Disjoint Union-Free 3-Uniform Hypergraphs

by

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Abstract

A k-uniform hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ of order n is a family of k-subsets $\mathcal{B}$ of an n-set $\mathcal{X}$. A k-uniform hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ is *disjoint union-free* (DUF) if all disjoint pairs of elements of $\mathcal{B}$ have distinct unions; that is, if for every $A, B, C, D \in \mathcal{B}$, $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$ implies $\{A, B\} = \{C, D\}$. DUF families of maximum size have been studied by Erdős and Füredi, and in the case $k = 3$ this maximum size has been conjectured to equal $\binom{n}{2}$. In this thesis, we study DUF 3-uniform hypergraphs with the main goals of presenting evidence to support this conjecture and studying the structures that have conjectured maximum size.

If each pair of distinct elements of $\mathcal{X}$ is covered exactly λ times in $\mathcal{B}$ then we call $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ an (n, k, λ)-design. Using a blend of graph- and design-theoretic techniques, we study the DUF $(n, 3, 3)$-designs that are the conjectured unique structures having maximum size. Central results of this thesis include substantially improving lower bounds on the maximum size for a large class of n, giving conditions on pair coverage in a DUF 3-uniform hypergraph that force an $(n, 3, 3)$-design, and providing constructions for DUF 3-uniform hypergraphs from families of DUF hypergraphs with smaller orders.

Let $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ be a DUF k-uniform hypergraph with the property that $\mathcal{H} \cup \{E\}$ is not DUF for any k-subset E of $\mathcal{X}$ not already in $\mathcal{H}$. Then $\mathcal{H}$ is *maximally* DUF. We introduce the problem of finding the minimum size of maximally DUF families and provide bounds on this quantity for $k = 3$.

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Chapter 1

Introduction

We begin with the necessary definitions:

Definition 1.1 *A* k-uniform hypergraph of order n *is a pair* $(\mathcal{X}, \mathcal{B})$ *where* $|\mathcal{X}| = n$ *and* $\mathcal{B}$ *is a collection of subsets of* $\mathcal{X}$ *of cardinality* k*. The set* $\mathcal{X}$ *will be referred to as the* vertex set *of the hypergraph, and the set* $\mathcal{B}$ *will be called the* edge set*. The cardinality of the edge set,* $|\mathcal{B}|$*, will be called the* size *of the hypergraph.*

We may abuse notation and identify a hypergraph with its edge set. We will assume throughout this thesis that all hypergraphs are *simple*, meaning that repeated edges are not permitted.

Definition 1.2 *A* subhypergraph *of a* k*-uniform hypergraph* $(\mathcal{X}, \mathcal{B})$ *is a pair* $(\mathcal{X}_1, \mathcal{B}_1)$ *where* $\mathcal{X}_1 \subset \mathcal{X}$*,* $\mathcal{B}_1 \subset \mathcal{B}$*, and* $\mathcal{B}_1$ *is a collection of subsets of* $\mathcal{X}_1$ *of cardinality* k.

Definition 1.3 *A* graph *is a 2-uniform hypergraph, and a subgraph is a subhypergraph of a graph.*

For other standard graph and hypergraph terminology we refer the reader to Bollobás [2]. This thesis will focus on 3-uniform hypergraphs.

Given a family of forbidden k-uniform hypergraphs $\mathcal{F}$, $EX_k(n;\mathcal{F})$ is the set of k-uniform hypergraphs of order n not containing any forbidden hypergraph as a subhypergraph and having maximum size. This maximum size was classically denoted $ex_k(n;\mathcal{F})$. Elements of $EX_k(n;\mathcal{F})$ are called the *extremal hypergraphs* for the family $\mathcal{F}$. Any hypergraph with a larger size has an unavoidable forbidden subhypergraph.

The determination of $ex_2(n;\mathcal{G})$ for a family of graphs $\mathcal{G}$ is referred to as a Turán-type problem. For arbitrary families, this is often an intractable problem. Most of the previous work on Turán-type problems in graphs has focused on forbidden cycles of even length, or on forbidden cycles of multiple lengths, at least one of which is even. It is in general the lower bounds on $ex_2(n;\mathcal{G})$ for families of cycles that are difficult to obtain.

Definition 1.4 *A k-uniform hypergraph $\mathcal{H}$ is disjoint union-free (DUF) if all disjoint pairs of edges of $\mathcal{H}$ have distinct unions, that is for $A, B, C, D \in \mathcal{H}$, $A \cap B = \emptyset = C \cap D$ and $A \cup B = C \cup D$ together imply that $\{A, B\} = \{C, D\}$.*

When we let $k = 3$, the forbidden disjoint union configuration, which we will refer to as a *forbidden union*, consists of six points which can be partitioned into two 3-sets in two different ways. Any such configuration is isomorphic to $\{\{x, y, a\}, \{x, y, b\}, \{u, v, a\}, \{u, v, b\}\}$.

Definition 1.5 *An (n, k, λ)-*design *is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a set of n points and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{X}$ (called* blocks*) in which each pair of distinct points in $\mathcal{X}$ occurs together in exactly λ elements of $\mathcal{B}$.*

An (n, k, λ)-design is essentially a k-uniform hypergraph of order n with a global regularity property dictating how often pairs of vertices occur together in edges. For more background on this topic, see the *Handbook of Combinatorial Designs* [6].

Definition 1.6 *An (n, k, λ)-*covering *is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a set of n points and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{X}$ in which each pair of distinct points in $\mathcal{X}$ occurs together in at least λ elements of $\mathcal{B}$.*

Definition 1.7 *An (n, k, λ)-*packing *is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a set of n points and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{X}$ in which each pair of distinct points in $\mathcal{X}$ occurs together in at most λ elements of $\mathcal{B}$.*

Here (n, k, λ)-designs, (n, k, λ)-coverings and (n, k, λ)-packings will be called *DUF* if the corresponding hypergraphs are.

Given a 3-uniform hypergraph $(\mathcal{X}, \mathcal{B})$, we can define a new system by looking at the third points occurring in $\mathcal{B}$ with each pair of distinct points in $\mathcal{X}$ as follows. For notational convenience, we define the set $\binom{\mathcal{X}}{i}$ to be the set of all subsets of cardinality i of a set $\mathcal{X}$. We note that any 3-uniform hypergraph with a finite edge set (or finite order) is an $(n, 3, \lambda)$-packing for some fixed $\lambda < \infty$.

Definition 1.8 *Let* $(\mathcal{X}, \mathcal{B})$ *be an* $(n, 3, \lambda)$*-packing. Define the mapping* $\theta : \binom{\mathcal{X}}{2} \rightarrow \cup_{i=0}^{\lambda} \binom{\mathcal{X}}{i}$ *by* $\theta(\{x, y\}) = A$ *if and only if* $\{a : \{x, y, a\} \in \mathcal{B}\} = A$. *Let* $\mathcal{A} = \theta(\binom{\mathcal{X}}{2})$ *be the multiset whose* $\binom{n}{2}$ *elements are the images of* θ *over all pairs in* $\mathcal{X}$.

We can restate the DUF property of a 3-uniform hypergraph as:

$$|\theta(\{u, v\}) \cap \theta(\{x, y\})| \geq 2 \text{ implies that } \{u, v\} \cap \{x, y\} \neq \emptyset.$$

This will be referred to as the *theta condition.*

The edge-set of a hypergraph will sometimes be referred to as the $\mathcal{B}$-system, especially in the context of the associated $\mathcal{A}$-system as constructed above.

Definition 1.9 *The* extremal size, $f_k(n)$, *represents the maximum size of a disjoint union-free* k*-uniform hypergraph of order* n.

In classical notation $f_k(n)$ would be represented as $ex_k(n; \mathcal{D}_k)$ where $\mathcal{D}_k$ represents the forbidden union in a k-uniform hypergraph. This thesis will study the extremal hypergraphs with respect to the conjectured value for $f_3(n)$:

Conjecture 1.10 *(Füredi's Conjecture [12]) For sufficiently large n, $|\mathcal{H}| \leq \binom{n}{2}$ for any simple DUF 3-uniform hypergraph $\mathcal{H}$ of order n.*

The study of this problem began in 1977 when Erdős [12] asked to determine $f_k(n)$, the maximum size of a DUF k-uniform hypergraph of order n. Füredi [12] conjectured that for sufficiently large n, $f_k(n) \leq \binom{n}{k-1}$. He further constructed an infinite family of 3-uniform hypergraphs of order n with $\binom{n}{2}$ edges, with the edge sets of these conjectured extremal hypergraphs being the block set of $(n, 3, 3)$-designs. Since a DUF hypergraph generates an infinite class of DUF families by the repetition of any number of sets already in the hypergraph, we will consider only *simple* hypergraphs, those without repeated edges.

In undertaking a study of the DUF 3-uniform hypergraphs of maximum size, it is useful to examine classes of 3-uniform hypergraphs that are maximal with respect to the forbidden union.

Definition 1.11 *A k-uniform hypergraph $\mathcal{H}$ is* maximally DUF *if it is DUF and the addition of any k-set not already an edge in the hypergraph produces a hypergraph which is not DUF.*

Any DUF k-uniform hypergraph is a subhypergraph of a maximally DUF one. It should be noted that elsewhere in the literature, the term *DU-*

saturated is also used to mean maximally DUF.

Three classes of 3-uniform hypergraphs will be referred to repeatedly throughout the thesis. The order of each hypergraph should be taken to be n, unless specified otherwise:

Construction A. The standard result against which any DUF 3-uniform hypergraph construction should be measured is Füredi's *pencil through a point* construction [12]. By considering all possible triples through a fixed point $*$ plus an arbitrary set of $\lfloor \frac{n-1}{3} \rfloor$ pairwise disjoint triples not containing $*$, we see that a DUF 3-uniform hypergraph of order n can always be constructed with size $\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$.

Construction B. The simple DUF $(n, 3, 3)$-design. These are unique up to isomorphism and order, and exist iff $n \equiv 1, 5 \pmod{20}$ [9]. The construction is described in Chapter Two.

Construction C. The simple $(n, 3, 2)$-design produced from an $(n, 4, 1)$-design by breaking up the blocks with $(4, 3, 2)$-designs. These exist iff $n \equiv 1, 4 \pmod{12}$ and have size $\frac{2}{3}\binom{n}{2}$.

Using techniques from design theory and graph theory, we study local and global properties of simple DUF 3-uniform hypergraphs to solve the following problems:

- For which values of λ do there exist simple DUF $(n, 3, \lambda)$-designs?

- Give upper and lower bounds on pair coverage in a DUF 3-uniform hypergraph that force an $(n, 3, 3)$-design.
- Construct a reasonably good upper bound on the extremal size using elementary techniques.
- Substantially improve the known lower bound on the extremal size for a large class of hypergraph orders.
- Provide examples of 3-uniform hypergraphs of large size that are maximal with respect to the forbidden union.
- Provide constructions for DUF 3-uniform hypergraphs from families of DUF 3-uniform hypergraphs with smaller orders.
- Give the order of the minimum size of 3-uniform hypergraphs that are maximal with respect to the forbidden union.
- Provide an example of a 3-uniform hypergraph of relatively small size that is maximal with respect to the forbidden union.

We further present partial results towards the following:

- Settle the existence problem for simple DUF $(n, 3, 2)$-designs.
- Present evidence that $(n, 3, 3)$-designs are the unique extremal hypergraphs for the DUF problem.

Chapter 2

DUF and Maximally DUF Designs

In this chapter DUF $(n, 3, \lambda)$-designs are investigated and partial existence results are given. We are particularly interested in the designs with $\lambda = 2$ as this is the only open nontrivial case.

If an $(n, 3, \lambda)$-design has $\binom{n}{2}$ blocks, then $\frac{\lambda\binom{n}{2}}{\binom{3}{2}} = \binom{n}{2}$. Thus $\lambda = 3$ in this case. The following result says that this is the largest possible λ for a DUF $(n, 3, \lambda)$-design:

Theorem 2.1 *For a DUF $(n, 3, \lambda)$-covering to exist, $\lambda \leq 3$.*

Proof: Take a DUF $(n, 3, \lambda)$-covering with vertex set $\mathcal{X}$. The $\mathcal{A}$-system (defined in the Introduction) consists of $\binom{n}{2}$ sets of cardinality at least λ. We have $|\{(\{a, b\}, A) : a, b \in \mathcal{X}, A \in \mathcal{A} \text{ and } \{a, b\} \subset A\}| \geq \binom{n}{2}\binom{\lambda}{2}$. Since there are $\binom{n}{2}$ pairs of vertices in $\mathcal{X}$, some pair occurs at least $\binom{\lambda}{2}$ times in $\mathcal{A}$.

