

The Ramsey Multiplicity Problem

by

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We acknowledge and respect the Lək'wəḡən (Songhees and X<sup>w</sup>sepsəm/Esquimalt) Peoples on whose territory the university stands, and the Lək'wəḡən and WSÁNEĆ Peoples whose historical relationships with the land continue to this day.

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## ABSTRACT

“Complete chaos is impossible” is a core concept of Ramsey theory. Ramsey’s Theorem formalizes this idea by showing that any sufficiently large graph, no matter how disordered, must inevitably contain a certain type of pattern. The Ramsey multiplicity problem extends this concept quantitatively: instead of asking whether a fixed pattern exists, it asks how many copies of a fixed pattern are guaranteed as the size of the graph increases. Combinatorial limit theory, one of the most significant developments in modern combinatorics, helps to understand these large discrete structures. It also provides a framework for viewing discrete structures as approximations of rich continuous objects, like measurable functions or measures, which facilitates the use of analytic tools to tackle the problems in extremal combinatorics including the Ramsey multiplicity problem. By solving such a problem, we want to deepen our understanding of large and seemingly chaotic graphs.

A graph  $H$  is said to be *common* if the number of monochromatic copies of  $H$  is asymptotically minimized by a random colouring. It is well known that the disjoint union of two common graphs may be uncommon; e.g.,  $K_2$  and  $K_3$  are common, but their disjoint union is not.

In Chapter 3, we investigate the commonality of disjoint unions of multiple copies of  $K_3$  and  $K_2$ . As a consequence of our results, we obtain the first example of a pair of uncommon graphs whose disjoint union is common. Our approach is to reduce the problem of showing that certain disconnected graphs are common to a constrained optimization problem in which the constraints are derived from supersaturation bounds related to Razborov’s Triangle Density Theorem. We also improve the bounds on the Ramsey multiplicity constant of a triangle with a pendant edge and the disjoint union of  $K_3$  and  $K_2$ .

Fox and Wigderson recently identified a large family of graphs whose Ramsey multiplicity constants are attained by sequences of “Turán colourings;” i.e. colourings in which one of the colour classes forms the edge set of a balanced complete multipartite graph. Each graph in their family comes from taking a connected non-3-colourable graph with a critical edge and adding many pendant edges. In Chapter 4, We extend their result to an off-diagonal variant of the Ramsey multiplicity constant which involves minimizing a weighted sum of red copies of one graph and blue copies of another.

In Chapter 5, we focus on finding smaller graphs whose Ramsey multiplicity constants are achieved by Turán colourings. While Fox and Wigderson provide many examples, their smallest constructions involve graphs with at least  $10^{66}$  vertices. In

contrast, we identify a graph on only 10 vertices whose Ramsey multiplicity constant is achieved by Turán colourings. To prove this, we apply the method developed in Chapter 3 and used a powerful technique known as the flag algebra method, assisted by semi-definite programming.

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# Symbols and Notation

$V(G)$	the set of vertices of $G$
$E(G)$	the set of edges of $G$
$v(G)$	the number of vertices in $G$
$e(G)$	the number of edges in $G$
$d_G(v)$	the degree of vertex $v$ in $G$
$\delta(G)$	the minimum degree of $G$
$N_G(v)$	the neighbourhood of vertex $v$ in $G$
$P_n$	the path graph with $n$ edges
$C_n$	the cycle graph with $n$ vertices
$K_n$	the complete graph with $n$ vertices
$K_{m,n}$	the complete bipartite graph with 2 parts of size $m$ and $n$
$\chi(G)$	the chromatic number of $G$
$\text{aut}(G)$	the number of automorphisms of $G$
$G(n, p)$	the Erdős–Rényi random graph on $n$ vertices with edge probability $p$
$k(G)$	the number of connected components of $G$
$\text{crit}(G)$	the number of critical edges of $G$
$G[S]$	the subgraph of $G$ induced by the set $S$

# Chapter 1

## Introduction

For any positive integer  $t$ , Ramsey's theorem [55] tells us that any 2-edge-colouring of the complete graph  $K_n$  contains a monochromatic copy of  $K_t$  when  $n$  is sufficiently large with respect to  $t$ . The smallest such  $n$  is called *Ramsey number*, denoted  $r(K_t)$ . This concept extends naturally to any graph  $H$ , where  $r(H)$  is the smallest  $n$  such that every 2-edge-colouring of  $K_n$  contains a monochromatic copy of  $H$ . The *Ramsey multiplicity problem* asks:

*what are the asymptotics of the minimum number of labelled monochromatic copies of a fixed graph  $H$  among all 2-edge-colourings of  $K_n$  as  $n \rightarrow \infty$ ?*

Let us introduce some terminology to make this more precise. For two graphs  $H$  and  $G$ , a *graph homomorphism* from  $H$  to  $G$  is a vertex mapping  $f : V(H) \rightarrow V(G)$  that preserves adjacency; that is,  $f(u)f(v) \in E(G)$  if  $uv \in E(H)$ . Let  $\text{hom}(H, G)$  denote the number of homomorphisms from  $H$  to  $G$ . We write  $\text{hom}_{\text{inj}}(H, G)$  for the number of injective homomorphisms, which corresponds to the number of labelled copies of  $H$  in  $G$ . Define

$$M(H, n) := \min_{v(G)=n} (\text{hom}_{\text{inj}}(H, G) + \text{hom}_{\text{inj}}(H, \overline{G})),$$

where  $\overline{G}$  denotes the complement graph and  $v(G)$  denotes the number of vertices of  $G$ . By thinking of edges of  $G$  as being red and  $\overline{G}$  as blue, we can see that  $M(H, n)$  is the minimum number of monochromatic labelled copies of  $H$  in a 2-edge-colouring of  $K_n$ .

The *Ramsey multiplicity constant*  $c(H)$  is defined as the asymptotic minimum density of monochromatic copies:

$$c(H) := \lim_{n \rightarrow \infty} \frac{M(H, n)}{n(n-1) \cdots (n-v(H)+1)} = \lim_{n \rightarrow \infty} \frac{M(H, n)}{n^{v(H)}}.$$

The limit in the definition of  $c(H)$  always exists; see Theorem 2.1.1.

To determine  $c(H)$ , it is necessary to find the best strategy for colouring the edges of a large clique to minimize monochromatic copies of  $H$ . One natural strategy is to colour the edges randomly. For  $n \in \mathbb{N}$  and  $p \in [0, 1]$ , the *Erdős–Rényi random graph*

$G(n, p)$  is the graph on  $n$  vertices where each edge is included independently with probability  $p$ . The expected number of injective homomorphisms of  $H$  in  $G(n, p)$  is

$$\mathbb{E}(\text{hom}_{inj}(H, G(n, p))) = p^{e(H)} \cdot n(n-1) \cdots (n - v(H) + 1).$$

So, if  $G = G(n, \frac{1}{2})$ , then

$$\mathbb{E}(\text{hom}_{inj}(H, G) + \text{hom}_{inj}(H, \bar{G})) = 2 \left(\frac{1}{2}\right)^{e(H)} \cdot n(n-1) \cdots (n - v(H) + 1).$$

By the First Moment Method, there must exist a colouring with at most as many monochromatic copies as the expected number. Thus, the expected number under a random colouring provides an upper bound on the Ramsey multiplicity constant:

$$c(H) \leq 2 \left(\frac{1}{2}\right)^{e(H)}. \tag{1.0.1}$$

In 1959, Goodman [32] determined the Ramsey multiplicity constant for a triangle, surprisingly showing that  $c(K_3) \geq \frac{1}{4}$ . This lower bound matches the upper bound obtained from random colouring (1.0.1), and thus  $c(K_3) = \frac{1}{4}$ . This result means that a random colouring, the most chaotic construction, turns out to be asymptotically optimal in minimizing the number of monochromatic triangles. Although this seems paradoxical, it inspired further conjectures. Erdős [19] conjectured that random colouring minimizes the number of monochromatic copies for all complete graphs; that is,  $c(K_t) = 2\left(\frac{1}{2}\right)^{\binom{t}{2}}$  for all  $t$ . Burr and Rosta [9] extended this idea further, conjecturing that  $c(H) = 2\left(\frac{1}{2}\right)^{e(H)}$  holds for all graphs.

However, both conjectures were eventually disproved. Sidorenko [60] showed that the paw graph  $P$  (see Figure 1), which consists of a triangle with a pendant edge, is not minimized by a random colouring;  $c(P) < 2\left(\frac{1}{2}\right)^4$ . Later, Thomason [65] proved that  $c(K_4) < 2\left(\frac{1}{2}\right)^6$ , thereby disproving Erdős' conjecture. Jagger, Štoviček, and Thomason [37] further generalized this result, showing that the number of monochromatic copies of any graph containing  $K_4$  is also not minimized by a random colouring. In particular, the number of monochromatic copies of almost every graph fails to be minimized by a random colouring. Moreover, Jagger, Štoviček, and Thomason [37] also proved that the Ramsey multiplicity of any non-bipartite graph with sufficiently many pendant edges is not attained by a random colouring, which generalizes Sidorenko's result for the paw graph. These results motivate the following question:

*which graphs  $H$  have the property that the number of monochromatic copies of  $H$  is asymptotically minimized by a random colouring?*

A graph  $H$  is called *common* if the number of monochromatic copies of  $H$  in any

2-edge-colouring of  $K_n$  is asymptotically minimized by a random colouring with edge density  $1/2$ . Formally, this means the following equality holds:

$$c(H) = 2 \left( \frac{1}{2} \right)^{e(H)}.$$

So, the paw graph  $P$ ,  $K_4$  and any graph containing  $K_4$  are not common. The concept of common graphs is closely related to one of the central conjectures in extremal graph theory, called Sidorenko’s Conjecture (which will be stated later as Conjecture 2.2.3). A graph  $H$  is said to be *Sidorenko* if it satisfies the inequality (2.2.4) for all graphs. Known families of Sidorenko graphs include complete bipartite graphs, paths, even cycles, trees, and others; see [12–15, 35, 39, 64]. It is straightforward to see that every bipartite graph satisfying Sidorenko’s conjecture is common (see Proposition 2.2.5). Several studies have focused on identifying new classes of common graphs, including odd cycles [59], even wheels [37], and triangle-trees [33], which are formed by gluing triangles along vertices or edges in a tree-like structure.

In 2012, Hatami, Hladký, Král’, Norine, and Razborov [36] proved that the 5-wheel, a graph with chromatic number four, is common. This was the first example of a non-3-colourable common graph. More recently, Král’, Volec, and Wei [46] constructed families of common graphs with arbitrarily high chromatic number. In recent years, there has been a steady flow of results proving that the members of certain families of graphs are common or uncommon [3, 4, 16, 17, 33, 34, 36, 40, 43–46, 54]. In spite of this progress, the task of classifying common graphs seems hopelessly difficult.

One focus of this thesis is on the commonality of disjoint unions of graphs. For graphs  $H_1$  and  $H_2$ , let  $H_1 \sqcup H_2$  denote their disjoint union; also, for a graph  $F$  and  $\ell \geq 1$ , let  $\ell \cdot F$  be the disjoint union of  $\ell$  copies of  $F$ . The original argument of Sidorenko [60] showing that the paw graph is uncommon also applies to prove that the disjoint union of  $K_3$  and  $K_2$  is uncommon (with the same proof). Interestingly,  $K_3 \sqcup K_2$  is not common even though each graph  $K_3$  and  $K_2$  individually is common. This raises a natural question: when is it possible for the disjoint union of two common graphs, two uncommon graphs or one of each to be common or uncommon?

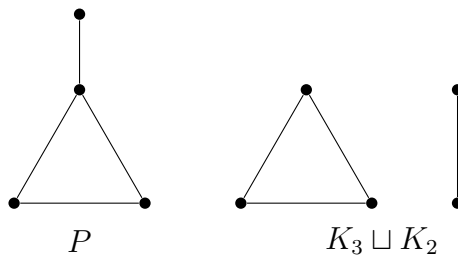


Figure 1.1: The graphs  $P$  and  $K_3 \sqcup K_2$  are uncommon by Sidorenko [60].

Trivially, there exist common graphs whose disjoint union is common, e.g.,  $K_2 \sqcup K_2$  is also common [60]. If  $H$  contains a  $K_4$ , then it is uncommon [37] and so is  $H \sqcup F$  for any graph  $F$ . Therefore, the disjoint union of a common graph and an uncommon graph or two uncommon graphs can be uncommon. Král', Volec and Wei [46] proved that if  $H$  is a graph with high girth and  $n$  is sufficiently large with respect to  $e(H)$ , then  $H \sqcup K_{n,n}$  is common. Complete bipartite graphs satisfy the Sidorenko's Conjecture and are common and  $H$  can be taken to be uncommon using the result of Jagger, Štoviček, and Thomason [37] that adding sufficiently many pendant edges to a non-bipartite graph results in an uncommon graph. This implies that the disjoint union of a common graph and an uncommon graph can also be common. In Chapter 3, we obtain the first example of two uncommon graphs whose disjoint union is common. In fact, we show that the disjoint union of two copies of a single uncommon graph can be common.

**Theorem 1.0.2.** *For  $0 \leq \ell \leq 2$ , the graph  $(2 \cdot K_3) \sqcup (\ell \cdot K_2)$  is common.*

**Corollary 1.0.3.** *There exists an uncommon graph  $H$  such that  $H \sqcup H$  is common.*

*Proof.* Consider  $H = K_3 \sqcup K_2$ . The fact that  $H$  is uncommon was shown by Sidorenko [60], and the fact that  $H \sqcup H$  is common follows from Theorem 1.0.2 with  $\ell = 2$ .  $\square$

We remark that our results also allow us to obtain new examples of graphs  $H_1$  and  $H_2$  such that  $H_1$  is common,  $H_2$  is uncommon and  $H_1 \sqcup H_2$  is common, which are much smaller than those of [46]. For example,  $H_1 = K_3$  and  $H_2 = K_3 \sqcup K_2$  or  $H_2 = K_3 \sqcup 2 \cdot K_2$  are such examples.

Furthermore, we also investigate the commonality of disjoint unions of several copies of  $K_3$  and  $K_2$ . This is motivated by a result of Jagger, Štoviček and Thomason [37]; see Theorem 2.1.3. Their result implies that for any non-bipartite common graph  $H$ , the graph  $H \sqcup t \cdot K_2$  becomes uncommon when  $t$  is sufficiently large in terms of  $e(H)$ . The bound coming from their result is exponential in  $e(H)$  in general and when  $H = K_3$ , a triangle which has three edges, it implies that approximately 134 edges are sufficient. However, it is already known that even a single edge ( $t = 1$ ) suffices to make  $K_3 \sqcup K_2$  uncommon. This discrepancy leads to a natural question:

*what is the smallest number of disjoint edges  $t$  required to make  $H \sqcup t \cdot K_2$  uncommon for a given graph  $H$ ?*

To formalize this, define

$$DE(H) := \inf\{\ell \geq 0 : H \sqcup (\ell \cdot K_2) \text{ is uncommon}\}.$$

Here,  $DE$  stands for “disjoint edges.” Note that, if  $H \sqcup (\ell \cdot K_2)$  remains common for all  $\ell \geq 0$ , then  $DE(H) = \infty$ . For certain graphs, upper and lower bounds on  $DE(H)$  are

known. For example, for a balanced complete tripartite graph  $K_{n,n,n}$ , a result of Fox [24] implies that  $DE(K_{n,n,n}) = O(n^2)$ . Grzesik, Lee, Lidický and Volec [33, Lemma 3.6] proved that for a triangle-tree graph with  $t$  gluing edges, we have  $DE(H) > t$ .

In Chapter 3, our research also focuses on determining the optimal number of disjoint edges required to make certain families of graphs uncommon, aiming to understand the boundary between common and uncommon graphs. For one triangle, Sidorenko [60] showed that  $DE(K_3) = 1$ . We show that  $DE(2 \cdot K_3) = 3$ ; see Proposition 3.5.3. For  $k \geq 3$ , the next result implies that  $DE(k \cdot K_3) > \frac{5}{3}k$ .

**Theorem 1.0.4.** *For  $k \geq 3$  and  $0 \leq \ell \leq \frac{5k}{3}$ , the graph  $(k \cdot K_3) \sqcup (\ell \cdot K_2)$  is common.*

In fact, this is derived from a more general result (Corollary 3.4.17) which applies to the disjoint union of a Sidorenko graph (see (2.2.4) below for a definition) and a graph built up from gluing together triangles in a tree-like fashion. While we do not believe that the bound on  $\ell$  in Theorem 1.0.4 is tight for all  $k \geq 3$ , the next theorem shows that it is tight for  $k = 3$  and within a factor  $1.1799 + o(1)$  of being tight for general  $k$ .

**Theorem 1.0.5.** *For  $k \geq 1$  and  $\ell = \lceil 1.9665k \rceil$ , the graph  $(k \cdot K_3) \sqcup (\ell \cdot K_2)$  is uncommon.*

As a consequence of the result  $c(H) < 2(\frac{1}{2})^{e(H)}$  for most of graphs by Jagger, Štoviček, and Thomason [37], there has been substantial interest in determining the Ramsey multiplicity constant for uncommon graphs, especially for complete graphs  $K_k$  with  $k > 3$ : see [11, 18, 19, 21, 22, 27, 28, 30, 33, 37, 52, 63, 65, 66]. Nevertheless, for most graphs, not only are the exact constants still unknown, but also the best known bounds remain far from tight. Our results include bounding the Ramsey multiplicity constant for the paw graph  $P$  and  $K_3 \sqcup K_2$  which are the first examples of uncommon graphs from Sidorenko [60]. We obtain, to our knowledge, the tightest known upper bounds on the Ramsey multiplicity constants of these two graphs; for the latter graph, we also obtain a reasonably tight lower bound which is proven “by hand” without the assistance of the flag algebra method; see Remark 3.2.9.

**Theorem 1.0.6.** *The paw graph  $P$  satisfies  $c(P) < 0.121415$ .*

**Theorem 1.0.7.**  $0.121423 < c(K_3 \sqcup K_2) < 0.121450$ .

Until recently, the Ramsey multiplicity constant  $c(H)$  was not known exactly for any uncommon graphs. In 2023, Fox and Wigderson [26] became the first to determine  $c(H)$  exactly for a fairly large family of uncommon graphs. Their result is achieved using a “Turán colouring,” a 2-edge colouring in which the blue colour class forms  $\chi(H) - 1$  disjoint cliques of almost equal size, where  $\chi(H)$  is the chromatic number of  $H$ , and the red colour class is its complement; i.e., a complete multipartite graph. After the random colouring, it is perhaps the next most natural strategy to “pack in”

as many red edges as possible without creating a red copy of  $H$ . The following general upper bound is then immediate;

$$c(H) \leq \left( \frac{1}{\chi(H) - 1} \right)^{v(H)-1}$$

for a connected graph  $H$ . While this bound is often weaker than the one obtained from random colouring, there are cases where it is stronger and even tight.

One particularly interesting family of graphs for which the Turán colouring is optimal consists of so called hairy graphs. For a graph  $F$  and  $t \geq 0$ , a  $t$ -hairy  $F$  is a graph with  $v(F) + t$  vertices that is created by adding  $t$  copies of  $K_2$  to  $F$ , one at a time, such that each added  $K_2$  has exactly one endpoint in  $V(F)$ . If  $H$  is a  $t$ -hairy  $F$  for some  $t$ , then we simply say that  $H$  is a hairy  $F$ . For example, the paw graph  $P$  is a 1-hairy  $K_3$ . An edge  $e$  of a graph  $F$  is called *critical* if  $\chi(F - e) < \chi(F)$ ; i.e. removing the edge  $e$  (and neither of its vertices) from  $F$  decreases the chromatic number. Fox and Wigderson [26] proved that the Turán colouring “uniquely” minimizes the number of monochromatic copies of  $H$  when a graph  $H$  is a  $t$ -hairy  $F$  for any connected non-3-colourable graph  $F$  with a critical edge and  $t$  is very large. In this case, we say that  $H$  is a *bonbon*.

In Chapter 4, we extend the notion of a bonbon to an off-diagonal setting. There has been recent interest in off-diagonal generalizations of basic questions in Ramsey multiplicity. Parczyk, Pokutta, Spiegel and Szabó [53] determined the Ramsey multiplicity constant on suitable linear combinations of the red copies of  $K_s$  and the blue copies of  $K_t$  for small  $s$  and  $t$ , and Behague, Morrison and Noel [3, 4] extended the notion of common graphs to an off-diagonal setting. Moss and Noel [51] recently introduced an off-diagonal notion of Ramsey multiplicity for general pairs of graphs. We find a family of pairs of graphs  $(H_1, H_2)$  such that the number of red copies of  $H_1$  and blue copies of  $H_2$  with the suitable coefficients of the off-diagonal Ramsey multiplicity which is formally defined in Chapter 2 is uniquely minimized by Turán colourings. We say such a pair  $(H_1, H_2)$  is a *bonbon pair*. Roughly speaking, we prove that for non-bipartite graphs  $F_1$  and  $F_2$ , each containing a critical edge and satisfying some additional restrictions, a pair of non-empty graphs  $(H_1, H_2)$  is a bonbon pair if  $H_1$  is a  $t_1$ -hairy of  $F_1$  and  $H_2$  is a  $t_2$ -hairy  $F_2$  for sufficiently large numbers  $t_1$  and  $t_2$ , and if the gap between  $v(H_1)$  and  $v(H_2)$  is not too large. One interesting phenomenon is if  $v(H_1)$  and  $v(H_2)$  differ too much, then the pair  $(H_1, H_2)$  is not a bonbon pair. So, if  $t_1$  is too large compared to  $t_2$ , then a pair  $(H_1, H_2)$  is not a bonbon pair even if  $H_1$  is a  $t_1$ -hairy  $F_1$  and  $H_2$  is a  $t_2$ -hairy  $F_2$  for sufficiently large  $t_1$  and  $t_2$ . For such a pair  $(H_1, H_2)$ , we find a colouring in which the total number of red copies of  $H_1$  and blue copies of  $H_2$  is smaller than in the Turán colouring; that is, the pair  $(H_1, H_2)$  is not a bonbon pair.

In Chapter 5, we are motivated by the discrepancy between the bound given in Theorem 2.1.3 and the actual number of disjoint edges (i.e.  $t = 1$ ) required to obtain an uncommon graph from a triangle. This raises a natural question:

*how many disjoint edges do we need to add to a graph  $H$  to obtain a bonbon?*

We start with the smallest non-3-colourable critical graph,  $K_4$ , and investigate the minimum value of  $s$  to make  $K_4 \sqcup s \cdot K_2$  a bonbon. We just focus exclusively on the Ramsey multiplicity constant in this chapter, without requiring the minimizer to be unique. To formalize this, we say that a graph  $H$  is *Ramsey multiplicity good* if the number of monochromatic copies of  $H$  is minimized by the Turán colouring, regardless of whether the Turán colouring is the unique minimizer. In Theorem 2.5.2, Fox and Wigderson showed that adding approximately  $10^{66}$  edges suffices to make  $K_4 \sqcup s \cdot K_2$  a bonbon. It is also the smallest known Ramsey multiplicity good graph. It is interesting to determine whether a smaller Ramsey multiplicity good graph exists. We obtain a smaller one; namely,  $K_4 \sqcup 3 \cdot K_2$  is Ramsey multiplicity good.

**Theorem 1.0.8.**

$$c(K_4 \sqcup 3 \cdot K_2) = \left(\frac{1}{3}\right)^6.$$

Furthermore,  $K_4 \sqcup s \cdot K_2$  is Ramsey multiplicity good for any  $s \geq 3$ .

**Theorem 1.0.9.** *For  $s \geq 3$ ,*

$$c(K_4 \sqcup s \cdot K_2) = \left(\frac{1}{3}\right)^{s+3}.$$

# Chapter 2

## Preliminaries

In this chapter, we introduce the technical definitions, prove some simple statements to warm up and give some preliminaries on graph theory and graph limit theory.

### 2.1 Ramsey Multiplicity Constant

We start with the standard proof that the Ramsey multiplicity constant exists. Given a graph  $G$  on vertex set  $V(G)$  and a set  $S \subseteq V(G)$ , let  $G[S]$  be the subgraph of  $G$  induced by  $S$ ; i.e. the graph with vertex set  $S$  and edge set  $\{uv \in E(G) : u, v \in S\}$ .

**Theorem 2.1.1** (Burr and Rosta [9]). *The Ramsey multiplicity constant  $c(H)$  exists for any graph  $H$ .*

*Proof.* Let  $t$  be the number of vertices of  $H$ . We aim to show that the sequence

$$a_n := \frac{M(H, n)}{n(n-1) \cdots (n-t+1)}$$

converges by the Monotone Convergence Theorem, which says that a sequence converges if it is upper-bounded and non-decreasing. Since  $a_n \in [0, 1]$  for all  $n$ , it suffices to show that the sequence  $(a_n)_{n \in \mathbb{N}}$  is non-decreasing, that is,  $a_{n-1} \leq a_n$ , which is equivalent to

$$\begin{aligned} \frac{M(H, n-1)}{(n-1) \cdots (n-t)} &\leq \frac{M(H, n)}{n(n-1) \cdots (n-t+1)} \\ \iff n \cdot M(H, n-1) &\leq (n-t) \cdot M(H, n). \end{aligned}$$

Let us use double counting. Fix an optimal colouring of  $K_n$  for  $M(H, n)$ . Consider all  $\binom{n}{n-1} = n$  induced subgraphs on  $(n-1)$  vertices. The sum of the number of monochromatic labelled copies of  $H$  over all  $(n-1)$ -vertex subgraphs is at least  $n \cdot M(H, n-1)$ . On the other hand, each monochromatic labelled copy of  $H$  of  $K_n$ , of

which there are  $M(H, n)$ , appears exactly  $n-t$  times in these induced subgraphs, which is  $(n-t) \cdot M(H, n)$ . Putting these together, we get the desired inequality.  $\square$

A natural question is whether  $c(H)$  can be zero. The next theorem provides an easy lower bound, which answers this question in the negative.

**Theorem 2.1.2.** *For every graph  $H$ ,*

$$c(H) \geq \frac{\text{aut}(H)}{t!} \cdot \frac{1}{\binom{r(H)}{t}}$$

where  $\text{aut}(H)$  is the number of automorphism of  $H$  and  $t = v(H)$ .

*Proof.* Let  $t$  be the number of vertices of  $H$ . For  $n \geq r(H)$ , we will get a lower bound on  $M(H, n)$  using double counting. For any colouring of  $K_n$ , consider all  $\binom{n}{r(H)}$  induced subgraphs on  $r(H)$  vertices. By Ramsey Theorem, there is at least one monochromatic copy of  $H$  as a subgraph, which means there are  $\text{aut}(H)$  labelled copies. So, the total number of monochromatic labelled copies of  $H$  in all  $r(H)$ -vertex subgraphs is at least  $\binom{n}{r(H)} \cdot \text{aut}(H)$ . For each monochromatic labelled copy of  $H$ , we count it exactly  $\binom{n-t}{r(H)-t}$  times. Then,

$$M(H, n) \geq \frac{\binom{n}{r(H)} \cdot \text{aut}(H)}{\binom{n-t}{r(H)-t}} = \frac{\frac{n! \cdot \text{aut}(H)}{r(H)! \cdot (n-r(H))!}}{\frac{(n-t)!}{(r(H)-t)! \cdot (n-r(H))!}} = \frac{n! \cdot \text{aut}(H)! \cdot (r(H) - t)!}{r(H)! \cdot (n - t)!}$$

Since the sequence  $\left( \frac{M(H, n)}{n(n-1) \cdots (n-t+1)} \right)_{n \in \mathbb{N}}$  is non-decreasing by the proof of Theorem 2.1.1, the following is true:

$$c(H) \geq \frac{M(H, n)}{n(n-1) \cdots (n-t+1)} > \frac{\frac{n! \cdot \text{aut}(H) \cdot (r(H)-t)!}{r(H)! \cdot (n-t)!}}{n(n-1) \cdots (n-t+1)} = \frac{\text{aut}(H) \cdot (r(H) - t)!}{r(H)!},$$

which is equal to what we desired.  $\square$

Lastly, we state the formal theorem that determines an upper bound on how many pendant edges are needed to make a given graph uncommon.

**Theorem 2.1.3** (Jagger, Šťovíček and Thomason [37]). *Let*

$$T_0 = \lceil 4^{e(H)} \cdot e(H) \cdot \log 2 \rceil.$$

*For any  $t \geq T_0$  and any non-bipartite graph  $H$ , the graph obtained from  $H$  by adding pendant trees with at least  $t - e(H)$  edges is uncommon.*

## 2.2 Homomorphism Density

To study the asymptotic behavior of quantities that arise from counting copies, it is convenient to work with normalized quantities. Homomorphism densities provide a natural way to express the Ramsey multiplicity constant. Given graphs  $H$  and  $G$ , the *homomorphism density*  $t(H, G)$  is the number of homomorphisms from  $H$  to  $G$  divided by the number of possible functions from  $V(H)$  to  $V(G)$ :

$$t(H, G) := \frac{\text{hom}(H, G)}{v(G)^{v(H)}}.$$

This number also represents the probability that a uniformly random map of  $V(H)$  into  $V(G)$  is a homomorphism. Similarly, define the *injective homomorphism density*  $t_{inj}(H, G)$  as follows:

$$t_{inj}(H, G) := \frac{\text{hom}_{inj}(H, G)}{v(G)(v(G) - 1) \cdots (v(G) - v(H) + 1)},$$

which is the probability that a uniform random injective mapping is a homomorphism. The Ramsey multiplicity constant can be written in terms of injective homomorphism density as follows:

$$c(H) = \lim_{n \rightarrow \infty} \min_{G_n} \{t_{inj}(H, G_n) + t_{inj}(H, \overline{G_n})\}$$

where  $G_n$  is an  $n$ -vertex graph. Let us now argue that  $c(H)$  can also be defined in terms of homomorphism density. Since the number of non-injective functions from  $V(H)$  to the vertex set of any  $n$ -vertex graph is  $O(n^{v(H)-1})$ , it follows that

$$t(H, G) = t_{inj}(H, G) + o(1) \tag{2.2.1}$$

where the  $o(1)$  term approaches zero as  $v(G) \rightarrow \infty$ . That is, for large graphs  $G$ , the homomorphism density approximates the injective homomorphism density. So far, we have defined the Ramsey multiplicity constant in terms of the number of injective homomorphisms from  $H$  to  $G$ . However, sometimes it will be convenient to count all homomorphisms without injectivity. By (2.2.1),

$$c(H) = \lim_{n \rightarrow \infty} \min_{G_n} \{t(H, G_n) + t(H, \overline{G_n})\}.$$

The following inequality is equivalent to the definition of a common graph:

$$t(H, G) + t(H, \overline{G}) \geq 2 \left(\frac{1}{2}\right)^{e(H)} - o(1), \tag{2.2.2}$$

where the  $o(1)$  term tends to zero as  $v(G) \rightarrow \infty$ .

Now that the homomorphism density has been introduced, we can formally state Sidorenko's Conjecture.

**Conjecture 2.2.3** (Sidorenko's Conjecture [61]). *If  $H$  is bipartite, then*

$$t(H, G) \geq t(K_2, G)^{e(H)} \tag{2.2.4}$$

for every graph  $G$ .

As mentioned in the introduction, we say a graph  $H$  is Sidorenko if it satisfies the inequality (2.2.4) and we will prove the following statement.

**Proposition 2.2.5** (Jagger, Štovíček and Thomason [37]). *If  $H$  is Sidorenko, then  $H$  is commom.*

*Proof.* Let  $p = t(K_2, G)$ , then  $t(K_2, \overline{G}) = 1 - p - o(1)$  and so

$$t(H, G) + t(H, \overline{G}) \geq p^{e(H)} + (1 - p)^{e(H)} - o(1) \geq 2 \cdot \left(\frac{1}{2}\right)^{e(H)} - o(1).$$

□

Currently, every bipartite graph known to be common is also known to be Sidorenko even though the Sidorenko property is formally stronger than the common property.

Several of the known results on common graphs including Sidorenko graphs actually establish stronger inequalities than (2.2.2) as above. Following [4], a non-empty graph  $H$  is said to be *strongly common* if

$$t(H, G) + t(H, \overline{G}) \geq t(K_2, G)^{e(H)} + t(K_2, \overline{G})^{e(H)} - o(1) \tag{2.2.6}$$

for every graph  $G$ . Every Sidorenko graph is strongly common and every strongly common graph is common. A classical example of a strongly common graph is  $K_3$ ; see Theorem 3.2.2. The only known examples of strongly common graphs which are not Sidorenko are the odd cycles [4, 32, 40].

Our strategy for obtaining new examples of common graphs relies on strong correlation inequalities, such as (2.2.4) and (2.2.6). Given this, it is natural to wonder whether all common graphs are strongly common; this question was raised in [4]. As it turns out, this is far from true. For example,  $K_3 \sqcup K_3$  is common but not strongly common, and there are many other examples as well.

**Theorem 2.2.7.** *There exists a common graph  $H$  which is not strongly common.*

After we found the first example of a common graph which is not strongly common, we learned that Chen and Ma [10] obtained a much more general result than

Theorem 2.2.7. Specifically, they prove that every graph containing a triangle, except for  $K_3$  itself, is not strongly common. This was generalized by Versteegen [68], who proved that a graph of odd girth is strongly common if and only if it is an odd cycle.

## 2.3 Spectral Graph Theory

Let  $G$  be a graph with vertices  $v_1, v_2, v_3, \dots, v_n$ . The *adjacency matrix*  $A_G$  of a graph  $G$  is a  $n \times n$  symmetric 0-1 matrix such that the entry of  $A_G(i, j)$  is 1 if  $(v_i, v_j) \in E(G)$  and 0 otherwise.

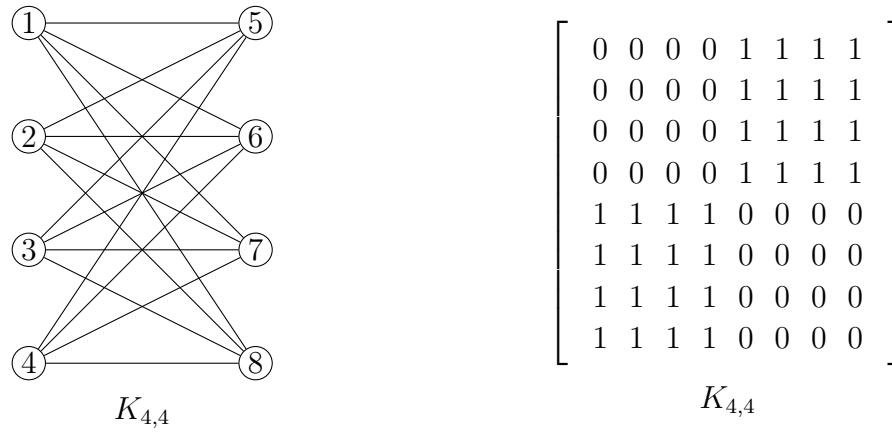


Figure 2.1: A complete bipartite graph  $K_{4,4}$  and its adjacency matrices  $A_{K_{4,4}}$

We could count the number of homomorphisms using the adjacency matrix: ;

$$\text{hom}(H, G) = \sum_{f: V(H) \rightarrow V(G)} \prod_{uv \in E(H)} A_G(f(u), f(v)).$$

The eigenvalues of  $G$  are the eigenvalues of its adjacency matrix  $A_G$ . Since the adjacency matrix is symmetric, all eigenvalues are real and say  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ .

**Theorem 2.3.1** (Lovász [49]). For  $k > 2$ ,

$$\text{hom}(C_k, G) = \sum_{i=1}^n (\lambda_i)^k$$

**Corollary 2.3.2.** For  $k > 2$ ,

$$t(C_k, G) = \frac{\sum_{i=1}^n (\lambda_i)^k}{n^k} = \sum_{i=1}^n \left( \frac{\lambda_i}{n} \right)^k$$

## 2.4 Graph Limits

In this section, we introduce some of the notations from the book “Large Networks and Graph Limits” by Lovász [49]. A sequence of graphs  $(G_n)_{n \in \mathbb{N}}$  where  $v(G_n)$  goes to infinity is said to be *convergent* if the sequence of homomorphism densities  $t(H, G_n)$  converges for every graph  $H$ . That is, for all fixed graphs  $H$ ,

$$\lim_{n \rightarrow \infty} t(H, G_n) \text{ exists.}$$

For example, following are convergent sequences: the sequence of complete graphs  $(K_n)_{n \in \mathbb{N}}$ , the sequence of complete balanced bipartite graphs  $(K_{n,n})_{n \in \mathbb{N}}$ , the sequence of complete balanced tripartite graphs  $(K_{n,n,n})_{n \in \mathbb{N}}$  and the sequence of random graphs  $(G(n, p))_{n \in \mathbb{N}}$  for fixed  $p \in [0, 1]$ .

