CERTAIN OPERATIONAL TECHNIQUES AND THEIR APPLICATIONS IN ANALYTIC AND UNIVALENT FUNCTION THEORY

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ABSTRACT

Various families of linear operators are becoming increasingly useful in *Geometric Function Theory* which is the study of the relationship between the analytic properties of a given function and the geometric properties of its image domain. Furthermore, an immensely useful class of special functions (namely, the generalized hypergeometric function) played a rather crucial rôle in Louis de Branges' proof of the celebrated Bieberbach, Robertson, and Milin conjectures in the theory of analytic and univalent functions. These latter developments in an area other than the so-called traditional areas of applications of generalized hypergeometric functions have naturally provided a new impetus for the study of such an important class of special functions. With these points in view, we first illustrate the usefulness (in the study of univalent, starlike, and convex generalized hypergeometric functions) of certain families of linear operators which are defined in terms of (for example) fractional derivatives and fractional integrals, Hadamard product or convolution, and so on. We also present a systematic discussion of some properties and theorems involving starlike functions and various families of integral operators considered here. Finally, we consider several inclusion theorems associated with the Hardy space of analytic functions, which hold true for various classes of generalized hypergeometric functions whose derivative has a positive real part. Relevent connections with recent works and developments on the subject are indicated throughout this presentation.

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1. Introduction, Definitions, and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are *univalent* in $\mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{S}$ consisting of all functions which are, respectively, *starlike* and *convex of order* α in $\mathcal{U}$ $(0 \leq \alpha < 1)$, that is,

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leq \alpha < 1; \quad z \in \mathcal{U}) \right\} \tag{1.2}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leq \alpha < 1; \quad z \in \mathcal{U}) \right\}. \tag{1.3}$$

It follows readily from the definitions (1.2) and (1.3) that

$$f(z) \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leq \alpha < 1), \tag{1.4}$$

whose special case, when $\alpha = 0$, is the familiar Alexander theorem (*cf.*, *e.g.*, Duren [11, p. 43, Theorem 2.12]). We note also that

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S} \qquad (0 \leq \alpha < 1), \tag{1.5}$$

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) =: \mathcal{S}^* \qquad (0 \leq \alpha < 1), \tag{1.6}$$

and

$$\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) =: \mathcal{K} \qquad (0 \leq \alpha < 1), \tag{1.7}$$

where $\mathcal{S}^*$ denotes the class of all functions in $\mathcal{A}$ which are starlike (with respect to the origin) in $\mathcal{U}$ and $\mathcal{K}$ denotes the class of all functions in $\mathcal{A}$ which are convex in $\mathcal{U}$ (that is, a function which maps $\mathcal{U}$ conformally onto a *convex* domain).

In statements like those involved in the definitions (1.2) and (1.3), and in analogous situations *throughout this paper,* it should be understood that functions such as

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad \frac{zf''(z)}{f'(z)},$$

which have *removable singularities* at $z = 0$, have had these singularities tacitly removed.

For the functions $f_j(z)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \qquad (j = 1, 2), \tag{1.8}$$

we denote by $(f_1 * f_2)(z)$ the *Hadamard product* or *convolution* of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1}\, z^{n+1}. \tag{1.9}$$

Thus, following the work of Ruscheweyh [58], a function $f(z) \in \mathcal{A}$ is said to be *prestarlike of order* α $(\alpha \leq 1)$ if and only if

$$\begin{cases} \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) & (\alpha < 1) \\ \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2} & (\alpha = 1; \quad z \in \mathcal{U}), \end{cases} \tag{1.10}$$

and we denote by $\mathcal{F}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all prestarlike functions of order α in $\mathcal{U}$.

Next, with a view to introducing another interesting family of analytic functions, we recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathcal{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $h(z)$, analytic in $\mathcal{U}$ with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1, \tag{1.11}$$

such that

$$f(z) = g(h(z)) \qquad (z \in \mathcal{U}). \tag{1.12}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.13}$$

In particular, if $g(z)$ is univalent in $\mathcal{U}$, the subordination (1.13) is *equivalent* to (*cf.* Goodman [16, p. 85])

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \tag{1.14}$$

The concept of subordination between analytic functions can be traced back to Lindelöf [33], although Littlewood ([34], [35]) and Rogosinski ([56], [57]) introduced the term and established the basic results involving subordination. Making use of this concept, we have

Definition 1 (*cf.* Janowski [18]). For $-1 \leq B < A \leq 1$, a function $p(z)$, analytic in $\mathcal{U}$ with $p(0) = 1$, is said to belong to the class $\mathcal{P}(A, B)$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz} \qquad (-1 \leq B < A \leq 1). \tag{1.15}$$

Definition 2 (*cf.* Janowski [18]). A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}(A, B) \qquad (-1 \leq B < A \leq 1). \tag{1.16}$$

We note from Definition 2 that

$$\mathcal{S}^*(1, -1) \equiv \mathcal{S}^*. \tag{1.17}$$

More generally, we recall

Definition 3 (*cf.* Noor [44]). A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{B}(A, B; \alpha)$ if and only if

$$\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) \in \mathcal{P}(A, B) \qquad (\alpha \geq 0; \quad -1 \leq B < A \leq 1). \tag{1.18}$$

Clearly, we have the following relationships:

$$\mathcal{B}(A, B; 0) = \mathcal{S}^*(A, B) \qquad (-1 \leq B < A \leq 1) \tag{1.19}$$

and

$$\mathcal{B}(1, -1; \alpha) = \mathcal{B}_1(\alpha) \qquad (\alpha \geq 0), \tag{1.20}$$

where $\mathcal{S}^*(A, B)$ is given by Definition 2, and $\mathcal{B}_1(\alpha)$ is a subclass of Bazilevič functions, which was introduced and studied by Singh [62].

