

Γ -Switchable 2-Colourings of (m, n) -Mixed Graphs

by

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B.Sc., Thompson Rivers University, 2018

A Thesis Submitted in Partial Fulfillment of the
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ABSTRACT

A (m, n) -mixed graph is a mixed graph whose edges are assigned one of m colours, and whose arcs are assigned one of n colours. Let G be a (m, n) -mixed graph and $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \dots, \gamma_n)$ be a $(n + 2)$ -tuple of permutations from $S_m \times S_n \times S_2^n$. We define *switching at a vertex v with respect to π* as follows. Replace each edge vw of colour ϕ by an edge vw of colour $\alpha(\phi)$, and each arc vx of colour ϕ by an arc $\gamma_\phi(vx)$ of colour $\beta(\phi)$.

In this thesis, we study the complexity of the question: “Given a (m, n) -mixed graph G , is there a sequence of switches at vertices of G with respect to the fixed group Γ so that the resulting (m, n) -mixed graph admits a homomorphism to some (m, n) -mixed graph on 2 vertices?”

We show the following: (1) When restricted to $(m, 0)$ -mixed graphs H on at most 2 vertices, the Γ -switchable homomorphism decision problem is solvable in polynomial time; (2) for each bipartite $(0, n)$ -mixed graph H , there is a bipartite $(2n, 0)$ -mixed graph such that the respective Γ -switchable homomorphism decision problems are polynomially equivalent; (3) For all (m, n) -mixed graphs and groups, when H has at most 2 vertices, the Γ -switchable homomorphism decision problem is polynomial time solvable; (4) For a yes-instance of the Γ -switchable homomorphism problem for $(m, 0)$ -mixed graphs, we can find in quadratic time a sequence of switches on G such that the resulting $(m, 0)$ -mixed graph admits a homomorphism to H .

By proving (1)-(4), we show that the Γ -switchable 2-colouring problem for (m, n) -mixed graphs is solvable in polynomial time for all finite permutation groups Γ and provide a step towards a dichotomy theorem for the complexity of the Γ -switchable homomorphism decision problem.

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Chapter 1

Introduction

1.1 Overview of Thesis

We use [7] for basic definitions and terminology in graph theory. In this thesis, we will assume all graphs are finite and simple.

We first state the problem we wish to solve. Definitions necessary to understand the statement will be introduced later in the chapter. Let k be a non-negative integer. The long term goal is to find a dichotomy theorem for the Γ -switchable k -colouring problem for all transitive permutation groups Γ . Such a theorem is known when Γ is Abelian, or belongs to some other classes of groups. We take a step towards this goal by showing that Γ -switchable 2-colouring of (m, n) -mixed graphs is solvable in polynomial time for all transitive permutation groups Γ .

A Γ -switchable k -colouring of a (m, n) -mixed graph G corresponds to a Γ -switchable homomorphism of G to a (m, n) -mixed graph H on k vertices. Thus, the problem of deciding whether a given (m, n) -mixed graph G has a Γ -switchable homomorphism to a fixed (m, n) -mixed graph H generalizes the Γ -switchable k -colouring problem. An even longer term goal is to determine the complexity of the Γ -switchable homomorphism decision problem for each (m, n) -mixed graph H .

In Chapter 1 of the thesis we trace the background of the problem we seek to solve, starting with vertex colouring and leading through various homomorphism problems. Definitions necessary to understand the problems are introduced along the way. In each case, we state the dichotomy theorems corresponding to the results we wish to prove, and its generalization. Our problem is introduced in Section 1.12. By the time the reader has reached this section, they should have an understanding of the necessary definitions and terminology, as well as the background and context of the problem. For the remainder of Chapter 1, we survey known results for our problem and its generalization for special classes of groups.

Chapter 2 begins with an illustrative example using the quaternions. The results in this section foreshadow the general theorems to be presented later. In the second section of Chapter 2 we outline the proof of the main result, which is then carried out in Sections 2.3-2.6. And finally, in Section 2.7 a related reconfiguration problem is considered.

1.2 Vertex Colouring

Vertex colouring is one of the oldest and most studied topics in graph theory.

Definition 1.2.1. *Given a graph G , and a positive integer k , a k -colouring of G is a function $c : V(G) \rightarrow \mathbb{Z}_k$ such that for $uv \in E(G)$, we have $c(u) \neq c(v)$.*

In a k -colouring of G , we assign k colours to the vertices of G such that adjacent vertices receive different colours. A graph is called *k -colourable* if there exists a k -colouring of G .

Let $k \geq 1$ be an integer. The decision problem k -COL is defined as follows:

k -COL

INPUT: A graph G .

QUESTION: Is G k -colourable?

There is a complete description of the complexity of the problem.

Theorem 1.2.2. [11] Let k be a fixed positive integer. If $k \leq 2$, then k -COL is solvable in polynomial time. If $k \geq 3$, then k -COL is NP-complete. \square

Theorem 1.2.2 is an interesting type of theorem known as a dichotomy theorem. For a family of decision problems that depend on a parameter P , a *dichotomy theorem* specifies the conditions on P under which the problem is solvable in polynomial time, and asserts that in all other cases the problem is NP-complete. In this case, the parameter is given as a threshold integer k . Determining if a graph is 2-colourable is easy, but determining if it is 3-colourable is hard.

What makes 2-COL solvable in polynomial time? One answer lies in the structure theorem below. The theorem itself makes it possible to describe an easily verifiable certificate that a graph is 2-colourable, and also an easily verifiable certificate that a graph is not 2-colourable. Further, the proof of the theorem uses a breadth first labelling of the graph to produce a 2-colouring or reveal an odd cycle.

Theorem 1.2.3. *A graph G is 2-colourable if and only if G contains no odd cycles.*

\square

1.3 Graph Homomorphisms

We begin with a fundamental definition.