A *graphon* is a measurable function  $W : [0, 1]^2 \rightarrow [0, 1]$  such that  $W(x, y) = W(y, x)$  (i.e.  $W$  is *symmetric*) for all  $x, y \in [0, 1]$ . It is a limit object for a convergent sequence of graphs. Intuitively, it is a continuous generalization of an adjacency matrix. We could represent each graph  $G$  to an associated graphon  $W_G$  by dividing  $[0, 1]$  into  $v(G)$  intervals  $I_1, \dots, I_{v(G)}$  of equal measure corresponding to the vertices of  $G$  and setting  $W_G$  equal to 1 on  $I_i \times I_j$  if the  $i$ th and  $j$ th vertices are adjacent and 0 otherwise.

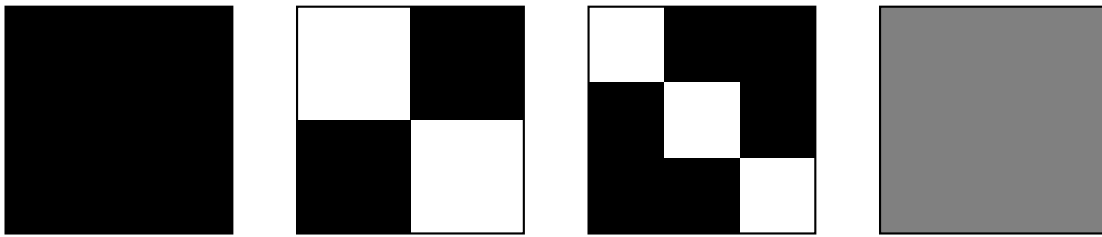


Figure 2.2: The graphons corresponding to the sequences  $(K_n)_{n \in \mathbb{N}}$ ,  $(K_{n,n})_{n \in \mathbb{N}}$ ,  $(K_{n,n,n})_{n \in \mathbb{N}}$  and  $(G(n, \frac{1}{2}))_{n \in \mathbb{N}}$

Given a graph  $H$  with vertex set  $V(H)$ , the *homomorphism density* of  $H$  in a graphon  $W$  is defined by

$$t(H, W) = \int_{[0,1]^{v(H)}} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{v \in V(H)} dx_v.$$

More generally, it is sometimes useful to consider generalizations of graphons with codomain  $\mathbb{R}$  as opposed to  $[0, 1]$ . A *kernel* is a bounded symmetric measurable function  $U : [0, 1]^2 \rightarrow \mathbb{R}$ . While a graphon is typically used to represent the limit of a sequence of simple (unweighted) graphs, a kernel can be viewed as its natural generalization to the setting of weighted graphs. Just as a graphon serves as a continuous analogue of an adjacency matrix, a kernel serves as a continuous analogue of a weighted adjacency matrix.

Every asymptotic inequality involving homomorphism density function in graphs is equivalent to an analogous inequality in graphons. For example, a non-empty graph  $H$  is

- *Sidorenko* if and only if for every graphon  $W$ ,

$$t(H, W) \geq t(K_2, W)^{e(H)},$$

- *common* if and only if for every graphon  $W$ ,

$$t(H, W) + t(H, 1 - W) \geq 2 \cdot \left(\frac{1}{2}\right)^{e(H)},$$

- *strongly common* if and only if for every graphon  $W$ ,

$$t(H, W) + t(H, 1 - W) \geq t(K_2, W)^{e(H)} + t(K_2, 1 - W)^{e(H)}.$$

## 2.5 Bonbon, Bonbon Pair and Ramsey Multiplicity Good

As mentioned in the introduction, we now present the formal definition of a bonbon.

**Definition 2.5.1** (Fox and Wigderson [26]). A non-empty graph  $H$  is said to be a *bonbon* if there exists  $n_0 = n_0(H)$  such that, if  $n \geq n_0$  and  $G$  is an  $n$ -vertex graph such that

$$t_{\text{inj}}(H, G) + t_{\text{inj}}(H, \overline{G})$$

is minimized over all  $n$ -vertex graphs, then either  $G$  or  $\overline{G}$  is a Turán graph with  $\chi(H) - 1$  parts.

We now state the theorem that determines how many pendants suffice to make a bonbon from a given graph.

**Theorem 2.5.2** (Fox and Wigderson [26, Theorem 1.2]). *For any connected graph  $F$  with chromatic number  $k > 3$  that contains a critical edge, there exists a number*

$$t_0 = (1000 \cdot v(F) \cdot k)^{10} \cdot v(F)^{10k}$$

*such that, for any  $t \geq t_0$ , every  $t$ -hairly  $F$  is a bonbon.*

The formal definition of the Ramsey multiplicity constant in an off-diagonal setting is as follows.

**Definition 2.5.3** (Moss and Noel [51]). For non-empty graphs  $H_1$  and  $H_2$  and  $\lambda \in [0, 2]$ , define

$$c_\lambda(H_1, H_2) := \lim_{n \rightarrow \infty} \left[ \min_{G: v(G)=n} (\lambda \cdot t(H_1, G) + (2 - \lambda) \cdot t(H_2, \overline{G})) \right].$$

**Lemma 2.5.4** (Moss and Noel [51, Lemma 2.11]). *For any non-empty graphs  $H_1$  and  $H_2$  and  $\lambda \in (0, 2)$  we have  $c_\lambda(H_1, H_2) > 0$ .*

The following is the formal definition of a bonbon pair, naturally derived from the definition of a bonbon. Before stating it, we define  $k(H)$  to be the number of connected components of a graph  $H$ .

**Definition 2.5.5.** A pair  $(H_1, H_2)$  of non-empty graphs is a *bonbon pair* if there exists  $n_0 = n_0(H_1, H_2)$  such that, if  $n \geq n_0$  and  $G$  is an  $n$ -vertex graph such that

$$(\chi(H_2) - 1)^{v(H_1) - k(H_1)} \cdot t_{\text{inj}}(H_1, G) + (\chi(H_1) - 1)^{v(H_2) - k(H_2)} \cdot t_{\text{inj}}(H_2, \overline{G})$$

is minimized over all  $n$ -vertex graphs, then either  $G$  is a Turán graph with  $\chi(H_1) - 1$  parts or  $\overline{G}$  is a Turán graph with  $\chi(H_2) - 1$  parts.

Note that a graph  $H$  is a bonbon if and only if  $(H, H)$  is a bonbon pair. The coefficients on  $t_{\text{inj}}(H_1, G)$  and  $t_{\text{inj}}(H_2, \overline{G})$  in the above definition are chosen so that the two different Turán colourings achieve the same value, asymptotically. That is, if  $G$  is a Turán graph with  $\chi(H_1) - 1$  parts or  $\overline{G}$  is a Turán graph with  $\chi(H_2) - 1$  parts, we have

$$(\chi(H_2) - 1)^{v(H_1) - k(H_1)} \cdot t_{\text{inj}}(H_1, G) + (\chi(H_1) - 1)^{v(H_2) - k(H_2)} \cdot t_{\text{inj}}(H_2, \overline{G}) = 1 - o(1).$$

**Remark 2.5.6.** The key definition in [51] is the *balanced Ramsey multiplicity constant* of  $(H_1, H_2)$ , defined by  $c(H_1, H_2) := \sup_{\lambda \in [0, 2]} c_\lambda(H_1, H_2)$ . For and  $\lambda \in [0, 2]$  and any graphs  $H_1$  and  $H_2$ , we have

$$c_\lambda(H_1, H_2) \leq \frac{2}{(\chi(H_2) - 1)^{v(H_1) - k(H_1)} + (\chi(H_1) - 1)^{v(H_2) - k(H_2)}}.$$

Indeed, this can be shown by taking one of the two Turán colourings from Definition 2.5.5, where the specific choice depends on whether

$$\lambda \leq \frac{2(\chi(H_2) - 1)^{v(H_1) - k(H_1)}}{(\chi(H_2) - 1)^{v(H_1) - k(H_1)} + (\chi(H_1) - 1)^{v(H_2) - k(H_2)}}$$

or not. Combining this with the definition of a bonbon pair, we get that, if  $(H_1, H_2)$

is a bonbon pair, then

$$c(H_1, H_2) = \frac{2}{(\chi(H_2) - 1)^{v(H_1) - k(H_1)} + (\chi(H_1) - 1)^{v(H_2) - k(H_2)}}.$$

Lastly, we formulate the definition of Ramsey multiplicity goodness. We say a graph  $H$  is *Ramsey multiplicity good* if and only if the following holds: for all graphs  $G$ ,

$$t(H, G) + t(H, \overline{G}) \geq \left( \frac{1}{\chi(H) - 1} \right)^{v(H) - k(H)} - o(1).$$

Equivalently, for all graphons  $W$ ,

$$t(H, W) + t(H, 1 - W) \geq \left( \frac{1}{\chi(H) - 1} \right)^{v(H) - k(H)}.$$

This definition can be naturally extended to the off-diagonal setting. Say that  $(H_1, H_2)$  is *Ramsey multiplicity good* if

$$(\chi(H_2) - 1)^{v(H_1) - k(H_1)} t(H_1, G) + (\chi(H_1) - 1)^{v(H_2) - k(H_2)} t(H_2, \overline{G}) \geq 1 - o(1)$$

for all graphs  $G$ . Clearly, every bonbon pair is Ramsey multiplicity good. Moreover, we say that a graph  $H$  is *Ramsey multiplicity good* if and only if the pair  $(H, H)$  is Ramsey multiplicity good.

# Chapter 3

## Disconnected Common Graphs

This chapter is organized as follows. In Section 3.1, we prove a general lower bound on  $c(H)$  for graphs  $H$  satisfying certain correlation inequalities involving homomorphism densities. This lower bound is written in terms of the minimum of a certain constrained optimization problem over two variables  $x$  and  $y$ . In Section 3.2, we illustrate this result by presenting a proof of the lower bound in Theorem 1.0.7. Our goal in Section 3.3 is to do a bit of calculus to remove the dependence on the variable  $y$  in the optimization problem. In Section 3.4, we showcase some applications of this approach involving graphs built up from triangles and edges via certain gluing operations. In particular, we prove Theorems 1.0.2 and 1.0.4. In Section 3.5, we turn our attention to negative examples; in particular, we prove Theorems 1.0.5 and 2.2.7 and the upper bounds in Theorems 1.0.7 and 1.0.6. The last section consists of open problems and further discussion. In the appendix, we derive some simpler sufficient conditions for bounding the minimum of the optimization problem, these are applied in Section 3.4.

### 3.1 A Bound on the Ramsey Multiplicity Constant

We now define a type of correlation inequality that appears frequently in the study of extremal problems on homomorphism densities; see, e.g., [4, 6, 33, 48].

**Definition 3.1.1.** Let  $H$  and  $J$  be non-empty graphs and let  $k$  and  $\ell$  be real numbers such that  $k \neq 0$ . We say that  $H$  is  $(J, k, \ell)$ -correlated if

$$(3.1.2) \quad e(H) = ke(J) + \ell \text{ and}$$

$$(3.1.3) \quad t(H, W) \geq t(J, W)^k t(K_2, W)^\ell \text{ for every graphon } W.^1$$

Our approach to obtaining new examples of common graphs relies on certain “supersaturation bounds” from extremal graph theory. That is, we require a lower bound on  $t(J, W)$  for a graph  $J$  in terms of  $t(K_2, W)$ .

---

<sup>1</sup>In the case that  $\ell = 0$  and  $t(K_2, W) = 0$ , we regard  $0^0$  as being equal to 1.

**Definition 3.1.4.** Given a non-empty graph  $J$ , let  $\rho_J : [0, 2] \rightarrow [0, \infty)$  be the function defined by

$$\rho_J(z) := 2^{e(J)} \inf\{t(J, W) : W \in \mathcal{W}_0 \text{ and } t(K_2, W) = z/2\}$$

for all  $0 \leq z \leq 2$ .

We pause for a few basic observations.

**Observation 3.1.5.** We have  $\rho_J(1+x) \leq (1+x)^{e(J)}$  for any non-empty graph  $J$  and  $-1 \leq x \leq 1$ .

*Proof.* Consider the constant graphon  $W$  which is equal to  $\frac{1}{2}(1+x)$  everywhere. Then  $\rho_J(1+x) \leq 2^{e(J)}t(J, W) = (1+x)^{e(J)}$ .  $\square$

**Observation 3.1.6.**  $\rho_J(1+x) = (1+x)^{e(J)}$  for all  $-1 \leq x \leq 1$  if and only if  $J$  is Sidorenko.

*Proof.* If  $J$  is Sidorenko and  $x \in [-1, 1]$ , then  $t(J, W) \geq \left(\frac{1}{2}(1+x)\right)^{e(J)}$  for every graphon with  $t(K_2, W) = \frac{1}{2}(1+x)$ . Combining this with the upper bound in Observation 3.1.5 yields  $\rho_J(1+x) = (1+x)^{e(J)}$ . If  $J$  is not Sidorenko, then there exists a graphon  $W$  such that  $t(J, W) < t(K_2, W)^{e(J)}$ . Thus, for  $x = 2t(K_2, W) - 1$ , we have  $\rho_J(1+x) < (1+x)^{e(J)}$ .  $\square$

All of the new examples of common graphs described in this chapter will be built up from known examples of strongly common graphs. However, it seems to us that the same approach may also work with graphs that are not strongly common, but satisfy weaker inequalities. This motivates the next definition.

**Definition 3.1.7.** Given a function  $g : [0, 2] \rightarrow [0, \infty)$ , we say that a graph  $H$  is  $g$ -bounded if, for  $-1 \leq x \leq 1$ , it holds that

$$2^{e(H)} (t(H, W) + t(H, 1 - W)) \geq g(1+x) + g(1-x)$$

for every graphon  $W$  such that  $t(K_2, W) = \frac{1}{2}(1+x)$ .

Let us tie this in with the notion of strongly common graphs.

**Observation 3.1.8.** Let  $H$  be a non-empty graph and let  $g : [0, 2] \rightarrow [0, \infty)$  be defined by  $g(z) = z^{e(H)}$ . Then  $H$  is strongly common if and only if it is  $g$ -bounded.

*Proof.* If  $H$  is strongly common, then, for any  $0 \leq x \leq 1$  and graphon  $W$  with  $t(K_2, W) = \frac{1}{2}(1+x)$ , we have

$$2^{e(H)} (t(H, W) + t(H, 1 - W)) \geq 2^{e(H)} (t(K_2, W)^{e(H)} + t(K_2, 1 - W)^{e(H)})$$

$$= 2^{e(H)} \left( \left( \frac{1+x}{2} \right)^{e(H)} + \left( \frac{1-x}{2} \right)^{e(H)} \right) = g(1+x) + g(1-x)$$

and so  $H$  is  $g$ -bounded.

On the other hand, suppose that  $H$  is  $g$ -bounded. Let  $W$  be a graphon and define  $x = 2t(K_2, W) - 1$ . We get

$$t(H, W) + t(H, 1-W) \geq 2^{-e(H)} (g(1+x) + g(1-x)) = t(K_2, W)^{e(H)} + t(K_2, 1-W)^{e(H)}$$

and so  $H$  is strongly common.  $\square$

Given a  $g$ -bounded graph  $J$  and  $\rho \leq \rho_J$ , the following lemma implies a bound on the Ramsey multiplicity constant  $c(H)$  in terms of  $g, k, \ell$  and  $\rho$  for any graph  $H$  which is  $(J, k, \ell)$ -correlated.

**Lemma 3.1.9.** *Let  $g, \rho : [0, 2] \rightarrow [0, \infty)$  such that  $\rho \leq g$ , let  $k$  and  $\ell$  be real numbers such that  $k \neq 0$  and let  $J$  be a non-empty  $g$ -bounded graph such that  $\rho_J \geq \rho$ . Then, for any graphon  $W$ ,  $2^{k e(J) + \ell} (t(K_2, W)^\ell t(J, W)^k + t(K_2, 1-W)^\ell t(J, 1-W)^k)$  is at least the minimum of*

$$f_{g,k,\ell}(x, y) := (1+x)^\ell (g(1+x) - y)^k + (1-x)^\ell (g(1-x) + y)^k \quad (3.1.10)$$

over all  $x$  and  $y$  such that  $-1 \leq x \leq 1$  and  $0 \leq y \leq g(1+x) - \rho(1+x)$ .

*Proof.* Define  $m := e(J)$ . Let  $W$  be a graphon and let  $x := 2t(K_2, W) - 1$ . By the symmetry between  $W$  and  $1-W$ , we may assume, without loss of generality, that

$$g(1+x) - 2^m t(J, W) \geq g(1-x) - 2^m t(J, 1-W). \quad (3.1.11)$$

Now, define

$$y := \max\{0, g(1+x) - 2^m t(J, W)\}.$$

Then, by definition of  $y$ , we have

$$2^m t(J, W) \geq g(1+x) - y \geq 0. \quad (3.1.12)$$

Since  $J$  is  $g$ -bounded and  $m = e(J)$ , it holds that

$$2^m (t(J, W) + t(J, 1-W)) \geq g(1+x) + g(1-x)$$

which is equivalent to

$$2^m t(J, 1-W) - g(1-x) \geq g(1+x) - 2^m t(J, W).$$

If  $y = g(1+x) - 2^m t(J, W)$ , then the above inequality translates to  $2^m t(J, 1-W) \geq$

$g(1-x) + y$ . On the other hand, if  $y = 0$ , then, by definition of  $y$ , we must have  $g(1+x) - 2^m t(J, W) \leq 0$ . By (3.1.11), this implies that  $g(1-x) - 2^m t(J, 1-W) \leq 0$ . So, in the case that  $y = 0$ , we again get  $2^m t(J, 1-W) \geq g(1-x) + y$ . Thus, regardless of the value of  $y$ , we have

$$2^m t(J, 1-W) \geq g(1-x) + y \geq 0. \quad (3.1.13)$$

Now, since  $x = 2t(K_2, W) - 1$  and  $t(K_2, W) + t(K_2, 1-W) = 1$ ,

$$\begin{aligned} & t(K_2, W)^\ell t(J, W)^k + t(K_2, 1-W)^\ell t(J, 1-W)^k \\ &= \left(\frac{1+x}{2}\right)^\ell t(J, W)^k + \left(\frac{1-x}{2}\right)^\ell t(J, 1-W)^k. \end{aligned}$$

By (3.1.12) and (3.1.13), the right side of the above equality is at least

$$2^{-km-\ell}(1+x)^\ell (g(1+x) - y)^k + 2^{-km-\ell}(1-x)^\ell (g(1-x) + y)^k = 2^{-ke(J)-\ell} f_{g,k,\ell}(x, y).$$

To complete the proof, we need to show that  $x$  and  $y$  satisfy the constraints  $-1 \leq x \leq 1$  and  $0 \leq y \leq g(1+x) - \rho(1+x)$ . The constraint  $-1 \leq x \leq 1$  follows simply from the fact that  $x = 2t(K_2, W) - 1$  and  $0 \leq t(K_2, W) \leq 1$ . Also,  $y \geq 0$  simply by definition. If  $y = 0$ , then the constraint  $y \leq g(1+x) - \rho(1+x)$  holds by the hypothesis  $\rho \leq g$  of the lemma. On the other hand, if  $y = g(1+x) - 2^m t(J, W)$ , then, since  $x = 2t(K_2, W) - 1$ , we have

$$\begin{aligned} y &= g(1+x) - 2^m t(J, W) \leq g(1+x) - 2^m \inf\{t(J, W) : t(K_2, W) = (1+x)/2\} \\ &= g(1+x) - \rho_J(1+x) \leq g(1+x) - \rho(1+x). \end{aligned}$$

The result follows.  $\square$

**Remark 3.1.14.** For most graphs  $J$  satisfying the hypotheses of Lemma 3.1.9, it seems unlikely that  $t(J, W)$  would be anywhere near  $2^{-e(J)}\rho(2t(K_2, W))$  for the graphon  $W$  which minimizes  $t(K_2, W)^\ell t(J, W)^k + t(K_2, 1-W)^\ell t(H, 1-W)^k$ . Thus, one may be able to tighten the upper bound constraint on  $y$  in certain cases (or even in general). However, while it is perhaps not the most powerful constraint possible, the bound  $y \leq g(1+x) - \rho(1+x)$  is sufficient to obtain nearly sharp bounds on the Ramsey multiplicity constant of certain graphs, as we shall demonstrate in the next section.

## 3.2 A Quick Application

As a simple application of Lemma 3.1.9, we prove the lower bound in Theorem 1.0.7, which we restate here for convenience.

**Proposition 3.2.1.**  $c(K_3 \sqcup K_2) > 0.121423$ .

Before proving this proposition, we require a few preparations. First, we need that  $K_3$  is strongly common, which essentially follows from an old result of Goodman; see, e.g., [4, Theorem 5.3] for a modern proof.

**Theorem 3.2.2** (Goodman’s Theorem [32]).  $K_3$  is strongly common.

We also require bounds on  $\rho_{K_3}(1+x)$ . The precise value of this function for all  $-1 \leq x \leq 1$  can be derived from the landmark “Triangle Density Theorem” of Razborov [57]. For the purposes of proving Proposition 3.2.1, it will be enough to have a tight bound on  $\rho_{K_3}(1+x)$  when  $x$  is close to zero and a more crude bound for larger  $x$ . For the former, we use the following theorem which was first announced by Fisher [23]; as mentioned in [57], the proof contained a hole that can be patched using a later result of [31]. A new proof was found by Razborov [56] prior to proving the general Triangle Density Theorem in [57].

**Theorem 3.2.3** (Fisher [23] and Goldwurm and Santini [31]; see also Razborov [56]). *Every graphon  $W$  with  $t(K_2, W) \leq \frac{2}{3}$  satisfies*

$$t(K_3, W) \geq \frac{1}{9} \left( -2 \left( 2 + \sqrt{4 - 6t(K_2, W)} \right) + 3t(K_2, W) \left( 3 + \sqrt{4 - 6t(K_2, W)} \right) \right)$$

For larger edge densities, we resort to the following convenient linear bound proved by Bollobás [7] (see also [8, Chapter VI.1]).

**Theorem 3.2.4** (Bollobás [7]). *Every graphon  $W$  satisfies*

$$t(K_3, W) \geq \frac{4}{3}t(K_2, W) - \frac{2}{3}.$$

We combine Theorems 3.2.3 and 5.1.2 to get the following general lower bound on  $\rho_{K_3}(1+x)$ .

**Corollary 3.2.5.** *We have*

$$\rho_{K_3}(1+x) \geq \begin{cases} 0 & \text{for } -1 \leq x < 0, \\ \frac{4}{9} (1 - \sqrt{1 - 3x} + 3x (3 + \sqrt{1 - 3x})) & \text{for } 0 \leq x \leq 1/3 \\ \frac{16}{3}x & \text{for } 1/3 < x \leq 1. \end{cases}$$

*Proof.* The fact that  $\rho_{K_3}(1+x) \geq 0$  for all  $x$  is simply by definition. The bound  $\rho_{K_3}(1+x) \geq \frac{16}{3}x$  for all  $-1 \leq x \leq 1$  comes from plugging  $x = 2t(K_2, W) - 1$  into

Theorem 5.1.2. Finally, for  $0 \leq x \leq \frac{1}{3}$ , plugging  $x = 2t(K_2, W) - 1$  into Theorem 3.2.3 yields

$$\rho_{K_3}(1+x) \geq \frac{4}{9} \left( 1 - \sqrt{1-3x} + 3x(3 + \sqrt{1-3x}) \right).$$

The result follows.  $\square$

We will also use the following basic fact about homomorphism densities.

**Observation 3.2.6.** For any graphs  $F$  and  $H$  and graphon  $W$ ,

$$t(F \sqcup H, W) = t(F, W)t(H, W).$$

We now present the proof of Proposition 3.2.1.

*Proof of Proposition 3.2.1.* By Observation 3.2.6, we have

$$t(K_3 \sqcup K_2, W) = t(K_3, W)t(K_2, W)$$

for every graphon  $W$ . By Theorem 3.2.2 and Observation 3.1.8,  $K_3$  is  $z^3$ -bounded. So, by Lemma 3.1.9,  $c(K_3 \sqcup K_2)$  is at least the minimum of

$$2^{-4}(1+x)((1+x)^3 - y) + 2^{-4}(1-x)((1-x)^3 + y)$$

over all  $x$  and  $y$  such that  $-1 \leq x \leq 1$  and  $0 \leq y \leq (1+x)^3 - \rho_{K_3}(1+x)$ . Expanding the above expression yields

$$2^{-4}(1+x)^4 + 2^{-4}(1-x)^4 - 2^{-3}xy. \tag{3.2.7}$$

It suffices to bound the expression in (3.2.7) below by 0.121423 for all  $x$  and  $y$  satisfying the constraints described above. First, if  $x \leq 0$ , then, since  $y \geq 0$ , the expression in (3.2.7) is at least 0.125. So, we assume that  $x > 0$ . In this case, the expression in (3.2.7) is minimized when  $y$  is as large as possible.

If  $\frac{1}{3} < x \leq 1$ , then Corollary 3.2.5 tells us that the expression in (3.2.7) is at least

$$2^{-4}(1+x)^4 + 2^{-4}(1-x)^4 - 2^{-3}x \left( (1+x)^3 - \frac{16}{3}x \right).$$

The derivative of this expression with respect to  $x$  is  $-\frac{1}{24}(27x^2 - 50x + 3)$ , which is positive for all  $x$  such that  $\frac{1}{3} \leq x \leq 1$ ; thus, when  $\frac{1}{3} < x \leq 1$  and  $0 \leq y \leq (1+x)^3 - \rho_{K_3}(1+x)$ , the expression in (3.2.7) is bounded below by

$$2^{-4} \left( 1 + \frac{1}{3} \right)^4 + 2^{-4} \left( 1 - \frac{1}{3} \right)^4 - 2^{-3} \left( \frac{1}{3} \right) \left( \left( 1 + \frac{1}{3} \right)^3 - \frac{16}{9} \right) = \frac{5}{27} > 0.121423.$$

Finally, we assume that  $0 \leq x \leq \frac{1}{3}$ . In this case, by Corollary 3.2.5, the expression in (3.2.7) is at least

$$2^{-4}(1+x)^4 + 2^{-4}(1-x)^4 - 2^{-3}x((1+x)^3 - \frac{4}{9}(1 - \sqrt{1-3x} + 3x(3 + \sqrt{1-3x}))).$$

If we let  $z = \sqrt{1-3x}$ , then the above expression can be rewritten as

$$\frac{1}{216}(3z^6 + 4z^5 + 12z^4 - 4z^3 - 28z^2 + 40). \quad (3.2.8)$$

Our final aim is to minimize this function over all  $0 \leq z \leq 1$ . The derivative is

$$\frac{1}{108}z(9z^4 + 10z^3 + 24z^2 - 6z - 28).$$

So, in particular,  $z = 0$  is a critical point. However, if  $z = 0$ , then the expression in (3.2.8) evaluates to  $\frac{5}{27} > 0.121423$ .

So, we focus on the other critical points; i.e. the roots of the function  $h(z) := 9z^4 + 10z^3 + 24z^2 - 6z - 28$ . Note that  $h(0) < 0$  and

$$\frac{d^2h}{dz^2} = \frac{119}{3} + \frac{1}{3}(5 + 18z)^2$$

which is positive for all  $z$ . Thus, there is at most one root of  $h$ , say  $z'$ , in the interval  $[0, 1]$ . The exact value of this root is rather complicated, so we will not attempt to write it down, but we can estimate it. Define

$$z_0 := 0.908638793,$$

$$z_1 := 0.908638794.$$

By plugging these values into  $h$ , we see  $h(z_0) < 0$  and  $h(z_1) > 0$ . So, the unique root of  $h$  in  $[0, 1]$  is between  $z_0$  and  $z_1$ . Thus, the minimum of the expression in (3.2.8) over all  $0 \leq z \leq 1$  is at least

$$\frac{1}{216}(3z_0^6 + 4z_0^5 + 12z_0^4 - 4z_1^3 - 28z_1^2 + 40) > 0.121423.$$

This completes the proof. □

**Remark 3.2.9.** By applying the flag algebra method of Razborov [56], one can obtain a lower bound on  $c(K_3 \sqcup K_2)$  of roughly 0.121449536 which is slightly better than that of Theorem 3.2.1; note that this is a raw floating point number from a semi-definite program that was approximately solved by a computer, and so it should be taken with a grain of salt. Thus, while it seems that Lemma 3.1.9 may not produce a tight bound in this case, it is within about 0.022% of the lower bound that one can prove using

some of the most powerful modern machinery available. It also has the advantage of being relatively short and human-checkable, which is rarely the case for proofs which use the flag algebra method.

### 3.3 Reducing the Optimization Problem

Our goal in this section is to use a bit of basic calculus to prove Proposition 3.3.1, stated below, which reduces the optimization problem in Lemma 3.1.9 to a single-variable problem. In our applications, we will always deal with  $k \geq 1$  and non-decreasing functions  $g$  and  $\rho$ , and so we will focus on this case. If  $k = 1$ , then  $f_{g,k,\ell}(x, y)$  is linear in the variable  $y$  and so, for any  $x$ , the minimum is achieved at either  $y = 0$  or  $y = g(1+x) - \rho(1+x)$ . In other words, when  $k = 1$ , the problem reduces to a single-variable problem trivially; see, e.g., the proof of Proposition 3.2.1 above. Thus, in the following proposition, we only deal with  $k > 1$ .

**Proposition 3.3.1.** *Let  $k$  and  $\ell$  be real numbers such that  $k > 1$  and let  $g, \rho : [0, 2] \rightarrow [0, \infty)$  be non-decreasing functions such that  $\rho \leq g$  and  $\rho(2) > 0$ . For  $0 < c \leq 2g(1)^k$ , if, for every  $0 \leq x \leq 1$ , either*

$$\frac{(1+x)^\ell(1-x)^\ell(g(1+x) + g(1-x))^k}{\left((1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}}\right)^{k-1}} \geq c. \quad (x \approx 0)$$

or both of the following hold

$$\frac{g(1+x)(1+x)^{\frac{\ell}{k-1}} - g(1-x)(1-x)^{\frac{\ell}{k-1}}}{(1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}}} \geq g(1+x) - \rho(1+x), \quad \text{and} \quad (x \approx 1)$$

$$(1+x)^\ell \rho(1+x)^k + (1-x)^\ell (g(1-x) + g(1+x) - \rho(1+x))^k \geq c, \quad (x \approx 1')$$

then  $f_{g,k,\ell}(x, y) \geq c$  for all  $-1 \leq x \leq 1$  and  $0 \leq y \leq g(1+x) - \rho(1+x)$ .

In order to prove this proposition, we require the following basic inequality.

**Lemma 3.3.2.** *For any real numbers  $a, b$  and  $k$  such that  $a \geq 0$ ,  $a + b \geq 0$  and  $k > 1$ ,*

$$(a+b)^k \geq a^k + kba^{k-1}.$$

*Proof.* If  $a = 0$ , then the inequality  $a + b \geq 0$  simply translates to  $b \geq 0$ . In this case, the left side is equal to  $b^k \geq 0$  and the right side is zero, and so the inequality holds. Now, if  $a > 0$ , then, by Bernoulli's Inequality (i.e. the fact that  $(1+z)^r \geq 1 + rz$

provided that  $1 + z \geq 0$  and  $r \geq 1$ ),

$$(a + b)^k = a^k \left(1 + \frac{b}{a}\right)^k \geq a^k \left(1 + \frac{kb}{a}\right) = a^k + kba^{k-1}.$$

This completes the proof.  $\square$

*Proof of Proposition 3.3.1.* We start by showing that the range of  $x$  in the optimization problem in Lemma 3.1.9 can be reduced to  $0 < x \leq 1$ . Fix  $-1 \leq x \leq 0$ . Applying Lemma 3.3.2 to both terms of  $f_{g,k,\ell}(x, y)$ , we get

$$f_{g,k,\ell}(x, y) = (1 + x)^\ell (g(1 + x) - y)^k + (1 - x)^\ell (g(1 - x) + y)^k$$

$$\geq (1 + x)^\ell g(1 + x)^k + (1 - x)^\ell g(1 - x)^k - yk (g(1 + x)^{k-1}(1 + x)^\ell - g(1 - x)^{k-1}(1 - x)^\ell).$$

Now, using the facts that  $y \geq 0$ ,  $g$  is non-decreasing and  $k > 1$ , we see that, if  $x \leq 0$ , then the above expression is at least

$$(1 + x)^\ell g(1 + x)^k + (1 - x)^\ell g(1 - x)^k.$$

However, this is precisely equal to  $f_{g,k,\ell}(-x, 0)$ . Therefore, under the constraint  $y \geq 0$ , the minimum of  $f_{g,k,\ell}(x, y)$  must be attained at a point  $(x, y)$  such that  $x \geq 0$ . If  $x = 0$ , then the above expression evaluates to  $2g(1)^k$  which is at least  $c$  by assumption. So, we can assume that  $0 < x \leq 1$ .

From here forward, fix  $0 < x \leq 1$ . We find and analyze the critical points with respect to  $y$ . Observe that

$$\frac{\partial f_{g,k,\ell}}{\partial y}(x, y) = -k(1 + x)^\ell (g(1 + x) - y)^{k-1} + k(1 - x)^\ell (g(1 - x) + y)^{k-1}. \quad (3.3.3)$$

Setting this equal to zero yields

$$(1 + x)^\ell (g(1 + x) - y)^{k-1} = (1 - x)^\ell (g(1 - x) + y)^{k-1}.$$

Taking the  $(k - 1)$ th root of both sides gives us

$$(1 + x)^{\frac{\ell}{k-1}} (g(1 + x) - y) = \pm (1 - x)^{\frac{\ell}{k-1}} (g(1 - x) + y)$$

$$\implies y \left( (1 + x)^{\frac{\ell}{k-1}} \pm (1 - x)^{\frac{\ell}{k-1}} \right) = (1 + x)^{\frac{\ell}{k-1}} g(1 + x) \mp (1 - x)^{\frac{\ell}{k-1}} g(1 - x).$$

Thus, for any choice of  $x$ , the partial derivative of  $f_{g,k,\ell}(x, y)$  with respect to  $y$  is zero

at  $y = y_0(x)$  where

$$y_0(x) := \frac{(1+x)^{\frac{\ell}{k-1}}g(1+x) - (1-x)^{\frac{\ell}{k-1}}g(1-x)}{(1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}}}.$$

There may also be a second such point  $y_1(x)$  where

$$y_1(x) := \frac{(1+x)^{\frac{\ell}{k-1}}g(1+x) + (1-x)^{\frac{\ell}{k-1}}g(1-x)}{(1+x)^{\frac{\ell}{k-1}} - (1-x)^{\frac{\ell}{k-1}}}.$$

However,

$$\begin{aligned} y_1(x) &= \frac{(1+x)^{\frac{\ell}{k-1}}g(1+x) + (1-x)^{\frac{\ell}{k-1}}g(1-x)}{(1+x)^{\frac{\ell}{k-1}} - (1-x)^{\frac{\ell}{k-1}}} \geq \frac{(1+x)^{\frac{\ell}{k-1}}g(1+x) + (1-x)^{\frac{\ell}{k-1}}g(1-x)}{(1+x)^{\frac{\ell}{k-1}}} \\ &\geq g(1+x) \geq g(1+x) - \rho(1+x). \end{aligned}$$

So,  $y_1(x) \geq g(1+x) - \rho(1+x)$ . We claim that this inequality is strict; indeed, if  $x < 1$ , then the first two inequalities above are both strict and, if  $x = 1$ , then we have  $g(1+x) > g(1+x) - \rho(1+x)$  because  $\rho(2) > 0$ . Thus, when  $y = y_1(x)$ , the constraint  $y \leq g(1+x) - \rho(1+x)$  is violated, and so we can ignore the possibility that  $y = y_1(x)$  in what follows.

