A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics

Srivastava, H. M.

2021

This article was originally published at:
https://doi.org/10.3390/sym13122294

symmetry

MDPI

Review

A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics

Hari Mohan Srivastava [1,2,3,4]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
[4] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

Abstract: Often referred to as special functions or mathematical functions, the origin of many members of the remarkably vast family of higher transcendental functions can be traced back to such widespread areas as (for example) mathematical physics, analytic number theory and applied mathematical sciences. Here, in this survey-cum-expository review article, we aim at presenting a brief introductory overview and survey of some of the recent developments in the theory of several extensively studied higher transcendental functions and their potential applications. For further reading and researching by those who are interested in pursuing this subject, we have chosen to provide references to various useful monographs and textbooks on the theory and applications of higher transcendental functions. Some operators of fractional calculus, which are associated with higher transcendental functions, together with their applications, have also been considered. Many of the higher transcendental functions, especially those of the hypergeometric type, which we have investigated in this survey-cum-expository review article, are known to display a kind of symmetry in the sense that they remain invariant when the order of the numerator parameters or when the order of the denominator parameters is arbitrarily changed.

Keywords: gamma; digamma and polygamma functions; hypergeometric functions and their generalizations and multivariate extensions; Riemann and Hurwitz (or generalized) Zeta functions; Lerch's transcendent and the Hurwitz-Lerch Zeta function; Mittag-Leffler type functions; Fox-Wright hypergeometric function; Riemann-Liouville and related fractional derivative operators; Liouville-Caputo fractional derivative operator; Fox-Wright hypergeometric function; generalized Fox-Wright function; operators of fractional calculus; quantum or basic (or $\mathfrak{q}$-) analysis; fractional-order quantum or basic (or $\mathfrak{q}$-) analysis

MSC: Primary 11M35; 26A33; 33C20; 33C45; 33E12; Secondary 11M06; 30C50; 33B15; 33C60; 33C65; 33C90; 44A10

Citation: Srivastava, H.M. A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics. *Symmetry* **2021**, *13*, 2294. https://doi.org/10.3390/sym13122294

Academic Editors: Ioan Raşa, Michel Planat and Sergei D. Odintsov

Received: 23 October 2021
Accepted: 22 November 2021
Published: 2 December 2021

1. Introduction and Motivation

Throughout this survey-cum-expository review article, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Additionally, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

By far the most useful and the most fundamental special function of mathematical analysis and applied mathematics happens to be the familiar (Euler's) Gamma function $\Gamma(z)$. Essentially, it stemmed from an attempt by Leonhard Euler to render a meaning to $x!$ when x is any positive real number. In the year 1729, Euler undertook the problem of interpolating $n!$ between the positive integer values of n. It led to what is widely known as the (Euler's) Gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\, dt & \big(\Re(z) > 0\big) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1} (z+j)} & \big(z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}\big). \end{cases} \tag{1}$$

Historically, of course, the origin of the Gamma function $\Gamma(z)$ defined by (1) can be traced back to two letters from Leonhard Euler (1707–1783) to Christian Goldbach (1690–1764), elaborating upon a simple desire to extend factorials to values between the integers. It is regrettable to see that, in several recent amateurish-type publications, some authors have trivially changed the variable t of integration in the integral definition in (1) and have thereby claimed to have produced a "generalization" of this classical Gamma function $\Gamma(z)$ (see, for details, [1], Section 3, pp. 1505–1506).

Among the numerous special functions, which are related rather closely to the Gamma function, we choose to mention the incomplete Gamma functions, the Beta and the incomplete Beta functions, the Error functions and the Probability integral, the Digamma and Polygamma functions, and so on.

Our main objective, in this survey-cum-expository review article, is to present a brief introductory overview and survey of some of the recent developments in the theory of a large variety of extensively-studied higher transcendental functions and their potential applications. The detailed plan of this review is as follows. In the next section (Section 2), we present a discussion of the hypergeometric type functions and their multivariate extensions and generalizations. In Section 3, we describe the Polygamma and related functions of analytic number theory. Section 4 is concerned with the Mittag-Leffler type functions and their applications to the modeling and analysis of applied problems by means of fractional calculus are presented in Section 5. Finally, in the concluding section (Section 6), we present a number of further remarks and observations.

We remark in passing that we have made this review article as much complete, comprehensive and self-contained as possible. Of course, wherever applicable, we have chosen to clearly and fully give due credits to the authors of the earlier publications and also included them in the list of references. This has, naturally, resulted in some necessary overlaps with earlier, but clearly and explicitly cited, publications.

2. The Hypergeometric Functions and Their Extensions and Multivariate Generalizations

We begin this section by recalling the following second-order homogeneous linear ordinary differential equation (popularly known as the *Gauss hypergeometric equation*):

$$z(1-z)\,\frac{d^2w}{dz^2} + [c - (a+b+1)\,z]\,\frac{dw}{dz} - abw = 0, \tag{2}$$

with three singularities at $z = 0, 1, \infty$, each of which is a regular singular point of (2). In fact, it happens to be the most celebrated differential equation of the *Fuchsian class* consisting of differential equations whose only singularities (including the point at ∞) are regular singular points. The importance of the differential Equation (2) arises from the following known observation in the theory of differential equations.

Every homogeneous linear differential equation of the second order, whose singularities (including the point at infinity) are regular and at most three in number, can be transformed into the hypergeometric Equation (2).

The general solution of the hypergeometric Equation (2), which is valid in a neighborhood of the origin ($z = 0$), can be expressed in terms of the Gauss hypergeometric function ${}_2F_1$, which is named after the famous German mathematician, Carl Friedrich Gauss (1777–1855), who introduced this function in the year 1812 and gave the F-notation for it. In terms of the general Pochhammer symbol or the *shifted factorial* $(\lambda)_\nu$, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the Gamma function in (1), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{3}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists, the Gauss hypergeometric function ${}_2F_1$ is defined by

$$\begin{aligned} {}_2F_1(a,b;c;z) &:= 1 + \frac{ab}{1\cdot c}\, z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}\, z^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\, \frac{z^n}{n!} \qquad (a,b \in \mathbb{C};\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{4}$$

or by its analytic continuation for all $z \in \mathbb{C} \setminus [1,\infty)$. The function ${}_2F_1$ does indeed provide the fundamental solution of the hypergeometric differential Equation (2) in the neighborhood of the singular point $z = 0$.

In the general case, which is usually credited to *Bishop* Ernest William Barnes (1874-1953) of the Church of England in Birmingham, who used the notation ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ analogously in the year 1907 for a generalized hypergeometric function, with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, p)$ and q denominator parameters $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \cdots, q)$, given by

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ z\right] &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\, \frac{z^n}{n!} \\ &=: {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z), \end{aligned} \tag{5}$$

in which the infinite series

(i) converges absolutely for $|z| < \infty$ if $p \leqq q$,
(ii) converges absolutely for $|z| < 1$ if $p = q + 1$, and
(iii) diverges for all z $(z \neq 0)$ if $p > q + 1$.

Furthermore, if we set

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

then it is known that the generalized hypergeometric ${}_pF_q$ series in (5) (with $p = q + 1$) is

I. absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$,
II. conditionally convergent for $|z| = 1$ $(z \neq 1)$ if $-1 < \Re(\omega) \leqq 0$, and
III. divergent for $|z| = 1$ if $\Re(\omega) \leqq -1$.

Remarkably, from the definition (5) of the generalized hypergeometric function ${}_pF_q$, it is clearly seen that the sets of the p numerator parameters $a_1, \cdots, a_p$ and the q denominator parameters $b_1, \cdots, b_q,$ are symmetric individually. Such potentially useful facts apply equally well to some of the further generalizations and extensions of the generalized hypergeometric function ${}_pF_q,$ which is defined above by (5).

In the particular case when $p - 1 = q = 2$, the function ${}_3F_2$ is known as the Clausen hypergeometric function in honor of the Danish mathematician and astronomer, Thomas Clausen (1801–1885) who, in the year 1828, established the following hypergeometric identity:

$$\left({}_2F_1 \left[\begin{array}{r} a, b; \\ a+b+\frac{1}{2}; \end{array} z \right] \right)^2 = {}_3F_2 \left[\begin{array}{r} 2a, a+b, 2b; \\ a+b+\frac{1}{2}, 2(a+b); \end{array} z \right]. \tag{6}$$

For $a, b, c, z \in \mathbb{R}$, the positivity of the function ${}_3F_2$ on the right-hand side of Clausen's identity (6) was instrumental in de Branges' proof of the 68-year-old Bieberbach conjecture that

$$|a_n| \leqq n \qquad (n \in \mathbb{N} \setminus \{1\})$$

for functions f in the normalized analytic and univalent function class $\mathcal{S}$ given by

$$\mathcal{S} := \left\{ f : f \in \mathcal{A}, \;\; f(z) = z + \sum_{n=0}^{\infty} a_n z^n \;\; (z \in \mathbb{U}) \;\text{ and }\; f \text{ is univalent in } \mathbb{U} \right\},$$

where $\mathcal{A}$ denotes the class of normalized functions which are univalent in the open unit disk $\mathbb{U}$ (see, for details, [2]).

