Lyapunov Exponents of Random Matrix Products

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Linear Algebra

The *real numbers*, $\mathbb{R}$, consists of every number that can be represented on the number line. An $m \times n$ matrix with real entries is an array of real numbers with m rows and n columns.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

We will only consider matrices with real entries. These matrices have their own algebraic operations such as addition, matrix multiplication, and multiplication by an element of $\mathbb{R}$.

A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that, for every $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $T(x+y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$. Each $m \times n$ matrix with real entries corresponds to a unique linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

An $n \times n$ matrix A is *orthogonal* if its associated linear transformation is an isometry of $\mathbb{R}^n$. That is, given any vector x in $\mathbb{R}^n$, the vector Ax has the same length as x.

Singular Value Theorem for Real Matrices

Let A be an $d \times d$ matrix with real entries. Then there is a factorization

$$A = U \begin{pmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_d \end{pmatrix} V$$

where U, V are orthogonal matrices and $s_1 \geq s_2 \geq \cdots \geq s_d \geq 0$. Moreover, the numbers $s_1, s_2, \dots, s_d \in \mathbb{R}$ are the same for any such decomposition of A. The number $s_j =: s_j(A)$ is called the *j-th singular value* of A.

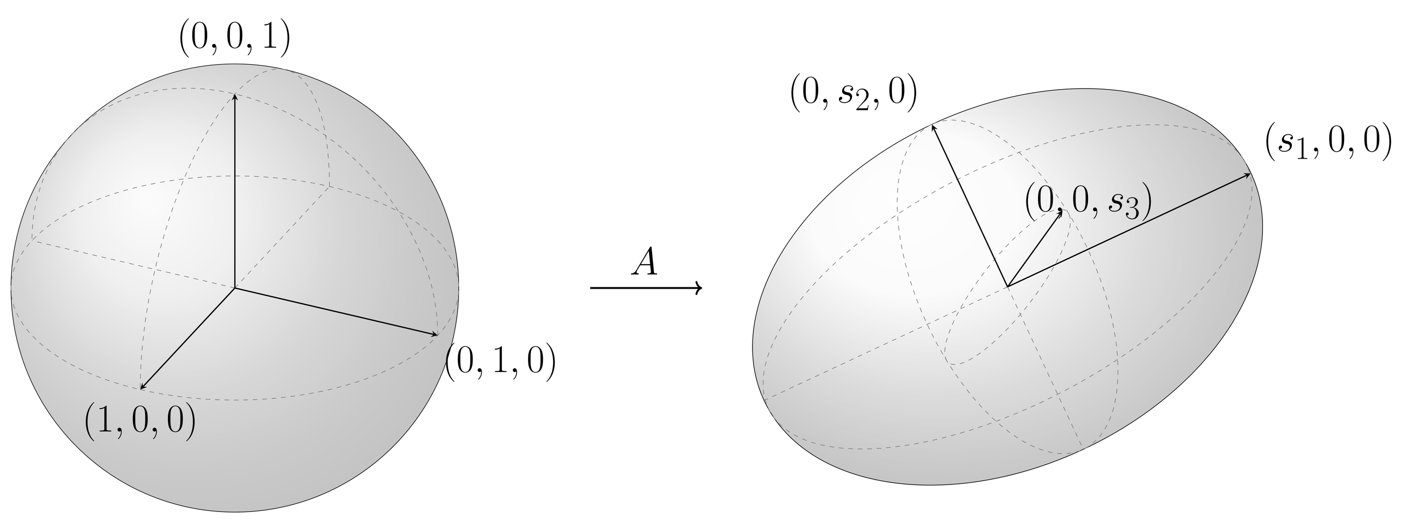

Figure 1: Suppose A is a 3×3 real-valued matrix, and apply the linear transformation A to the unit ball in $\mathbb{R}^3$. The length of the longest vector Ax for x in the unit ball corresponds to the first singular value s_1 of A. The value of s_2 is the longest length of a vector Ay for y in the unit ball which is perpendicular to x. The smallest singular value, s_3, is also equal to the smallest value of Az for z in the ball.

