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TO PARABOLIC STARLIKE & UNIFORMLY
CONVEX FUNCTIONS***

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APPLICATIONS OF FRACTIONAL CALCULUS TO PARABOLIC STARLIKE AND UNIFORMLY CONVEX FUNCTIONS

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Abstract

Let \mathcal{A} be the class of analytic functions in the open unit disk \mathcal{U} . Given $0 \leq \lambda < 1$, let Ω^λ be the operator defined on \mathcal{A} by

$$\left(\Omega^\lambda f\right)(z) = \Gamma(2 - \lambda)z^\lambda D_z^\lambda f(z),$$

where $D_z^\lambda f$ is the fractional derivative of f of order λ . A function f in \mathcal{A} is said to be in the class SP_λ if $\Omega^\lambda f$ is a parabolic starlike function. In this paper several basic properties and characteristics of the class SP_λ are investigated. These include subordination, inclusion, and growth theorems, class-preserving operators, Fekete-Szegő problems, and sharp estimates for the first few coefficients of the inverse function.

1. Introduction

Let \mathcal{A} be the class of functions analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and let \mathcal{A}_0 be the family of functions f in \mathcal{A} satisfying the normalization condition (*cf.* [3]):

$$f(0) = f'(0) - 1 = 0.$$

A function f in \mathcal{A}_0 is said to be *uniformly convex* in \mathcal{U} if f is a *univalent convex* function along with the property that, for every circular arc γ contained in \mathcal{U} , with centre ζ also in \mathcal{U} , the image

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curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV (see, for details, [5]). It is well known from [8] and [14] that

$$f \in UCV \iff \left| \frac{zf''(z)}{f'(z)} \right| < \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \quad (z \in \mathcal{U}). \quad (1.1)$$

The condition (1.1) implies that

$$1 + \frac{zf''(z)}{f'(z)}$$

lies in the interior of the parabolic region

$$\mathcal{R} := \{w : w = u + iv \quad \text{and} \quad v^2 < 2u - 1\}$$

for every value of $z \in \mathcal{U}$. Let

$$P := \{p : p \in \mathcal{A}, p(0) = 1, \quad \text{and} \quad \operatorname{Re} \{p(z)\} > 0 \quad (z \in \mathcal{U})\}$$

and

$$PAR := \{p : p \in P \quad \text{and} \quad p(\mathcal{U}) \subset \mathcal{R}\}.$$

A function $f \in \mathcal{A}_0$ is said to be in the class of *parabolic starlike functions*, denoted by SP (cf. [14]), if

$$\frac{zf'(z)}{f(z)} \in \mathcal{R} \quad (z \in \mathcal{U}).$$

Let

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in \mathcal{U}; c \neq 0, -1, -2, \dots), \quad (1.2)$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \dots\}). \end{cases}$$

Further, let (cf. [2] and [12])

$$\mathcal{L}(a, c) f(z) = \varphi(a, c; z) * f(z) \quad (f \in \mathcal{A}), \quad (1.3)$$

in terms of the Hadamard product (or convolution). Note that $\mathcal{L}(a, a)$ is the identity operator and

$$\mathcal{L}(a, c) = \mathcal{L}(a, b) \mathcal{L}(b, c) \quad (b, c \neq 0, -1, -2, \dots).$$

It is well known that, if $c > a > 0$, then \mathcal{L} maps \mathcal{A} into itself.

We also need the following definitions of a *fractional derivative*.

Definition 1 (cf. [11] and [12], see also [18] and [19]) Let the function $f(z)$ be analytic in a simply-connected region of the z -plane containing the origin. The *fractional derivative of f of order λ* is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z - \zeta)^\lambda$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Using Definition 1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [12] introduced the operator $\Omega^\lambda : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\Omega^\lambda f(z) := \Gamma(2 - \lambda) z^\lambda D_z^\lambda f(z) \quad (\lambda \neq 2, 3, 4, \dots). \quad (1.4)$$

Definition 2. Suppose that the functions f and F are in \mathcal{A} . We say that f is *subordinate* to F in \mathcal{U} , written as $f \prec F$, if F is univalent in \mathcal{U} ,

$$f(0) = F(0), \quad \text{and} \quad f(\mathcal{U}) \subseteq F(\mathcal{U}).$$

In a recent paper [20], the authors studied the class $\mathcal{S}_\lambda(\alpha)$ ($0 \leq \alpha < 1$; $0 \leq \lambda < 1$) of functions f such that $\Omega^\lambda f$ is a *univalent starlike function of order* α . In the present sequel to our earlier paper [20], we study a class of analytic functions, related to *UCV* and *SP*, using the operator Ω^λ defined by (1.4).

Definition 3. Let SP_λ ($0 \leq \lambda \leq 1$) be the class of functions $f \in \mathcal{A}_0$ satisfying the inequality:

$$\left| \frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right| < \operatorname{Re} \left\{ \frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \right\} \quad (z \in \mathcal{U}). \quad (1.5)$$

It follows that

$$SP_1 \equiv UCV \quad \text{and} \quad SP_0 \equiv SP. \quad (1.6)$$

We investigate here several basic properties and characteristics of the general class SP_λ . These include inclusion, subordination, and growth theorems, class-preserving operators (like the Hadamard product and various integral transforms), Fekete-Szegő problems, and sharp estimates for the first few coefficients of the inverse function.

