BOUNDS FOR THE DOMINATION NUMBER OF GRID GRAPHS

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ABSTRACT

It is shown that γ_n, the domination number of the 2-dimensional grid graph satisfies $\frac{1}{5}(n^2+n-3) \le \gamma_n \le \frac{1}{5}(n^2+4n-2)$.

1. Introduction

The 2-dimensional $n \times n$ grid graph G_n has vertex set $V = \{(i,j) \mid i,j \in 1,\ldots,n\}$ and vertices (i,j) and (i',j') are adjacent if they are consecutive on a row or column, i.e., if $i = i'$ and $j = j' + 1$ or or $j = j'$ and $i = i' + 1$.

Grid graphs clearly model city street layouts and more recently they have emerged as one of the most frequently studied models of processor connections in multiprocessor systems, hence the importance of their graph-theoretic properties. Farley and Hedetniemi [ ], Ko [ ], and G.F. Peck [ ] have studied the problem of fast transmission of information through grid graphs. In this paper we give upper and lower bounds for γ_n the domination number of the graph G_n, i.e., the minimum number of vertices in a set D such that each vertex of V - D is adjacent to at least one vertex of D.

2. An Upper Bound for γ_n

Let Z be the set of integers and G_∞ be the infinite grid graph with vertex set $Z \times Z$ and vertices adjacent if they are consecutive on a row or column. We define a dominating set D_∞ of G_∞ as follows (see Fig. 1, in which vertices of D_∞ are solid circles).

Let $S = \{(2,1), (4,2), (1,3), (3,4), (5,5)\}$

$T = \{(i,j+5k) : (i,j) \in S \text{ and } k \in Z\}$

and $D_\infty = \{(i+5k,j) : (i,j) \in T \text{ and } k \in Z\}$.

The set D_∞ is a very efficient dominating set in that each vertex is dominated exactly once.

For the remainder of this section G_n will be considered as the induced subgraph of G_∞ whose vertex set is the set of vertices of G_∞ in the rows and columns labelled 1 to n (G_7 is thus depicted within the dotted square of Fig. 1). The vertex set $E_n = D_\infty \cap V(G_n)$ does not dominate G_n, but E_n may be extended to a dominating set (not necessarily of smallest size) of G_n, by adding vertices which are either in the first or last row or column of G_n. (These vertices have open circles around them for the graph G_7 of Fig. 1).

A dominating set of G_n of this type is called *-dominating. We suspect that one can get close to the minimum size dominating set for G_n by such a construction. Let γ_n^* be the smallest cardinality of a *-dominating set of G_n.

Theorem 1 For all $n \geq 2$, $\gamma_{n+5}^* \leq \gamma_n^* + 2n + 9$. (1)

Proof We show how to augment a *-dominating of G_n to form a *-dominating set of G_{n+5}.

For ease of description, we consider the vertex set of G_∞ as an array A of 0's and 1's, where the vertex (i,j) is assigned a 1 if and only if $(i,j) \in D_\infty$. Let A_n be the array of 0's and 1's corresponding to the vertex set of G_n, where vertex (i,j) is 1 if and only if (i,j) is in D_n, a *-dominating set of G_n. We insert 5 additional rows (columns) between the first and second rows (columns) of A_n to form a new array B_{n+5}. We refer to Fig. 2, in which the original array A_n is now split into the four corner portions, Y is a $5 \times (n-1)$ subarray of A with consecutive rows and cols. whose first column is (0,0,1,0,0) and Z is an $(n-1) \times 5$ subarray of A whose first row is (0,0,0,0,1). The $(n+5) \times 5$ $(5\times(n+5))$ array consisting of cols. (rows) 2, ..., 6 of B_{n+5} has precisely one 1 per row (col.). Hence the number of 1's of B_{n+5} is equal to