Remarkably, all of such widely used special functions of mathematical physics and applied mathematics as (for example) the Bessel, Legendre and Lommel functions, Whittaker functions, Elliptic integrals, and so on, are expressible in terms of the generalized hypergeometric function ${}_pF_q$. Moreover, the Jacobi, Laguerre, Hermite and other associated orthogonal and non-orthogonal polynomials (such as, for example, the Bessel polynomials) are essentially hypergeometric functions ${}_pF_q \;\; (p, q \in \mathbb{N}_0)$ in which one or more of the numerator parameters $\alpha_j \;\; (j = 1, \cdots, p)$ is a negative integer. In particular, the special orthogonal polynomials $\{\mathcal{S}_n(x)\}_{n \in \mathbb{N}_0}$, which are related rather closely to the Bessel polynomials as well as the Laguerre polynomials, occurred in the investigation of energy spectral functions for a certain family of isotropic turbulence fields (see, for details, [3] and the references cited therein). Quite recently in [4], a modified form of these special orthogonal polynomials $\{\mathcal{S}_n(x)\}_{n \in \mathbb{N}_0}$ has provided a novel set of (orthogonal) basis functions along with some suitable collocation points in a certain matrix technique for computationally treating a class of multi-order fractional pantograph differential equations by using matrix techniques. Thus, clearly, the potential for the usefulness of the generalized hypergeometric function ${}_pF_q \;\; (p, q \in \mathbb{N}_0)$ and its considerably wide variety of special cases cannot be overemphasized. Some of the books, monographs and tables along these lines include those by Abramowitz and Stegun [5], Andrews [6], Andrews et al. [7], Carlson [8], (Erdélyi et al. [9], Volumes I and II), Rainville [10], Lebedev [11], Luke (see [12,13]), Magnus et al. [14], Miller [15], Srivastava et al. (see [16,17]), Temme [18], Szegö [19], and others.

Various extensions and multivariate generalizations of hypergeometric functions have stemmed from, and are somewhat motivated by, the potential for their applications in solutions of systems of partial differential equations. Some of the multivariate generalizations of hypergeometric functions include the two-variable Appell and Kampé de Fériet functions and the Lauricella functions in several variables, the Srivastava-Daoust hypergeometric functions in two and more variables, and so on. For the theory and applications of many of these univariate and multivariate hypergeometric functions, the interested reader is referred to the monographs by (for example) Appell and Kampé de Fériet [20], Bailey [21], (Erdélyi et al. [9], Vol. I), Slater [22], Seaborn [23], Srivastava et al. (see, for details, [16]), and other authors. In particular, for the theory and applications of the substantially more general G and H functions in one, two and more variables, the reader is referred to the monographs by Mathai et al. [24] and Srivastava et al. (see [17,25]), and others. It should be remarked in passing that a nontrivial *and* non-vacuous generalization of the H-function was encountered in a systematic study of a class of Feynman integrals, the $\overline{H}$-function was introduced and investigated rather widely and extensively (see, for details, [26–30]).

We choose to conclude this section by recalling the following notable remark which was once made by Gian-Carlo Rota (1932–1999):

> *The families of univariate and multivariate hypergeometric functions are capable of encompassing just about everything in sight.*

3. The Polygamma and Related Functions of Analytic Function Theory

The *Polygamma functions* $\psi^{(n)}(z)$ $(n \in \mathbb{N}_0)$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\{\log \Gamma(z)\} = \frac{d^n}{dz^n}\{\psi(z)\} \qquad (n \in \mathbb{N}_0;\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{7}$$

of which $\psi^{(0)}(z) = \psi(z)$ is called the Digamma function. In terms of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, which is defined by

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \qquad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \tag{8}$$

and by its meromorphic continuation to the whole complex s-plane except for a simple pole at $s = 1$ with the residue 1, it is easily seen that

$$\psi^{(n)}(z) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} = (-1)^{n+1}\, n!\, \zeta(n+1, z) \tag{9}$$

$$(n \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

which may be used to deduce the properties of the Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ from those of the Hurwitz (or generalized) Zeta function $\zeta(s,z)$ $(s = n+1;\ n \in \mathbb{N})$. The special case of the Hurwitz (or generalized) Zeta function $\zeta(s,z)$ when $z = 1$ is the familiar Riemann Zeta function $\zeta(s)$, which is involved in the following Taylor-Maclaurin series expansion of $\log \Gamma(1+z)$ about the origin $z = 0$:

$$\log \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n\, \zeta(n)\, \frac{z^n}{n} \qquad (|z| < 1), \tag{10}$$

where γ denotes the Euler-Mascheroni constant. The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \left(\Re(s) > 1\right) \\[2ex] \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \left(\Re(s) > 0;\ s \neq 1\right) \end{cases} \tag{11}$$

and by its meromorphic continuation to the whole complex s-plane except for a simple pole at $s = 1$ with the residue 1.

Just as in its obvious special cases $\zeta(s)$ and $\zeta(s,a)$, the general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{12}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right),$$

can be continued meromorphically to the whole complex s-plane except for a simple pole at $s = 1$ with the residue 1. In fact, the general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (12) contains, as its *special* cases, not only the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ and the Lerch Zeta function $\ell_s(\xi)$ defined by

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\, \Phi\left(e^{2\pi i\xi}, s, 1\right) \qquad \left(\xi \in \mathbb{R};\ \Re(s) > 1\right), \tag{13}$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\, \Phi(z,s,1) \tag{14}$$

$$\left(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right)$$

and the Lipschitz-Lerch Zeta function $\phi(\xi,a,s)$:

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) =: L(\xi,s,a) \tag{15}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\right),$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see, for details, [31]).

For various further extensions of the Hurwitz-Lerch Zeta function $\Phi(s,z,a)$, including its *multi-parameter* extensions, the reader is referred to the recent monograph by Srivastava and Choi [32] (see also the other more recent developments reported in [33,34]). In the monograph [32], one can also find a detailed systematic presentation of the *double*, *triple* and *multiple* Gamma functions which were studied by Barnes and others (see also (Whittaker and Watson [35], p. 264) and (Gradshteyn and Ryzhik [36], p. 661, Entry 6.441(4); p. 937, Entry 8.333)).

An interesting and potentially useful family of the λ-generalized Hurwitz-Lerch Zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa)$$

defined by ([37], p. 503, Equation (6.2)) (see also [32])

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+\kappa)^s} \tag{16}$$

$$\Bigg(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1,\cdots,p);\ \kappa,\mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1,\cdots,q);$$

$$\rho_j,\sigma_k \in \mathbb{R}^+\ (j = 1,\cdots,p;\ k = 1,\cdots,q); \Delta > -1 \ \text{when}\ s,z \in \mathbb{C};$$

$$\Delta = -1 \ \text{and}\ s \in \mathbb{C} \ \text{when}\ |z| < \nabla^*;$$

$$\Delta = -1 \ \text{and}\ \Re(\Xi) > \frac{1}{2} \ \text{when}\ |z| = \nabla^*\Bigg),$$

where, for convenience,

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2} \tag{17}$$

and

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right), \tag{18}$$

It was introduced and investigated systematically in a recent paper by Srivastava [38], who also discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and, in addition, considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch Zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [38]). For the convenience of the interested reader in pursuing some of the related open problems, we choose to reproduce here the definition of the λ-generalized Hurwitz-Lerch Zeta function whose investigation was initiated by Srivastava [38]:

$$\begin{aligned} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp\left(-at - \frac{b}{t^\lambda}\right) \\ &\cdot {}_p\Psi_q^* \left[\begin{matrix} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{matrix} \; ze^{-t} \right] \mathrm{d}t \end{aligned} \tag{19}$$

$$\big(\min\{\Re(a),\Re(s)\} > 0;\ \Re(b) \geqq 0;\ \lambda \geqq 0\big),$$

so that, obviously, we have the following relationship with the multi-parameter Hurwitz-Lerch Zeta function defined by (16):

$$\begin{aligned} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;0,\lambda) &= \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) \\ &= e^b\, \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,0). \end{aligned} \tag{20}$$

4. The Mittag-Leffler Type Functions: Extensions and Generalizations

We begin this section by recalling the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$, which are defined, respectively, by (see [39–41])

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \qquad \text{and} \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{21}$$

$$\big(z,\alpha,\beta \in \mathbb{C};\ \Re(\alpha) > 0\big).$$

The one-parameter function $E_\alpha(z)$ was first considered by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1903 and its two-parameter version $E_{\alpha,\beta}(z)$ was introduced by Anders Wiman (1865–1959) in 1905 (see also [42]).

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are *natural* extensions of the exponential, hyperbolic and trigonometric functions. In fact, it is easily observed that

$$E_1(z) = e^z, \quad E_2\left(z^2\right) = \cosh z, \quad E_2\left(-z^2\right) = \cos z,$$

$$E_{1,2}(z) = \frac{e^z - 1}{z} \qquad \text{and} \qquad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

Some of the potentially useful properties, extensions and applications of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ were presented by Gorenflo et al. [43], Haubold et al. [44]

and Kilbas et al. ([45,46] and ([47], Chapter 1)). The Mittag-Leffler function $E_\alpha(z)$ in (21) and its several generalized forms were computed numerically in the recent works in [48,49].

