placed in columns $0, 1, \ldots, k$. We leave all odd columns open, except for column k if k is odd, and place queens in some even columns and column k in such a way that each d-diagonal $d = k-2, \ldots, d = -k+1$, contains exactly one queen. This will give a placement of $2k-2$ queens. First we use even columns to place queens on the d-diagonals $d = k-2, \ldots, d = 1$, in the following way:

Place four queens (if possible) in column 2, namely in rows k, $k-1$, $k-2$, and $k-3$; four queens (if possible) in column 4, namely in rows $k-2$, $k-3$, $k-4$ and $k-5$, and so on, until a queen has been placed on a square which lies on the diagonal $d=1$. (The last column used may contain one, two, three or four queens.) For $k \neq 3$, we place queens on the d-diagonals $d=0,\ldots,d=-k+1$ by placing k queens in column k, one each on rows 1 to k. If $k=3$, place three queens in rows 0 to 2 in column 2. The precise coordinates of the queens are as follows:

For $k=3$: $(2,0)$, $(2,1)$, $(2,2)$, $(2,3)$

For $k \neq 3$: $(i,\ k-i-1)$, $(i,\ k-i)$, $(i,\ k-i+1)$, $(i,\ k-i+2)$ for each even integer $i=2,\ldots,2(m-1)$, where $m=\lceil (k-2)/4 \rceil \geq 1$;
$(2m,\ 2m+1),\ldots,(2m,\ k-2m+2)$, that is,
$k+2-4m \in \{1,2,3,4\}$ queens;
$(k,1),\ldots,(k,k)$.

We complete the construction of D_n by repeating the above procedure for the left hand side of the board and by placing a queen on $(0,k)$. Clearly, $|D_n| = 4k-3 = 2n-5$. The set D_{25} is illustrated in Figure 3, where the dominating queens are shown as black dots, the private neighbours as circles and where each queen is joined to a private neighbour by a line. A line (row, column or diagonal) of the chessboard containing a queen and any of its private neighbours is called a *PN-line* (*private neighbourhood line*).

Note that each d-diagonal $d=k-2,\ldots,d=-k+1$ (each s-diagonal $s=k-2,\ldots,s=-k+1$) contains exactly one queen, while each d-diagonal $d=k-1,\ldots,d=2k$ (each s-diagonal $s=k-1,\ldots,s=2k$) contains at least one queen.

Theorem 3. *The set D_n is a minimal dominating set of Q_n, where $n \geq 5$ is odd.*

Proof. Clearly, each square of Q_n on or above the diagonal $d=-k+1$ ($s=-k+1$, respectively) is dominated by a queen on a d-diagonal (s-diagonal, respectively). This leaves the square $(0,-k)$, which is dominated (only) by the queen on $(0,k)$. Hence D_n dominates Q_n.

