IRREDUNDANT RAMSEY NUMBERS FOR GRAPHS

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ABSTRACT

The irredundant Ramsey Number $s(m,n)$ is the least value of p such that for any p–vertex graph G, either G has an irredundant set of at least n vertices or its complement $\bar{G}$ has an irredundant set of at least m vertices. The existence of these numbers is guaranteed by Ramsey's theorem. We prove that $s(3,3) = 6$, $s(3,4) = 8$ and $s(3,5) = 12$.

1. Introduction

Let $G = (V,E)$ be a simple undirected graph. The *open neighbourhood* of the vertex v, denoted by $N(v)$, is given by $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighbourhood* $N[v] = \{v\} \cup N(v)$. More generally we define the *closed neighbourhood of a subset* $X \subseteq V$ by $N[X] = \bigcup_{x \in X} N[x]$.

A subset X of V is called *independent* if for any $u,v \in X$, $uv \notin E$. If X, Y are subsets of V, then X *dominates* Y if and only if $Y \subseteq N[X]$. If X dominates V, then X is called a *dominating set* of G.

A vertex x is called *redundant in a subset* X if $N[x] \subseteq N[X-\{x\}]$ and a set X is called *irredundant* if it contains no vertices which are redundant in X. Thus, a set X is irredundant if and only if for each $x \in X$, $N[x]$ has a vertex which is not in the union of closed neighbourhoods of the remainder of X.

We will use the following facts about irredundance which are immediate from the definition:

(i) The *private neighbourhood* $I(x,X)$ of the element x of X is defined by

$$I(x,X) = N[x] - N[X-\{x\}],$$

and the elements of $I(x,X)$ are called the *private neighbours* of x (relative to X). A set X is irredundant if and only if for each $x \in X$, $I(x,X) \neq \emptyset$.

(ii) Any independent set Y is also irredundant since for each $y \in Y$, $y \in I(y,Y)$.

(iii) Any element x of an irredundant set X of G which is not isolated in $G[X]$ (the subgraph induced by X in G) has a private neighbour outside the set X.

Extremal sets of these types are related by the following two well–known results:

PROPOSITION 1 (Berge [1]). If X is maximal independent, then X is minimal dominating.

PROPOSITION 2 (Cockayne, Hedetniemi and Miller [2]). If X is minimal dominating, then X is maximal irredundant.

We now define six parameters concerning these types of vertex subsets.

The *lower (upper) independence numbers* $i(G)(\beta(G))$, *domination numbers* $\gamma(G)(\Gamma(G))$ and *irredundance numbers* $ir(G)$ $(IR(G))$ are respectively the smallest (largest) cardinalities of maximal independent, minimal dominating and maximal irredundant sets of vertices of G.

It follows from the above propositions that for any graph G,

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Since independent sets of a graph G are precisely cliques in the complement $\bar{G}$, the classical Ramsey numbers $r(m,n)$ for graphs which are usually defined using cliques, may also be defined using independence. In fact, $r(m,n)$ is the least integer p such that any p–vertex graph G satisfies either $\beta(\bar{G}) \geq m$ or $\beta(G) \geq n$.

We now define analogous numbers for irredundance.

The *irredundant Ramsey number* $s(m,n)$ is the least integer p such that any p–vertex graph G satisfies $IR(\bar{G}) \geq m$ or $IR(G) \geq n$.

Since $IR(G) \geq \beta(G)$ for all G, it follows that $s(m,n) \leq r(m,n)$ for all m and n. The same recurrence inequality which holds for Ramsey numbers also holds for irredundant Ramsey numbers. The proofs are analogous and are omitted.

PROPOSITION 3. For all positive integers $m \geq 2$, $n \geq 2$,

$$s(m,n) \le s(m-1,n) + s(m,n-1).$$

The classical Ramsey numbers have proved extremely difficult to evaluate. All known values and some bounds are given in Table 1. (See [3].)

	3	4	5	6	7	8	9
3	6	9	14	18	23	28/29	36
4		18	25/28	34/44			
5			42/55	57/94			

Table 1. Ramsey Numbers $r(m,n)$

Since irredundance is a more complex property than independence, we predict that there will be similar difficulties in the evaluation of $s(m,n)$. In this paper we evaluate $s(3,3)$, $s(3,4)$ and $s(3,5)$.