Now $|\theta(\{u,v\}) \cap \theta(\{x,y\})| \geq 2$ implies that $\{u,v\} \cap \{x,y\} \neq \emptyset$ for all $u,v,x,y \in \mathcal{X}$ since the covering is DUF. Consider a pair $\{a,b\}$ occurring together in the $\mathcal{A}$-system $\binom{\lambda}{2}$ times. If no vertex is common to all pairs $\{x,y\}$ having $\theta(\{x,y\}) = \{a,b,*\}$, then $|\{\{x,y\} : \theta(\{x,y\}) = \{a,b,*\}\}| \leq 3$ because four or more pairs intersecting pairwise must intersect in a vertex. Otherwise all pairs $\{x,y\}$ having $\theta(\{x,y\}) = \{a,b,*\}$ must be of the form $\{w, w_i\}$, for some fixed w. In this case, the number of sets in the $\mathcal{A}$-system containing the pair $\{a,b\}$ is bounded by the number of triples in the $\mathcal{B}$-system containing the pair $\{w,a\}$ (or $\{w,b\}$ similarly). In either case, $\lambda \leq 3$ or $\binom{\lambda}{2} \leq \lambda$, and the result follows. ☐

We examine $(n,3,3)$-designs before looking at the case for $\lambda = 1,2$. A result of Hanani (as cited in [12]) states that $(n,5,1)$-designs exist whenever $n \equiv 1,5 \pmod{20}$. Replacing each block of this design with all of its 3-subsets produces an $(n,3,3)$-design. Dukes and Ling [9] showed that simple $(n,3,3)$-designs are DUF if and only if they come from this construction. These are the unique designs meeting the conjectured bound for $k = 3$, and will be referred to as Construction B, for appropriate order n. Here we see that the designs described by Dukes and Ling cannot be extended to a larger DUF hypergraph:

Proposition 2.2 *The $(n,3,3)$-designs coming from Construction B are maximally DUF.*

Proof: Let $\mathcal{B}$ be the block set of a simple DUF $(n,3,3)$-design. Adjoin a new set $S=\{s_1,s_2,s_3\}$ to $\mathcal{B}$. We will show that the larger family $\mathcal{B}\cup\{S\}$ is not DUF, establishing the result. The set S is not contained in any block of the original $(n,5,1)$-design as it is not in $\mathcal{B}$. Choose any pair of points of S (without loss of generality say s_1 and s_2) and say this pair lies in block B of the $(n,5,1)$-design. Now let $x\in B-\{s_1,s_2\}$. The points x and s_3 lie in a block C of the $(n,5,1)$ design. Note that $C\neq B$ since if $s_3\in B$ then $S\subset B$, a contradiction. Now $C-\{x,s_3\}$ contains three points disjoint from B; choose two of these, say c_1 and c_2. We have:

$$M_1=\{x,s_1,s_2\}\in\mathcal{B} \text{ since } M_1\subset B;$$
$$M_2=\{s_3,c_1,c_2\}\in\mathcal{B} \text{ since } M_2\subset C;$$
$$M_3=\{x,c_1,c_2\}\in\mathcal{B} \text{ since } M_3\subset C;$$

Thus $M_1,M_2,M_3,S\in\mathcal{B}\cup\{S\}$ and $M_1\cap M_2=\emptyset$, $M_3\cap S=\emptyset$
with $M_1\cup M_2=M_3\cup S$ so that $\mathcal{B}\cup\{S\}$ is not DUF. □

DUF $(n,3,3)$-designs are a restricted class of $(n,3,3)$-designs. The following example gives an infinite family of $(n,3,3)$-designs with a forbidden union.

Definition 2.3 *Let G be an additive abelian group of order n. Then t k-element subsets of G, $B_i=\{b_{i,1},\ldots,b_{i,k}\}$ $(1\leq i\leq t)$ form an (n,k,λ)-* difference family *if every nonzero element of G occurs λ times among the differences $b_{i,x}-b_{i,y}$ $(i=1,\ldots,t;x,y=1,\ldots,k,x\neq y)$.*

Example 2.4 *A family of* $(n,3,3)$*-designs which are not DUF.*

When $n = 2t+1$ $(t \geq 2)$, $\{\{0,i,2i\} : 1 \leq i \leq t\}$ is an $(n,3,3)$-difference family over $\mathbb{Z}_n$ [6]. Taking the translates $b_i + g$, where $b_i = \{0,i,2i\}$, $1 \leq i \leq t$, and $g \in \mathbb{Z}_n$, we get an $(n,3,3)$-design. When n is not divisible by three and $n \geq 7$, this is an infinite family of simple $(n,3,3)$-designs with a forbidden union since {0,1,2}, {3,4,5} and {0,2,4}, {1,3,5} are always in the block sets of these designs.

Note that when n is divisible by three this family is not simple because the base block $\{0, \frac{n}{3}, \frac{2n}{3}\}$ generates three copies of each block in its orbit under translations by group elements g. The following result describes the $(n,3,3)$-designs which can be constructed as an $\mathcal{A}$-system to a $\mathcal{B}$-system:

Proposition 2.5 *Every simple DUF* $(n,3,3)$*-design can be constructed as an* $\mathcal{A}$*-system to a* $\mathcal{B}$*-system. Moreover, any simple DUF* $(n,3,3)$*-design has itself as its* $\mathcal{A}$*-system.*

Proof: Let $(\mathcal{X},\mathcal{B})$ be a simple DUF $(n,3,3)$-design. We know that $(\mathcal{X},\mathcal{B})$ comes from an underlying $(n,5,1)$-design $(\mathcal{X},\mathcal{S})$. The block set $\mathcal{B} = \cup_{S \in \mathcal{S}} \binom{S}{3}$. Let $S^* = \{s_1,\ldots,s_5\} \in \mathcal{S}$. Then $\theta(\binom{S^*}{2}) = \binom{S^*}{3}$. Since each pair of points in $\mathcal{X}$ occurs exactly once in a block of $\mathcal{S}$, and since the block S^* was chosen arbitrarily, it follows that $\mathcal{A} = \theta(\binom{\mathcal{X}}{2}) = \{\theta(\binom{S}{2}) : S \in \mathcal{S}\} = \{\binom{S}{3} : S \in \mathcal{S}\} = \mathcal{B}$, as desired. $\square$

The theta condition on a DUF 3-uniform hypergraph restricts the $\mathcal{B}$-system relative to the $\mathcal{A}$-system. It is therefore useful to study DUF $\mathcal{B}$-systems together with the $\mathcal{A}$-systems they generate. The above result is a first step in that direction, saying that one possibility for a $\mathcal{B}$-system which generates a DUF $(n, 3, 3)$-design as its $\mathcal{A}$-system is the design itself. While any such $\mathcal{B}$-system must be a simple $(n, 3, 3)$-design, it is not clear that the system must also be DUF.

In order to consider DUF hypergraphs with cover index one, we require the following:

Definition 2.6 *A* Steiner triple system *of order n, denoted $STS(n)$, is an $(n, 3, 1)$-design.*

A Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod{6}$. It is an easy fact that any 3-uniform hypergraph in which any pair of vertices appear in at most one edge is DUF. This is true because a forbidden union requires two disjoint pairs to be covered twice. We can deduce an even stronger result:

Observation 2.7 *Any 3-uniform hypergraph in which any pair of vertices appears in at most one edge is DUF, but not maximally DUF if $n > 3$.*

Thus any $STS(n)$ is DUF. If $n > 3$, an $STS(n)$ is not maximally DUF. We now consider the more interesting case of simple DUF $(n, 3, 2)$-designs.

When $n \equiv 0, 1 (\bmod 3)$ and $n > 3$, there is always a simple $(n, 3, 2)$-design [6]. Such designs are often called twofold triple systems of order n, or $TTS(n)$.

Example 2.8 *The unique DUF* $(4, 3, 2)$*-design:*

Four is the smallest number of points which admits a simple index-two design with block size three. To construct this design, simply take all possible 3-sets of the set of points. This design is trivially DUF because the point set has fewer than six elements.

Example 2.9 *A DUF* $(6, 3, 2)$*-design that is not maximally DUF:*

Construct a regular pentagon and place an additional point in its centre. The vertices, together with the additional point, form the point set of the design. The block set of this design consists of the sets of vertices of all triangles containing exactly one edge of the pentagon [3]. There are $\frac{2\binom{6}{2}}{\binom{3}{2}} = 10$ blocks out of a possible $\binom{6}{3} = 20$. Notice that no block has a block as its complement in the set of six points. Thus a forbidden union occurs only if at least two sets are added to the family. We can therefore add any set to this family and it will still be DUF. (This design is unique up to isomorphism).

Proposition 2.10 *All simple* $(7, 3, 2)$*-designs are maximally DUF.*

Proof: We construct the unique simple $(7, 3, 2)$-design, up to isomorphism, by taking the vertices of a regular heptagon H as the set of points and choosing as blocks the sets of vertices of all scalene triangles in H [3]. We will see that the addition of any triple to the block set causes a forbidden union.

Note that the block set is symmetric with respect to orientation and rotation of the heptagon.

Any triple that is a subset of the vertices of H can be regarded as a triangle, and as such, has associated with it the multiset of shortest distances between the three pairs of vertices. These distances are either 1, 2 or 3, as measured by the number of points to be travelled around the circle circumscribing the heptagon. The configuration $[1, 2, 3]$ represents a scalene triangle, and any vertices in such a configuration form a block of the design. Since the multisets $[1, 1, 1]$, $[2, 2, 2]$, $[3, 3, 3]$, $[1, 1, 3]$, $[1, 2, 2]$ and $[2, 3, 3]$ are all geometrically inadmissible, we need to check that the configurations $[1, 1, 2]$, $[1, 3, 3]$ and $[2, 2, 3]$ all result in forbidden unions when a triple of this form is added to the block set.

Let $H = \{0, 1, 2, \ldots, 6\}$ so that distances may be calculated modulo 7.

Case I: Configuration $[1, 1, 2]$. Without loss of generality, add the triple $\{0, 1, 2\}$ to the block set. The triples $\{1, 5, 6\}$ and $\{0, 2, 3\}$ are blocks of the design, resulting in a forbidden union together with the block $\{3, 5, 6\}$.

Case II: Configuration $[1, 3, 3]$. Without loss of generality, add the triple $\{0, 1, 4\}$ to the block set. The triples $\{0, 1, 3\}$ and $\{2, 4, 5\}$ are blocks of the design, resulting in a forbidden union together with the block $\{2, 3, 5\}$.

Case III: Configuration $[2, 2, 3]$. Again without loss of generality, add the triple $\{0, 2, 4\}$ to the block set. The triples $\{2, 3, 5\}$ and $\{0, 4, 6\}$ are blocks of the design, resulting in a forbidden union together with the block $\{3, 5, 6\}$. The three other non-isomorphic $(7, 3, 2)$-designs are not simple. ☐

Observation 2.11 *There are no simple DUF* $(9, 3, 2)$*-designs.*

In the Appendix, we have listed the nonisomorphic simple $(9, 3, 2)$-designs and exhibited a forbidden union in each instance.

Definition 2.12 *A* group divisible design *(GDD) is a triple* $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ *which satisfies the following properties:*
(1) $\mathcal{G}$ *is a partition of a set* $\mathcal{X}$ *of points into subsets called groups;*
(2) $\mathcal{B}$ *is a set of subsets of* $\mathcal{X}$ *called blocks such that a group and a block contain at most one common point;*
(3) every pair of points from distinct groups occurs in exactly λ *blocks.*

A GDD is said to be a k-GDD if for every block B in $\mathcal{B}$, $|B| = k$. We note that an (n, k, λ)-design is a k-GDD where the groups in $\mathcal{G}$ consist of the individual points of $\mathcal{X}$. Chee, Colbourn and Ling [4] defined a 3-GDD $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ to be *weakly union-free* (WUF) if:
(1) whenever $\{\{a, b, x\}, \{a, b, y\}\} \subset \mathcal{B}$, the points x and y are in different groups, and;
(2) whenever four distinct blocks B_1, B_2, B_3, B_4 are chosen from $\mathcal{B}$, it does not happen that $B_1 \cup B_2 = B_3 \cup B_4$.

The second condition rules out blocks of the following four forms:

C1: $\{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$;
C2: $\{\{x, a, b\}, \{x, a, c\}, \{x, b, d\}, \{x, c, d\}\}$;

$$\text{C3: } \{\{x,a,b\},\{x,a,c\},\{x,b,d\},\{a,c,d\}\};$$

$$\text{C4: } \{\{x,a,b\},\{x,c,d\},\{y,a,b\},\{y,c,d\}\}.$$

The configuration C4 is a forbidden union, so that a weakly union-free family is a (more restricted) DUF family.

The following theorem due to Chee, Colbourn and Ling provides a strong existence result for DUF twofold triple systems (TTS):

Theorem 2.13 *(Chee, Colbourn, Ling [4]) A WUF TTS(n) exists whenever* $n \equiv 0, 1 \pmod{3}$ *except when* $n \in \{3, 4, 6, 7, 9, 10\}$ *and possibly when* $n \in \{12, 15, 18, 22\}$.

It was later established by Denny and Gibbons [7] that $n = 12$ is a definite exception to the existence of WUF $TTS(n)$.

Because the only $(3, 3, 2)$-design contains a repeated block, a simple $TTS(3)$ does not exist. Simple DUF $TTS(n)$ were exhibited earlier for $n \in \{4, 6, 7\}$, and it was shown that simple DUF $TTS(9)$ do not exist. The last definite exception to explore then is the case $n = 10$.

Dukes [8] defined an *anti-firm* $TTS(n)$, also called an $AFTTS(n)$, as a $TTS(n)$ with the following properties:

(1) it has no repeated blocks;

(2) it contains no forbidden union (*fifa* in his terminology), i.e. avoids con-

figuration C4 above;

(3) it contains no *minififa*, which is obtained from a fifa by identifying two nonadjacent points.

Configuration C2 above is an example of a minififa. Given a $TTS(n)$, say $\mathcal{B}$, we have the following chain of implications:

$\mathcal{B}$ WUF $\Longrightarrow$ $\mathcal{B}$ is an $AFTTS(n)$ $\Longrightarrow$ $\mathcal{B}$ is a simple DUF $TTS(n)$.

To illustrate the minififa, the four blocks in configuration C4 above become $\{\{x,*,b\},\{x,*,d\},\{y,*,b\},\{y,*,d\}\}$ when the nonadjacent points a and c are identified. (This can be transformed into C2 under the isomorphism sending $(b,d,x,y,*)$ to (b,c,a,d,x)).