Yet another subclass of analytic functions is given by

Definition 4. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\gamma)$ if it satisfies the inequality:

$$\Re\{f'(z)\} > \gamma \qquad (0 \leq \gamma < 1; \quad z \in \mathcal{U}). \tag{1.21}$$

Evidently, we have

$$\mathcal{R}(\gamma) \subseteq \mathcal{R}(0) =: \mathcal{R} \qquad (0 \leq \gamma < 1). \tag{1.22}$$

The class $\mathcal{R}$ was studied rather systematically by MacGregor [37] who did indeed refer to numerous earlier works (by, for example, Alexander [2], Wolff [80], Noshiro [45], Warschawski [77], Tims [75], and Herzog and Piranian [17]) investigating various properties of functions whose derivative has a positive real part. In fact, a more general class of functions than those satisfying the inequality:

$$\Re\{f'(z)\} > 0 \qquad (z \in \mathcal{U}) \tag{1.23}$$

is the class of *close-to-convex* functions considered by Kaplan [21]. (See also Duren [11].) More recently, several interesting subclasses of $\mathcal{A}$ associated with the class $\mathcal{R}(\gamma)$ were considered elsewhere by (among others) Sarangi and Uralegaddi [61], Owa and Uralegaddi [53], and Srivastava and Owa [70].

Finally, let $\mathcal{H}^p$ $(0 < p \leq \infty)$ denote the *Hardy space* of analytic functions $f(z)$ in $\mathcal{U}$, and define the *integral means* $M_p(r, f)$ by

$$M_p(r, f) := \begin{cases} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p dp\right)^{1/p} & (0 < p < \infty) \\ \sup\limits_{0 \leq \theta < 2\pi} \left|f\left(re^{i\theta}\right)\right| & (p = \infty). \end{cases} \tag{1.24}$$

Definition 5. A function $f(z)$, analytic in $\mathcal{U}$, is said to belong to the Hardy space $\mathcal{H}^p (0 < p \leq \infty)$ if

$$\lim_{r \to 1-} \{M_p(r, f)\} < \infty \qquad (0 < p \leq \infty). \tag{1.25}$$

For $1 \leq p \leq \infty, \mathcal{H}^p$ is a *Banach space* with the norm $\|f\|_p$ defined by (*cf.*, *e.g.*, Duren [10, p. 23])

$$\|f\|_p := \lim_{r \to 1-} \{M_p(r, f)\} \qquad (1 \leq p \leq \infty). \tag{1.26}$$

Furthermore, $\mathcal{H}^\infty$ is the familiar class of *bounded* analytic functions in $\mathcal{U}$, whereas $\mathcal{H}^2$ is the class of power series $\sum a_n z^n$ with

$$\sum |a_n|^2 < \infty. \tag{1.27}$$

2. The Class $\mathcal{S}$ and Its Encounter with a Family of Generalized Hypergeometric Functions

In *Geometric Function Theory*, which indeed is (as we remarked at the outset) the study of the relationship between the analytic properties of a given function $f(z)$ and the geometric properties of its image domain

$$\mathcal{D} = f(\mathcal{U}), \tag{2.1}$$

it is an extremely difficult open problem to find a (useful) set of conditions on the Taylor coefficients a_n $(n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\})$ that are both *necessary* and *sufficient* for the function $f(z)$ to be in the class $\mathcal{S}$. One of the several partial results in connection with this problem is provided by de Branges' theorem (*cf.*, *e.g.*, [9]) which asserts the truth of the **Milin conjecture** of 1971:

$$f(z) \in \mathcal{S} \quad \text{and} \quad \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n\, z^n \tag{2.2}$$

$$\Rightarrow \sum_{k=1}^{n} (n-k+1)\left(k\,|\gamma_k|^2 - \frac{1}{k}\right) \leqq 0 \qquad (n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}).$$

In fact, in view of the second *Lebedev-Milin inequality* (*cf.* [11, p. 143]), it is not difficult to show that (2.2) implies the **Robertson conjecture** of 1936:
$f(z)$ is odd and in $\mathcal{S}$

$$\Rightarrow \sum_{k=1}^{n} |a_{2k-1}|^2 \leqq n \qquad (n \in \mathbb{N} \setminus \{1\};\ a_1 \equiv 1), \tag{2.3}$$

which, in turn, implies the celebrated **Bieberbach conjecture** of 1916:

$$f(z) \in \mathcal{S} \Rightarrow |a_n| \leqq n \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{2.4}$$

where the equality holds true for all integers $n \geqq 2$ only if

$$\begin{aligned} f(z) = K_\phi(z) &:= \frac{z}{(1 - z e^{i\phi})^2} \\ &= \sum_{n=1}^{\infty} n\, e^{i(n-1)\phi}\, z^n \qquad (\phi \in \mathbb{R}), \end{aligned} \tag{2.5}$$

$K_\phi(z)$ being a rotation of the Koebe function:

$$K(z) := K_0(z) = \sum_{n=1}^{\infty} n\, z^n = \frac{z}{(1-z)^2}. \tag{2.6}$$

The key ingredients in de Branges' proof of the Milin conjecture (2.2), and hence also of the Robertson conjecture (2.3) and the Bieberbach conjecture (2.4), include Löwner's differential equation (*cf.* [11, p. 83]) and the following nonnegativity result due to Askey and Gasper [3, p. 713, Theorem 3]:

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \geq 0 \qquad (\alpha \geq -2; \quad -1 < x \leq 1), \tag{2.7}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial of index or order (α, β) and degree n in x (*cf.* Szegö [74]). In fact, we have

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}, \tag{2.8}$$

where, in terms of *Gamma functions,*

$$\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)\Gamma(\mu+1)} = \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}),$$

so that

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \qquad (\lambda \in \mathbb{C}; \quad n \in \mathbb{N}). \tag{2.9}$$

Equivalently, (2.8) may be written in the form:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} \; {}_2F_1\left(-n, \alpha+\beta+n+1; \, \alpha+1; \, \frac{1-x}{2}\right), \tag{2.10}$$

in terms of the *Gaussian case*

$$\ell - 1 = m = 1$$

of the *generalized hypergeometric function* ${}_\ell F_m$ defined below.