From this, we can conclude that, for fixed  $0 < x \leq 1$ , the minimum of the function  $f_{g,k,\ell}(x, y)$  on  $0 \leq y \leq g(1+x) - \rho(1+x)$  is achieved at either  $y = 0$ ,  $y = y_0(x)$  or  $y = g(1+x) - \rho(1+x)$ . However, plugging  $y = 0$  into (3.3.3) and using the fact that  $x > 0$  and  $g$  is non-decreasing reveals that  $f_{g,k,\ell}(x, y)$  is decreasing with respect to  $y$  at the point  $y = 0$ . Thus, the only possibilities to consider are  $y = y_0(x)$  or  $y = g(1+x) - \rho(1+x)$ . To determine which of the two is the minimum, we use the second derivative; we have

$$\frac{\partial^2 f_{g,k,\ell}}{\partial y^2}(x, y) = k(k-1)(1+x)^\ell (g(1+x) - y)^{k-2} + k(k-1)(1-x)^\ell (g(1-x) + y)^{k-2}$$

which is clearly positive for all  $y$  in the range  $0 \leq y \leq g(1+x)$ . Therefore, we conclude that the minimum is attained at  $y = y_0(x)$  whenever  $y_0(x) < g(1+x) - \rho(1+x)$  and it is attained at  $y = g(1+x) - \rho(1+x)$  otherwise. The inequality in  $(x \approx 0)$  simply translates to  $f(x, y_0(x)) \geq c$ , whereas  $(x \approx 1)$  and  $(x \approx 1')$  are equivalent to  $y_0(x) \geq g(1+x) - \rho(1+x)$  and  $f(x, g(1+x) - \rho(1+x)) \geq c$ , respectively. This completes the proof.  $\square$

Let us briefly remark on the way that the interplay between the two conditions in Proposition 3.3.1 tends to work in practice. For fixed  $g$  and  $\ell$ , when  $k$  is large, one can often get that  $(x \approx 0)$  holds for all  $x$  sufficiently close to zero, i.e., in an interval of

the form  $[0, x_0]$ . Conversely,  $(x \approx 1)$  and  $(x \approx 1')$  tend to hold when  $x$  is close to one, i.e., in an interval of the form  $[x_1, 1]$ ; this explains the way that we have labelled of the three inequalities of Proposition 3.3.1. To prove that that  $f_{g,k,\ell}(x, y) \geq c$  for all  $x$  and  $y$  satisfying the constraints, it is therefore sufficient to show that  $x_1 \leq x_0$ .

### 3.4 Triangles, Edges and Triangle-Trees

Our goal in this section is to prove the following two theorems, which imply Theorems 1.0.2 and 1.0.4. After proving them, we will provide several applications to concrete families of graphs built up from gluing together triangles and edges in a “tree-like” way.

**Theorem 3.4.1.** *Let  $0 \leq \ell \leq 2$  be an integer. If  $H$  is  $(K_3, 2, \ell)$ -correlated, then  $H$  is common.*

**Theorem 3.4.2.** *Let  $k \geq 3$  and  $0 \leq \ell \leq \frac{5}{3}k$  be integers. If  $H$  is  $(K_3, k, \ell)$ -correlated, then  $H$  is common.*

We deduce Theorems 1.0.2 and 1.0.4 from these two results, after which we will turn our attention to their proofs.

*Proof of Theorem 1.0.2.* By Observation 3.2.6, for  $0 \leq \ell \leq 2$ , the graph  $H := (2 \cdot K_3) \sqcup (\ell \cdot K_2)$  is  $(K_3, 2, \ell)$ -correlated. Thus, it is common by Theorem 3.4.1.  $\square$

*Proof of Theorem 1.0.4.* Let  $k \geq 3$  and  $0 \leq \ell \leq \lfloor \frac{5}{3}k \rfloor$  be integers. By Observation 3.2.6, the graph  $H := (k \cdot K_3) \sqcup (\ell \cdot K_2)$  is  $(K_3, k, \ell)$ -correlated. Thus, it is common by Theorem 3.4.2.  $\square$

*Proof of Theorem 3.4.1.* Let  $0 \leq \ell \leq 2$  and let  $H$  be  $(K_3, 2, \ell)$ -correlated. By Theorem 3.2.2 and Observation 3.1.8,  $K_3$  is  $g$ -bounded where  $g(z) = z^3$  for  $z \in [0, 2]$ . Let  $\rho : [0, 2] \rightarrow [0, \infty)$  be defined by  $\rho(z) = \max\{0, \frac{16}{3}(z - 1)\}$  for  $0 \leq z \leq 2$  and observe that  $\rho \leq g$  and  $\rho \leq \rho_{K_3}$  by Corollary 3.2.5. By Lemma 3.1.9,  $2^{6+\ell}c(H)$  is at least the minimum of  $f_{g,2,\ell}(x, y)$  over all  $-1 \leq x \leq 1$  and  $0 \leq y \leq (1+x)^3 - \rho(1+x)$ . We will be done if we can show that this minimum is at least 2.

By Proposition 3.3.1, it suffices to show that every  $x \in [0, 1]$  satisfies  $(x \approx 0)$  or  $(x \approx 1)$  and  $(x \approx 1')$  for  $c = 2$  and the functions  $g$  and  $\rho$  defined above. By Lemma A.0.5 with  $\ell_0 = 2$ , we see that, to prove  $(x \approx 0)$  for a particular value of  $x$ , it suffices to show that  $(x \approx 0^*)$  holds. In this case, this translates to

$$((1+x)^3 + (1-x)^3)^2 \geq \frac{2((1+x)^2 + (1-x)^2)}{(1+x)^2(1-x)^2}. \quad (3.4.3)$$

The above inequality is equivalent to  $h(x) \geq 2$  where  $h : [0, 1) \rightarrow \mathbb{R}$  is defined by

$$h(x) := \frac{(1-x)^2(1+x)^2((1-x)^3 + (1+x)^3)^2}{(1-x)^2 + (1+x)^2}.$$

Note that  $h(0) = 2$ . Also,

$$\frac{dh}{dx}(x) = \frac{4x(x-1)(x+1)(1+3x^2)(-3+10x^2+9x^4)}{(1+x^2)^2}$$

which is non-negative for all  $0 \leq x \leq \frac{1}{3}\sqrt{-5+2\sqrt{13}} \approx 0.495$ . Thus, (3.4.3) holds for all  $0 \leq x \leq 0.49$ .

So, it suffices to show that  $(x \approx 1)$  and  $(x \approx 1')$  both hold whenever  $0.49 \leq x \leq 1$ . By Lemma A.0.2, to prove that a particular  $x$  satisfies  $(x \approx 1)$ , it is enough to show that  $(x \approx 1^*)$  holds. That is,

$$2\rho(1+x) \geq (1+x)^3 + (1-x)^3.$$

Plugging in  $\rho(1+x) = \frac{16}{3}x$ , the inequality becomes  $\frac{32}{3}x \geq (1+x)^3 + (1-x)^3$ . This holds for all  $x_1 \leq x \leq 1$  where  $x_1 = \frac{1}{9}(8 - \sqrt{37}) \approx 0.21 < 0.49$ . Finally, we analyze  $(x \approx 1')$ . It simplifies to

$$(1+x)^\ell \left(\frac{16}{3}x\right)^2 + (1-x)^\ell \left((1-x)^3 + (1+x)^3 - \frac{16}{3}x\right)^2 \geq 2.$$

For  $x \geq 0.49$  and  $\ell \geq 0$ , we have

$$(1+x)^\ell \left(\frac{16}{3}x\right)^2 + (1-x)^\ell \left((1-x)^3 + (1+x)^3 - \frac{16}{3}x\right)^2 \geq \left(\frac{16}{3}x\right)^2 > 6.8 > 2$$

and so  $(x \approx 1')$  holds for all  $x \geq 0.49$ . The result follows.  $\square$

Next, we prove the following lemma from which Theorem 3.4.2 will be derived. The proof is similar to that of Theorem 3.4.1, except that we sometimes need to deal with the more complicated inequalities  $(x \approx 0)$  and  $(x \approx 1)$  themselves as opposed to the sufficient conditions from Appendix A.

**Lemma 3.4.4.** *For any integer  $r$  such that  $0 \leq r \leq 15$  and any graphon  $W$ ,*

$$2^{9+\frac{r}{3}} \left( t(K_2, W)^{\frac{r}{3}} t(K_3, W)^3 + t(K_2, 1-W)^{\frac{r}{3}} t(K_3, 1-W)^3 \right) \geq 2. \quad (3.4.5)$$

*Proof.* We apply Lemma 3.1.9 and Proposition 3.3.1 with  $J = K_3$ ,  $\rho(z) = \frac{16}{3}(z-1)$ ,

$g(z) = z^3$ ,  $k = 3$  and  $\ell = r/3$ . The inequality ( $x \approx 0$ ) becomes

$$\frac{(1+x)^{\frac{r}{3}}(1-x)^{\frac{r}{3}}((1+x)^3 + (1-x)^3)^3}{\left((1+x)^{\frac{r}{6}} + (1-x)^{\frac{r}{6}}\right)^2} \geq 2. \quad (3.4.6)$$

By Lemma A.0.6, to prove (3.4.6), it suffices to show that

$$(1+x)^{\frac{r}{3}}(1-x)^{\frac{r}{3}} \left( \frac{(1+x)^3 + (1-x)^3}{2} \right)^{3-\frac{r}{9}} \geq 1. \quad (3.4.7)$$

For each integer  $r$  such that  $0 \leq r \leq 13$ , one can verify numerically that (3.4.7) is satisfied for all  $x$  in the range  $0 \leq x \leq x_{0,r}$  where

$$\begin{aligned} x_{0,0} = 1, \quad x_{0,1} \geq 0.99, \quad x_{0,2} \geq 0.99, \quad x_{0,3} \geq 0.98, \quad x_{0,4} \geq 0.95, \quad x_{0,5} \geq 0.91, \\ x_{0,6} \geq 0.86, \quad x_{0,7} \geq 0.79, \quad x_{0,8} \geq 0.72, \quad x_{0,9} \geq 0.64, \quad x_{0,10} \geq 0.55, \quad x_{0,11} \geq 0.46, \\ x_{0,12} \geq 0.34, \quad x_{0,13} \geq 0.19. \end{aligned}$$

Also, for  $r \in \{14, 15\}$ , one can verify numerically that (3.4.6) is satisfied whenever  $x$  is in the range  $0 \leq x \leq x_{0,r}$  where  $x_{0,r} \geq 0.14$ ; the worst case is  $r = 15$ .

So, to complete the proof, we need to show that, for each  $r$ , ( $x \approx 1$ ) and ( $x \approx 1'$ ) hold for all  $x_{0,r} \leq x \leq 1$ . Plugging in the values of  $k, \ell, \rho, g$ , these inequalities translate to

$$\frac{(1+x)^3(1+x)^{\frac{r}{6}} - (1-x)^3(1-x)^{\frac{r}{6}}}{(1+x)^{\frac{r}{6}} + (1-x)^{\frac{r}{6}}} \geq (1+x)^3 - \frac{16}{3}x \quad (3.4.8)$$

and

$$(1+x)^{\frac{r}{3}} \left( \frac{16}{3}x \right)^3 + (1-x)^{\frac{r}{3}} \left( (1-x)^3 + (1+x)^3 - \frac{16}{3}x \right)^3 \geq 2 \quad (3.4.9)$$

respectively. For each integer  $0 \leq r \leq 15$ , (3.4.9) holds for all  $0 \leq x \leq 1$ ; again, we are only able to verify this numerically. By Lemma A.0.2, for a given  $0 \leq x \leq 1$ , if

$$\frac{32}{3}x \geq (1+x)^3 + (1-x)^3,$$

then (3.4.8) holds automatically. The above inequality holds for all  $0.22 \leq x \leq 1$ . Thus, since  $x_{0,r} > 0.22$  for  $r \in \{0, \dots, 12\}$ , the proof is complete in these cases.

The last thing to do is to verify that (3.4.8) holds for all  $x_{0,r} \leq x \leq 1$  in the cases  $r \in \{13, 14, 15\}$ . One can check numerically that the inequality holds for all  $0.1397 \leq x \leq 1$  for each  $r \in \{13, 14, 15\}$ ; the worst case is  $r = 13$ . Since  $x_{0,r} > 0.1397$  for  $r \in \{13, 14, 15\}$ , the proof is complete.  $\square$

Next, we use Lemma 3.4.4 and Hölder's Inequality to prove Theorem 3.4.2.

*Proof of Theorem 3.4.2.* Since  $H$  is  $(K_3, k, \ell)$ -correlated, for any graphon  $W$ , we have

$$t(H, W) + t(H, 1 - W) \geq t(K_2, W)^\ell t(K_3, W)^k + t(K_2, 1 - W)^\ell t(K_3, 1 - W)^k.$$

Define

$$\begin{aligned} p &:= \frac{k}{3}, & q &:= \frac{k}{k-3}, \\ x_1 &:= t(K_2, W)^{\frac{3\ell}{k}} t(K_3, W)^3, & x_2 &:= t(K_2, 1 - W)^{\frac{3\ell}{k}} t(K_3, 1 - W)^3, \\ \alpha &:= \frac{1}{3} \left( \left\lceil \frac{9\ell}{k} \right\rceil - \frac{9\ell}{k} \right), & r &:= \left\lceil \frac{9\ell}{k} \right\rceil, \\ y_1 &:= t(K_2, W)^\alpha, & y_2 &:= t(K_2, 1 - W)^\alpha. \end{aligned}$$

By Lemma A.0.3,

$$(x_1^p + x_2^p)^{\frac{1}{p}} (y_1^q + y_2^q)^{\frac{1}{q}} \geq x_1 y_1 + x_2 y_2.$$

In other words,

$$\begin{aligned} & t(K_2, W)^\ell t(K_3, W)^k + t(K_2, 1 - W)^\ell t(K_3, 1 - W)^k \\ & \geq \left( \frac{t(K_2, W)^{\frac{3\ell}{k} + \alpha} t(K_3, W)^3 + t(K_2, 1 - W)^{\frac{3\ell}{k} + \alpha} t(K_3, 1 - W)^3}{(t(K_2, W)^{\alpha q} + t(K_2, 1 - W)^{\alpha q})^{\frac{k-3}{k}}} \right)^{\frac{k}{3}}. \end{aligned}$$

By definition of  $\alpha$ , we have  $\frac{3\ell}{k} + \alpha = \frac{r}{3}$ . Since  $0 \leq \ell \leq \frac{5}{3}k$ , we have that  $r$  is an integer satisfying  $0 \leq r \leq 15$ . So, by Lemma 3.4.4, we get that the right side of the above inequality is at least

$$\frac{2^{(-8 - \frac{r}{3})(\frac{k}{3})}}{(t(K_2, W)^{\alpha q} + t(K_2, 1 - W)^{\alpha q})^{\frac{k-3}{3}}}.$$

Note that

$$\alpha q = \frac{1}{3} \left( \left\lceil \frac{9\ell}{k} \right\rceil - \frac{9\ell}{k} \right) \left( \frac{k}{k-3} \right) \leq \frac{1}{3} \left( \frac{k-1}{k} \right) \left( \frac{k}{k-3} \right) = \frac{1}{3} \left( \frac{k-1}{k-3} \right) \leq 1$$

where the last inequality uses that  $k \geq 4$ . Thus, since  $t(K_2, W) + t(K_2, 1 - W) = 1$ , we have that  $t(K_2, W)^{\alpha q} + t(K_2, 1 - W)^{\alpha q} \leq (\frac{1}{2})^{\alpha q} + (\frac{1}{2})^{\alpha q}$  by Jensen's Inequality. So, we get

$$\begin{aligned} & \frac{2^{(-8 - \frac{r}{3})(\frac{k}{3})}}{(t(K_2, W)^{\alpha q} + t(K_2, 1 - W)^{\alpha q})^{\frac{k-3}{3}}} \geq \frac{2^{-\frac{8k}{3} - \frac{kr}{9}}}{((\frac{1}{2})^{\alpha q} + (\frac{1}{2})^{\alpha q})^{\frac{k-3}{3}}} = \frac{2^{-\frac{8k}{3} - \frac{kr}{9}}}{2^{\frac{k-3}{3}} (\frac{1}{2})^{\alpha(\frac{k}{3})}} \\ & = \frac{2^{-\frac{8k}{3} - \frac{kr}{9}}}{2^{k/3 - 1 - \alpha(k/3)}} = \frac{2^{1 - 3k - \frac{kr}{9}}}{2^{-\frac{1}{3}(r - \frac{9\ell}{k})(k/3)}} = \frac{2^{1 - 3k - \frac{kr}{9}}}{2^{-\frac{kr}{9} + \ell}} = 2 \left( \frac{1}{2} \right)^{3k + \ell} = 2 \left( \frac{1}{2} \right)^{e(H)} \end{aligned}$$

where the last equality follows from (3.1.2) and the fact that  $H$  is  $(K_3, k, \ell)$ -correlated.

Therefore,  $H$  is common and we are done.  $\square$

To close this section, we provide some concrete families of  $(K_3, k, \ell)$ -correlated graphs. Following [33], the class  $\mathcal{T}$  of *triangle-trees* is defined as follows. We have  $K_3 \in \mathcal{T}$  and, if  $H \in \mathcal{T}$ , then any graph obtained from  $H$  by either adding two vertices  $u, v$  to  $H$  and edges  $uv, uw, vw$  for some  $w \in V(H)$ , or adding one vertex  $x$  to  $H$  and edges  $xy$  and  $xz$  for some  $yz \in E(H)$ , is also in  $\mathcal{T}$ . In other words, the graphs in  $\mathcal{T}$  are built up by gluing triangles together on vertices or edges in a tree-like manner. Our results in this section apply to a slightly more general class of graphs (see Example 3.4.13), which is a special case of the class considered in [4, Section 6]. For any set  $X$ , let  $2^X$  denote the power set of  $X$ .

**Definition 3.4.10.** Given a graph  $F$ , define an  $F$ -tree to be a pair  $(T, \varphi)$  such that  $T$  is a tree and  $\varphi : V(T) \cup E(T) \rightarrow 2^{V(F)}$  satisfies  $\varphi(st) \subsetneq \varphi(s) \cap \varphi(t)$  for all  $st \in E(T)$ .

**Definition 3.4.11.** Let  $F$  be a graph and  $(T, \varphi)$  be an  $F$ -tree. Define  $H(T, \varphi)$  to be the graph constructed by taking pairwise disjoint graphs  $J_t$  for  $t \in V(T)$ , where  $J_t$  is isomorphic to  $F[\varphi(t)]$ , and then, for each  $st \in E(T)$ , identifying the vertex corresponding to  $w$  in the  $J_s$  with the vertex corresponding to  $w$  in  $J_t$  for all  $w \in \varphi(st)$ .

**Example 3.4.12.** Let  $F$  be a graph, let  $T$  be any tree with  $k$  vertices and suppose that  $\varphi$  maps every vertex of  $T$  to  $V(F)$  and every edge of  $T$  to  $\emptyset$ . Then  $H(T, \varphi)$  is nothing more than the graph  $k \cdot F$ .

**Example 3.4.13.** Let  $(T, \varphi)$  be a  $K_3$ -tree such that  $|\varphi(t)| = 3$  for all  $t \in V(T)$  and  $1 \leq |\varphi(e)| \leq 2$  for all  $e \in E(T)$ . Then  $H(T, \varphi)$  is a triangle-tree. Moreover, every triangle-tree can be represented as  $H(T, \varphi)$  for some such  $T$  and  $\varphi$ .

**Definition 3.4.14.** Given a  $K_3$ -tree  $(T, \varphi)$  and  $1 \leq j \leq 3$ , define

$$v_j(T, \varphi) := |\{t \in V(T) : |\varphi(t)| = j\}|.$$

Also, for  $0 \leq j \leq 2$ , define

$$e_j(T, \varphi) := |\{e \in E(T) : |\varphi(e)| = j\}|.$$

The statement of the following lemma is technically slightly more general than [33, Lemma 3.6]. However, the argument in the proof of [33, Lemma 3.6] applies, with virtually no modifications, to prove it; for this reason, we attribute it to [33]. A more general (but also more technical) statement for odd cycles which implies this lemma can be found in [4, Corollary 6.32].

**Lemma 3.4.15** (Grzesik, Lee, Lidický and Volec [33, Lemma 3.6]). *Let  $(T, \varphi)$  be a  $K_3$ -tree, let  $k = v_3(T, \varphi)$ , let  $\gamma = e_2(T, \varphi) - v_2(T, \varphi) \geq 0$ . If  $H = H(T, \varphi)$ , then  $H$  is  $(K_3, k, -\gamma)$ -correlated.*

We derive two consequences of Theorems 3.4.1 and 3.4.2. Note that, since  $\ell \cdot K_2$  is Sidorenko for all  $\ell$ , these results imply lower bounds on the number of disjoint edges that can be added to  $H$  while maintaining the property that it is common whenever  $(T, \varphi)$  is a  $K_3$ -tree and  $H = H(T, \varphi)$ .

**Corollary 3.4.16.** *Let  $(T, \varphi)$  be a  $K_3$ -tree such that  $v_3(T, \varphi) = 2$  and  $e_2(T, \varphi) \geq v_2(T, \varphi)$ . If  $H = H(T, \varphi)$  and  $F$  is a Sidorenko graph such that*

$$e(F) \leq 2 + e_2(T, \varphi) - v_2(T, \varphi),$$

*then  $H \sqcup F$  is common.*

*Proof.* Define  $\ell = e(F) - e_2(T, \varphi) + v_2(T, \varphi)$ . By Observation 3.2.6, Lemma 3.4.15 and the fact that  $F$  is Sidorenko, we have that, for any graphon  $W$ ,

$$\begin{aligned} t(H \sqcup F, W) &= t(H, W)t(F, W) \geq t(K_3, W)^2 t(K_2, W)^{-e_2(T, \varphi) + v_2(T, \varphi)} t(K_2, W)^{e(F)} \\ &= t(K_3, W)^2 t(K_2, W)^\ell. \end{aligned}$$

Also, since  $H$  is  $(K_3, 2, -e_2(T, \varphi) + v_2(T, \varphi))$ -correlated by Lemma 3.4.15, we have

$$\begin{aligned} e(H \sqcup F) &= e(H) + e(F) = 2e(K_3) - e_2(T, \varphi) + v_2(T, \varphi) + e(F) \\ &= 2e(K_3) + \ell. \end{aligned}$$

Thus,  $H \sqcup F$  is  $(K_3, 2, \ell)$ -correlated, where  $\ell \leq 2$  by hypothesis. The result now follows from Theorem 3.4.1.  $\square$

**Corollary 3.4.17.** *Let  $(T, \varphi)$  be a  $K_3$ -tree such that  $v_3(T, \varphi) = k \geq 3$  and  $e_2(T, \varphi) \geq v_2(T, \varphi)$ . If  $H = H(T, \varphi)$  and  $F$  is a Sidorenko graph such that*

$$e(F) \leq \frac{5}{3}k + e_2(T, \varphi) - v_2(T, \varphi),$$

*then  $H \sqcup F$  is common.*

*Proof.* The proof is analogous to that of Corollary 3.4.16.  $\square$

## 3.5 Negative Results

In this section, we turn our attention to obtaining upper bounds on  $c(H)$  for various graphs  $H$ . The graphons that we will use are all of the same general form. For  $n \geq 1$ , let  $\Delta_n$  be the set of all vectors  $\vec{z}$  of length  $n$  with non-negative entries that sum to one. Given  $\vec{z} \in \Delta_n$  and an  $n \times n$  symmetric matrix  $A$  with entries in  $[0, 1]$ , let  $W_{\vec{z}, A}$  be defined as follows. First, divide  $[0, 1]$  into  $n$  intervals  $I_1, \dots, I_n$  such that the measure

of  $I_i$  is equal to the  $i$ -th component of  $\vec{z}_i$ . Next, for each  $1 \leq i, j \leq n$ , define  $W_{\vec{z}, A}$  to be equal to  $A_{i,j}$  for all  $(x, y) \in I_i \times I_j$ . It is easily observed that, for any graph  $H$ ,

$$t(H, W_{\vec{z}, A}) = \sum_{f: V(H) \rightarrow [n]} \prod_{v \in V(H)} \vec{z}_{f(v)} \prod_{uv \in E(H)} A_{f(u), f(v)}. \quad (3.5.1)$$

**Remark 3.5.2.** In the statistical physics literature, the pair  $(\vec{z}, A)$  is often referred to as a “spin system” and  $t(H, W_{\vec{z}, A})$  is the “partition function” of that spin system.

The constructions used to prove Theorem 1.0.6, the upper bound in Theorem 1.0.7 and to show that Theorem 1.0.2 is best possible are similar to one another; we present them next.

**Proposition 3.5.3.** *The graph  $(2 \cdot K_3) \sqcup (3 \cdot K_2)$  is uncommon*

*Proof.* We prove that the graph  $H = (2 \cdot K_3) \sqcup (3 \cdot K_2)$  is uncommon. For  $z \in [0, 1/2]$  and  $y \in [0, 1]$ , we define  $W_{z,y} := W_{\vec{z}, A}$  where  $\vec{z} = (1 - 2z, z, z) \in \Delta_3$  and  $A$  is the symmetric  $3 \times 3$  matrix in which  $A(1, 2) = A(1, 3) = 1$ ,  $A(2, 3) = y$  and  $A(i, i) = 0$  for  $1 \leq i \leq 3$ . Using (3.5.1), one can compute

$$t(K_2, W_{z,y}) = 4z(1 - 2z) + 2z^2y,$$

$$t(K_2, 1 - W_{z,y}) = (1 - 2z)^2 + 2z^2(2 - y),$$

$$t(K_3, W_{z,y}) = 6z^2(1 - 2z)y$$

and

$$t(K_3, 1 - W_{z,y}) = (1 - 2z)^3 + z^3(2 + 6(1 - y)^2).$$

Therefore, by Observation 3.2.6,  $t(H, W_{z,y}) + t(H, 1 - W_{z,y}) = h(z, y)$  where

$$\begin{aligned} h(z, y) &= (4z(1 - 2z) + 2z^2y)^3 \cdot (6z^2(1 - 2z)y)^2 \\ &\quad + ((1 - 2z)^2 + 2z^2(2 - y))^3 \cdot ((1 - 2z)^3 + z^3(2 + 6(1 - y)^2))^2. \end{aligned}$$

Thus,  $c(H)$  is at most the minimum of  $h(z, y)$  over all  $0 \leq z \leq \frac{1}{2}$  and  $0 \leq y \leq 1$ . Setting  $z = 0.28$  and  $y = 0.42$  yields  $h(z, y) = 0.00390226 < 2 \left(\frac{1}{2}\right)^9$ , which completes the proof.  $\square$

*Proof of Theorem 1.0.7.* See Proposition 3.2.1 for the lower bound. For the upper bound, we need to prove  $c(K_3 \sqcup K_2) < 0.12145$ . We let  $W_{z,y}$  be the graphon as in the proof of Proposition 3.5.3 above. By Observation 3.2.6,  $t(K_3 \sqcup K_2, W_{z,y}) + t(K_3 \sqcup K_2, 1 - W_{z,y}) = h(z, y)$  where

$$h(z, y) = (4z(1 - 2z) + 2z^2y) \cdot (6z^2(1 - 2z)y) \\ + ((1 - 2z)^2 + 2z^2(2 - y)) \cdot ((1 - 2z)^3 + z^3(2 + 6(1 - y)^2)).$$

Thus,  $c(K_3 \sqcup K_2)$  is at most the minimum of  $h(z, y)$  over all  $0 \leq z, y \leq 1$ . Setting  $z = 0.263661$  and  $y = 0.2177$  yields  $h(z, y) = 0.12145$ , which completes the proof.  $\square$

*Proof of Theorem 1.0.6.* Let  $P$  be the paw graph and let  $W_{z,y}$  be the graphon as in the proofs of Proposition 3.5.3 and Theorems 1.0.7. Using (3.5.1), we get that

$$t(P, W_{z,y}) = 2z^2(1 - 2z)y(2z + 2(1 - 2z) + 2zy) = 4z^2(1 - z)y(1 - z(1 - y))$$

and

$$t(P, 1 - W_{z,y}) = (1 - 2z)^4 + 2z^4(2 - y) + 6z^4 \cdot (1 - y)^2 \cdot (2 - y) = (1 - 2z)^4 + z^4(2 - y)(2 + 6(1 - y)^2).$$

Thus,  $t(P, W_{z,y}) + t(P, 1 - W_{z,y})$  is equal to  $h(z, y)$ , where

$$h(z, y) = 4z^2(1 - z)y(1 - z(1 - y)) + (1 - 2z)^4 + z^4(2 - y)(2 + 6(1 - y)^2).$$

Therefore,  $c(P) < h(0.266491, 0.2187477) = 0.121415$ , which completes the proof.  $\square$

Recall that, by Theorem 1.0.4, the graph obtained from  $3 \cdot K_3$  by adding five disjoint edges is common. Our next example shows that a very closely related graph, obtained from  $3 \cdot K_3$  by adding three pendant edges in different components and two disjoint edges, is uncommon

**Proposition 3.5.4.**  $(3 \cdot P) \sqcup (2 \cdot K_2)$  is uncommon.

*Proof.* Let  $H = (3 \cdot P) \sqcup (2 \cdot K_2)$ . Once again, we use the graphon  $W_{z,y}$  from the previous three proofs. This time, we set  $z = 0.429919$  and  $y = 0.43222$ . We get

$$t(K_2, W_{z,y}) = 0.560411,$$

$$t(K_2, 1 - W_{z,y}) = 0.439589,$$

$$t(P, W_{z,y}) = 0.0506164,$$

$$t(P, 1 - W_{z,y}) = 0.074879.$$

Thus,  $t(H, W_{z,y}) + t(H, 1 - W_{z,y}) < 0.000121856 < 2(\frac{1}{2})^{14}$  and the result follows.  $\square$

Next, we provide an example which demonstrates that not all common graphs are strongly common; the question of whether such graphs exist was raised in [4]. Note that the fact that  $K_3 \sqcup K_3$  is common can be easily deduced from Theorem 3.2.2.

**Proposition 3.5.5.**  $K_3 \sqcup K_3$  is not strongly common.

*Proof.* Let  $\vec{z} = (\frac{1}{2}, \frac{1}{2})$ , let  $A$  be the  $2 \times 2$  symmetric matrix such that  $A(1, 1) = A(2, 2) = 1/3$  and  $A(1, 2) = 1$  and define  $W = W_{\vec{z}, A}$ . Then, by (3.5.1),

$$t(K_2, W) = \frac{2}{3},$$

$$t(K_2, 1 - W) = \frac{1}{3},$$

$$t(K_3, W) = \left(\frac{1}{3}\right)^3 \left(\frac{1}{2}\right)^3 + 6 \left(\frac{1}{3}\right) \left(\frac{1}{2}\right)^3 = \frac{55}{216}$$

and

$$t(K_3, 1 - W) = \left(\frac{2}{3}\right)^3 \left(\frac{1}{2}\right)^3 = \frac{1}{27}.$$

Therefore,

$$\begin{aligned} t(K_3 \sqcup K_3, W) + t(K_3 \sqcup K_3, 1 - W) &= \left(\frac{55}{216}\right)^2 + \left(\frac{1}{27}\right)^2 = \frac{3089}{46656} < \frac{65}{729} \\ &= t(K_2, W)^6 + t(K_2, 1 - W)^6 \end{aligned}$$

and so  $K_3 \sqcup K_3$  is not strongly common.  $\square$

The first common graph with chromatic number four which was discovered is the 5-wheel [36]; i.e. the graph  $W_5$  obtained from a cycle of length five by adding a vertex joined to everything on the cycle. The following proposition implies, for example, that  $W_5$  is not strongly common.

**Proposition 3.5.6.** Let  $H$  be a graph with  $m$  components and chromatic number  $k$ . If

$$\left(\frac{1}{k-1}\right)^{v(H)-m} < \left(\frac{k-2}{k-1}\right)^{e(H)} + \left(\frac{1}{k-1}\right)^{e(H)},$$

then  $H$  is not strongly common.

*Proof.* Let  $W = W_{K_{k-1}}$ ; i.e. the graphon corresponding to the complete graph  $K_{k-1}$ . Then,  $t(K_2, W) = (k-2)/(k-1)$  and  $t(K_2, 1 - W) = 1/(k-1)$ . Since  $H$  has chromatic number  $k$ ,

$$t(H, W) = 0.$$

Also,

$$t(H, 1 - W) = \left( \frac{1}{k-1} \right)^{v(H)-m}.$$

So, by the hypothesis of the proposition,  $H$  is not strongly common.  $\square$

**Corollary 3.5.7.**  $W_5$  is not strongly common.

*Proof.* The chromatic number of  $W_5$  is four and

$$\left( \frac{1}{4-1} \right)^{6-1} = \frac{1}{243} < \frac{1025}{59049} = \left( \frac{4-2}{4-1} \right)^{10} + \left( \frac{1}{4-1} \right)^{10}.$$

Thus, the result follows by Proposition 3.5.6.  $\square$

Finally, we prove Theorem 1.0.5.

*Proof of Theorem 1.0.5.* Let  $k \geq 1$  and  $\ell = \lceil 1.9665k \rceil$ . We show that  $H = (k \cdot K_3) \sqcup (\ell \cdot K_2)$  is uncommon. Define  $\alpha = \ell/k$  and note that  $1.9665 \leq \alpha \leq 2$ . Let

$$p := 1 - 2^{-\frac{1}{3+\alpha}}.$$

We let  $W$  be the graphon  $W_{\vec{z}, A}$  where  $\vec{z} = (\frac{1}{2}, \frac{1}{2})$  and  $A$  is a  $2 \times 2$  matrix whose diagonal entries are  $p$  and off-diagonal entries are 1. We have

$$\begin{aligned} t(K_2, W) &= \frac{1}{2}(1+p), \\ t(K_2, 1 - W) &= \frac{1}{2}(1-p), \\ t(K_3, W) &= \frac{1}{4}p^3 + \frac{3}{4}p, \\ t(K_3, 1 - W) &= \frac{1}{4}(1-p)^3. \end{aligned}$$

Thus, by Observation 3.2.6,

$$t(H, W) = \left[ \left( \frac{1}{2}(1+p) \right)^\alpha \left( \frac{1}{4}p^3 + \frac{3}{4}p \right) \right]^k = \frac{1}{2^{3k+\ell}} \left[ 2^{3+\alpha} \left( \frac{1}{2}(1+p) \right)^\alpha \left( \frac{1}{4}p^3 + \frac{3}{4}p \right) \right]^k.$$

Plugging in the value of  $p$  defined above yields

$$\frac{1}{2^{3k+\ell}} \left[ 2 \left( 1 - 2^{-\frac{1}{3+\alpha}} \right) \left( 2 - 2^{-\frac{1}{3+\alpha}} \right)^\alpha \left( 3 + \left( 2^{-\frac{1}{3+\alpha}} - 1 \right)^2 \right) \right]^k$$

The expression within the square brackets is decreasing in  $\alpha$  for  $\alpha \in [0, 2]$ . Therefore,

for all  $\alpha \geq 1.9665$  it is at most

$$\begin{aligned} & \frac{1}{2^{3k+\ell}} \left[ 2 \left( 1 - 2^{-\frac{1}{4.9665}} \right) \left( 2 - 2^{-\frac{1}{4.9665}} \right)^{1.9665} \left( 3 + \left( 2^{-\frac{1}{4.9665}} - 1 \right)^2 \right) \right]^k \\ & < \frac{1}{2^{3k+\ell}} (0.9999994)^k. \end{aligned}$$

Next, we compute

$$t(H, 1 - W) = \left[ \left( \frac{1}{2}(1-p) \right)^\alpha \frac{1}{4}(1-p)^3 \right]^k = \frac{1}{2^{3k+\ell}} \left[ 2^{3+\alpha} \left( \frac{1}{2}(1-p) \right)^\alpha \frac{1}{4}(1-p)^3 \right]^k.$$

When substituting the value of  $p$  chosen above into this, the expression inside of the square brackets evaluates to 1. Putting all of this together, we get

$$t(H, W) + t(H, 1 - W) < \frac{1}{2^{3k+\ell}} (0.9999994)^k + \frac{1}{2^{3k+\ell}} < 2 \left( \frac{1}{2} \right)^{e(H)}.$$

This completes the proof.  $\square$

Note that the graphon  $W$  used in the proof of Theorem 1.0.5 above is vertex-transitive. Thus, if  $H$  is a graph obtained from the disjoint union of two graphs  $F_1$  and  $F_2$  by identifying one vertex of  $F_1$  with one vertex of  $F_2$ , then  $t(H, W) = t(F_1, W)t(F_2, W)$  and  $t(H, 1 - W) = t(F_1, 1 - W)t(F_2, 1 - W)$  (for the specific graphon  $W$  used in the proof). Thus, the same construction can be used to obtain the following.