A theorem of Dukes [8] states that there is no $AFTTS(10)$. This was proved by listing, up to isomorphism, the $TTS(10)$ without repeated blocks and exhibiting a forbidden union in each. This gives the following more general result that there is no simple DUF $TTS(10)$, closing the case of the six definite exceptions raised by Chee, Colbourn and Ling.

An existence result among the possible exceptions is also due to Dukes:

Theorem 2.14 (*Dukes [8]*) *Let* $\varepsilon_0 = \{9, 10, 12, 15, 18, 24, 33, 45, 75, 87\}$. *For all* $n \equiv 0, 1 \pmod 3$, $n \notin \varepsilon_0$, *there exists an* $AFTTS(n)$.

This implies that there exists a simple DUF $TTS(22)$. Of the remaining cases, $n = 12, 15, 18$, the case $n = 15$ is special because there exist $STS(15)$. A twofold triple system $TTS(n)$ is said to be *separable* if its block set is the union of the block sets of two $STS(n)$.

Two Steiner triple systems, $S_1 = (V, \mathcal{B}_1)$ and $S_2 = (V, \mathcal{B}_2)$, are said to be *orthogonal* if $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and if $\{u, v\} \neq \{x, y\}$, $\{u, v, w\}, \{x, y, w\} \in \mathcal{B}_1$, $\{u, v, s\}, \{x, y, t\} \in \mathcal{B}_2$, then $s \neq t$. The pair is then referred to as an *orthogonal Steiner triple system*, or $OSTS(n)$, for appropriate n. Here is a partial non-existence result for simple DUF $TTS(15)$:

The orthogonality of S_1 and S_2 is necessary if the union of the two blocksets is to be DUF. Dukes and Mendelsohn [10] showed that each $OSTS(15)$ contains a forbidden union in its blockset, implying that if a simple DUF $TTS(15)$ exists, then it is not separable.

Example 2.15 *A simple DUF* $TTS(18)$*:*

Let the point set $\mathcal{X}$ be given by $\{0, 1, \ldots, 16\} \cup \{\infty\}$, where the first 17 elements are understood to be elements of $\mathbb{Z}_{17}$. The collection of blocks $\{\infty, 0, 1\}, \{0, 1, 4\}, \{0, 2, 8\}, \{0, 2, 12\}, \{0, 3, 8\}, \{0, 4, 11\}$ developed with the automorphisms of $\mathbb{Z}_{17} \cup \{\infty\}$ yield a simple DUF $TTS(18)$.

To see that a $TTS(18)$ results, we note that every unordered pair of points corresponds to a difference in the set $\{\infty, 1, 2, 3, 4, 5, 6, 7, 8\}$, and the blocks have differences $\{\infty, 1, \infty\}, \{1, 3, 4\}, \{2, 6, 8\}, \{2, 5, 7\}, \{3, 5, 8\}, \{4, 7, 6\}$ so that each difference in $\{\infty, 1, 2, 3, 4, 5, 6, 7, 8\}$ appears exactly twice. The resulting design is simple since none of the six blocks above are in the orbit of any other under any automorphism (again the differences in the blocks will verify this).

To see that the design is DUF, we list the pairs of the form $\{0, x\}$ and the corresponding pairs $\theta(\{0, x\})$. We then calculate the *θ-difference* of $\theta(\{0, x\})$, in other words the quantity $\overline{a-b} \in \{\infty, \overline{1}, \ldots, \overline{8}\}$, where $\theta(\{0, x\}) = \{a, b\}$. Forbidden unions will result if two pairs $\theta(\{0, x\})$ and $\theta(\{0, y\})$, with $x \neq \pm y$, have the same θ-difference and $\{\alpha, x+\alpha\} \cap \{0, y\} = \emptyset$ where $\theta(\{0, x\}) + \alpha = \theta(\{0, y\})$. For the list of pairs and associated calculations, see the Appendix.

The results extending the work of Chee, Colbourn and Ling can be summarized as follows:

Theorem 2.16 *There exist simple DUF* $TTS(n)$ *for all* $n \equiv 0, 1 \pmod 3$ *except when* $n \in \{3, 9, 10\}$ *and possibly when* $n \in \{12, 15\}$.

It is not clear which of the simple DUF $TTS(n)$ are maximally DUF. We give an infinite class of examples in either case. Breaking up the blocks of any $(n, 6, 1)$-design with the blocks of the unique (simple DUF) $(6, 3, 2)$-design produces a simple DUF $(n, 3, 2)$-design. Plenty of blocks can be added to this design without creating a forbidden union. Since $(n, 6, 1)$-designs exist asymptotically for $n \equiv 1, 6 \pmod{15}$, we have an infinite class of such examples. On the other hand, breaking up the blocks of an $(n, 4, 1)$-design with the blocks of the unique (simple DUF) $(4, 3, 2)$-design produces a simple

DUF $(n, 3, 2)$-design which is maximally DUF, as we will see in Chapter Six. Since $(n, 4, 1)$-designs exist whenever $n \equiv 1, 4 \pmod{12}$, we have an infinite class of such examples.

Chapter 3

The Structure of DUF $\lambda = 3$ Coverings

In this chapter, the main structural characterisation of the thesis is presented. The techniques involve studying DUF 3-uniform hypergraphs whose covering and packing indices are close together. The hypergraphs can then be reduced to graphs encoding the same structural information.

Theorem 3.1 *A simple DUF* $(n, 3, 3)$*-covering which is also an* $(n, 3, 4)$*-packing is an* $(n, 3, 3)$*-design.*

Proof: Let $(\mathcal{X}, \mathcal{B})$ be a simple DUF $(n, 3, 3)$-covering which is also an $(n, 3, 4)$-packing. Define the graph of excess $\mathcal{G}_{\mathcal{B}}$ as the graph with vertex set $\mathcal{X}$ in which an edge occurs between vertices a and b with multiplicity m if and only if the pair $\{a, b\}$ is covered $m + 3$ times in $\mathcal{B}$. Note that $\mathcal{G}_{\mathcal{B}}$ is a simple graph since pairs occur only with multiplicities three and four in $\mathcal{B}$. Define the graph of excess $\mathcal{G}_{\mathcal{A}}$ similarly (the $\mathcal{A}$-system is defined in the Introduction). Note that if $\mathcal{G}_{\mathcal{B}}$ is a simple graph, then so is $\mathcal{G}_{\mathcal{A}}$. (Suppose not. Say the edge $\{x, y\}$ occurs twice in $\mathcal{G}_{\mathcal{A}}$. Then this pair is covered at least five times in the $\mathcal{A}$-system. By the theta condition, the pairs in the $\mathcal{B}$-system whose images under the θ mapping include both a and b must all intersect pairwise. So these pairs in the $\mathcal{B}$-system are of the form $\{w, w_i\}$ for $i = 1, \ldots, 5$. This means that the pairs $\{a, w\}$ and $\{b, w\}$ occur with multiplicity at least five in the $\mathcal{B}$-system, so that $\mathcal{G}_{\mathcal{B}}$ is not simple.)

Each edge in $\mathcal{G}_{\mathcal{B}}$ represents a pair whose image under the θ-mapping has cardinality four. Counting pairs with multiplicity in the $\mathcal{A}$-system gives $3\binom{n}{2} + 3|\mathcal{G}_{\mathcal{B}}|$, where $|\mathcal{G}_{\mathcal{B}}|$ is the number of edges in $\mathcal{G}_{\mathcal{B}}$. Note that a three-fold covering of the $\mathcal{B}$-system does not a priori imply a three-fold covering of the $\mathcal{A}$-system. However, if some pairs are covered fewer than three times in the $\mathcal{A}$-system, this will mean additional edges in $\mathcal{G}_{\mathcal{A}}$. Thus $|\mathcal{G}_{\mathcal{A}}| \geq 3|\mathcal{G}_{\mathcal{B}}|$.

Fix a point x in $\mathcal{X}$ and let its degree in $\mathcal{G}_{\mathcal{A}}$ be d, with neighbours $y_1, \ldots, y_d$. This means that x and each y_i occur together four times as third points of pairs in $\mathcal{B}$. Since the covering is DUF, the theta condition means that these four pairs intersect pairwise (say in z_i, so that x and each y_i are third points of pairs of the form $\{z_i, z_{i_j}\}$ for $i = 1, \ldots, d$ and $j = 1, 2, 3, 4$). A pair can be covered at most four times in $\mathcal{A}$. This means that there are at most eight choices, not necessarily distinct, for third and fourth points in the sets $\{x, y_i, *\}$ in $\mathcal{A}$. Thus there must be *at least* a third as many z_i's as y_i's; it is not possible for more than three of the y_i's to contribute to the same edge $\{x, z_i\}$ in $\mathcal{G}_\mathcal{B}$ since otherwise the pair $\{x, z_i\}$ would be covered more than four times in $\mathcal{B}$. The degree of x in $\mathcal{G}_\mathcal{B}$ is therefore at least $\frac{d}{3}$. Since x was arbitrary, $|\mathcal{G}_\mathcal{B}| = \frac{1}{2}\sum_x \deg_{\mathcal{G}_\mathcal{B}}(x) \geq \frac{1}{6}\sum_x \deg_{\mathcal{G}_\mathcal{A}}(x) = \frac{1}{3}|\mathcal{G}_\mathcal{A}|$. Combining this with the bound above we have $3|\mathcal{G}_\mathcal{B}| = |\mathcal{G}_\mathcal{A}|$.

This equality forces the following $\frac{d}{3}$ sets to occur with multiplicity four in the $\mathcal{A}$-system: $\{x, y_1, y_2, y_3\}, \{x, y_4, y_5, y_6\}, \ldots, \{x, y_{d-2}, y_{d-1}, y_d\}$, producing $\frac{d}{3}$ K_4's in $\mathcal{G}_\mathcal{A}$. Since x was arbitrary, each edge in $\mathcal{G}_\mathcal{A}$ is part of a K_4. Note that the K_4 in $\mathcal{G}_\mathcal{A}$ containing the edge $\{x, y_i\}$ is unique since if $\{x, y_i, y_j, y_k\} \in \mathcal{G}_\mathcal{A}$ with multiplicity four and $\langle x, y_i, * \rangle$ induces a K_4 in $\mathcal{G}_\mathcal{A}$ then $* = \{y_j, y_k\}$ since otherwise the pair $\{x, y_i\}$ is repeated more than four times in $\mathcal{A}$. Again, this argument can be made for any vertex so that $\mathcal{G}_\mathcal{A}$ consists of edge-disjoint copies of K_4 (along with possible isolated vertices).

Now for $0 \leq k \leq \frac{d}{3} - 1$ and $j = 1, 2, 3, 4$, let z_k be the common point in the four pairs $\{z_k, z_{k_j}\}$ such that $\theta(\{z_k, z_{k_j}\}) = \{x, y_{k+1}, y_{k+2}, y_{k+3}\}$. These z_k's are distinct since if $z_i = z_j$, $0 \leq i < j \leq \frac{d}{3} - 1$, the pair $\{x, z_i\}$ would be covered more than four times in $\mathcal{B}$.

The $\frac{d}{3}$ K_4's in $\mathcal{G}_\mathcal{A}$ containing vertex x contribute $2d$ edges to $\mathcal{G}_\mathcal{A}$, where d is the degree of x in $\mathcal{G}_\mathcal{A}$. The $\frac{d}{3}$ z_k's contribute $\frac{4d}{3}$ edges of the form $\{x, z_k\}, \{y_{k+1}, z_k\}, \{y_{k+2}, z_k\}$, or $\{y_{k+3}, z_k\}$ to $\mathcal{G}_\mathcal{B}$. We know that an edge in $\mathcal{G}_\mathcal{B}$ cannot have both its endpoints acting as a z_k since if, say, $\theta(\{e, e_i\}) = \{a, b, c, d\}$ and $\theta(\{a, a_i\}) = \{e, f, g, h\}$ for $i = 1, 2, 3, 4$ then $\theta(\{a, e\}) = \{e_i\} = \{a_i\}$ so that $\theta(\{a, h\}) = \theta(\{a, g\}) = \theta(\{a, f\}) = \theta(\{a, e\}) = \theta(\{b, e\}) =$

$\theta(\{c, e\}) = \theta(\{d, e\})$. Any pair in $\{e_i\}$ is therefore covered more than four times in the $\mathcal{A}$-system unless $\{b, c, d\} = \{f, g, h\}$ and $a = e$. This contradicts the existence of such an edge in $\mathcal{G}_{\mathcal{B}}$.

Summing over all vertices and dividing by four to account for the four vertices in a K_4, we get: $|\mathcal{G}_{\mathcal{B}}| \geq \frac{1}{4} \sum_{x \in \mathcal{X}} \frac{4}{3} \deg_{\mathcal{A}}(x) = \frac{2}{3} |\mathcal{G}_{\mathcal{A}}|$.
This means $|\mathcal{G}_{\mathcal{A}}| = 3|\mathcal{G}_{\mathcal{B}}| \geq 2|\mathcal{G}_{\mathcal{A}}|$, so that $|\mathcal{G}_{\mathcal{A}}| = |\mathcal{G}_{\mathcal{B}}| = 0$.

Since $\mathcal{B}$ is an $(n, 3, 3)$-covering with no pairs covered more than three times, it is necessarily an $(n, 3, 3)$-design. We require the hypothesis that the covering of the $\mathcal{B}$-system be simple because the repetition of triples already in the $\mathcal{B}$-system cannot cause a forbidden union to occur. □

We observed in the proof of the theorem that if $\mathcal{G}_{\mathcal{B}}$ is simple, then so is $\mathcal{G}_{\mathcal{A}}$. In general, by a similar argument, if the $\mathcal{B}$-system is a DUF $(n, 3, \lambda)$-packing for some $\lambda \geq 4$, then the $\mathcal{A}$-system is also an $(n, 3, \lambda)$-packing. This is again due to the theta condition (defined in the Introduction).