A large amount of research has been done recently concerning domination, independence and irredundance. The reader is referred to [4] for an excellent bibliography.

2. Calculation of $s(3,3)$, $s(3,4)$ and $s(3,5)$.

For ease of explanations we will abbreviate $IR(G)$, $IR(\bar{G})$ to IR, $\overline{IR}$, etc. and will refer to edges of G, $\bar{G}$ as red edges and blue edges respectively. Denote by $C(v_1,\cdots,v_n)$ the cycle of G with vertex sequence $v_1,\cdots,v_n$. Extensive use will be made of the following result.

Lemma 1. $\bar{G}$ has an irredundant set of size 3 if and only if there is a red K_3 or there is a red six cycle $C(v_1v_2\cdots v_6)$ and the edges v_1v_4, v_2v_5 and v_3v_6 are blue.

Proof. Let $X = \{x,y,z\}$ be irredundant in $\bar{G}$. There are three cases to consider.

(i) X is independent in $\bar{G}$ and hence forms a red K_3.

(ii) $\bar{G}[X]$ (the subgraph induced by X in $\bar{G}$) has exactly one isolate x in $\bar{G}$ and thus xy, xz are red and yz is blue. Since y, z have private neighbours in $\bar{G}$, say y_1, z_1 respectively, xy_1 and zy_1 are red, making xzy_1 a red K_3.

(iii) $\bar{G}[X]$ has no isolates. Then, in $\bar{G}$, x, y, z have distinct private neighbours x_1,y_1,z_1 in $V - X$. It follows that xy_1, xz_1, yx_1, yz_1, zx_1, zy_1 are all red while xx_1, yy_1, zz_1 are blue. Therefore $C(x,y_1,z,x_1,y,z_1)$ is the required cycle.

The converse is obvious. ■

THEOREM 1. $s(3,3) = 6$.

Proof. Firstly, $s(3,3) \le r(3,3) = 6$ and secondly, the graph C_5 has neither an induced K_3 nor a 6–cycle. Therefore, by Lemma 1, $IR(\bar{C}_5) \le 2$. Since C_5 is self–complementary, $IR(C_5) \le 2$ and hence $s(3,3) > 5$. ■

Using Lemmas 2, 3 and 4 we now prove that there is no 8–vertex graph G with $IR < 4$ and $\overline{IR} < 3$. In each of these lemmas, G is an assumed counterexample to this statement. We remark that the non–existence of G could also be established by showing that each of the 8–vertex graphs which have $\bar{\beta} \le 2$ and $\beta \le 3$ (see [5], p. 363), have $IR = 4$. However, we will illustrate the use of Lemma 1 instead.

LEMMA 2. Each vertex of G has degree 2 or 3.

Proof. If G has a vertex u with degree at least four, consider the set $U = \{u_1,u_2,u_3,u_4\} \subseteq N(u)$. If U is independent in G, then U is irredundant in G and $IR \geq 4$; otherwise some u_iu_j is red, uu_iu_j is a red K_3 and (by Lemma 1) $\overline{IR} \geq 3$.

If G has a vertex u of degree one or less, then there is a set U of six vertices which are not adjacent to u. Since $r(3,3) = 6$, either U contains a red K_3 (impossible by Lemma 1) or a blue K_3. In the latter case the blue K_3 together with u constitutes a blue K_4 whose vertex set is irredundant in G. ∎

Lemma 3. G does not have adjacent vertices of degree 3.

Proof. Let $V = \{1,\cdots,8\}$ and 1, 2 be adjacent vertices of degree 3. Since there is no red K_3, $N(1) \cap N(2) = \emptyset$ and we take $N(1) = \{8,7\}$ and $N(2) = \{3,4\}$. It follows that 34 and 78 are blue, as are all remaining edges joining 1 and 2.

Case (a). Suppose 56 is red. Vertex 5 is adjacent in G to a vertex of $\{3,4\}$, otherwise $\{1,3,4,5\}$ is a blue K_4. We assume 45 is red. Similarly, to avoid the blue K_4 $\{2,5,7,8\}$, we take 57 to be red. Vertex 6 must send a red edge to $\{3,4\}$ to avoid the blue K_4 $\{1,3,4,6\}$. If 46 were red, there is a red K_3, hence 36 is red and 46 is blue. Similarly 68 is red and 67 is blue. To avoid red K_3's we deduce that 38, 35, 47 are blue.