Lyapunov Exponents

In the study of dynamical systems, Lyapunov exponents measure how small changes in initial conditions change the behaviour of a system over time. Positive Lyapunov exponents correspond to the system being sensitive to small initial changes. Chaos, when it occurs, corresponds to a positive leading Lyapunov exponent. When the Lyapunov exponents are all negative, the system is stable.

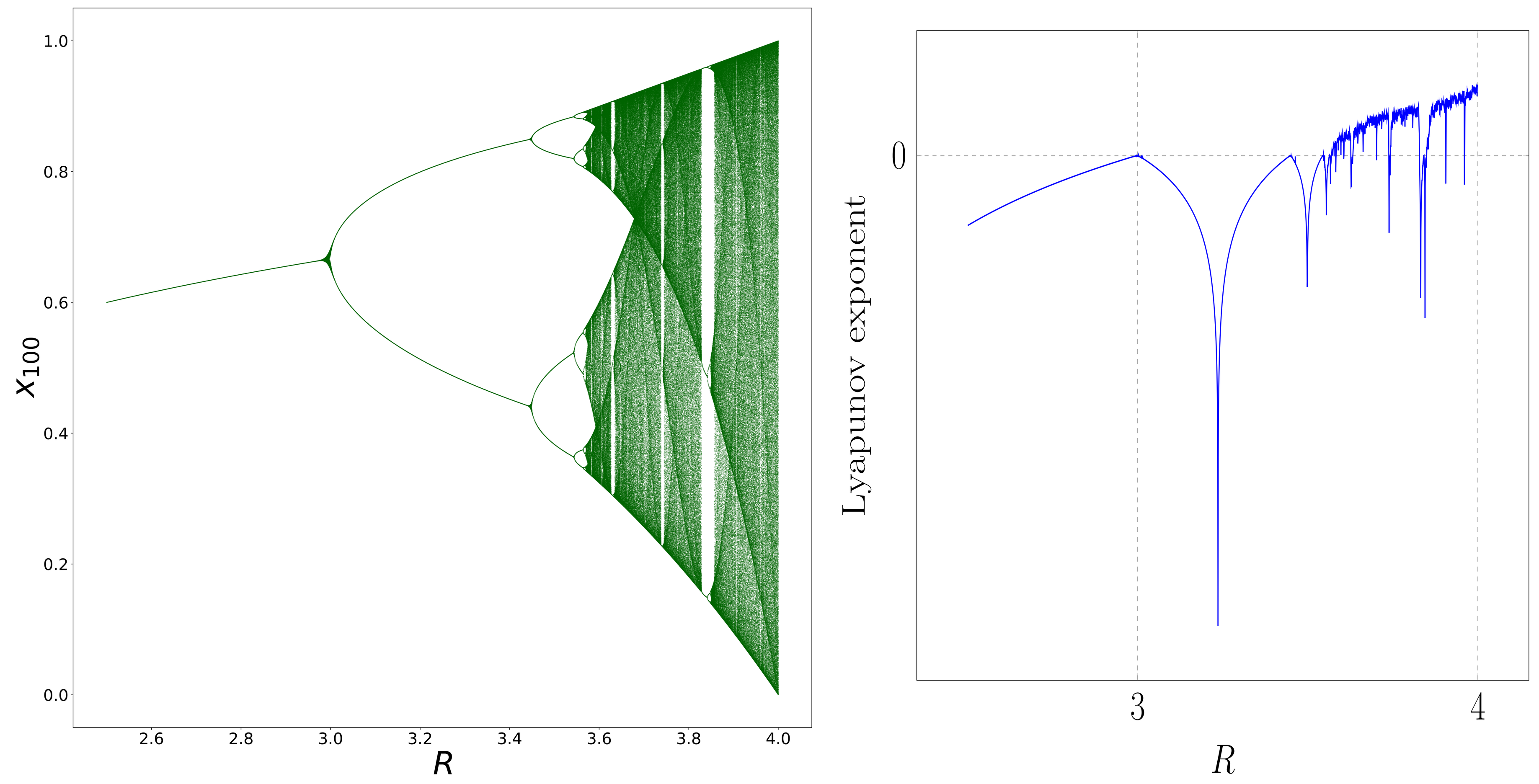

Figure 2: The logistic map is an example of a one-dimensional dynamical system with a single Lyapunov exponent. The map is given by the recurrence $x_{n+1} = R \cdot x_n(1 - x_n)$ where $0 < x_0 < 1$. On the left is plotted the value of x_{100} for random values of $x_0 \in (0,1)$ and fixed R. On the right is the Lyapunov exponent for each corresponding value of R. When $R < 3$ the value of x_n will eventually approach some limiting value, regardless of the choice of $x_0 \in (0,1)$. For some values of $R > 3$, small changes in initial conditions lead to wildly different outcomes. For these values of R, the Lyapunov exponent is positive.