**Theorem 3.5.8.** *Let  $(T, \varphi)$  be a  $K_3$ -tree such that  $e_2(T, \varphi) = 0$  and let  $H = H(T, \varphi)$ . If*

$$v_2(T, \varphi) \geq 1.9665 \cdot v_3(T, \varphi),$$

*then  $H$  is uncommon.*

A graph  $H$  is said to be *positive* if  $t(H, W) \geq 0$  for every kernel  $W$ . If  $H$  is a graph obtained from two copies of a graph  $F$  by gluing them on an independent set, then  $H$  is easily seen to be positive, and it is conjectured that all positive graphs arise in this way [2]. Theorem 3.5.8 provides examples of connected positive graphs that are uncommon, e.g. the graph obtained by taking two triangles and four edges all glued on a single vertex, which answers [40, Question 5.1] in the affirmative (in a strong sense).

The construction in the proof of Theorem 1.0.5 can also be used for longer odd cycles, as we show next.

**Theorem 3.5.9.** *For integers  $k, r \geq 1$ , the graph  $(k \cdot C_{2r+1}) \sqcup (2rk \cdot K_2)$  is uncommon.*

*Proof.* Let  $H = (k \cdot C_{2r+1}) \sqcup (2rk \cdot K_2)$  and let

$$p := 1 - 2^{-\frac{1}{4r+1}}.$$

Let  $W$  be the graphon  $W_{\vec{z}, A}$  where  $\vec{z} = (\frac{1}{2}, \frac{1}{2})$  and  $A$  is a  $2 \times 2$  matrix whose diagonal entries are  $p$  and off-diagonal entries are 1. Note that the eigenvalues of  $A$  are  $p + 1$  and  $p - 1$  and  $\mathbb{1}_{2,2} - A$  has eigenvalue  $1 - p$  with multiplicity two, where  $\mathbb{1}_{2,2}$  is the  $2 \times 2$  all-ones matrix. So, by Corollary 2.3.2, we get

$$t(C_{2r+1}, W) = \left(\frac{p+1}{2}\right)^{2r+1} + \left(\frac{p-1}{2}\right)^{2r+1},$$

$$t(C_{2r+1}, 1 - W) = 2 \left(\frac{1-p}{2}\right)^{2r+1}.$$

Thus, by Observation 3.2.6,

$$\begin{aligned} t(H, W) &= \left[ \left(\frac{p+1}{2}\right)^{2r} \left( \left(\frac{p+1}{2}\right)^{2r+1} + \left(\frac{p-1}{2}\right)^{2r+1} \right) \right]^k \\ &= \frac{1}{2^{(4r+1)k}} [(p+1)^{2r} ((p+1)^{2r+1} + (p-1)^{2r+1})]^k = \frac{1}{2^{(4r+1)k}} [(p+1)^{4r+1} + (p-1)^{2r+1}(p+1)^{2r}]^k. \end{aligned}$$

Substituting in the value of  $p$  yields

$$\frac{1}{2^{(4r+1)k}} \left[ \left(2 - 2^{-\frac{1}{4r+1}}\right)^{4r+1} - 2^{-\frac{2r+1}{4r+1}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^{2r} \right]^k.$$

Our next goal is to show that the expression inside of the square brackets is less than one for all  $r \geq 1$ . That is, we want to prove that

$$\left(2 - 2^{-\frac{1}{4r+1}}\right)^{4r+1} - 2^{-\frac{2r+1}{4r+1}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^{2r} < 1$$

or, in other words,

$$\left(2 - 2^{-\frac{1}{4r+1}}\right)^{4r+1} < 1 + 2^{-\frac{2r+1}{4r+1}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^{2r}. \quad (3.5.10)$$

By the AM-GM Inequality, the right side of (3.5.10) can be bounded below as follows:

$$\begin{aligned} 1 + 2^{-\frac{2r+1}{4r+1}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^{2r} &> 2 \sqrt{2^{-\frac{2r+1}{4r+1}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^{2r}} \\ &= 2^{\frac{6r+1}{8r+2}} \left(2 - 2^{-\frac{1}{4r+1}}\right)^r. \end{aligned}$$

Thus, to prove (3.5.10), it suffices to show that

$$2^{\frac{6r+1}{8r+2}} \geq \left(2 - 2^{-\frac{1}{4r+1}}\right)^{3r+1} \quad (3.5.11)$$

or, in other words, that

$$2^{x-y} + 2^{-x} \geq 2$$

where  $x = \frac{1}{4r+1}$  and  $y = \frac{1}{2(3r+1)(4r+1)}$ . This holds if and only if

$$x \geq y + \frac{\log(1 + \sqrt{1 - 2^{-y}})}{\log(2)}. \quad (3.5.12)$$

The right side of (3.5.12) can be bounded above as follows:

$$\begin{aligned} y + \frac{\log(1 + \sqrt{1 - 2^{-y}})}{\log(2)} &\leq y + \frac{\sqrt{1 - 2^{-y}}}{\log(2)} = y + \frac{\sqrt{1 - e^{-\log(2)y}}}{\log(2)} \\ &\leq y + \frac{\sqrt{1 - (1 - \log(2)y)}}{\log(2)} = y + \sqrt{\frac{y}{\log(2)}}. \end{aligned}$$

Thus, to prove (3.5.12), it suffices to show that

$$\frac{1}{2(3r+1)(4r+1)} + \sqrt{\frac{1}{\log(8)(3r+1)(4r+1)}} < \frac{1}{4r+1}.$$

This inequality holds for all  $r \geq 1$ . Therefore, we have that  $t(H, W) < \frac{1}{2e(H)}$ .

Next, we compute

$$t(H, 1 - W) = \left[ \left( \frac{1-p}{2} \right)^{2r} \left( 2 \left( \frac{1-p}{2} \right)^{2r+1} \right) \right]^k = \frac{1}{2^{(4r+1)k}} [2(1-p)^{4r+1}]^k.$$

When substituting the value of  $p$  chosen above into this, the expression inside of the square brackets evaluates to 1. Putting all of this together, we get

$$t(H, W) + t(H, 1 - W) < 2 \left( \frac{1}{2} \right)^{(4r+1)k} = 2 \left( \frac{1}{2} \right)^{e(H)}$$

which completes the proof. □

Of course, the above theorem can also be extended to graphs built up by gluing together odd cycles and edges on single vertices, analogous to the way that Theorem 3.5.8 follows from the same proof as Theorem 1.0.5.

Theorem 2.1.3 says that an exponential number of edges to make a graph uncommon suffices. We can improve this for non-3-colourable graphs.

**Proposition 3.5.13.** *For a connected graph  $H$  with chromatic number  $k > 3$ , if  $t$*

satisfies the following inequality

$$\left(\frac{1}{k-1}\right)^{v(H)-1+t} < \left(\frac{1}{2}\right)^{e(H)-1+t},$$

then  $DE(H) \leq t$ .

*Proof.* Let  $W = W_{k-1}$ . Since  $H$  has chromatic number  $k$ ,

$$t(H, W) = 0, \quad t(H, 1 - W) = \left(\frac{1}{k-1}\right)^{v(H)-1}.$$

Note that  $t(K_2, 1 - W) = \frac{1}{k-1}$ . Then,

$$t(H, W) \cdot t(K_2, W)^t + t(H, 1 - W) \cdot t(K_2, 1 - W)^t = \left(\frac{1}{k-1}\right)^{v(H)-1+t}.$$

By the assumption, the inequality

$$\left(\frac{1}{k-1}\right)^{v(H)-1+t} < \left(\frac{1}{2}\right)^{e(H)-1+t}$$

holds. So,  $H \sqcup t \cdot K_2$  is uncommon. □

**Corollary 3.5.14.**  $DE(H) \leq 2e(H)$  for a non-3-colourable connected graph  $H$ .

*Proof.* If we put  $t = 2e(H)$ , then

$$\begin{aligned} & t(H, W) \cdot t(K_2, W)^{2e(H)} + t(H, 1 - W) \cdot t(K_2, 1 - W)^{2e(H)} \\ &= \left(\frac{1}{\chi(H) - 1}\right)^{3e(H)-1} \\ &\leq \left(\frac{1}{3}\right)^{3e(H)-1} \end{aligned}$$

It suffices to show

$$\begin{aligned} & \left(\frac{1}{3}\right)^{3e(H)-1} < \left(\frac{1}{2}\right)^{3e(H)-1} \\ \iff & 3 \cdot \left(\frac{1}{9}\right)^{e(H)} < 2 \cdot \left(\frac{1}{8}\right)^{e(H)}. \end{aligned}$$

This inequality holds if  $e(H) > 3$ . Since  $H$  is not 3-colourable,  $e(H) > 3$ . So, the graph  $H \sqcup (2e(H)) \cdot K_2$  is uncommon. □

Cororally 3.5.14 says that a linear number of edges in terms of  $e(H)$  suffices to make

a non-3-colourable graph uncommon. In [33], the function  $UC(H)$  defined to be the minimum number of edges in a tree  $T$  such that gluing a pendant copy of  $T$  to  $H$  yields an uncommon graph; when no such tree exists,  $UC(H) := \infty$ . The graphon we used to prove in Proposition 3.5.13 is vertex-transitive. So,  $UC(H)$  can be also bounded by the same number.

**Corollary 3.5.15.**  $UC(H) \leq 2e(H) - 3v(H) + 1$  for a non-3-colourable connected graph  $H$ .

The next result is about the disjoint union of two different odd cycles.

**Theorem 3.5.16.**  $C_a \sqcup C_b$  for any odd  $a < b$  is not common.

*Proof.*  $W_x := W_{\vec{z}, A}$  where  $\vec{z} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$  and

$$A = \begin{bmatrix} x & 1 & 1 & 0 & 0 & 1-x \\ 1 & x & 1 & 0 & 1-x & 0 \\ 1 & 1 & x & 1-x & 0 & 0 \\ 0 & 0 & 1-x & x & 1 & 1 \\ 0 & 1-x & 0 & 1 & x & 1 \\ 1-x & 0 & 0 & 1 & 1 & x \end{bmatrix}.$$

The eigenvalues of  $A$  are  $3, 2x+1, 0, 0, -2(1-x), -2(1-x)$ . We have

$$\mathbb{1} - A = \begin{bmatrix} 1-x & 0 & 0 & 1 & 1 & x \\ 0 & 1-x & 0 & 1 & x & 1 \\ 0 & 0 & 1-x & x & 1 & 1 \\ 1 & 1 & x & 1-x & 0 & 0 \\ 1 & x & 1 & 0 & 1-x & 0 \\ x & 1 & 1 & 0 & 0 & 1-x \end{bmatrix}.$$

The eigenvalues of  $\mathbb{1} - A$  are  $3, 2(1-x), 2(1-x), 0, 0, -2x-1$ . By the Corollary 2.3.2,

$$\begin{aligned} t(C_a, W_x) &= \left(\frac{3}{6}\right)^a + \left(\frac{2x+1}{6}\right)^a + 2\left(\frac{-2(1-x)}{6}\right)^a \\ &= \left(\frac{3}{6}\right)^a + \left(\frac{2x+1}{6}\right)^a - 2\left(\frac{2(1-x)}{6}\right)^a \\ t(C_a, \mathbb{1} - W_x) &= \left(\frac{3}{6}\right)^a + 2\left(\frac{2(1-x)}{6}\right)^a + \left(\frac{-2x-1}{6}\right)^a \\ &= \left(\frac{3}{6}\right)^a - \left(\left(\frac{2x+1}{6}\right)^a - 2\left(\frac{2(1-x)}{6}\right)^a\right) \end{aligned}$$

Define

$$C(x) := \left(\frac{2x+1}{6}\right)^a - 2\left(\frac{2(1-x)}{6}\right)^a, D(x) := \left(\frac{2x+1}{6}\right)^b - 2\left(\frac{2(1-x)}{6}\right)^b.$$

Thus,  $t(C_a, W) \cdot t(C_b, W) + t(C_a, 1-W) \cdot t(C_b, 1-W)$  equal to

$$\begin{aligned} & \left(\left(\frac{1}{2}\right)^a + C(x)\right) \cdot \left(\left(\frac{1}{2}\right)^b + D(x)\right) + \left(\left(\frac{1}{2}\right)^a - C(x)\right) \cdot \left(\left(\frac{1}{2}\right)^b - D(x)\right) \\ &= 2\left(\frac{1}{2}\right)^{a+b} + 2C(x) \cdot D(x). \end{aligned}$$

**Claim 3.5.17.** *There exist  $x \in [0, 1]$  such that  $C(x) < 0$  and  $D(x) > 0$ .*

*Proof of Claim 3.5.17.*

$$\begin{aligned} C(x) < 0 &\iff \left(\frac{2x+1}{6}\right)^a - 2\left(\frac{2(1-x)}{6}\right)^a < 0 \\ &\iff (2x+1)^a < 2(2(1-x))^a \\ &\iff 2x+1 < 2^{\frac{1}{a}}(2(1-x)) \\ &\iff (2^{1+\frac{1}{a}}+2)x < 2^{\frac{1}{a}+1}-1 \\ &\iff x < \frac{2^{\frac{1}{a}+1}-1}{2^{1+\frac{1}{a}}+2} \\ &\iff x < 1 - \frac{3}{2^{1+\frac{1}{a}}+2} \end{aligned}$$

which is in the interval  $[0, 1]$ . Since  $a < b$ , we have

$$1 - \frac{3}{2^{1+\frac{1}{a}}+2} > 1 - \frac{3}{2^{1+\frac{1}{b}}+2}.$$

So, there exists  $x$  such that

$$1 - \frac{3}{2^{1+\frac{1}{b}}+2} < x < 1 - \frac{3}{2^{1+\frac{1}{a}}+2},$$

and so  $C(x) < 0$  and  $D(x) > 0$  which completes the proof.  $\square$

Therefore, there exist  $x \in [0, 1]$  such that

$$t(C_a, W) \cdot t(C_b, W) + t(C_a, 1-W) \cdot t(C_b, 1-W) < 2\left(\frac{1}{2}\right)^{a+b}.$$

$\square$

## 3.6 Related Directions for Future Work

As mentioned in the introduction, one of the most classical examples of an uncommon graph is the paw graph  $P$  obtained from  $K_3$  by adding a pendant edge. The problem of computing the Ramsey multiplicity constant of  $P$  seems to be intimately linked to the analogous problem for  $K_3 \sqcup K_2$ . Given that  $2 \cdot (K_3 \sqcup K_2)$  is common by Theorem 1.0.2, we believe that the same may be true for two disjoint copies of  $P$ .

**Conjecture 3.6.1.** *If  $P$  is the paw graph, then  $P \sqcup P$  is common.*

If true, then it would be particularly interesting to have a flag algebra-free proof of this; however, beggars can't be choosers.

It would also be nice to have a linear bound  $DE(H)$  in terms of  $e(H)$  also for non-bipartite 3-colourable graphs. However, it seems unlikely. So, we are curious finding a family satisfying a following question.

**Question 3.6.2.** *Is it possible that for every  $n$ , the disjoint union  $H_n \sqcup f(e(H_n))$  with an exponential function in terms of the number of edges, is common, where  $(H_n)$  is a family of non-bipartite 3-colourable graphs?*

If it is true,  $K_3 \sqcup K_{n,n}$  might be one of them. The more precise bound for  $DE(H)$ , Theorems 1.0.2, 1.0.4 and 1.0.5 can be viewed as bounds on  $DE(k \cdot K_3)$  for certain values of  $k$ . We ask the following.

**Question 3.6.3.** *Does the sequence  $(DE(k \cdot K_3)/k)_{k=1}^{\infty}$  converge? If so, what is its limit?*

It would also be interesting to compute or bound  $DE(H)$  for families of graphs which are not covered by the results in this chapter. In particular, the results proved in Section 3.4 may extend nicely to longer odd cycles. Some, but not all, of the ingredients needed for this already exist in the literature. Specifically, the fact that odd cycles are strongly common was proven very recently in [40] (the case of the 5-cycle was proven in [4]) and a correlation inequality generalizing Lemma 3.4.15 can be derived from a result of [4]. A supersaturation theorem for  $C_5$  in graphons of specific edge densities was obtained in [5], but the supersaturation problem for odd cycles is not well understood in general; this is the main barrier in generalizing our results on  $K_3$  to longer odd cycles.

The quantity  $DE(H)$  is likely to be closely related to the function  $UC(H)$ . However, as Proposition 3.5.4 and the case  $k = 3$  of Theorem 1.0.4 show, adding pendant edges can have a different effect than adding disjoint edges; see Figure 3.1 below. The authors of [33] raise the question of bounding  $UC(H)$  for non-bipartite graphs. Theorem 3.5.8 provides an upper bound on  $UC(H)$  when  $H$  is, for example, a *triangle-vertex tree*; i.e.

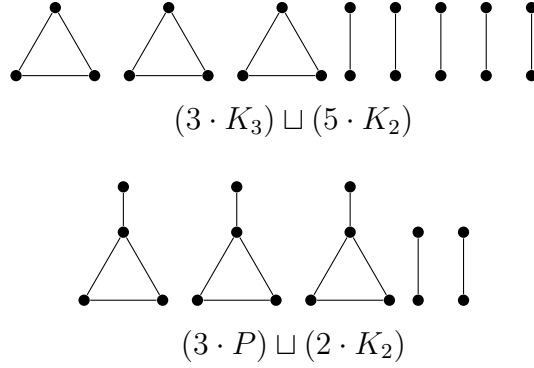


Figure 3.1: The graph  $(3 \cdot K_3) \sqcup (5 \cdot K_2)$  is common by Theorem 1.0.4 and the graph  $(3 \cdot P) \sqcup (2 \cdot K_2)$  is uncommon by Proposition 3.5.4.

a triangle-tree in which one is only allowed to glue together two triangles on a single vertex (not on an edge).

An inherent limitation of Lemma 3.1.9 is that we do not know of many graphs  $H$  which are  $g$ -bounded for a reasonably large function  $g$ . In particular, all of the known examples of strongly common graphs are Sidorenko graphs and odd cycles. It would be interesting to find new examples of non-bipartite strongly common graphs (if they exist) or examples of graphs that are  $g$ -bounded for other (non-trivial) functions  $g$ .

The proof of Theorem 1.0.5 actually establishes a stronger statement. Specifically, if  $H = k \cdot K_3 \sqcup \ell \cdot K_2$  where  $k \geq 1$  and  $\ell = \lceil 1.9665k \rceil$ , then there is a graphon  $W$  such that

$$\max\{t(H, W), t(H, 1 - W)\} \leq \left(\frac{1}{2}\right)^{e(H)}$$

$$\min\{t(H, W), t(H, 1 - W)\} < \left(\frac{1}{2}\right)^{e(H)}.$$

If  $H$  satisfies these conditions, one immediately gets that the disjoint union of any positive number of copies of  $H$  is uncommon. It would be interesting to get a better understanding of the class of graphs  $H$  with the property that such a graphon  $W$  exists.

# Chapter 4

## Turán Colourings in Off-Diagonal Ramsey Multiplicity

Our main result in this chapter (Theorem 4.0.1 below) extends this theorem of Fox and Wigderson Theorem 2.5.1 to an off-diagonal setting. In Chapter 2, we provided a formal definition of bonbon pairs and an off-diagonal variant of the Ramsey multiplicity constant. The proof of Theorem 4.0.1, which is inspired by the proof of Theorem 2.5.2 in [26], is provided in Sections 4.1 and 4.2. First, in Section 4.1, we show that an optimal colouring has the “approximate” structure of a Turán colouring; i.e. the vertices can be partitioned into a small number of classes such that edges within the classes are nearly monochromatic. Then, in Section 4.2, we refine the structure of the colouring until it precisely matches that of a Turán colouring. In Section 4.3, we discuss various constructions of colourings which we use to prove a strong form of Proposition 4.0.2 (Theorem 4.3.4).

**Theorem 4.0.1.** *Let  $q \in (0, 1]$  and let  $F_1$  and  $F_2$  be non-bipartite graphs, each of which contains a critical edge, such that  $\chi(F_1) + \chi(F_2) \geq 7$ . Then there exists  $t_0 = t_0(F_1, F_2, q)$  such that if  $H_1$  is a  $t_1$ -hairly  $F_1$  and  $H_2$  is a  $t_2$ -hairly  $F_2$  with  $t_1, t_2 \geq t_0$  and*

$$\min\{v(H_1), v(H_2)\} \geq q \cdot \max\{v(H_1), v(H_2)\},$$

*then  $(H_1, H_2)$  is a bonbon pair.*

One may wonder whether the presence of critical edges and the dependence of  $t_0$  on the parameter  $q$  are essential in Theorem 4.0.1. The next proposition implies that both conditions are necessary. Let  $\text{crit}(F)$  denote the number of critical edges in a graph  $F$ . An explicit form of the function  $g$  in the following proposition will be provided in Section 4.3 (see Theorem 4.3.4).

**Proposition 4.0.2.** *There exists a function  $g : \mathbb{R}^4 \rightarrow \mathbb{R}$  with the property that, if  $H_1$  and  $H_2$  are non-empty graphs such that*

$$e(H_1) > \text{crit}(H_2) \cdot g(\chi(H_1), \chi(H_2), v(H_2), k(H_2)),$$

then  $(H_1, H_2)$  is not a bonbon pair.

The other conditions in Theorem 4.0.1—namely, that  $F_1$  and  $F_2$  are non-bipartite and  $\chi(F_1) + \chi(F_2) \geq 7$ —are also necessary. First, if  $F_2$  is bipartite and  $H_2$  is a  $t_2$ -hairy  $F_2$  for some  $t_2 \geq 1$ , then  $H_2$  does not contain a critical edge and so Proposition 4.0.2 implies that there cannot exist a graph  $H_1$  with non-empty edge set such that  $(H_1, H_2)$  is a bonbon pair. Also, Fox and Wigderson [26] observed that the conclusion of Theorem 2.5.2 is false for every 3-chromatic graph  $F$ . Therefore, the conclusion of Theorem 4.0.1 is false for  $F_1 = F_2 = F$  whenever  $F$  is a 3-chromatic graph.

## 4.1 Proof of Theorem 4.0.1: Rough Structure

The focus of this section is on obtaining an approximate version of Theorem 4.0.1 (Lemma 4.1.7 below) which will be refined in the next section to complete the proof of the theorem.

**Remark 4.1.1.** Fox and Wigderson [26] cleverly avoided using the Graph Removal Lemma in their proof of Theorem 2.5.2. Doing so added a few steps to their argument, but resulted in much better bounds on  $t_0$ . To differentiate from [26], we have chosen to present a shorter argument which uses the Removal Lemma at the expense of having poorer control over  $t_0$ . We remark that better bounds on our  $t_0$  could be obtained by following the proof of [26, Theorem 1.2] more closely.

Throughout the next two sections, we let  $q \in (0, 1]$  and let  $F_1$  and  $F_2$  be non-bipartite graphs, each of which contains a critical edge, such that  $\chi(F_1) + \chi(F_2) \geq 7$ . Define  $f := \max\{v(F_1), v(F_2)\}$  and  $\chi := \max\{\chi(F_1), \chi(F_2)\}$ . We let  $t_0$  be an integer chosen large with respect to  $F_1, F_2$  and  $q$ , which will be specified later. Actually,  $t_0$  is defined in terms of a throng of other parameters  $\theta, \varepsilon, \delta, \beta, \xi, \gamma$  and  $\tau$ , where each parameter depends on  $F_1, F_2$  and  $q$  and the parameters that come before it in the list. The relationships between  $F_1, F_2, q, \theta, \varepsilon, \delta, \beta, \xi, \gamma, \tau$  and  $t_0$  will be revealed “as needed” throughout this section and the next, and will be summarized in the final proof of Theorem 4.0.1 at the end of Section 4.2.

Let  $t_1, t_2 \geq t_0$  and let  $H_1$  be a  $t_1$ -hairy  $F_1$  and  $H_2$  be a  $t_2$ -hairy  $F_2$  satisfying

$$\min\{v(H_1), v(H_2)\} \geq q \cdot \max\{v(H_1), v(H_2)\}. \quad (4.1.2)$$

Note that  $\chi(H_i) = \chi(F_i)$  and  $k(H_i) = k(F_i)$  for  $i \in \{1, 2\}$ . For the sake of brevity, let

$$\rho_1 := (\chi(H_2) - 1)^{v(H_1) - k(H_1)}$$

$$\rho_2 := (\chi(H_1) - 1)^{v(H_2) - k(H_2)}.$$

Note that, by definition,

$$\rho_1 \cdot \left( \frac{1}{\chi(F_2) - 1} \right)^{v(H_1) - k(F_1)} = \rho_2 \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - k(F_2)} = 1. \quad (4.1.3)$$

We may assume that  $H_1$  and  $H_2$  have no singleton components, since any such components do not affect injective homomorphism densities into large enough graphs, nor do they affect  $\rho_1$  or  $\rho_2$  (since adding a singleton component to a graph  $H$  increases both of  $v(H)$  and  $k(H)$  by one and does not affect  $\chi(H)$ ).

Let  $n_0$  be a large integer which may depend on  $H_1, H_2$  and all of the parameters discussed so far, and assume that  $n \geq n_0$ . For any graph  $G$ , define

$$m(H_1, H_2; G) := \rho_1 \cdot t_{\text{inj}}(H_1, G) + \rho_2 \cdot t_{\text{inj}}(H_2, \overline{G}).$$

Here, the letter  $m$  stands for “monochromatic.” Let  $G_1$  be a graph on  $n$  vertices chosen so that  $\min_{G: v(G)=n} m(H_1, H_2; G) = m(H_1, H_2; G_1)$  and let  $G_2 := \overline{G_1}$ . Our goal in the proof of Theorem 4.0.1 is to show that either  $G_1$  is a Turán graph with  $\chi(F_1) - 1$  parts or  $G_2$  is a Turán graph with  $\chi(F_2) - 1$  parts. Since  $m(H_1, H_2; G_1)$  is at most the value of  $m(H_1, H_2; G)$  when  $G$  is an  $n$ -vertex Turán graph with  $\chi(F_1) - 1$  parts, we have that

$$m(H_1, H_2; G_1) \leq (1 - o(1)) \rho_2 \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - k(F_2)} = 1 - o(1) \quad (4.1.4)$$

where the last equality follows from (4.1.3). Note that  $m(H_2, H_1; G_2) = m(H_1, H_2; G_1)$  and so it is also at most  $1 - o(1)$ .

It is useful to classify vertices based on their degrees in  $G_1$  and  $G_2$ . Let  $V := V(G_1) = V(G_2)$ . For a graph  $G$  with vertex set  $V$  and a vertex  $v \in V$ , the *degree* of  $v$  in  $G$  is the number of edges of  $G$  that are connected to  $v$ , denoted by  $d_G(v)$ . For  $i \in \{1, 2\}$ , we let  $d_i(v) := d_{G_i}(v)$ . When interpreting the next definition, recall that  $\xi$  is one of the many parameters that appears throughout this section and the next and will be specified in the final proof of Theorem 4.0.1.

**Definition 4.1.5.** For  $i \in \{1, 2\}$ , define

$$V_i := \left\{ v \in V : d_i(v) \geq \left( 1 - \frac{1 + 2\xi}{\chi(F_i) - 1} \right) (n - 1) \right\}.$$

Also, let  $V_0 := V \setminus (V_1 \cup V_2)$  and  $V_3 = V_1 \cap V_2$ .

We may assume the following, without loss of generality.

**Assumption 4.1.6.**  $|V_1| \leq |V_2|$ .

The focus of the rest of this section is on proving the following lemma which determines the “rough structure” of  $G_1$  and  $G_2$ . For any two subsets  $S, T \subseteq V$  and a graph

$G$  with vertex set  $V$ , define  $e_G(S, T)$  to be the number of ordered pairs  $(u, v) \in S \times T$  such that  $uv \in E(G)$  and let  $e_G(S) := \frac{1}{2}e_G(S, S)$ . For any  $S, T \subseteq V$  and  $i \in \{1, 2\}$ , we let  $e_i(S, T) := e_{G_i}(S, T)$  and  $e_i(S) := e_{G_i}(S)$ .

**Lemma 4.1.7.** *There exists a partition  $A_1, A_2, \dots, A_{\chi(F_2)-1}$  of  $V$  such that*

$$\sum_{i=1}^{\chi(F_2)-1} e_2(A_i) \leq \varepsilon n^2.$$

It is worth noting that the proof of Lemma 4.1.7 does not require  $F_1$  and  $F_2$  to have critical edges, nor does it require the inequality (4.1.2); these conditions come into play when seeking the exact structure of an optimal colouring in the next section.

A high-level overview of the proof of Lemma 4.1.7 is as follows. We first show that  $V_3 = \emptyset$  and that  $V_0$  and  $V_1$  are both quite small; specifically  $|V_0| \leq \xi n$  and  $|V_1| \leq 25\xi n$ . Therefore, most of the vertices reside in  $V_2$ , and thus have a large degree in  $G_2$ . If the density of  $F_2$  in  $G_2$  is sufficiently far from zero, then there must be several copies of  $F_2$  whose vertices are contained in  $V_2$ , and each of these copies can be “extended” to a copy of  $H_2$  in  $G_2$  in many ways due to the high  $G_2$ -degree of vertices in  $V_2$ . This would lead to a large density of  $H_2$  in  $G_2$ , which would violate (4.1.4). The Graph Removal Lemma then implies that  $G_2$  can be made  $F_2$ -free by deleting a small proportion of its edges. After deleting these edges, we obtain a graph with close to  $\left(1 - \frac{1}{\chi(F_2)-1}\right) \binom{n}{2}$  edges which is  $F_2$ -free. The classical Erdős–Simonovits Stability Theorem then states that such a graph must be “close” to a complete  $(\chi(F_2) - 1)$ -partite graph, which gives us Lemma 4.1.7. The rest of the section is devoted to fleshing out the details of these arguments.

### 4.1.1 Analyzing Degrees

We show that  $V_1$  and  $V_2$  have empty intersection. The following assumption is useful for proving this, and will be used again later as well:

$$0 < \xi < \frac{1}{39}. \tag{4.1.8}$$

**Lemma 4.1.9.**  $V_3 = V_1 \cap V_2 = \emptyset$ .

*Proof.* Suppose not and let  $v \in V_1 \cap V_2$ . Then, since  $\chi(F_1), \chi(F_2) \geq 3$  and  $\chi(F_1) +$

$\chi(F_2) \geq 7$ , we have

$$\begin{aligned} n-1 &= d_1(v) + d_2(v) \\ &\geq \left(1 - \frac{1+2\xi}{\chi(F_1)-1}\right)(n-1) + \left(1 - \frac{1+2\xi}{\chi(F_2)-1}\right)(n-1) \\ &\geq \left(1 - \frac{1+2\xi}{2}\right)(n-1) + \left(1 - \frac{1+2\xi}{3}\right)(n-1) = \left(\frac{7}{6} - \frac{5\xi}{3}\right)(n-1). \end{aligned}$$

This implies that  $\xi \geq 1/10$ ; however, this contradicts (4.1.8).  $\square$

We obtain a bound on the degrees of vertices in  $V_i$  for  $i \in \{1, 2\}$  via a similar argument.

**Observation 4.1.10.** Let  $\{i, j\} = \{1, 2\}$ . If  $v \in V_i$ , then  $d_i(v) > \left(\frac{5}{4} \cdot \frac{1+\xi}{\chi(F_j)-1}\right)n$ .

*Proof.* If not, then, since  $v \in V_i$  and  $n$  is large,

$$\left(1 - \frac{1+3\xi}{\chi(F_i)-1}\right)n \leq \left(1 - \frac{1+2\xi}{\chi(F_i)-1}\right)(n-1) \leq d_i(v) \leq \left(\frac{5}{4} \cdot \frac{1+\xi}{\chi(F_j)-1}\right)n.$$

Recall that  $\chi(F_1), \chi(F_2) \geq 3$  and  $\chi(F_1) + \chi(F_2) \geq 7$ . In the case that  $\chi(F_i) = 3$ , then we must have  $\chi(F_j) \geq 4$  and so the above inequality implies that

$$1 - \frac{1+3\xi}{2} \leq \frac{5}{4} \cdot \frac{1+\xi}{3}.$$

However, this can only hold if  $\xi \geq 1/23$  which contradicts (4.1.8). On the other hand, if  $\chi(F_1) \geq 4$ , then we instead get

$$1 - \frac{1+3\xi}{3} \leq \frac{5}{4} \cdot \frac{1+\xi}{2}.$$

This can only hold if  $\xi \geq 1/39$  which again contradicts (4.1.8).  $\square$

Next, we prove that both  $V_0$  and  $V_1$  are small. The next lemma says that, for  $i \in \{1, 2\}$ , there cannot be a fairly sizeable set  $S$  such that  $t(F_i, G_i[S])$  is bounded away from zero and  $d_i(v)$  is relatively large for every  $v \in S$ . To prove this, we assume that  $t_0$  is chosen large enough so that the following holds:

$$(1 + \xi)^{t_0} > 3/\tau. \tag{4.1.11}$$

**Lemma 4.1.12.** Let  $\{i, j\} = \{1, 2\}$ . If  $S$  is a non-empty subset of  $V$  such that

$$d_i(v) \geq \left(\frac{1+\xi}{\chi(F_j)-1}\right)n$$

for all  $v \in S$ , then  $t(F_i, G_i[S]) \leq \tau \cdot (n/|S|)^{v(F_i)}$ .

*Proof.* Suppose to the contrary that the hypotheses hold but

$$t(F_i, G_i[S]) > \tau \cdot (n/|S|)^{v(F_i)}.$$

The probability that a uniformly random function  $\varphi$  from  $V(H_i)$  to  $V$  is a homomorphism from  $H_i$  to  $G_i$  is at least the probability that the restriction of  $\varphi$  to  $V(F_i)$  is a homomorphism from  $F_i$  to  $G_i[S]$  multiplied by the probability that, for every  $w \in V(H_i) \setminus V(F_i)$ , if  $v$  is the unique neighbour of  $w$  in  $H_i$ , then  $\varphi(w)$  is adjacent to  $\varphi(v)$  in  $G_i$ . Thus,

$$\begin{aligned} t(H_i, G_i) &\geq (|S|/n)^{v(F_i)} t(F_i, G_i[S]) \left( \frac{1 + \xi}{\chi(F_j) - 1} \right)^{v(H_i) - v(F_i)} \\ &> \tau \left( \frac{1 + \xi}{\chi(F_j) - 1} \right)^{v(H_i) - v(F_i)} \\ &= \tau(1 + \xi)^{t_i} \left( \frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - v(F_i)} \\ &> 3 \left( \frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - k(F_i)} \end{aligned}$$

where the last inequality follows from (4.1.11) and the facts that  $t_1, t_2 \geq t_0$  and  $v(F_i) \geq k(F_i)$ . So, by (2.2.1) and the fact that  $n$  is large, we have

$$t_{\text{inj}}(H_i, G_i) = t(H_i, G_i) - o(1) > 2 \left( \frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - k(F_i)}.$$

Consequently, by (4.1.3),

$$m(H_1, H_2; G_1) = \rho_1 \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2) > 2$$

which contradicts (4.1.4) and thus completes the proof.  $\square$

Next, we prove that  $V_0$  is quite small. For this, we assume the following bound on  $\tau$ :

$$0 < \tau < \frac{c_1(F_1, F_2) \cdot \xi^f}{4}. \quad (4.1.13)$$

Note that  $c_1(F_1, F_2) > 0$  by Lemma 2.5.4 and so it is possible to choose  $\tau$  to satisfy this condition. The next lemma is analogous to [26, Claim 3.3].

**Lemma 4.1.14.**  $|V_0| < \xi n$ .

*Proof.* Suppose that  $|V_0| \geq \xi n$ . Our goal is to obtain a contradiction via an application of Lemma 4.1.12 with  $S = V_0$ . By definition of  $V_0$ , for each  $i \in \{1, 2\}$ , every  $v \in V_0$

satisfies

$$d_i(v) \leq \left(1 - \frac{1 + 2\xi}{\chi(F_i) - 1}\right) (n - 1).$$

Since  $d_1(v) + d_2(v) = n - 1$ , this tells us that

$$d_1(v) \geq \left(\frac{1 + 2\xi}{\chi(F_2) - 1}\right) (n - 1) > \left(\frac{1 + \xi}{\chi(F_2) - 1}\right) n$$

and

$$d_2(v) \geq \left(\frac{1 + 2\xi}{\chi(F_1) - 1}\right) (n - 1) > \left(\frac{1 + \xi}{\chi(F_1) - 1}\right) n$$

for every vertex  $v \in V_0$  and large enough  $n$ . Therefore, by Lemma 4.1.12, for each  $i \in \{1, 2\}$ , we must have

$$t(F_i, G_i[V_0]) \leq \tau(n/|V_0|)^{v(F_i)} \leq \tau/\xi^{v(F_i)}.$$

According to (4.1.13), this last expression is less than  $c_1(F_1, F_2)/4$ . Thus, for  $i \in \{1, 2\}$ ,

$$t(F_i, G_i[V_0]) < c_1(F_1, F_2)/4.$$

On the other hand, by definition of  $c_1(F_1, F_2)$ , we have

$$t(F_1, G_1[V_0]) + t(F_2, G_2[V_0]) \geq c_1(F_1, F_2) - o(1)$$

where the  $o(1)$  term tends to 0 as  $\xi n \rightarrow \infty$ . Since  $c_1(F_1, F_2) > 0$  by Lemma 2.5.4 this implies that, for large  $n$ ,

$$t(F_1, G_1[V_0]) + t(F_2, G_2[V_0]) \geq c_1(F_1, F_2)/2.$$

Therefore, we can let  $i \in \{1, 2\}$  such that

$$t(F_i, G_i[V_0]) \geq c_1(F_1, F_2)/4.$$

Combining the upper and lower bound on  $t(F_i, G_i[V_0])$  that we have proven leads to a contradiction, thereby completing the proof.  $\square$

Next, we show that  $V_1$  is small which, when combined with the fact that  $|V_0| < \xi n$ , implies that the vast majority of the vertices are in  $V_2$ . To do this, we use the following form of the Erdős–Simonovits Supersaturation Theorem.