If the $\mathcal{B}$-system is an $(n, 3, 3)$-covering, then $\mathcal{B}$ consists of at least $\binom{n}{2}$ blocks. We now consider the case of a simple, DUF $(n, 3, 4)$-packing and $(n, 3, 2)$-covering with at least $\binom{n}{2}$ blocks. Taking the simplest case when $|\mathcal{B}| = \binom{n}{2}$, the *pair deficit* to the $(n, 3, 3)$-covering, in other words the number of pairs which are covered only twice in the $\mathcal{B}$-system, is equal to $|\mathcal{G}_{\mathcal{B}}|$. In the $\mathcal{A}$-system, we have $\binom{n}{2} - 2|\mathcal{G}_{\mathcal{B}}|$ triples, $|\mathcal{G}_{\mathcal{B}}|$ pairs, and $|\mathcal{G}_{\mathcal{B}}|$ four-tuples, contributing a total of $3\binom{n}{2} - 6|\mathcal{G}_{\mathcal{B}}| + |\mathcal{G}_{\mathcal{B}}| + 6|\mathcal{G}_{\mathcal{B}}| = 3\binom{n}{2} + |\mathcal{G}_{\mathcal{B}}|$ pairs occurring together, counting multiplicity.

We know from Proposition 2.5 that a simple DUF $(n, 3, 3)$-design has as its $\mathcal{A}$-system an $(n, 3, 3)$-design. In the proof, the hypothesis of the $(n, 3, 3)$-covering is used only to establish $|\mathcal{G}_{\mathcal{A}}| \geq 3|\mathcal{G}_{\mathcal{B}}|$. Unfortunately, a counting argument cannot be used to get this bound when the hypotheses are changed to an $(n, 3, 2)$-covering together with $|\mathcal{B}| \geq \binom{n}{2}$. We also need a structural result showing that there is at least a $2|\mathcal{G}_{\mathcal{B}}|$ pair deficit to the three-cover in the $\mathcal{A}$-system. We worry that the $\mathcal{A}$-system may fill better than this, in the sense that the pair deficit may not be as big as $2|\mathcal{G}_{\mathcal{B}}|$, so that $|\mathcal{G}_{\mathcal{A}}| < 3|\mathcal{G}_{\mathcal{B}}|$. In this case, the technique of the proposition's proof could not be applied.

Conjecture 3.2 *The only simple DUF $(n, 3, 4)$-packings with at least $\binom{n}{2}$ blocks are $(n, 3, 3)$-designs.*

Note that the conjecture is formulated with a lower bound on the number of blocks despite the fact that any $(n, 3, 3)$-design has exactly $\binom{n}{2}$ blocks because the DUF $(n, 3, 3)$-designs were shown earlier to be maximally DUF.

Another question is whether the result stated in the proposition might be true if the $(n, 3, 4)$-packing hypothesis is weakened to an $(n, 3, 5)$-packing. This is a weaker hypothesis because less is known about the graphs $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{B}}$ in this case; they are no longer even required to be simple graphs. In order to understand this situation better, we introduce an extension of the graphs of excess introduced earlier.

Definition 3.3 *Let $m \geq 4$. Let $\mathcal{G}_\mathcal{B}^m$ be the simple graph with vertex set $\mathcal{X}$ in which an edge occurs between vertices a and b if and only if the pair $\{a, b\}$ is covered exactly m times in $\mathcal{B}$. Define $\mathcal{G}_\mathcal{A}^m$ similarly, based on the number of times a pair is covered in $\mathcal{A}$.*

Suppose the $\mathcal{B}$-system is a simple DUF $(n, 3, 5)$-packing and $(n, 3, 3)$-covering. The $\mathcal{A}$-system has $3\binom{n}{2} + 3|\mathcal{G}_\mathcal{B}^4| + 7|\mathcal{G}_\mathcal{B}^5|$ pairs occurring together, counting multiplicities. This gives $|\mathcal{G}_\mathcal{A}^4| + 2|\mathcal{G}_\mathcal{A}^5| \geq 3|\mathcal{G}_\mathcal{B}^4| + 7|\mathcal{G}_\mathcal{B}^5|$. Thus either $|\mathcal{G}_\mathcal{A}^4| \geq 3|\mathcal{G}_\mathcal{B}^4|$ or $|\mathcal{G}_\mathcal{A}^5| \geq \frac{7}{2}|\mathcal{G}_\mathcal{B}^5| \geq 3|\mathcal{G}_\mathcal{B}^5|$. So the lower bounds on the $|\mathcal{G}_\mathcal{A}^i|$ are in line with what we had in the proof of the proposition.

In general, for an $(n, 3, \lambda)$-packing and $(n, 3, 3)$-covering with $\lambda \geq 4$ we have: $\sum_{i=4}^{\lambda}(i-3)|\mathcal{G}_\mathcal{A}^i| \geq \sum_{i=4}^{\lambda}[\binom{i}{2} - 3]|\mathcal{G}_\mathcal{B}^i|$. This means that
$|\mathcal{G}_\mathcal{A}^i| \geq (1 + \frac{i}{2})|\mathcal{G}_\mathcal{B}^i| \geq 3|\mathcal{G}_\mathcal{B}^i|$ for at least one i such that $4 \leq i \leq \lambda$.

Obtaining upper bounds on the $|\mathcal{G}_\mathcal{A}^i|$ based on the proof of the proposition is not as fruitful.

Proposition 3.4 *Let the $\mathcal{B}$-system be a simple DUF $(n, 3, \lambda)$-packing and let $4 \leq m \leq \lambda$. Then $\sum_{i=m}^{\lambda} |\mathcal{G}_\mathcal{B}^i| \geq \frac{m}{\lambda(\lambda-1)}|\mathcal{G}_\mathcal{A}^m|$.*

Proof: Fix m such that $4 \leq m \leq \lambda$. Fix a point x in $\mathcal{X}$ and let its degree in $\mathcal{G}_\mathcal{A}^m$ be d, with neighbours $y_1, \ldots, y_d$. This means that x and each y_i occur together exactly m times as third points of pairs in $\mathcal{B}$. Since the covering is DUF, the theta condition means that these m pairs intersect pairwise (say in z_i, so that x and each y_i are third points of pairs of the form $\{z_i, z_{i_j}\}$ for $i = 1, \ldots, d$ and $j = 1, 2, \ldots, m$). A pair can be covered at most λ times in $\mathcal{B}$. This means that there are at most $m(\lambda - 2)$ choices, not necessarily distinct, for filling in the sets $\{x, y_i, *\}$ in $\mathcal{A}$. Thus there must be *at least* $\frac{m}{\lambda(\lambda-1)}$ as many z_i's as y_i's; it is not possible for more than $\frac{\lambda(\lambda-1)}{m}$ of the y_i's to contribute to the same edge $\{x, z_i\}$ in $\mathcal{G}_{\mathcal{B}}$ since otherwise the pair $\{x, z_i\}$ would be covered more than λ times in $\mathcal{B}$. The point x therefore occurs in at least $\frac{dm}{\lambda(\lambda-1)}$ pairs occurring together in $\mathcal{B}$, each with multiplicity at least m. Since x was arbitrary,

$\sum_{i=m}^{\lambda} |\mathcal{G}_{\mathcal{B}}^{i}| = \frac{1}{2} \sum_{i=m}^{\lambda} \sum_x \deg_{\mathcal{G}_{\mathcal{B}}^{i}}(x) \geq \frac{m}{2\lambda(\lambda-1)} \sum_x \deg_{\mathcal{G}_{\mathcal{A}}^{m}}(x) = \frac{m}{\lambda(\lambda-1)} |\mathcal{G}_{\mathcal{A}}^{m}|.$ $\square$

On the left-hand side, the same edge cannot be counted more than once, by definition of the $\mathcal{G}_{\mathcal{A}}^{m}$'s. Knowing, for instance, exactly how many times x and each z_i occur together as pairs in $\mathcal{B}$ would allow us to improve the above bound by not having to allow for the possibility that these points could occur together anywhere from m to λ times in $\mathcal{B}$.

As an example, in an $(n, 3, 5)$-packing, the above bound yields:
$|\mathcal{G}_{\mathcal{B}}^{4}| + |\mathcal{G}_{\mathcal{B}}^{5}| \geq \frac{1}{5} |\mathcal{G}_{\mathcal{A}}^{4}|$ and $|\mathcal{G}_{\mathcal{B}}^{5}| \geq \frac{1}{4} |\mathcal{G}_{\mathcal{A}}^{5}|$. The techniques used in the proof of the proposition do not allow us to force enough structure to conclude that $|\mathcal{G}_{\mathcal{B}}^{m}| = 0$ for all $4 \leq m \leq \lambda$ when we cannot compare the sizes of the $\mathcal{G}_{\mathcal{A}}^{m}$'s among themselves, or the $\mathcal{G}_{\mathcal{B}}^{m}$'s among themselves, for the various values of m.

Conjecture 3.5 *The only simple DUF $(n, 3, 3)$-coverings (for any $(n, 3, \lambda)$-packing, $\lambda \geq 4$) are $(n, 3, 3)$-designs.*

Based on the two previous conjectures, this leads to another structural conjecture:

Conjecture 3.6 *The only simple DUF 3-uniform hypergraphs of size at least $\binom{n}{2}$ are $(n, 3, 3)$-designs.*

Recall Füredi's Conjecture (FC), Conjecture 1.10 in this thesis:
For sufficiently large n, $|\mathcal{H}| \leq \binom{n}{2}$ for any simple DUF 3-uniform hypergraph $\mathcal{H}$ of order n.

Proposition 3.7 *Conjecture 3.6 implies (FC), for all values of n.*

Proof: Suppose for contradiction that $\mathcal{H}$ is a simple DUF 3-uniform hypergraph of order n and size $> \binom{n}{2}$. Then there exists a subhypergraph of $\mathcal{H}$ with $\binom{n}{2}$ edges; this subfamily is necessarily both simple and DUF. By Conjecture 3.6 this subfamily is an $(n, 3, 3)$-design; DUF $(n, 3, 3)$-designs were shown earlier to be maximally DUF. This is a contradiction to $|\mathcal{H}| > \binom{n}{2}$ as an $(n, 3, 3)$-design contains exactly $\binom{n}{2}$ edges. $\square$

The desire to prove (FC) motivates the work and conjectures in this section.

Conjecture 3.6 would imply that DUF 3-uniform hypergraphs of order n of size $\binom{n}{2}$ exist only when $n \equiv 1, 5 \pmod{20}$. A small example below shows that no 3-uniform hypergraph of order 6 can be simple and DUF with $\binom{6}{2} = 15$ edges:

Consider a simple DUF 3-uniform hypergraph of order 6. At most one pair of edges which are complements in the vertex set can co-exist in this hypergraph, otherwise the DUF condition is violated. This hypergraph therefore has at most $\frac{1}{2}\binom{6}{3} + 1 = 11$ edges.

Chapter 4

Upper Bounds on Extremal Size

An unpublished result of Erdős and Frankl from 1975 [12] states that $f_r(n) < O(n^{r-0.5})$ for all $r \geq 2$. For completeness, we give a proof that $f_3(n) \in O(n^{\frac{5}{2}})$. A much better upper bound established by Füredi [12] is $f_r(n) < \frac{7}{2}\binom{n}{r-1}$ for all $r \geq 3$. In this chapter upper bounds are obtained on the extremal size by using the concept of variance to study pair coverage in DUF 3-uniform hypergraphs.

Let $(\mathcal{X}, \mathcal{B})$ be a DUF 3-uniform hypergraph with $|\mathcal{X}| = n$ and $|\mathcal{B}| = c\binom{n}{2}$, for some real constant $c > 1$. We wish to study upper bounds on the extremal size in any potential cases where Füredi's Conjecture (see Conjecture 1.10) does not hold. Let $b : \binom{\mathcal{X}}{2} \to \mathbb{Z}^+ \cup \{0\}$ be given by $b(T) = |\theta(T)|$. We will write $\overline{b}$ for $\frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} b(T)$.

The *variance* of b is by definition $Var(b) = \frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} (b(T) - \overline{b})^2$. We will write V for $Var(b)$ throughout.

Let $a : \binom{\mathcal{X}}{2} \to \mathbb{Z}^+ \cup \{0\}$ be defined by $a(T) = \binom{b(T)}{2}$, recalling that $\binom{1}{2} = \binom{0}{2} = 0$. Again we write $\overline{a}$ for $\frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} a(T)$.

Suppose the 3-uniform hypergraph $(\mathcal{X}, \mathcal{B})$ has packing index λ. Then $\frac{b(T)}{a(T)} \geq \frac{\lambda}{\binom{\lambda}{2}} = \frac{2}{\lambda - 1}$ for all $T \in \binom{\mathcal{X}}{2}$. It follows that $\overline{b} \geq \frac{2}{\lambda-1}\overline{a}$. We can also express $\overline{a}$ in terms of the variance, V:

Proposition 4.1 *The relation* $\overline{a} = \frac{1}{2}\left[V + (9c^2 - 3c)\right]$ *holds for all DUF* 3-*uniform hypergraphs with size* $c\binom{n}{2}$, $c > 1$.