$$|D_n| + 2(n+5) - 5. \tag{2}$$

By construction B_{n+5} contains all the 1's corresponding to E_{n+5}. Also since D_n is *-dominating, the set of vertices having 1's in B_{n+5} dominates all vertices of G_{n+5} except the fourth vertices in W and X and (possibly) some vertices among the first seven of the last row or last column. We show that one extra vertex (i.e., extra 1) in the last row is sufficient to ensure that the first seven entries in this row are dominated. This is obvious if both (n+5, 1) and (n+5, 7) are dominated by 1's of B_{n+5}, since there is already one 1 among the remaining five vertices. Suppose (n+5, 1) is not dominated. It follows that (n+5, 2) is zero and (n+5, 7) is a 1, otherwise the vertex which is now at (n+5, 1), would not have been dominated in G_n. The seven vertices may be dominated by changing vertex (n+5, 2) from a zero to a 1. To see this, we consider (n+5, y) the location of the 1 in the last row of Z and use the fact that B_{n+5} contains the 1's of E_{n+5}. If $y = 4$ or 5 then $\{(n+5, y), (n+5, 2), (n+5, 7)\}$ dominates the seven vertices. If $y = 3$, then (n+4, 5) is also 1 and the set $\{(n+5, 3), (n+5, 2), (n+5, 7), (n+4, 5)\}$ dominates the set. The remaining case $y = 6$ is impossible, since this would imply (n+5, 1) is 1 and hence dominated, contrary to assumption. The argument for the case (n+5, 7) undominated, is similar. Thus one more vertex suffices to dominate the first seven vertices of the last row. A similar proof holds for the first seven vertices of the last column.

From the above we see that four vertices may be added to B_{n+5} to form a *-dominating set of G_{n+5}. From (2), there is a *-dominating set of G_{n+5} of size $|D_n| + (2n+9)$. This completes the proof of Theorem 1.

Corollary 1 For all $n \geq 7$,

$$\gamma_n \leq \frac{n^2 + 4n - 2}{5} .$$

Proof Suppose $n = 5q + \lambda$ where $\lambda \in \{2,\ldots,6\}$. The solution of the recurrence $u_{n+5} = u_n + 2n + 9$ is

$$u_n = q(2n-5q+4) + u_\lambda . \qquad (3)$$

From (1) and (3)

$$\gamma_n \leq \gamma_n^* \leq q(2n-5q+4) + \gamma_\lambda^* .$$

This gives $\gamma_n \leq \frac{n^2 + 4n}{5} + \left(\gamma_\lambda^* - \frac{\lambda^2}{5} - \frac{4\lambda}{5}\right)$. The bracketed constant is at most -2/5 in all five cases.

3. A Lower Bound for γ_n

Theorem 2 If $n \geq 4$, $\gamma_n \geq \frac{1}{5}(n^2+n-3)$.

Proof Let D be the subset of vertices of a minimum dominating set which are in rows or columns labelled 1, 2, n-1 or n. An element of D is called *deficient* if

1) it is in row or column labelled 1 or n,

or 2) its closed neighborhood intersects the closed neighborhood of some other vertex of D.

We obtain a lower bound on the number of deficient vertices in D′, the subset of vertices of D which are in the first two rows. Let f(k) be the number of deficient elements of D' which are in the first k columns. Clearly $f(j+1) \geq f(j)$ for $j = 1,\ldots,n-1$. We now show that $f(k+4) \geq f(k) + 1$ for all $k \in 3,\ldots,n-4$.

Let x be an element of D' in the j^{th} column, where $j \leq k$ and $(k-j)$ is minimum. We note that $j \in \{k-2, k-1, k\}$ for otherwise $(1, k-1)$ is not dominated. There are six short cases to consider.

Case 1. $x = (1, k-2)$. Since $(1,k)$ is dominated, $(1, k+1) \in D'$ and is deficient. Hence

$$f(k+1) \geqq f(k) + 1. \tag{4}$$

Case 2. $x = (1, k-1)$. Since $(1, k+1)$ is dominated, either one of $(1, k+1)$, $(1, k+2)$ is in D' in which case

$$f(k+2) \geq f(k) + 1, \tag{5}$$

or $(2, k+1) \in D'$. If the latter holds, since $(1, k+2)$ must be dominated, one of $(1, k+2)$, $(1, k+3)$, $(2, k+2)$ is in D'. The first two of these possibilities imply

$$f(k+3) \geq f(k) + 1. \tag{6}$$

Otherwise both $(2, k+1)$ and $(2, k+2)$ are in D' and are deficient and (4) holds.

Case 3. $x = (1,k)$. The arguments of Case 2 are used with k replaced by $k+1$ and we deduce

$$f(k+4) \geq f(k) + 1. \tag{7}$$

Case 4. $x = (2, k-2)$. Impossible since $(1, k-1)$ is not dominated.

Case 5. $x = (2, k-1)$. Since $(1,k)$ must be dominated, $(1, k+1) \in D'$ and (4) is true.

Case 6. $x = (2,k)$. Since $(1, k+1)$ must be dominated, one of $(1, k+1)$, $(1, k+2)$, $(2, k+1)$ is in D'. In each case either (4) or (5) is true.