**Theorem 4.1.15** (Erdős–Simonovits Supersaturation Theorem [20]). *For every non-empty graph  $F$  and  $\xi > 0$  there exists  $\gamma = \gamma(F, \xi) > 0$  such that if  $G$  is a graph with  $t(K_2, G) \geq 1 - \frac{1-\xi}{\chi(F)-1}$ , then  $t(F, G) \geq \gamma$ .*

Using Theorem 4.1.15, we define  $\gamma$  by

$$\gamma := \min\{\gamma(F_1, \xi), \gamma(F_2, \xi)\}. \quad (4.1.16)$$

We also assume that  $\tau$  is chosen so that

$$0 < \tau < \gamma \cdot (25\xi)^f. \quad (4.1.17)$$

The next lemma is analogous to [26, Claim 3.4].

**Lemma 4.1.18.**  $|V_1| < 25\xi n$ .

*Proof.* Let us begin by establishing the following claim.

**Claim 4.1.19.** *For each  $i \in \{1, 2\}$ , if  $|V_i| \geq 25\xi n$ , then*

$$e_i(V_i) < \left(1 - \frac{1 - \xi}{\chi(F_i) - 1}\right) \frac{|V_i|^2}{2}.$$

*Proof of Claim 4.1.19.* Suppose not. Then there exists  $i \in \{1, 2\}$  such that

$$t(K_2, G_i[V_i]) = \frac{2e_i(V_i)}{|V_i|^2} \geq 1 - \frac{1 - \xi}{\chi(F_i) - 1}.$$

Consequently, Theorem 4.1.15 implies that  $t(F_i, G_i[V_i]) \geq \gamma$ . Using the hypothesis  $|V_i| \geq 25\xi n$ ,

$$\gamma \geq \gamma \cdot \left(\frac{25\xi n}{|V_i|}\right)^{v(F_i)} \geq \gamma \cdot (25\xi)^{v(F_i)} \left(\frac{n}{|V_i|}\right)^{v(F_i)}$$

which, by (4.1.17), is greater than  $\tau \cdot (n/|V_i|)^{v(F_i)}$ . By Observation 4.1.10, we have  $d_i(v) \geq \left(\frac{1+\xi}{\chi(F_j)-1}\right)n$  for all  $v \in V_i$ , where  $j \in \{1, 2\} \setminus \{i\}$ . So, the set  $S = V_i$  contradicts Lemma 4.1.12. Therefore, the claim holds.  $\square$

We now use Claim 4.1.19 to complete the proof of the lemma. If  $|V_1| < 25\xi n$ , then we are done; so, we assume  $|V_1| \geq 25\xi n$ . By Assumption 4.1.6,  $|V_2| \geq 25\xi n$  as well. In particular, both  $V_1$  and  $V_2$  are non-empty and satisfy the hypothesis, and therefore the conclusion, of Claim 4.1.19. For  $i \in \{1, 2\}$ , define

$$\eta_i := \frac{e_i(V_1, V_2)}{|V_1||V_2|}$$

and note that  $\eta_1 + \eta_2 = 1$  because  $V_1$  and  $V_2$  are disjoint (by Lemma 4.1.9) and  $G_2$  is the complement of  $G_1$ . By definition of  $V_1$ , we have

$$\sum_{v \in V_1} d_1(v) \geq |V_1| \left(1 - \frac{1 + 2\xi}{\chi(F_1) - 1}\right) (n - 1) \geq |V_1| \left(1 - \frac{1 + 3\xi}{\chi(F_1) - 1}\right) n.$$

On the other hand,

$$\begin{aligned}
\sum_{v \in V_1} d_1(v) &= 2e_1(V_1) + e_1(V_1, V \setminus V_1) \\
&= 2e_1(V_1) + e_1(V_1, V_2) + e_1(V_1, V_0) \\
&\leq 2e_1(V_1) + \eta_1|V_1||V_2| + |V_1||V_0|.
\end{aligned}$$

Claim 4.1.19 tells us that the above expression is less than

$$\left(1 - \frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1|^2 + \eta_1|V_1||V_2| + |V_1||V_0|.$$

Combining the lower and upper bounds on  $\sum_{v \in V_1} d_1(v)$  obtained above and cancelling a factor of  $|V_1|$ , we get

$$\left(1 - \frac{1 + 3\xi}{\chi(F_1) - 1}\right) n \leq \left(1 - \frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1| + \eta_1|V_2| + |V_0|.$$

By Lemma 4.1.9, we have  $|V_0| + |V_1| + |V_2| = n$  and so this inequality becomes

$$\begin{aligned}
\left(1 - \frac{1}{\chi(F_1) - 1} - \frac{3\xi}{\chi(F_1) - 1}\right) n &\leq n - \left(\frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1| + (\eta_1 - 1)|V_2| \\
&= n - \frac{|V_1|}{\chi(F_1) - 1} + \frac{\xi|V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2|.
\end{aligned}$$

Adding and subtracting  $\frac{|V_2| + |V_0|}{\chi(F_1) - 1}$  in this final expression and using  $|V_0| + |V_1| + |V_2| = n$  again yields

$$\begin{aligned}
n - \frac{|V_1|}{\chi(F_1) - 1} + \frac{\xi|V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2| &+ \frac{|V_2| + |V_0|}{\chi(F_1) - 1} - \frac{|V_2| + |V_0|}{\chi(F_1) - 1} \\
&= n - \frac{n}{\chi(F_1) - 1} + \frac{\xi|V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2| + \frac{|V_2| + |V_0|}{\chi(F_1) - 1}.
\end{aligned}$$

Since  $|V_1| \leq n$  trivially and  $|V_0| < \xi n$  by Lemma 4.1.14, we get that this last expression is strictly less than

$$n - \frac{n}{\chi(F_1) - 1} + \left(\eta_1 - 1 + \frac{1}{\chi(F_1) - 1}\right) |V_2| + \frac{2\xi n}{\chi(F_1) - 1}.$$

To recap, the inequality that we have just derived is

$$\left(1 - \frac{1}{\chi(F_1) - 1} - \frac{3\xi}{\chi(F_1) - 1}\right) n < n - \frac{n}{\chi(F_1) - 1} + \left(\eta_1 - 1 + \frac{1}{\chi(F_1) - 1}\right) |V_2| + \frac{2\xi n}{\chi(F_1) - 1}.$$

By rearranging, we get

$$\left(1 - \eta_1 - \frac{1}{\chi(F_1) - 1}\right) |V_2| < \frac{5\xi n}{\chi(F_1) - 1}. \quad (4.1.20)$$

Applying the same argument, but with the roles of  $(F_1, V_1, \eta_1)$  and  $(F_2, V_2, \eta_2)$  reversed, we get that

$$\left(1 - \eta_2 - \frac{1}{\chi(F_2) - 1}\right) |V_1| < \frac{5\xi n}{\chi(F_2) - 1}. \quad (4.1.21)$$

We now divide the proof into cases depending on the values of  $\eta_1$  and  $\eta_2$ .

*Case 1.*  $\eta_i \leq \frac{2}{5}$  for some  $i \in \{1, 2\}$ .

Let  $j \in \{1, 2\} \setminus \{i\}$ . Since  $\chi(F_i) \geq 3$ , we get the following by applying (4.1.20) or (4.1.21):

$$\frac{5\xi n}{2} \geq \frac{5\xi n}{\chi(F_i) - 1} > \left(1 - \eta_i - \frac{1}{\chi(F_i) - 1}\right) |V_j| \geq \left(1 - \frac{2}{5} - \frac{1}{2}\right) |V_j| = \frac{|V_j|}{10}$$

and so  $|V_j| < 25\xi n$ . Since  $|V_1| \leq |V_2|$  by Assumption 4.1.6, this implies that  $|V_1| < 25\xi n$ .

*Case 2.*  $\eta_1, \eta_2 > \frac{2}{5}$ .

Since  $\chi(F_1) + \chi(F_2) \geq 7$ , we can let  $j \in \{1, 2\}$  so that  $\chi(F_j) \geq 4$  and let  $i \in \{1, 2\} \setminus \{j\}$ . Since  $\eta_i > \frac{2}{5}$  and  $\eta_i + \eta_j = 1$ , we have  $\eta_j < \frac{3}{5}$ . Now, by applying (4.1.20) or (4.1.21),

$$\frac{5\xi n}{3} \geq \frac{5\xi n}{\chi(F_j) - 1} > \left(1 - \eta_j - \frac{1}{\chi(F_j) - 1}\right) |V_i| > \left(1 - \frac{3}{5} - \frac{1}{3}\right) |V_i| = \frac{|V_i|}{15}$$

and so  $|V_i| < 25\xi n$ . Since  $|V_1| \leq |V_2|$  by Assumption 4.1.6, this completes the proof.  $\square$

## 4.1.2 Obtaining the Partition

Now that we know that most vertices are in  $V_2$  and, thus, have high degree in  $G_2$ , the next step is to show that  $G_2$  can be made  $F_2$ -free by deleting a small proportion of its edges. For this, we apply the well-known Graph Removal Lemma of Alon, Duke, Lefmann, Rödl and Yuster [1] and Füredi [29]. As discussed in Remark 4.1.1, this is one place in which our argument deviates from that of [26].

**Theorem 4.1.22** (Graph Removal Lemma [1, 29]). *For every graph  $F$  and any given  $\delta > 0$ , there is a  $\beta = \beta(F, \delta) > 0$  such that if  $G$  is a graph with  $t(F, G) \leq \beta$ , then there is a spanning subgraph  $G'$  of  $G$  with  $t(F, G') = 0$  and  $t(K_2, G') \geq t(K_2, G) - \delta$ .*

Using Theorem 4.1.22, define

$$\beta := \min\{\beta(F_1, \delta/2), \beta(F_2, \delta/2)\}. \quad (4.1.23)$$

We also assume that  $\xi$  and  $\tau$  satisfy the following:

$$0 < \xi < \frac{\beta}{52 \cdot f}, \quad (4.1.24)$$

$$0 < \tau < \beta/2. \quad (4.1.25)$$

The following lemma allows us to apply Theorem 4.1.22.

**Lemma 4.1.26.**  $t(F_2, G_2) < \beta$ .

*Proof.* Suppose, to the contrary, that  $t(F_2, G_2) \geq \beta$ . The number of homomorphisms from  $F_2$  to  $G_2$  which map a vertex to  $V_0 \cup V_1$  is at most  $v(F_2) \cdot n^{v(F_2)-1} \cdot |V_0 \cup V_1|$  which, by Lemmas 4.1.14 and 4.1.18, is no more than  $26v(F_2)\xi n^{v(F_2)}$ . Therefore,

$$t(F_2, G_2[V_2]) \geq \frac{t(F_2, G_2)n^{v(F_2)} - 26v(F_2)\xi n^{v(F_2)}}{|V_2|^{v(F_2)}} \geq (n/|V_2|)^{v(F_2)}(\beta - 26v(F_2)\xi).$$

By (4.1.24) and (4.1.25), this is greater than  $\tau \cdot (n/|V_2|)^{v(F_2)}$ . Recall that, by Observation 4.1.10, we have  $d_2(v) \geq \left(\frac{1+\xi}{\chi(F_1)-1}\right)n$  for all  $v \in V_2$ . So, the set  $S = V_2$  contradicts Lemma 4.1.12, and the proof is complete.  $\square$

The last step in verifying Lemma 4.1.7 involves utilizing the Erdős–Simonovits Stability Theorem [62] in the following form.

**Theorem 4.1.27** (Erdős–Simonovits Stability Theorem [62]). *For every non-empty graph  $F$  and  $\varepsilon > 0$ , there exists  $\delta = \delta(F, \varepsilon) > 0$  such that if  $G$  is a graph with  $t(F, G) = 0$  and  $t(K_2, G) \geq 1 - \frac{1}{\chi(F)-1} - \delta$ , then there exists a partition  $A_1, \dots, A_{\chi(F)-1}$  of  $V(G)$  such that  $\sum_{i=1}^{\chi(F)-1} e(A_i) \leq \varepsilon n^2$ .*

Using Theorem 4.1.27, we define

$$\delta := \min \{ \delta(F_1, \varepsilon/2), \delta(F_2, \varepsilon/2), 2\varepsilon \} \quad (4.1.28)$$

We also assume that  $\xi$  satisfies

$$0 < \xi < \frac{\delta}{52}. \quad (4.1.29)$$

The following lemma establishes a lower bound on  $t(K_2, G_2)$  which will facilitate our application of Theorem 4.1.27.

**Lemma 4.1.30.**  $t(K_2, G_2) \geq 1 - \frac{1}{\chi(F_2)-1} - \frac{\delta}{2}$ .

*Proof.* By Lemmas 4.1.14 and 4.1.18, we have  $|V_2| = n - |V_0 \cup V_1| \geq (1 - 26\xi)n$ .

Therefore,

$$\begin{aligned}
t(K_2, G_2) &= \frac{2e(G_2)}{n^2} \\
&= \frac{1}{n^2} \sum_{v \in V} d_2(v) \\
&\geq \frac{1}{n^2} \sum_{v \in V_2} d_2(v) \\
&\geq \frac{|V_2|}{n^2} \left(1 - \frac{1 + 2\xi}{\chi(F_2) - 1}\right) (n - 1) \\
&\geq \frac{|V_2|}{n^2} \left(1 - \frac{1 + 3\xi}{\chi(F_2) - 1}\right) n \\
&\geq (1 - 26\xi) \left(1 - \frac{1 + 3\xi}{\chi(F_2) - 1}\right) \\
&\geq 1 - \frac{1}{\chi(F_2) - 1} - 26\xi.
\end{aligned}$$

By (4.1.29), this is at least  $1 - \frac{1}{\chi(F_2) - 1} - \frac{\delta}{2}$  and so the proof is complete.  $\square$

Finally, we present the proof of Lemma 4.1.7, thereby accomplishing our primary objective of this section.

*Proof of Lemma 4.1.7.* By Lemma 4.1.26, Theorem 4.1.22 and (4.1.23), there exists a spanning subgraph  $G'_2$  of  $G_2$  such that  $t(F_2, G'_2) = 0$  and  $t(K_2, G'_2) \geq t(K_2, G_2) - \delta/2$ . So, Lemma 4.1.30 implies that  $t(K_2, G'_2) \geq 1 - \frac{1}{\chi(F_2) - 1} - \delta$ . By Theorem 4.1.27 and (4.1.28), there is a partition  $A_1, A_2, \dots, A_{\chi(F_2) - 1}$  of  $V$  such that  $\sum_{i=1}^{\chi(F_2) - 1} e_{G'_2}(A_i) \leq (\varepsilon/2)n^2$ . Since  $t(K_2, G) = 2e(G)/v(G)^2$  for any graph  $G$ , the inequality  $t(K_2, G'_2) \geq t(K_2, G_2) - \delta/2$  is equivalent to  $e(G'_2) \geq e(G_2) - (\delta/4)n^2$ . Therefore,

$$\sum_{i=1}^{\chi(F_2) - 1} e_2(A_i) \leq \sum_{i=1}^{\chi(F_2) - 1} e_{G'_2}(A_i) + (\delta/4)n^2 \leq (\varepsilon/2 + \delta/4)n^2 \leq \varepsilon n^2$$

where the last inequality follows from (4.1.28).  $\square$

## 4.2 Proof of Theorem 4.0.1: Exact Structure

The aim of this section is to complete the proof of Theorem 4.0.1. The way that this breaks down is as follows. We start by obtaining control over the number of copies of  $H_1$  in  $G_1$  and  $H_2$  in  $G_2$  that contain any given vertex  $v \in V$ . In particular, we show that any two vertices in  $V$  “contribute” roughly the same amount to  $m(H_1, H_2; G_1)$ . Thus,

if one vertex contributes “too much,” then all vertices do, which leads to a violation of (4.1.4).

After this, we refine the rough structure afforded to us by Lemma 4.1.7 until we get that  $G_2$  is simply a Turán graph with  $\chi(F_2) - 1$  parts. The first step in this process is to show that the parts have nearly the same size and almost all edges between pairs of parts are in  $G_2$ . We then show that the  $G_2$ -neighbourhood of every vertex  $v \notin V_1$  roughly “respects” the partition. Next, we prove that the  $G_2$ -degree of any vertex is within a small window around  $\left(1 - \frac{1}{\chi(F_2)-1}\right)n$ , which implies that  $V_1 = \emptyset$ . After that, we can use the critical edge in  $F_2$  to show that all edges within  $A_i$  must be in  $G_1$  for all  $1 \leq i \leq \chi(F_2) - 1$ . From this point, the theorem is easily deduced via a convexity argument.

### 4.2.1 Every Vertex Contributes the Same

For a graph  $H$ , a graph  $G$  on vertex set  $V$  and  $v \in V$ , define  $t_{\text{inj}}(H, G)(v)$  to be the probability that a random function from  $V(H)$  to  $V$  is an injective homomorphism from  $H$  to  $G$  whose image contains  $v$ . The main idea of the next lemma is that, if  $u$  and  $w$  are vertices such that an appropriate weighted sum of  $t_{\text{inj}}(H_1, G_1)(u)$  and  $t_{\text{inj}}(H_2, G_2)(u)$  is significantly smaller than the analogous sum for  $w$ , then one can get a better colouring by “deleting”  $w$  and “cloning”  $u$ . This is a standard idea in extremal combinatorics going back at least as far as Zykov’s proof of Turán’s Theorem [69]. This lemma is analogous to [26, Lemma 2.1].

**Lemma 4.2.1.** *There exists a constant  $C = C(H_1, H_2) > 0$  such that, for any  $u, w \in V$ ,*

$$\rho_1 \cdot t_{\text{inj}}(H_1, G_1)(u) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(u) \geq \rho_1 \cdot t_{\text{inj}}(H_1, G_1)(w) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(w) - \frac{C}{n^2}.$$

*Proof.* Suppose, to the contrary, that the inequality does not hold for some  $u, w \in V$ . If we remove all edges incident to the vertex  $w$  from  $G_1$ , then we lose all of the injective homomorphisms from  $H_1$  to  $G_1$  which map at least one vertex to  $w$ . Likewise, if we delete all edges incident to  $w$  from  $G_2$ , then we lose all of the injective homomorphisms from  $H_2$  to  $G_2$  which map at least one vertex to  $w$ . (Note that, here, we are subtly using the assumption that the graphs  $H_1$  and  $H_2$  have no singleton components.)

After deleting all such edges from  $G_1$  and  $G_2$ , suppose that we add to  $G_1$  all edges of the form  $wv$  such that  $uv \in E(G_1)$  and  $v \neq w$  to form a new graph  $G'_1$ . Similarly, add to  $G_2$  all edges of the form  $wv$  such that  $v$  is a vertex with  $uv \notin E(G_1)$  and  $v \neq w$  to get a graph  $G'_2$ . Note that  $G'_2$  is the complement of  $G'_1$ . In adding these edges, we gain one injective homomorphism from  $H_1$  to  $G'_1$  per injective homomorphism from  $H_1$  to  $G_1$  that includes  $u$  and not  $w$ . Similarly, we gain one injective homomorphism from  $H_2$  to  $G'_2$  per injective homomorphism from  $H_2$  to  $G_2$  that includes  $u$  and not  $w$ . Additionally,

for each  $i \in \{1, 2\}$ , we may also gain  $O(n^{v(H_i)-2})$  injective homomorphisms which map to both  $u$  and  $w$ . Thus, for each  $i \in \{1, 2\}$ ,

$$t_{\text{inj}}(H_i, G'_i) \leq t_{\text{inj}}(H_i, G_i) - t_{\text{inj}}(H_i, G_i)(w) + t_{\text{inj}}(H_i, G_i)(u) + O(1/n^2)$$

where the constant factor on the  $O(1/n^2)$  term is bounded by a function of  $H_i$ . Thus, assuming that the inequality in the lemma is not true, we have that  $m(H_1, H_2; G'_1)$  is at most  $m(H_1, H_2; G_1)$  plus a  $O(1/n^2)$  term, where the constant factor depends on  $H_1$  and  $H_2$ , minus  $C/n^2$ . So, if  $C$  is chosen large enough with respect to  $H_1$  and  $H_2$ , we get that  $G'_1$  contradicts our choice of  $G_1$ . Thus, the lemma holds.  $\square$

Analogous to the definition of  $t_{\text{inj}}(H, G)(v)$ , let  $t(H, G)(v)$  be the probability that a uniformly random function from  $V(H)$  to  $V$  is a homomorphism from  $H$  to  $G$  whose image contains  $v$ . The following lemma restricts  $t(H_i, G_i)(v)$  for every vertex  $v$ .

**Lemma 4.2.2.** *Suppose that  $\{i, j\} = \{1, 2\}$ . For every  $v \in V$ ,*

$$t(H_i, G_i)(v) \leq \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left( \frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - k(F_i)}.$$

*Proof.* Suppose, to the contrary, that there exists  $v \in V$  such that for some  $i \in \{1, 2\}$ ,

$$t(H_i, G_i)(v) > \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left( \frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - k(F_i)}. \quad (4.2.3)$$

By Lemma 4.2.1, (4.1.3), (4.2.3) and the fact that  $t_{\text{inj}}(H_i, G_i)(v) = t(H_i, G_i)(v) + O(1/n^2)$ , we get that, for large  $n$ , every  $u \in V$  satisfies

$$\rho_1 \cdot t_{\text{inj}}(H_1, G_1)(u) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(u) > \frac{2 \max\{v(H_1), v(H_2)\}}{n}.$$

Summing this inequality over all  $u \in V$  yields

$$\begin{aligned} 2 \max\{v(H_1), v(H_2)\} &< \sum_{u \in V} (\rho_1 \cdot t_{\text{inj}}(H_1, G_1)(u) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(u)) \\ &= \rho_1 \cdot v(H_1) \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot v(H_2) \cdot t_{\text{inj}}(H_2, G_2) \\ &\leq \max\{v(H_1), v(H_2)\} (\rho_1 \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)). \end{aligned}$$

This contradicts (4.1.4), and thus the proof is complete.  $\square$

## 4.2.2 Refining the Partition

We assume, throughout the remainder of this section, that  $A_1, \dots, A_{\chi(F_2)-1}$  is a partition of  $V$  as in Lemma 4.1.7. Let us show that the sets  $A_1, \dots, A_{\chi(F_2)-1}$  have ap-

proximately the same size and that  $G_1$  contains almost no edges between different parts. In order to prove this, we make the following assumption on  $\varepsilon$ . Recall that  $\chi = \max\{\chi(F_1), \chi(F_2)\}$ .

$$0 < \varepsilon < \frac{1}{12\chi^4}. \quad (4.2.4)$$

The next lemma is analogous to [26, Claim 3.8].

**Lemma 4.2.5.** *For  $1 \leq i \neq j \leq \chi(F_2) - 1$ ,*

$$(a) \quad \left| |A_i| - \frac{n}{\chi(F_2)-1} \right| \leq \sqrt{3\varepsilon} \cdot n \text{ and}$$

$$(b) \quad e_2(A_i, A_j) \geq (1 - 2\chi(F_2)^2\varepsilon)|A_i||A_j|.$$

*Proof.* First observe that, since  $\sum_{i=1}^{\chi(F_2)-1} |A_i| = n$ ,

$$\begin{aligned} & \sum_{i=1}^{\chi(F_2)-1} \left( \frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1} \right)^2 \\ &= \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} - 2 \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|}{n(\chi(F_2)-1)} + \sum_{i=1}^{\chi(F_2)-1} \left( \frac{1}{\chi(F_2)-1} \right)^2 \\ &= \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} - \frac{1}{\chi(F_2)-1} \end{aligned}$$

and, also,

$$1 = \left( \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|}{n} \right)^2 = \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} + 2 \left( \sum_{1 \leq i < j \leq \chi(F_2)-1} \frac{|A_i||A_j|}{n^2} \right).$$

Solving for  $\sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2}$  in one of these two equations and substituting into the other yields

$$\sum_{i=1}^{\chi(F_2)-1} \left( \frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1} \right)^2 + \frac{1}{\chi(F_2)-1} = 1 - 2 \left( \sum_{1 \leq i < j \leq \chi(F_2)-1} \frac{|A_i||A_j|}{n^2} \right)$$

which is equivalent to

$$1 - \frac{1}{\chi(F_2)-1} = \sum_{i=1}^{\chi(F_2)-1} \left( \frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1} \right)^2 + 2 \left( \sum_{1 \leq i < j \leq \chi(F_2)-1} \frac{|A_i||A_j|}{n^2} \right). \quad (4.2.6)$$

Also, by (4.1.28) and Lemma 4.1.30, we have  $t(K_2, G_2) \geq 1 - \frac{1}{\chi(F_2)-1} - \varepsilon$ . So,

$$\begin{aligned}
1 - \frac{1}{\chi(F_2) - 1} - \varepsilon &\leq t(K_2, G_2) \\
&= \frac{2e(G_2)}{n^2} \\
&= \sum_{i=1}^{\chi(F_2)-1} \frac{2e_2(A_i)}{n^2} + \sum_{1 \leq i < j \leq \chi(F_2)-1} \frac{2e_2(A_i, A_j)}{n^2} \\
&\leq 2\varepsilon + 2 \left( \sum_{1 \leq i < j \leq \chi(F_2)-1} \frac{e_2(A_i, A_j)}{n^2} \right)
\end{aligned}$$

where the last inequality is by Lemma 4.1.7. Substituting the expression for  $1 - \frac{1}{\chi(F_2)-1}$  in (4.2.6) into this inequality and rearranging yields

$$\sum_{i=1}^{\chi(F_2)-1} \left( \frac{|A_i|}{n} - \frac{1}{\chi(F_2) - 1} \right)^2 + 2 \sum_{1 \leq i < j \leq \chi(F_2)-1} \left( \frac{|A_i||A_j|}{n^2} - \frac{e_2(A_i, A_j)}{n^2} \right) \leq 3\varepsilon.$$

Since all summands on the left side are non-negative, we get

$$\left( \frac{|A_i|}{n} - \frac{1}{\chi(F_2) - 1} \right)^2 \leq 3\varepsilon$$

for all  $i$ , which proves (a). Similarly, for each  $i \neq j$ , the above inequality implies that

$$2 \left( \frac{|A_i||A_j|}{n^2} - \frac{e_2(A_i, A_j)}{n^2} \right) \leq 3\varepsilon$$

and so

$$e_2(A_i, A_j) \geq |A_i||A_j| - (3\varepsilon/2)n^2 = \left( 1 - \frac{3\varepsilon n^2}{2|A_i||A_j|} \right) |A_i||A_j|.$$

By (a), the right side is at least

$$\left( 1 - \frac{3\varepsilon}{2 \left( \frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon} \right)^2} \right) |A_i||A_j|$$

and by (4.2.4), this is at least  $(1 - 2\chi(F_2)^2\varepsilon)|A_i||A_j|$  (with room to spare). Therefore, (b) holds.  $\square$

Next, we show that the  $G_2$ -neighbourhood of every vertex that is not in  $V_1$  roughly “respects” the partition  $A_1, \dots, A_{\chi(F_2)-1}$  (see Lemma 4.2.10 below). We assume that  $\varepsilon$

satisfies the following condition:

$$0 < \varepsilon < \frac{\theta^2}{4 \cdot \chi^2 f^2}. \quad (4.2.7)$$

Also, we assume that  $t_0$  is chosen large enough so that, for all  $t \geq t_0$ , we have

$$\left(\frac{5}{4}\right)^{t/2} > \frac{6(t+f)}{q \left(\frac{\theta}{2\chi}\right)^f}. \quad (4.2.8)$$

**Definition 4.2.9.** Say that a vertex  $v \in V$  is *bad* if, for all  $1 \leq i \leq \chi(F_2) - 1$ , the number of  $G_2$ -neighbours of  $v$  in  $A_i \cap V_2$  is at least  $\theta|A_i|$ . Let  $B$  be the set of all bad vertices.

**Lemma 4.2.10.**  $B \subseteq V_1$ .

*Proof.* Suppose, to the contrary, that there exists a bad vertex  $v \notin V_1$ . For each vertex  $x$  of  $F_2$ , let  $p(x)$  be the number of pendant edges incident to  $x$  which were added during the construction of  $H_2$  from  $F_2$ . Let  $v_0z$  be a critical edge of  $F_2$  where we assume, without loss of generality, that  $p(v_0) \leq p(z)$ . Then, in particular, at most half of the pendant edges added in the construction of  $H_2$  are incident to  $v_0$ ; i.e.  $p(v_0) \leq \frac{v(H_2) - v(F_2)}{2}$ . Let  $F'_2 := F_2 \setminus \{v_0\}$  and note that, since  $v_0z$  is a critical edge,  $\chi(F'_2) < \chi(F_2)$ . Fix a proper colouring  $\psi : V(F'_2) \rightarrow \{1, \dots, \chi(F_2) - 1\}$  of  $F'_2$ . Our aim is to prove a lower bound on  $t(H_2, G_2)(v)$  which is large enough to contradict Lemma 4.2.2.

Let  $S$  be the  $G_2$ -neighbourhood of  $v$ . The probability that a random function  $\varphi$  from  $V(H_2)$  to  $V$  is a homomorphism from  $H_2$  to  $G_2$  such that  $\varphi(v_0) = v$  is at least the probability that the restriction of  $\varphi$  to  $V(F'_2)$  is a homomorphism from  $F'_2$  to  $G_2[S \cap V_2]$ , times  $1/n$  (the probability that  $\varphi(v_0) = v$ ), times the probability that every vertex of  $V(H_2) \setminus V(F_2)$  is mapped to a  $G_2$ -neighbour of the image of its unique neighbour in  $H_2$ . Taking into account that  $v \notin V_1$ , we have

$$d_2(v) = n - 1 - d_1(v) \geq \left(\frac{1 + 2\xi}{\chi(F_1) - 1}\right) (n - 1) > \left(\frac{1}{\chi(F_1) - 1}\right) n.$$

Thus, since  $d_2(w) \geq \left(\frac{5}{4} \cdot \frac{1}{\chi(F_1) - 1}\right) n$  for all  $w \in V_2$  by Observation 4.1.10, we get that  $t(H_2, G_2)(v)$  is greater than

$$\left(\frac{|S \cap V_2|}{n}\right)^{v(F'_2)} t(F'_2, G[S \cap V_2]) \frac{1}{n} \left(\frac{1}{\chi(F_1) - 1}\right)^{p(v_0)} \left(\frac{5}{4} \cdot \frac{1}{\chi(F_1) - 1}\right)^{v(H_2) - v(F_2) - p(v_0)}. \quad (4.2.11)$$

Next, we bound  $(|S \cap V_2|/n)^{v(F'_2)} t(F'_2, G[S \cap V_2])$  from below. First, since  $v$  is bad, we have that  $|S \cap A_i \cap V_2| \geq \theta|A_i|$  for all  $1 \leq i \leq \chi(F_2) - 1$ . So, if we map  $V(F'_2)$  randomly

to  $V$ , then the probability that every vertex  $w$  of  $F'_2$  is mapped to  $S \cap A_{\psi(w)} \cap V_2$  is at least  $\prod_{w \in V(F'_2)} (\theta |A_{\psi(w)}|/n)$  which, by Lemma 4.2.5 (a), is at least  $\theta^{v(F'_2)} \left( \frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon} \right)^{v(F'_2)}$ . By Lemma 4.2.5 (b), the number of non-edges in  $G_2$  from  $S \cap A_i \cap V_2$  to  $S \cap A_j \cap V_2$  for  $i \neq j$  is at most  $2\chi(F_2)^2\varepsilon|A_i||A_j|$  which, since  $v$  is bad, is at most

$$\frac{2\chi(F_2)^2\varepsilon|S \cap A_i \cap V_2||S \cap A_j \cap V_2|}{\theta^2}.$$

Thus, for any fixed edge  $wy$  of  $F'_2$ , the conditional probability that  $\varphi(w)$  is not adjacent to  $\varphi(y)$  given that  $\varphi(w) \in S \cap A_{\psi(w)} \cap V_2$  and  $\varphi(y) \in S \cap A_{\psi(y)} \cap V_2$  is at most  $\frac{2\chi(F_2)^2\varepsilon}{\theta^2}$ . By taking a union bound over all edges of  $F'_2$ , we get that the probability that every vertex  $w$  of  $F'_2$  is mapped to  $S \cap A_{\psi(w)} \cap V_2$  and no edge of  $F'_2$  is mapped to a non-edge of  $G_2$  is at least

$$\theta^{v(F'_2)} \left( \frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon} \right)^{v(F'_2)} \left( 1 - \frac{2e(F'_2)\chi(F_2)^2\varepsilon}{\theta^2} \right).$$

By (4.2.4), the product of the first two factors is at least  $\left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)}$  and, by (4.2.7), the third factor is at least  $1/2$ . So, the expression in (4.2.11) is at least

$$\begin{aligned} \frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{1}{\chi(F_1)-1} \right)^{p(v_0)} \left( \frac{5}{4} \cdot \frac{1}{\chi(F_1)-1} \right)^{v(H_2)-v(F_2)-p(v_0)} \\ = \frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{4}{5} \right)^{p(v_0)} \left( \frac{5}{4} \cdot \frac{1}{\chi(F_1)-1} \right)^{v(H_2)-v(F_2)}. \end{aligned}$$

Now, since  $p(v_0) \leq \frac{v(H_2)-v(F_2)}{2}$ , we get that this is at least

$$\begin{aligned} \frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{4}{5} \right)^{\frac{v(H_2)-v(F_2)}{2}} \left( \frac{5}{4} \cdot \frac{1}{\chi(F_1)-1} \right)^{v(H_2)-v(F_2)} \\ = \frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{5}{4} \right)^{\frac{v(H_2)-v(F_2)}{2}} \left( \frac{1}{\chi(F_1)-1} \right)^{v(H_2)-v(F_2)} \\ = \frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{5}{4} \right)^{t_2/2} \left( \frac{1}{\chi(F_1)-1} \right)^{v(H_2)-v(F_2)}. \end{aligned}$$

By (4.2.8) and the fact that  $t_2 \geq t_0$ , this is at least

$$\begin{aligned}
\frac{1}{2} \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)} \frac{1}{n} \left( \frac{6v(H_2)}{q \left( \frac{\theta}{\chi(F_2)} \right)^{v(F_2)}} \right) \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - v(F_2)} \\
&> \frac{3v(H_2)}{q \cdot n} \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - k(F_2)} \\
&\geq \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - k(F_2)}
\end{aligned}$$

where the penultimate step uses  $v(F_2) > k(F_2)$  and the last step uses (4.1.2). This contradicts Lemma 4.2.2 and completes the proof.  $\square$

Using the above lemma, it follows relatively easily that  $d_2(v)$  cannot be too large for any vertex  $v \in V$ . To verify this, we use the following assumption:

$$0 < \xi < \frac{\theta}{26\chi}. \quad (4.2.12)$$

**Lemma 4.2.13.** *For every  $v \in V$ ,*

$$d_2(v) \leq \left( 1 - \frac{1 - 3\theta}{\chi(F_2) - 1} \right) (n - 1).$$

*Proof.* If  $v \in V_1$ , then, by Observation 4.1.10,

$$\begin{aligned}
d_2(v) &= n - 1 - d_1(v) \\
&\leq n - \left( \frac{5}{4} \cdot \frac{1 + \xi}{\chi(F_2) - 1} \right) n \\
&< \left( 1 - \frac{1}{\chi(F_2) - 1} \right) n \\
&< \left( 1 - \frac{1 - 3\theta}{\chi(F_2) - 1} \right) (n - 1).
\end{aligned}$$

On the other hand, if  $v \notin V_1$ , then, by Lemma 4.2.10, we have that  $v \notin B$ . So, there exists  $i$  such that  $v$  has fewer than  $\theta|A_i|$  neighbours in  $A_i \cap V_2$ . Since  $|V \setminus V_2| \leq 26\xi n$  by Lemmas 4.1.9, 4.1.14 and 4.1.18, we have

$$d_2(v) \leq \sum_{j \neq i} |A_j| + \theta \cdot |A_i| + |A_i \setminus V_2| \leq n - (1 - \theta)|A_i| + 26\xi n$$

By Lemma 4.2.5 (a) this is at most

$$n - (1 - \theta) \left( \frac{1}{\chi(F_2) - 1} - \sqrt{3\varepsilon} \right) n + 26\xi n \leq \left( 1 - \frac{1 - \theta}{\chi(F_2) - 1} + \sqrt{3\varepsilon} + 26\xi \right) n.$$

Note that (4.2.7) implies that  $\varepsilon < \frac{\theta^2}{3(\chi(F_2) - 1)^2}$ . Using this bound, together with (4.2.12), tells us that the above expression is at most  $\left( 1 - \frac{1 - 3\theta}{\chi(F_2) - 1} \right) (n - 1)$  as desired.  $\square$

Next, we show that  $d_2(v)$  is reasonably large for every vertex  $v \in V$ . This will then be used to show that  $V_1 = \emptyset$ . To prove it, we assume the following:

$$0 < \varepsilon < \frac{1}{4f^2\chi^4}. \quad (4.2.14)$$

Also, we assume that  $t_0$  is chosen large enough that, for all  $t \geq t_0$ ,

$$e^{\theta t} \geq \frac{6\chi^{2f}(t + f)}{q}. \quad (4.2.15)$$

**Lemma 4.2.16.** *For every  $v \in V$ ,*

$$d_2(v) \geq \left( 1 - \frac{1 + 15\theta f}{\chi(F_2) - 1} \right) (n - 1).$$

*Proof.* Suppose that the lemma is false. Then there exists  $v \in V$  such that

$$\begin{aligned} d_1(v) = n - 1 - d_2(v) &> n - 1 - \left( 1 - \frac{1 + 15\theta f}{\chi(F_2) - 1} \right) (n - 1) \\ &\geq \left( \frac{1 + 14\theta f}{\chi(F_2) - 1} \right) n. \end{aligned}$$

Our goal is to show that  $v$  is contained in a large number of copies of  $H_1$  in  $G_1$  which will contradict Lemma 4.2.2. For each  $w \in V(F_1)$ , let  $p(w)$  be the number of pendant edges incident to  $w$  added in the construction of  $H_1$  from  $F_1$  and let  $v_0 \in V(F_1)$  so that  $p(v_0)$  is maximum. Then, by the Pigeonhole Principle,

$$p(v_0) \geq \frac{v(H_1) - v(F_1)}{v(F_1)}. \quad (4.2.17)$$

Let  $F'_1 := F_1 \setminus \{v_0\}$ .