2. Basic Properties of the Class SP_λ

We need the following results in our investigation of the class SP_λ .

Lemma 1 (cf. Ruscheweyh and Sheil-Small [16]). *Let F and G be univalent convex functions in \mathcal{U} . Then the Hadamard product $F * G$ is also univalent convex in \mathcal{U} .*

Lemma 2 (cf. Ruscheweyh and Stankiewicz [17]). *Let the functions F and G be univalent convex in \mathcal{U} . Also let $f \prec F$ and $g \prec G$. Then $f * g \prec F * G$.*

Lemma 3 (cf. Ruscheweyh and Sheil-Small [16]). *Let each of the functions f and g be univalent starlike of order $\frac{1}{2}$. Then, for every function $F \in \mathcal{A}$,*

$$\frac{f(z) * g(z) F(z)}{f(z) * g(z)} \in \overline{\text{CH}} \{F(\mathcal{U})\} \quad (z \in \mathcal{U})$$

where $\overline{\text{CH}}$ denotes the closed convex hull.

Theorem 1. *If $0 \leq \mu < \lambda < 1$, then*

$$SP_\lambda \subset SP_\mu.$$

Proof. Let $f \in SP_\lambda$. Then

$$\begin{aligned} \Omega^\mu f &= \mathcal{L}(2, 2 - \mu) f = \mathcal{L}(2 - \lambda, 2 - \mu) \Omega^\lambda f \\ &= \varphi(2 - \lambda, 2 - \mu; z) * \Omega^\lambda f \end{aligned}$$

and

$$\begin{aligned} z(\Omega^\mu f)' &= \mathcal{L}(2, 1) \mathcal{L}(2 - \lambda, 2 - \mu) \Omega^\mu f \\ &= \varphi(2 - \lambda, 2 - \mu; z) * \{z(\Omega^\mu f)'\}. \end{aligned}$$

Also it is known that (cf. [6])

$$\varphi(2 - \lambda, 2 - \mu; z) \in \mathcal{S}^* \left(\frac{1}{2} \right).$$

Since \mathcal{R} is a convex region, using Lemma 3, we get

$$\frac{z(\Omega^\mu f(z))}{\Omega^\mu f(z)} = \frac{\varphi(2 - \lambda, 2 - \mu; z) * \frac{z(\Omega^\mu f(z))'}{\Omega^\mu f(z)} \Omega^\lambda f(z)}{\varphi(2 - \lambda, 2 - \mu; z) * \Omega^\lambda f(z)} \in \mathcal{R}.$$

Thus $f \in SP_\mu$. This completes the proof of Theorem 1.

Corollary 1. *If $0 \leq \lambda < 1$, then the functions in SP_λ are parabolic starlike and $UCV \subset SP$.*

Proof. Clearly, we have [cf. Equation (1.6)]

$$SP_\lambda \subset SP_0 \equiv SP \quad \text{and} \quad UCV \equiv SP_1 \subset SP_0 \equiv SP.$$

It can be verified that the Riemann map q of \mathcal{U} onto the region \mathcal{R} , satisfying $q(0) = 1$ and $q'(0) > 0$, is given by

$$q(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \quad (2.1)$$

$$\begin{aligned} &= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2k+1} \right) z^n \\ &= \sum_{n=0}^{\infty} B_n z^n = 1 + \frac{8}{\pi^2} \left(z + \frac{2}{3} z^2 + \frac{23}{45} z^3 + \frac{44}{105} z^4 + \dots \right) \quad (z \in \mathcal{U}). \end{aligned} \quad (2.2)$$

We define the function G by

$$G(z) := \frac{1}{z} \left\{ \mathcal{L}(2 - \lambda, 2) z \exp \left(\int_0^z \frac{q(s) - 1}{s} ds \right) \right\} \quad (z \in \mathcal{U}). \quad (2.3)$$

Theorem 2. *Let $0 \leq \lambda < 1$ and let $G(z)$ be defined by (2.3). Then $G(z)$ is a convex univalent function. Furthermore, if $f \in SP_\lambda$, then*

$$\frac{f(z)}{z} \prec G(z).$$

Proof. We first note that

$$G(z) = \frac{\varphi(2 - \lambda, 2; z)}{z} * \exp \left(\int_0^z \frac{q(s) - 1}{s} ds \right) \quad (z \in \mathcal{U}), \quad (2.4)$$

where each member of the Hadamard product in (2.4) is known to be a convex univalent function (cf. [6] and [8]). Therefore, by Lemma 1, $G(z)$ is a univalent convex function.