It should be remarked in passing that the families of Mittag-Leffler type functions have the demonstrated potential for application in a wide range of problems in applied as well as engineering sciences. Moreover, their several extended and generalized forms, including those involving multiple indices, are known to be needed in solving fractional differential and integro-differential equations (see, for example, [50]; see also [51–53], as well as the references to the related earlier works which are cited therein). For a detailed presentation of some of these recent developments, one is referred to a recent survey-cum-expository review article [54].

Most (if not all) of the above-mentioned multi-parameter generalizations and extensions of the Mittag-Leffler function $E_\alpha(z)$ in (21) are contained, as obvious special cases, of the general Fox-Wright function ${}_p\Psi_q \;\; (p,q \in \mathbb{N}_0)$ or ${}_p\Psi_q^* \;\; (p,q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C} \quad (j = 1, \cdots, p) \qquad \text{and} \qquad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q).$$

These general Fox-Wright functions ${}_p\Psi_q$ and ${}_p\Psi_q^*$ are defined by (see, for details, ([9], Volume I, p. 183) and ([16], p. 21); see also ([47], p. 56), ([55], p. 65) and ([25], p. 19))

$$\begin{aligned} {}_p\Psi_q^*\left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z\right] &:= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \\ &=: \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \; {}_p\Psi_q\left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z\right] \end{aligned} \tag{22}$$

$$\left(\Re(A_j) > 0 \; (j = 1, \cdots, p); \; \Re(B_j) > 0 \; (j = 1, \cdots, q); \; \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geqq -1\right),$$

where, and in what follows, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial defined already by (3), and the equality in the convergence condition holds true only for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j}\right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

Clearly, the generalized hypergeometric function ${}_pF_q \;\; (p,q \in \mathbb{N}_0)$ in (23), with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, is a widely and extensively investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q \;\; (p,q \in \mathbb{N}_0)$ when

$$A_j = 1 \quad (j = 1, \cdots, p) \qquad \text{and} \qquad B_j = 1 \quad (j = 1, \cdots, q),$$

given by

$$
\begin{aligned}
{}_pF_q\left[\begin{array}{c} a_1,\cdots,a_p; \\ \\ b_1,\cdots,b_q; \end{array} z\right] &:= \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{z^n}{n!} \\
&= {}_p\Psi_q^*\left[\begin{array}{c} (a_1,1),\cdots,(a_p,1); \\ \\ (b_1,1),\cdots,(b_q,1); \end{array} z\right] \\
&= \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\;{}_p\Psi_q\left[\begin{array}{c} (a_1,1),\cdots,(a_p,1); \\ \\ (b_1,1),\cdots,(b_q,1); \end{array} z\right].
\end{aligned} \tag{23}
$$

Just as we have already remarked above, various families of special functions of the Mittag-Leffler and Fox-Wright types are known to play important roles in the theory of fractional calculus and operational calculus as well as in their applications in the basic processes of evolution, relaxation, diffusion, oscillation and wave propagation. Furthermore, the Mittag-Leffler type functions have only recently been calculated numerically in the whole complex plane (see, for example, [48,49]; see also [5,56]). Some other general families of Mittag-Leffler type functions have been investigated and applied recently (see, for details, [54] and the references to earlier works cited therein).

It was my proud privilege to have met many times and discussed mathematical researches, especially on various families of higher transcendental functions and related topics, with my Canadian colleague, Charles Fox (1897–1977) of birth and education in England, both at McGill University and Sir George Williams University (*now* Concordia University) in Montréal, mainly during the 1970s (see, for details, [57]). Another remarkable mathematical scientist of modern times happens to be *Sir* Edward Maitland Wright (1906–2005), with whom I had the privilege to meet and discuss research emerging from his publications on hypergeometric and related higher transcendental functions during my visit to the University of Aberdeen in Scotland in the year 1976. We recall here a series of monumental works by Wright (see, for example, [58–60]), in which he introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [58], p. 424):

$$
\mathfrak{E}_{\alpha,\beta}(\phi;z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n+\beta)}\,z^n \qquad (\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0), \tag{24}
$$

where $\phi(t)$ is a function satisfying suitable conditions. Wright's work was motivated substantially by the earlier developments involving simpler cases, which were reported by Mittag-Leffler in 1905, Wiman in 1905, Ernest William Barnes (1874–1953) in 1906, Godfrey Harold Hardy (1877–1947) in 1905, George Neville Watson (1886–1965) in 1913, Fox in 1928, and many other authors. In particular, the aforementioned work [61] by *Bishop* Ernest William Barnes (1874–1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class defined below:

$$
E_{\alpha,\beta}^{(\kappa)}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\kappa)^s\,\Gamma(\alpha n+\beta)} \qquad (\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0) \tag{25}
$$

for suitably restricted parameters κ and s. It is easy to deduce, from the definition (25), the following relationships with the Mittag-Leffler type function $E_{\alpha,\beta}^{(\kappa)}(s;z)$ of Barnes [61]:

$$
E_\alpha(z) = \lim_{s\to 0}\left\{E_{\alpha,1}^{(\kappa)}(s;z)\right\} \qquad \text{and} \qquad E_{\alpha,\beta}(z) = \lim_{s\to 0}\left\{E_{\alpha,\beta}^{(\kappa)}(s;z)\right\}. \tag{26}
$$

More interestingly, we also have the following relationship:

$$\lim_{\alpha\to 0}\left\{E^{(\kappa)}_{\alpha,\beta}(s;z)\right\} = \frac{1}{\Gamma(\beta)}\ \Phi(z,s,\kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch Zeta function) $\Phi(z,s,\kappa)$ defined by (12).

As we have indicated above, Barnes [61] systematically presented asymptotic expansions of many functions such as those in the class of the Mittag-Leffler type function $E^{(\kappa)}_{\alpha,\beta}(s;z)$ defined by (25), and the classical Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ defined by (21). On the other hand, the multi-parameter Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa),$$

defined by (16), obviously provides a natural unification and generalization of the Fox-Wright function ${}_p\Psi^*_q$ defined by (22) as well as the Hurwitz-Lerch Zeta function $\Phi(z,s,\kappa)$ defined by (12).

We conclude this section by recalling the following interesting unification of the definitions in (24) and (16) for suitably restricted function $\varphi(\tau)$ (see, for details, [1,54]):

$$\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+\kappa)^s\ \Gamma(\alpha n+\beta)}\ z^n \qquad \big(\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\big), \tag{27}$$

where the parameters α, β, s and κ are appropriately constrained as above. If, upon replacing the sequence $\{\varphi(n)\}_{n=0}^{\infty}$ in the definition (27) by the sequence $\{\phi(n)\}_{n=0}^{\infty}$, we readily observe that

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) = \lim_{s\to 0}\{\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa)\}\big|_{\varphi\equiv\phi}. \tag{28}$$

Moreover, if we put $\alpha=\beta=1$ and

$$\varphi(n) = \frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}} \qquad (n\in\mathbb{N}_0) \tag{29}$$

in the definition (27), then the definition (27) will immediately yield the definition (16) of the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa).$$

Alternatively, in the special case of (27) when $\alpha\to 0$, $\beta=1$ and

$$\varphi(n) = \frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\cdot\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}} \qquad (n\in\mathbb{N}_0) \tag{30}$$

or, more simply, by setting

$$\varphi(n) = \frac{\Gamma(\alpha n+\beta)\ \prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\cdot\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}} \qquad (n\in\mathbb{N}_0), \tag{31}$$

we are led to the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,\kappa)$$

defined by (16).

5. Operators of Fractional Calculus and Their Applications

The current literature is loaded by fractional-order modeling and analysis of real-world and other problems which arise in the mathematical, physical, biological, statistical and engineering sciences. It is truly amazing to see how fast the subject of fractional calculus has grown in recent years, the concept of which is rooted essentially in a question raised in the year 1695 by Marquis de l'Hôpital (1661–1704) to Gottfried Wilhelm Leibniz (1646–1716), which sought the meaning of Leibniz's (currently popular) notation

$$\frac{d^n y}{dx^n}$$

for the derivative of order $n \in \mathbb{N}_0$ when $n = \frac{1}{2}$, and indeed also in Leibniz's reply dated 30 September 1695 to l'Hôpital as follows:

> "$\cdots$ *This is an apparent paradox from which, one day, useful consequences will be drawn.* $\cdots$"

In widespread applications of fractional calculus, use is made of fractional-order derivatives of different (and, occasionally, ad hoc) kinds (see, for example, [62–74]). It is fairly traditional to define the fractional-order integrals and fractional-order derivatives by means of the following right-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a+}$ and the left-sided Riemann-Liouville fractional integral operator ${}^{\mathrm{RL}}I^{\mu}_{a-}$, and the corresponding Riemann-Liouville fractional derivative operators ${}^{\mathrm{RL}}D^{\mu}_{a+}$ and ${}^{\mathrm{RL}}D^{\mu}_{a-}$, as follows (see, for example, ([9], Volume II, Chapter 13), ([47], pp. 69–70) and [75]):

$$\left({}^{\mathrm{RL}}I^{\mu}_{a+}f\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\, dt \qquad \big(x > a;\ \Re(\mu) > 0\big), \tag{32}$$

$$\left({}^{\mathrm{RL}}I^{\mu}_{a-}f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^a (t-x)^{\mu-1} f(t)\, dt \qquad \big(x < a;\ \Re(\mu) > 0\big) \tag{33}$$

and

$$\left({}^{\mathrm{RL}}D^{\mu}_{a\pm}f\right)(x) = \left(\pm\frac{d}{dx}\right)^n \left(I^{n-\mu}_{a\pm}f\right)(x) \qquad \big(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\big), \tag{34}$$

where the function f is assumed to be at least locally integrable and $[\Re(\mu)]$ indicates the greatest integer in $\Re(\mu)$. It is worthwhile to remark in passing that, in the current literature, there is an unfortunate trend to trivially and inconsequentially translate known theory and known results based upon the classical Riemann-Liouville and other types of fractional integrals and fractional derivatives in terms of the corresponding theory and the corresponding results by forcing in some obviously redundant (or superfluous) parameters and variables in the widely and extensively investigated definitions and results (see, for details, [1], Section 2, pp. 1504–1505).