There is now a red 6–cycle $C(1,2,3,6,5,7)$ and by Lemma 1 we deduce that 16, 25 and 37 cannot all be blue, otherwise $\overline{IR} \geq 3$. Hence 37 is red. In future uses of this 6–cycle structure with Lemma 1, we will abbreviate the previous sentences to "By $C(1,2,3,6,5,7)$, 37 is red". In fact, the next deduction is of this type. By $C(1,2,4,5,6,8)$, 48 is red. Every vertex in G is now saturated (i.e. has degree 3). Therefore no more red edges can be added. However, the set $\{1,2,3,7\}$ is irredundant in G and Case (a) is finished.

Case (b). Suppose 56 is blue. Each vertex of {3,4,7,8} sends a red edge to either 5 or 6. (For example, if 35 and 36 are blue, then {1,3,5,6} is a blue K_4.) Further, there is a red edge between {3,4} and {7,8}, otherwise these vertices form a blue K_4. We assume without losing generality that 35 and 37 are red and deduce that 57 is blue. Since 7 sends a red edge to {5,6}, the edge 67 is red and all remaining edges to 3 and 7 are blue. The set $X = \{4,5,6,8\}$ is irredundant in G since $2 \in I(4,X)$, $3 \in I(5,X)$, $7 \in I(6,X)$ and $1 \in I(8,X)$. This completes the proof. ■

Lemma 4. G does not have adjacent vertices of degree two.

Proof. Suppose the contrary and let 1 and 2 be adjacent in G and have degree two. Assume 18, 23 are red. It follows that all edges from {1,2} to {4,5,6,7} and the edges 13 and 28 are blue. Let $Q = \{v \in V \mid v3 \text{ and } v1 \text{ are blue}\}$. Since 3 sends at most two red edges to {4,5,6,7}, $|Q| \geq 2$. If there is a blue edge $q_1q_2 \in Q$, then $\{q_1,q_2,1,3\}$ is a blue K_4. Hence Q is a red clique and $|Q| \leq 2$. Thus $|Q| = 2$ and we assume that 34 and 35 are red, 36 and 37 are blue and deduce that 67 is red. Similarly 8 sends exactly two red edges to {4,5,6,7} and by Lemma 2 , 83 is blue. If both 48 and 58 were red, then each vertex of $Y = \{1,2,3,4,5,8\}$ either has degree three in G or is adjacent to a vertex of degree three in G. Therefore by Lemma 3, no further red edges may be added to vertices of Y. This implies 6 and 7 have degree one in G, contrary to Lemma 2. The edges 87 and 86 cannot both be red and so we may assume without losing generality that 87 and 85 are red while 86 and 84 are blue. To avoid the blue K_4 {2,4,6,8}, we deduce that 46 is red. By Lemma 3, all remaining edges joining vertices of {4,5,6,7} are blue. By C(3,4,6,7,8,5), $\overline{IR} \geq 3$. ■

Theorem 2. $s(3,4) = 8$.

Proof. Suppose the 8–vertex graph G has $IR \leq 3$ and $\overline{IR} \leq 2$. Lemmas 2, 3, 4 imply that G is bipartite with independent sets U_1, U_2, the sets of vertices of degree two and degree three respectively. Either U_1 or U_2 has at least four vertices and is irredundant which shows that $IR \geq 4$. Therefore no such G can exist and $s(3,4) \leq 8$.

The graph C_7 has $IR = 3$ and $\overline{IR} = 2$, hence $s(3,4) > 7$. ∎

THEOREM 3. $s(3,5) = 12$.

Proof. As a first step in showing that $s(3,5) \leq 12$, we prove that any counterexample, i.e. a 12–vertex graph G with $IR < 5$ and $\overline{IR} < 3$, is regular of degree four. Suppose $X = \{v_1,\cdots,v_5\} \subseteq N(v)$. Then either X is independent, in which case $IR \geq 5$, or some v_iv_j is red, in which case vv_iv_j is a red K_3, i.e. $\overline{IR} \geq 3$.