By the lower bound on  $d_1(v)$  proven above, there must exist some  $1 \leq i \leq \chi(F_2) - 1$  such that  $v$  has at least

$$\frac{1}{\chi(F_2) - 1} \cdot \left( \frac{1 + 14\theta f}{\chi(F_2) - 1} \right) n \geq \frac{n}{(\chi(F_2) - 1)^2}$$

$G_1$ -neighbours in  $A_i$ . Let  $S$  be the set of  $G_1$ -neighbours of  $v$  in  $A_i$ . Recall that, by Lemma 4.1.7, the number of non-edges of  $G_1$  in  $S$  is at most

$$\varepsilon n^2 \leq \varepsilon n^2 \left( \frac{|S|}{n/(\chi(F_2) - 1)^2} \right)^2 = \varepsilon(\chi(F_2) - 1)^4 |S|^2.$$

Therefore, for large enough  $n$ ,

$$t(K_2, G_1[S]) = \frac{2e_1(S)}{|S|^2} \geq \frac{2\binom{|S|}{2} - 2\varepsilon(\chi(F_2) - 1)^4 |S|^2}{|S|^2} \geq 1 - 2\varepsilon\chi(F_2)^4.$$

Thus, if  $V(F'_1)$  is mapped to  $S$  randomly, then the probability that any individual edge of  $F'_1$  is mapped to a non-edge of  $G_1$  is at most  $2\varepsilon\chi(F_2)^4$ . So, by a union bound and (4.2.14), we have that

$$t(F'_1, G_1[S]) \geq 1 - 2\varepsilon e(F'_1)\chi(F_2)^4 > 1/2. \quad (4.2.18)$$

Now, if  $\varphi$  is a random function from  $V(H_1)$  to  $V$ , then the probability that  $\varphi$  is a homomorphism mapping  $v_0$  to  $v$  is at least the probability that the restriction of  $\varphi$  to  $V(F'_1)$  is a homomorphism from  $F'_1$  to  $G[S]$ , times  $1/n$  (the probability that  $v_0$  maps to  $v$ ) times the probability that every vertex of  $V(H_1) \setminus V(F_1)$  is mapped to a  $G_1$ -neighbour of the image of its unique neighbour in  $H_1$ . So, by Lemma 4.2.13,  $t(H_1, G_1)(v)$  is at least

$$(|S|/n)^{v(F'_1)} t(F'_1, G_1[S]) \cdot \frac{1}{n} \left( \frac{1 + 14\theta f}{\chi(F_2) - 1} \right)^{p(v_0)} \left( \frac{1 - 3\theta}{\chi(F_2) - 1} \right)^{v(H_1) - v(F_1) - p(v_0)}.$$

Using the fact that  $|S| \geq n/(\chi(F_2) - 1)^2 > n/\chi(F_2)^2$  and (4.2.18), we get that this is at least

$$\frac{1}{2\chi(F_2)^{2v(F_1)}} \frac{1}{n} \left( \frac{1}{\chi(F_2) - 1} \right)^{v(H_1) - v(F_1)} (1 + 14\theta f)^{p(v_0)} (1 - 3\theta)^{v(H_1) - v(F_1) - p(v_0)}.$$

Using the inequalities  $1 + r \geq e^{r/2}$  and  $1 - r \geq e^{-2r}$ , which are valid for all  $r \in [0, 1/2]$ , we can bound the product of the last two factors as follows:

$$\begin{aligned} & (1 + 14\theta f)^{p(v_0)} (1 - 3\theta)^{v(H_1) - v(F_1) - p(v_0)} \\ & \geq \exp(7\theta f p(v_0) - 6\theta(v(H_1) - v(F_1) - p(v_0))). \end{aligned}$$

By (4.2.17), this is at least

$$\exp(7\theta(v(H_1) - v(F_1)) - 6\theta(v(H_1) - v(F_1) - p(v_0))) \geq e^{\theta t_1}.$$

So, by (4.2.15) and the fact that  $t_1 \geq t_0$ , we have that

$$\begin{aligned} t(H_1, G_1)(v) &> \frac{1}{n} \left( \frac{1}{\chi(F_2) - 1} \right)^{v(H_1) - v(F_1)} \left( \frac{3(t_1 + f)}{q} \right) \\ &> \frac{3v(H_1)}{q \cdot n} \left( \frac{1}{\chi(F_2) - 1} \right)^{v(H_1) - k(F_1)} \\ &\geq \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left( \frac{1}{\chi(F_2) - 1} \right)^{v(H_1) - k(F_1)} \end{aligned}$$

where the penultimate step used  $v(F_1) > k(F_1)$  and the last step applied (4.1.2). This contradicts Lemma 4.2.2 and completes the proof.  $\square$

As a consequence of the previous lemma, we will show next that  $V_1 = \emptyset$ . This also implies  $B = \emptyset$  by virtue of Lemma 4.2.10. For this, we assume

$$0 < \theta < \frac{1}{60f}. \quad (4.2.19)$$

**Lemma 4.2.20.** *We have  $V_1 = \emptyset$ . Consequently,  $B = \emptyset$ .*

*Proof.* Assuming  $v \in V_1$ , Lemma 4.2.16 implies that

$$d_1(v) = n - 1 - d_2(v) \leq \left( \frac{1 + 15\theta f}{\chi(F_2) - 1} \right) (n - 1)$$

which, by (4.2.19), is less than

$$\left( \frac{5}{4} \cdot \frac{1}{\chi(F_2) - 1} \right) n.$$

This contradicts Observation 4.1.10, and so  $V_1$  must be empty. Lemma 4.2.10 then implies that  $B$  is also empty.  $\square$

From here forward, we impose an additional assumption that  $\sum_{i=1}^{\chi(F_2)-1} e_2(A_i)$  is minimum among all partitions  $A_1, \dots, A_{\chi(F_2)-1}$  of  $V$ . This allows us to prove the next lemma, which is analogous to [26, Claim 3.11]. We assume that  $\xi$  satisfies

$$0 < \xi < \theta \left( \frac{1}{f-1} - \sqrt{3\varepsilon} \right). \quad (4.2.21)$$

Note that the expression on the right side of the rightmost inequality above is positive by (4.2.14), and so it is possible to choose  $\xi$  in this way. We use the assumption on the choice of partition to show that, for each  $i$ , every vertex in  $A_i$  has few  $G_2$ -neighbours in  $A_i \cap V_2$ .

**Lemma 4.2.22.** *For  $1 \leq i \leq \chi(F_2) - 1$ , every  $v \in A_i$  is adjacent in  $G_2$  to fewer than  $\theta|A_i|$  vertices of  $A_i \cap V_2$ .*

*Proof.* Let  $v \in A_i$ . Then  $v$  is not bad by Lemma 4.2.20, so there must exist an index  $i'$  such that the number of  $G_2$ -neighbours of  $v$  in  $A_{i'} \cap V_2$  is at most  $\theta|A_{i'}|$ . If  $i' = i$ , then we are done; so, we assume that  $i' \neq i$ . Since  $V = V_0 \sqcup V_2$  by Lemmas 4.1.9 and 4.2.20, the number of  $G_2$ -neighbours of  $v$  in  $A_{i'}$  overall is at most  $\theta|A_{i'}| + |A_{i'} \cap V_0|$  which, by Lemma 4.1.14, is at most

$$\theta|A_{i'}| + \xi n < 2\theta|A_{i'}|$$

where the last step applies Lemma 4.2.5 (a) and (4.2.21). Since  $i \neq i'$ , the vertex  $v$  must have at most  $2\theta|A_{i'}|$  neighbours in  $A_i$  as well; otherwise, moving  $v$  from  $A_i$  to  $A_{i'}$  would decrease  $\sum_{i=1}^{\chi(F_2)-1} e_2(A_i)$ , contradicting our choice of partition. Thus,

$$d_2(v) \leq \sum_{j \notin \{i, i'\}} |A_j| + 4\theta|A_{i'}|.$$

By Lemma 4.2.5 (a) and (4.2.19), this is at most

$$(\chi(F_2) - 3 + 4\theta) \left( \frac{1}{\chi(F_2) - 1} + \sqrt{3\varepsilon} \right) n \leq \left( 1 - \frac{2}{\chi(F_2) - 1} + 4\theta + \chi(F_2)\sqrt{3\varepsilon} \right) n.$$

Using (4.2.7), we can bound this above by

$$\begin{aligned} \left( 1 - \frac{2}{\chi(F_2) - 1} + 5\theta \right) n &< \left( 1 - \frac{2}{\chi(F_2) - 1} + \frac{5\theta\chi(F_2)}{\chi(F_2) - 1} \right) (n - 1) \\ &= \left( 1 - \frac{1 + (1 - 5\theta\chi(F_2))}{\chi(F_2) - 1} \right) (n - 1). \end{aligned}$$

By (4.2.19), we have  $\theta \leq \frac{1}{15f + 5\chi(F_2)}$ . Plugging this into the above expression yields an upper bound of

$$\left( 1 - \frac{1 + 15\theta f}{\chi(F_2) - 1} \right) (n - 1)$$

contradicting Lemma 4.2.16 and completing the proof of the claim.  $\square$

Next, let us show that every vertex  $v \in A_i$  has many neighbours in  $A_j$  for  $j \neq i$ .

**Lemma 4.2.23.** *For  $1 \leq i \neq j \leq \chi(F_2) - 1$ , every  $v \in A_i$  is adjacent in  $G_2$  to at least  $(1 - 33\theta f)|A_j|$  vertices of  $A_j$ .*

*Proof.* Let  $v \in A_i$ . Suppose that  $v$  has fewer than  $(1 - 33\theta f)|A_j|$   $G_2$ -neighbours in  $A_j$ . Let  $S$  be the  $G_2$ -neighbourhood of  $v$ . Then

$$d_2(v) = \sum_{\ell=1}^{\chi(F_2)-1} |A_\ell \cap S| \leq |S \cap A_i \cap V_2| + |S \cap A_j| + |V_0| + \sum_{\ell \notin \{i, j\}} |A_\ell \cap S|$$

which, by Lemmas 4.1.14 and 4.2.22, is at most

$$\theta|A_i| + (1 - 33\theta f)|A_j| + \xi n + n - |A_i| - |A_j| = \left(1 + \xi - \frac{(1 - \theta)|A_i| + 33\theta f|A_j|}{n}\right) n.$$

Using the lower bound on  $|A_i|$  and  $|A_j|$  in Lemma 4.2.5 (a) yields an upper bound of

$$\begin{aligned} & \left(1 + \xi - (1 - \theta + 33\theta f) \left(\frac{1}{\chi(F_2) - 1} - \sqrt{3\varepsilon}\right)\right) n \\ & \leq \left(1 - \frac{1}{\chi(F_2) - 1} + \xi + \sqrt{3\varepsilon} - 32\theta f \left(\frac{1}{\chi(F_2) - 1} - \sqrt{3\varepsilon}\right)\right) n. \end{aligned}$$

By (4.2.4), (4.2.7) and (4.2.21), this is less than

$$\left(1 - \frac{1 + 15\theta f}{\chi(F_2) - 1}\right) (n - 1)$$

which contradicts Lemma 4.2.16 and completes the proof.  $\square$

Next, we prove that, in fact, there are no edges within  $G_2[A_i]$  for any  $1 \leq i \leq \chi(F_2) - 1$ . The presence of critical edges in  $F_1$  and  $F_2$  is crucial in this step. After this, the proof of Theorem 4.0.1 will follow relatively easily. The following lemma is analogous to [26, Claim 3.12]. Assume that  $t_0$  is large enough that the following holds for all  $t \geq t_0$ :

$$\left(\frac{10}{9}\right)^{qt} > 4\chi \left(\frac{4\chi}{3}\right)^f (t + f)^2. \quad (4.2.24)$$

Also, choose  $\theta$  small enough so that

$$0 < \theta < \frac{1}{66f^2}. \quad (4.2.25)$$

and

$$(1 - 8 \cdot \theta f) \geq (5/6) \cdot (1 + 15\theta f)^{1/q}. \quad (4.2.26)$$

Note that such a  $\theta$  exists because the limit as  $\theta \rightarrow 0$  of the left side is 1 and the limit of the right side is  $5/6$ .

**Lemma 4.2.27.**  $e_2(A_i) = 0$  for  $1 \leq i \leq \chi(F_2) - 1$ .

*Proof.* Suppose that the lemma is not true; without loss of generality, the set  $A_1$  contains an edge of  $G_2$ . Let  $u_0$  and  $v_0$  be the endpoints of such an edge. Let  $G'_2$  be the graph obtained from  $G_2$  by deleting the edge  $u_0v_0$  and let  $G'_1 = \overline{G'_2}$ . We estimate the number of copies of  $H_1$  in  $G_1$  that are “gained” and the number of copies of  $H_2$  in  $G_2$  that are “lost” when replacing  $(G_1, G_2)$  by  $(G'_1, G'_2)$  with a goal of contradicting the choice of  $G_1$ .

We begin by bounding from above the number of injective homomorphisms of  $H_1$  to  $G'_1$  which are not homomorphisms from  $H_1$  to  $G_1$ . Any such homomorphism can be described as follows. First, we pick an edge  $e = wz$  of  $H_1$  and map its endpoints to  $u_0v_0$  (in one of two possible ways). Now, imagine that we list the vertices of  $H_1$  so that  $w$  and  $z$  are listed first (in this order), followed by the other vertices of the component of  $H_1$  containing  $w$  and  $z$ , and then the vertices of another (arbitrary) component, and so on, so that each vertex in the list is either the first vertex of its component or has a neighbour which comes before it in the list, which we refer to as its “parent.” Then, in a homomorphism, each vertex in the list after  $w$  and  $z$  must be mapped to a  $G_1$ -neighbour of its parent (if it has one). Thus, since  $k(H_1) = k(F_1)$  and each vertex has at most  $\left(\frac{1+15\theta f}{\chi(F_2)-1}\right) (n-1)$  neighbours in  $G_1$  by Lemma 4.2.16, the number of such mappings is at most

$$\begin{aligned} & 2e(H_1)n^{k(F_1)-1} \left(\frac{1+15\theta f}{\chi(F_2)-1}\right)^{v(H_1)-2-(k(F_1)-1)} n^{v(H_1)-2-(k(F_1)-1)} \\ & = 2e(H_1)(\chi(F_2)-1)(1+15\theta f)^{v(H_1)-k(F_1)-1} \left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-k(F_1)} n^{v(H_1)-2}. \end{aligned}$$

Thus, by (4.1.3),

$$\begin{aligned} \rho_1(t_{\text{inj}}(H_1, G'_1) - t_{\text{inj}}(H_1, G_1)) & \leq 2e(H_1)(\chi(F_2)-1)(1+15\theta f)^{v(H_1)-k(F_1)-1} n^{-2} + O(n^{-3}) \\ & \leq 2e(H_1)(\chi(F_2)-1)(1+15\theta f)^{v(H_1)} n^{-2} + O(n^{-3}). \end{aligned}$$

Next, let us bound from below the number of injective homomorphisms of  $H_2$  to  $G'_2$  which are not homomorphisms from  $H_2$  to  $G_2$ . Let  $e_0 = w_0z_0$  be a critical edge of  $F_2$ , let  $F'_2 = F_2 \setminus \{e_0\}$  and let  $\psi : V(F'_2) \rightarrow \{1, \dots, \chi(F_2) - 1\}$  be a proper colouring of  $F'_2$  such that  $\psi(w_0) = \psi(z_0) = 1$ . Now, suppose that  $\varphi$  is a function that maps  $w_0$  to  $u_0$  and  $z_0$  to  $v_0$  and then maps every other vertex of  $F'_2$  to  $V$  randomly. The probability that every other vertex  $u$  of  $F'_2$  is mapped by  $\varphi$  to  $A_{\psi(u)}$  is

$$\prod_{u \in V(F'_2) \setminus \{w_0, z_0\}} \left(\frac{|A_{\psi(u)}|}{n}\right) \geq \left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)^{v(F'_2)-2} \geq \left(\frac{1}{\chi}\right)^f.$$

by Lemma 4.2.5 (a) and (4.2.4). Given this, by Lemma 4.2.23, the probability that every edge of  $F'_2$  maps to an edge of  $G_2$  is, by a union bound, at least

$$1 - 33\theta e(F'_2)f \geq 1/2$$

where the inequality is by (4.2.25). Finally, given these two events, if each vertex of

$V(H_2) \setminus V(F_2)$  is mapped randomly to  $V$ , an application of Lemma 4.2.16 combined with the above inequalities tells us that the probability that the final function is a homomorphism is at least

$$\frac{1}{2} \left( \frac{1}{\chi} \right)^f \left( 1 - \frac{1 + 16\theta f}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)} = \frac{1}{2} \left( \frac{1}{\chi} \right)^f \left( \frac{\chi(F_2) - 2 - 16\theta f}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)}.$$

The last factor can be bounded as follows:

$$\begin{aligned} & \left( \frac{\chi(F_2) - 2 - 16\theta f}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)} \\ &= \frac{(\chi(F_1) - 1)^{v(H_2) - v(F_2)}}{(\chi(F_1) - 1)^{v(H_2) - v(F_2)}} \left( \frac{\chi(F_2) - 2 - 16\theta f}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)} \\ &= \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - v(F_2)} \left( \frac{(\chi(F_1) - 1)(\chi(F_2) - 2 - 16\theta f)}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)} \\ &\geq \left( \frac{1}{\chi(F_1) - 1} \right)^{v(H_2) - k(F_2)} \left( \frac{(\chi(F_1) - 1)(\chi(F_2) - 2 - 16\theta f)}{\chi(F_2) - 1} \right)^{v(H_2) - v(F_2)} \end{aligned}$$

where in the inequality we used that  $v(F_2) \geq k(F_2)$ . Now, since  $\chi(F_1) \geq 3$  and  $\chi(F_1) + \chi(F_2) \geq 7$ , we have

$$\frac{(\chi(F_1) - 1)(\chi(F_2) - 2 - 16\theta f)}{\chi(F_2) - 1} \geq \frac{4}{3}(1 - 8 \cdot \theta f).$$

Putting this all together and applying (4.1.3), we get that

$$\begin{aligned} \rho_2(t_{\text{inj}}(H_2, G_2) - t_{\text{inj}}(H_2, G'_2)) &\geq \frac{1}{2} \left( \frac{1}{\chi} \right)^f \left( \frac{4}{3}(1 - 8 \cdot \theta f) \right)^{v(H_2) - v(F_2)} n^{-2} - O(n^{-3}) \\ &\geq \frac{1}{2} \left( \frac{3}{4\chi} \right)^f \left( \frac{4}{3}(1 - 8 \cdot \theta f) \right)^{v(H_2)} n^{-2} - O(n^{-3}) \end{aligned}$$

which, by (4.1.2), is at least

$$\frac{1}{2} \left( \frac{3}{4\chi} \right)^f \left( \frac{4}{3}(1 - 8 \cdot \theta f) \right)^{qv(H_1)} n^{-2} - O(n^{-3}).$$

Now, by (4.2.26), this is at least

$$\begin{aligned} & \frac{1}{2} \left( \frac{3}{4\chi} \right)^f \left( \frac{4}{3} \cdot \frac{5}{6} (1 + 15\theta f)^{1/q} \right)^{qv(H_1)} n^{-2} - O(n^{-3}) \\ &= \frac{1}{2} \left( \frac{3}{4\chi} \right)^f \left( \frac{10}{9} \right)^{qv(H_1)} (1 + 15\theta f)^{v(H_1)} n^{-2} - O(n^{-3}). \end{aligned}$$

Combining the upper bound that we have proven on  $\rho_1(t_{\text{inj}}(H_1, G'_1) - t_{\text{inj}}(H_1, G_1))$  and the lower bound on  $\rho_2(t_{\text{inj}}(H_2, G_2) - t_{\text{inj}}(H_2, G'_2))$ , we get that

$$\begin{aligned} & n^2(m(H_1, H_2; G_1) - m(H_1, H_2; G'_1)) \\ & \geq \frac{1}{2} \left(\frac{3}{4\chi}\right)^f \left(\frac{10}{9}\right)^{qv(H_1)} (1 + 15\theta f)^{v(H_1)} - 2e(H_1)(\chi(F_2) - 1)(1 + 15\theta f)^{v(H_1)} - O(n^{-1}) \end{aligned}$$

which is positive for large  $n$  by (4.2.24). This contradicts the definition of  $G_1$  and completes the proof.  $\square$

Finally, we present the proof of Theorem 4.0.1.

*Proof of Theorem 4.0.1.* Given  $F_1, F_2$  and  $q$  satisfying the hypotheses of the theorem, we select our parameters in the following order, subject to the given conditions:

- choose  $\theta$  to satisfy (4.2.19), (4.2.25) and (4.2.26),
- choose  $\varepsilon$  so that (4.2.4), (4.2.7) and (4.2.14) hold,
- choose  $\delta$  as in (4.1.28),
- choose  $\beta$  as in (4.1.23),
- choose  $\xi$  so that (4.1.8), (4.1.24), (4.1.29), (4.2.12) and (4.2.21) all hold,
- choose  $\gamma$  as in (4.1.16),
- choose  $\tau$  to satisfy (4.1.13), (4.1.17) and (4.1.25),
- choose  $t_0$  large enough so that (4.1.11), (4.2.8), (4.2.15) and (4.2.24) all hold.

Let  $t_1, t_2 \geq t_0$  and let  $H_1$  be a  $t_1$ -hairy  $F_1$  and  $H_2$  be a  $t_2$ -hairy  $F_2$ . We may assume that  $H_1$  and  $H_2$  have no singleton components. Let  $n_0$  be large with respect to  $H_1$  and  $H_2$  and the parameters chosen in the previous paragraph and let  $G_1$  be a graph on  $n$  vertices minimizing  $m(H_1, H_2; G_1)$  and  $G_2 = \overline{G_1}$ . Without loss of generality,  $|V_1| \leq |V_2|$ . As a result of our parameter choices, all of the statements in Sections 4.1 and 4.2 hold. In particular, Lemma 4.2.27 implies that there is a partition  $A_1, \dots, A_{\chi(F_2)-1}$  of  $V = V(G_1)$  such that  $G_2$  contains no edge with endpoints in  $A_i$  for  $1 \leq i \leq \chi(F_2) - 1$ , and Lemma 4.2.5 ((a)) guarantees that all of the sets of the partition have approximately the same size,  $\frac{n}{\chi(F_2)-1}$ .

We assert that  $G_1$  has no edges between  $A_i$  and  $A_j$  for  $i \neq j$ . To prove this, suppose that such an edge exists in  $G_1$ . If we move this edge from  $G_1$  to  $G_2$ , it would destroy at least one injective homomorphism from  $H_1$  to  $G_1$  (since  $|A_i| > v(H_1)$  for large  $n$  and  $H_1$  has at least  $t_1 \geq 1$  vertices of degree one). At the same time, this would not create any injective homomorphism from  $H_2$  to  $G_2$ , since  $G_2$  is still  $(\chi(F_2) - 1)$ -partite

after adding such an edge to it. This contradicts our choice of  $G_1$ . Therefore,  $G_2$  is a complete  $(\chi(F_2) - 1)$ -partite graph. In particular,  $t(H_2, G_2) = 0$  and  $G_1$  is a disjoint union of  $\chi(F_2) - 1$  cliques.

Finally, we show that the cardinalities of any two sets  $A_i$  and  $A_j$  differ by at most one. Each homomorphism from  $H_1$  to  $G_1$  gives rise to a partition of  $V(H_1)$  into at most  $\chi(F_2) - 1$  classes such that each partition class is a union of components of  $H_1$  and all vertices of each class are mapped to the same component of  $G_1$ . We think of these partitions as being “unlabelled” in the sense that they contain information about which components of  $H_1$  are mapped to the same component of  $G_1$  but not about which component of  $G_1$  they are mapped to. Given such a partition  $\mathcal{P} = \{P_1, \dots, P_{\chi(F_2)-1}\}$  (where we allow some of the sets  $P_j$  to be empty), we show that the number of injective homomorphisms of  $H_1$  to  $G_1$  giving rise to the partition  $\mathcal{P}$  is minimized when the cardinalities any two of the sets  $A_i$  and  $A_j$  differ by at most one. For each  $1 \leq i \leq \chi(F_2) - 1$ , let us count the number of choices for the mapping of vertices in  $P_i$  given the mapping of the vertices in  $\bigcup_{j=1}^{i-1} P_j$ . Let  $T_i$  be the set of indices  $t$  such that there does not exist  $1 \leq j \leq i - 1$  such that the vertices of  $P_j$  are mapped to  $A_t$ . Then the number of choices for the mapping of  $P_i$  given that of  $P_j$  for all  $j < i$  is

$$\sum_{t \in T_i} \frac{|A_t|!}{(|A_t| - |P_i|)!} \quad (4.2.28)$$

For an integer  $c \geq 2$ , define  $f_c : \mathbb{R} \rightarrow \mathbb{R}$  by  $f_c(z) = z(z - 1) \cdots (z - c + 1)$ . Then  $f_c$  has  $c - 1$  distinct (integer) roots in the interval  $[0, c - 1]$ . The derivative  $f'_c(z)$  is a polynomial of degree  $c - 1$  with  $c - 1$  real roots which interlace the roots of  $f_c$ ; in particular, its roots are also contained in the interval  $[0, c - 1]$ . By similar logic, the roots of the second derivative  $f''_c$  are in  $[0, c - 1]$  as well. From this, we see that  $f''_c$  is positive on  $[c, \infty)$ , and so  $f_c$  is strictly convex on this set. Thus, by Jensen’s Inequality, for any  $i$  such that  $|P_i| \geq 2$ , the sum in (4.2.28) is uniquely minimized when the cardinalities of the sets  $A_t$  for  $t \in T_i$  are as similar as possible. Thus, the number of injective homomorphisms from  $H_1$  to  $G_1$  is minimized by taking  $G_2$  to be a  $(\chi(F_2) - 1)$ -partite Turán graph.  $\square$

### 4.3 Beating the Turán Colouring

In this section, we show that if  $(H_1, H_2)$  is a bonbon pair, then  $e(H_1)$  cannot be “excessively large;” see Theorem 4.3.4 below. This result will then be used to derive Proposition 4.0.2. We will use the following result of [25] which was previously known as Tomescu’s Graph Colouring Conjecture [67].

**Theorem 4.3.1** (Fox, He and Manners [25]). *For  $m \neq 3$ , every connected  $m$ -chromatic graph on  $n$  vertices has at most  $m!(m - 1)^{n-m}$  proper  $m$ -colourings.*

The case  $m = 2$  of the above theorem is trivial, as every connected bipartite graph has precisely two proper 2-colourings. It is also necessary to exclude the case  $m = 3$ , as an odd cycle of length  $k \geq 5$  has more than  $3!2^{k-3}$  proper 3-colourings. Knox and Mohar [41, 42] established the cases  $m = 4$  and  $m = 5$  before the full conjecture was proven by Fox, He and Manners [25]. Note that every hairy  $K_m$  on  $n$  vertices has exactly  $m!(m-1)^{n-m}$  proper  $m$ -colourings and so Theorem 4.3.1 is tight. We will use the following corollary of Theorem 4.3.1. Given a graph  $H$ , say that a vertex colouring  $f : V(H) \rightarrow [\chi(H) - 1]$  of  $H$  is *nearly proper* if there is a unique edge of  $H$  whose endpoints are the same colour.

**Corollary 4.3.2.** *If  $H$  is a graph such that  $\chi(H) \neq 4$ , then the number of nearly proper colourings of  $H$  is at most*

$$\text{crit}(H) \cdot (\chi(H) - 1)! \cdot (\chi(H) - 2)^{v(H) - \chi(H) - k(H) + 1} \cdot (\chi(H) - 1)^{k(H)}.$$

*Proof.* Given an edge  $e$  of  $H$ , let  $H/e$  be the graph obtained by *contracting*  $e$ ; i.e. by identifying the two endpoints of  $e$  and removing any multi-edges that arise. Let  $z_e$  be the vertex formed by contracting the edge  $e$ . The number of nearly proper colourings of  $H$  is equal to the number of ways to select

- a critical edge  $e$  of  $H$ ,
- a proper  $(\chi(H) - 1)$ -colouring of the component of  $H/e$  containing  $z_e$  and
- a proper  $(\chi(H) - 1)$ -colouring of the components of  $H/e$  that do not contain  $z_e$ .

The number of choices in the first step is clearly  $\text{crit}(H)$ .

Assuming that a critical edge  $e$  has been chosen, let  $p$  denote the number of vertices in the component of  $H/e$  containing  $z_e$ . Note that the chromatic number of this component is exactly  $\chi(H) - 1$  which, since  $\chi(H) \neq 4$ , is not equal to three. So, by Theorem 4.3.1, the number of choices in the second step is at most  $(\chi(H) - 1)! (\chi(H) - 2)^{p - (\chi(H) - 1)}$ .

In the last step, for each component that does not contain  $z_e$ , there are at most  $\chi(H) - 1$  choices for the colour of an arbitrary “root” vertex of this component and then at most  $\chi(H) - 2$  choices for each subsequent vertex. Since  $H/e$  has  $v(H) - 1$  vertices, the number of vertices in the components of  $H/e$  that do not contain  $z_e$  is  $v(H) - 1 - p$ . Thus, the number of choices in the last step is at most

$$(\chi(H) - 1)^{k(H) - 1} (\chi(H) - 2)^{v(H) - 1 - p - (k(H) - 1)}.$$

Putting this all together, we get that the number of nearly proper colourings of  $H$  is

at most

$$\text{crit}(H) \cdot (\chi(H) - 1)! \cdot (\chi(H) - 2)^{v(H) - \chi(H) - k(H) + 1} \cdot (\chi(H) - 1)^{k(H) - 1}$$

as desired.  $\square$

We also need the following simple bound on the number of nearly proper colourings in the case that  $\chi(H) = 4$ . The proof is analogous to that of the previous corollary, except that, instead of Theorem 4.3.1, we use the (trivial) fact that every connected 3-chromatic graph on  $n$  vertices has at most  $3 \cdot 2^{n-1}$  proper 3-colourings.

**Lemma 4.3.3.** *If  $H$  is a 4-chromatic graph, then the number of nearly proper colourings of  $H$  is at most*

$$\text{crit}(H) \cdot 3^{k(H)} \cdot 2^{v(H) - k(H) - 1}.$$

Next, we use Corollary 4.3.2 and Lemma 4.3.3 to prove the following result which restricts the number of edges in a graph contained in a bonbon pair. In fact, it applies to a slightly more general class of graphs, namely those that are Ramsey multiplicity good.

**Theorem 4.3.4.** *Let  $H_1$  and  $H_2$  be graphs such that, if  $\chi(H_2) \neq 4$ , then*

$$e(H_1) > \frac{\text{crit}(H_2) \cdot (\chi(H_2) - 2)! \cdot (\chi(H_2) - 2)^{v(H_2) - \chi(H_2) - k(H_2) + 1} \cdot (\chi(H_1) - 1)^{v(H_2) - k(H_2)}}{(\chi(H_2) - 1)^{v(H_2) - k(H_2)}}$$

and, otherwise,

$$e(H_1) > \frac{\text{crit}(H_2) \cdot 2^{v(H_2) - k(H_2) - 1} (\chi(H_1) - 1)^{v(H_2) - k(H_2)}}{3^{v(H_2) - k(H_2)}}.$$

Then  $(H_1, H_2)$  is not Ramsey multiplicity good.

*Proof.* Suppose that  $H_1$  and  $H_2$  are graphs satisfying the hypotheses of the theorem. Let  $\varepsilon > 0$  be very small and, for each  $n \geq \chi(H_2) - 1$ , let  $G_{n,\varepsilon}$  be a graph on  $n$  vertices obtained from the complement of the Turán graph with  $n$  vertices and  $\chi(H_2) - 1$  parts by jettisoning each edge of this graph with probability  $\varepsilon$  independently of all other such edges. Define

$$f_1(\varepsilon) := \lim_{n \rightarrow \infty} t(H_1, G_{n,\varepsilon}), \quad f_2(\varepsilon) := \lim_{n \rightarrow \infty} t(H_2, \overline{G_{n,\varepsilon}})$$

and note that both of these limits exist with probability 1. Moreover, with probability one,

$$(\chi(H_2) - 1)^{v(H_1) - k(H_1)} \cdot f_1(\varepsilon) = (1 - \varepsilon)^{e(H_1)} = 1 - e(H_1)\varepsilon + O(\varepsilon^2)$$

where the asymptotics here (and throughout the proof) are as  $\varepsilon \rightarrow 0$ .

Let  $K$  be the number of nearly proper colourings of  $H_2$ . Then, with probability one,

$$(\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot f_2(\varepsilon) = (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot \varepsilon K \left( \frac{1}{\chi(H_2) - 1} \right)^{v(H_2)} + O(\varepsilon^2). \quad (4.3.5)$$

At this point, we divide the proof into cases.

*Case 1.*  $\chi(H_2) \neq 4$ .

By Corollary 4.3.2, the linear term of (4.3.5) (with respect to  $\varepsilon$ ) is at most

$$\begin{aligned} & (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot \varepsilon \cdot \text{crit}(H_2) \cdot (\chi(H_2) - 2)! \\ & \quad \cdot (\chi(H_2) - 2)^{v(H_2)-\chi(H_2)-k(H_2)+1} \left( \frac{1}{\chi(H_2) - 1} \right)^{v(H_2)-k(H_2)} \end{aligned}$$

which is equal to

$$\frac{\varepsilon \cdot \text{crit}(H_2) \cdot (\chi(H_2) - 2)! \cdot (\chi(H_2) - 2)^{v(H_2)-\chi(H_2)-k(H_2)+1} \cdot (\chi(H_1) - 1)^{v(H_2)-k(H_2)}}{(\chi(H_2) - 1)^{v(H_2)-k(H_2)}}.$$

Therefore, the lower bound on  $e(H_1)$  assumed at the beginning of the proof implies that the linear term with respect to  $\varepsilon$  in  $(\chi(H_2) - 1)^{v(H_1)-k(H_1)} \cdot f_1(\varepsilon) + (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot f_2(\varepsilon)$  has a negative coefficient. So, for  $\varepsilon$  sufficiently small, we have that

$$(\chi(H_2) - 1)^{v(H_1)-k(H_1)} t(H_1, G_{n,\varepsilon}) + (\chi(H_1) - 1)^{v(H_2)-k(H_2)} t(H_2, \overline{G_{n,\varepsilon}}) = 1 - \Omega(\varepsilon)$$

as  $n \rightarrow \infty$  which implies that  $(H_1, H_2)$  is not Ramsey multiplicity good.