Proof: Clearly $\overline{b} = \frac{3|\mathcal{B}|}{\binom{n}{2}} = 3c$.

$$
\begin{aligned}
V &= \frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} \left[(b(T) - \overline{b})^2\right] \\
&= \frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} \left[b(T)^2 - 2b(T)\overline{b} + \overline{b}^2\right] \\
&= \frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} \left[b(T)^2 - b(T) - 2b(T)\left(\overline{b} - \frac{1}{2}\right) + \overline{b}^2\right] \\
&= \frac{2}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} \binom{b(T)}{2} + \frac{1}{\binom{n}{2}} \sum_{T \in \binom{\mathcal{X}}{2}} \left[-2b(T)\left(\overline{b} - \frac{1}{2}\right) + \overline{b}^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\binom{n}{2}} \sum_{T \in \binom{X}{2}} \binom{b(T)}{2} - 2\overline{b}\left(\overline{b} - \frac{1}{2}\right) + \overline{b}^2 \\
&= \frac{2}{\binom{n}{2}} \sum_{T \in \binom{X}{2}} \binom{b(T)}{2} - \overline{b}^2 + \overline{b} \\
&= \frac{2}{\binom{n}{2}} \sum_{T \in \binom{X}{2}} \binom{b(T)}{2} - (9c^2 - 3c)
\end{aligned}
$$

This implies that $\overline{a} = \frac{1}{\binom{n}{2}} \sum_{T \in \binom{X}{2}} \binom{b(T)}{2} = \frac{1}{2}\left[V + (9c^2 - 3c)\right]$. □

We can also deduce further inequalities relating $\overline{a}, \overline{b}$ and V by exploiting the DUF hypothesis:

Proposition 4.2 *The inequalities* $\lambda \geq \overline{a} \geq \binom{\overline{b}}{2} \geq \overline{b}$ *hold for all DUF* 3-*uniform hypergraphs with packing index* λ *and size* $c\binom{n}{2}$, $c > 1$.

Proof: The existence of a pair covered more than three times in the $\mathcal{A}$-system implies the existence of a pair covered more than three times in the $\mathcal{B}$-system by the theta condition. This gives the first inequality. The second inequality is deduced from the previous proposition with $V \geq 0$ and $\overline{b} = 3c$. The last inequality is trivial since $\overline{b} > 3$. □

Corollary 4.3 *If* $\lambda < 50$, *a DUF* 3-*uniform hypergraph of order* n *with packing index* λ *has size less than* $\frac{7}{2}\binom{n}{2}$.

Proof: By the previous proposition, $\lambda \geq \binom{\overline{b}}{2} = \binom{3c}{2}$. If $\lambda < 50$, then $\binom{3c}{2} < 50$, which gives $c < \frac{7}{2}$. □

This means that we have an improvement on Füredi's upper bound when $\lambda < 50$.

Corollary 4.4 *The extremal size* $f_3(n) \in O(n^{\frac{5}{2}})$.

Proof: The packing index λ is at most $n-2$ by definition. Thus $\binom{3c}{2} < \lambda$ gives $c \in O(n^{\frac{1}{2}})$, for any choice of λ. But $f_3(n) \le c\binom{n}{2}$ by definition of the extremal size, so that $f_3(n) \in O(n^{\frac{5}{2}})$, as desired. ☐

Note that a lower bound on V improves Corollary 4.3. Suppose we could show that $V \ge (\frac{s}{\lambda})(\overline{a} - \overline{b})^2$ for some positive constant s and for any DUF 3-uniform hypergraph of order n with packing index λ and having variance of pair coverage V. Then by Proposition 4.1, we could solve a quadratic equation in $\sqrt{V}$ by taking square roots of both sides and get an upper bound on c as follows:

$$\begin{aligned} \sqrt{V} &\ge \sqrt{\frac{s}{\lambda}}(\overline{a} - \overline{b}) \\ &= \sqrt{\frac{s}{\lambda}}\left(\frac{1}{2}\left[V + (9c^2 - 3c)\right] - 3c\right) \\ &= \sqrt{\frac{s}{\lambda}}\left(\frac{1}{2}\left[V + (9c^2 - 9c)\right]\right) \\ &= \sqrt{\frac{s}{\lambda}}\left(\frac{V}{2} + 9\binom{c}{2}\right) \end{aligned}$$

giving $\frac{1}{2}V - \sqrt{\frac{\lambda}{s}}\sqrt{V} + 9\binom{c}{2} \le 0$. Solving this in $\sqrt{V}$ yields:

$$\sqrt{\tfrac{\lambda}{s}} - \sqrt{\tfrac{\lambda}{s} - 18\tbinom{c}{2}} \le \sqrt{V} \le \sqrt{\tfrac{\lambda}{s}} + \sqrt{\tfrac{\lambda}{s} - 18\tbinom{c}{2}},$$

which implies that $\frac{\lambda}{s} \geq 18\binom{c}{2}$ since the discriminant must be positive. Thus $\binom{c}{2} \leq \frac{\lambda}{18s}$, and we have an upper bound on the extremal size.

The possibility of finding a constant s to prove such a statement about the variance lends hope for an eventual lowering of the constant $\frac{7}{2}$ in Füredi's upper bound.

Chapter 5

Lower Bounds on Extremal Size

In this chapter we will be interested in bounding $f_3(n)$, the maximum size of a DUF 3-uniform hypergraph of order n, from below. We have already seen that $f_3(n) \geq \binom{n}{2}$ when $n \equiv 1, 5 \pmod{20}$. This construction comes from the simple DUF $(n, 3, 3)$-designs. Known lower bounds are improved here for some congruence classes modulo 20 using design and packing constructions. For a table of results, see the end of this chapter.

Definition 5.1 *A* pairwise-balanced design $PBD(n, K; \lambda)$ *of order* n *with block sizes in* $K = \{k_i\}$ *is a pair* $(\mathcal{X}, \mathcal{B})$ *where* $\mathcal{X}$ *is a set of cardinality* n *and* $\mathcal{B}$ *is a family of subsets of* $\mathcal{X}$ *which satisfy the following properties:*

(i) If $B \in \mathcal{B}$ *then* $|B| \in K$*;*

(ii) Every pair of distinct points of $\mathcal{X}$ *occurs in exactly* λ *blocks of* $\mathcal{B}$.

The notation $PBD(n, K)$ *will be used in the case* $\lambda = 1$.

Proposition 5.2 *Suppose there exists a* $PBD(n, K)$, *say* $(\mathcal{X}, \mathcal{B})$, *and for all* $k \in K$ *there exists a DUF* 3-*uniform hypergraph* $\mathcal{H}_k = (\mathcal{X}_k, \mathcal{B}_k)$ *with* $|\mathcal{X}_k| = k$. *Then the* 3-*uniform hypergraph constructed from* $(\mathcal{X}, \mathcal{B})$ *by mapping each block* B *to* $\mathcal{H}_k$ *on vertex set* $\mathcal{X}_k = B$, *where* $|B| = k$, *is DUF.*

Example 5.3 *As an illustration before proving the proposition, consider the following* $PBD(10, \{3, 4\})$ *with blocks in columns:*

$$\left|\begin{array}{cccccccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\
2 & 5 & 8 & 5 & 6 & 7 & 5 & 6 & 7 & 5 & 6 & 7 \\
3 & 6 & 9 & 8 & 9 & 10 & 10 & 8 & 9 & 9 & 10 & 8 \\
4 & 7 & 10 & & & & & & & & &
\end{array}\right|$$

together with the following two 3-uniform hypergraphs of orders 3 *and* 4 *respectively:* $\mathcal{H}_1 = \{\{1, 2, 3\}\}$; $\mathcal{H}_2 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

The DUF 3-uniform hypergraph $\mathcal{H}$ *is below, with four edges coming from each block of size four in the PBD, and one edge coming from each block of size three in the PBD:*

$$\left|\begin{array}{ccccccccccccccccccccc}
1 & 1 & 1 & 2 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 8 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\
2 & 2 & 3 & 3 & 5 & 5 & 6 & 6 & 8 & 8 & 9 & 9 & 5 & 6 & 7 & 5 & 6 & 7 & 5 & 6 & 7 \\
3 & 4 & 4 & 4 & 6 & 7 & 7 & 7 & 9 & 10 & 10 & 10 & 8 & 9 & 10 & 10 & 8 & 9 & 9 & 10 & 8
\end{array}\right|$$

Proof: (Proposition) Suppose for contradiction that the 3-uniform hypergraph $\mathcal{H}$ contains a forbidden union so that $\{u,v,a\},\{u,v,b\},\{x,y,a\},$ $\{x,y,b\}\in\mathcal{H}(\mathcal{B})$ with $\{u,v,a\}\cap\{x,y,b\}=\emptyset=\{u,v,b\}\cap\{x,y,a\}$. If two edges of $\mathcal{H}$ share two vertices, these edges must belong to $\mathcal{H}(B_j)$ for some j since $(\mathcal{X},\cup_j B_j)$ has index one. Thus $\{u,v,a\}$ and $\{u,v,b\}$ belong to, say, $\mathcal{H}(B_r)$; $\{x,y,a\}$ and $\{x,y,b\}$ belong to, say, $\mathcal{H}(B_s)$. But now $\{u,v,a,b\}\subseteq B_r$ and $\{x,y,a,b\}\subseteq B_s$ so that $\{a,b\}\subseteq B_r\cap B_s$, implying that $B_r=B_s$. Now $\{u,v,a\},\{u,v,b\},\{x,y,a\},\{x,y,b\}\in\mathcal{H}(B_r)$. Therefore the hypergraph $\mathcal{H}_r$ is not DUF, contrary to the hypothesis. □

This proposition says that the DUF property is *PBD-closed*, meaning that the DUF structure is preserved by building a larger-order hypergraph from smaller-order hypergraphs and a PBD of index one.

The above result can be generalized by noting that the only structural fact that we used about the PBD was that no pair of points ever appears in distinct blocks:

Corollary 5.4 *Construct a 3-uniform hypergraph* $\mathcal{H}=(\mathcal{X},H)$ *from a hypergraph* $\mathcal{G}=(\mathcal{X},G)$ *in which no pair of vertices appears in more than one edge of* $\mathcal{G}$ *(acting as the PBD in the previous proposition) by breaking up the blocks in* G *with DUF 3-uniform hypergraphs of the appropriate orders (as in the previous proposition) and taking the union of the edges of the resulting hypergraphs to be* H*, the edge set of* $\mathcal{H}$*. Then* $\mathcal{H}$ *is DUF.*

Necessary conditions for the existence of a $PBD(n, K)$ are:

$$n - 1 \equiv 0 \pmod{\alpha(K)} \text{ and } n(n-1) \equiv 0 \pmod{\beta(K)}$$

$$\text{where } \alpha(K) = \gcd\{k - 1 : k \in K\} \text{ and } \beta(K) = \gcd\{k(k-1) : k \in K\}.$$

When $K = \{4, 5\}$, these conditions imply that $n(n-1) \equiv 0 \pmod 4$, so that $n \equiv 0, 1 \pmod 4$. These necessary conditions are asymptotically sufficient and have only three exceptions for $n \geq 4$, namely when $n = 8, 9$, or 12 no $PBD(n, \{4, 5\})$ exists [6]. This existence result, together with the previous result that the DUF property is PBD-closed, will allow us to construct DUF 3-uniform hypergraphs of large size and with arbitrary order.

Proposition 5.5 *Construction A (defined in the Introduction) is DUF.*

Proof: We argue by contradiction. Let $*$ be a fixed point used for the construction. Suppose $\{u, v, a\} \cap \{x, b, y\} = \emptyset = \{u, v, b\} \cap \{x, y, a\}$ is a forbidden union in the edge set. If three or more of these edges are pairwise disjoint, then more than the allowed six vertices are involved in the forbidden union, so at least two edges must be triples through $*$. Since each point of the forbidden union appears in exactly two edges, there are exactly two edges not passing through $*$. These will be disjoint, forcing the edges through $*$ to be disjoint, a contradiction. □

Now consider a $PBD(n, \{4, 5\})$, where the block set is composed of b_4 blocks of cardinality 4 and b_5 blocks of cardinality 5. By counting the occurrences of pairs in the design, we have $6b_4 + 10b_5 = \binom{n}{2}$.

When all blocks are of cardinality 5, $10b_5 = \binom{n}{2}$. Taking all possible triples in each block therefore gives $\binom{n}{2}$ triples forming an $(n, 3, 3)$-design. This can only occur when $n \equiv 1, 5 \pmod{20}$, yielding Construction B.

On the other hand, when all blocks are of cardinality 4, $6b_4 = \binom{n}{2}$ and each block yields only four triples. We therefore end up with $\frac{2}{3}\binom{n}{2}$ triples. This situation occurs whenever $n \equiv 1, 4 \pmod{12}$, yielding Construction C.

It is easy to verify that

$$\binom{n}{2} > \binom{n-1}{2} + \lfloor \tfrac{n-1}{3} \rfloor$$
$$\text{and } \binom{n-1}{2} + \lfloor \tfrac{n-1}{3} \rfloor > \tfrac{2}{3}\binom{n}{2} \text{ for } n \geq 5.$$

These inequalities justify the further study of these $PBD(n, \{4, 5\})$.

Definition 5.6 *The packing number $D(n, k, \lambda)$ is the maximum number of blocks in an (n, k, λ)-packing.*

An upper bound on $D(n, k, \lambda)$ is $D(n, k, \lambda) \leq \lfloor \frac{n}{k} \lfloor \frac{\lambda(n-1)}{k-1} \rfloor \rfloor$; let $B(n, k, \lambda)$ represent this upper bound. This is known as the Schoenheim bound. Note that for admissible parameters (n, k, λ) for a design, $B(n, k, \lambda)$ is the number of blocks in an (n, k, λ)-design [6].

Ling [14] showed that when $n \equiv 2, 6, 10 \pmod{20}$ there exists an *optimal* $(n, 5, 1)$-packing, (a packing having $B(n, 5, 1)$ blocks), with at most 32 possible exceptions. When $n \equiv 14, 18 \pmod{20}$ there exists an optimal $(n, 5, 1)$-packing with a finite (large) number of possible exceptions.