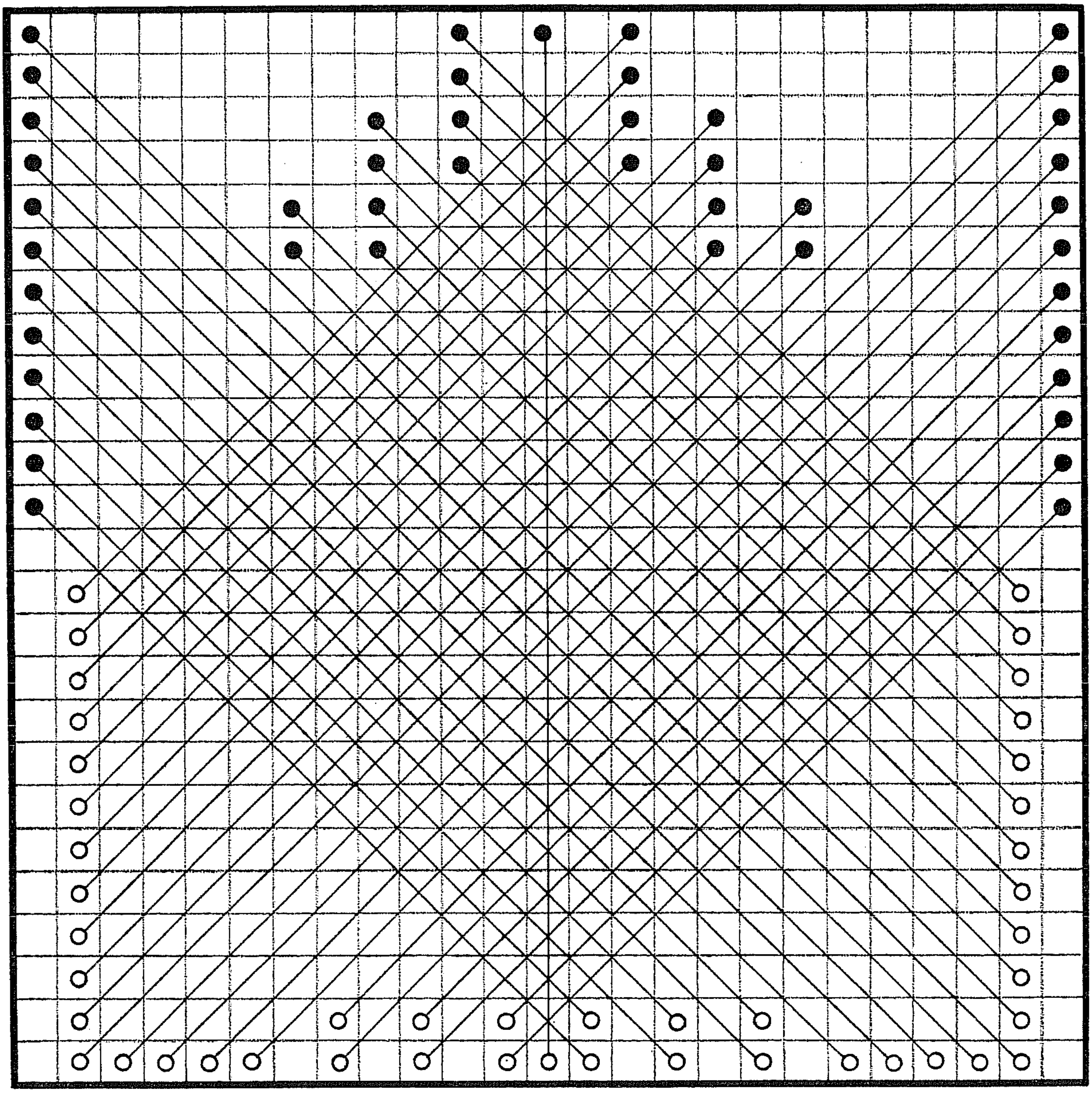

Figure 3

To show that D_n is minimal, we show that $PN(u, D_n) \neq \emptyset$ for each $u \in D_n$. Clearly, $(0, -k) \in PN\big((0, k),\ D_n\big)$. Each square below the diagonal $d = -k + 1$ ($s = -k + 1$, respectively) not in a column containing a queen is a private neighbour, along an s-diagonal (a d-diagonal, respectively) of a queen on the left (right, respectively) hand side of the board. Also, each of these queens has a private neighbour along one of the diagonals described. For example, for $k \neq 3$, each square $(-k + 1, i)$, where $i \in \{-k, \ldots, -1\}$, is a private neighbour of the queen on the d-diagonal $i + k - 1$; each square $(j, -k)$, where $j \in \{-k+2, \ldots, -2m-1\}$, is a private neighbour of the queen on the diagonal $d = -k-j$, while for each odd integer $\ell \in \{-2m+1, \ldots, -1\}$, the squares $(\ell, -k)$ and $(\ell, -k+1)$ are private neighbours of the queens on $d = -k - \ell$ and $d = -k - \ell + 1$ respectively. Private neighbours of the queens on the left hand side of the board are obtained by symmetry. ∎

Corollary 4. *For $n \geq 5$ odd, $IR(Q_n) \geq \Gamma(Q_n) \geq 2n - 5$.*

We now describe the placement of queens in columns 1 to k to form a minimal dominating set D_n of Q_n for $n \geq 6$ even. In this case we leave all odd columns open, except for column 1 and, if k is odd, column k, and place queens in some even columns and columns 1 and k in such a way that each d-diagonal $d = k-1, \ldots, d = -k+3$, contains exactly one queen. This will account for $2k - 3$ queens. First we use column 1 and the even columns to place queens on the diagonals $d = k - 1, \ldots, d = 1$, as follows:

Place one queen in column 1, row k and three queens (if possible) in column 2, rows k, $k - 1$, and $k - 2$; four queens (if possible) in column 4, in rows $k - 1$, $k - 2$, $k - 3$ and $k - 4$; four queens (if possible) in column 6, rows $k - 3, \ldots, k - 6$, etc., until a queen has been placed on a square which lies on the diagonal $d = 1$. As before, the last column used may contain one, two, three or four queens. For $k \neq 3$, we place one queen on each of the diagonals $d = 0, \ldots, d = -k + 3$ by placing $k - 2$ queens in column k, one each in rows 3 to k. If $k = 3$, place a queen in row 2 of column 2. The precise coordinates of the queens are as follows:

For $k = 3$: $(1,3)$, $(2,2)$, $(2,3)$

For $k = 4$: $(1,4)$, $(2,3)$, $(2,4)$, $(4,3)$, $(4,4)$

For $k \geq 5$: $(1,k)$, $(2, k-2)$, $(2, k-1)$, $(2,k)$
$(i, k-i)$, $(i, k-i+1)$, $(i, k-i+2)$, $(i, k-i+3)$ for each even integer $i = 4, \ldots, 2(m-1)$, where $m = \lceil (k-1)/4 \rceil \geq 2$;
$(2m, 2m+1), \ldots, (2m, k-2m+3)$, that is,
$k - 4m + 3 \in \{1,2,3,4\}$ queens;
$(k,3), \ldots, (k,k)$.

The construction of D_n is completed by repeating the above procedure for the left hand side of the board. Thus, $|D_n| = 4k-6 = 2n-6$. Further, each d-diagonal $d = k-2, \ldots, d = -k+3$ (s-diagonal $s = k-2, \ldots, s = -k+3$) contains exactly one queen, while each d-diagonal $d = k-1, \ldots, d = 2k-1$ (s-diagonal $s = k-1, \ldots, s = 2k-1$) contains at least one queen.