Each of the inequalities (4), (5), (6), (7) imply

$$f(k+4) \geq f(k) + 1. \tag{8}$$

Similar arguments but with less cases show that (8) is also true for $k = 1,2$. Using (8) and $f(1) \geq 0$, $f(2), f(3), f(4) \geq 1$ we obtain $f(n) \geq \left\lfloor \frac{n+2}{4} \right\rfloor$.

Similarly there are at least $\left\lfloor \frac{n+2}{4} \right\rfloor$ deficients in the last two rows. Next, we show that there are at least $\left\lfloor \frac{n-4}{4} \right\rfloor$ extra deficients on the "left edge" of G_n i.e., in the vertex set $L = \{(i,j) : i \in \{3,\ldots,n-2\},\ j \in \{1,2\}\}$. Let $g(k)$ be the number of deficients in the first k rows of L. An argument identical to that which established (8), shows that

$$g(k+4) \geq g(k) + 1 \quad \text{for } k = 1, \ldots, n-6. \tag{9}$$

In this case it is easy to show that $g(4) \geq 1$ and of course $g(1), g(2), g(3) \geq 0$. Hence from (9) the number $g(n-4)$ of deficients in L is at least $\left\lfloor \frac{n-4}{4} \right\rfloor$.

Hence there are at least

$$2\left\lfloor \frac{n+2}{4} \right\rfloor + 2\left\lfloor \frac{n-4}{4} \right\rfloor \geq n-3 \tag{10}$$

deficients in D.

We claim that the union of closed neighborhoods of m deficients, contains at most $4m$ vertices. Let $Y(Z)$ be the set of deficients in row or column 1 or n (row or column 2 or $n-1$). Certainly

$$\left| \bigcup_{v \in Y} N[y] \right| \leq 4|Y|. \tag{11}$$

Let Z_1 be the subset of vertices of Z whose closed neighbourhood intersects the closed neighbourhood of some vertex of Y. The addition of a vertex of Z_1 adds at most 4 more vertices to the left hand side of (11) and we deduce

$$\left| \bigcup_{v \in Y \cup Z_1} N[y] \right| \leq 4\left(|Y| + |Z_1|\right). \tag{12}$$

We now consider the extra contribution that vertices of $Z-Z_1$ make to (12). It will be seen that either one vertex adds in at most 4 vertices to the union or a pair of these vertices contributes at most 8 extra vertices.

Let $z \in Z-Z_1$. Then for some $z' \in Z$, $N[z]$ intersects $N[z']$. If this intersection contained only 1 vertex (we refer to Fig. 3) then, since x is dominated and $z \notin Z_1$, we deduce $z' \in Z_1$. The set $N[z']$ is already counted in (12) and hence $N[z]$ adds at most 4 extra vertices to the union. Otherwise the intersection of $N[z]$ and $N[z']$ has 2 vertices (i.e. z' is adjacent to z). Either $z' \in Z_1$, in which case $N[z]$ contributes at most 4 vertices to the union or we can add in $N[z] \cup N[z']$ which contributes at most 8 extra vertices.

We can proceed in this way until all elements of $Z-Z_1$ have been added into (12). This establishes the claim.

The total number of vertices dominated by γ_n vertices, m of which are deficient, is at most $4m + 5(\gamma_n - m) = 5\gamma_n - m$. Hence