In any case, these $(4m+2, 5, 1)$-packings consist of
$\lfloor\frac{(4m+2)}{5}\lfloor\frac{(4m+1)}{4}\rfloor\rfloor = \lfloor\frac{m(4m+2)}{5}\rfloor$ blocks and so can generate
$2m(4m+2)$ edges when $m \equiv 0, 2 \pmod{5}$, or $\geq 2m(4m+2) - 4$ edges when $m \equiv 1, 3, 4 \pmod{5}$, in a DUF 3-uniform hypergraph of order $4m + 2$.

Let $n = 4m + 2$. Then $2m(4m+2) - 4 = \frac{(n-2)}{2}n - 4$.
When $n \geq 28$,

$$\begin{aligned}
\binom{n-1}{2} + \lfloor\frac{n-1}{3}\rfloor &\leq \frac{(n-2)}{2}\left[n - 1 + \frac{2(n-1)}{3(n-2)}\right] \\
&= \frac{(n-2)}{2}\left[n - 1 + \frac{3(n-2)}{3(n-2)} - \frac{(n-4)}{3(n-2)}\right] \\
&= \frac{n-2}{2}n - \frac{(n-4)}{6} \\
&\leq \frac{n-2}{2}n - 4.
\end{aligned}$$

In this case the DUF 3-uniform hypergraph construction via $(4m+2, 5, 1)$-packings is at least as good as Construction A.

We want to compare this result with those coming from a $PBD(n, \{4,5\})$ construction. Suppose:

$$\begin{aligned} 6b_4 + 10b_5 &= \tbinom{n}{2} \\ 4b_4 + 10b_5 &= \tfrac{(n-2)}{2}n. \end{aligned}$$

Then $b_4 = \frac{1}{2}(\binom{n}{2} - \frac{(n-2)}{2}n) = \frac{1}{4}n$. This means any $PBD(n, \{4,5\})$ with $\frac{n}{4}$ blocks of cardinality 4 will beat the bound above on $f_3(n)$.

To construct an example meeting the bound above we return to the notion of GDD's which were defined previously; the *group type* of a GDD $\mathcal{G}$ is the multiset $\{|G| : G \in \mathcal{G}\}$, and is usually expressed in exponential notation $1^{i_1}2^{i_2}3^{i_3}\cdots$, which denotes the occurrence of i_j groups of cardinality j. A GDD is said to be *uniform* when all groups are of the same cardinality, i.e. when the group type is g^u.

Necessary conditions for the existence of a uniform k-GDD of type $g^u (u > 1)$ are [1]:

$$\begin{gathered} u \geq k; \\ (u-1)g \equiv 0 \pmod{k-1}; \\ u(u-1)g^2 \equiv 0 \pmod{k(k-1)}. \end{gathered}$$

We will consider 5-GDDs of type 4^u. The necessary conditions for existence are: $u \geq 5$, $u \equiv 0, 1 \pmod 5$. These conditions have been shown to be sufficient in this case [1].

Any 5-GDD of type 4^u, $(u \equiv 0, 1 \pmod 5, u \geq 5)$, produces a $PBD(4u, \{4, 5\})$ with exactly u blocks of cardinality 4. The blocks of cardinality 5 contain exactly $\frac{1}{2}4u(4u-4)$ pair occurrences, so that there are $\frac{1}{5}u(4u-4)$ blocks of cardinality 5. We can use this PBD to produce a DUF 3-uniform hypergraph with $4u + 2u(4u-4) = 4u(2u-1)$ edges. Letting $n = 4u$ this gives $n\frac{(n-2)}{2}$ edges as in the case $n \equiv 2, 10 \pmod{20}$. Thus we have an infinite family of DUF 3-uniform hypergraphs which are not $(n, 3, 3)$-designs but which do have more than $\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$ edges.

When $n \equiv 3 \pmod 4$, there are no specific results on $(4m+3, 5, 1)$-packings to give a lower bound on $f_3(n)$. We might consider using the DUF construction for $n-1 \equiv 2 \pmod 4$ on $n-1$ points to give $\frac{(n-3)}{2}(n-1)$ triples and compare this to Constructions A and C (as defined in the Introduction).

Since $\frac{(n-1)(n-3)}{2} = \frac{(n-3)}{n}\binom{n}{2} > \frac{2}{3}\binom{n}{2}$ for $n > 9$, this is better than Construction C. However, $\frac{(n-3)(n-1)}{2} < \binom{n-1}{2} < \binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$, so that Construction A is better than the one coming from the $(n-1, 5, 1)$-packing.

In this case where no general $PBD(4m+3, \{4, 5\})$ structure is known, the lower bound on $f_3(n)$ can best be stated as:

$$\max\{\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor, \alpha(n)\},$$

where $\alpha(n) = \max\{4b_4 + 10b_5 : \exists PBD(n, \{4,5\})$ with b_4 blocks of cardinality 4 and b_5 blocks of cardinality $5\}$.

As is easily seen, $\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor \geq \frac{3n-7}{3n-3}\binom{n}{2}$.
We also have $\frac{(n-2)}{2}n = \frac{n-2}{n-1}\binom{n}{2}$ and $\frac{(n-2)}{2}n - 4 = \left(\frac{n-2}{n-1} - \frac{8}{n(n-1)}\right)\binom{n}{2}$. Since

$$\lim_{n\to\infty} \frac{3n-7}{3n-3} = 1 = \lim_{n\to\infty} \frac{n-2}{n-1} = \lim_{n\to\infty} \left(\frac{n-2}{n-1} - \frac{8}{n(n-1)}\right),$$

both Construction A and the $(4m+2, 5, 1)$-packing construction can produce DUF 3-uniform hypergraphs of sizes arbitrarily close to the conjectured bound of $\binom{n}{2}$, for sufficiently large n.

This is particularly interesting in the former case since Construction A is structurally quite different from the simple DUF $(n, 3, 3)$-designs which we have conjectured to be the unique structures meeting the conjectured bound.

Even better results in the cases $n \equiv 9, 13, 17 \pmod{20}$ give bounds of $f_3(n) \geq \binom{n}{2} - c_i$, where c_i $(i = 9, 13, 17)$ are constants which vary with the congruence class of n.

Consider the following special $PBD(n, K)$ where the block set consists of blocks of size five along with exactly one block of a different size:

Table 5.7 *[6] $PBD(n, \{5, k^*\})$ with exactly one block of size k*

k	$n \equiv x \pmod{20}$	n	exceptions(*)
9	$n \equiv 9, 17 \pmod{20}$	$n \geq 9$	$2d$; $1p$
13	$n \equiv 13 \pmod{20}$	$n \geq 13$	$1d$
17	$n \equiv 9, 17 \pmod{20}$	$n \geq 17$	$4d$; $13p$

(*) d, p indicates the number of definite and possible exceptions respectively

Let $F(n) = \binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$.

We will use

$$F(9) = \binom{9-1}{2} + \lfloor \tfrac{9-1}{3} \rfloor = 28 + 2 = 30;\ \binom{9}{2} = 36$$
$$F(13) = \binom{13-1}{2} + \lfloor \tfrac{13-1}{3} \rfloor = 66 + 4 = 70;\ \binom{13}{2} = 78$$
$$F(17) = \binom{17-1}{2} + \lfloor \tfrac{17-1}{3} \rfloor = 120 + 5 = 125;\ \binom{17}{2} = 136$$

In the case of the $PBD(n, \{5, 9^*\})$,

$$\binom{20m+9}{2} = 10b_5 + \binom{9}{2} \text{ and}$$
$$f_3(20m+9) \geq 10b_5 + F(9) = \binom{20m+9}{2} - 6.$$

This is in contrast to the case of the $PBD(n, \{5, 17^*\})$, which gives

$$\binom{20m+9}{2} = 10b_5 + \binom{17}{2} \text{ and}$$
$$f_3(20m+9) \geq 10b_5 + F(17) = \binom{20m+9}{2} - 11.$$

Similarly from the $PBD(n, \{5, 9^*\})$,

$$\binom{20m+17}{2} = 10b_5 + \binom{9}{2} \text{ and}$$
$$f_3(20m+17) \geq 10b_5 + F(9) = \binom{20m+9}{2} - 6.$$

Using the $PBD(n, \{5, 13^*\})$, we have

$$\binom{20m+13}{2} = 10b_5 + \binom{13}{2} \text{ and}$$
$$f_3(20m+13) \geq 10b_5 + F(13) = \binom{20m+13}{2} - 8.$$

A final point is that for $n \equiv 0, 1 \pmod 4$, $PBD(n, \{4, 5\})$ exist except for $n = 8, 9, 12$. For these values of n, Construction A gives the best lower bound on $f_3(n)$. We have $F(8) = 23, F(9) = 30, F(12) = 58$.

The following table summarizes the lower bounds on $f_3(n)$ by the methods described in this section.

Table 5.8 *Lower bounds on $f_3(n)$ for various n*

n	$f_3(n) \geq$
$\equiv 1, 5 \pmod{20}$	$\binom{n}{2}$
$\equiv 9, 17 \pmod{20}, n \geq 9(*)$	$\binom{n}{2} - 6$
$\equiv 13 \pmod{20}, n \geq 13(**)$	$\binom{n}{2} - 8$
$\equiv 14, 18 \pmod{20}, n \geq 66(***)$	$(\frac{n-2}{n})n - 4 = \binom{n}{2} - \frac{n}{2} - 4$
$\equiv 2, 10 \pmod{20}(****)$	$(\frac{n-2}{2})n = \binom{n}{2} - \frac{n}{2}$
$\equiv 6 \pmod{20}, n \geq 66(****)$	$(\frac{n-2}{n})n - 4 = \binom{n}{2} - \frac{n}{2} - 4$
$\equiv 0, 4 \pmod{20}, n \geq 20$	$(\frac{n-2}{n})n = \binom{n}{2} - \frac{n}{2}$
all other n	$\binom{n-1}{2} + \lfloor \frac{n-1}{3} \rfloor$

(*): except $n = 17, 29$ and possibly 49

(**): except $n = 33$

(***): with a finite (large) number of exceptions, all at most 17578

(****): with at most 32 exceptions ranging between 10 and 1186

Chapter 6

The Minimum Saturation Problem

This chapter presents an asymptotic result on the minimum size of maximally DUF 3-uniform hypergraphs. The dual restrictions of minimum size and maximality combine to establish the upper and lower bounds necessary for the result. Before presenting the main theorem, we first study several maximally DUF structures to provide further evidence in support of previous structural conjectures.

Observation 6.1 *Construction A (described in the Introduction) comes from listing* 3*-sets lexicographically:* $\{1,2,3\},\ldots,\{n-2,n-1,n\}$ *and choosing sets as edges for the hypergraph in this order if and only if they do not cause a forbidden union with the edges previously chosen.*

Clearly, there exists an ordering for the sets $\{1, 2, 3\}, \ldots, \{n-2, n-1, n\}$ for which the greedy algorithm above will produce a simple DUF 3-uniform hypergraph of maximum size. We make the observation above to point out that there is hope that such an ordering has a structure that is easily described, and therefore easily studied.

Proposition 6.2 *Adding an edge to Construction B produces exactly* 27 *forbidden unions.*

Proof: Suppose we add the edge $\{x, y, z\}$ to a simple DUF $(n, 3, 3)$-design. In the underlying $(n, 5, 1)$-design, say the pair $\{x, y\}$ occurs in the block $\{x, y, a, b, c\}$. The pair $\{a, z\}$ also occurs in exactly one block of the underlying $(n, 5, 1)$-design, say $\{a, z, u, v, w\}$. We then have the following three forbidden unions in the new system:

$$\{x, y, z\}, \{x, y, a\}, \{a, u, v\}, \{z, u, v\} ;$$
$$\{x, y, z\}, \{x, y, a\}, \{a, u, w\}, \{z, u, w\} ;$$
$$\{x, y, z\}, \{x, y, a\}, \{a, v, w\}, \{z, v, w\}.$$

We had three choices for which pair to isolate in the beginning ($\{x, y\}$, $\{x, z\}$ or $\{y, z\}$) and also three choices for the second pair to choose (in the case of the pair $\{x, y\}$, we could have paired z with a, b, or c) and finally three choices for the pair which will share two third points with the first pair. Each of these choices results in a distinct forbidden union otherwise a contradiction to the structure of the underlying $(n, 5, 1)$-design arises. Thus the addition of any edge produces 27 forbidden unions in a simple DUF $(n, 3, 3)$-design.□

Proposition 6.3 *Adding an edge to Construction A produces an average of nine forbidden unions as $n \to \infty$, where the average is taken over all edges not present in the hypergraph.*

Proof: Suppose we add an edge to Construction A. Let the points contained in the $\lfloor \frac{n-1}{3} \rfloor$ disjoint edges be denoted P_1 and the points in $\mathcal{X} - P_1$ that are not equal to $*$ be denoted P_2. Denote the set of $\lfloor \frac{n-1}{3} \rfloor$ disjoint edges by $\mathcal{D}$. Recall that $*$ is the point with all possible edges through it.

A new edge may contain up to three points of P_1 and up to two points of P_2. There are five types of edges which can be added:

(i) those with three points in P_1 which intersect exactly two elements of $\mathcal{D}$;

(ii) those with three points in P_1 which intersect exactly three elements of $\mathcal{D}$;

(iii) those with two points in P_1 which intersect exactly two elements of $\mathcal{D}$;

(iv) those with two points in P_1 which intersect exactly one element of $\mathcal{D}$;

(v) those with one point in P_1.

Below is a table listing the configuration of the new edge and the total number of forbidden unions produced by adding this edge. The constructions that give the forbidden union are outlined following the table.