If we are given a sequence of matrices $A_1, A_2, \dots$, this can be thought of as a discrete dynamical system. Given some initial vector, we first apply the transformation A_1, followed by A_2, and so on. We define the Lyapunov exponent of a sequence of $d \times d$ matrices $A_1, A_2, \dots$ to be the limiting values $\lambda_j := \limsup_{n\to\infty} \frac{1}{n} s_j(A_n \cdot A_{n-1} \cdots A_1)$ for $1 \leq j \leq d$. If this quantity exists, then for large values of n we essentially have that

$$s_j(A_n A_{n-1} \cdots A_1) \approx e^{\lambda n}$$

We are interested in these quantities λ_j when $A_1, A_2, \dots$ are matrices with random entries. In general it is not easy to find explicit expressions for these quantities. A theorem due to Goldsheid and Margulis [1] shows that they exist for many distributions of matrices, with λ_1 positive and $\lambda_1 > \lambda_2 > \cdots > \lambda_d$. We find the values of λ for some special distributions of A_i where this theorem applies.

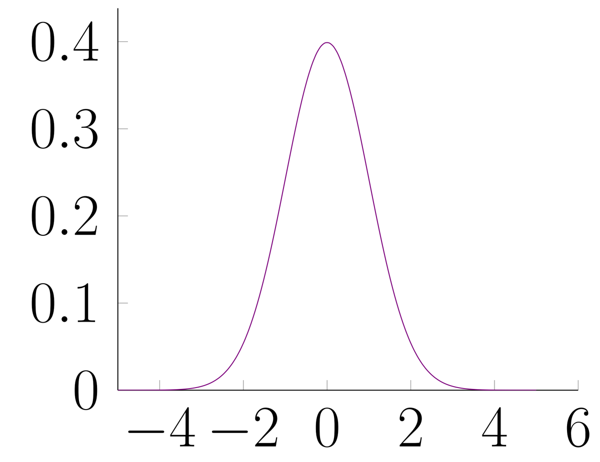

Figure 3: The standard normal distribution.

Some Results

For every $i \geq 1$, let N_i be a $d \times d$ matrix whose entries are independently distributed according to a standard normal distribution. Let U be either an orthogonal matrix or the zero matrix and $\epsilon > 0$. Then an expression for the j-th Lyapunov exponent of the sequence $U + \epsilon N_1, U + \epsilon N_2, \dots$ is given by

$$\lambda_j = \mathbb{E} \log \left\| c_j^{\perp} \right\|$$

where $c_j^{\perp}$ is the part of the j-th column of $U + N_1$ which is perpendicular to the first $j-1$ columns.

This gives us an expression for λ_j, but is not practical to use since it is defined recursively. For U orthogonal, we found a simpler approximation for λ_j which is close to λ_j for small values of ϵ. Using this approximation, we get that

$$\lambda_j = \frac{1}{2}(d - 2k)\epsilon^2 + C\epsilon^{2.9}$$

for some $C \in \mathbb{R}$ bounded by a constant which depends only on d.

Further Research

A motivation for this research was to answer the following question. Given a sequence $A_1, A_2, \dots$ of matrices with norm 1, how large are the gaps between Lyapunov exponents of the sequence $A_i + \epsilon N_i$ where we perturb each matrix A_i by a matrix with independent standard normal entries? We hypothesize that a lower bound is given by the difference between Lyapunov exponents of the sequence $I_d + \epsilon N_i$, where I_d is the $d \times d$ identity matrix.

References

[1] I Ya Gol'dsheid and G A Margulis, *Lyapunov indices of a product of random matrices*, Russian Mathematical Surveys **44** (1989), no. 5, 11.

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