Definition 6. Let $\lambda_j \, (j = 1, \cdots, \ell)$ and $\mu_j \, (j = 1, \cdots, m)$ be complex numbers such that

$$\mu_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, m).$$

Then the generalized hypergeometric function ${}_\ell F_m(z)$ is defined by

$$\begin{aligned} {}_\ell F_m(z) &\equiv {}_\ell F_m(\lambda_1, \cdots, \lambda_\ell; \, \mu_1, \cdots, \mu_m; \, z) \\ &= {}_\ell F_m \left[\begin{array}{cc} \lambda_1, \cdots, \lambda_\ell; & \\ & z \\ \mu_1, \cdots, \mu_m; & \end{array}\right] \\ &:= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_\ell)_n}{(\mu_1)_n \cdots (\mu_m)_n} \frac{z^n}{n!} \qquad (\ell \leq m+1), \end{aligned} \tag{2.11}$$

where $(\lambda)_n$ denotes the *Pochhammer symbol* defined, again in terms of Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} \tag{2.12}$$

We note in passing that

$$z\,{}_\ell F_m\left(\lambda_1,\cdots,\lambda_\ell;\;\mu_1,\cdots,\mu_m;\;z\right)\in\mathcal{A}, \tag{2.13}$$

since the ${}_\ell F_m$ series in (2.11) converges absolutely for (*cf.*, *e.g.*, Erdélyi *et al.* [14, Chapter 4])

(i) $|z|<\infty$ if $\ell<m+1$;
(ii) $z\in\mathcal{U}$ if $\ell=m+1$;
(iii) $z\in\partial\mathcal{U}:=\{z:z\in\mathbb{C}$ and $|z|=1\}$ if $\ell=m+1$,

provided further that

$$\Re\left(\sum_{j=1}^{m}\mu_j-\sum_{j=1}^{\ell}\lambda_j\right)>0,$$

unless (of course) the series terminates.

Making use of the hypergeometric representation (2.10), it is not difficult to rewrite the inequality (2.7) in the generalized hypergeometric form [3, p. 717, Equation (3.1)]:

$$\frac{(\alpha+2)_n}{n!}\,{}_3F_2\left[\begin{array}{c}-n,\;\alpha+n+2,\;\tfrac{1}{2}(\alpha+1);\\ \\ \alpha+1,\;\tfrac{1}{2}(\alpha+3);\end{array}\;x\right]\geqq 0 \tag{2.14}$$

$$(0\leqq x<1;\quad \alpha\geqq -2;\quad n\in\mathbb{N}_0).$$

The theory of special functions has so far remained unavoidable in proving the aforementioned conjectures in Geometric Function Theory (see also Aleksandrov [1] *and* Mitrinović [41, p. 289 *et seq.*]). Even the relatively more recent attempt by Weinstein [77] to prove the Bieberbach conjecture (2.4) *directly* is based rather heavily upon the addition theorem for the Legendre (or spherical) polynomials $P_n(x)$, where [*cf.* Equation (2.10)]

$$P_n(x)=P_n^{(0,0)}(x)={}_2F_1\left(-n,n+1;1;\frac{1-x}{2}\right). \tag{2.15}$$

(See also Todorov [76], Wilf [79], and Koepf and Schmersau [29], *and* the references cited therein.)

3. A Set of Linear Operators Defined on the Space $\mathcal{A}$

Several linear operators (whose usefulness, in the study of such subclasses of analytic functions as those defined in Section 1, will be considered in this paper) are given below:

I. Carlson-Shaffer Operator. The *Carlson-Shaffer operator* $\mathcal{L}(a,c)$ is defined by the convolution (*cf.* Carlsson and Shaffer [6]; see also Owa *et al.* [45]):

$$\mathcal{L}(a,c)f(z):=\phi(a,c;z)*f(z)\qquad (f(z)\in\mathcal{A}), \tag{3.1}$$

where $\phi(a,c;z)$ is an *incomplete Beta function* defined by

$$\phi(a,c;z):=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\,z^{n+1}=z\,{}_2F_1(1,a;c;z) \tag{3.2}$$

$$(c\neq 0,-1,-2,\cdots;\;z\in\mathcal{U}).$$

The operator $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself. Furthermore, if we let

$$a\neq 0,-1,-2,\cdots,$$

then $\mathcal{L}(c,a)$ is an inverse of $\mathcal{L}(a,c)$. Observe also that (*cf.* Owa and Srivastava [48, p. 1067])

$$\mathcal{K}(\alpha)=\mathcal{L}(1,2)\,\mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{S}^*(\alpha)=\mathcal{L}(2,1)\mathcal{K}(\alpha) \tag{3.3}$$

$$(0 \leq \alpha < 1).$$

II. Generalized Bernardi-Libera-Livingston Integral Operator. An interesting generalization of the *Bernardi-Libera-Livingston integral operator*, denoted here by $\mathcal{J}_\gamma$, is defined by

$$\mathcal{J}_\gamma f(z) := \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt \qquad (\gamma > -1; \quad f(z) \in \mathcal{A}), \tag{3.4}$$

which, for various further constraints on the parameter γ, was used recently by several authors (see, *e.g.*, Srivastava and Owa [71, pp. 66, 154, 181, and 338]; see also Bernardi [4], Libera [31], and Livingston [36]).

III. Miller-Mocanu-Reade-Integral Operator. The *Miller-Mocanu-Reade integral operator* $\mathcal{I}$ is defined (for suitable analytic functions $\phi(z)$ and $\Phi(z)$, and for suitable constants $\alpha, \beta \neq 0, \gamma$, and δ) by

$$\mathcal{I} f(z) := \left(\frac{\beta+\gamma}{z^\gamma \Phi(z)}\int_0^z \{f(t)\}^\alpha \,\phi(t)t^{\delta-1}dt\right)^{1/\beta} \qquad (f(z) \in \mathcal{A}), \tag{3.5}$$

which, in the *special* case when

$$\alpha=\beta=1, \quad \delta=\gamma, \quad \text{and} \quad \phi(z)=\Phi(z)=1, \tag{3.6}$$

reduces at once to the generalized Bernardi-Libera-Livingston operator $\mathcal{J}_\gamma$ $(\gamma > -1)$ (see Miller *et al.* [39], [40]).

4. Fractional Calculus Operators and Their Generalizations

Numerous operators of *fractional calculus* (that is, *fractional integral* and *fractional derivative*) have indeed been studied in the literature rather extensively (*cf.*, *e.g.*, [15, Chapter 13], [43], [46], [59], [60], [65, p. 21 *et seq.*], [68, Chapter 5], and [47]). We choose to recall here the following operators of fractional calculus.