Definition 1.3.1. *A homomorphism of a graph G to a graph H is a function $h : V(G) \rightarrow V(H)$ such that if uv is an edge of G , then $h(u)h(v)$ is an edge of H .*

An in-depth study of graph homomorphisms can be found in [15].

Suppose h is a homomorphism of G to H . If the vertices of H are regarded as colours, then by the definition of a homomorphism, the function h is an assignment of these colours to the vertices of G so that adjacent vertices receive adjacent colours. Since H is loop-free, the colours will be distinct. Conversely, if c is a k -colouring of G , and H is the graph obtained by identifying each set of vertices assigned the same colour, then c is a homomorphism of G to H . Thus an equivalent definition of a k -colouring of a graph G is a homomorphism of G to some graph H on k -vertices. Further, since whenever there is a homomorphism of G to H there is a homomorphism of G to any supergraph of H , another equivalent definition of a k -colouring of a graph G is a homomorphism of G to K_k .

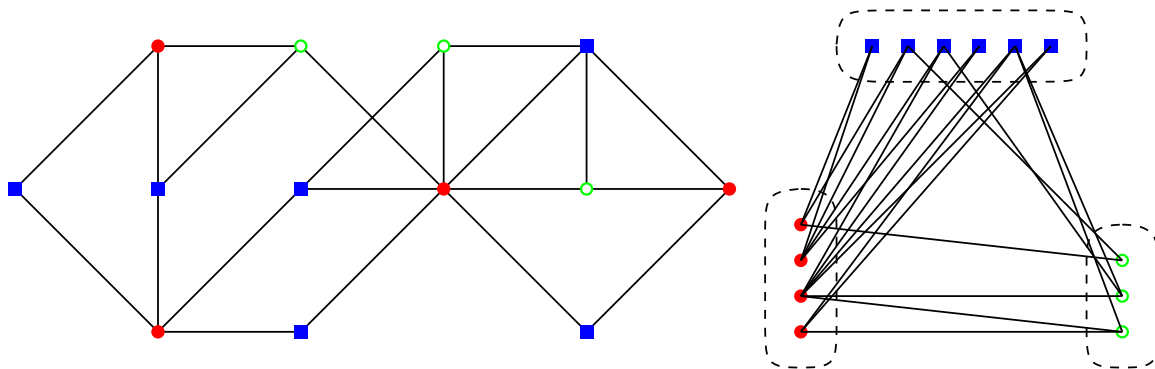


Figure 1.1: Illustration of an equivalence of vertex colourings and homomorphisms. Each partition is identified leaving a K_3 .

Let H be a fixed graph. The homomorphism decision problem is defined as follows:

HOM- H

INPUT: A graph G .

QUESTION: Does G admit a homomorphism to H ?

There is a complete description of the complexity of the problem.

Theorem 1.3.2. [14] If H is bipartite, then $\text{HOM-}H$ is solvable in polynomial time. If H has an odd cycle, then $\text{HOM-}H$ is NP-complete. \square

We note that $\text{HOM-}H$ is a more general problem than $k\text{-COL}$. Theorem 1.2.2 is a corollary of Theorem 1.3.2: take H to be K_k .

1.4 Oriented Graph Homomorphisms

We recall that an *oriented graph* is a directed graph in which there are no directed cycles of length at most 2. Such a directed graph can be obtained from a simple graph by assigning each edge an orientation (or direction). A survey of oriented graph homomorphisms is found in [29].

Definition 1.4.1. A homomorphism from an oriented graph G to an oriented graph H is a function $h : V(G) \rightarrow V(H)$ such that, if uv is an arc of G , then $h(u)h(v)$ is an arc of H .

Given an oriented graph G and a positive integer k , an *oriented k -colouring* of G is a homomorphism to some oriented graph H with k vertices. Similarly to the situation for undirected graphs, H can be assumed to be a tournament (the underlying graph is complete). We define the oriented k -colouring decision problem as follows:

ORIENTED k -COL

INPUT: An oriented graph G .

QUESTION: Is G k -colourable?

There is a complete description of the complexity of the problem.

Theorem 1.4.2. [19] Let k be a fixed positive integer. If $k \leq 3$, then ORIENTED

k -COL is solvable in polynomial time. If $k \geq 4$, then ORIENTED k -COL is NP-complete. \square

We have another structural theorem.

Theorem 1.4.3. *Let G be an oriented graph. Then G has an oriented 2-colouring if and only if G contains no directed path of length 2.* \square

This structure theorem makes it possible to describe an easily verifiable certificate that a graph is oriented 2-colourable, and also an easily verifiable certificate that a graph is not oriented 2-colourable. Further, the proof of the theorem uses a breadth first labelling of the graph to produce an oriented 2-colouring or reveal a directed path of length 2.

We also refer to the homomorphism decision problem for an oriented graph H as HOM- H . There is a dichotomy theorem, however it is not straightforward to describe. See Section 1.8 for a discussion.

1.5 2-Edge Coloured Graph Homomorphisms

We begin with a basic definition.

Definition 1.5.1. *A 2-edge coloured graph is a graph $G = (V, E)$ together with a function $c : E(G) \rightarrow \mathbb{Z}_2$ that assigns each edge one of the two colours 0 or 1.*

A 2-edge coloured graph can be obtained from a simple graph by assigning each edge a colour. Note that a vertex can be incident with many edges of the same colour.

The idea of a homomorphism extends to 2-edge coloured graphs.

Definition 1.5.2. *A homomorphism of a 2-edge coloured graph G to a 2-edge coloured graph H is a function $h : V(G) \rightarrow V(H)$ such if uv is an edge of G of colour ϕ , then $h(u)h(v)$ is an edge of H of colour ϕ .*

An in-depth study of homomorphisms of 2-edge coloured graphs and their complexity can be found in [2, 3].