Next, if $f \in SP_\lambda$, then (cf. Definition 2)

$$\frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \prec q(z).$$

Thus there exists a function ψ satisfying the Schwarz lemma such that

$$\frac{\Omega^\lambda f(z)}{z} = \exp \left(\int_0^z \frac{q(\psi(s)) - 1}{s} ds \right) \quad (z \in \mathcal{U}).$$

Since $q(z) - 1$ is a univalent convex function, a result of Goluzin ([4]; see also [13, p. 50]) yields

$$\frac{\Omega^\lambda f(z)}{z} \prec \exp \left(\int_0^z \frac{q(s) - 1}{s} ds \right).$$

It now follows from a known result of Ling and Ding [6, p. 508, Theorem 2] that

$$\frac{f(z)}{z} \prec G(z).$$

The proof Theorem 2 is evidently completed.

Remark 1. Taking $\lambda = 1$ in Theorem 2, we immediately obtain a subordination result due to Ma and Minda [8, p. 169, Theorem 3].

Theorem 3. Let $0 \leq \lambda < 1$. If $f \in SP_\lambda$, then

$$G(-r) \leq \left| \frac{f(z)}{z} \right| \leq G(r) \quad (|z| = r) \quad (2.5)$$

and

$$\left| \arg \left(\frac{f(z)}{z} \right) \right| \leq \max_{\theta \in [0, 2\pi]} \left\{ \arg \left(G \left(r e^{i\theta} \right) \right) \right\} \quad (z = r e^{i\theta}), \quad (2.6)$$

where $G(z)$ is defined by (2.3). Equality holds true in (2.5) and (2.6) for some $z \neq 0$ if and only if f is a rotation of $zG(z)$.

Proof. Let $f \in SP_\lambda$. Then, by Theorem 2 and Lindelöf's principle of subordination, we get

$$\begin{aligned} \inf_{|z|=r} \operatorname{Re} \{G(z)\} &\leq \inf_{|z|\leq r} \operatorname{Re} \left\{ \frac{f(z)}{z} \right\} \leq \sup_{|z|\leq r} \operatorname{Re} \left\{ \frac{f(z)}{z} \right\} \\ &\leq \sup_{|z|\leq r} \left| \frac{f(z)}{z} \right| \leq \sup_{|z|\leq r} \operatorname{Re} \{G(z)\}. \end{aligned} \quad (2.7)$$

Since $G(z)$ is a univalent convex function and has real coefficients, $G(\mathcal{U})$ is a convex region symmetric with respect to real axis. Hence

$$\inf_{|z|\leq r} \operatorname{Re} \{G(z)\} = \inf_{-r \leq x \leq r} G(x) = G(-r) \quad (2.8)$$

and

$$\sup_{|z|\leq r} \operatorname{Re} \{G(z)\} = \sup_{-r \leq x \leq r} G(x) = G(r). \quad (2.9)$$

Thus (2.7) gives the assertion (2.5) of Theorem 3. Also we readily have the assertion (2.6) of Theorem 3.

The sharpness in the inequalities (2.5) and (2.6) is also a consequence of the principle of subordination. This completes the proof of Theorem 3.

Corollary 2. *Let $f \in SP_\lambda$. Then*

$$\{w : |w| \leq G(-1)\} \subseteq f(\mathcal{U}).$$

The result is sharp.

Remark 2. Taking $\lambda = 1$ in Theorem 3, we get a result of Ma and Minda [8, p. 170, Corollary 3].

3. Class-Preserving Operators and Transforms

Theorem 4. *If $f \in \mathcal{S}_\lambda(\frac{1}{2})$ and $g \in SP_\mu$ ($\lambda \leq \mu$), then*

$$\Omega^\lambda f * \Omega^\mu g \in SP_\mu.$$

In particular, if $f \in \mathcal{S}_\lambda(\frac{1}{2})$ and $g \in SP_\lambda$, then

$$\Omega^\lambda f * \Omega^\lambda g \in SP_\lambda.$$

Proof. Let $f \in \mathcal{S}_\lambda(\frac{1}{2})$ and $g \in SP_\mu$ ($\lambda \leq \mu$). By definition,

$$f \in \mathcal{S}^*\left(\frac{1}{2}\right) \quad \text{and} \quad \Omega^\mu g \in SP \subset \mathcal{S}^*\left(\frac{1}{2}\right).$$

The commutative and associative properties of the Hadamard product yield

$$\begin{aligned} z \left(\Omega^\lambda f * \Omega^\mu g \right)' &= \mathcal{L}(2, 1) \left(\Omega^\lambda f * \Omega^\mu g \right) \\ &= \Omega^\lambda f * \mathcal{L}(2, 1) \Omega^\mu g \\ &= \Omega^\lambda f * \{z (\Omega^\mu g)'\}. \end{aligned} \tag{3.1}$$

Therefore, using Lemma 3, we get

$$\frac{z \left(\Omega^\lambda f * \Omega^\mu g \right)'}{\Omega^\lambda f * \Omega^\mu g} = \frac{\Omega^\lambda f * \frac{z (\Omega^\mu g)'}{\Omega^\mu g}}{\Omega^\lambda f * \Omega^\mu g} \in \mathcal{R} \quad (z \in \mathcal{U}). \tag{3.2}$$

This completes the proof of Theorem 4.