Definition 7 (Fractional Integral Operator). The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta \qquad (\lambda > 0), \tag{4.1}$$

where $f(z)$ is an analytic function in a *simply-connected* region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be *real* when $z-\zeta > 0$.

Definition 8 (Fractional Derivative Operator). The *fractional derivative of order* λ is defined, for a function $f(z)$ by

$$D_z^\lambda f(z) := \begin{cases} \dfrac{1}{\Gamma(1-\lambda)}\dfrac{d}{dz}\int_0^z \dfrac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta & (0 \leq \lambda < 1) \\ \dfrac{d^n}{dz^n}D_z^{\lambda-n}f(z) & (n \leq \lambda < n+1; \quad n \in \mathbb{N}), \end{cases} \tag{4.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 7.

Definition 9 (Generalized Fractional Integral Operator). Under the hypotheses of Definition 7, the *generalized fractional integral of order* λ is defined, for a function $f(z)$, by

$$I_{0,z}^{\lambda,\mu,\nu} f(z) := \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\nu; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta)d\zeta \tag{4.3}$$

$$(\lambda > 0; \qquad \kappa > \max\{0, \mu-\nu\} - 1),$$

provided further that

$$f(z) = O\left(|z|^{\kappa}\right) \qquad (z \to 0). \tag{4.4}$$

It follows readily from Definition 7 and Definition 9 that

$$D_z^{-\lambda} f(z) = I_{0,z}^{\lambda,-\lambda,\nu} f(z) \qquad (\lambda > 0). \tag{4.5}$$

Furthermore, since

$${}_2F_1(a,b;b;z) = {}_0F_1(a;-;z) = (1-z)^{-\alpha} \qquad (z \in \mathcal{U}), \tag{4.6}$$

we have the relationship:

$$I_{0,z}^{\lambda,\mu,-\lambda} f(z) = D_z^{-\lambda}\, z^{-\lambda-\mu} f(z) \qquad (\lambda > 0). \tag{4.7}$$

The fractional calculus operator D_z^{λ}, given by Definition 7 and Definition 8, is related rather closely to the Carlson-Shaffer operator $\mathcal{L}(a,c)$ defined by (3.1); in fact, we have

$$\mathcal{L}(2,c)f(z) = \Gamma(c) z^{2-c} D_z^{2-c} f(z) \qquad (c \neq 0, -1, -2, \cdots) \tag{4.8}$$

or, equivalently,

$$D_z^{\lambda} f(z) = \frac{z^{-\lambda}}{\Gamma(2-\lambda)} \mathcal{L}(2, 2-\lambda) f(z) \qquad (\lambda \neq 2, 3, 4, \cdots). \tag{4.9}$$

On the other hand, the operator $I_{0,z}^{\lambda,\mu,\nu}$ is a generalization of the fractional integral operator which was studied by Saigo [59] and applied subsequently by Srivastava and Saigo [72] in solving various boundary value problems involving the Euler-Darboux equation:

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{x-y}\left(\beta\frac{\partial u}{\partial x} - \alpha\frac{\partial u}{\partial y}\right) = 0 \tag{4.10}$$

$$(\alpha > 0;\ \beta > 0;\ \alpha + \beta < 1).$$

Definition 10 (Generalized Fractional Derivative Operator). Under the hypotheses of Definition 8, the *generalized fractional derivative of order* λ is defined, for a function $f(z)$, by

$$\begin{aligned} J_{0,z}^{\lambda,\mu,\nu} f(z) :=& \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} \right. \\ & \left. \cdot\ {}_2F_1\left(\mu-\lambda, -\nu; 1-\lambda; 1-\frac{\zeta}{z}\right) f(\zeta)d\zeta \right\} \end{aligned} \tag{4.11}$$

$$(0 \leq \lambda < 1; \quad \kappa > \max\{0, \mu-\nu-1\} - 1),$$

where κ is given, as before, by the order estimate (4.4).

It follows readily from Definition 10 that

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1), \tag{4.12}$$

where the fractional calculus operator D_z^{λ} is, in fact, given by Definitions 7 and 8 for all values of λ (see, *e.g.*, Srivastava and Owa [71, p. 343]). Furthermore, in terms of Gamma functions, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu} \tag{4.13}$$

$$(\rho + 2 > \mu - \nu).$$

(See also Sohi [63], Srivastava *et al.* [73], and Owa *et al.* [50].)

5. Applications of Operational Techniques on the Space $\mathcal{A}$

By applying the Carlson-Shaffer operator $\mathcal{L}(a,c)$ defined by (3.1), Owa and Srivastava [48] proved

Theorem 1. *For the generalized hypergeometric function* ${}_{\ell}F_m(z)$ *defined by* (2.11), *let,*

$$\left| \frac{z\, {}_{\ell}F_m''(z)}{{}_{\ell}F_m'(z)} \right| < (1-\alpha)^{-1} \left(1 - \frac{3}{2}\alpha + \alpha^2\right) \tag{5.1}$$

$$\left(0 \leq \alpha \leq \frac{1}{2};\ \lambda_1 \cdots \lambda_{\ell} \neq 0;\ z \in \mathcal{U}\right).$$

Then

$$z\, {}_{\ell+1}F_{m+1}(\lambda_1+1, \cdots, \lambda_{\ell}+1, 1; \mu_1+1, \cdots, \mu_m+1, 2; z) \in \mathcal{S}^*(\alpha) \tag{5.2}$$

$$(0 \leq \alpha \leq \frac{1}{2}).$$

For the generalized Bernardi-Libera-Livingston operator $\mathcal{J}_{\gamma}$ defined by (3.4), it is known that

$$f(z) \in \mathcal{S}^* \Rightarrow \mathcal{J}_{\gamma} f(z) \in \mathcal{S}^* \qquad (0 \leq \gamma \leq 1). \tag{5.3}$$

Making use of this last inclusion property (5.3) and the definition (3.4), it is not difficult to apply Theorem 1 (with $\alpha = 0$) *iteratively* in order to deduce