In a series of papers, Hilfer et al. (see [62–64]; see also [66,67,70]) introduced and studied an interesting family of generalized Riemann-Liouville fractional derivatives of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \nu \leqq 1)$. The right-sided Hilfer fractional derivative operator ${}^{\mathrm{H}}D^{\mu,\nu}_{a+}$ and the left-sided Hilfer fractional derivative operator ${}^{\mathrm{H}}D^{\alpha,\beta}_{a-}$ of order μ $(0 < \mu < 1)$ and type ν $(0 \leqq \beta \leqq 1)$ with respect to x are defined by

$$\left({}^{\mathrm{H}}D^{\mu,\nu}_{a\pm}f\right)(x) = \left(\pm\, {}^{\mathrm{H}}I^{\nu(1-\mu)}_{a\pm} \frac{d}{dx}\left({}^{\mathrm{H}}I^{(1-\nu)(1-\mu)}_{a\pm} f\right)\right)(x), \tag{35}$$

in which the second member is tacitly assumed to exist.

For $\nu = 0$, (35) reduces to the familiar Riemann-Liouville fractional derivative operator. When $\nu = 1$, (35) yields the fractional derivative operator that was introduced by (Liouville [76], p. 10) which, quite frequently, is attributed obviously incorrectly to Caputo [77] now-a-days, but which should more appropriately be called the *Liouville-Caputo fractional derivative*, giving due credit to the originator, Joseph Liouville (1809–1882), who considered such fractional derivatives many decades earlier in 1832 (see [76]). The general operators in (35) are referred to as the Hilfer fractional derivative operators (see, for example, [68]). The Hilfer fractional derivative operator $D_{a\pm}^{\alpha,\beta}$ was applied in [64] (see also [50,78]).

The Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (22), and also such more general functions as Meijer's G-function and Fox's H-function, were used as kernels of many different classes of operators of fractional calculus (see, for details, [25,72,79], as well as the references cited therein). It should be noted here that Srivastava et al. [72] used fractional integrals of the Riemann-Liouville type with kernels involving the Fox H-function and the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$. Moreover, they also considered applications of their results to the general $\overline{H}$-function (see, for example, [29]).

The Wright function $\mathfrak{E}_{\alpha,\beta}(\varphi; z)$ in (24), which was introduced and studied in [58] as long ago as 1940, has appeared recently in [80] in connection with fractional calculus, but without giving due credit to Wright [58]. Closely following the recent works [1,54], the general right-sided fractional integral operator $\mathcal{I}_{a+}^{\mu}(\varphi; s, \kappa)$ and the general left-sided fractional integral operator $\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, and the corresponding fractional derivative operators $\mathcal{D}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu)$ and $\mathcal{D}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu)$, each of the Riemann-Liouville type, are defined by

$$\left(\mathcal{I}_{a+}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(x-t)^{\nu}, s, \kappa\big)\, f(t)\, dt \tag{36}$$

$$\big(x > a;\ \Re(\mu) > 0\big),$$

$$\left(\mathcal{I}_{a-}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^a (t-x)^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; z(t-x)^{\nu}, s, \kappa\big)\, f(t)\, dt \tag{37}$$

$$\big(x < a;\ \Re(\mu) > 0\big)$$

and

$$\left(\mathcal{D}_{a\pm}^{\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) = \left(\pm\frac{d}{dx}\right)^n \left(\mathcal{I}_{a\pm}^{n-\mu}(\varphi; z, s, \kappa, \nu) f\right)(x) \tag{38}$$

$$\big(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\big),$$

where f is in the space $L(\mathfrak{a}, \mathfrak{b})$ of Lebesgue integrable functions on an interval $[\mathfrak{a}, \mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of $\mathbb{R}$, which is given explicitly by

$$L(\mathfrak{a}, \mathfrak{b}) = \left\{ f:\ \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)| dx < \infty \right\}. \tag{39}$$

For potential applications based upon the general fractional-calculus operators defined by the Equations (36) to (38), we list below several useful properties of the kernel $\mathcal{E}_{\alpha,\beta}(\varphi; zx^{\nu}, s, \kappa)$ involved therein.

$$\begin{aligned} &\frac{d^n}{dx^n}\left\{ x^{\mu-1}\, \mathcal{E}_{\alpha,\beta}\big(\varphi; zx^{\nu}, s, \kappa\big) \right\} \\ &\qquad = x^{\mu-n-1} \sum_{j=0}^{\infty} \frac{\varphi(j)}{(j+\kappa)^s\, \Gamma(\alpha j + \beta)} \frac{\Gamma(\nu j + \mu)}{\Gamma(\nu j + \mu - n)} \left(zx^{\nu}\right)^j \end{aligned} \tag{40}$$

$$\big(n \in \mathbb{N}_0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

which, in the special case when $\mu = \beta$ and $\nu = \alpha$, yields

$$\frac{d^n}{dx^n}\Big\{x^{\beta-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;zx^{\alpha},s,\kappa\big)\Big\} = x^{\beta-n-1}\,\mathcal{E}_{\alpha,\beta-n}\big(\varphi;zx^{\alpha},s,\kappa\big) \tag{41}$$

$$\big(n \in \mathbb{N}_0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big),$$

provided that each member of the Equations (40) and (41) exists.

$$\begin{aligned}
&\mathfrak{I}\Big\{t^{\mu-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;zt^{\nu},s,\kappa\big)\Big\}(x)\\
&\quad := \int_0^x t^{\mu-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;zt^{\nu},s,\kappa\big)\,dt\\
&\quad = x^{\mu}\sum_{j=0}^{\infty}\frac{\varphi(j)}{(j+\kappa)^s\,\Gamma(\alpha j+\beta)}\,\frac{\Gamma(\nu j+\mu)}{\Gamma(\nu j+\mu+1)}\,\big(zx^{\nu}\big)^j
\end{aligned} \tag{42}$$

$$\big(\Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

provided that the integral exists. More generally, we have

$$\begin{aligned}
&\mathfrak{I}^n\Big\{t^{\mu-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;zt^{\nu},s,\kappa\big)\Big\}(x)\\
&\quad = x^{\mu+n-1}\sum_{j=0}^{\infty}\frac{\varphi(j)}{(j+\kappa)^s\,\Gamma(\alpha j+\beta)}\,\frac{\Gamma(\nu j+\mu+n-1)}{\Gamma(\nu j+\mu+n)}\,\big(zx^{\nu}\big)^j
\end{aligned} \tag{43}$$

$$\big(n \in \mathbb{N};\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

which, in the special case when $\mu = \beta$ and $\nu = \alpha$, yields

$$\mathfrak{I}^n\Big\{t^{\beta-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;zt^{\alpha},s,\kappa\big)\Big\}(x) = x^{\beta+n-1}\,\mathcal{E}_{\alpha,\beta+n}\big(\varphi;zx^{\alpha},s,\kappa\big) \tag{44}$$

$$\big(n \in \mathbb{N};\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big),$$

provided that each member of the Equations (43) and (44) exists.

For the operator $\mathcal{L}$ of the Laplace transform given by

$$\mathcal{L}\{f(\tau):\mathfrak{s}\} := \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\tau}\,f(\tau)\,d\tau =: F(\mathfrak{s}) \qquad \big(\Re(\mathfrak{s}) > 0\big), \tag{45}$$

where the function $f(\tau)$ is so constrained that the integral exists, we get

$$\mathcal{L}\Big\{\tau^{\mu-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;z\tau^{\nu},s,\kappa\big):\mathfrak{s}\Big\} = \frac{1}{\mathfrak{s}^{\mu}}\sum_{j=0}^{\infty}\frac{\varphi(j)\,\Gamma(\nu j+\mu)}{(j+\kappa)^s\,\Gamma(\alpha j+\beta)}\left(\frac{z}{\mathfrak{s}^{\nu}}\right)^j \tag{46}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\mu) > 0;\ \Re(\nu) > 0;\ \Re(\alpha) > 0\big),$$

provided that each member of (46) exists. For $\mu = \beta$ and $\nu = \alpha$, the Laplace transform Formula (46) assumes the following simpler form:

$$\mathcal{L}\Big\{\tau^{\beta-1}\,\mathcal{E}_{\alpha,\beta}\big(\varphi;z\tau^{\alpha},s,\kappa\big):\mathfrak{s}\Big\} = \frac{1}{\mathfrak{s}^{\mu}}\sum_{j=0}^{\infty}\frac{\varphi(j)}{(j+\kappa)^s}\left(\frac{z}{\mathfrak{s}^{\alpha}}\right)^j \tag{47}$$

$$\big(\Re(\mathfrak{s}) > 0;\ \Re(\alpha) > 0;\ \Re(\beta) > 0\big).$$

Unfortunately, by trivially changing, in the definition of the classical Laplace transform (45), the index $\mathfrak{s}$ or the integration variable t or both the index $\mathfrak{s}$ *and* the integration variable t, many mainly amateurish-type researchers have made and continue to make the obviously false claim to have "generalized" the classical Laplace transform (45) itself. Some of the examples in this connection include, but are not limited to, the so-called Sumudu transform, the so-called natural transform, the so-called Shehu transform, the so-called $\mathcal{P}_\delta$-transform, the so-called k-Laplace transform, and so on (see, for details, [1], Section 4, pp. 1508–1510).