Corollary 3 (cf. Rønning [14]). *If $f \in \mathcal{S}^*(\frac{1}{2})$ and $g \in SP$, then $f * g \in SP$. In particular, if $f \in SP$ and $g \in SP$, then $f * g \in SP$.*

Theorem 5. *Let $f \in SP$ and $g \in SP_\mu$ ($0 \leq \mu \leq 1$). Then $f * g \in SP_\mu$. In particular, if $f \in UCV$ and $g \in UCV$, then $f * g \in UCV$.*

Proof. The proof of Theorem 5 is similar to that of Theorem 4. Let $f \in SP$ and $g \in SP_\mu$ ($0 \leq \mu \leq 1$). We first note that

$$z \left(\Omega^\mu (f * g) \right)' = \Omega^\mu g(z) * z f'(z) \quad \text{and} \quad \Omega^\mu (f * g)(z) = \Omega^\mu g(z) * f(z). \tag{3.3}$$

Therefore, using Lemma 3, we get

$$\frac{z(\Omega^\mu(f * g)(z))'}{\Omega^\mu(f * g)(z)} = \frac{\Omega^\mu g(z) * \frac{zf'(z)}{f(z)}}{\Omega^\mu g(z) * f(z)} \in \mathcal{R} \quad (z \in \mathcal{U}). \quad (3.4)$$

Thus $f * g \in SP_\mu$. Next, by Corollary 1, $UCV \subset SP$. Thus, by taking $\mu = 1$, we see that the Hadamard product of two uniformly convex functions is a uniformly convex function. This completes the proof of Theorem 5.

Theorem 6. Let $f_j \in SP_\lambda$ ($j = 1, \dots, n$). Also let

$$\alpha_j > 0 \quad \text{and} \quad \sum_{j=1}^n \alpha_j = 1. \quad (3.5)$$

Define a function g by

$$\Omega^\alpha g = \prod_{j=1}^n (\Omega^\lambda f_j)^{\alpha_j}. \quad (3.6)$$

Then $g \in SP_\lambda$.

Proof. Let $f_j \in SP_\lambda$ ($j = 1, \dots, n$) and let g be defined by (3.6). Direct calculation gives

$$\begin{aligned} \left| \frac{z(\Omega^\lambda g)'}{\Omega^\lambda g} - 1 \right| &= \left| \sum_{j=1}^n \alpha_j \frac{z(\Omega^\lambda f_j)'}{\Omega^\lambda f_j} - 1 \right| \\ &< \sum_{j=1}^n \alpha_j \operatorname{Re} \left(\frac{z(\Omega^\lambda f_j)'}{\Omega^\lambda f_j} \right) \\ &= \operatorname{Re} \left(\frac{z(\Omega^\lambda g)'}{\Omega^\lambda g} \right). \end{aligned} \quad (3.7)$$

Thus, by Definition 2, $g \in SP_\lambda$. This completes the proof of Theorem 6.

Theorem 7. Let $f \in SP_\lambda$ ($0 \leq \lambda \leq 1$). Then the function $F(z)$ defined by the integral transform:

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (z \in \mathcal{U}; \quad c > -1) \quad (3.8)$$

is also in the class SP_λ .

Proof. We begin by noting that

$$F(z) = \mathcal{L}(c+1, c+2)f(z)$$

and

$$\begin{aligned} z \left(\Omega^\lambda f(z) \right)' &= \mathcal{L}(2, 1) \mathcal{L}(2, 2-\lambda) \mathcal{L}(c+1, c+2)f(z) \\ &= \mathcal{L}(c+1, c+2) \left\{ z \left(\Omega^\lambda f(z) \right)' \right\} = \varphi(c+1, c+2; z) * \left\{ z \left(\Omega^\lambda f(z) \right)' \right\}. \end{aligned}$$

Using a result of Bernardi [1], it can be verified that

$$\varphi(c+1, c+2; z) \in \mathcal{S}^* \left(\frac{1}{2} \right).$$

Also, by hypothesis, $\Omega^\lambda f(z) \in SP \subseteq \mathcal{S}^* \left(\frac{1}{2} \right)$. Thus, using Lemma 3, we get

$$\frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} = \frac{\varphi(c+1, c+2; z) * \frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} \Omega^\lambda f(z)}{\varphi(c+1, c+2; z) * \Omega^\lambda f(z)} \in \mathcal{R}, \quad (3.9)$$

which completes the proof of Theorem 7.