Given a 2-edge coloured graph G , and a positive integer k , a k -colouring of G is a homomorphism to some 2-edge coloured graph H on k vertices. As before, the underlying graph of H can be assumed to be complete.

We define the 2-edge coloured k -colouring decision problem as follows:

2EC- k -COL

INPUT: An 2-edge coloured graph G .

QUESTION: Is G k -colourable?

There is a complete description of the complexity of the problem. It is an immediate consequence of Theorem 1.2.2.

Theorem 1.5.3. *Let k be a fixed positive integer. If $k \leq 2$, then 2EC- k -COL is solvable in polynomial time. If $k \geq 3$, then 2EC- k -COL is NP-complete. \square*

In particular, a 2-edge coloured graph is vertex 2-colourable if and only if it is bipartite (as the underlying graph must be 2-colourable) and monochromatic (all edges are of the same colour). Thus there are easily checkable certificates that there is a 2-colouring, or is not a 2-colouring. Further, a breadth first search can be used to either produce the 2-colouring, or a certificate that the given 2-edge coloured graph is not 2-colourable.

We refer to the homomorphism decision problem for a 2-edge coloured graph H as HOM- H . As before, there is a dichotomy theorem, however it is not straightforward to describe. See Section 1.8 for a discussion.

1.6 Mixed Graph Homomorphisms

We begin with a basic definition.

Definition 1.6.1. A mixed graph is an ordered triple $G = (V(G), E(G), A(G))$ where $V(G)$ is a set of objects called vertices, $E(G)$ is a set of unordered pairs of vertices called edges, and $A(G)$ is a set of ordered pairs of vertices called arcs.

A mixed graph can be viewed as arising from an undirected graph (called its *underlying graph*) by choosing an orientation for some of its edges. A mixed graph is called *simple* if its underlying graph is a simple graph. Recall that we consider only simple graphs, hence we only consider simple mixed graphs.

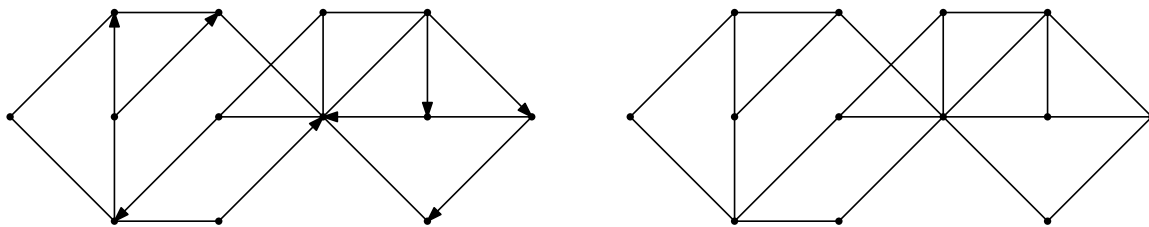


Figure 1.2: A mixed graph and its undirected underlying graph.

The definition of a homomorphism can be extended to mixed graphs.

Definition 1.6.2. A homomorphism of a mixed graph G to a mixed graph H is a function $h : V(G) \rightarrow V(H)$ such that, if uv is an edge of G , then $h(u)h(v)$ is an edge of H , and if wx is an arc of G , then $h(w)h(x)$ is an arc of H .

Given a mixed graph G and a positive integer k , a *mixed k -colouring* of G is a homomorphism to some mixed graph H with k vertices. Similarly to the situation for oriented graphs, the underlying graph of H can be assumed to be complete.

We define the mixed k -colouring decision problem as follows:

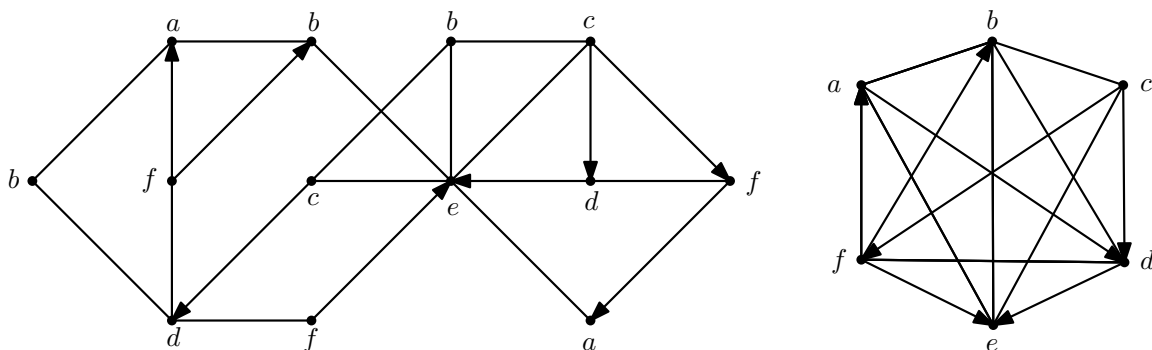


Figure 1.3: A homomorphism of the mixed graph from Figure 1.2 showing it to be 6-colourable.

MIXED k -COL

INPUT: A mixed graph G .

QUESTION: Is G mixed k -colourable?

There is a complete description of the complexity of the problem. It is a consequence of Theorem 1.2.2 and Theorem 1.4.2.

Theorem 1.6.3. *Let k be a fixed positive integer. If $k \leq 2$, then MIXED k -COL is solvable in polynomial time. If $k \geq 3$, then MIXED k -COL is NP-complete. \square*

We have a structural theorem for 2-colourings of mixed graphs. It is a consequence of Theorem 1.2.3 and Theorem 1.4.3.