Theorem 2. *For the generalized hypergeometric function* ${}_{\ell}F_m(z)$ *defined by* (2.11), *let*

$$\left| \frac{z\, {}_{\ell}F_m''(z)}{{}_{\ell}F_m'(z)} \right| < 1 \tag{5.4}$$

$$(\lambda_1 \cdots \lambda_{\ell} \neq 0; \qquad z \in \mathcal{U}).$$

Then

$$z\, {}_{\ell+s+1}F_{m+s+1} \left[\begin{array}{cc} \lambda_1+1, \cdots, \lambda_{\ell}+1, 1, \sigma_1+1, \cdots, \sigma_s+1; & \\ & z \\ \mu_1+1, \cdots, \mu_m+1, 2, \sigma_1+2, \cdots, \sigma_s+2; & \end{array} \right] \in \mathcal{S}^* \tag{5.5}$$

$$(0 \leq \sigma_j \leq 1; \quad j = 1, \cdots, s).$$

From amongst the various special cases of Theorem 2, which are worthy of note, we consider here the case when

$$\sigma_j = 1 \quad (j = 1, \cdots, s).$$

The assertion (5.5) reduces, in this case, to the inclusion relation:

$$z\,{}_{\ell+s}F_{m+s}\left[\begin{array}{c}\lambda_1+1,\cdots,\lambda_\ell+1,1,2,\cdots,2;\\ \\ \mu_1+1,\cdots,\mu_m+1,3,3,\cdots,3;\end{array}\ z\ \right]\in\mathcal{S}^*, \tag{5.6}$$

provided that the relevant hypotheses of Theorem 2 hold true.

The following result is analogous to the inclusion relation (5.3); it holds true for the substantially more general Miller-Mocanu-Reade operator $\mathcal{I}$ defined by (3.5).

Theorem 3. *Let the functions $f(z)$ and $\phi(z)$ be in the class $\mathcal{S}^*(\rho)$ $(0\leq\rho<1)$. Then the function $F(z)$ defined by* [*cf. Equation* (3.5) *with $\Phi(z)=1$, $\beta=\alpha+1$, and $\delta=\gamma$*]

$$\begin{aligned}F(z) &:= \left(\frac{\gamma+\alpha+1}{z^\gamma}\int_0^z\{f(t)\}^\alpha\phi(t)t^{\gamma-1}dt\right)^{1/(\alpha+1)}\\ &= z+\sum_{n=2}^\infty b_nz^n\end{aligned} \tag{5.7}$$

$$(\gamma>0;\quad \alpha>0)$$

is also in the class $\mathcal{S}^*(\rho)$ $(0\leq\rho<1)$.

The proof of Theorem 3, detailed elsewhere by Kim *et al.* [22], would make use of a number of results associated with the Miller-Mocanu-Reade integral operator $\mathcal{I}$ defined by (3.5). As a matter of fact, by applying the integral operator $\mathcal{I}$ and several properties and characteristics of the function spaces $\mathcal{P}(A,B)$ and $\mathcal{B}(A,B;\alpha)$ (see Definition 1 and Definition 3, respectively), Kim *et al.* [22] also proved Theorem 4 below, and established Theorem 5 below in the *special* case when $n=1$ and $\alpha\in\mathbb{N}$.

Theorem 4. *Let the functions $f(z)$ and $\phi(z)$ be in the class $\mathcal{S}^*(A,B)$ given by Definition* 2. *Then the function $F(z)$ defined by* (5.7) *is also in the class $\mathcal{S}^*(A,B)$.*

Theorem 5. *Let the function $f(z)$ be in the class $\mathcal{B}(A,B;\alpha)$ given by Definition* 3. *Then the function $G_n(z)$ defined by*

$$\begin{aligned}G_n(z) &:= \left(\frac{\gamma+\alpha+n}{z^\gamma}\int_0^z\{f(t)\}^\alpha\, t^{\gamma+n-1}dt\right)^{1/(\alpha+n)}\\ &= z+\sum_{n=2}^\infty c_nz^n\end{aligned} \tag{5.8}$$

$$(\gamma>-\alpha-n;\qquad n\in\mathbb{N})$$

(*that is, by* (5.7) *with $\phi(z)=z^n$, $\gamma>-\alpha-n$, and $n\in\mathbb{N}$*) *is in the class $\mathcal{B}(A,B;\alpha+n)$.*

The general result (Theorem 5 above) for $\alpha\geq0$ and $n\in\mathbb{N}$ was also given by Kim *et al.* [22].

More recently, Owa *et al.* [51] proved a generalization of Theorem 3, which may be recalled here as

Theorem 6. *Let $f(z)\in\mathcal{S}^*(\eta_1)$ and $g(z)\in\mathcal{S}^*(\eta_2)$. Then the function $F(z)$ defined by*

$$F(z) := \left\{\frac{\gamma+\alpha-\eta+1}{z^\gamma}\int_0^z\{f(t)\}^\alpha g(t)t^{\gamma-\eta-1}dt\right\}^{1/(\alpha-\eta+1)} \tag{5.9}$$

$$(\alpha \geq 0; \quad \gamma > 0; \quad 0 \leq \eta \leq \eta_2; \quad \alpha\eta_1 + \eta_2 - \eta \leq 1)$$

belongs to the class:

$$\mathcal{S}^*\left(\frac{\alpha\eta_1 + \eta_2 - \eta}{\alpha - \eta + 1}\right).$$

It is easily seen that, by setting $\eta_1 = \eta_2$ and $\eta = 0$ in Theorem 6, we arrive at Theorem 3. As a matter of fact, if we make use of a result of Mocanu *et al.* [42], we can establish the following result (Theorem 7 below) which will weaken the hypothesis of Theorem 3 while sharpening the conclusion (*cf.* Li and Srivastava [32]):

Theorem 7. *Let* $\alpha, \beta, \gamma, \delta,$ *and* σ *be real numbers satisfying*

$$\alpha \geq 0, \quad \beta > 0, \quad \text{and} \quad \beta + \gamma = \alpha + \delta > 0. \tag{5.10}$$

Suppose also that the function $\Phi(z)$ *is analytic in* $\mathcal{U}$ *and satisfies the condiitons:*

$$\Phi(0) = 1 \quad \text{and} \quad \Phi(z) \neq 0 \qquad (z \in \mathcal{U}). \tag{5.11}$$