4. The Fekete-Szegő Problem for the Class SP_λ

Let the function f , given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (z \in \mathcal{U}), \quad (4.1)$$

be in the class SP_λ . Then there exists a function $w \in \mathcal{A}$, satisfying

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$\frac{z (\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} = q(w(z)) \quad (z \in \mathcal{U}). \quad (4.2)$$

Let the function p_1 in P be defined by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \quad (4.3)$$

Then, by using (2.2), (4.2), and (4.3) in the form:

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1},$$

we find that

$$a_2 = \frac{4\Gamma(3-\lambda)}{\pi^2\Gamma(3)\Gamma(2-\lambda)} c_1, \quad (4.4)$$

$$a_3 = \frac{2\Gamma(4-\lambda)}{\pi^2\Gamma(4)\Gamma(2-\lambda)} \left[c_2 - \frac{1}{6} \left(1 - \frac{24}{\pi^2} \right) c_1^2 \right], \quad (4.5)$$

and

$$a_4 = \frac{4\Gamma(5-\lambda)}{3\pi^2\Gamma(2-\lambda)\Gamma(5)} \left[c_3 - \frac{1}{3} \left(1 - \frac{18}{\pi^2} \right) c_1 c_2 + \frac{2}{45} \left(1 - \frac{45}{2\pi^2} + \frac{180}{\pi^4} \right) c_1^3 \right]. \quad (4.6)$$

These expressions shall be used throughout the rest of the paper.

Define the functions $k(z, \tau, \nu)$ in SP_λ by

$$k(z, \tau, \nu) = \mathcal{L}(2-\lambda, 2) z \exp \left(\int_0^z \left[q \left(\frac{e^{i\tau} \zeta (\zeta + \nu)}{1 + \nu \zeta} \right) - 1 \right] \frac{d\zeta}{\zeta} \right) \quad (0 \leq \tau \leq 2\pi; 0 \leq \nu \leq 1). \quad (4.7)$$

Note that $k(z, 0, 1) = zG(z)$ defined by (2.3) and that $k(z, \tau, 0)$ is an odd function. We also need the following lemma in our investigation.

Lemma 4. Let $g \in P$, where

$$g(z) = 1 + c_1z + c_2z^2 + \cdots = 1 + G(z). \quad (4.8)$$

Then

$$|c_n| \leq 2 \quad (n \in \mathbb{N}) \quad (4.9)$$

and

$$\left| c_2 - \frac{1}{2}\mu c_1^2 \right| \leq 2 + \frac{1}{2}(|\mu - 1| - 1)|c_1|^2. \quad (4.10)$$

Furthermore, if we define the sequence $\{A_n\}_{n=1}^{\infty}$ by

$$\sum_{n=1}^{\infty} (-1)^{n-1} \gamma_{n-1} \{G(z)\}^n = \sum_{n=1}^{\infty} A_n z^n, \quad (4.11)$$

where

$$\gamma_0 = 1 \quad \text{and} \quad \gamma_n = \frac{1}{2^n} \left[1 + \frac{1}{2} \sum_{j=1}^n \binom{n}{j} B_j \right], \quad (4.12)$$

and the sequence $\{B_n\}_{n=1}^{\infty}$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad (4.13)$$

then

$$|A_n| \leq 2 \quad (n \in \mathbb{N}). \quad (4.14)$$

Remark 3. The inequalities (4.9) and (4.10) are well known (cf. [3]) and the estimate (4.14) can be found in [10]. For $n = 2$ and $n = 3$, (4.10) reduces to

$$|A_2| = |c_2 - \gamma_1 c_1^2| \leq 2 \quad (4.15)$$

and

$$|A_3| = |c_3 - 2\gamma_1 c_1 c_2 + \gamma_2 c_1^3| \leq 2, \quad (4.16)$$

respectively.

Theorem 8. Let the function f , given by (4.1), be in the class SP_{λ} . Then

$$|\mu a_2^2 - a_3| \leq \begin{cases} \frac{4}{3\pi^2} (3-\lambda)(2-\lambda) \left(\frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{4}{\pi^2} - \frac{1}{3} \right) & (\mu \geq \sigma_1) \\ \frac{2}{3\pi^2} (3-\lambda)(2-\lambda) & (\sigma_2 \leq \mu \leq \sigma_1) \\ \frac{4}{3\pi^2} (3-\lambda)(2-\lambda) \left(\frac{1}{3} + \frac{4}{\pi^2} - \frac{12(2-\lambda)\mu}{\pi^2(3-\lambda)} \right) & (\mu \leq \sigma_2), \end{cases} \quad (4.16)$$

where, for convenience,

$$\sigma_1 := \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} + \frac{5\pi^2}{72} \right) \quad \text{and} \quad \sigma_2 := \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} - \frac{\pi^2}{72} \right). \quad (4.17)$$

Each of the estimates in (4.16) is sharp.

Proof. Using (4.4) and (4.5), we write

$$|\mu a_2^2 - a_3| = \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left| \left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} + \frac{1}{3} - \frac{8}{\pi^2} \right) c_1^2 - 2c_2^2 \right| \quad (4.18)$$

$$\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left(\left| \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{5}{3} - \frac{8}{\pi^2} \right| |c_1|^2 + 2|c_1^2 - c_2|^2 \right). \quad (4.19)$$

If $\mu \geq \sigma_1$, then the expression inside the first modulus on the right-hand side of (4.19) is nonnegative. Thus, by applying Lemma 4, we get

$$|\mu a_2^2 - a_3| \leq \frac{4(2-\lambda)(3-\lambda)}{3\pi^2} \left(\frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{4}{\pi^2} - \frac{1}{3} \right), \quad (4.20)$$

which is the first part of the assertion (4.16). Equality in (4.20) [or, equivalently, in (4.19)] holds true if and only if $|c_1| = 2$. Thus the function f is $k(z, 0, 1)$ or one of its rotations for $\mu > \sigma_1$.

