EIGENVALUES AND SIGNED DIGRAPHS

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When $A = [a_{ij}]$ is a real matrix of order n, let $Q(A)$ denote the set of all matrices with the same sign pattern as A. The aim is to characterize $Q(A)$ which either require or allow a particular eigenvalue property. This is done by using graphs on the node set $\{1,\ldots,n\}$. The digraph $D(A)$ has edge set $\{(j,i)\colon a_{ij} \neq 0\}$, the signed digraph $SD(A)$ has the same edge set as $D(A)$ with edge (j,i) signed as $\operatorname{sgn} a_{ij}$, and the (undirected) graph $G(A)$ has edge set $\{\{i,j\}\colon i \neq j,\ a_{ij} \neq 0 \neq a_{ji}\}$. Note that, for a given matrix A, each matrix in $Q(A)$ has $SD(A)$ as its signed digraph.

If each eigenvalue of A has nonpositive <negative> real part, then A is <u>semistable</u> <<u>stable</u>>. If A is semistable and each eigenvalue with zero real part is a simple root of the minimum polynomial of A, then A is <u>quasistable</u>. A is <u>sign-semistable</u> <<u>sign-quasistable</u>, <u>sign-stable</u>> if each member of $Q(A)$ is semistable <quasistable, stable>. These matrix properties are also important in the stability analysis of dynamical systems associated with $\dot{x} = Ax$.

Sign-semistable matrices were originally characterized in 1965 by Quirk and Ruppert [3].

THEOREM 1. ([3]) A is sign-semistable iff it satisfies the following three cycle conditions:

(α) each 1-cycle in SD(A) is negative;

(β) each 2-cycle in SD(A) is negative;

(γ) D(A) has no p-cycle for $p \geq 3$. ∎

Jeffries et al. in 1977 established in detail and in 1987 [1] reformulated the characterization of sign-stability in terms of the three cycle conditions and two colouring conditions on G(A). In the presence of (γ), G(A) is a forest, and node i is called <u>distinguished</u> if $a_{ii} \neq 0$. For G(A) a forest with a set of distinguished nodes, a <u>$\delta\langle\epsilon\rangle$-colouring</u> is a scheme for colouring all nodes black or white, such that each distinguished node is black, no black has exactly one white neighbour, each <no> white has a white neighbour. The <u>$\delta\langle\epsilon\rangle$-rim</u> is the set of all nodes that are white in at least one $\delta\langle\epsilon\rangle$-colouring. A nonempty $\delta\langle\epsilon\rangle$-rim implies the existence of a (nonzero) purely imaginary <zero> eigenvalue for some <all> $\tilde{A} \in Q(A)$; and this leads to the following result.

THEOREM 2. ([1]) A is sign-stable iff it is sign-semistable and both the δ-rim and ϵ-rim of G(A) are empty. ∎

When A is irreducible, sign-quasistability is equivalent to sign-semistability; but, in the general case, sign-quasistability depends in a complex way on paths between rim nodes of D(A).

THEOREM 3. ([1]) Assume A is sign-semistable. Then A is sign-quasistable if there is no path in D(A) from a node j to a node i such that i and j are in different components of D(A) and are both in the δ-rim or both in the ϵ-rim. ∎

This gives a sufficient condition for sign-quasistability; a complete characterization of $Q(A)$ requiring quasistability is more intricate and involves the ideas of δ- and ϵ- driving using linear theory of differential equations. The characterization of δ-drivers, identification of ϵ-drivers and algorithmic recognition of sign-quasistability is given in [1]. When matrix A of order n is presented by means of adjacency lists and sign lists for $SD(A)$, it can be tested for sign-semistability, sign-quasistability and sign-stability in time $O(n + \text{number of nonzero entries in } A)$.

In the above problems, eigenvalues on the imaginary axis play a crucial role. So it is of interest to characterize $Q(A)$ which allow a zero or a (nonzero) purely imaginary eigenvalue. In all that follows, A is assumed to be an irreducible matrix with a tree graph. With $G(A)$ a tree, $SD(A)$ is called <u>λ-consistent</u> if there exist nonzero constants $\{\lambda_1,\ldots,\lambda_n\}$ such that $\lambda_i a_{ij} = -\lambda_j a_{ji}$ for all $i \neq j$, $\lambda_i a_{ii} \geq 0$ for all i, and $\lambda_i a_{ii} > 0$ for some i. For example, if A is a sign-stable matrix, then $SD(A)$ is λ-consistent (with all $\lambda_i < 0$). This is used to introduce two new colouring conditions which generalize δ- and ϵ-colourings above. For $G(A)$ a tree, an <u>Im<0>-colouring</u> is a scheme for colouring all nodes of $SD(A)$ black or white, such that:

(i) no black has exactly one white neighbour;

(ii) each maximal white block as a subgraph is not λ-consistent and contains at least one negative 2-cycle <and is either a single undistinguished node or a digraph on at least 2 nodes with each end node distinguished>.

These conditions lead to the following result.

THEOREM 4. ([2]) Suppose that A is an irreducible matrix of order ≥ 2, and $G(A)$ is a tree. Then there exists a sinusoidal <constant> trajectory satisfying $\dot{x} = Ax$, $x \not\equiv 0$, for some $\tilde{A} \in Q(A)$ iff $SD(A)$ admits an Im<0>-colouring with at least one white node. ■

Multiple eigenvalues on the imaginary axis can be considered qualitatively by using branching in the (undirected) block graphs corresponding to Im- and 0-colourings. These results and application to sign-controllability, as well as proofs of the statements in Theorem 4, are given in [2].

REFERENCES

[1] C. JEFFRIES, V. KLEE and P. VAN DEN DRIESSCHE, Qualitative stability of linear systems, Lin. Alg. Appl. (1987).

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