Theorem 5. *For each even integer $n \geq 6$, the set D_n is a minimal dominating set of Q_n.*

Proof. It is easy to see, as in the proof of Theorem 3, that D_n dominates Q_n. Also, to prove minimality, private neighbours of the dominating queens can be obtained similarly. ∎

Corollary 6. *For $n \geq 6$ even, $IR(Q_n) \geq \Gamma(Q_n) \geq 2n-5$.*

As stated in [5], $IR(Q_i) = \Gamma(Q_i) = i$ for $i = 1$, 4 and 5, and $IR(Q_i) = \Gamma(Q_i) = i-1$ for $i = 2, 3$. Also mentioned in [5] was the fact that W.D. Weakley showed that $\Gamma(Q_6) \geq 7$. The dominating set (indicated by solid black dots, with private neighbours indicated by circles) of Q_8 in Figure 4 shows that $IR(Q_8) \geq \Gamma(Q_8) \geq 11$. Also, the dominating set of Q_{14} in Figure 5, generated by McRae [7] using computer techniques, shows that $IR(Q_{14}) \geq \Gamma(Q_{14}) \geq 24$.

We now use the sets D_n constructed above to show that $IR(Q_n) \geq 5n/2 - O(1)$ if $n \geq 18$ and $\Gamma(Q_n) \geq 5n/2 - O(1)$ if $n \geq 19$. The main technique we employ is to symmetrically remove two queens from each side of the board, while adding three or more queens along the PN-line of one removed queen on each side, using the PN-line of the other removed queen to obtain private neighbours for the new queens, and taking care not

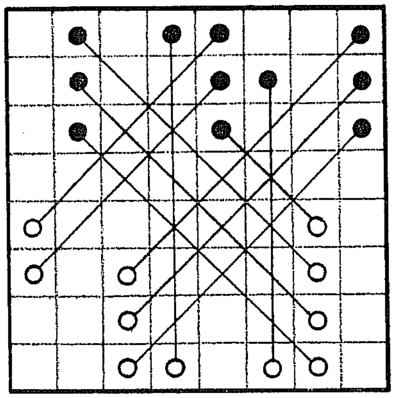

Figure 4

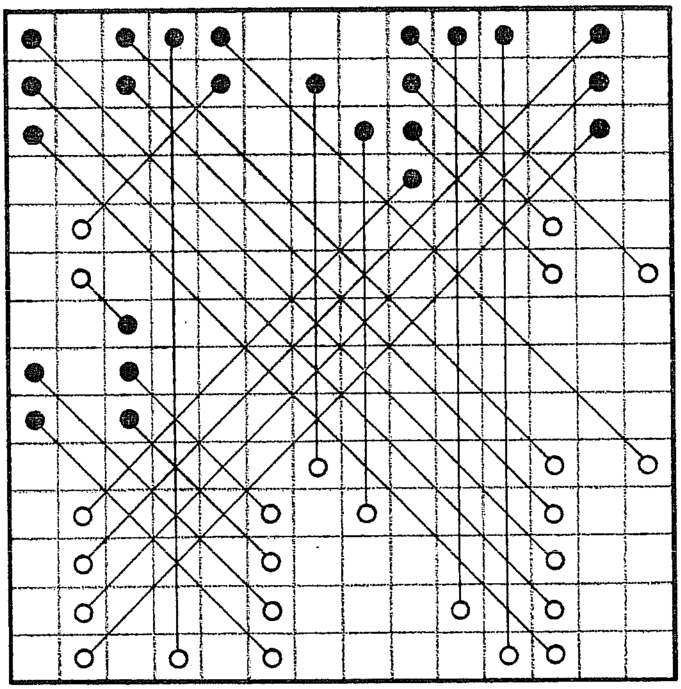

Figure 5

to destroy the private neighbourhoods of other queens. We consider eight cases, depending on the congruency of n modulo 8. In all cases except $n \equiv 2 \pmod 4$ the largest irredundant sets found are also dominating, while for $n \equiv 2 \pmod 4$, the largest minimal dominating sets obtained have cardinality two less than the largest irredundant sets.

We first assume that n is odd and use the same terminology as in the description of D_n. Again we only describe the placement of queens on the right hand side of the board.

Case 1. $n \equiv 1 \pmod 8$, $n \geq 25$. Then $k = 4m$.