If $f(z) \in \mathcal{S}^*(\eta_1)$ *and* $g(z) \in \mathcal{S}^*(\eta_2)$, *then the function* $F(z)$ *defined by*

$$F(z) := J(f,g)(z) = \left\{\frac{\beta+\gamma}{z^{\gamma}\Phi(z)} \int_0^z \{f(t)\}^{\alpha}\{g(t)\}^{\sigma}\, t^{\delta-\sigma-1} dt\right\}^{1/\beta} \tag{5.12}$$

with

$$\delta + \alpha\eta_1 + (\eta_2 - 1)\,\sigma \geq 0, \tag{5.13}$$

satisfies the inequality:

$$\Re\left\{\frac{zF'(z)}{F(z)} + \frac{1}{\beta}\cdot\frac{z\Phi'(z)}{\Phi(z)}\right\} \geq W(\rho; \beta, \gamma) \qquad (z \in \mathcal{U}), \tag{5.14}$$

where

$$\rho = \frac{\delta + \alpha\eta_1 + (\eta_2 - 1)\,\sigma - \gamma}{\beta} \tag{5.15}$$

and $W(\rho; \beta, \gamma)$ *is given by*

$$W(\rho; \beta, \gamma) := \inf_{|z|<1} \Re\{H(z)\}, \tag{5.16}$$

where

$$H(z) := \frac{(1-z)^{2(\rho-1)\beta}}{\beta \int_0^1 t^{\beta+\gamma+1}(1-zt)^{2(\rho-1)\beta} dt} - \frac{\gamma}{\beta}. \tag{5.17}$$

This result is sharp with the extremal function given by

$$F_0(z) := J(k_1, k_2)(z) \tag{5.18}$$

$$\left(k_1(z) := z(1-z)^{2(\eta_1-1)}; \quad k_2(z) := z(1-z)^{2(\eta_2-1)}\right).$$

It should be remarked in passing that, in the case when

$$\max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\} = \rho_0 \leq \rho < 1, \tag{5.19}$$

the value of $W(\rho;\beta,\gamma)$ given by (5.16) can be replaced by

$$W(\rho;\beta,\gamma)=H(-1)=\frac{1}{\beta}\left\{\frac{(\beta+\gamma)z^{-2\beta(1-\rho)}}{{}_2F_1\left[2\beta(1-\rho),\beta+\gamma;\beta+\gamma+1;-1\right]}-\gamma\right\}, \tag{5.20}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function defined by (2.11) with

$$\ell-1=m=1.$$

Furthermore, by assigning appropriate special values to the various parameters involved in Theorem 7, we can derive several interesting consequences of Theorem 7. For example, if we set

$$\Phi(z)\equiv 1,\quad \sigma=1,\quad \beta=\alpha-\eta+1,\quad \text{and}\quad \delta=\gamma-\eta+1$$

in Theorem 7, we obtain the following

Corollary. *Let* $f(z)\in\mathcal{S}^*(\eta_1)$ *and* $g(z)\in\mathcal{S}^*(\eta_2)$*. Then the function* $F(z)$ *defined by* (5.10)*, with*

$$\alpha\geq 0,\quad \gamma\geq 0,\quad \textit{and}\quad \eta\leq\alpha\eta_1+\eta_2,$$

belongs to the class:

$$\mathcal{S}^*\left(W(\rho;\alpha-\eta+1,\eta)\right)\qquad\left(\rho:=\frac{\alpha\eta_1+\eta_2-\eta}{\alpha-\eta+1}\right),$$

where $W(\rho;\alpha-\eta+1,\gamma)$ *is given by* (5.16) *with* $\beta=\alpha-\eta+1$*. This result is sharp.*

The above Corollary extends and improves both Theorem 3 and Theorem 6. Other interesting consequences of Theorem 7 (considered by Li and Srivastava [32]) would improve the corresponding results of Miller and Mocanu [38].

Next we turn to an application of the foregoing operational techniques involving the classes $\mathcal{F}(\alpha)$ and $\mathcal{K}(\alpha)$, that is, the classes of prestarlike and convex functions of order α. Indeed it is easily verified from the definitions (1.10) and (3.1) that

$$\mathcal{F}(\alpha)=\mathcal{L}(1-2-2\alpha)\mathcal{S}^*(\alpha)\qquad(\alpha<1). \tag{5.21}$$

Moreover, we have

$$\mathcal{F}(1)=\left\{f:f\in\mathcal{A}\quad\text{and}\quad\Re\left(\frac{f(z)}{z}\right)>\frac{1}{2}\qquad(z\in\mathcal{U})\right\}. \tag{5.22}$$

In order to present a connection theorem involving the classes $\mathcal{F}(\alpha)$ and $\mathcal{K}(\alpha)$, we introduce the operator $\Omega_z^{\lambda,\mu,\nu}$ defined by

$$\Omega_z^{\lambda,\mu,\nu}f(z):=\frac{\Gamma(2-\mu)\Gamma(3-\lambda+\nu)}{\Gamma(3-\mu+\nu)}z^{\mu}\,J_{0,z}^{\lambda,\mu,\nu}f(z), \tag{5.23}$$

where the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\nu}$ is given by Definition 10.

In view of the formula (4.13), it is fairly straightforward to relate the operator $\Omega_z^{\lambda,\mu,\nu}$ with the Carlson-Shaffer operator $\mathcal{L}(a,c)$ as follows:

$$\Omega_z^{\lambda,\mu,\nu}f(z)=\mathcal{L}(2,2-\mu)\mathcal{L}(3-\mu+\nu,3-\lambda+\nu)f(z) \tag{5.24}$$

$$(0\leq\lambda<1;\quad \mu-\nu<3;\quad f(z)\in\mathcal{A}).$$

Making use of the relationships (5.24) and (3.3), and also the following inclusion relation (due essentially to Carlson and Shaffer [6]):

$$\mathcal{L}(2-2\beta, 2-2\alpha)\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(\beta) \subset \mathcal{S}^*(\alpha) \tag{5.25}$$

$$(0 \leq \alpha \leq \beta < 1),$$

it can be shown that

$$\mathcal{L}(3-\lambda+\nu, 3-\mu+\nu)\Omega_z^{\lambda,\mu,\nu}\mathcal{K}(\frac{1}{2}) \subset \mathcal{S}^*(\frac{1}{2}) \tag{5.26}$$

$$(0 \leq \lambda < 1; \quad \mu-\nu < 3; \quad 0 \leq \mu < 1).$$

Finally, if we rewrite a special case $\left(\beta = \frac{1}{2}\right)$ of (5.25) in the form:

$$\mathcal{L}(1, 2-\alpha)\mathcal{S}^*\left(\frac{\alpha}{2}\right) \subset \mathcal{S}^*(\frac{1}{2}) \subset \mathcal{S}^*\left(\frac{\alpha}{2}\right) \tag{5.27}$$

$$(0 \leq \alpha < 2),$$

we shall obtain the following connection theorem involving the classes $\mathcal{F}(\alpha)$ and $\mathcal{K}(\alpha)$ (*cf.* Owa and Srivastava [49]):