Next, if $\mu \leq \sigma_2$, we rewrite (4.18) as

$$\begin{aligned} |\mu a_2^2 - a_3| &= \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left| 2c_2 + \left(\frac{8}{\pi^2} - \frac{1}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) c_1^2 \right| \\ &\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left\{ 2|c_2| + \left(\frac{8}{\pi^2} - \frac{1}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) |c_1|^2 \right\}. \end{aligned}$$

The estimates $|c_2| \leq 2$ and $|c_1| \leq 2$, after simplification, yield the second part of the assertion (4.16), in which equality holds true if and only if f is a rotation of $k(z, 0, 1)$ for $\mu < \sigma_2$.

If $\mu = \sigma_2$, then equality holds true if and only if $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+\nu}{2} \left(\frac{1+z}{1-z} \right) + \frac{1-\nu}{2} \left(\frac{1-z}{1+z} \right) \quad (0 < \nu < 1; z \in \mathcal{U}).$$

Thus the function f is $k(z, 0, \nu)$ or one of its rotations.

If $\mu = \sigma_1$, then

$$\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{5}{3} - \frac{8}{\pi^2} = 0.$$

Therefore, equality holds true if and only if $|c_1^2 - c_2| = 2$. This happens if and only if

$$\frac{1}{p_1(z)} = \frac{1+\nu}{2} \left(\frac{1+z}{1-z} \right) + \frac{1-\nu}{2} \left(\frac{1-z}{1+z} \right) \quad (0 < \nu < 1; z \in \mathcal{U}).$$

Thus the function f is $k(z, \pi, \nu)$ or one of its rotations.

Finally, we see that

$$|\mu a_2^2 - a_3| = \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left| 2 \left(c_2 - \frac{1}{2} c_1^2 \right) + \left(\frac{8}{\pi^2} + \frac{2}{3} - \frac{24(2-\lambda)\mu}{3-\lambda} \right) c_1^2 \right|$$

and

$$\max \left| \frac{8}{\pi^2} + \frac{2}{3} - \frac{24(2-\lambda)\mu}{3-\lambda} \right| \leq 1 \quad (\sigma_2 \leq \mu \leq \sigma_1).$$

Therefore, using Lemma 4, we get

$$\begin{aligned} |\mu a_2^2 - a_3| &\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left\{ 2 \left(2 - \frac{1}{2} |c_1|^2 \right) + |c_1|^2 \right\} \\ &= \frac{2}{3\pi^2} (2-\lambda)(3-\lambda) \quad (\sigma_2 \leq \mu \leq \sigma_1). \end{aligned}$$

If $\sigma_2 < \mu < \sigma_1$, then equality holds true if and only if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1 + \nu z^2}{1 - \nu z^2} \quad (0 \leq \nu \leq 1; z \in \mathcal{U}).$$

Thus the function f is $k(z, 0, 0)$ or one of its rotations. The proof of Theorem 8 is evidently completed.

Remark 4. The second part of the assertion (4.16) can be improved as follows:

$$\begin{aligned} |\mu a_2^2 - a_3| + \left\{ \mu - \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} - \frac{\pi^2}{72} \right) \right\} |a_2|^2 \\ \leq \frac{2}{3\pi^2} (3-\lambda)(2-\lambda) \quad (\sigma_2 \leq \mu \leq \sigma_3) \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} |\mu a_2^2 - a_3| + \left\{ \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} + \frac{5\pi^2}{72} \right) - \mu \right\} |a_2|^2 \\ \leq \frac{2}{3\pi^2} (3-\lambda)(2-\lambda) \quad (\sigma_3 \leq \mu \leq \sigma_1), \end{aligned} \quad (4.22)$$

where σ_1 and σ_2 are given, as before, by (4.17), and

$$\sigma_3 := \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} - \frac{\pi^2}{36} \right). \quad (4.23)$$