Theorem 1.6.4. *Let G be a mixed graph. Then G has a 2-colouring if and only if $A(G) = \emptyset$ and G has no odd cycle, or $E(G) = \emptyset$ and G has no directed path of length 2. \square*

This structure theorem makes it possible to describe an easily verifiable certificate that a mixed graph is 2-colourable, and also an easily verifiable certificate that a mixed graph is not 2-colourable. Further, the proof of the theorem uses a breadth first labelling of the mixed graph to produce a 2-colouring or a certificate that no 2-colouring exists.

We also refer to the homomorphism decision problem for a mixed graph H as $\text{HOM-}H$. There is a dichotomy theorem, however it is not straightforward to describe. See Section 1.8 for a discussion.

1.7 (m, n) -Mixed Graphs

We now introduce one of the main definitions of the thesis.

Definition 1.7.1. *Let m and n be non-negative integers. A (m, n) -mixed graph is a mixed graph $G = (V(G), E(G), A(G))$ together with functions $c : E(G) \rightarrow \mathbb{Z}_m$ and $d : A(G) \rightarrow \mathbb{Z}_n$ that assign each edge one of m colours, and each arc one of n colours.*

Note that a vertex may be incident with several edges of the same colour, or several arcs of the same colour.

We refer to the mixed graph G in the definition as the *underlying mixed graph*. We also may refer to the *underlying graph* of a (m, n) -mixed graph; it is the underlying graph of G . In this thesis, we often will refer to the structural properties of a (m, n) -mixed graph interchangeably with those of its underlying graph. For example, a (m, n) -mixed graph G is bipartite if and only if its underlying graph is bipartite.

Mixed graphs were introduced by Nešetřil and Raspaud [26] as an attempt to unify the theory of colourings of 2-edge coloured graphs and the theory of colourings of oriented graphs. There are a number of results for which the statements and proofs are identical (for example, [1] vs [27], also [20]), but no direct translation seems possible because of examples in Sen's PhD thesis [28]. Also, compare Theorem 1.4.2 and Theorem 1.5.3.

In what follows we will be interested in referring to an edge or arc along with its colour. When the colour assigned to the edge or arc uv equals ϕ , we will say uv is of colour ϕ .

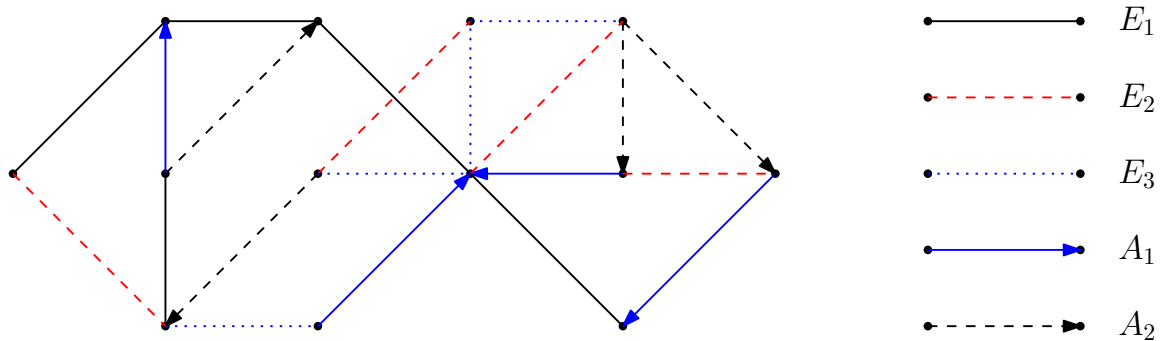


Figure 1.4: A (3,2)-mixed graph.

We refer to a $(m, 0)$ -mixed graph as a m -edge coloured graph, and we refer to a $(0, n)$ -mixed graph as a n -arc coloured graph. It is clear that a simple $(1, 0)$ -mixed graph is the same as a simple graph, and a simple $(0, 1)$ -mixed graph is the same as an oriented graph.

When G is a m -edge coloured graph and has all edges of colour ϕ , we say G is *monochromatic* of colour ϕ , and similarly for n -arc coloured graphs. We refer to a m -edge coloured graph or a n -arc coloured graph as monochromatic when there exists ϕ such that it is monochromatic of colour ϕ .

1.8 Homomorphisms of (m, n) -Mixed Graphs

We begin by introducing another one of the most important definitions in the thesis.

Definition 1.8.1. A homomorphism of a (m, n) -mixed graph G to a (m, n) -mixed graph H is a function $h : V(G) \rightarrow V(H)$ such that, if uv is an edge of G of colour ϕ , then $h(u)h(v)$ is an edge of H of colour ϕ , and if uv is an arc of G of colour ϕ , then $h(u)h(v)$ is an arc of H of colour ϕ .

Note that a homomorphism of a (m, n) -mixed graph G to a (m, n) -mixed graph H is also a homomorphism of the underlying mixed graph of G to the underlying mixed graph of H and similarly for the underlying graphs.

Homomorphisms of (m, n) -mixed graphs compose.

Proposition 1.8.2. *Let F, G and H be (m, n) -mixed graphs. Suppose f is a homomorphism of F to G and g is a homomorphism of G to H . Then, $g \circ f$ is a homomorphism of F to H . \square*

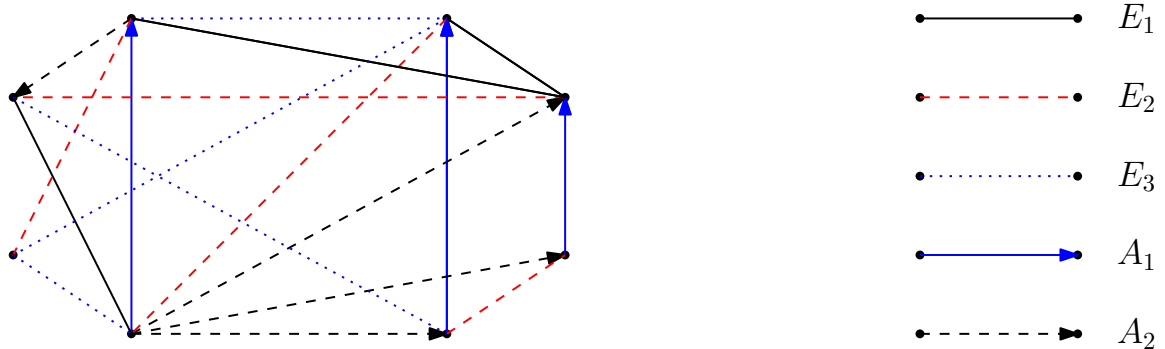


Figure 1.5: A $(3, 2)$ -mixed graph to which the $(3, 2)$ -mixed graph in Figure 1.4 has a homomorphism.