We begin by moving the four queens of D_n in column $2(m-1)$, which occur on the diagonals $d = k-1-4(m-1) = 3, \ldots, d = k+2-4(m-1) = 6$, one column to the right along these diagonals, and the $k+2-4m = 2$ queens in column $2m$, on the diagonals $d = 1$ and $d = 2$, one column to the left along these diagonals. Delete the queens on $(k,\, k-3)$ and $(k,\, k-1)$, and add queens on the diagonal $d = -1$, in columns $2m, \ldots, k-3$. Elementary arithmetic shows that since $n \equiv 1 \pmod 8$, $(n-1)/4 - 2$ queens are added, a gain of $(n-1)/4 - 4$ queens. The precise coordinates of the queens are:

$$(i,\, k-i-1),\ (i,\, k-i),\ (i,\, k-i+1),\ (i,\, k-i+2) \text{ for each even integer } i = 2, \ldots, 2(m-2);$$

$$(2m-1,\, 2m), \ldots, (2m-1,\, 2m+5);$$

$$(k, 1), \ldots, (k,\, k-4),\ (k,\, k-2),\ (k, k);$$

$$(2m,\, 2m-1), \ldots, (k-3,\, k-4).$$

The construction of the set I_n is completed by repeating this procedure for the left hand side of the board. See Figure 6 for the case $n = 25$. Note that

$$|I_n| = 2n - 5 + 2[(n-1)/4 - 4] = \frac{5n-27}{2}.$$

Theorem 7. *If $n \equiv 1 \pmod 8$, $n \geq 25$, then I_n is a minimal dominating set of Q_n.*

Proof. Each d-diagonal $d = k-2, \ldots, d = -k+1$, except $d = -1$ and $d = -3$, contains exactly one queen, each d-diagonal $d = k-1, \ldots, d = 2k-4$, $d = 2k-2$, $d = 2k$ and $d = -1$ contains at least one queen, while $d = 2k-3$, $d = 2k-1$ and $d = -3$ are open. Hence each square of Q_n on or above the diagonal $d = -k+1$ except the squares on $d = 2k-3$, $d = 2k-1$ and $d = -3$, is dominated by a queen on a d-diagonal. The squares