Table 6.4 *Adding an edge to Construction A*

edge type	*new forbidden unions*
(i)	$n-4$
(ii)	3
(iii)	2
(iv)	$n-5$
(v)	1

Consider the new edge, together with each element of $\mathcal{D}$, one at a time. In case (i) we either have a configuration of four points, which forms a forbidden union involving the two other points $\{*, a\}$, where a is any point not equal to $*$ and not in the set of four points, or a configuration of five points which forms a forbidden union with the point $*$.

Each configuration in cases (ii), (iii), (v) produces a single forbidden union (on the five points together with $*$) for each choice of element of $\mathcal{D}$. In case (iv), we again have a forbidden union involving the two other points $\{*, a\}$, where a is any point not equal to $*$ and not in the set of four points.

Summing the number of possible edges of each type times the number of forbidden unions resulting from each type of addition we have:

$$[(n-4)(9)\lfloor \tfrac{n-1}{3} \rfloor \lfloor \tfrac{n-4}{3} \rfloor + (3)(27)\binom{\lfloor \frac{n-1}{3} \rfloor}{3} + (9)\lfloor \tfrac{n-1}{3} \rfloor \lfloor \tfrac{n-4}{3} \rfloor (\alpha)$$

$$+(n-5)(3)\lfloor \tfrac{n-1}{3} \rfloor (\alpha) + (1)(3)\lfloor \tfrac{n-1}{3} \rfloor (\beta)]$$

where $\alpha \equiv n-1 \pmod 3$ and $\beta = 1$ if $n-1 \equiv 2 \pmod 3$ and 0 otherwise. Call this quantity $M(n)$. Dividing by the number of edges missing in the original construction, we find that approximately 9 forbidden unions are added on average as $n \to \infty$:

$$lim_{n\to\infty}\frac{M(n)}{\binom{n-1}{3}-\lfloor\frac{n-1}{3}\rfloor} = \frac{6}{n^3}[n^3+\frac{n^3}{2}] = 9.$$

$\square$

Proposition 6.5 *Construction C is maximally DUF.*

Proof: This hypergraph is DUF by Proposition 5.2. Suppose $\{x, y, z\}$ is not an edge of this hypergraph. We will see that adding this edge produces a forbidden union. The pair $\{x, y\}$ occurs in exactly one block of the underlying $(n, 4, 1)$-design, say $\{x, y, a, b\}$. The pair $\{a, z\}$ also occurs in exactly one block of the underlying $(n, 4, 1)$-design, say $\{a, z, u, v\}$. By the construction of the hypergraph all triples in $\{x, y, a, b\}$ and $\{a, z, u, v\}$ are edges, so in particular $\{x, y, a\}$, $\{a, u, v\}$ and $\{z, u, v\}$ are all edges of the hypergraph. These three edges together with $\{x, y, z\}$ form a forbidden union. The two pairs of blocks are disjoint as required, otherwise a contradiction arises to the structure of the underlying $(n, 4, 1)$-design. $\square$

Proposition 6.6 *Adding a new edge to Construction C produces exactly six forbidden unions.*

Proof: Suppose we add the edge $\{x, y, z\}$ to the hypergraph $\mathcal{H}$ coming from Construction C. Each pair of vertices occurs in exactly two edges of

$\mathcal{H}$: say the pair $\{x, y\}$ occurs with third points a and b. In the underlying $(n, 4, 1)$-design both a and b occur in distinct blocks with z, say $\{a, a_1, a_2, z\}$ and $\{b, b_1, b_2, z\}$ respectively. Thus $\{\{x, y, a\}, \{x, y, z\}, \{a, a_1, z\}, \{a, a_2, z\}\}$ and $\{\{x, y, b\}, \{x, y, z\}, \{b, b_1, z\}, \{b, b_2, z\}\}$ are two forbidden unions in $\mathcal{H} \cup \{x, y, z\}$. We obtain two other forbidden unions for each of the pairs $\{x, z\}$ and $\{y, z\}$. The six forbidden unions are distinct otherwise a contradiction arises to the structure of the underlying $(n, 4, 1)$-design. $\square$

In order to compare various maximally DUF constructions, we have looked at the average number of forbidden unions formed when an edge is added. The simple $(n, 3, 3)$-designs pack in significantly more potentially forbidden union-forming structures per block, giving more evidence to support our earlier structural conjecture that they are the unique extremal 3-uniform hypergraphs with respect to the forbidden union.

Interestingly, we can show that a maximally DUF 3-uniform hypergraph of minimum size is not a design: we will show the existence of a maximally DUF 3-uniform hypergraph of order n with size less than $\frac{2}{3}\binom{n}{2}$. This, together with the fact that $STS(n)$ are not maximally DUF, will be sufficient.

Definition 6.7 *Let the* 3-saturation number *of n, denoted $\phi_3(n)$, be the minimum size of a maximally DUF 3-uniform hypergraph of order n.*

The previous proposition shows that $\phi_3(n) \leq \frac{2}{3}\binom{n}{2}$. We will now examine the asymptotic behaviour of $\frac{\phi_3(n)}{n^2}$.

We begin with a result due to Mantel [2] in 1907:

Lemma 6.8 *The maximum size of a triangle-free graph of order* n *is* $\lfloor \frac{n^2}{4} \rfloor$.

Proposition 6.9 *The* 3*-saturation number* $\phi_3(n)$ *has* $\liminf_{n\to\infty} \frac{\phi_3(n)}{n^2} \geq \frac{1}{12}$.

Proof: Given a simple maximally DUF 3-uniform hypergraph $\mathcal{H}$, we wish to show that its size is at least $\frac{n^2}{12} + O(n^\alpha)$, for $\alpha < 2$. We proceed by showing that any simple DUF hypergraph with size less than $\frac{n^2-2n}{12}$ is not maximal. Let $\mathcal{H}$ be such a hypergraph. Then $|\mathcal{H}| < \frac{n^2-2n}{12} = \frac{n^2-2n}{4}\frac{2}{n(n-1)}\frac{n(n-1)}{6}$. Thus we can write $|\mathcal{H}| < g(n)\frac{n(n-1)}{6}$, where $g(n) = \left(\binom{n}{2} - \frac{n^2}{4}\right)\binom{n}{2}^{-1}$. Since $\frac{n(n-1)}{6}$ is the size of an $STS(n)$, a 3-uniform hypergraph of this size will cover each pair of vertices exactly once, on average. We associate $\mathcal{H}$ with the multigraph $\mathcal{G}$ on the same vertex set by adding λ edges between two vertices in $\mathcal{G}$ if and only if they occur together exactly λ times in edges of $\mathcal{H}$. Thus $\mathcal{G}$ has fewer than $\binom{n}{2} - \frac{n^2}{4}$ edges. Define the complement $\overline{\mathcal{G}}$ of this multigraph as follows: the edge xy occurs in $\overline{\mathcal{G}}$ if and only if the edge xy does not appear in $\mathcal{G}$. Thus the complement $\overline{\mathcal{G}}$ is a graph, and has size strictly greater than $\frac{n^2}{4}$. It follows, by Mantel's result, that $\overline{\mathcal{G}}$ contains a triangle, say xy, yz, xz. The triple $\{x, y, z\}$ is by definition a triple in $\mathcal{H}$ in which no pair of vertices appear together in an edge of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is not maximally DUF because if a forbidden union were to result from the addition of the edge $\{x, y, z\}$, at least one of these pairs would have to be covered in another edge of $\mathcal{H}$. □

Proposition 6.10 *The* 3*-saturation number* $\phi_3(n)$ *has*

$$\limsup_{n\to\infty} \frac{\phi_3(n)}{n^2} \leq \frac{1}{12}.$$

Proof: We first construct a simple DUF 3-uniform hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ of order $n \equiv 2, 4 \pmod 6$ with size $\frac{n^2}{12} + O(n)$ and show that we can add a linear number of edges to make the hypergraph maximally DUF. Let $\mathcal{X} = \{A, B, u_0, \ldots, u_{\frac{n-4}{2}}, v_0, \ldots, v_{\frac{n-4}{2}}\}$. Form two $STS(\frac{n-2}{2})$ on the point sets $\{u_0, \ldots, u_{\frac{n-4}{2}}\}$ and $\{v_0, \ldots, v_{\frac{n-4}{2}}\}$. We then add additional edges $\{A, u_0, v_0\}$, $\{B, u_0, v_0\}$, $\{A, B, u_0\}$, $\{A, B, v_0\}$ and additional edges of the form $\{A, B, u_i\}, \{A, B, v_i\}, \{A, u_0, u_i\}, \{B, v_0, v_i\}$, where $1 \leq i \leq \frac{n-4}{2}$.

To check that this 3-uniform hypergraph is DUF, we first list the pairs that are covered at least twice, together with their images under the theta mapping:

Table 6.11 *Pair coverage in a maximally DUF hypergraph of size* $\frac{n^2}{12} + O(n)$

pair covered at least twice	$\theta(pair)$
$\{A, B\}$	$\{u_0, \ldots, u_{\frac{n-4}{2}}, v_0, \ldots, v_{\frac{n-4}{2}}\}$
$\{A, u_0\}$	$\{u_i, v_0, B\}$
$\{A, u_i\}$	$\{u_0, B\}$
$\{B, v_0\}$	$\{A, u_0, v_i\}$
$\{B, v_i\}$	$\{A, v_0\}$
$\{u_0, u_i\}$	$\{A, \theta_u(\{u_0, u_i\})\}$
$\{v_0, v_i\}$	$\{B, \theta_v(\{v_0, v_i\})\}$
$\{u_0, v_0\}$	$\{A, B\}$
$\{A, v_0\}$	$\{B, u_0\}$
$\{B, u_0\}$	$\{A, v_0\}$

where $\theta_u(\{u_0, u_i\})$ refers to the third point of $\{u_0, u_i\}$ in the $STS(\frac{n-2}{2})$ on points $\{u_0, \ldots, u_{\frac{n-4}{2}}\}$, and similarly for $\theta_v(\{v_0, v_i\})$. As before $1 \leq i \leq \frac{n-4}{2}$. Looking at the theta images, we see that no two disjoint pairs have images that intersect in more than one point. We now consider the edges that can be added to this hypergraph. Adding an edge that is a subset of $\{u_0, \ldots, u_{\frac{n-4}{2}}, v_0, \ldots, v_{\frac{n-4}{2}}\}$ causes a forbidden union as this will cause a pair in one of the STS to be covered twice, and this pair together with $\{A, B\}$ yields disjoint pairs sharing the same two third points. The only option is to add an edge containing exactly one of A or B. By symmetry, we will assume we are adding an edge through A. We consider adding edges of the form $\{A, u_0, v_i\}, \{A, u_i, v_k\}, \{A, u_i, u_j\}, \{A, v_i, v_j\}, \{A, u_i, v_0\}, \{A, v_0, v_i\}$, where $1 \leq i, k \leq \frac{n-4}{2}$ or $1 \leq i < j \leq \frac{n-4}{2}$, as appropriate.

The edge $\{A, u_0, v_i\}$ forms a forbidden union with

$$\{A, u_0, B\}, \{v_i, v_0, \theta_v(\{v_i, v_0\})\}, \{B, v_0, \theta_v(\{v_i, v_0\})\}.$$

The edge $\{A, u_i, v_k\}$ forms a forbidden union with

$$\{A, u_i, u_0\}, \{v_k, v_0, B\}, \{u_0, v_0, B\}.$$

The edge $\{A, u_i, u_j\}$ forms a forbidden union with

$$\{\theta_u(\{u_i, u_j\}), u_i, u_j\}, \{A, u_0, \theta_u(\{\theta_u(\{u_i, u_j\}), u_0\})\},$$

$$\{\theta_u(\{u_i, u_j\}), u_0, \theta_u(\{\theta_u(\{u_i, u_j\}), u_0\})\}$$

unless $\theta_u(\{u_i, u_j\}) = u_0$. Define M as the set of edges that can be added to the hypergraph to preserve the DUF structure. The set M consists of three types of edges. Firstly, $\{\{A, *, **\} : \{u_0, *, **\}$ is in the first STS $\} \subset M$. The edge $\{A, v_i, v_j\}$ forms a forbidden union with

$$\{\theta_v(\{v_i, v_j\}), v_i, v_j\}, \{A, v_0, B\}, \{\theta_v(\{v_i, v_j\}), v_0, B\}$$

unless $\theta_v(\{v_i, v_j\}) = v_0$. Secondly, $\{\{A, *, **\} : \{v_0, *, **\}$ is in the second STS $\} \subset M$. Finally, all edges of the form $\{A, u_i, v_0\}$ or $\{A, v_i, v_0\}$ are contained in M. There are no other edges in M except for the symmetric case of edges through B. The set M of edges that can be added to the hypergraph to preserve the DUF structure is therefore linear in cardinality. Since some subset $\overline{M}$ of M has the property that the new hypergraph $(\mathcal{X}, \mathcal{B} \cup \overline{M})$ is maximally DUF, the result follows in this case.

When $n \equiv 0, 1, 3$ or $5 \pmod 6$, we add at most three additional points so the new order falls into one of the congruence classes 2 or 4 modulo 6. Again a linear number of edge additions suffices to make the new hypergraph maximally DUF. We can possibly add an edge on the new points, an edge through a pair of new points, an edge through one of the new points and at least one of A or B, or an edge through a new point and a pair of points in the set $\{u_0, \ldots, u_{\frac{n-4}{2}}, v_0, \ldots, v_{\frac{n-4}{2}}\}$. Since in the latter case the pairs chosen from the set above and associated with any new point must be pairwise disjoint to ensure that the hypergraph is DUF, the upper bound on total possible edge additions is linear, and the result follows. □

We can summarise the results on this minimal saturation problem as follows:

Theorem 6.12 *The asymptotic behaviour of* $\frac{\phi_3(n)}{n^2}$ *is given by*

$$\lim_{n \to \infty} \frac{\phi_3(n)}{n^2} = \frac{1}{12}.$$

Definition 6.13 *A hypergraph is said to be* linear *if any pair of edges share at most one vertex in common. In other words, a linear hypergraph is a* 1*-packing.*

Corollary 6.14 *The minimum size of a maximal linear* 3*-uniform hypergraph of order* n *is* $\frac{n^2}{12} + O(n)$.