Given a (m, n) -mixed graph G and a positive integer k , a k -colouring of G is a homomorphism to some (m, n) -mixed graph H on k vertices. As before, the underlying graph of H can be assumed to be complete.

We note that if G is k -colourable, then the underlying graph of G is k -colourable. The converse does not hold in general. Similar comments apply to the underlying mixed graphs.

We define the (m, n) -mixed k -colouring decision problem as follows:

mn - k -COL

INPUT: An (m, n) -mixed graph G .

QUESTION: Is G k -colourable?

There is a complete description of the complexity of the problem. It is a consequence of Theorem 1.2.2 and Theorem 1.4.2.

Theorem 1.8.3. *Let k be a fixed positive integer. If $k \leq 2$, then mn - k -COL is solvable in polynomial time. If $k \geq 3$, then mn - k -COL is NP-complete. \square*

Again we have a structural theorem. It is a consequence of Theorem 1.2.3 and Theorem 1.4.3.

Theorem 1.8.4. *Let G be a (m, n) -mixed graph. Then G is 2-colourable if and only if it has no arcs, is bipartite, and is monochromatic or it has no edges, has no directed path of length 2, and is monochromatic. \square*

This structure theorem makes it possible to describe an easily verifiable certificate that a (m, n) -mixed graph is 2-colourable, and also an easily verifiable certificate that a (m, n) -mixed graph is not 2-colourable. Further, the proof of the theorem uses a breadth first labelling of the (m, n) -mixed graph to produce a 2-colouring or a certificate that no 2-colouring exists.

By definition, a (m, n) -mixed graph is 2-colourable if and only if it has a homomorphism to a (m, n) -mixed graph on 2 vertices. If there are neither edges nor arcs, then there is a homomorphism to the (m, n) -mixed graph with 2 vertices and neither edges nor arcs. Otherwise, a (m, n) -mixed graph is 2-colourable if and only if there is a homomorphism to a monochromatic K_2 or to a monochromatic T_2 (the unique tournament on 2 vertices).

We refer to the homomorphism decision problem for a (m, n) -mixed graph H as HOM- H . As before, there is a dichotomy theorem, however it is not straightforward to describe. We now explain why.

All of the homomorphism problems mentioned thus far are examples of *constraint satisfaction problems* (CSP). A *relational system* H is a set of vertices $V(H)$ together with relations R_1, R_2, \dots, R_k of arity s_1, s_2, \dots, s_k respectively on $V(H)$. Note a graph has a single symmetric binary relation, an oriented graph has a single irreflexive, antisymmetric binary relation, and a (m, n) -mixed graph has m symmetric

binary relations and n irreflexive, antisymmetric binary relations. If G and H are relational systems with the same number of relations with the same arities, then a homomorphism from G to H is again a function $h : V(G) \rightarrow V(H)$ that preserves the relations in the natural way. A constraint satisfaction problem is precisely any problem of the form $\text{HOM-}H$ for a fixed relation system H . In their landmark paper, Feder and Vardi [10] conjectured that every CSP is solvable in polynomial time or is NP-Complete. In 2017, Bulatov [6] and Zhuk [31] independently settled the conjecture in the positive.

There are several descriptions of the dichotomy, but all are equivalent to the existence of a *weak near unanimity function* (WNUF) for H , see MacGillivray and Swarts [24] and the references therein. There is no simple test to see if the fixed target H admits a WNUF. Thus, the general CSP dichotomy differs from those described above in that a simple description of the dichotomy (in terms of some easy to check property of H) is not known.

Of interest to us is the fact that Feder and Vardi showed that a dichotomy theorem for a restricted class of oriented graphs would imply a dichotomy for CSP. Hence a simple description of the dichotomy, even for oriented graphs, is unlikely. Using this result, in [3], the authors show every CSP is equivalent to a homomorphism problem on 2-edge coloured graphs, and thus a simple description of the dichotomy on 2-edge coloured graphs, is unlikely as well. On the other hand, once vertex switching (as defined below) is allowed, the dichotomy for 2-edge coloured graphs becomes simple to state; and appears similarly to the Hell-Nešetřil dichotomy for graphs [5].

1.9 Pushing Vertices in Oriented Graphs

We begin by defining an operation on oriented graphs.

Definition 1.9.1. *Let G be an oriented graph. The operation of pushing a vertex v transforms G to the oriented graph G' obtained from G by reversing the orientation of all arcs incident with v .*

An in-depth study of vertex pushing is found in [19]. In particular, two oriented graphs G and G' are *push equivalent* if there exists a sequence of vertex pushing operations that when applied to G , transforms G into an oriented graph isomorphic to G' . Clearly push equivalence is an equivalence relation on the set of all oriented graphs.

Definition 1.9.2. *Let G be an oriented graph and k be a positive integer. We say that G has a pushable k -colouring if there exists a sequence of vertex pushes which, when applied to G , results in an oriented graph which has an oriented k -colouring.*

Given an oriented graph G , and a positive integer k , G is *pushable k -colourable* if and only if G can be pushed so the resulting oriented graph has a homomorphism to some oriented graph H on k vertices. As before, the underlying graph of H can be assumed to be complete.