$$5\gamma_n - m \geq n^2,$$

Therefore

$$\gamma_n \geq \frac{n^2 + m}{5}$$

$$\geq \frac{n^2 + n - 3}{5} \quad \text{from (10).}$$

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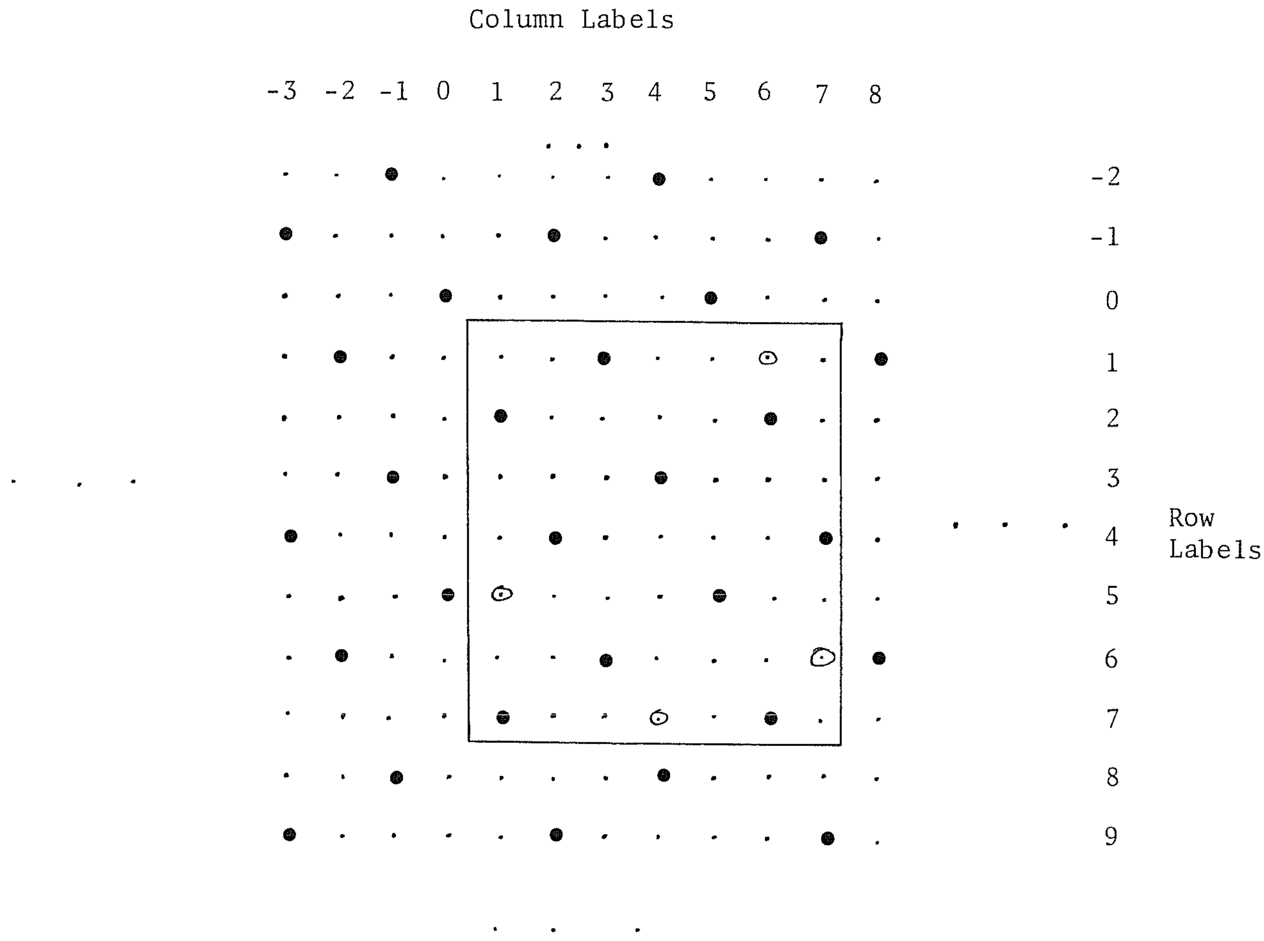

Figure 1. The graph G_∞ with dominating set D_∞ and a *-dominating set for G_7.

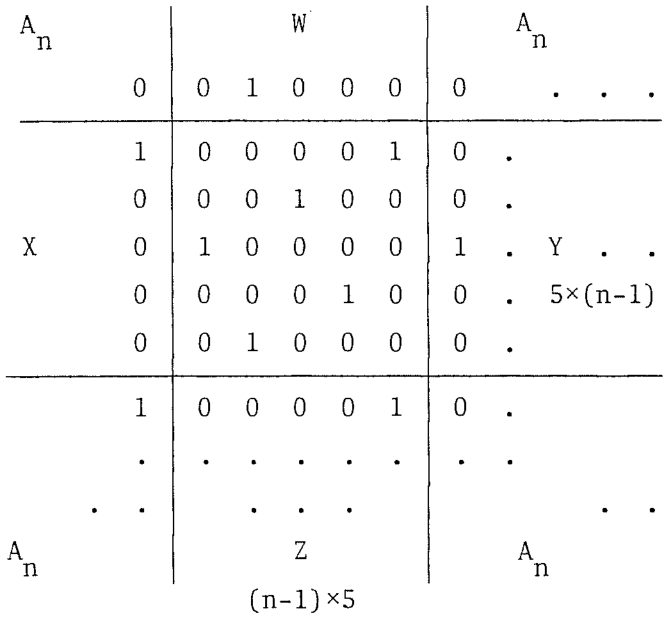

Figure 2. Construction of B_{n+5} from A_n.

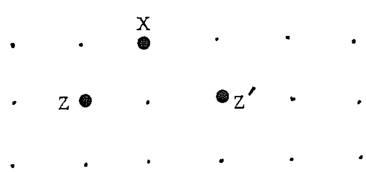

Figure 3. One point intersection of N[z], N[z′].