Various special cases and consequences of the above key results, involving simpler functions of the types which we have considered in this and the preceding sections, can be deduced fairly easily. We turn instead to the following examples of applications involving fractional-order derivatives of one kind or the other. In this connection, we note the following relationship between the Riemann-Liouville fractional derivative ${}^{\mathrm{RL}}D^{\mu}_{0+}$ and the Liouville-Caputo fractional derivative ${}^{\mathrm{LC}}D^{\mu}_{0+}$ of order μ (see, for example, ([47], p. 91, Equation (2.4.1)) *with* $a = 0$):

$$\left({}^{\mathrm{LC}}D^{\mu}_{0+}f\right)(x) = \left({}^{\mathrm{RL}}D^{\mu}_{0+}\left\{f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, t^k\right\}\right)(x), \tag{48}$$

where n is given by

$$n = \begin{cases} [\Re(\mu)] + 1 & (\mu \neq \mathbb{N}_0) \\ \mu & (\mu \in \mathbb{N}_0). \end{cases} \tag{49}$$

Equivalently, since

$$\left({}^{\mathrm{RL}}D^{\mu}_{0+}\left\{t^{\lambda-1}\right\}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}\, x^{\lambda-\mu-1} \qquad \left(\Re(\lambda) > 0;\ \Re(\mu) \geqq 0\right), \tag{50}$$

the relationship (48) can be written as follows:

$$\left({}^{\mathrm{LC}}D^{\mu}_{0+}f\right)(x) = \left({}^{\mathrm{RL}}D^{\mu}_{0+}f\right)(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\mu+1)}\, x^{k-\mu}, \tag{51}$$

where n is given, as before in (48), by (49).

I. The basic processes of relaxation, diffusion, oscillations and wave propagation were revisited by Gorenflo et al. [43] who introduced the Liouville-Caputo type fractional-order derivatives in the governing (ordinary or partial) differential equations and considered each of the following fractional differential equations:

$$\frac{d^\alpha u}{dt^\alpha} + c^\alpha\, u(t;\alpha) = 0 \qquad (c > 0;\ 0 < \alpha \leqq 2) \tag{52}$$

and

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = k\, \frac{\partial^2 u}{\partial x^2} \qquad (k > 0;\ -\infty < x < \infty;\ 0 < \beta \leqq 1), \tag{53}$$

where the aforementioned Liuoville-Caputo fractional derivative of order $\mu > 0$ of a *causal* function $f(t)$, that is,

$$f(t) = 0 \qquad (t < 0),$$

is given by

$$\begin{aligned} \frac{d^\mu}{dx^\mu}\{f(x)\} &= \left({}^{\mathrm{LC}}D^{\mu}_{0+}f\right)(x) \\ &:= \begin{cases} f^{(n)}(x) & (\mu = n \in \mathbb{N}_0) \\ \dfrac{1}{\Gamma(n-\mu)} \displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}}\, dt & \left(n - 1 < \Re(\mu) < n;\ n \in \mathbb{N}\right). \end{cases} \end{aligned} \tag{54}$$

Here, as usual, n is given by (49) and $f^{(n)}(t)$ denotes the ordinary derivative of $f(t)$ of order n.

The Equation (52) represents fractional relaxation when $0 < \alpha \leqq 1$ under the initial condition $u(0+;\alpha) = u_0$. Moreover, it can be viewed as fractional oscillation when $1 < \alpha \leqq 2$ under the following initial conditions:

$$u(0+;\alpha) = u_0 \qquad \text{and} \qquad \dot{u}(0+;\alpha) = v_0,$$

where $v_0 \equiv 0$ for continuous dependence of the solution on the parameter α also in the transition from $\alpha = 1-$ to $\alpha = 1+$, and $\dot{u}$ is the time-derivative of u.

In the Equation (53), $u = u(x,t;\beta)$ is assumed to be a *causal* function of time $(t > 0)$ such that

$$u(\mp\infty, t; \beta) = 0.$$

Clearly, the Equation (53) represents fractional diffusion when $0 < \beta \leqq \frac{1}{2}$ under the initial condition $u(x,0+;\beta) = f(x)$. It can also be viewed as fractional wave equation when $\frac{1}{2} < \beta \leqq 1$ under the following initial conditions:

$$u(x,0+;\beta) = f(x) \qquad \text{and} \qquad \dot{u}(x,0+;\beta) = g(x),$$

where $g(x) \equiv 0$ for continuous dependence of the solution on the parameter β also in the transition from $\beta = \frac{1}{2}-$ to $\beta = \frac{1}{2}+$.

In terms of the Mittag-Leffler function $E_\alpha(z)$ defined by (21), the explicit solution of the initial-value problem involving the fractional differential Equation (52) is given by

$$u(t;\alpha) = u_0\, E_\alpha\big(-(ct)^\alpha\big). \tag{55}$$

On the other hand, the explicit solution of the initial-value problem involving the fractional differential Equation (53) can be expressed as follows:

$$u(x,t;\beta) = \int_{-\infty}^{\infty} \mathcal{G}_c(\xi,t;\beta)\, f(x-\xi)\, d\xi, \tag{56}$$

where $\mathcal{G}_c(x,t;\beta)$ denotes the Green function given by

$$|x|\, \mathcal{G}_c(x,t;\beta) = \frac{z}{2} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\, \Gamma(1-\beta-\beta n)} \qquad \left(z = \frac{|x|}{\sqrt{k}t^\beta};\ 0 < \beta < 1\right), \tag{57}$$

which, in turn, can easily be expressed in terms of Wright's generalized Bessel function or the Bessel-Wright function $J_\nu^\mu(z)$ defined by (see, for example, [17], p. 42, Equation II.5 (22))

$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\, \Gamma(\mu n + \nu + 1)} = {}_0\Psi_2\left[\begin{matrix} \overline{\qquad\qquad};\\ (1,1), (\nu+1,\mu); \end{matrix}\ -z\right]. \tag{58}$$

II. In recent years, various different forms of kinetic equations of fractional order have been widely used, especially in the modeling and analysis of a number of important problems of physics and astrophysics (see, for details, [81]). Particularly, in the past decade or so, kinetic equations of fractional order have apparently gained popularity, mainly because of the discovery of their relation with the theory of CTRW (Continuous-Time Random Walks) (see [66]). These equations have been and are being investigated with the aim to first determine and then interpret certain physical phenomena which are known to govern such processes as (for example) diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on (see also [63,82]).

Theorems 1 to 3 below, each of which was established in [54], are sufficiently general key results, which are capable of being appropriately and suitably specialized with a view to including solutions of the corresponding (known or new) fractional-order kinetic equations associated with a large variety of simpler functions than those involved herein. With a view to making our presentation accessible readily to the interested reader, we have chosen to essentially reproduce each of these fundamental results (Theorems 1 to 3 below) from our earlier work [54].

Theorem 1. *Let $c, \mu, \nu, \rho, \sigma \in \mathbb{R}^+$. Suppose also that the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined by (27), exists. Then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0 \, t^{\mu-1} \, \mathcal{E}_{\alpha,\beta}(\varphi; zt^{\nu}, s, \kappa) = -c^{\rho} \left({}^{\mathrm{RL}}I_{0+}^{\sigma} \, N \right)(t) \tag{59}$$

is given by

$$N(t) = N_0 \, t^{\mu-1} \sum_{r=0}^{\infty} (-c^{\rho} t^{\sigma})^r \cdot \sum_{j=0}^{\infty} \frac{\varphi(j) \, \Gamma(\nu j + \mu)}{(j+\kappa)^s \, \Gamma(\alpha j + \beta) \, \Gamma(\nu j + \sigma r + \mu)} (zt^{\nu})^j \qquad (t > 0), \tag{60}$$

provided that the right-hand side of the solution asserted by (60) exists.

Theorem 2. *Let $c, \mu, \nu, \rho, \sigma \in \mathbb{R}^+$. Suppose also that the general function $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, defined by (24), exists. Then the solution of the following generalized fractional kinetic equation*:

$$N(t) - N_0 \, t^{\mu-1} \, \mathfrak{E}_{\alpha,\beta}(\phi; zt^{\nu}) = -c^{\rho} \left({}^{\mathrm{RL}}I_{0+}^{\sigma} \, N \right)(t) \tag{61}$$

is given by

$$N(t) = N_0 \, t^{\mu-1} \sum_{r=0}^{\infty} (-c^{\rho} t^{\sigma})^r \cdot \sum_{j=0}^{\infty} \frac{\phi(j) \, \Gamma(\nu j + \mu)}{\Gamma(\alpha j + \beta) \, \Gamma(\nu j + \sigma r + \mu)} (zt^{\nu})^j \qquad (t > 0), \tag{62}$$

provided that the right-hand side of the solution asserted by (62) exists.