Proof. For the values of μ prescribed in (4.21), we have

$$\begin{aligned}
|\mu a_2^2 - a_3| &+ \left\{ \mu - \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} - \frac{\pi^2}{72} \right) \right\} |a_2|^2 \\
&= \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left\{ \left| 2 \left(c_2 - \frac{1}{2} c_1^2 \right) \left(\frac{8}{\pi^2} + \frac{2}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) c_1^2 \right| \right. \\
&\quad \left. + \left(\frac{24(2-\lambda)\mu}{\pi^2(3-\lambda)} + \frac{1}{3} - \frac{8}{\pi^2} \right) |c_1|^2 \right\} \\
&\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2} \left\{ 4 - |c_1|^2 + \left(\frac{8}{\pi^2} + \frac{2}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) |c_1|^2 \right. \\
&\quad \left. + \left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} + \frac{1}{3} - \frac{8}{\pi^2} \right) |c_1|^2 \right\} \\
&= \frac{2}{3\pi^2} (3-\lambda)(2-\lambda) \quad (\sigma_2 \leq \mu \leq \sigma_3),
\end{aligned}$$

which establishes (4.21). Similarly, for the values of μ prescribed in (4.22), we have

$$\begin{aligned}
|\mu a_2^2 - a_3| &+ \left\{ \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} + \frac{5\pi^2}{72} \right) - \mu \right\} |a_2|^2 \\
&= \frac{(3-\lambda)(2-\lambda)}{6\pi^2} \left\{ \left| 2 \left(c_2 - \frac{1}{2} c_1^2 \right) + \left(\frac{8}{\pi^2} + \frac{2}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) c_1^2 \right| \right. \\
&\quad \left. + \left(\frac{8}{\pi^2} + \frac{5}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) |c_1|^2 \right\} \\
&\leq \frac{(3-\lambda)(2-\lambda)}{6\pi^2} \left\{ 4 - |c_1|^2 + \left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{8}{\pi^2} - \frac{2}{3} \right) |c_1|^2 \right. \\
&\quad \left. + \left(\frac{8}{\pi^2} + \frac{5}{3} - \frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) |c_1|^2 \right\} \\
&= \frac{2}{3\pi^2} (3-\lambda)(2-\lambda) \quad (\sigma_3 \leq \mu \leq \sigma_1),
\end{aligned}$$

which proves (4.23).

Remark 5. A special case of Theorem 8 when $\lambda = 1$ would yield a result due to Ma and Minda [9, p. 280, Theorem 2]. Furthermore, by setting $\lambda = 0$ in Theorem 8, we readily obtain

Corollary 4. *Let the function f , given by (4.1), be in the class SP . Then*

$$|\mu a_2^2 - a_3| \leq \begin{cases} \frac{8}{\pi^2} \left(\frac{4(2\mu-1)}{\pi^2} - \frac{1}{3} \right) & (\mu \geq \delta_1) \\ \frac{4}{\pi^2} & (\delta_2 \leq \mu \leq \delta_1) \\ \frac{8}{\pi^2} \left(\frac{1}{3} - \frac{4(2\mu-1)}{\pi^2} \right) & (\mu \leq \delta_2), \end{cases}$$

where, for convenience,

$$\delta_1 := \frac{1}{2} + \frac{5\pi^2}{48} \quad \text{and} \quad \delta_2 := \frac{1}{2} - \frac{\pi^2}{48}. \quad (4.24)$$

5. Coefficient Bounds for the Inverse Functions of SP_λ

We first state

Theorem 9. *Let the function f , given by (4.1), be in the class SP_λ . Also let the function f^{-1} , defined by*

$$f^{-1}(f(z)) = z = f(f^{-1}(z)), \quad (5.1)$$

be the inverse of f . If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \quad \left(|w| < r_0; r_0 > \frac{1}{4} \right), \quad (5.2)$$

then

$$|d_2| \leq \frac{4(2-\lambda)}{\pi^2} \quad (0 \leq \lambda \leq 1), \quad (5.3)$$

$$|d_3| \leq \begin{cases} \frac{2}{3\pi^2}(3-\lambda)(2-\lambda) & (\rho_1 \leq \lambda \leq 1) \\ \frac{4}{3\pi^2}(3-\lambda)(2-\lambda) & (\rho_2 \leq \lambda \leq 1), \end{cases} \quad (5.4)$$

and

$$|d_4| \leq \frac{(4-\lambda)(3-\lambda)(2-\lambda)}{9\pi^2} \quad (\rho_2 \leq \lambda \leq 1), \quad (5.5)$$

where, for convenience,

$$\rho_1 := \frac{216 - 15\pi^2}{120 - 5\pi^2} \cong 0.9618\dots \quad \text{and} \quad \rho_2 := \frac{4(72 - 5\pi^2)}{162 - 5\pi^2} \cong 0.8043\dots \quad (5.6)$$

Each of the estimates in (5.3), (5.4), and (5.5) is sharp.

Proof. The relation (5.2) gives

$$d_2 = -a_2, \quad d_3 = 2a_2^2 - a_3, \quad \text{and} \quad d_4 = -a_4 + 5a_3a_2 - 5a_2^3. \quad (5.7)$$

Thus, making use of (4.4), we get

$$|d_2| = \left| -\frac{2(2-\lambda)}{\pi^2} c_1 \right| \leq \frac{4(2-\lambda)}{\pi^2} \quad (0 \leq \lambda \leq 1),$$

in which equality holds true if and only if h is the inverse of $k(z, 0, 1)$ or one of its rotations.

An application of Theorem 8 (with $\mu = 2$) gives the estimate for $|d_3|$. Since the estimates in Theorem 8 are all sharp, the bounds on $|d_3|$ are also the best possible.