We define the pushable k -colouring decision problem as follows:

PUSH k -COL

INPUT: An oriented graph G .

QUESTION: Does G have a pushable k -colouring?

There is a complete description of the complexity of the problem.

Theorem 1.9.3. [19] Let k be a fixed positive integer. If $k \leq 2$, then PUSH k -COL is solvable in polynomial time. If $k \geq 3$, then PUSH k -COL is NP-complete. \square

We have structural theorems.

Theorem 1.9.4. [19] An oriented graph G has a pushable 2-colouring if and only if there exists a homomorphism of G to the directed cycle on 4 vertices. \square

Let G be an oriented graph. Let C be an oriented cycle in G . The *net length* of C equals the number of arcs whose tail precedes their head on C minus the number of arcs whose head precedes their tail on C .

Theorem 1.9.5. [23] Let G be an oriented graph. There exists a homomorphism of G to the directed cycle on 4 vertices if and only if the net length of every oriented cycle in G is a multiple of 4. \square

These structural theorems make it possible to describe an easily verifiable certificate that an oriented graph has a pushable 2-colouring, and also an easily verifiable certificate that an oriented graph does not have a pushable 2-colouring. Further, the proof of the theorem uses a breadth first labelling of the oriented graph to produce a 2-colouring or a certificate that no 2-colouring exists.

Definition 1.9.6. *Let G and H be oriented graphs. We say that G has a pushable homomorphism to H if there exists a sequence of vertex pushes which, when applied to G , results in an oriented graph which has a homomorphism to H .*

Let H be a fixed oriented graph. The pushable homomorphism decision problem is defined as follows:

PUSH HOM- H

INPUT: An oriented graph G .

QUESTION: Does G have a pushable homomorphism to H ?

There is a dichotomy theorem for PUSH HOM- H .

Theorem 1.9.7. [19] Let H be a fixed oriented graph. If H admits a homomorphism to the directed cycle on 4 vertices, then PUSH HOM- H is solvable in polynomial time. If H does not admit a homomorphism to the directed cycle on 4 vertices, then PUSH HOM- H is NP-complete. \square

1.10 Switching 2-Edge Coloured Graphs

We begin by defining an operation on 2-edge coloured graphs.

Definition 1.10.1. *Let G be a 2-edge coloured graph. The operation of switching at a vertex v transforms G to the 2-edge coloured graph G' , obtained from G , by replacing all edges of colour 0 incident with v with edges of colour 1 and all edges of colour 1 incident with v with edges of colour 0.*

An in-depth study of switching 2-edge coloured graphs is found in [3]. In particular, 2-edge coloured graphs G and G' are *switch equivalent* if there exists a sequence of vertex switching operations that when applied to G , transforms G into a 2-edge coloured graph isomorphic to G' . Clearly switch equivalence is an equivalence relation on the set of all 2-edge coloured graphs.

Definition 1.10.2. *Let G be a 2-edge coloured graph and k be a positive integer. We say that G has a switchable k -colouring if there exists a sequence of vertex switches which, when applied to G , results in a 2-edge coloured graph which has a 2-edge coloured k -colouring.*

We define the switchable k -colouring decision problem as follows:

SWITCH k -COL

INPUT: A 2-edge coloured graph G .

QUESTION: Does G have a switchable k -colouring?

There is a complete description of the complexity of the problem.

Theorem 1.10.3. [3] Let k be a fixed positive integer. If $k \leq 2$, then SWITCH k -COL is solvable in polynomial time. If $k \geq 3$, then SWITCH k -COL is NP-complete. \square

We have a structural theorem.

Theorem 1.10.4. [3] A 2-edge coloured graph G has a switchable 2-colouring if and only if there exists a homomorphism of G to the cycle on 4 vertices with alternating edge colours. \square

The structural theorem makes it possible to describe an easily verifiable certificate that a 2-edge coloured graph has a switchable 2-colouring, and also an easily verifiable certificate that a 2-edge coloured graph does not have a switchable 2-colouring. Further, the proof of the theorem uses a breadth first labelling of the 2-edge coloured graph to produce a 2-colouring or a certificate that no 2-colouring exists.

Definition 1.10.5. *Let G and H be 2-edge coloured graphs. We say that G has a switchable homomorphism to H if there exists a sequence of vertex switches which, when applied to G , results in a 2-edge coloured graph which has a homomorphism to H .*

Observe a 2-edge coloured graph G , is switchable k -colourable if and only if G can be switched so the resulting 2-edge coloured graph has a homomorphism to a 2-edge coloured graph H on k vertices. As before, the underlying graph of H can be assumed to be complete.

Let H be a fixed 2-edge coloured graph. The switchable homomorphism decision problem is defined as follows:

SWITCH HOM- H

INPUT: A 2-edge coloured graph G .

QUESTION: Does G have a switchable homomorphism to H ?

There is a dichotomy theorem for SWITCH HOM- H .

Theorem 1.10.6. [3] Let H be a fixed 2-edge coloured graph and C be a 4-cycle with alternating edge colours. If H has a homomorphism to C , then SWITCH HOM- H is solvable in polynomial time. If H does not have a homomorphism to C , then SWITCH HOM- H is NP-complete. \square

1.11 Switching in (m, n) -Mixed graphs

Let G be a (m, n) -mixed graph and let $\Gamma \leq S_m \times S_n \times S_2^n$ be a permutation group acting on the edge colours and arc colours and directions of G . We restrict to groups such that Γ permutes the edge colours of G , permutes the arc colours of G , and permutes the arc directions of G . That is, we do not allow Γ to move colours from edges to arcs or vice versa. As such, we can think of a permutation as an ordered $(n+2)$ -tuple $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \dots, \gamma_n)$ where α acts on the edge colours, β acts on the arc colours, and γ_ϕ acts on the arc direction of arcs of colour ϕ . Equivalently, we can regard π as an element $(\alpha, \beta, \gamma) \in S_m \times S_n \times S_2^n$. For the remainder of the thesis Γ will be assumed to be a permutation group as described above.