Theorem 3. *For $c, \mu, \nu, \rho, \sigma \in \mathbb{R}^+$, let the extended Hurwitz-Lerch Zeta function*:

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s, \kappa),$$

defined by (16), exist. Then the solution of the following generalized fractional kinetic equation:

$$N(t) - N_0 \, t^{\mu-1} \, \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(zt^{\nu}, s, \kappa) = -c^{\rho} \left({}^{\mathrm{RL}}I_{0+}^{\sigma} \, N \right)(t) \tag{63}$$

is given by

$$N(t) = N_0 \, t^{\mu-1} \sum_{r=0}^{\infty} (-c^{\rho} t^{\sigma})^r \frac{\Gamma(\mu)}{\Gamma(\sigma r + \mu)} \cdot \Phi_{\mu,\lambda_1,\cdots,\lambda_p;\sigma r+\mu,\mu_1,\cdots,\mu_q}^{(\nu,\rho_1,\cdots,\rho_p;\nu,\sigma_1,\cdots,\sigma_q)}(zt^{\nu}, s, \kappa) \qquad (t > 0), \tag{64}$$

provided that the right-hand side of the solution asserted by (64) exists.

Remarkably, the general function $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$, defined by (27), occurring in the non-homogeneous term of the kinetic Equation (59) of fractional order is distinctly advantageous because of its generality. Naturally, therefore, solutions of other kinetic equations involving other simpler non-homogeneous terms can be derived as appropriate special cases of the solution (60) given by Theorem 1. Similar remarks would apply equally strongly to the results (62) and (64), which are provided by Theorems 2 and 3, respectively.

III. The unified fractional derivative operator ${}^{\mathrm{H}}D_{0+}^{\alpha,\beta}$ of order $\alpha \quad (0 < \alpha < 1)$ and type $\beta \quad (0 \leqq \beta \leqq 1)$, defined by the Equation (35), was considered by Hilfer (see [63]) in order to derive the solution of the following general fractional differential equation:

$$\left({}^{\mathrm{H}}D_{0+}^{\alpha,\beta} f\right)(x) = \lambda f(x) \qquad (x > 0) \tag{65}$$

under the initial condition given, in terms of the corresponding *two-parameter* fractional integral operator ${}^{\mathrm{H}}I_{0+}^{\alpha,\beta}$, by

$$\left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+) = c_0, \tag{66}$$

where it is tacitly assumed that

$$\left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+) := \lim_{x \to 0+} \left\{ \left({}^{\mathrm{H}}I_{0+}^{(1-\beta)(1-\alpha)} f\right)(x) \right\},$$

c_0 is a given constant and the parameter λ is the eigenvalue. Hilfer's solution of the above initial-value problem is given by (see [63], p. 115, Equation (124)):

$$f(x) = c_0 \, x^{(1-\beta)(\alpha-1)} \, E_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^{\alpha}), \tag{67}$$

where $E_{\alpha,\beta}(z)$ denotes the two-parameter Mittag-Leffler function defined by (21).

If we put $\beta = 0$ and $c_0 = 1$ in Hilfer's solution (66), we can deduce the corrected version of the claimed solution (see [74], p. 802, Equation (3.1); see also [78]) of the following initial-value problem:

$$\left({}^{\mathrm{RL}}D_{0+}^{\alpha} f\right)(x) = \lambda f(x) \qquad (x > 0), \tag{68}$$

under the initial condition given by

$$\left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(0+) = 1, \tag{69}$$

where, as also in the equation (66),

$$\left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(0+) := \lim_{x \to 0+} \left\{ \left({}^{\mathrm{RL}}I_{0+}^{1-\alpha} f\right)(x) \right\}$$

in the form given by

$$f(x) = x^{\alpha-1} \, E_{\alpha,\alpha}(\lambda x^{\alpha}), \tag{70}$$

in terms of the two-parameter Mittag-Leffler function defined by (21).

In concluding this section, we remark that, in recent years, various real-world problems and issues in many areas of mathematical, physical and engineering sciences have been modeled and analyzed by appealing to several powerful tools, one of which involves applications of the operators of fractional calculus. Notably, a number of important and potentially useful definitions have been introduced and used for fractional-order derivatives. These include, among others, the fractional-order derivative operators which stem from the Riemann-Liouville, the Grünwald-Letnikov, the Liouville-Caputo, the Caputo-Fabrizio and the Atangana-Baleanu fractional-order derivatives (see, for example, [47,83–85]).

The Riemann-Liouville fractional derivative is known to involve the convolution of a given function and a power-law kernel (see, for details, [47,85]). On the other hand, the Liouville-Caputo (LC) fractional derivative involves the convolution of the local derivative of a given function with a power-law function [86]. The fractional-order derivatives proposed by Caputo and Fabrizio [84] and Atangana and Baleanu [83] are based upon the exponential decay law which is a generalized power-law function (see [87–92]). The Caputo-Fabrizio (CFC) fractional-order derivative as well as the Atangana-Baleanu (ABC) fractional-order derivative allow us to describe complex physical problems that follow, at the same time, the power law and the exponential decay law (see, for details, [87–92]). Such items of information as those presented in many of these works are believed to have the potential to generate further developments on fractional-order modeling and analysis of interesting applied problems. Many experiments and theories have shown that a fairly large number of abnormal phenomena, which occur in the engineering and applied sciences, can be described well by using discrete fractional calculus (see, for example, [93]). In particular, fractional difference equations have been found to provide powerful tools in the modeling and analysis of various phenomena in many different fields of science and engineering including those in mathematical physics, fluid mechanics, heat conduction, and so on.

6. Concluding Remarks and Observations

Commonly known as special functions or mathematical functions, the origin of many members of the remarkably vast family of higher transcendental functions can be traced back to such widespread areas as (for example) mathematical physics, analytic number theory and applied mathematical sciences. Here, in this survey-cum-expository review article, our objective has been to briefly present an introductory overview and survey of some important recent developments in the theory of several extensively studied families of higher transcendental functions (or, more popularly, special functions) and their potential applications in (for example) mathematical physics, analytic number theory and applied mathematical sciences. For further reading and researching by those who are interested in pursuing this subject, we have chosen to provide references to various useful monographs and textbooks on the theory and applications of higher transcendental functions. We have also considered several operators of fractional calculus, which are associated with higher transcendental functions, briefly indicating their applications as well.

The bibliography in this review article includes a number of recently published journal articles which have dealt with the extensively investigated subject of fractional calculus and its widespread applications. In fact, having an overview of the on-going contributions to the theory and applications of fractional calculus, which are continually appearing in some of the leading journals devoted to mathematical and physical sciences, biological sciences, statistical sciences, engineering sciences, and so on, the subject-matter, which we have dealt with in this review article, is remarkably important and potentially useful. Moreover, the interested future researchers will surely benefit from the listing of references to some of the other applications of various fractional-calculus operators in the mathematical and other sciences, which we have not considered in the preceding sections (see, for example, [94–111]).