Next, using the expressions for a_2, a_3 , and a_4 given by (4.4), (4.5), and (4.6), respectively, we have

$$\begin{aligned} |d_4| &= \frac{\Gamma(5-\lambda)}{18\pi^2\Gamma(2-\lambda)} \left| c_3 - \frac{1}{3} \left\{ 1 - \frac{18}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} \right) \right\} c_1 c_2 \right. \\ &\quad \left. + \left[\frac{2}{45} + \frac{1}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) + \frac{8}{\pi^4} \left\{ 1 + \frac{30(2-\lambda)}{4-\lambda} \left(\frac{3(2-\lambda)}{3-\lambda} - 1 \right) \right\} \right] c_1^3 \right| \\ &= \frac{\Gamma(5-\lambda)}{18\pi^2\Gamma(2-\lambda)} |c_3 - 2\gamma_1 c_1 c_2 + \gamma_2 c_1^3|, \end{aligned} \quad (5.8)$$

where

$$\gamma_1 = \left\{ \frac{1}{6} - \frac{18}{\pi^2} \left(1 - \frac{10(2-\lambda)}{4-\lambda} \right) \right\} = \frac{1}{2} \left(1 + \frac{1}{2}\beta_1 \right) \quad (5.9)$$

and

$$\begin{aligned} \gamma_2 &= \frac{2}{45} - \frac{1}{\pi^2} \left(1 - \frac{10(2-\lambda)}{4-\lambda} \right) + \frac{8}{\pi^4} \left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)} \right) \\ &= \frac{1}{4} \left(1 + \beta_1 + \frac{1}{2}\beta_2 \right). \end{aligned} \quad (5.10)$$

Thus

$$\beta_1 = \frac{12}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) - \frac{4}{3} \quad (5.11)$$

and

$$\begin{aligned} \frac{1}{2}\beta_2 &= \frac{23}{45} - \frac{8}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) \\ &\quad + \frac{32}{\pi^4} \left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)} \right). \end{aligned} \quad (5.12)$$

In order for the inequality:

$$|c_3 - 2\gamma_1 c_1 c_2 + \gamma_2 c_1^3| \leq 2$$

of Lemma 4 to be applicable here, we must have $|\beta_1| \leq 2$ and $|\beta_2| \leq 2$. The condition $|\beta_1| \leq 2$ simplifies to

$$\rho_2 \leq \lambda \leq 1, \quad (5.13)$$

where ρ_2 is given by (5.6). Using the estimates:

$$\frac{2-\lambda}{3-\lambda} \geq \frac{1}{2} \quad \text{and} \quad \frac{2-\lambda}{4-\lambda} \leq \frac{1}{2} \quad (0 \leq \lambda \leq 1),$$

it can be verified that $\beta_2 > 0$. Thus, in order to show that $|\beta_2| \leq 2$, it is sufficient to establish that

$$-\frac{4}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) + \frac{16}{\pi^4} \left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)} \right) \leq \frac{11}{45}. \quad (5.14)$$

The left-hand side of the inequality (5.14) can be rewritten as

$$\frac{4}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) \left[\frac{4}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) - 1 \right] + \left(\frac{4}{\pi^2} \cdot \frac{10(2-\lambda)}{4-\lambda} \right)^2 \left(\frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)} \right),$$

which, for the values of λ given by (5.12), is less than or equal to

$$\frac{10}{81} + \left(\frac{10}{9} + \frac{4}{\pi^2} \right)^2 \frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)}.$$

Now it is sufficient to show that

$$\frac{10}{81} + \left(\frac{10}{9} + \frac{4}{\pi^2} \right)^2 \frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)} \leq \frac{11}{45}.$$

This is equivalent to

$$\left[49\pi^2 + 4(5\pi^2 + 18)^2 \right] \lambda^2 - \left[245\pi^4 + 6(5\pi^4 + 18)^2 \right] \lambda + 294\pi^4 \geq 0,$$

which is true for all real values of λ . Thus $|\beta_2| \leq 2$ for the values of λ given by (5.14).

Finally, by applying (4.16), we arrive at (5.5).

In order to establish that our estimates are sharp, we need to find a function f corresponding to (4.13) with β_1 and β_2 given as above. One such function is

$$f(z) = \left\{ \frac{6}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) - \frac{2}{3} \right\} \frac{1+z}{1-z} - \left\{ \frac{6}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) - \frac{5}{3} \right\} \frac{1+\kappa z^2}{1-\kappa z^2}, \quad (5.15)$$

where

$$\kappa = \frac{\frac{53}{45} - \frac{14}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right) + \frac{32}{\pi^4} \left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)} \right)}{\frac{5}{3} - \frac{6}{\pi^2} \left(\frac{10(2-\lambda)}{4-\lambda} - 1 \right)}. \quad (5.16)$$

This completes the proof of Theorem 9.

Remark 6. The choice $\lambda = 1$ in Theorem 9 gives a recent result of Ma and Minda [9, p. 283, Theorem 3].

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