If v has degree at most three, then let $Y = V - N[v]$. Since $s(3,4) = 8$ and $|Y| \geq 8$, it follows that $\overline{IR}(G[Y]) \geq 3$ which implies $\overline{IR} \geq 3$, or $IR(G[Y]) \geq 4$. In the latter case v together with an irredundant set of G[Y] forms an irredundant set of G, i.e. $IR \geq 5$.

Let $V = \{1,\cdots,12\}$ and without loss of generality assume that $N(1) = \{2,10,11,12\}$ and $N(2) = \{1,3,4,5\}$. It follows that $\{2,10,11,12\}$ and $\{1,3,4,5\}$ are blue K_4's and that all edges joining $\{1,2\}$ to $\{6,7,8,9\}$ are blue. The four vertices 6, 7, 8, 9 cannot be independent (otherwise $\{1,6,7,8,9\}$ is a blue K_5) nor can these vertices contain a red K_3. Further, since each vertex of $\{6,7,8,9\}$ sends at least one red edge to each of the sets $\{10,11,12\}$ and $\{3,4,5\}$ (to avoid blue K_5's), no vertex of $G[\{6,7,8,9\}]$ has degree three. Therefore this induced (red) subgraph is one of the following five shown in Figure 1. In all five cases the edges 68 and 79 are blue and 89 is red.

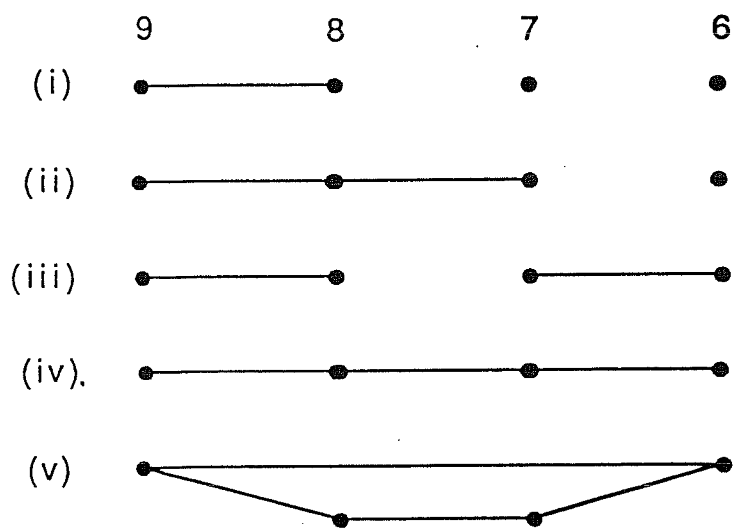

Figure 1. Possibilities for G[{6,7,8,9}].

Vertex 9 sends a red edge to {3,4,5} to avoid the blue K_5 {1,3,4,5,9}. Hence we assume 59 is red and by a similar argument to avoid the blue K_5 {2,9,10,11,12}, we may suppose 9 10 is red. The vertex 8 also sends a red edge to both {3,4,5} and {10,11,12} but cannot have a red neighbour in common with 9, otherwise a red K_3 results. We assume, without losing generality, that 84 and 8 11 are both red and deduce that 9 11, 8 10, 5 10, 4 11, 49, 58 are all blue.

We next use C(1,11,8,9,5,2) and C(1,10,9,8,4,2) to deduce that 5 11 and 4 10 are red.

Two cases are now required to show that each of the possibilities of Figure 1 is impossible.

Case 1. Suppose $G[\{6,7,8,9\}]$ is one of the graphs of Figure 1(i) or (ii). In $G[\{6,7,8,9\}]$ we know 89 is red, 78 may be either blue or red and the remaining edges are blue.

If 9 12 were red then C(1,12,9,8,4,2) implies 12 4 is red which gives vertex 4 degree four in G. Then {1,4,6,7,9} is a blue K_5 and we deduce that 9 12 is blue.

Similarly, if 39 were red, C(2,3,9,8,11,1) requires 3 11 red. Now 11 is saturated and {2,6,7,9,11} is a blue K_5. Therefore 39 is blue.

Hence only the three red edges 59, 10 9 and 89 are incident with 9, contrary to the 4–regular property established above.