There is considerable literature investigating and applying the quantum or basic (or $\mathfrak{q}$-) calculus not only in the area of higher transcendental functions, which we have presented in Section 2, and Geometric Function Theory of Complex Analysis (see, for a detailed historical and introductory overview, the recently published survey-cum-expository review article [102]), but also in the modeling and analysis of applied problems as well as in extending the well-established theory and applications of various rather classical mathematical functions and mathematical inequalities. It is regretful, however, to see that a large number of mostly amateurish-type researchers on these and other related topics continue to produce and publish obvious and inconsequential variations and straightforward translations of the known $\mathfrak{q}$-results in terms of the so-called $(\mathfrak{p}, \mathfrak{q})$-calculus by unnecessarily forcing-in an obviously superfluous (or redundant) parameter $\mathfrak{p}$ into the classical $\mathfrak{q}$-calculus and thereby falsely claiming "generalization" (see [102], p. 340 and [1], Section 5, pp. 1511–1512). Such tendencies to produce and flood the literature with trivialities should be discouraged by all means.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: It is my pleasure to express my appreciation and sincere thanks to the colleagues in the Symmetry Editorial Office (especially to the Managing Editor, Celina Ci) for kindly inviting me to contribute this survey-cum-expository review article to *Symmetry*.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
2. De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
3. Srivastava, H.M. Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulence fields. *Z. Angew. Math. Mech.* **1984**, *64*, 255–257. [CrossRef]
4. Izadi, M.; Srivastava, H.M. A novel matrix technique for multi-order pantograph differential equations of fractional order. *Proc. R. Soc. Lond. Ser. A Math. Phys. Engrgy Sci.* **2021**, *477*, 2021031. [CrossRef]
5. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Applied Mathematics Series; National Bureau of Standards: Washington, DC, USA, 1964; Reprinted by Dover Publications: New York, NY, USA, 1965.
6. Andrews, L.C. *Special Functions for Engineers and Applied Mathematicians*; Macmillan Company: New York, NY, USA, 1984.
7. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1999; Volume 71.
8. Carlson, B.C. *Special Functions of Applied Mathematics*; Academic Press: New York, NY, USA; San Francisco, CA, USA; Londin, UK, 1977.
9. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA; Toronto, ON, Canada, 1953; London, UK, 1955; Volumes I, II and III.
10. Rainville, E.D. *Special Functions*; Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company: Bronx, NY, USA, 1971.
11. Lebedev, N.N. *Special Functions and Their Applications*; Translated from the Russian by R. A. Silverman; Prentice-Hall: Englewood Cliffs, NJ, USA, 1965.
12. Luke, Y.L. *The Special Functions and Their Approximations*; Mathematics in Science and Engineering; A Series of Monographs and Textbooks; Academic Press: New York, NY, USA; London, UK, 1969; Volumes I (53-I) and II (53-II).
13. Luke, Y.L. *Mathematical Functions and Their Approximations*; Academic Press: New York, NY, USA; San Francisco, CA, USA; London, UK, 1975.
14. Magnus, W.; Oberhettinger, F.; Soni, R.P. *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed.; Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, Bd; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1966; Volume 52.
15. Miller, W., Jr. *Lie Theory and Special Functions*; Mathematics in Science and Engineering; A Series of Monographs and Textbooks; Academic Press: London, UK; New York, NY, USA, 1968; Volume 43.
16. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press: Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
17. Srivastava, H.M.; Kashyap, B.R.K. *Special Functions in Queuing Theory and Related Stochastic Processes*; Academic Press: New York, NY, USA; London, UK, 1982.
18. Temme, N.M. *Special Functions*: *An Introduction to Classical Functions of Mathematical Physics*; A Wiley-Interscience Publication, John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1996.
19. Szegö, G. *Orthogonal Polynomials*, 4th ed.; American Mathematical Society: Providence, RI, USA, 1975; Volume 23,
20. Appell, P.; Kampé de Fériet, J. *Fonctions Hypergéométriques et Hypersphériques*; *Polynômes d'Hermite*; Gauthier-Villars: Paris, France, 1926.
21. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge Tracts in Mathematics and Mathematical Physics; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1935; Volumw 32; Reprinted by Stechert-Hafner Service Agency: New York, NY, USA; London, UK, 1964.
22. Slater, L.J. *Generalized Hypergeometric Functions*; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1966.
23. Seaborn, J.B. *Hypergeometric Functions and Their Applications*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1991.
24. Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-Function*: *Theory and Applications*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2010.
25. Srivastava, H.M.; Gupta, K.C.; Goyal, S.P. *The H-Functions of One and Two Variables with Applications*; South Asian Publishers: New Delhi, India; Madras, India, 1982.
26. Inayat-Hussain, A.A. New properties of hypergeometric series derivable from Feynman integrals. I : Transformation and reduction formulae. *J. Phys. A Math. Gen.* **1987**, *20*, 4109–4118. [CrossRef]

27. Inayat-Hussain, A.A. New properties of hypergeometric series derivable from Feynman integrals. II: A generalisation of the H-function. *J. Phys. A Math. Gen.* **1987**, *20*, 4119–4128. [CrossRef]
28. Buschman, R.G.; Srivastava, H.M. The $\overline{H}$ function associated with a certain class of Feynman integrals. *J. Phys. A Math. Gen.* **1990**, *23*, 4707–4710. [CrossRef]
29. Srivastava, H.M.; Lin, S.-D.; Wang, P.-Y. Some fractional-calculus results for the $\overline{H}$-function associated with a class of Feynman integrals. *Russian J. Math. Phys.* **2006**, *13*, 94–100. [CrossRef]
30. Srivastava, H.M.; Saxena, R.K.; Parmar, R.K. Some families of the incomplete H-functions and the incomplete $\bar{H}$-functions and associated integral transforms and operators of fractional calculus with applications. *Russian J. Math. Phys.* **2018**, *25*, 116–138. [CrossRef]
31. Titchmarsh, E.C. *The Theory of the Riemann Zeta-Function*; Clarendon (Oxford University) Press: Oxford, UK; London, UK; New York, NY, USA, 1951; 2nd ed.; (Revised by D. R. Heath-Brown), 1986.
32. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2012.
33. Srivastava, H.M. The Zeta and related functions: Recent developments. *J. Adv. Engrgy Comput.* **2019**, *3*, 329–354. [CrossRef]
34. Srivastava, H.M. Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches. *Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan* **2019**, *45*, 234–269. [CrossRef]
35. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, 4th ed.; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 1927.
36. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series, and Products*; Corrected and Enlarged Edition; Jeffrey, A., Ed.; Academic Press: New York, NY, USA; London, UK, 1980; 6th ed.; 2000.
37. Srivastava, H.M.; Saxena, R.K.; Pogány, T.K.; Saxena, R. Integral and computational representations of the extended Hurwitz-Lerch Zeta function. *Integr. Transf. Spec. Funct.* **2011**, *22*, 487–506. [CrossRef]
38. Srivastava, H.M. A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications. *Appl. Math. Inform. Sci.* **2014**, *8*, 1485–1500. [CrossRef]
39. Mittag-Leffler, G.M. Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris* **1903**, *137*, 554–558.
40. Wiman, A. Über den Fundamentalsatz in der Theorie der Funcktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 191–201. [CrossRef]
41. Wiman, A. Über die Nullstellen der Funktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 217–234. [CrossRef]
42. Mainardi, F. Why the Mittag-Leffler function can be considered the queen function of the fractional calculus? *Entropy* **2020**, *22*, 1359. [CrossRef] [PubMed]
43. Gorenflo, R.; Mainardi, F.; Srivastava, H.M. Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. In Proceedings of the Eighth International Colloquium on Differential Equations, Plovdiv, Bulgaria, 18–23 August 1997; Bainov, D., Ed.; VSP Publishers: Utrecht, The Netherlands; Tokyo, Japan, 1998; pp. 195–202.
44. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Mittag-Leffler functions and their applications. *J. Appl. Math.* **2011**, *2011*, 298628. [CrossRef]
45. Kilbas, A.A.; Saigo, M.; Saxena, R.K. Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integr. Transf. Spec. Funct.* **2004**, *15*, 31–49. [CrossRef]
46. Kilbas, A.A.; Saigo, M.; Saxena, R.K. Solution of Volterra integro-differential equations with generalized Mittag-Leffler functions in the kernels. *J. Integral Equ. Appl.* **2002**, *14*, 377–396. [CrossRef]
47. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.
48. Hilfer, R.; Seybold, H.J. Computation of the generalized Mittag-Leffler function and its inverse in the complex plane. *Integr. Transf. Spec. Funct.* **2006**, *17*, 637–652. [CrossRef]
49. Seybold, H.J.; Hilfer, R. Numerical results for the generalized Mittag-Leffler function. *Fract. Calc. Appl. Anal.* **2005**, *8*, 127–139.
50. Srivastava, H.M. Some families of Mittag-Leffler type functions and associated operators of fractional calculus. *TWMS J. Pure Appl. Math.* **2016**, *7*, 123–145.
51. Srivastava, H.M.; Fernandez, A.; Baleanu, D. Some new fractional-calculus connections between Mittag-Leffler functions. *Mathematics* **2019**, *7*, 485. [CrossRef]
52. Srivastava, H.M.; Bansal, M.K.; Harjule, P. A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function. *Math. Methods Appl. Sci.* **2018**, *41*, 6108–6121. [CrossRef]
53. Srivastava, H.M.; Bansal, M.K.; Harjule, P. A class of fractional integral operators involving a certain general multi-index Mittag-Leffler function. *Ukrainian Math. J.* **2021**, in press.
54. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Engrgy Comput.* **2021**, *5*, 135–166.
55. Kilbas, A.A.; Saigo, M. *H-Transforms*: *Theory and Applications*; Analytical Methods and Special Functions: An International Series of Monographs in Mathematics; Chapman and Hall (A CRC Press Company): Boca Raton, MA, USA; London, UK; New York, NY, USA, 2004; Volume 9.

56. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; With 1 CD-ROM (Windows, Macintosh and UNIX); U. S. Department of Commerce, National Institute of Standards and Technology: Washington, DC, USA; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 2010.
57. Srivastava, H.M. Charles Fox. *Bull. Lond. Math. Soc.* **1980**, *12*, 67–70. [CrossRef]
58. Wright, E.M. The asymptotic expansion of integral functions defined by Taylor series. I. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1940**, *238*, 423–451.
59. Wright, E.M. The asymptotic expansion of integral functions defined by Taylor series. II. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1941**, *239*, 217–232.
60. Wright, E.M. The asymptotic expansion of integral functions and of the coefficients in their Taylor series. *Trans. Am. Math. Soc.* **1948**, *64*, 409–438. [CrossRef]
61. Barnes, E.W. The asymptotic expansion of integral functions defined by Taylor's series. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1906**, *206*, 249–297.
62. Hilfer, R. (Ed.) *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000.
63. Hilfer, R. Fractional time evolution. In *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore; New Jersey, NJ, USA; London, UK; Hong Kong, China, 2000; pp. 87–130.
64. Hilfer, R. Experimental evidence for fractional time evolution in glass forming materials. *J. Chem. Phys.* **2002**, *284*, 399–408. [CrossRef]
65. Hilfer, R. Threefold introduction to fractional derivatives. In *Anomalous Transport*: *Foundations and Applications*; Klages, R., Radons, G., Sokolov, I.M., Eds.; Wiley-VCH Verlag: Weinheim, Germany, 2008; pp. 17–73.
66. Hilfer, R.; Anton, L. Fractional master equations and fractal time random walks. *Phys. Rev. E* **1995**, *51*, R848–R851. [CrossRef]
67. Hilfer, R.; Luchko, Y.; Tomovski, Ž. Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. *Fract. Calc. Appl. Anal.* **2009**, *12*, 299–318.
68. Mainardi, F.; Gorenflo, R. Time-fractional derivatives in relaxation processes: A tutorial survey. *Fract. Calc. Appl. Anal.* **2007**, *10*, 269–308.
69. Mokhtary, P.; Ghoreishi, F.; Srivastava, H.M. The Müntz-Legendre Tau method for fractional differential equations. *Appl. Math. Model.* **2016**, *40*, 671–684. [CrossRef]
70. Sandev, T.; Tomovski, Ž. General time fractional wave equation for a vibrating string. *J. Phys. A Math. Theoret.* **2010**, *43*, 055204. [CrossRef]
71. Saxena, R.K.; Kalla, S.L. Solution of Volterra-type integro-differential equations with a generalized Lauricella confluent hypergeometric function in the kernels. *Int. J. Math. Math. Sci.* **2005**, *2005*, 1155–1170. [CrossRef]
72. Srivastava, H.M.; Harjule, P.; Jain, R. A general fractional differential equation associated with an integral operator with the *H*-function in the kernel. *Russian J. Math. Phys.* **2015**, *22*, 112–126. [CrossRef]
73. Srivastava, H.M.; Saxena, R.K. Some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel. *J. Integral Equ. Appl.* **2005**, *17*, 199–217. [CrossRef]
74. Tomovski, Ž.; Hilfer, R.; Srivastava, H.M. Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. *Integr. Transf. Spec. Funct.* **2010**, *21*, 797–814. [CrossRef]
75. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*: *Theory and Applications*; Gordon and Breach Science Publishers: Yverdon, Switzerland, 1993.
76. Liouville, J. Mémoire sur quelques quéstions de géometrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces quéstions. *J. École Polytech.* **1832**, *13*, 1–69.
77. Caputo, M. *Elasticità e Dissipazionne*; Zanichelli: Bologna, Italy, 1969.
78. Srivastava, H.M. Remarks on some fractional-order differential equations. *Integr. Transf. Spec. Funct.* **2017**, *28*, 560–564. [CrossRef]
79. Srivastava, H.M.; Saxena, R.K. Operators of fractional integration and applications. *Appl. Math. Comput.* **2001**, *118*, 1–52. [CrossRef]
80. Raina, R.K. On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Math. J.* **2005**, *21*, 191–203.
81. Haubold, H.J.; Mathai, A.M. The fractional kinetic equation and thermonuclear functions. *Astrophys. Space Sci.* **2000**, *273*, 53–63. [CrossRef]
82. Srivastava, H.M. Fractional-order derivatives and integrals: Introductory overview and recent developments. *Kyungpook Math. J.* **2020**, *60*, 73–116.
83. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Thermal Sci.* **2016**, *20*, 763–769. [CrossRef]
84. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **2015**, *1*, 73–85.
85. Podlubny, I. *Fractional Differential Equations*: *An Introduction to Fractional Derivatives*, *Fractional Differential Equations*, *to Methods of Their Solution and Some of Their Applications*; Mathematics in Science and Engineering; Academic Press: New York, NY, USA; London, UK; Sydney, Australia; Tokyo, Japan; Toronto, ON, Canada, 1999; Volume 198.
86. Caputo, M.; Mainardi, F. A new dissipation model based on memory mechanism. *Pure Appl. Geophys.* **1971**, *91*, 134–147. [CrossRef]

87. Alsaedi, A.; Baleanu, D.; Etemad, S.; Rezapour, S. On coupled systems of time-fractional differential problems by using a new fractional derivative. *J. Funct. Spaces* **2016**, *2016*, 4626940. [CrossRef]
88. Atangana, A. On the new fractional derivative and application to nonlinear Fisher's reaction diffusion equation. *Appl. Math. Comput.* **2016**, *273*, 948–956. [CrossRef]
89. Atangana, A.; Alkahtani, B.S.T. Extension of the RLC electrical circuit to fractional derivative without singular kernel. *Adv. Mech. Eng.* **2015**, *7*, 1687814015591937.
90. Atangana, A.; Alkahtani, B.S.T. Analysis of the Keller-Segel model with a fractional derivative without singular kernel. *Entropy* **2015**, *17*, 4439–4453. [CrossRef]
91. Atangana, A.; Alkahtani, B.S.T. New model of groundwater flowing within a confine aquifer: Application of Caputo-Fabrizio derivative. *Arabian J. Geosci.* **2016**, *9*, 8. [CrossRef]
92. Atangana, A.; Nieto, J.J. Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. *Adv. Mech. Engrgy* **2015**, *7*, 1687814015613758. [CrossRef]
93. Goodrich, C.; Peterson, A.C. *Discrete Fractional Calculus*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2015.
94. Ahmad, B.; Henderson, J.; Luca, R. *Boundary Value Problems for Fractional Differential Equations and Systems*; Trends in Abstract and Applied Analysis; World Scientific Publishing Company: Singapore; Hoboken, NJ, USA; London, UK; Hong Kong, China, 2021; Volume 9.
95. Ahmadova, A.; Huseynov, I.T.; Fernandez, A.; Mahmudov, N.I. Trivariate Mittag-Leffler functions used to solve multi-order systems of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *97*, 105735. [CrossRef]
96. Anastassiou, G.A. *Generalized Fractional Calculus*: *New Advancements and Applications*; Studies in Systems, Decision and Control; Springer: Cham, Switzerland, 2021; Volume 305.
97. Badík, A.; Fečkan, M. Applying fractional calculus to analyze final consumption and gross investment influence on GDP. *J. Appl. Math. Stat. Inform.* **2021**, *17*, 65–72. [CrossRef]
98. Chen, S.-B.; Jahanshahi, H.; Abba, O.A.; Solís-Pérez, J.E.; Bekiros, S.; Gómez-Aguilar, J.F.; Yousefpour, A.; Chu, Y.-M. The effect of market confidence on a financial system from the perspective of fractional calculus: Numerical investigation and circuit realization. *Chaos Solitons Fractals* **2020**, *140*, 110223. [CrossRef]
99. Garrappa, R.; Giusti, A.; Mainardi, F. Variable-order fractional calculus: A change of perspective. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *102*, 105904. [CrossRef]
100. Gómez-Aguilar, J.F.; Atangana, A. New chaotic attractors: Application of fractal-fractional differentiation and integration. *Math. Methods Appl. Sci.* **2020**, *130*, 3036–3065. [CrossRef]
101. Guariglia, E. Fractional calculus, zeta functions and Shannon entropy. *Open Math.* **2021**, *19*, 87–100. [CrossRef]
102. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
103. Saad, K.M. New fractional derivative with non-singular kernel for deriving Legendre spectral collocation method. *Alexandria Engrgy J.* **2020**, *59*, 1909–1917. [CrossRef]
104. Saad, K.M.; Al-Shareef, E.H.F.; Alomari, A.K.; Baleanu, D.; Gómez-Aguilar, J.F. On exact solutions for time-fractional Korteweg-de Vries and Korteweg-de Vries-Burgers equations using homotopy analysis transform method. *Chin. J. Phys.* **2020**, *63*, 149–162. [CrossRef]
105. Saad, K.M.; Gómez-Aguilar, J.F.; Almadiy, A.A. A fractional numerical study on a chronic hepatitis C virus infection model with immune response. *Chaos Solitons Fractals* **2020**, *139*, 110062. [CrossRef]
106. Shishkina, E.; Sitnik, S. *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*; Mathematics in Science and Engineering; Academic Press (Elsevier Science Publishers): New York, NY, USA; London, UK; Toronto, ON, Canada, 2020.
107. Srivastava, H.M.; Alomari, A.-K.N.; Saad, K.M.; Hamanah, W.M. Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method. *Fractal Fract.* **2021**, *5*, 131. [CrossRef]
108. Srivastava, H.M.; Deniz, S.; Saad, K.M. An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator. *J. King Saud Univ. Sci.* **2021**, *33*, 101345. [CrossRef]
109. Srivastava, H.M.; Jan, R.; Jan, A.; Deebani, W.; Shutaywi, M. Fractional-calculus analysis of the transmission dynamics of the dengue infection. *Chaos* **2021**, *31*, 53130. [CrossRef]
110. Wang, W.-T.; Khan, M.A. Analysis and numerical simulation of fractional model of bank data with fractal-fractional Atangana-Baleanu derivative, *J. Comput. Appl. Math.* **2020**, *369*, 112646. [CrossRef]
111. You, Z.-L.; Fečkan, M.; Wang, J.-R. Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions. *J. Comput. Appl. Math.* **2020**, *378*, 112939. [CrossRef]