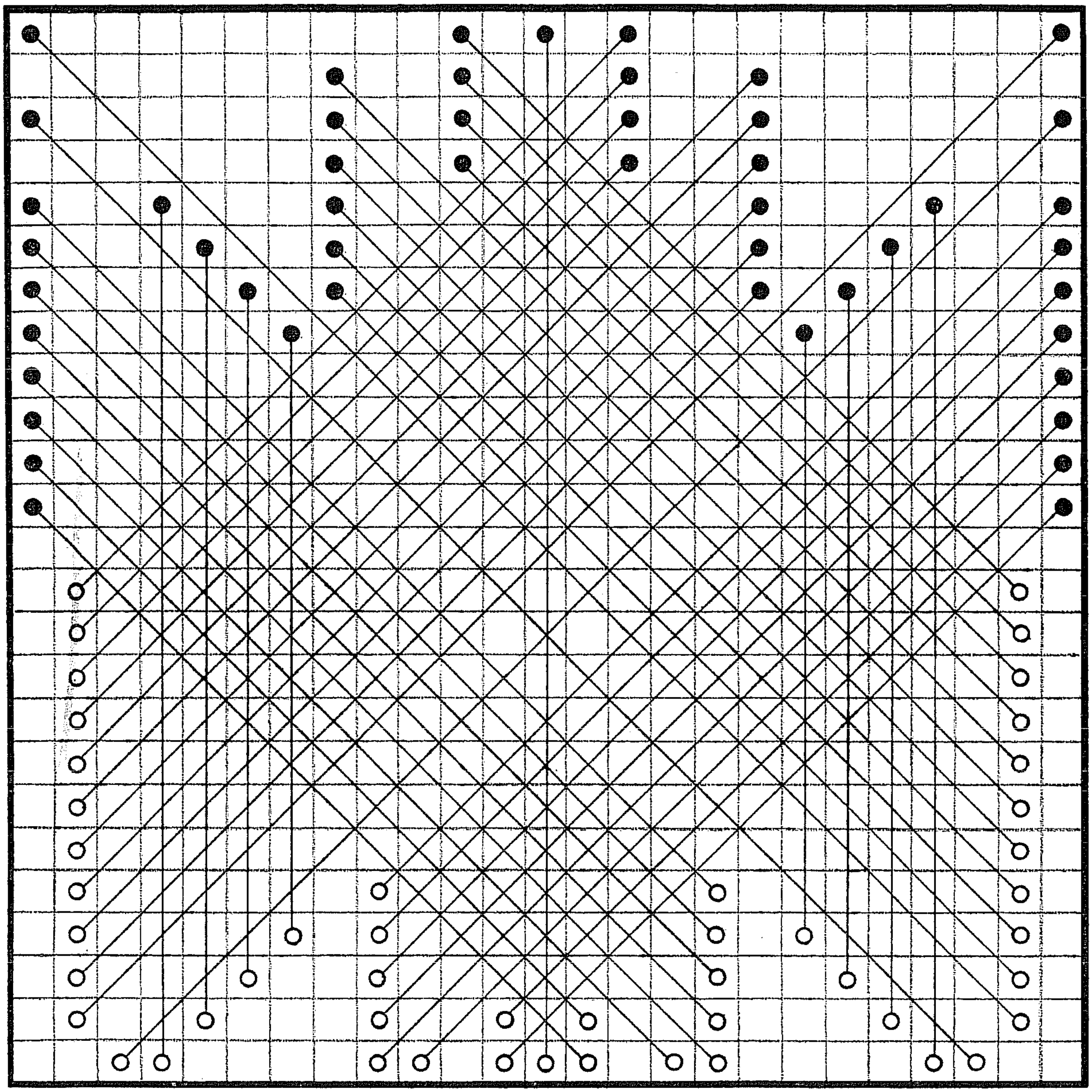

Figure 6

on $d = 2k - 3$ and $d = 2k - 1$ are dominated by the queens in column 2. The squares on $d = -3$, on or above $s = -k + 1$, are dominated by queens on s-diagonals, except for the square $(0, -3)$, which is dominated by the queen on $(0, k)$. Thus, consider the diagonal $d = -3$ below $s = -k+1$, namely the squares $(i, i-3)$, where $i \in \{-k+3, \ldots, (-k+2)/2\}$. Since $n \equiv 1 \pmod 8$, $(-k+2)/2 = -2m+1$. Therefore, these squares are dominated by the queens in columns $-2m+1, -2m, \ldots, -k+3$. In a similar way, it can be shown that all squares on or above $s = -k+1$ are dominated. Finally, $(0, -k)$ is dominated by the queen on $(0, k)$. Hence I_n dominates Q_n.

To show that I_n is irredundant, we list a private neighbour for each queen in I_n:

For each $i \in \{0, \ldots, k-2\}$, a private neighbour of the queen on the diagonal $d = i$ is given by $(-k+1,\ -k+1+i)$.

For each $j \in \{2m, \ldots, k-3\}$, the square $(j,\ -j-3)$ is a private neighbour of the queen $(j,\ j-1)$ on the diagonal $d = -1$.

The square $(-k+2,\ -k)$ is a private neighbour of the queen on $(k,\ k-2)$.

For each $\ell \in \{-k, \ldots, -k+2m-2\}$, the square $(-2m+2,\ \ell)$ is a private neighbour of the queen $(k,\ \ell + k + 2m - 2)$ on the diagonal $d = \ell + 2m - 2$.

For each odd integer $p \in \{-2m+1, \ldots, -1\}$, the squares $(p, -k)$ and $(p, -k+1)$ are private neighbours of the queens on $d = -k - p$ and $d = -k - p + 1$, respectively.

Private neighbours of the queens on the left hand side of the board are obtained by symmetry, while $(0,\ -k)$ is the private neighbour of the queen on $(0,\ k)$.

Thus I_n is irredundant and it follows that I_n is a minimal dominating set of Q_n. ∎

Case 2. $n \equiv 3 \pmod 8$, $n \geq 19$. Then $k = 4m + 1$.

Delete the queens of D_n on $(k,\ k-1)$ and $(k,\ k-2)$ and add queens on the diagonal $d = -1$, in columns $2m+1, \ldots, k-2$. Thus $2m - 1 = (n-3)/4 - 1$ queens are added, a gain of $(n-3)/4 - 3$ queens. The construction of I_n is completed by symmetry. Thus,

$$\begin{aligned} |I_n| &= 2n - 5 + 2\left[(n-3)/4 - 3\right] \\ &= \frac{5n - 25}{2}. \end{aligned}$$

Theorem 8. *If $n \equiv 3$ (mod 8), $n \geq 19$, then I_n is a minimal dominating set of Q_n.*

Proof. The only private neighbours of the queen on $(k,\ k-2)$, relative to D_n, are the squares $(i,\ i-2)$ for $i = -k, \ldots, -2m-1$. In I_n, each of these squares is dominated by the queen in column i, on the diagonal $s = -1$. The rest of the proof is similar to the proofs of Theorems 3 and 7 and details are omitted. ∎

Case 3. $n \equiv 5 \pmod 8$, $n \geq 21$. Then $k = 4m + 2$.

Delete the queens of D_n on $(k,\ k-1)$ and $(k,\ k-3)$ and add queens on the diagonal $d = -1$, in columns $2m+1, \cdots, k-3$. This gives a gain of $2m - 3 = (n-1)/4 - 4$ queens. Complete the construction of I_n symmetrically and note that

$$|I_n| = 2n - 5 + 2\left[(n-1)/4 - 4\right]$$
$$= \frac{5n-27}{2}.$$

The proof of the next theorem is similar to that of Theorem 8 and is omitted.

Theorem 9. *If $n \equiv 5 \pmod 8$, $n \geq 21$, then I_n is a minimal dominating set of Q_n.*

Case 4. $n \equiv 7 \pmod 8$, $n \geq 23$. Then $k = 4m - 1$.

Move the four queens of D_n in column $2(m-1)$, which occur on the diagonals $d = 2, \ldots, d = 5$, one column to the right along these diagonals. Delete the queen of D_n on $(k,\ k-2)$ and add queens on the diagonal $d = 1$, in column $2m+1, \ldots, k-2$. Thus $2m-3$ queens are added, providing a gain of $2m-4$ queens. Do the same on the left hand side of the board. Thus

$$|I_n| = 2n - 5 + 2\left[(n-3)/4 - 3\right]$$
$$= \frac{5n-25}{2}.$$

The proof of Theorem 10 is similar to that of Theorem 7 and is omitted.