*Case 2.*  $\chi(H_2) = 4$ .

In this case, by Lemma 4.3.3, the linear term of (4.3.5) with respect to  $\varepsilon$  is at most

$$\begin{aligned} & (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \varepsilon \cdot \text{crit}(H_2) \cdot 2^{v(H_2)-k(H_2)-1} \left( \frac{1}{3} \right)^{v(H_2)-k(H_2)} \\ & = \frac{\varepsilon \cdot \text{crit}(H_2) \cdot 2^{v(H_2)-k(H_2)-1} (\chi(H_1) - 1)^{v(H_2)-k(H_2)}}{3^{v(H_2)-k(H_2)}}. \end{aligned}$$

Thus, analogous to the previous case, by taking  $\varepsilon$  sufficiently close to zero, we get a certificate that  $(H_1, H_2)$  is not Ramsey multiplicity good.  $\square$

*Proof of Proposition 4.0.2.* The proposition follows immediately from Theorem 4.3.4.  $\square$

## 4.4 Related Directions for Future Work

We conclude this chapter by stating some open problems. A result of Goodman [32] implies that  $c(K_3) = \frac{1}{4}$  and so  $K_3$  is Ramsey multiplicity good. However, for odd  $n$ , the quantity  $\text{hom}_{\text{inj}}(K_3, G) + \text{hom}_{\text{inj}}(K_3, \overline{G})$  is minimized among all  $n$ -vertex graphs by every  $n$ -vertex graph  $G$  which is  $(\frac{n-1}{2})$ -regular; therefore,  $K_3$  is Ramsey multiplicity good but not a bonbon. We are currently unaware of any non-3-colourable graph which is Ramsey multiplicity good but not a bonbon, which leads us to the following question.

**Question 4.4.1.** *Is it true that every non-3-colourable Ramsey multiplicity good graph is a bonbon?*

It would also be interesting to explore off-diagonal variants of the above question, such as the following.

**Question 4.4.2.** *Suppose that  $(H_1, H_2)$  is Ramsey multiplicity good such that  $H_1$  and  $H_2$  are non-bipartite and  $\chi(H_1) + \chi(H_2) \geq 7$ . Does it follow that  $(H_1, H_2)$  is a bonbon pair?*

Currently, all of the known examples of bonbons contain vertices of degree one. It is unclear whether a bonbon of minimum degree at least two can exist. The analogous question for non-3-colourable Ramsey multiplicity good graphs is also intriguing (the case of chromatic number three is settled, since  $K_3$  is Ramsey multiplicity good).

**Question 4.4.3.** *Does there exist a bonbon  $H$  such that  $\delta(H) \geq 2$ ?*

**Question 4.4.4.** *Does there exist a non-3-colourable Ramsey multiplicity good graph  $H$  such that  $\delta(H) \geq 2$ ?*

Let us conclude with an observation that was shared with us by an anonymous referee. We claim that, if  $H$  is a bonbon, then  $H$  must contain a bridge. Indeed, consider the colouring obtained from a Turán colouring where the red edges form a complete  $(\chi(H) - 1)$ -partite graph and change one of the red edges to blue. If  $H$  does not have a bridge, then this colouring has the same number of monochromatic copies of  $H$  as the Turán colouring and so  $H$  is not a bonbon. In particular, this implies that the answer to Question 4.4.3 would be “no” if the condition  $\delta(H) \geq 2$  was replaced with the stronger condition that  $H$  is 2-edge connected. However, this example does not seem to preclude the existence of 2-edge connected Ramsey multiplicity good graphs.

**Question 4.4.5.** *Does there exist a non-3-colourable Ramsey multiplicity good graph  $H$  that is 2-edge connected?*

# Chapter 5

## Small Ramsey Multiplicity Good Graphs

Our main result of this chapter is to prove Theorem 5.0.1 below. Lemma 5.0.2 allows us to extend our result to Theorem 1.0.9. In Section 5.1, we prove a general homomorphism density inequality using the ideas from Chapter 3 reducing the problem into a certain constrained optimization problem. In Section 5.2 and Section 5.3, we introduce the flag algebra method and use it to prove a lower bound on Ramsey multiplicity constants under a restriction on edge density. It is easy to see that the following upper bound on Ramsey multiplicity constant for  $K_4 \sqcup 3 \cdot K_2$  can be obtained by the Turán colouring;

$$c(K_4 \sqcup 3 \cdot K_2) \leq \left(\frac{1}{3}\right)^6.$$

Therefore, to show Theorem 1.0.8, it suffices to prove the following theorem.

**Theorem 5.0.1.** *For every graphon  $W$ ,*

$$t(K_2, W)^3 t(K_4, W) + t(K_2, 1 - W)^3 t(K_4, 1 - W) \geq \left(\frac{1}{3}\right)^6.$$

We will deduce Theorem 5.0.1 from the following four lemmas, the fourth of which comes from [33]. After that, we will turn our attention to proving the first three lemmas.

**Lemma 5.0.2.** *Let  $k \geq 4$ . For any  $s \geq t \geq 0$  and any graphon  $W$ ,  $t(K_2, W)^s t(K_k, W) + t(K_2, 1 - W)^s t(K_k, 1 - W)$  is at least*

$$\left(\frac{1}{k-1}\right)^{s-t} (t(K_2, W)^t t(K_k, W) + t(K_2, 1 - W)^t t(K_k, 1 - W)).$$

**Lemma 5.0.3.** *If  $W$  is a graphon such that  $t(K_2, W) \leq \frac{1}{3}$ , then*

$$t(K_4, W) + t(K_4, 1 - W) \geq \left(\frac{1}{3}\right)^3.$$

**Lemma 5.0.4.** *If  $\frac{1}{3} \leq p \leq \frac{13}{36}$  and  $W$  is a graphon such that  $t(K_2, W) \leq p$ , then*

$$p^3 t(K_4, W) + (1-p)^3 t(K_4, 1-W) \geq \left(\frac{1}{3}\right)^6.$$

**Lemma 5.0.5** (Grzesik, Lee, Lidický and Volec [33]).  $c(K_4) \geq \frac{1}{33.77}$ .

Now, let us show how Theorem 5.0.1 follows from these four lemmas.

*Proof of Theorem 5.0.1.* Let  $W$  be any graphon. Without loss of generality, we may assume that  $t(K_2, W) \leq \frac{1}{2}$ ; otherwise, we simply exchange  $W$  with  $1-W$ . We divide the proof into three cases.

*Case 3.*  $t(K_2, W) \leq \frac{1}{3}$ .

In this case, applying Lemma 5.0.2 with  $s = 3$  and  $t = 0$  gives us

$$\begin{aligned} & t(K_2, W)^3 t(K_4, W) + t(K_2, 1-W)^3 t(K_4, 1-W) \\ & \geq (1/3)^3 (t(K_4, W) + t(K_4, 1-W)) \end{aligned}$$

which, by Lemma 5.0.3, is at least  $(1/3)^6$ . This completes the proof in this case.

*Case 4.*  $\frac{1}{3} < t(K_2, W) \leq \frac{13}{36}$ .

The result follows immediately from an application of Lemma 5.0.4 with  $p = t(K_2, W)$ .

*Case 5.*  $\frac{13}{36} < t(K_2, W) \leq \frac{1}{2}$ .

In this case, since  $t(K_2, 1-W) \geq t(K_2, W)$ , we have

$$\begin{aligned} & t(K_2, W)^3 t(K_4, W) + t(K_2, 1-W)^3 t(K_4, 1-W) \\ & \geq t(K_2, W)^3 (t(K_4, W) + t(K_4, 1-W)) \\ & > \left(\frac{13}{36}\right)^3 \left(\frac{1}{33.77}\right) \\ & > \left(\frac{1}{3}\right)^6 \end{aligned}$$

where Lemma 5.0.5 and the fact that  $t(K_2, W) > \frac{13}{36}$  were applied in the penultimate line. This completes the proof.  $\square$

## 5.1 Monotonicity

To prove Lemma 5.0.2, we apply the following consequence of the Kruskal–Katona Theorem [38, 47].

**Theorem 5.1.1** (Katona [38] and Kruskal [47]). *For every  $k \geq 2$ , every graphon  $W$  satisfies  $t(K_k, W) \leq t(K_2, W)^{k/2}$ .*

We also use the following precursor to the Clique Density Theorem of Reiher [58] which was first proved (in a stronger form) by Bollobás [7].

**Theorem 5.1.2** (Bollobás [7]). *For any  $k \geq 2$ , every graphon  $W$  satisfies*

$$t(K_k, W) \geq \frac{k(k-1)k!}{k^k} \left( t(K_2, W) - \frac{k-2}{k-1} \right).$$

We now prove Lemma 5.0.2.

*Proof of Lemma 5.0.2.* If  $t(K_2, W) > \frac{1}{k-1}$  and  $t(K_2, 1-W) > \frac{1}{k-1}$ , then the desired inequality holds trivially. So, assume that

$$t(K_2, W) \leq \frac{1}{k-1}. \tag{5.1.3}$$

Set

$$\begin{aligned} x &:= t(K_2, W), \\ y &:= t(K_k, W), \\ z &:= t(K_k, 1-W). \end{aligned}$$

Note that  $t(K_2, 1-W) = 1-x$ . We define

$$f(x, y, z) := x^s y + (1-x)^s z - \left( \frac{1}{k-1} \right)^{s-t} (x^t y + (1-x)^t z).$$

We will be done if we can show that  $f(x, y, z) \geq 0$  for all possible choices of  $x, y, z$ . By doing a bit of rearranging, we get

$$f(x, y, z) = x^t \left( x^{s-t} - \left( \frac{1}{k-1} \right)^{s-t} \right) y + (1-x)^t \left( (1-x)^{s-t} - \left( \frac{1}{k-1} \right)^{s-t} \right) z.$$

We compute the partial derivatives with respect to  $y$  and  $z$ . We have

$$\frac{\partial f}{\partial y} = x^t \left( x^{s-t} - \left( \frac{1}{k-1} \right)^{s-t} \right)$$

which is non-positive since  $0 \leq x \leq \frac{1}{k-1}$  by (5.1.3). Also,

$$\frac{\partial f}{\partial z} = (1-x)^t \left( (1-x)^{s-t} - \left( \frac{1}{k-1} \right)^{s-t} \right)$$

which is positive for all  $0 \leq x \leq \frac{1}{k-1}$  since  $k \geq 4$ . Therefore, the function  $f$  is minimized when the variable  $y$  is as large as possible and  $z$  is as small as possible. By applying Theorem 5.1.1 to  $W$  and Theorem 5.1.2 to  $1 - W$ , we know that

$$y \leq x^{k/2}$$

and

$$z \geq \frac{k(k-1)k!}{k^k} \left(1 - x - \frac{k-2}{k-1}\right) = \frac{k(k-1)k!}{k^k} \left(\frac{1}{k-1} - x\right).$$

Therefore,  $f(x, y, z)$  is at least  $f\left(x, x^{k/2}, \frac{k(k-1)k!}{k^k} \left(\frac{1}{k-1} - x\right)\right)$  which is equal to

$$x^t \left(x^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) x^{\frac{k}{2}} + (1-x)^t \left((1-x)^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) \frac{k(k-1)k!}{k^k} \left(\frac{1}{k-1} - x\right).$$

Using the fact that  $0 \leq x \leq \frac{1}{k-1}$ , the above expression can be bounded from below by

$$x^{t+\frac{k}{2}} \left(x^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) + \left(\frac{k-2}{k-1}\right)^t \left(\left(\frac{k-2}{k-1}\right)^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) \frac{k(k-1)k!}{k^k} \left(\frac{1}{k-1} - x\right).$$

Let  $g(x)$  be the above quantity. We show that  $g$  is decreasing. We have

$$\begin{aligned} \frac{dg}{dx} &= \left(t + \frac{k}{2}\right) x^{t+\frac{k}{2}-1} \left(x^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) + (s-t)x^{s+\frac{k}{2}-1} \\ &\quad - \left(\frac{k-2}{k-1}\right)^t \left(\left(\frac{k-2}{k-1}\right)^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) \frac{k(k-1)k!}{k^k} \\ &\leq (s-t) \left(\frac{1}{k-1}\right)^{s+\frac{k}{2}-1} - \left(\frac{k-2}{k-1}\right)^t \left(\left(\frac{k-2}{k-1}\right)^{s-t} - \left(\frac{1}{k-1}\right)^{s-t}\right) \frac{k(k-1)k!}{k^k} \\ &= \left(\frac{1}{k-1}\right)^{s-1} \left(-\frac{k \cdot k!}{k^k} (k-2)^s + s \left(\frac{1}{k-1}\right)^{\frac{k}{2}} + \frac{k \cdot k!}{k^k} (k-2)^t - t \left(\frac{1}{k-1}\right)^{\frac{k}{2}}\right). \end{aligned}$$

We prove that the final quantity in the above calculation is non-positive for all  $s \geq t \geq 0$ . To do this, it suffices to show that the function  $h : [0, \infty) \rightarrow \mathbb{R}$ , defined as follows, is increasing

$$h(w) := \frac{k \cdot k!}{k^k} (k-2)^w - w \left(\frac{1}{k-1}\right)^{\frac{k}{2}}.$$

To this end, we compute

$$\begin{aligned} \frac{dh}{dw} &= \log(k-2) \frac{k \cdot k!}{k^k} (k-2)^w - \left( \frac{1}{k-1} \right)^{\frac{k}{2}} \\ &\geq \log(k-2) \frac{k \cdot k!}{k^k} - \left( \frac{1}{k-1} \right)^{\frac{k}{2}}. \end{aligned}$$

For  $k \in \{4, 5, 6, 7, 8\}$ , this quantity can be shown to be positive by direct computation. Now, for  $k \geq 9$ , we use the bound  $k! > \left(\frac{k}{e}\right)^k$  and the fact that  $\log(k-2) \cdot k > 1$  to bound this quantity below by

$$\frac{1}{e^k} - \left( \frac{1}{k-1} \right)^{\frac{k}{2}} > \frac{1}{e^k} - \left( \frac{1}{e^2} \right)^{\frac{k}{2}} = 0$$

where, here, we used that  $k-1 \geq 8 \geq e^2$ . Therefore,  $h$  is increasing. This implies that  $g$  is non-increasing. Putting all of this together, we get that

$$f(x, y, z) \geq g\left(\frac{1}{k-1}\right) = 0$$

which completes the proof. □

## 5.2 Low Edge Density

The proofs in this section and the one that follows it will apply the flag algebra method of Razborov [56]. To avoid unnecessary abstraction, we will not discuss flag algebras in full generality. Instead, we will only build up the ideas from flag algebras that we require.

For each graph  $H$  and graphon  $W$ , let

$$t_{\text{ind}}(H, W) = \int_{[0,1]^{V(H)}} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{uv \in E(\overline{H})} (1 - W(x_u, x_v)) \prod_{v \in V(H)} dx_v.$$

This is similar to the homomorphism density, except that we additionally require every non-edge of  $H$  to be mapped to a non-edge. There are precisely  $2^{\binom{4}{2}} = 64$  distinct labelled graphs on four vertices, which we label by  $H_1, \dots, H_{64}$  in any order.

**Lemma 5.2.1.** *Every graphon  $W$  satisfies*

$$\sum_{i=1}^{64} t_{\text{ind}}(H_i, W) = 1.$$

*Proof.* Consider that

$$\begin{aligned}
1 &= \int_{[0,1]^4} dx_1 dx_2 dx_3 dx_4 \\
&= \int_{[0,1]^4} \prod_{1 \leq i \neq j \leq 4} (W(x_i, x_j) + (1 - W(x_i, x_j))) dx_1 dx_2 dx_3 dx_4 \\
&= \int_{[0,1]^4} W(x_1, x_2)W(x_1, x_3)W(x_1, x_4)W(x_2, x_3)W(x_2, x_4)W(x_3, x_4) dx_1 dx_2 dx_3 dx_4 \\
&+ \int_{[0,1]^4} (1 - W(x_1, x_2))W(x_1, x_3)W(x_1, x_4)W(x_2, x_3)W(x_2, x_4)W(x_3, x_4) dx_1 dx_2 dx_3 dx_4 \\
&\quad \vdots \\
&+ \int_{[0,1]^4} \prod_{1 \leq i \neq j \leq 4} (1 - W(x_i, x_j)) dx_1 dx_2 dx_3 dx_4 \\
&= t_{\text{ind}}(H_1, W) + \cdots + t_{\text{ind}}(H_{64}, W).
\end{aligned}$$

□

Next, for each  $H$ , define

$$d(H, W) := \frac{v(H)!}{\text{aut}(H)} t_{\text{ind}}(H, W)$$

where  $\text{aut}(H)$  is the number of automorphisms of  $H$ . Up to isomorphism, there are 11 graphs on four vertices, which we label by  $J_1, \dots, J_{11}$  as indicated in Figure 5.1.

The next lemma is an easy consequence of Lemma 5.2.1.

**Lemma 5.2.2** (See, e.g., [51, Lemma 5.3]). *Every graphon  $W$  satisfies*

$$\sum_{m=1}^{11} d(J_m, W) = 1.$$

*Proof.* There are  $\frac{4!}{\text{aut}(J_m)}$  different ways to label the graph  $J_m$ . It is easy to see that if  $H_i$  and  $H_j$  are isomorphic for some  $i, j \in \{1, 2, \dots, 64\}$ , then  $t_{\text{ind}}(H_i, W) = t_{\text{ind}}(H_j, W)$ . By Lemma 5.2.2,

$$\begin{aligned}
1 &= \sum_{i=1}^{64} t_{\text{ind}}(H_i, W) = \sum_{m=1}^{11} \frac{4!}{\text{aut}(J_m)} \cdot t_{\text{ind}}(J_m, W) \\
&= \sum_{m=1}^{11} d(J_m, W)
\end{aligned}$$

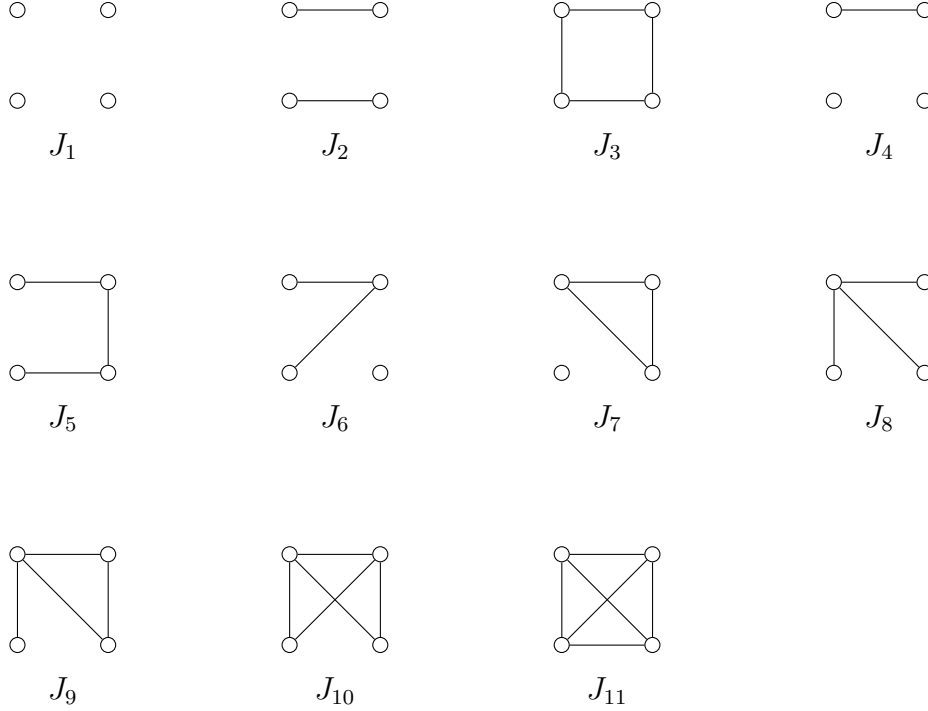


Figure 5.1: The graphs  $J_1, \dots, J_{11}$

□

**Lemma 5.2.3** (See, e.g., [51, Lemma 5.5]). *For any graphon  $W$  and any graph  $K$  with at most four vertices,*

$$t(K, W) = \sum_{m=1}^{11} t_{\text{inj}}(K, J_m) \cdot d(J_m, W).$$

*Proof.* For a graphon  $W$ , by standard results in graph limits [49, Proposition 11.32], we can take a sequence of graphs  $(G_n)_{n \in \mathbb{N}}$  such that for any graph  $H$ ,

$$t_{\text{inj}}(H, G_n) \rightarrow t(H, W) \text{ and } d(H, G_n) \rightarrow d(H, W)$$

as  $n \rightarrow \infty$ . For each  $m$ , let  $d(J_m, G_n)$  denote the probability that a uniformly chosen labelled 4-vertex subset  $S \subset V(G_n)$  is isomorphic to  $J_m$ . Then, the expectation of  $t_{\text{inj}}(K, G_n[S])$  for a uniformly chosen labelled 4-vertex subset  $S$  is  $t_{\text{inj}}(K, G_n)$ . On the

other hand,

$$\begin{aligned}\mathbb{E}(t_{inj}(K, G_n[S])) &= \sum_{m=1}^{11} t_{inj}(K, G_n[S]) \cdot \mathbb{P}(G_n[S] \text{ is isomorphic to } J_m) \\ &= \sum_{m=1}^{11} t_{inj}(K, G_n[S]) \cdot d(J_m, G).\end{aligned}$$

Therefore,

$$t_{inj}(K, G_n) = \sum_{m=1}^{11} t_{inj}(K, J_m) \cdot d(J_m, G_n).$$

Taking  $n \rightarrow \infty$  yields the desired.  $\square$

It is natural to get the following statement when the edge density is restricted.

**Corollary 5.2.4.** *Let  $p \in [0, 1]$ . If  $W$  is a graphon such that  $t(K_2, W) \leq p$ , then*

$$\sum_{m=1}^{11} (t_{inj}(K_2, J_m) - p) d(J_m, W) \leq 0.$$

*Proof.* Observe that by Lemma 5.2.2 and Lemma 5.2.3,

$$\begin{aligned}\sum_{m=1}^{11} (t_{inj}(K_2, J_m) - p) d(J_m, W) &= \sum_{m=1}^{11} t_{inj}(K_2, J_m) \cdot d(J_m, W) - \sum_{m=1}^{11} p \cdot d(J_m, W) \\ &= t(K_2, W) - p \sum_{m=1}^{11} d(J_m, W) \\ &= t(K_2, W) - p,\end{aligned}$$

which is non-positive by the assumption.  $\square$

Let  $F_1^1, \dots, F_4^1$  be the four distinct labeled graphs with vertex set  $\{1, 2, 3\}$  in which vertices 1 and 2 are non-adjacent and let  $F_1^2, \dots, F_4^2$  be the four distinct labeled graphs with vertex set  $\{1, 2, 3\}$  in which vertices 1 and 2 are adjacent. For all eight of these graphs, we view vertices 1 and 2 as being the *root* vertices. These eight graphs are depicted in Figure 5.2 with the root vertices as squares and the non-root vertex (i.e. vertex 3) as a circle.

For each  $F_i^\ell$  for  $1 \leq i \leq 4$  and  $\ell \in \{1, 2\}$ , define a function  $t_{ind,2}(F_i^\ell, W) : [0, 1]^2 \rightarrow [0, 1]$  by

$$t_{ind,2}(F_i^\ell, W)(x_1, x_2) = \int_0^1 \prod_{uv \in E(F_i^\ell)} W(x_u, x_v) \prod_{uv \in E(\overline{F_i^\ell})} (1 - W(x_u, x_v)) dx_3.$$

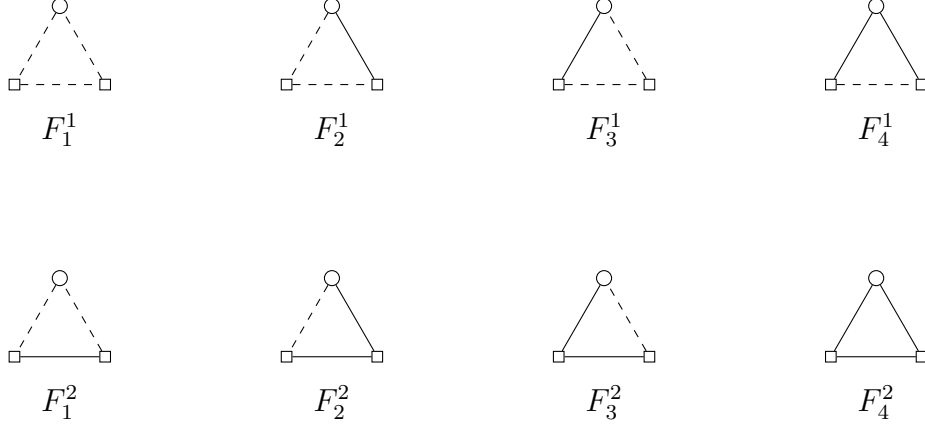


Figure 5.2: The graphs  $F_1^1, \dots, F_4^1, F_1^2, \dots, F_4^2$ . In each case, the vertices 1 and 2 are depicted by square nodes where the vertex 1 is on the left and 2 is on the right. Vertex 3 is depicted by a round node. Dashed lines represent non-adjacent pairs of vertices.

In other words, this is the same as  $t_{\text{ind}}(F_i^\ell, W)$  except that we only integrate over the choices of the variable corresponding to the non-root vertex (vertex 3). Now, for each pair  $i, j \in \{1, \dots, 4\}$ , we define a function  $t_{\text{ind},2}(F_i^1 \cdot F_j^1, W) : [0, 1]^2 \rightarrow [0, 1]$  by

$$t_{\text{ind},2}(F_i^1 \cdot F_j^1, W)(x_1, x_2) := \begin{cases} \frac{t_{\text{ind},2}(F_i^1, W)(x_1, x_2) t_{\text{ind},2}(F_j^1, W)(x_1, x_2)}{1 - W(x_1, x_2)} & \text{if } 1 - W(x_1, x_2) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, for  $F_i^2$  and  $F_j^2$ , we define

$$t_{\text{ind},2}(F_i^2 \cdot F_j^2, W)(x_1, x_2) := \begin{cases} \frac{t_{\text{ind},2}(F_i^2, W)(x_1, x_2) t_{\text{ind},2}(F_j^2, W)(x_1, x_2)}{W(x_1, x_2)} & \text{if } W(x_1, x_2) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A key idea is that the integral of each of the functions  $t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2)$  over  $[0, 1]^2$  can be expressed as a linear combination of the quantities  $d(J_m, W)$  for  $1 \leq m \leq 11$ . For example,

$$\int_{[0,1]^2} t_{\text{ind},2}(F_1^1 \cdot F_1^1, W)(x_1, x_2) dx_1 dx_2 = 0$$

if  $W(x_1, x_2) = 1$ . Otherwise,

$$\begin{aligned}
& \int_{[0,1]^2} t_{\text{ind},2}(F_1^1 \cdot F_1^1, W)(x_1, x_2) dx_1 dx_2 \\
&= \int_{[0,1]^2} \frac{t_{\text{ind},2}(F_1^1, W)(x_1, x_2) t_{\text{ind},2}(F_1^1, W)(x_1, x_2)}{1 - W(x_1, x_2)} dx_1 dx_2 \\
&= \int_{[0,1]^2} (1 - W(x_1, x_2)) \left( \int_0^1 (1 - W(x_1, x_3))(1 - W(x_2, x_3)) dx_3 \right) \cdot \left( \int_0^1 (1 - W(x_1, x_4))(1 - W(x_2, x_4))(1 - W(x_3, x_4)) dx_4 \right) dx_1 dx_2 \\
&= \int_{[0,1]^4} (1 - W(x_1, x_2))(1 - W(x_1, x_3))(1 - W(x_2, x_3))(1 - W(x_1, x_4))(1 - W(x_2, x_4)) dx_1 dx_2 dx_3 dx_4 \\
&= \int_{[0,1]^4} (1 - W(x_1, x_2))(1 - W(x_1, x_3))(1 - W(x_2, x_3))(1 - W(x_1, x_4))(1 - W(x_2, x_4))(W(x_3, x_4))(1 - W(x_3, x_4)) dx_1 dx_2 dx_3 dx_4 \\
&= \int_{[0,1]^4} (1 - W(x_1, x_2))(1 - W(x_1, x_3))(1 - W(x_2, x_3))(1 - W(x_1, x_4))(1 - W(x_2, x_4)) W(x_3, x_4) dx_1 dx_2 dx_3 dx_4 \\
&+ \int_{[0,1]^4} (1 - W(x_1, x_2))(1 - W(x_1, x_3))(1 - W(x_2, x_3))(1 - W(x_1, x_4))(1 - W(x_2, x_4))(1 - W(x_3, x_4)) dx_1 dx_2 dx_3 dx_4 \\
&= t_{\text{ind}}(J_1, W) + t_{\text{ind}}(J_4, W) \\
&= \frac{\text{aut}(J_1)}{24} d(J_1, W) + \frac{\text{aut}(J_4)}{24} d(J_4, W) \\
&= d(J_1, W) + \frac{1}{6} d(J_4, W).
\end{aligned}$$

For each  $1 \leq i, j \leq 4$ ,  $\ell \in \{1, 2\}$  and  $1 \leq m \leq 11$ , let  $a(F_i^\ell, F_j^\ell; J_m)$  denote the coefficient of  $d(J_m, W)$  that one obtains when integrating  $t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2)$  over  $[0, 1]^2$ . The following lemma is the key to all of our flag algebra proofs.

We say an  $n \times n$  symmetric matrix  $A$  with real entries is *positive semi-definite* (PSD) if, for every  $n \times 1$  column vector  $\mathbf{x}$ , where  $\mathbf{x}^T$  is the transpose of  $\mathbf{x}$ , the real number  $\mathbf{x}^T A \mathbf{x} \geq 0$ . Observe that for a real vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and an  $n \times n$  matrix  $A$ , we have

$$\begin{aligned} \mathbf{x}^T A \mathbf{x} &= (x_1, x_2, \dots, x_n) A (x_1, x_2, \dots, x_n)^T \\ &= \sum_{i=1}^n \sum_{j=1}^n A(i, j) \cdot x_i \cdot x_j. \end{aligned}$$

Thus, if a real symmetric matrix  $A$  is PSD, then the last expression is non-negative for all  $x_1, x_2, \dots, x_n \in \mathbb{R}$ . Moreover, it is known that if an  $n \times n$  real matrix  $A$  is symmetric, then it is diagonalizable. Let the eigenvalues of  $A$  be  $\lambda_1, \lambda_2, \dots, \lambda_n$  and its corresponding eigenvectors be  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ . If  $A$  is PSD, then all eigenvalues of  $A$  are non-negative. Suppose not; say  $\lambda_1 < 0$ . Then,

$$\mathbf{y}_1^T A \mathbf{y}_1 = \lambda_1 \mathbf{y}_1^T \mathbf{y}_1 < 0,$$

which is contradiction. Conversely, every real symmetric matrix with non-negative eigenvalues is PSD.

**Lemma 5.2.5.** *Let  $K$  be a graph with at most four vertices, let  $\alpha, \beta \in \mathbb{R}$  and let  $p \in [0, 1]$ . Suppose that  $A_1$  and  $A_2$  are positive semi-definite  $4 \times 4$  matrices and that  $\eta \geq 0$ . Then, for every graphon  $W$  with  $t(K_2, W) \leq p$ , the quantity  $\alpha \cdot t(K, W) + \beta \cdot t(K, 1 - W)$  is at least*

$$\min_{1 \leq m \leq 11} \left\{ \alpha \cdot t_{\text{inj}}(K, J_m) + \beta \cdot t_{\text{inj}}(K, \overline{J_m}) + (t_{\text{inj}}(K_2, J_m) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_m) \right\}.$$

Before proving the lemma, the following claim will be useful.

**Claim 5.2.6.** *For any  $4 \times 4$  PSD matrix  $A$  and any  $\ell \in \{1, 2\}$ ,*

$$\sum_{m=1}^{11} \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) a(F_i^\ell, F_j^\ell; J_m) d(J_m, W) \geq 0.$$

*Proof.* By the definition of  $a(F_i^\ell, F_j^\ell; J_m)$ , we have

$$\sum_{m=1}^{11} a(F_i^\ell, F_j^\ell; J_m) d(J_m, W) = \sum_{i=1}^4 \sum_{j=1}^4 \int_{[0,1]^2} t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2) dx_1 dx_2.$$

Consider that

$$\begin{aligned}
& \sum_{m=1}^{11} \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) a(F_i^\ell, F_j^\ell; J_m) d(J_m, W) \\
&= \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) \sum_{m=1}^{11} a(F_i^\ell, F_j^\ell; J_m) d(J_m, W) \\
&= \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) \int_{[0,1]^2} t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2) dx_1 dx_2. \\
&= \int_{[0,1]^2} \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) \cdot t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2) dx_1 dx_2.
\end{aligned}$$

We will prove the stronger statement that for any  $\ell$  and every  $(x_1, x_2) \in [0, 1]^2$ ,

$$\sum_{i=1}^4 \sum_{j=1}^4 A(i, j) \cdot t_{\text{ind},2}(F_i^\ell \cdot F_j^\ell, W)(x_1, x_2) \geq 0.$$

If  $\ell = 1$  and  $1 - W(x_1, x_2) = 0$  or  $\ell = 2$  and  $W(x_1, x_2) = 0$ , then the LHS of the last inequality is zero. Thus, we are done. Otherwise,

$$\begin{aligned}
& \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) \frac{t_{\text{ind},2}(F_i^\ell, W)(x_1, x_2) \cdot t_{\text{ind},2}(F_j^\ell, W)(x_1, x_2)}{1 - W(x_1, x_2)} \\
&= \frac{1}{1 - W(x_1, x_2)} \sum_{i=1}^4 \sum_{j=1}^4 A(i, j) [t_{\text{ind},2}(F_i^1, W)(x_1, x_2)] [t_{\text{ind},2}(F_j^1, W)(x_1, x_2)]
\end{aligned}$$

which is non-negative because  $A$  is positive semi-definite. □

*Proof of Lemma 5.2.5.* Observe that

$$\begin{aligned}
& \alpha \cdot t(K, W) + \beta \cdot t(K, 1 - W) \\
&= \sum_{m=1}^{11} (\alpha \cdot t_{\text{inj}}(K, J_m) + \beta \cdot t_{\text{inj}}(K, \overline{J_m})) d(J_m, W)
\end{aligned}$$

By Lemma 5.2.4 and Claim 5.2.6, the last expression is at least

$$\begin{aligned}
& \sum_{m=1}^{11} (\alpha \cdot t_{\text{inj}}(K, J_m) + \beta \cdot t_{\text{inj}}(K, \overline{J_m})) d(J_m, W) \\
& + \sum_{m=1}^{11} (t_{\text{inj}}(K_2, J_m) - p) d(J_m, W) \cdot \eta - \sum_{m=1}^{11} \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_m) d(J_m, W) \\
& = \sum_{m=1}^{11} \left( \alpha \cdot t_{\text{inj}}(K, J_m) + \beta \cdot t_{\text{inj}}(K, \overline{J_m}) + (t_{\text{inj}}(K_2, J_m) - p) \eta - \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_m) \right) d(J_m, W).
\end{aligned}$$

Since  $\sum_{m=1}^{11} d(J_m, W) = 1$  and  $d(J_m, W) \geq 0$  for all  $1 \leq m \leq 11$ , the last expression is at least the minimum term as desired.  $\square$

We can now prove Lemma 5.0.3.