Definition 1.11.1. Let Γ be a group, G be a (m, n) -mixed graph, and $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \dots, \gamma_n) \in \Gamma$ be a permutation. We define $G^{(v, \pi)}$ as the (m, n) -mixed graph arising from

G by switching at a vertex v with respect to π as follows. Replace each edge vw of G of colour ϕ by an edge vw of G of colour $\alpha(\phi)$, and each arc vx of G of colour ϕ by an arc $\gamma_\phi(vx)$ of colour $\beta(\phi)$.

We take a moment to observe that vertex pushing corresponds to switching $(0, 1)$ -mixed graphs with respect to Γ that reverses the orientations of arcs (hence Γ is isomorphic to S_2). Further, switching in 2-edge coloured graphs corresponds to switching $(2, 0)$ -mixed graphs with respect to S_2 .

Example 1.11.2. Let G be the $(3, 0)$ -mixed graph and π be the permutation shown in Figure 1.6. When the edge colours at vertex u are switched with respect to π , the outside 4-cycle is unchanged, and the colours of the interior edges change.

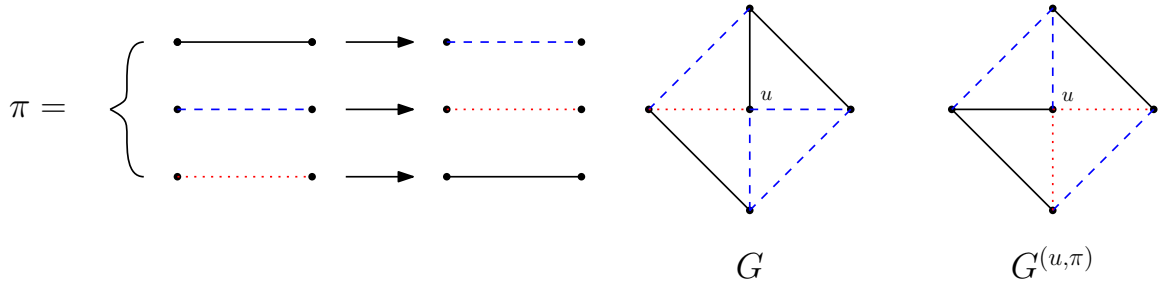


Figure 1.6: A (u, π) -switch.

Definition 1.11.3. Let G be a (m, n) -mixed graph and $\Sigma = (v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_1, \pi_1)$ be a sequence of ordered pairs from $V(G) \times \Gamma$. We define switching G with respect to the sequence Σ as follows:

$$G^\Sigma = (G)^{(v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_1, \pi_1)} = (G^{(v_1, \pi_1)})^{(v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_2, \pi_2)}$$

It is important to note that this is a sequence and not a set. The order of switches matters when Γ is not Abelian.

Example 1.11.4. Let G be a monochromatic 3-edge coloured graph of colour 0 for which the underlying graph is a path with 2 edges and centre vertex labelled v . Let Γ be

S_3 , and $\pi_1 = (01)$ and $\pi_2 = (12)$. Consider $\Sigma_1 = (v, \pi_1)(v, \pi_2)$ and $\Sigma_2 = (v, \pi_2)(v, \pi_1)$. Then G^{Σ_1} is monochromatic of colour 1 whereas G^{Σ_2} is monochromatic of colour 2.

We can also switch a set of vertices X of $V(G)$ with a particular permutation π . This operation is well defined since each edge or arc with an end in X is acted on by either π or π^2 . This is denoted as $G^{(X, \pi)}$.

We now state another important definition.

Definition 1.11.5. *Two (m, n) -mixed graphs G and H are switch equivalent with respect to a group Γ if there exists a sequence of switches Σ such that G^Σ is isomorphic to H .*

Let G and H be (m, n) -mixed graphs. If G is switch equivalent to H with respect to Γ , we write $G \cong_\Gamma H$.

Proposition 1.11.6. *Switch equivalence is an equivalence relation on the set of (m, n) -mixed graphs.*

Proof. To show \cong_Γ is reflexive, we observe that the identity element belongs to every group and so for a (m, n) -mixed graph G , $G \cong_\Gamma G$.

To show \cong_Γ is symmetric, we observe that every group is closed with respect to inverses, and so for (m, n) -mixed graphs G and G' , if there exists a sequence

$$\Sigma = (v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_1, \pi_1)$$

such that $G^\Sigma = G'$, then the sequence

$$\Sigma^{-1} = (v_1, \pi_1^{-1})(v_2, \pi_2^{-1}) \dots (v_k, \pi_k^{-1})$$

applied to G' results in $G'^{\Sigma^{-1}} = G$. And so if $G \cong_\Gamma G'$, then $G' \cong_\Gamma G$.

Lastly, to show \cong_Γ is transitive, consider three (m, n) -mixed graphs G, G', G'' , such that there exists a sequence

$$\Sigma_1 = (v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_1, \pi_1)$$

such that $G^{\Sigma_1} = G'$, and there exists a sequence

$$\Sigma_2 = (v'_k, \pi'_k)(v'_{k-1}, \pi'_{k-1}) \dots (v'_1, \pi'_1)$$

such that $G'^{\Sigma_2} = G''$. Then the sequence

$$\Sigma'' = \Sigma_2 \Sigma_1 = (v'_k, \pi'_k)(v'_{k-1}, \pi'_{k-1}) \dots (v'_1, \pi'_1)(v_k, \pi_k)(v_{k-1}, \pi_{k-1}) \dots (v_1, \pi_1)$$

applied to G gives $G^{\Sigma''} = G^{\Sigma_2 \Sigma_1} = G'^{\Sigma_2} = G''$. And so if $G \cong_\Gamma G'$ and $G' \cong_\Gamma G''$, then $G \cong_\Gamma G''$. \square

The switch equivalence class of H with respect to Γ is the set of all (m, n) -mixed graphs G such that $G \cong_\Gamma H$, and is denoted by $[H]_\Gamma$.