Theorem 10. *If $n \equiv 7 \pmod 8$, $n \geq 23$, then I_n is a minimal dominating set of Q_n.*

We now assume that n is even and first consider the two cases where $n \equiv 0 \pmod 4$.

Case 5. $n \equiv 0 \pmod 8$, $n \geq 24$. Then $k = 4m$.

Move the four queens of D_n in column $2(m-1)$, on the diagonals $d = 4, \ldots, d = 7$, one column to the right along these diagonals. Delete the queens on $(k,\ k-2)$ and $(k,\ k-1)$ and add queens on the diagonal $d = -1$, in columns $2m+1, \ldots, k-2$. Thus $2m - 2 = n/4 - 2$ queens are added, a gain of $n/4 - 4$ queens. Repeating this procedure for the left hand side of the board yields a set I_n with

$$\begin{aligned}|I_n| &= 2n - 6 + 2(n/4 - 4)\\ &= \frac{5n-28}{2}.\end{aligned}$$

Theorem 11. *If* $n \equiv 0 \pmod 8$, $n \geq 24$, *then* I_n *is a minimal dominating set of* Q_n.

Proof. The squares of Q_n which are not dominated by $D_n - \{(k,\ k-2)\}$ are precisely the squares on the diagonal $d = -2$ which occur below the diagonal $s = -k+3$ and which are not in a column containing a queen of D_n. These are the squares with coordinates $(-k+2,\ -k), (-k+3,\ -k+1), \ldots, (-2m-1,\ -2m-3)$, which are dominated by the queens of I_n in columns $-k+2, \ldots, -2m-1$ on the diagonal $s = -1$, as well as the square $(-2m+1,\ -2m-1)$, which is dominated by the four queens of I_n in column $-2m+1$. Similarly, all squares of Q_n not dominated by $D_n - \{(-k,\ k-2)\}$ are dominated by vertices of I_n. The squares not dominated by $D_n - \{(-k,\ k-1),\ (k,\ k-1)\}$ are dominated by the queens of I_n on the diagonals $s = -1$ and $d = -1$. Hence I_n dominates Q_n. It is easy to show that I_n is irredundant by listing private neighbours for each of its queens and details are omitted. Hence I_n is a minimal dominating set of Q_n. ∎

Case 6. $n \equiv 4 \pmod 8$, $n \geq 20$. Then $k = 4m - 2$.

Delete the queen of D_n on $(k,\ k-2)$ and add queens on the diagonal $d = 1$ in columns $2m-1,\ 2m+1, 2m+2, \ldots, k-2$. Repeat the procedure symmetrically for the left hand side of the board and note that $2m - 3 = n/4 - 2$ queens are added on each side, resulting in a total gain of $n/2 - 6$ queens. Therefore

$$\begin{aligned}|I_n| &= 2n - 6 + n/2 - 6\\ &= \frac{5n-24}{2}.\end{aligned}$$

Theorem 12. *If $n \equiv 4 (\text{mod } 8)$, $n \geq 20$, then I_n is a minimal dominating set of Q_n.*

The proof is omitted.

Case 7. $n \equiv 2$ (mod 8), $n \geq 18$. Then $k = 4m + 1$.
Construct the set I_n by deleting the queens of D_n on $(k,\ k)$ and $(k,\ k-1)$, adding queens on the diagonal $d = 0$, in columns $2m+1, \ldots, k-2$, and by repeating the procedure for the left hand side. Note that I_n does not dominate the squares $(-k+1,\ -k)$ and $(k-1,\ -k)$. For $n \geq 26$, also construct the set X_n as follows. Move the four queens of D_n in column $2(m-1)$, on the diagonals $d = 5, \ldots, d = 8$, one column to the right along these diagonals. Delete the queens on $(k,\ k-3)$ and $(k,\ k-1)$ and add queens on the diagonal $d = -1$, in the columns $2m+1, \ldots, k-3$. Complete the construction of X_n by symmetry. Observe that

$$|I_n| = 2n - 6 + n/2 - 7$$
$$= \frac{5n - 26}{2}$$

while

$$|X_n| = 2n - 6 + n/2 - 9$$
$$= \frac{5n - 30}{2}.$$

Theorem 13. *If $n \equiv 2$ (mod 8) and $n \geq 18$, then I_n is a maximal irredundant set of Q_n and if $n \geq 26$, then X_n is a minimal dominating set of Q_n.*

Proof. It is easy to find private neighbours for each queen in I_n and in X_n to show that these sets are irredundant. The maximality of I_n follows from the fact that any square which dominates (say) $(k-1,\ -k)$, also dominates the private neighbourhood of a queen in I_n. That X_n dominates Q_n is also easily established and details are omitted. ∎

Case 8. $n \equiv 6$ (mod 8), $n \geq 22$. Then $k = 4m - 1$.
Construct I_n by deleting the queen of D_n on $(2m,\ 2m+1)$, adding queens on the diagonal $d = 2$, in columns $2m+1, \ldots, k-2$, and by doing the same on the left hand side of the board. The squares $(-k+1,\ -k+2)$ and $(k-1,\ -k+2)$ are not dominated by I_n. Construct X_n by moving the queen of D_n on $(2m,\ 2m+2)$ one column to the left, to