Theorem 8. *For the classes* $\mathcal{K}(\alpha)$ *and* $\mathcal{F}(\alpha)$ *defined by* (1.3) *and* (1.10), *respectively,*

$$\mathcal{L}(3-\lambda+\nu, 3-\mu+\nu)\Omega_z^{\lambda,\mu,\nu}\mathcal{K}\left(\frac{\mu}{2}\right) = \mathcal{F}\left(\frac{\mu}{2}\right) \tag{5.28}$$

$$(0 \leq \lambda < 1; \quad \mu-\nu < 3; \quad 0 \leq \mu < 2).$$

In its special cases when $\mu = 0$ and $\mu = 1$, the assertion (5.28) of Theorem 8 would simplify considerably, and we have

$$\mathcal{L}(3-\lambda+\nu, 3+\nu)\Omega_z^{\lambda,0,\nu}\mathcal{K} = \mathcal{F}(0) \tag{5.29}$$

$$(0 \leq \lambda < 1; \quad \nu > -3)$$

and

$$\mathcal{L}(3-\lambda+\nu, 2+\nu)\Omega_z^{\lambda,1,\nu}\mathcal{K}\left(\frac{1}{2}\right) = \mathcal{F}\left(\frac{1}{2}\right) \tag{5.30}$$

$$(0 \leq \lambda < 1; \quad \nu > -2),$$

respectively.

6.Classes of Generalized Hypergeometric Functions Associated with the Hardy Space

Several inclusion theorems associated with the Hardy space of analytic functions (see Definition 5) were proven for various families of generalized hypergeometric functions whose derivative has a positive real part (see Definition 4). In this section we aim at developing a relatively simpler proof of a unification (and generalization) of these inclusion theorems.

We begin by recalling the following inclusion theorem which was proven by Jung *et al.* [19] by applying the *one-parameter* family of integral operators defined by (3.4):

Theorem 9. *Let the function*

$$z\,{}_\ell F_m\left(\lambda_1,\cdots,\lambda_\ell;\,\mu_1,\cdots,\mu_m;\,z\right)\qquad(\ell\leq m+1)$$

be in the class $\mathcal{R}$ *defined by* (1.22).

Then the function

$$z\,{}_{\ell+s}F_{m+s}\left[\begin{array}{lc}\lambda_1,\cdots,\lambda_\ell,\ \alpha_1+1,\cdots,\alpha_s+1; & \\ & z\\ \mu_1,\cdots,\mu_m,\ \alpha_1+2,\cdots,\alpha_s+2; & \end{array}\right]$$

is in $\mathcal{H}^\infty$ *at least for* $\alpha_j>0\quad(j=1,\cdots,s)$.

Another inclusion theorem for generalized hypergeometric functions, involving the class $\mathcal{R}(\gamma)$ given by Definition 4, is contained in

Theorem 10. *Let the function*

$$z\,{}_\ell F_m\left(\lambda_1,\cdots,\lambda_\ell;\,\mu_1,\cdots,\mu_m;z\right)\qquad(\ell\leq m+1)$$

be in the class $\mathcal{R}(\gamma)\,(0\leq\gamma<1)$.

Then the function

$$z\,{}_{\ell+s}F_{m+s}\left[\begin{array}{rc}\lambda_1,\cdots,\lambda_\ell,2,\cdots,2; & \\ & z\\ \mu_1,\cdots,\mu_m,\alpha_1+2,\cdots,\alpha_s+2; & \end{array}\right]$$

is in $\mathcal{H}^\infty$ *for* $\alpha_j\in\mathbb{N}\;(j=1,\cdots,s)$.

The proof of Theorem 10 by Kim *et al.* [25] makes use of the generalized fractional integral operator $I_{0,z}^{\lambda,\mu,\nu}$ given by Definition 9.

We now give a simple and direct proof of the following unification (and generalization) of Theorem 9 and Theorem 10, *without* using the integral operators $\mathcal{J}_\gamma$ and $I_{0,z}^{\lambda,\mu,\nu}$ defined by (3.4) and (4.3), respectively.

Theorem 11. *Let the function*

$$z\,{}_\ell F_m\left(\lambda_1,\cdots,\lambda_\ell;\,\mu_1,\cdots,\mu_m;z\right)\qquad(\ell\leq m+1)$$

be in the class $\mathcal{R}(\gamma)\,(0\leq\gamma<1)$. *Suppose also that the function* $\Psi(z)$ *is defined, in terms of a generalized hypergeometric function, by*

$$\Psi(z):=z\,{}_{\ell+s}F_{m+s}\left[\begin{array}{lc}\lambda_1,\cdots,\lambda_\ell,\alpha_1,\cdots,\alpha_s; & \\ & z\\ \mu_1,\cdots,\mu_m,\beta_1,\cdots,\beta_s; & \end{array}\right]\qquad(\ell\leq m+1;\quad s\in\mathbb{N})\tag{6.1}$$

for (real or complex) parameters $\alpha_1,\cdots,\alpha_s$ *and* $\beta_1,\cdots,\beta_s$ *such that*

$$\beta_j\neq0,-1,-2,\cdots\qquad(j=1,\cdots,s).$$

Then

$$\Psi(z) \in \mathcal{H}^{\infty} \tag{6.2}$$

and, more precisely,

$$|\Psi(z)| < \infty \qquad \left(z \in \mathcal{U} := \mathcal{U} \cup \partial\overline{\mathcal{U}} = \{z : z \in \mathbb{C} \quad \mathit{in}\ |z| \leq 1\}\right), \tag{6.3}$$

provided that

$$\Re\left(\sum_{j=1}^{s} \beta_j - \sum_{j=1}^{s} \alpha_j\right) > 0. \tag{6.4}$$