Proof: Suppose $\mathcal{H}$ is a linear 3-uniform hypergraph of order n with fewer than $\frac{n^2-2n}{12}$ edges. As in the proof of Proposition 6.9, we can add an edge without violating linearity. This gives the lower bound. On the other hand, for $n \equiv 2, 4, 6 \pmod{12}$, we can take two disjoint STS of as close to the same order as possible. This gives a construction for the upper bound. For the other congruence classes of n, the minimum size is still $\frac{n^2}{12} + O(n)$ since, as before, if we add a small number of points, only a linear number of edge additions are required to make the resulting hypergraph maximally linear.□

While this corollary may not be a useful result in a design-theoretic setting, where we usually look for the best packings, it is useful in the study of 3-saturation numbers. For many forbidden subgraph problems in 3-uniform hypergraphs the construction for the upper bound will rely on a result such as this one.

The quantity $\mathrm{sat}(n, \mathcal{H})$ has been used by Erdös, Füredi and others [11] to denote the minimum size of a maximally $\mathcal{H}$-free k-uniform hypergraph on n vertices for any forbidden k-uniform hypergraph $\mathcal{H}$. Here we use the notation $\mathrm{sat}_k(n, \mathcal{H})$ to make the cardinality of the hyperedges explicit. In this context, $\phi_3(n) = \mathrm{sat}_3(n, \mathcal{D}_3)$, where $\mathcal{D}_3$ is the forbidden union formed by four hyperedges in a 3-uniform hypergraph. In general, let $\mathcal{D}_k$ represent the forbidden union in a k-uniform hypergraph.

It is a theorem due to Ollmann, as cited in [16], that

$$\mathrm{sat}_2(n, C_4) = \lfloor \tfrac{3n-5}{2} \rfloor \text{ for } n \geq 5$$

where C_4 is the forbidden union $\mathcal{D}_2$. We have established here that $\mathrm{sat}_3(n, \mathcal{D}_3) = \frac{n^2}{12} + O(n)$. For arbitrary k-uniform hypergraphs $\mathcal{H}$, the conjecture [11]

$$\mathrm{sat}_k(n, \mathcal{H}) \leq O(n^{k-1})$$

remains open, even for the case $k = 3$. In light of this, we note that a study of the quantity $\lim_{n \to \infty} \frac{\mathrm{sat}_k(n, \mathcal{D}_k)}{n^{k-1}}$ may prove instructive for $k \geq 4$.

Chapter 7

Further Attacks and Summary

In Chapter Three a structural characterisation was made of DUF 3-uniform hypergraphs with specific packing and covering indices. In Chapter Five lower bounds on the extremal size were improved. These results support both Füredi's Conjecture (Conjecture 1.10) and the structural conjectures made earlier in this thesis. In this chapter, two possible lines of further attack on Füredi's Conjecture are presented and future research directions are highlighted.

Proposition 7.1 *Let* $(\mathcal{X}, \mathcal{B})$ *be a* 3*-uniform hypergraph of order* n *with* $|\mathcal{B}| = \binom{n}{2} + \alpha$, $\alpha \geq 0$. *Then the number of pairs occurring together in the* $\mathcal{A}$*-system of* $\mathcal{B}$ *is at least* 6α *more than the number of pairs occurring together in* $\mathcal{B}$.

Proof: Let x_i be the number of pairs in the $\mathcal{B}$-system that are covered exactly i times, where $0 \leq i \leq \lambda$ (λ represents the maximum pair coverage, which is finite since $|\mathcal{B}|$ is finite). We have:

$$\begin{aligned}
x_0 + \ldots + x_\lambda &= \binom{n}{2} && (7.1)\\
x_1 + \ldots + \lambda x_\lambda &= 3\binom{n}{2} + 3\alpha && (7.2)\\
(7.2) - (7.1) \rightarrow x_2 + \ldots + (\lambda - 1)x_\lambda &\geq 2\binom{n}{2} + 3\alpha && (7.3)\\
(7.1) \rightarrow x_2 + \ldots + x_\lambda &\leq \binom{n}{2} && (7.4)\\
(7.3) - (7.4) \rightarrow x_3 + \ldots + (\lambda - 2)x_\lambda &\geq \binom{n}{2} + 3\alpha && (7.5)\\
(7.5) \rightarrow 3x_3 + \ldots + (3\lambda - 6)x_\lambda &\geq 3\binom{n}{2} + 9\alpha && (7.6)\\
(7.6) \rightarrow 3x_3 + \ldots + \binom{\lambda}{2}x_\lambda &\geq 3\binom{n}{2} + 9\alpha && (7.7)
\end{aligned}$$

since $\binom{i}{2} \geq 3i - 6$ for all $i \geq 3$. This implies that the number of pairs occurring together in the $\mathcal{A}$-system of $\mathcal{B}$ is at least 6α more than the number of pairs occurring together in $\mathcal{B}$. □

Definition 7.2 *Let $\mathcal{B}_2$ be the multiset of pairs occurring together in the $\mathcal{B}$-system. Let $\overline{\mathcal{B}_2}$ be the multiset of pairs occurring together in a maximum 3-packing of the $\mathcal{B}$-system. Define $\mathcal{A}_2$ and $\overline{\mathcal{A}_2}$ similarly with respect to the $\mathcal{A}$-system coming from $\mathcal{B}$.*

Proposition 7.3 *Let $\mathcal{B}$ be an $(n, 3, \lambda)$-packing for some $\lambda \geq 3$, and let $|\mathcal{B}| = \binom{n}{2} + \alpha, (\alpha \geq 0)$. Then $|\overline{\mathcal{A}_2}| \leq \frac{\lambda}{3}|\overline{\mathcal{B}_2}|$.*

Proof: Suppose $|\overline{\mathcal{A}_2}| > \frac{\lambda}{3}|\overline{\mathcal{B}_2}|$. Since $|\overline{\mathcal{A}_2}| \leq 3\binom{n}{2}$ by definition, $3\binom{n}{2} > \frac{\lambda}{3}|\overline{\mathcal{B}_2}|$, implying that $|\overline{\mathcal{B}_2}| < \frac{9}{\lambda}\binom{n}{2}$. We have $|\mathcal{B}_2| \leq \frac{\lambda}{3}|\overline{\mathcal{B}_2}|$ so that $\frac{3}{\lambda}|\mathcal{B}_2| < \frac{9}{\lambda}\binom{n}{2}$, contradicting $|\mathcal{B}_2| \geq 3\binom{n}{2}$. □

It may be possible to use this result, or a similar one, together with the proposition above to get an upper bound on α. This may prove difficult, however, as the results here derive from counting arguments and give little structural information.

Another way of studying the problem of DUF 3-uniform hypergraphs $(\mathcal{X}, \mathcal{B})$ is to use a matrix formulation of the DUF property. Index the rows and columns of a matrix B with $\binom{\mathcal{X}}{2}$ and define $B_{(xy,ab)}$ to be 1 if $\{x, y, a\}$ and $\{x, y, b\}$ are in $\mathcal{B}$ and 0 otherwise. If a column ab has two 1's in rows corresponding to disjoint pairs (say $\{u, v\}$ and $\{x, y\}$) then a forbidden union is present in $\mathcal{B}$ (namely $\{u, v, a\}$, $\{u, v, b\}$, $\{x, y, a\}$, $\{x, y, b\}$). If the dot product of any two rows corresponding to disjoint pairs is greater than 0 then a forbidden union is present in the system. The 0's in $B^T B$ may tell us something about the number of 0's in B but further investigation using techniques from linear algebra is required to answer this question.

It is natural to ask the following: If a 3-uniform hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{B})$ is maximally DUF must every pair be covered in its $\mathcal{A}$-system? This is still an open question. We constructed a maximally DUF hypergraph of size less than $\frac{2}{3}\binom{n}{2}$, the size of an $(n, 3, 2)$-design, in the proof of Proposition 6.10. All pairs are covered in the $\mathcal{A}$-system of this hypergraph despite its small size due to a few pairs being covered a large number of times in the original hypergraph.

We close with a list of additional open problems, ranging from specific structural questions to generalized versions of the problem studied here:

- Show whether a 3-uniform hypergraph that is a maximally DUF 2-packing must necessarily be an $(n, 3, 2)$-design.
- Give the spectrum of size values for maximally DUF 3-uniform hypergraphs.
- The original disjoint union problem for r-uniform hypergraphs with $r > 3$ is still wide open. No r-uniform hypergraphs of order n and of conjectured extremal size $\binom{n}{r-1}$ are known to have been constructed.
- An even bigger challenge lies ahead: in 1994 Erdős [5] asked for $ex_k(n; \mathcal{M})$ where the forbidden k-uniform hypergraph $\mathcal{M}$ is a collection of m disjoint pairs having the same union, with $k \geq 3, m > 2$.

Chapter 8

Appendix

The $(9, 3, 2)$-designs

The nonisomorphic simple $(9, 3, 2)$-designs [6] with a forbidden union exhibited:

000000001111112222233333
112445672445674455644556
233567883567887868778687
(01) 012, 013, 247, 347

000000001111112222233333
112445672445674455644556
233567883568787867878687
(02) 045, 057, 146, 167

000000001111112222233334
112445672345673455644565
233567884576886878778876
(03) 057, 078, 156, 168

000000001111112222233334
112445672345673455644565
233567884567888768778678
(04) 012, 013, 247, 347

000000001111112222233334
112445672345673455644565
233567884576888678778678
(05) 012, 013, 267, 367

000000001111112222233334
112445672345673455644565
233567884568787868778687
(06) 045, 057, 146, 167

000000001111112222233334
112445672345673455644565
233567884586787768868877
(07) 012, 013, 258, 358

000000001111112222233334
112445672345673455644565
233567884568788767878678
(08) 012, 013, 247, 347

000000001111112222233334
112445672345673455644565
233567884586788768767788
(09) 012, 013, 247, 347

000000001111112222233344
112355672344673345645655
234467885568787868776878
(10) 237, 238, 457, 458

000000001111112222233344
112355672344673345645556
234467885658786778887867
(11) 012, 013, 258, 358

000000001111112222233344
112345672345673345645655
234567885468787876886778
(12) 237, 238, 457, 458

000000001111112222233344
112345672345673345645655
234567885468787886776878
(13) 012, 013, 256, 356

A simple DUF $TTS(18)$

We list the pairs $\{0, x\}, \theta(\{0, x\})$ and the associated θ-differences.

Table 8.1 *The θ-differences associated with $\theta(\{0, x\})$*

$\{0, x\}$	$\theta(\{0, x\})$	$\theta - difference$
$\{0, \infty\}$	$\{1, 16\}$	2
$\{0, 1\}$	$\{\infty, 4\}$	∞
$\{0, 2\}$	$\{8, 12\}$	4
$\{0, 3\}$	$\{8, 16\}$	8
$\{0, 4\}$	$\{1, 11\}$	7
$\{0, 5\}$	$\{14, 7\}$	7
$\{0, 6\}$	$\{15, 10\}$	5
$\{0, 7\}$	$\{5, 13\}$	8
$\{0, 8\}$	$\{2, 3\}$	1
$\{0, 9\}$	$\{11, 12\}$	1
$\{0, 10\}$	$\{15, 6\}$	8
$\{0, 11\}$	$\{9, 4\}$	5
$\{0, 12\}$	$\{9, 2\}$	7
$\{0, 13\}$	$\{14, 7\}$	7
$\{0, 14\}$	$\{5, 13\}$	8
$\{0, 15\}$	$\{6, 10\}$	4
$\{0, 16\}$	$\{\infty, 3\}$	∞

It is necessary to check the two cases where θ-differences are repeated, that is 7 and 8. Recall that forbidden unions will result if two pairs $\theta(\{0, x\})$ and $\theta(\{0, y\})$ have the same θ-difference and $\{\alpha, x+\alpha\} \cap \{0, y\} = \emptyset$ where $\theta(\{0, x\}) + \alpha = \theta(\{0, y\})$.

Observe that if $\theta(\{0, x\}) + \alpha = \theta(\{0, y\})$ then $\theta(\{0, x\}) + (\alpha - y) = \theta(\{-y, 0\})$ since $\theta(\{-y, 0\}) + y = \theta(\{0, y\})$. Now $\{\alpha, x + \alpha\} \cap \{0, y\} = y + (\{\alpha - y, x + \alpha - y\} \cap \{-y, 0\}$ so we need only check two sets of pairs for each θ-difference.

For θ-difference 7:

$$\theta(\{0,5\}) + 4 = \theta(\{0,4\})\text{but}\{4,9\} \cap \{0,4\} \neq \emptyset$$
$$\theta(\{0,12\}) + 5 = \theta(\{0,13\})\text{but}\{0,5\} \cap \{0,13\} \neq \emptyset$$

For θ-difference 8:

$$\theta(\{0,7\}) + 3 = \theta(\{0,3\})\text{but}\{3,10\} \cap \{0,3\} \neq \emptyset$$
$$\theta(\{0,10\}) + 7 = \theta(\{0,14\})\text{but}\{7,0\} \cap \{0,14\} \neq \emptyset$$

Finally, no pair other than $\{x, \infty\}$ has a θ-difference of 2, so we know that no pair of the form $\{x, \infty\}$ can be covered twice to form a forbidden union. We have thus exhibited a simple DUF $TTS(18)$.

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