$(2m-1,\, 2m+1)$, deleting the queens on $(k,\, k-3)$ and $(k,\, k-2)$ and adding queens on the diagonal $d=-3$, in columns $2m+1,\ldots,k-2$. Then

$$|I_n| = 2n - 6 + n/2 - 7 = \frac{5n-26}{2}$$

and

$$|X_n| = 2n - 6 + n/2 - 9 = \frac{5n-30}{2}.$$

Theorem 14. *If $n \equiv 6 \pmod 8$, $n \geq 2$, then I_n is a maximal irredundant set and X_n a minimal dominating set of Q_n.*

Proof. Similar to the proof of Theorem 13. ∎

The next result now follows directly from Theorems 7-14.

Theorem 15. *If $n \geq 18$, then*

(i) $IR(Q_n) \geq \Gamma(Q_n) \geq \frac{5n-28+i}{2}$ *if n is odd and $n \equiv i \pmod 4$,*
(ii) $IR(Q_n) \geq \frac{5n}{2} - 13$ *and* $\Gamma(Q_n) \geq 5n/2 - 15$ *if $n \equiv 2 \pmod 4$,*
(iii) $IR(Q_n) \geq \Gamma(Q_n) \geq \frac{5n}{2} - 12$ *if $n \equiv 4 \pmod 8$,*
(iv) $IR(Q_n) \geq \Gamma(Q_n) \geq \frac{5n}{2} - 14$ *if $n \equiv 0 \pmod 8$.*

For $n = 50$, McRae [7] found a computer generated irredundant (but not dominating) set of cardinality 116, which shows that the bound for IR in Theorem 15(ii) is not exact. We suspect that the other bounds are also not exact.

Another interesting problem is to determine the largest number of queens in a minimal dominating or a maximal irredundant set of Q_n using only one type of diagonals, say d-diagonals, as PN-lines. Hence we define the parameters $\Gamma_d(Q_n)$ and $IR_d(Q_n)$ as the maximum cardinalities among the minimal dominating and maximal irredundant sets, respectively, of Q_n, such that every queen in the set has a private neighbour using a d-diagonal as PN-line. It is easy to see that $IR_d(Q_i) \geq \Gamma_d(Q_i) \geq i-1$ for $2 \leq i \leq 5$, while the coordinates given below show that $IR_d(Q_n) \geq \Gamma_d(Q_n) \geq 2n-7$ for $n \geq 6$.

$n = 2k,\ \ k \geq 3$

The coordinates are:

$$\begin{array}{ccc} (1,\,-k) & ,\ldots, & (k-1,\,-k) \\ (-k,\,1) & ,\ldots, & (-k,\,k-1) \\ (2,\,k-1) & ,\ldots, & (k-1,\,k-1) \\ (k-1,\,2) & ,\ldots, & (k-1,\,k-2). \end{array}$$

$n = 2k+1,\ \ k \geq 3$

The coordinates are:

$$\begin{array}{ccc} (1,\,-k) & ,\ldots, & (k-1,\,-k) \\ (-k,\,1), & ,\ldots, & (-k,\,k-1) \\ (1,\,k-1), & ,\ldots, & (k-1,\,k-1) \\ (k-1,\,1), & ,\ldots, & (k-1,\,k-2). \end{array}$$

A private neighbour of each of these queens using a d-diagonal as PN-line is given by the reflection about the diagonal $s = 0$.

5. An upper bound for $IR(Q_n)$. In [3] it was shown that for $n \geq 6$, $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n+3} \rfloor$. We now establish a slightly better upper bound. For a vertex v of Q_n, let $r(v)$, $c(v)$, $d(v)$, $s(v)$ denote respectively the row, column, d-diagonal and s-diagonal which contain v.

Theorem 16. *For $n \geq 9$, $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n + \sqrt{n} + 1} \rfloor$.*

Proof. Let I be an irredundant set of Q_n, A the set of isolated vertices of $Q_n[I]$, where $|A| = \alpha \leq n$ (since $\beta(Q_n) = n$) and $X = \{x_1, \ldots, x_t\} = I - A$. Since I is irredundant, each x_i, $i = 1, \ldots, t$, has a private neighbour y_i relative to I. Vertices x_i and y_i are on a line ℓ_i. Let $Y = \{y_1, \ldots, y_t\}$ and $Z = V - (I \cup Y)$. The private neighbour property implies that $\ell_1, \ldots, \ell_t$ are distinct. Define $U = \{\ell_1, \ldots, \ell_t\}$. Now suppose that U contains r, c, s, d rows, columns, s-diagonals and d-diagonals respectively, where $s \geq d$. We first state a few lemmas, the proofs of which can be found in [3] as indicated.

Lemma 1. *If $v \in A$, then $\{r(v),\, c(v),\, d(v),\, s(v)\} \cap U = \emptyset$.*

Proof. Proposition 4 of [3].

Lemma 2. *If $v \in A$, then $N(v) \subseteq Z$.*

Proof. Proposition 5 of [3].