*Proof of Lemma 5.0.3.* We apply Lemma 5.2.5 with  $\alpha = \beta = 1$  and  $p = 1/3$ . Define  $\eta := \frac{17}{54}$  and let  $A_1$  and  $A_2$  be the following matrices

$$A_1 := \frac{1}{20736} \begin{bmatrix} 17216 & -8608 & -8608 & -6456 \\ -8608 & 12800 & -4192 & 3228 \\ -8608 & -4192 & 12800 & 3228 \\ -6456 & 3228 & 3228 & 2421 \end{bmatrix}$$

$$A_2 := \frac{1}{2916} \begin{bmatrix} 855 & 285 & 285 & -1710 \\ 285 & 906 & -716 & -570 \\ 285 & -716 & 906 & -570 \\ -1710 & -570 & -570 & 3420 \end{bmatrix}.$$

These two matrices are positive semi-definite; in particular, the eigenvalues of  $A_1$  are  $0, 0, \frac{59}{72}, \frac{9415}{6912}$  and the eigenvalues of  $A_2$  are  $0, 0, \frac{811}{1458}, \frac{4465}{2916}$ .

All that remains is to compute the quantity

$$t_{\text{inj}}(K_4, J_m) + t_{\text{inj}}(K_4, \overline{J_m}) + \left( t(K_2, J_m) - \frac{1}{3} \right) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_m) \tag{5.2.7}$$

for each  $m \in \{1, \dots, 11\}$  and show that it is at least  $1/27$  in every case. Note that  $t_{\text{inj}}(K_4, J_m) + t_{\text{inj}}(K_4, \overline{J_m}) = 1$  if  $m \in \{1, 11\}$  and it is zero otherwise. For  $m = 1$ , the

quantity in (5.2.7) simplifies as follows:

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_1) + t_{\text{inj}}(K_4, \overline{J_1}) + \left( t_{\text{inj}}(K_2, J_1) - \frac{1}{3} \right) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_{\ell}(i, j) a(F_i^{\ell}, F_j^{\ell}; J_1) \\
&= 1 + \left( 0 - \frac{1}{3} \right) \frac{17}{54} - A_1(1, 1) \frac{24}{24} \\
&= \frac{145}{162} - \frac{17216}{20736} \\
&= \frac{7}{108} \geq \frac{1}{27}.
\end{aligned}$$

Now,  $m = 2$ . Then (5.2.7) becomes

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_2) + t_{\text{inj}}(K_4, \overline{J_2}) + \left( t_{\text{inj}}(K_2, J_2) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_{\ell}(i, j) a(F_i^{\ell}, F_j^{\ell}; J_2) \\
&= 0 + \left( \frac{1}{3} - \frac{1}{3} \right) \frac{17}{54} - A_1(2, 3) \frac{8}{24} - A_1(3, 2) \frac{8}{24} - A_2(1, 1) \frac{8}{24} \\
&= \frac{4192}{20736} \cdot \frac{8}{24} + \frac{4192}{20736} \cdot \frac{8}{24} - \frac{855}{2916} \cdot \frac{8}{24} \\
&= \frac{1}{27} \geq \frac{1}{27}.
\end{aligned}$$

Next,  $m = 3$ . We have

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_3) + t_{\text{inj}}(K_4, \overline{J_3}) + \left( t_{\text{inj}}(K_2, J_3) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_{\ell}(i, j) a(F_i^{\ell}, F_j^{\ell}; J_3) \\
&= 0 + \left( \frac{2}{3} - \frac{1}{3} \right) \frac{17}{54} - A_1(4, 4) \frac{8}{24} - A_2(2, 3) \frac{8}{24} - A_2(3, 2) \frac{8}{24} \\
&= \frac{17}{162} - \frac{2421}{20736} \frac{8}{24} + \frac{716}{2916} \frac{8}{24} + \frac{716}{2916} \frac{8}{24} \\
&= \frac{128611}{559872} \geq \frac{1}{27}.
\end{aligned}$$

Now, we check th case  $m = 4$ .

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_4) + t_{\text{inj}}(K_4, \overline{J_4}) + \left( t_{\text{inj}}(K_2, J_4) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_{\ell}(i, j) a(F_i^{\ell}, F_j^{\ell}; J_4) \\
&= 0 + \left( \frac{1}{6} - \frac{1}{3} \right) \frac{17}{54} - A_1(1, 1) \frac{4}{24} - A_1(1, 2) \frac{4}{24} - A_1(1, 3) \frac{4}{24} - A_1(2, 1) \frac{4}{24} - A_1(3, 1) \frac{4}{24} - A_2(1, 1) \frac{4}{24} \\
&\quad - \frac{17}{324} - \frac{17216}{20736} \cdot \frac{4}{24} + \frac{8608}{20736} \cdot \frac{4}{24} + \frac{8608}{20736} \frac{4}{24} + \frac{8608}{20736} \cdot \frac{4}{24} + \frac{8608}{20736} \cdot \frac{4}{24} - \frac{855}{2916} \cdot \frac{4}{24} \\
&= \frac{1}{27} \geq \frac{1}{27}.
\end{aligned}$$

Let  $m = 5$ ; (5.2.7) gives us

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_5) + t_{\text{inj}}(K_4, \overline{J_5}) + \left( t_{\text{inj}}(K_2, J_5) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_5) \\
&= 0 + \left( \frac{1}{2} - \frac{1}{3} \right) \frac{17}{54} - A_1(2, 3) \frac{2}{24} - A_1(2, 4) \frac{2}{24} - A_1(3, 2) \frac{2}{24} - A_1(3, 4) \frac{2}{24} - A_1(4, 2) \frac{2}{24} \\
&\quad - A_1(4, 3) \frac{2}{24} - A_2(1, 2) \frac{2}{24} - A_2(1, 3) \frac{2}{24} - A_2(2, 1) \frac{2}{24} - A_2(2, 3) \frac{2}{24} - A_2(3, 1) \frac{2}{24} - A_2(3, 2) \frac{2}{24} \\
&= \frac{17}{324} + \frac{4192}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} + \frac{4192}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} \\
&\quad - \frac{3228}{20736} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} + \frac{716}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} + \frac{716}{2916} \cdot \frac{2}{24} \\
&= \frac{5965}{139968} \geq \frac{1}{27}.
\end{aligned}$$

Consider  $m = 6$ ; (5.2.7) becomes

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_6) + t_{\text{inj}}(K_4, \overline{J_6}) + \left( t_{\text{inj}}(K_2, J_6) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_6) \\
&= 0 + \left( \frac{1}{3} - \frac{1}{3} \right) \frac{17}{54} - A_1(1, 2) \frac{2}{24} - A_1(1, 3) \frac{2}{24} - A_1(1, 4) \frac{2}{24} - A_1(2, 1) \frac{2}{24} - A_1(2, 2) \frac{2}{24} - A_1(3, 1) \frac{2}{24} \\
&\quad - A_1(3, 3) \frac{2}{24} - A_1(4, 1) \frac{2}{24} - A_2(1, 2) \frac{2}{24} - A_2(1, 3) \frac{2}{24} - A_2(2, 1) \frac{2}{24} - A_2(3, 1) \frac{2}{24} \\
&= 0 + \frac{8608}{20736} \cdot \frac{2}{24} + \frac{8608}{20736} \cdot \frac{2}{24} + \frac{6456}{20736} \cdot \frac{2}{24} + \frac{8608}{20736} \cdot \frac{2}{24} - \frac{12800}{20736} \cdot \frac{2}{24} + \frac{8608}{20736} \cdot \frac{2}{24} \\
&\quad - \frac{12800}{20736} \cdot \frac{2}{24} + \frac{6456}{20736} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} - \frac{285}{2916} \cdot \frac{2}{24} \\
&= \frac{2557}{46656} \geq \frac{1}{27}.
\end{aligned}$$

Next, for  $m = 7$ , (5.2.7) simplifies to

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_7) + t_{\text{inj}}(K_4, \overline{J_7}) + \left( t_{\text{inj}}(K_2, J_7) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_7) \\
&= 0 + \left( \frac{1}{2} - \frac{1}{3} \right) \frac{17}{54} - A_1(2, 2) \frac{6}{24} - A_1(3, 3) \frac{6}{24} - A_2(1, 4) \frac{6}{24} - A_2(4, 1) \frac{6}{24} \\
&= \frac{17}{324} - \frac{12800}{20736} \cdot \frac{6}{24} - \frac{12800}{20736} \cdot \frac{6}{24} + \frac{1710}{2916} \cdot \frac{6}{24} + \frac{1710}{2916} \cdot \frac{6}{24} \\
&= \frac{1}{27} \geq \frac{1}{27}.
\end{aligned}$$

Now we check the case  $m = 8$ .

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_8) + t_{\text{inj}}(K_4, \overline{J_8}) + \left( t_{\text{inj}}(K_2, J_8) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_8) \\
&= 0 + \left( \frac{1}{2} - \frac{1}{3} \right) \frac{17}{54} - A_1(1, 4) \frac{6}{24} - A_1(4, 1) \frac{6}{24} - A_2(2, 2) \frac{6}{24} - A_2(3, 3) \frac{6}{24} \\
&= \frac{17}{324} + \frac{6456}{20736} \cdot \frac{6}{24} + \frac{6456}{20736} \cdot \frac{6}{24} - \frac{906}{2916} \cdot \frac{6}{24} - \frac{906}{2916} \cdot \frac{6}{24} \\
&= \frac{821}{15552} \geq \frac{1}{27}.
\end{aligned}$$

Next,  $m = 9$ . We have

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_9) + t_{\text{inj}}(K_4, \overline{J_9}) + \left( t_{\text{inj}}(K_2, J_9) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_9) \\
&= 0 + \left( \frac{2}{3} - \frac{1}{3} \right) \cdot \frac{17}{54} - A_1(2, 4) \frac{2}{24} - A_1(3, 4) \frac{2}{24} - A_1(4, 2) \frac{2}{24} - A_1(4, 3) \frac{2}{24} - A_2(1, 4) \frac{2}{24} - A_2(2, 2) \frac{2}{24} \\
&\quad - A_2(2, 4) \frac{2}{24} - A_2(3, 3) \frac{2}{24} - A_2(3, 4) \frac{2}{24} - A_2(4, 1) \frac{2}{24} - A_2(4, 2) \frac{2}{24} - A_2(4, 3) \frac{2}{24} \\
&= \frac{17}{162} - \frac{3228}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} - \frac{3228}{20736} \cdot \frac{2}{24} + \frac{1710}{2916} \cdot \frac{2}{24} - \frac{906}{2916} \cdot \frac{2}{24} \\
&\quad + \frac{570}{2916} \cdot \frac{2}{24} - \frac{906}{2916} \cdot \frac{2}{24} + \frac{570}{2916} \cdot \frac{2}{24} + \frac{1710}{2916} \cdot \frac{2}{24} + \frac{570}{2916} \cdot \frac{2}{24} + \frac{570}{2916} \cdot \frac{2}{24} \\
&= \frac{851}{5184} \geq \frac{1}{27}.
\end{aligned}$$

For  $m = 10$ , We get that (5.2.7) is

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_{10}) + t_{\text{inj}}(K_4, \overline{J_{10}}) + \left( t_{\text{inj}}(K_2, J_{10}) - \frac{1}{3} \right) \frac{17}{54} - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_{10}) \\
&= 0 + \left( \frac{5}{6} - \frac{1}{3} \right) \frac{17}{54} - A_1(4, 4) \frac{4}{24} - A_2(2, 4) \frac{4}{24} - A_2(3, 4) \frac{4}{24} - A_2(4, 2) \frac{4}{24} - A_2(4, 3) \frac{4}{24} - A_2(4, 4) \frac{4}{24} \\
&= \frac{17}{108} - \frac{2421}{20736} \cdot \frac{4}{24} + \frac{570}{2916} \cdot \frac{4}{24} + \frac{570}{2916} \cdot \frac{4}{24} + \frac{570}{2916} \cdot \frac{4}{24} + \frac{570}{2916} \cdot \frac{4}{24} + \frac{570}{2916} \cdot \frac{4}{24} - \frac{3420}{2916} \cdot \frac{4}{24} \\
&= \frac{27169}{373248} \geq \frac{1}{27}.
\end{aligned}$$

Next,  $m = 11$ , we get that (5.2.7) is

$$\begin{aligned}
& t_{\text{inj}}(K_4, J_{11}) + t_{\text{inj}}(K_4, \overline{J_{11}}) + \left( t_{\text{inj}}(K_2, J_{11}) - \frac{1}{3} \right) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_{\ell}(i, j) a(F_i^{\ell}, F_j^{\ell}; J_1) \\
&= 1 + \left( 1 - \frac{1}{3} \right) \frac{17}{54} - A_2(4, 4) \frac{24}{24} \\
&= \frac{98}{81} - \frac{3420}{2916} \\
&= \frac{1}{27} \geq \frac{1}{27}.
\end{aligned}$$

□

### 5.3 Middle Edge Density

To prove Lemma 5.0.4, we apply Lemma 5.2.5 again, but this time the matrices will depend on the value of  $p$ .

*Lemma 5.0.4.* Let  $p \in [1/3, 13/36]$ . We apply Lemma 5.2.5 with  $\alpha = p^3$  and  $\beta = (1 - p)^3$ . We define  $a(p), b(p), c(p)$  and  $d(p)$  as follows

$$\begin{aligned}
a(p) &:= -\frac{1}{46656} + \frac{23}{20736}p + \frac{1}{256}p^3, \\
b(p) &:= \frac{11}{11664} - \frac{5}{2592}p + \frac{1}{32}p^3, \\
c(p) &:= \frac{2}{6561} - \frac{1}{2916}p + \frac{1}{36}p^3, \\
d(p) &:= -\frac{473}{17496} + \frac{665}{5832}p.
\end{aligned}$$

Let  $\eta = \frac{1}{81}$  and let  $A_1$  and  $A_2$  be the matrices

$$A_1 := \begin{bmatrix} 64a(p) & -32a(p) & -32a(p) & -24a(p) \\ -32a(p) & 16b(p) & 32a(p) - 16b(p) & 12a(p) \\ -32a(p) & 32a(p) - 16b(p) & 16b(p) & 12a(p) \\ -24a(p) & 12a(p) & 12a(p) & 9a(p) \end{bmatrix}$$

$$A_2 := \begin{bmatrix} 9c(p) & 3c(p) & 3c(p) & -18c(p) \\ 3c(p) & d(p) & 2c(p) - d(p) & -6c(p) \\ 3c(p) & 2c(p) - d(p) & d(p) & -6c(p) \\ -18c(p) & -6c(p) & -6c(p) & 36c(p) \end{bmatrix}.$$

The eigenvalues of  $A_1$  are

- 0 with eigenvector  $(3/8, 0, 0, 1)$ ,
- 0 with eigenvector  $(1, 1, 1, 0)$ ,
- $105a(p)$  with eigenvector  $(-8/3, 4/3, 4/3, 1)$ ,
- $32(b(p) - a(p))$  with eigenvector  $(0, -1, 1, 0)$

while the eigenvalues of  $A_2$  are

- 0 with eigenvector  $(2, 0, 0, 1)$
- 0 with eigenvector  $(-2/3, 1, 1, 0)$
- $47c(p)$  with eigenvector  $(-1/2, -1/6, -1/6, 1)$
- $2(d(p) - c(p))$  with eigenvector  $(0, -1, 1, 0)$ .

Therefore, to certify that  $A_1$  and  $A_2$  are positive semi-definite, it suffices to prove that  $a(p) \geq 0, b(p) \geq a(p), c(p) \geq 0$  and  $d(p) \geq c(p)$  for all  $1/3 \leq p \leq 13/36$ . It is easily observed that  $a(p) \geq 0$  because  $a(1/3) \geq 0$  and the derivative of  $a(p)$  is positive for all  $p \in \mathbb{R}$ . Likewise,  $c(1/3) \geq 0$  and the derivative of  $c(p)$  is positive for all  $1/3 \leq p \leq 13/32$ . Similarly,  $b(1/3) - a(1/3) \geq 0$  and the derivative of  $b(p) - a(p)$  is positive for all  $1/3 \leq p \leq 13/32$ . Finally,  $d(1/3) - c(1/3) \geq 0$  and the derivative of  $d(p) - c(p)$  is positive for all  $1/3 \leq p \leq 13/32$  as well. So,  $A_1$  and  $A_2$  are positive semi-definite.

We need to show that, for each  $1 \leq m \leq 11$ , the quantity

$$p^3 t_{\text{inj}}(K, J_m) + (1-p)^3 t_{\text{inj}}(K, \overline{J_m}) + (t(K_2, J_m) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_m) \quad (5.3.1)$$

is at least  $(1/3)^6$ . Note that  $p^3 t_{\text{inj}}(K, J_m) + (1-p)^3 t_{\text{inj}}(K, \overline{J_m})$  is  $(1-p)^3$  if  $m = 1$ , it is  $p^3$  if  $m = 11$  and it is zero for all other  $m$ . For  $m = 1$ , the quantity in (5.3.1) simplifies as follows:

$$\begin{aligned} & p^3 t_{\text{inj}}(K, J_1) + (1-p)^3 t_{\text{inj}}(K, \overline{J_1}) + (t(K_2, J_1) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_1) \\ &= (1-p)^3 + (0-p) \cdot \frac{1}{81} - A_1(1, 1) \frac{24}{24} \\ &= (1-p)^3 - \frac{1}{81} p - 64 \left( -\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3 \right) \\ &= \frac{730}{729} - \frac{37}{12} p + 3p^2 - \frac{5}{4} p^3. \end{aligned}$$

The derivative of the above cubic with respect to  $p$  is negative for all  $p \in \mathbb{R}$ . So, for  $1/3 \leq p \leq 13/36$ , the minimum is attained at  $p = 13/36$ . Plugging in  $p = 13/36$  yields  $\frac{41111}{186624} > (1/3)^6$ .

Next, for  $m = 2$ , (5.3.1) simplifies to

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_2) + (1-p)^3 t_{\text{inj}}(K, \overline{J}_2) + (t(K_2, J_2) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_2) \\
&= \left(\frac{1}{3} - p\right) \frac{1}{81} - A_1(2, 3) \frac{8}{24} - A_1(3, 2) \frac{8}{24} - A_2(1, 1) \frac{8}{24} \\
&= \left(\frac{1}{3} - p\right) \frac{1}{81} - \frac{16}{24} \left( 32 \left( -\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3 \right) - 16 \left( \frac{11}{11664} - \frac{5}{2592} p + \frac{1}{32} p^3 \right) \right) \\
&\quad - \frac{8}{24} \cdot 9 \cdot \left( \frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3 \right) \\
&= \frac{10}{729} - \frac{1}{18} p + \frac{1}{6} p^3.
\end{aligned}$$

The derivative of the above cubic with respect to  $p$  is non-negative for all  $p \geq 1/3$ . So, its minimum is attained at  $p = 1/3$ . Plugging in  $p = 1/3$  yields precisely  $(1/3)^6$ .

Consider  $m = 3$ ; (5.3.1) becomes

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_3) + (1-p)^3 t_{\text{inj}}(K, \overline{J}_3) + (t(K_2, J_3) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_3) \\
&= \left(\frac{2}{3} - p\right) \frac{1}{81} - A_1(4, 4) \frac{8}{24} - A_2(2, 3) \frac{8}{24} - A_2(3, 2) \frac{8}{24} \\
&= \left(\frac{2}{3} - p\right) \frac{1}{81} - \frac{8}{24} 9 \left( -\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3 \right) \\
&\quad - \frac{16}{24} \left( 2 \left( \frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3 \right) - \left( -\frac{473}{17496} + \frac{665}{5832} p \right) \right) \\
&= -\frac{12767}{1259712} + \frac{11347}{186624} p - \frac{337}{6912} p^3
\end{aligned}$$

This cubic has positive derivative for  $1/3 \leq p \leq 13/36$  and is larger than  $(1/3)^6$  at  $p = 1/3$  and so it satisfies the desired inequality.

Let  $m = 4$ ; (5.3.1) gives us

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_4) + (1-p)^3 t_{\text{inj}}(K, \overline{J}_4) + (t(K_2, J_4) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_4) \\
&= \left(\frac{1}{6} - p\right) \frac{1}{81} - A_1(1, 1) \frac{4}{24} - A_1(1, 2) \frac{4}{24} - A_1(1, 3) \frac{4}{24} - A_1(2, 1) \frac{4}{24} - A_1(3, 1) \frac{4}{24} - A_2(1, 1) \frac{4}{24} \\
&= \left(\frac{1}{6} - p\right) \frac{1}{81} - \frac{4}{24} 64 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{16}{24} (-32) \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) \\
&\quad - \frac{4}{24} 9 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= (1/3)^6.
\end{aligned}$$

Next, we check the case  $m = 5$ .

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_5) + (1-p)^3 t_{\text{inj}}(K, \overline{J}_5) + (t(K_2, J_5) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_5) \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - A_1(2, 3) \frac{2}{24} - A_1(2, 4) \frac{2}{24} - A_1(3, 2) \frac{2}{24} - A_1(3, 4) \frac{2}{24} - A_1(4, 2) \frac{2}{24} \\
&\quad - A_1(4, 3) \frac{2}{24} - A_2(1, 2) \frac{2}{24} - A_2(1, 3) \frac{2}{24} - A_2(2, 1) \frac{2}{24} - A_2(2, 3) \frac{2}{24} - A_2(3, 1) \frac{2}{24} - A_2(3, 2) \frac{2}{24} \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - \frac{4}{24} \left(32 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - 16 \left(\frac{11}{11664} - \frac{5}{2592} p + \frac{1}{32} p^3\right)\right) \\
&\quad - \frac{8}{24} 12 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{8}{24} 3 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&\quad - \frac{4}{24} \left(2 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) - \left(-\frac{473}{17496} + \frac{665}{5832} p\right)\right) \\
&= \frac{313}{78732} - \frac{391}{46656} p + \frac{17}{1728} p^3.
\end{aligned}$$

This cubic has negative derivative for  $1/3 \leq p \leq 13/36$ . So, it is minimize in this range at  $p = 13/36$ . At this value, the cubic is slightly larger than  $(1/3)^6$ .

Next,  $m = 6$ . We have

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_6) + (1-p)^3 t_{\text{inj}}(K, \overline{J_6}) + (t(K_2, J_6) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_6) \\
&= \left(\frac{1}{3} - p\right) \frac{1}{81} - A_1(1, 2) \frac{2}{24} - A_1(1, 3) \frac{2}{24} - A_1(1, 4) \frac{2}{24} - A_1(2, 1) \frac{2}{24} - A_1(2, 2) \frac{2}{24} - A_1(3, 1) \frac{2}{24} \\
&\quad - A_1(3, 3) \frac{2}{24} - A_1(4, 1) \frac{2}{24} - A_2(1, 2) \frac{2}{24} - A_2(1, 3) \frac{2}{24} - A_2(2, 1) \frac{2}{24} - A_2(3, 1) \frac{2}{24} \\
&= \left(\frac{1}{3} - p\right) \frac{1}{81} - \frac{8}{24} (-32) \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{4}{24} (-24) \left(\frac{11}{11664} - \frac{5}{2592} p + \frac{1}{32} p^3\right) \\
&\quad - \frac{4}{24} 16 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{8}{24} 3 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= \frac{389}{52488} - \frac{253}{23328} p + \frac{37}{288} p^3.
\end{aligned}$$

This function is increasing in  $p$  for  $1/3 \leq p \leq 13/36$  and it is at least  $(1/3)^6$  when  $p = 1/3$ , and so the desired inequality holds in this case.

Now, consider  $m = 7$ . Then (5.3.1) becomes

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_7) + (1-p)^3 t_{\text{inj}}(K, \overline{J_7}) + (t(K_2, J_7) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_7) \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - A_1(2, 2) \frac{6}{24} - A_1(3, 3) \frac{6}{24} - A_2(1, 4) \frac{6}{24} - A_2(4, 1) \frac{6}{24} \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - \frac{12}{24} 16 \left(\frac{11}{11664} - \frac{5}{2592} p + \frac{1}{32} p^3\right) - \frac{12}{24} (-18) \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= (1/3)^6.
\end{aligned}$$

Next,  $m = 8$ . Then (5.3.1) becomes

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_8) + (1-p)^3 t_{\text{inj}}(K, \overline{J_8}) + (t(K_2, J_8) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_8) \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - A_1(1, 4) \frac{6}{24} - A_1(4, 1) \frac{6}{24} - A_2(2, 2) \frac{6}{24} - A_2(3, 3) \frac{6}{24} \\
&= \left(\frac{1}{2} - p\right) \frac{1}{81} - \frac{12}{24} (-24) \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{12}{24} \left(-\frac{473}{17496} + \frac{665}{5832} p\right).
\end{aligned}$$

The derivative of this cubic is negative for  $1/3 \leq p \leq 13/36$  and the value at  $p = 13/36$  is slightly larger than  $(1/3)^6$ .

Now,  $m = 9$ . We get that (5.3.1) equals

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_9) + (1-p)^3 t_{\text{inj}}(K, \overline{J_9}) + (t(K_2, J_9) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_9) \\
&= \left(\frac{2}{3} - p\right) \frac{1}{81} - A_1(2, 4) \frac{2}{24} - A_1(3, 4) \frac{2}{24} - A_1(4, 2) \frac{2}{24} - A_1(4, 3) \frac{2}{24} - A_2(1, 4) \frac{2}{24} - A_2(2, 2) \frac{2}{24} \\
&\quad - A_2(2, 4) \frac{2}{24} - A_2(3, 3) \frac{2}{24} - A_2(3, 4) \frac{2}{24} - A_2(4, 1) \frac{2}{24} - A_2(4, 2) \frac{2}{24} - A_2(4, 3) \frac{2}{24} \\
&= \left(\frac{2}{3} - p\right) \frac{1}{81} - \frac{8}{24} 12 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{4}{24} (-18) \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&\quad - \frac{4}{24} \left(-\frac{473}{17496} + \frac{665}{5832} p\right) - \frac{8}{24} (-6) \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= \frac{251}{17496} - \frac{5249}{139968} + \frac{71}{576} p^3
\end{aligned}$$

which is increasing for all  $1/3 \leq p \leq 13/36$  and is greater than  $(1/3)^6$  at  $p = 1/3$ .

Next,  $m = 10$ . We get that (5.3.1) is

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_{10}) + (1-p)^3 t_{\text{inj}}(K, \overline{J_{10}}) + (t(K_2, J_{10}) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_{10}) \\
&= \left(\frac{5}{6} - p\right) \frac{1}{81} - A_1(4, 4) \frac{4}{24} - A_2(2, 4) \frac{4}{24} - A_2(3, 4) \frac{4}{24} - A_2(4, 2) \frac{4}{24} - A_2(4, 3) \frac{4}{24} - A_2(4, 4) \frac{4}{24} \\
&= \left(\frac{5}{6} - p\right) \frac{1}{81} - \frac{4}{24} 9 \left(-\frac{1}{46656} + \frac{23}{20736} p + \frac{1}{256} p^3\right) - \frac{16}{24} (-6) \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&\quad - \frac{4}{24} 36 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= \frac{8155}{839808} - \frac{4973}{373248} p - \frac{283}{4608} p^3.
\end{aligned}$$

This is clearly decreasing for all  $p \in \mathbb{R}$ . At  $p = 13/36$ , it is greater than  $(1/3)^6$  and so the same is true for all  $1/3 \leq p \leq 13/36$ .

For  $m = 11$ , (5.3.1) simplifies to

$$\begin{aligned}
& p^3 t_{\text{inj}}(K, J_{11}) + (1-p)^3 t_{\text{inj}}(K, \overline{J_{11}}) + (t(K_2, J_{11}) - p) \eta - \sum_{\ell=1}^2 \sum_{i=1}^4 \sum_{j=1}^4 A_\ell(i, j) a(F_i^\ell, F_j^\ell; J_{11}) \\
&= p^3 + (1-p) \cdot \frac{1}{81} - A_2(4, 4) \frac{24}{24} \\
&= p^3 + \frac{1}{81} (1-p) - 36 \left(\frac{2}{6561} - \frac{1}{2916} p + \frac{1}{36} p^3\right) \\
&= (1/3)^6.
\end{aligned}$$

□

## 5.4 Related Directions for Future Work

We finish the chapter with some questions. We showed that the Ramsey multiplicity constant for  $K_4 \sqcup 3 \cdot K_2$  was achieved by Turán colouring. However, we do not know whether it is a unique construction or not.

**Question 5.4.1.** *Is  $K_4 \sqcup 3 \cdot K_2$  a bonbon?*

We are interested in whether  $K_4 \sqcup K_2$  or  $K_4 \sqcup 2 \cdot K_2$  is Ramsey multiplicity good or not.

**Question 5.4.2.** *What is the minimum  $s$  such that  $K_4 \sqcup s \cdot K_2$  is Ramsey multiplicity good?*

All of the results in this chapter involved  $K_4$ . It would be nice if we could extend these results to all complete graphs.

**Conjecture 5.4.3.** *There exists a polynomial function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that  $K_k \sqcup p(k) \cdot K_2$  is Ramsey multiplicity good.*

# Appendix A

## Simpler But Less Powerful Sufficient Conditions

The inequalities  $(x \approx 0)$  and  $(x \approx 1)$  seem to be fairly unwieldy in general. Our goal in this appendix is to obtain sufficient conditions for these inequalities that may be easier to analyze in practice. To simplify  $(x \approx 1)$  for  $\ell \geq 0$ , we apply the following simple inequality for real numbers.

**Lemma A.0.1.** *Let  $a, b, c, d, s, t \geq 0$  such that  $b \geq d$  and  $s \geq t$ . Then*

$$(ab^s - cd^s)(b^t + d^t) \geq (ab^t - cd^t)(b^s + d^s).$$

*Proof.* Expand both sides of the inequality. After doing a bit of cancellation, it becomes

$$ab^s d^t - cd^s b^t \geq ab^t d^s - cd^t b^s$$

which is the same as

$$ab^s d^t + cd^t b^s \geq ab^t d^s + cd^s b^t.$$

Since  $s \geq t$  and  $b \geq d$ , we have  $ab^s d^t \geq ab^t d^s$  and  $cd^t b^s \geq cd^s b^t$ ; so, the inequality holds.  $\square$

**Lemma A.0.2.** *Let  $k, \ell, g$  and  $\rho$  satisfy the hypotheses of Proposition 3.3.1. For  $0 \leq x \leq 1$ , if  $\ell \geq \ell_0$  and*

$$\frac{g(1+x)(1+x)^{\frac{\ell_0}{k-1}} - g(1-x)(1-x)^{\frac{\ell_0}{k-1}}}{(1+x)^{\frac{\ell_0}{k-1}} + (1-x)^{\frac{\ell_0}{k-1}}} \geq g(1+x) - \rho(1+x)$$

*then  $(x \approx 1)$  holds. In particular, if  $\ell \geq 0$  and*

$$2\rho(1+x) \geq g(1+x) + g(1-x), \quad (x \approx 1^*)$$

*then  $(x \approx 1)$  holds.*

*Proof.* Applying Lemma A.0.1 with  $a = g(1+x)$ ,  $b = (1+x)^{\frac{1}{k-1}}$ ,  $c = g(1-x)$ ,  $d = (1-x)^{\frac{1}{k-1}}$ ,  $s = \ell$  and  $t = \ell_0$  yields

$$\frac{g(1+x)(1+x)^{\frac{\ell}{k-1}} - g(1-x)(1-x)^{\frac{\ell}{k-1}}}{(1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}}} \geq \frac{g(1+x)(1+x)^{\frac{\ell_0}{k-1}} - g(1-x)(1-x)^{\frac{\ell_0}{k-1}}}{(1+x)^{\frac{\ell_0}{k-1}} + (1-x)^{\frac{\ell_0}{k-1}}}.$$

Thus, if the right side of this inequality is at least  $g(1+x) - \rho(1+x)$ , then so is the left.

For the “in particular” part of the lemma, if  $\ell \geq 0$ , then we can apply the first part of the lemma with  $\ell_0 = 0$  to get that the following inequality implies ( $x \approx 1$ ):

$$\frac{g(1+x) - g(1-x)}{2} \geq g(1+x) - \rho(1+x).$$

Rearranging this inequality gives us ( $x \approx 1^*$ ). □

Next, we use Hölder’s Inequality to find simpler sufficient conditions for ( $x \approx 0$ ).

**Lemma A.0.3** (Hölder’s Inequality, see [50, Section 2.8]). *For  $n \geq 1$ , if  $x_1, \dots, x_n, y_1, \dots, y_n$  are positive real numbers and  $p, q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then*

$$\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^p \right)^{1/p} \left( \sum_{i=1}^n y_i^q \right)^{1/q}.$$

**Claim A.0.4.** *For any non-negative real numbers  $b_1, b_2, s, m$  such that  $m \geq s > 0$ ,*

$$(b_1^s + b_2^s)^m \leq 2^{m-s} (b_1^m + b_2^m)^s.$$

*Proof.* If  $m = s$ , then both sides of the inequality evaluate to  $(b_1^s + b_2^s)^m$  and there is nothing to prove; so, assume that  $m > s$ . Define  $p = m/s$  and  $q = m/(m-s)$  and note that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $x_1 = b_1^s, x_2 = b_2^s$  and  $y_1 = y_2 = 1$ . By Lemma A.0.3,

$$x_1 y_1 + x_2 y_2 \leq (x_1^p + x_2^p)^{1/p} (y_1^q + y_2^q)^{1/q}.$$

In other words,

$$b_1^s + b_2^s \leq (b_1^m + b_2^m)^{s/m} \left( 1^{m/(m-s)} + 1^{m/(m-s)} \right)^{(m-s)/m}.$$

The desired inequality now follows by raising both sides to the power  $m$ . □

**Lemma A.0.5.** *Let  $k, \ell, g$  and  $\rho$  satisfy the hypotheses of Proposition 3.3.1. For  $0 \leq$*

$x < 1$ , if  $\ell_0 \geq \ell$  and

$$(g(1+x) + g(1-x))^k \geq \frac{c2^{k-2} ((1+x)^{\ell_0} + (1-x)^{\ell_0})}{(1+x)^{\ell_0}(1-x)^{\ell_0}} \quad (x \approx 0^*)$$

then  $(x \approx 0)$  holds.

*Proof.* Applying Claim A.0.4 with  $s = \frac{\ell}{k-1}$ ,  $m = \ell$ ,  $b_1 = 1+x$  and  $b_2 = 1-x$  yields

$$\left( (1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}} \right)^\ell \leq 2^{\ell - \frac{\ell}{k-1}} ((1+x)^\ell + (1-x)^\ell)^{\frac{\ell}{k-1}}.$$

Raising both sides to the power  $\frac{k-1}{\ell}$  gives us

$$\left( (1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}} \right)^{k-1} \leq 2^{k-2} ((1+x)^\ell + (1-x)^\ell).$$

Thus, to prove  $(x \approx 0)$ , it is sufficient to show that

$$(g(1+x) + g(1-x))^k \geq \frac{c2^{k-2} ((1+x)^\ell + (1-x)^\ell)}{(1+x)^\ell(1-x)^\ell}.$$

Note that the function  $\frac{(1+x)^\ell + (1-x)^\ell}{(1+x)^\ell(1-x)^\ell}$  is increasing in  $\ell$ . So, for  $\ell_0 \geq \ell$ , the inequality  $(x \approx 0^*)$  is also sufficient for proving  $(x \approx 0)$ .  $\square$

**Lemma A.0.6.** *Let  $k, \ell, g$  and  $\rho$  satisfy the hypotheses of Proposition 3.3.1. Furthermore, let  $m \geq \frac{\ell}{k-1}$  and assume that  $g(z) = z^m$  for  $z \in [0, 2]$ . For  $0 \leq x < 1$ , if*

$$\left( \frac{(1+x)^m + (1-x)^m}{2} \right)^{k - \frac{\ell}{m}} \geq \frac{c}{2(1+x)^\ell(1-x)^\ell} \quad (x \approx 0^\dagger)$$

then  $(x \approx 0)$  holds.

*Proof.* Applying Claim A.0.4 with  $s = \frac{\ell}{k-1}$ ,  $b_1 = 1+x$  and  $b_2 = 1-x$  yields

$$\left( (1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}} \right)^m \leq 2^{m - \frac{\ell}{k-1}} ((1+x)^m + (1-x)^m)^{\frac{\ell}{k-1}}.$$

Raising both sides to the power  $\frac{k-1}{m}$  gives us

$$\left( (1+x)^{\frac{\ell}{k-1}} + (1-x)^{\frac{\ell}{k-1}} \right)^{k-1} \leq 2^{k-1 - \frac{\ell}{m}} ((1+x)^m + (1-x)^m)^{\frac{\ell}{m}}.$$

Thus, to prove  $(x \approx 0)$ , it is sufficient to show that

$$\frac{((1+x)^m + (1-x)^m)^k}{2^{k-1 - \frac{\ell}{m}} ((1+x)^m + (1-x)^m)^{\frac{\ell}{m}}} \geq \frac{c}{(1+x)^\ell(1-x)^\ell}.$$

This inequality is equivalent to  $(x \approx 0^\dagger)$ .

□

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