Case 2. Suppose $G[\{6,7,8,9\}]$ is one of the graphs of Fig. 1(iii), (iv) or (v). In $G[\{6,7,8,9\}]$ we assume 76 is red, 78 and 96 may be either blue or red and the remaining edges 68 and 79 are blue. These assumptions cover all the three cases of Figure 1(iii), (iv) and (v).

Vertices 7 and 6 must each send a red edge to {10,11,12} to avoid the blue K_5's {2,10,11,12,6} and {2,10,11,12,7}. Further 7 and 6 cannot join the same vertex of {10,11,12} with red edges, otherwise a red K_3 results. Therefore there is a red edge from {7,6} to {10,11} and we assume 7 10 is red. It follows that 6 10 and 47 are blue.

Vertex 10 now has degree four in G. Hence 3 10 is blue and C(2,1,10,7,6,3) implies 63 is blue. Also, C(2,1,10,7,6,5) implies that 65 is blue. We deduce that 64 is red to avoid the blue K_5 {1,3,4,5,6}.

Vertex 4 now has four red edges and so 4 12 is blue. This requires 7 11 to be blue by C(4,10,7,11,5,2) and 7 12 to be blue by C(7,12,1,2,4,6).

We now deduce that 9 12 is red to avoid the blue K_5 {2,7,9,11,12}. But this is impossible by C(4,2,1,12,9,8). This completes the proof of Case 2 and hence $s(3,5) \leq 12$.

It is easy to verify that the 11–vertex graph depicted in Figure 2 has $\mathrm{IR} = 4$ and $\overline{\mathrm{IR}} = 2$, hence $s(3,5) > 11$. ■

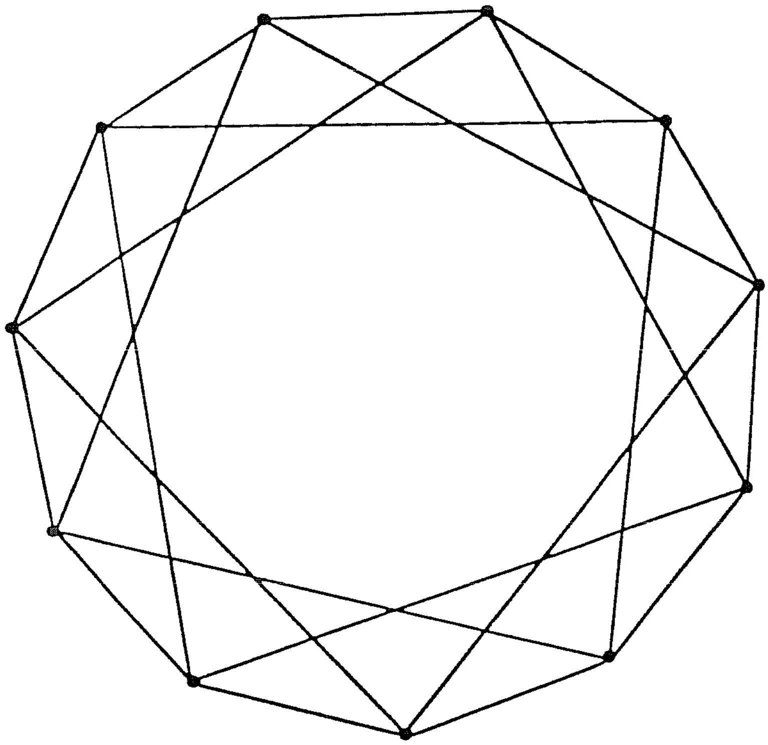

Figure 2. An 11–vertex graph with $\mathrm{IR} = 4$, $\overline{\mathrm{IR}} = 2$.

3. Further research

Similar methods could possibly be used to evaluate exactly or bound $s(m,n)$ for other values of m and n. We suggest that exact values can be found for those m and n for which $r(m,n)$ is known. Perhaps computational techniques similar to those of [3] will be useful.

Ramsey numbers are defined for more colours than two. The only non–trivial three colour classical Ramsey number known is $r(3,3,3) = 17$. What is the value of the obvious irredundant analogue?

Finally, irredundance in hypergraphs whose edges are k–subsets of the vertex sets, and associated irredundant Ramsey numbers (i.e. analogues of the classical Ramsey numbers for such hypergraphs) could prove to be fruitful areas for further investigations.

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