Lemma 3. *For each $i = 1, \ldots, t$, $\ell_i - \{x_i, y_i\} \subseteq Z$.*

Proof. Proposition 6 of [3].

Lemma 4. *If $r + \alpha \geq n - 4$ or if $c + \alpha \geq n - 4$, then $|I| \leq 3n$.*

Proof. Proposition 7 of [3].

We now prove the following lemma:

Lemma 5. *If $r + c + \alpha \geq 2n - 2 - 4\sqrt{n}$, then $|I| \leq 4n - 5$.*

Proof. We may assume that $r + \alpha = n - k$ where $k \geq 5$, for otherwise $r + \alpha \geq n - 4$ and by Lemma 4, $|I| \leq 3n \leq 4n - 5$. Suppose that $c = \lambda n$. We establish a lower bound for $|Z|$. Let

$$R_U = \bigcup_{\ell_i \in U \text{ is a row}} \{\ell_i - \{x_i, y_i\}\}$$

$$C_U = \bigcup_{\ell_i \in U \text{ is a column}} \{\ell_i - \{x_i, y_i\}\}$$

$$R_A = \bigcup_{a \in A} \{r(a) - \{a\}\}$$

$$C_A = \bigcup_{a \in A} \{c(a) - \{a\}\}.$$

By Lemmas 2 and 3, $R_U \cup C_U \cup R_A \cup C_A \subseteq Z$. Observe that $|R_U| = r(n-2)$, $|C_U| = \lambda n(n-2)$, $|R_A| = |C_A| = \alpha(n-1)$, $|R_U \cap R_A| = |C_U \cap C_A| = 0$ (by Lemma 1), $|C_U \cap R_U| + |C_U \cap R_A| \leq \lambda n(r + \alpha)$, $|R_A \cap C_A| \leq \alpha(\alpha - 1)$ and $|R_U \cap C_A| \leq \alpha r$. The Principle of Inclusion-Exclusion then shows that

$$|Z| \geq |R_U \cup C_U \cup R_A \cup C_A|$$

$$\geq (r + \lambda n)(n-2) + 2\alpha(n-1) - \lambda n(r + \alpha) - \alpha(r + \alpha - 1).$$

Substituting $r + \alpha = n - k$ and simplifying, we obtain

$$|Z| \geq (n-k)(n-2) + \lambda n(k-2) + \alpha(1+k). \tag{1}$$

But

$$|Z| = |V(Q_n) - (X \cup Y \cup A)| = n^2 - 2t - \alpha. \tag{2}$$

From (1) and (2) follows

$$|I| = t + \alpha \leq \frac{n}{2}\left[(k+2) - \lambda(k-2)\right] - k. \tag{3}$$

Suppose first that $\lambda \geq (k-6)/(k-2)$. Then, using (3),

$$\begin{aligned} |I| &\leq \frac{n}{2}\left[(k+2) - \left(\frac{k-6}{k-2}\right)(k-2)\right] - k \\ &= 4n - k \\ &\leq 4n - 5. \end{aligned}$$

We have thus proved that

$$r + \alpha = n - k (k \geq 5) \quad \text{and} \quad c \geq (k-6)n/(k-2) \Longrightarrow |I| \leq 4n - 5. \tag{4}$$

Let

$$\begin{aligned} M &= \max_{5 \leq k \leq n} \left\{(n-k) + \frac{(k-6)n}{k-2}\right\} \\ &= 2n - 2 - 4\sqrt{n} \qquad \text{(by elementary calculus).} \end{aligned}$$

Let $r + c + \alpha \geq M$. Since $r + \alpha = n - k$ $(k \geq 5)$, it follows that $n - k + c \geq M \geq n - k + (k-6)n/(k-2)$ and thus $c \geq (k-6)n/(k-2)$. By (4), $|I| \leq 4n - 5$ and Lemma 5 is proved.

We need one further lemma from [3].

Lemma 6. $|I| \leq 6n - 2 - 4\sqrt{2s}$.

Proof. Inequality (1) of [3].

We now continue with the proof of Theorem 16. By Lemma 5, if $r+c+\alpha \geq 2n-2-4\sqrt{n}$, then $|I| \leq 4n-5 < 6n+6-8\sqrt{n+\sqrt{n}+1}$ (since $n \geq 9$). Hence we only need to consider sets I with $r+c+\alpha < 2n-2-4\sqrt{n}$. For such sets,

$$|I| = r+c+\alpha+s+d < 2n-2-4\sqrt{n}+2s = f_1(s) \text{ (say)}. \quad (5)$$

By Lemma 6,

$$|I| \leq 6n-2-4\sqrt{2s} = f_2(s) \text{ (say)}. \quad (6)$$

Since $f_1(s)$ is increasing and $f_2(s)$ is decreasing,

$$|I| \leq \max_{1 \leq s \leq 2n-3} \{\min\{f_1(s), f_2(s)\}\}.$$

The maximum occurs where $f_1(s) = f_2(s)$, *i.e.*, where $\sqrt{2s} = \sqrt{4n+4\sqrt{n}+4}-2$, so that by (6),

$$|I| \leq 6n+6-8\sqrt{n+\sqrt{n}+1}. \blacksquare$$

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