In place of the integral operators $\mathcal{J}_{\gamma}$ and $I_{0,z}^{\lambda,\mu,\nu}$ (which were applied by Jung *et al.* [19] and Kim *et al.* [25] to prove Theorem 9 and Theorem 10, respectively), our proof of Theorem 11 is based upon the coefficient inequality asserted by the following (*cf.* MacGregor [37, p 533, Theorem 1])

Lemma. *Let the function $f(z)$ be in the class $\mathcal{R}(\gamma)$ $(0 \leq \gamma < 1)$.*
Then

$$|a_n| \leq \frac{2}{n} \qquad (n \in \mathbb{N}^* := \mathbb{N} \setminus \{1\}). \tag{6.5}$$

Proof. Since $0 \leq \gamma < 1$, the hypothesis $f(z) \in \mathcal{R}(\gamma)$ implies that

$$\Re\{f'(z)\} > \gamma \geq 0 \quad (z \in \mathcal{U}).$$

Thus the assertion (6.5) of the Lemma can be deduced fairly readily by setting $g(z) = f'(z)$ in the following well-known (*rather classical*) result due to Constantin Carathéodory (1873-1950): *If*

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

is analytic in the open unit disk $\mathcal{U}$, and if

$$\Re\{g(z)\} > 0 \qquad (z \in \mathcal{U}),$$

then

$$|b_n| \leq 2 \qquad (n \in \mathbb{N}).$$

(*Cf.* Carathéodory [5]; see also Pólya and Szegö [54, pp. 150 and 355, Problem 235].)

Proof of Theorem 11. For the sake of convenience, we put

$$\Omega_n = \frac{(\lambda_1)_n \cdots (\lambda_\ell)_n}{(\mu_1)_n \cdots (\mu_m)_n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{6.6}$$

Then, by the assertion (6.5) of the Lemma, the hypothesis

$$z\, {}_{\ell}F_m(\lambda_1, \cdots, \lambda_\ell; \mu_1, \cdots, \mu_m; z) = z + \sum_{n=2}^{\infty} \frac{\Omega_{n-1}}{(n-1)!} z^n \in \mathcal{R}(\gamma) \qquad (0 \leq \gamma < 1)$$

implies that

$$\left|\frac{\Omega_{n-1}}{(n-1)!}\right| \leq \frac{2}{n} \qquad (n \in \mathbb{N}^*). \tag{6.7}$$

From the definition (2.12) and Stirling's asymptotic expansion for the Gamma function (*cf.*, *e.g.*, Erdélyi *et al.* [14, p. 47, Section 1.18]), it is not difficult to show for

$$\Delta_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_s)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}} \qquad (n \in \mathbb{N})$$

with *fixed* parameters α_j and β_j $(j = 1, \cdots, s)$ that

$$\Delta_n = K^{-1} n^{-\omega} \left[1 + O\left(n^{-1}\right)\right] \qquad (n \to \infty), \tag{6.8}$$

where, for convenience,

$$K := \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_s)}{\Gamma(\beta_1) \cdots \Gamma(\beta_s)}$$

and

$$\omega := \sum_{j=1}^{s} \beta_j - \sum_{j=1}^{s} \alpha_j. \tag{6.9}$$

Now, for the function $\Psi(z)$ defined by (6.1), we readily have

$$|\Psi(z)| \leq |z| + \sum_{n=2}^{\infty} \left| \frac{\Omega_{n-1}}{(n-1)!} \right| |\Delta_n| \, |z|^n, \tag{6.10}$$

which, for $z \in \overline{\mathcal{U}}$, yields

$$|\Psi(z)| \leq 1 + \sum_{n=2}^{\infty} |c_n|, \tag{6.11}$$

where

$$|c_n| = \left| \frac{\Omega_{n-1}}{(n-1)!} \right| |\Delta_n| \qquad (n \in \mathbb{N}^*). \tag{6.12}$$

Applying the results (6.7) and (6.8), we find from (6.12) that

$$|c_n| \leq \frac{2M}{|K|} \frac{1}{n^{1+\Re(\omega)}} \qquad (n \geq N \in \mathbb{N}; \quad M > 0), \tag{6.13}$$

which proves that the power series for the function $\Psi(z)$ converges absolutely for *each* $z \in \overline{\mathcal{U}}$, provided that the real part of ω defined by (6.9) is positive, that is, that the condition (6.4) of Theorem 11 is satisfied.

This evidently completes our direct proof of *both* the assertions (6.2) and (6.3) of Theorem 11.

In its special case when $\gamma = 0$ *and*

$$\beta_j = \alpha_j + 1 \qquad (j = 1, \cdots, s), \tag{6.14}$$

the assertion (6.2) of Theorem 11 would correspond to Theorem 9 *without* the inequalities required there to be satisfied by the parameters $\alpha_1, \cdots, \alpha_s$. Furthermore, a special case of the assertion (6.2) of Theorem 11 when

$$\alpha_j = 2 \quad \text{and} \quad \beta_j = \alpha_j + 2 \qquad (j = 1, \cdots, s) \tag{6.15}$$

would yield Theorem 11 with the relataively less stringent condition:

$$\Re(\alpha_1 + \cdots + \alpha_s) > 0.$$

We should like to conclude by remarking that, under the aforementioned special cases (6.14) and (6.15), our proof of the assertion (6.3) of Theorem 11 would show that not only are the functions (involved in Theorem 9 and Theorem 10) unbounded, but their power series are also absolutely convergent, for *each* $z \in \partial\mathcal{U}$.

Some of the recent developments on the applications of various families of generalized fractional calculus and other linear operators in analytic and univalent function theory are reported also in (for example) [7], [8], [12], [13], [20], [24], [26] to [28], [30], [55], and [69]. The interested reader may refer to these earlier works on the subject for further details.

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