1.12 Γ -Switchable Homomorphism of (m, n) -Mixed Graphs

We begin by introducing another one of the most important definitions in the thesis.

Definition 1.12.1. *Let Γ be a permutation group. A (m, n) -mixed graph G has a Γ -switchable homomorphism to a (m, n) -mixed graph H if there exists sequence of switches Σ such that G^Σ has a homomorphism to H .*

We denote the existence of a Γ -switchable homomorphism from G to H with

respect to a group Γ as $G \rightarrow_{\Gamma} H$. When the name of the function is important, we may refer to a homomorphism $f : G \rightarrow_{\Gamma} H$.

Let G be a (m, n) -mixed graph, k be a positive integer, and Γ be a permutation group acting on the edge colours, arc colours, and arc directions of G . We say that G has a Γ -switchable k -colouring if there exists a sequence of switches Σ such that G^{Σ} is k -colourable. We define the Γ -switchable k -colouring decision problem as follows:

Γ -SWITCHABLE k -COL

INPUT: A (m, n) -mixed graph G .

QUESTION: Is G Γ -switchable k -colourable?

The main result of this thesis is a step toward proving the conjecture below.

Conjecture 1.12.2. *Let $k \geq 1$ be an integer. If $k \leq 2$, then Γ -SWITCHABLE k -COL is solvable in polynomial time. If $k \geq 3$ is an integer and Γ is any group, then Γ -SWITCHABLE k -COL is NP-complete.*

It is trivial to decide if a (m, n) -mixed graph is 1-colourable. Thus a natural next step in approaching the conjecture is to consider the case $k = 2$. Results are known when Γ belongs to certain families of groups. These are reviewed in the next section.

The Γ -switchable homomorphism problem is a generalization of Γ -switchable k -colouring.

Definition 1.12.3. *Let G and H be (m, n) -mixed graphs. We say that G has a Γ -switchable homomorphism to H if there exists a sequence of switches Σ such that G^{Σ} has a homomorphism to H .*

Definition 1.12.4. *Let G and H be m -edge coloured graphs. A homomorphism h of G to H is called complete if it maps $V(G)$ onto $V(H)$, and the induced mapping of*

$E(G)$ to $E(H)$ is also onto. If there is a complete homomorphism of G to H , then H is called a homomorphic image of G .

Definition 1.12.5. Let G and H be m -edge coloured graphs. Suppose h is a complete homomorphism of G to H . Then lifting a switch from H to G is obtained by the rule: if we switch $h(v)$ in H with π , then we switch v in G with π .

By the use of lifting, we observe homomorphisms of Γ -switchable (m, n) -mixed graphs compose.

Proposition 1.12.6. Let F, G and H be (m, n) -mixed graphs. Suppose f is a Γ -switchable homomorphism of F to G and g is a Γ -switchable homomorphism of G to H . Then, $g \circ f$ is a Γ -switchable homomorphism of F to H . \square

We formally define the Γ -switchable homomorphism decision problem. Let H be a (m, n) -mixed graph, and let Γ be a permutation group.

Γ -HOM- H

INPUT: A (m, n) -mixed graph G .

QUESTION: Does G have a Γ -switchable homomorphism to H ?

The complexity of Γ -HOM- H has been previously investigated for the same families of groups as Γ -switchable k -colouring. Results for this general problem will be discussed later in the thesis.

The main result in this thesis is that if H is a monochromatic K_2 or a monochromatic T_2 , then for any group Γ , the problem Γ -HOM- H is solvable in polynomial time. Further, we also describe an easily checkable certificate that the desired Γ -switchable homomorphism exists, or does not exist. Our results imply polynomial time algorithms for switching a (m, n) -mixed graph with respect to Γ so that the resulting

(m, n) -mixed graph has a homomorphism to a monochromatic K_2 or a monochromatic T_2 . Equivalently, our results imply a polynomial time algorithm for switching a bipartite (m, n) -mixed graph so that it is monochromatic, if this is possible. That is, we show that the Γ -switchable 2-colouring problem for (m, n) -mixed graphs is solvable in polynomial time for all finite permutation groups Γ .

1.13 Known Results

In this section we discuss the history of the 2-colouring problem for (m, n) -mixed graphs.

1.13.1 Γ is Abelian

In [21], the authors initiated the study of Γ -switchable homomorphisms of general (m, n) -mixed graphs. We review their results and the related literature.

As noted above, if a (m, n) -mixed graph G is 2-colourable, then it has edges or arcs, but not both. We first study (m, n) -mixed graphs with only edges, that is, m -edge coloured graphs.

We now introduce a construction that reduces the Γ -switchable homomorphism problem to a homomorphism problem. This construction generalizes the work of [4, 19].

Definition 1.13.1. *Let H be a m -edge coloured graph and Γ be an Abelian group acting on \mathbb{Z}_m . The switch graph of H with respect to Γ , is the m -edge coloured graph $P_\Gamma(H)$ such that: $V(P_\Gamma(H))$ equals $V(G) \times \Gamma$. And two vertices (u, α) and (v, β) are joined by an edge of colour i if u and v are joined in G by an edge of colour j , and $\alpha\beta(j) = i$.*

An example of a switch graph is shown in Figure 1.8

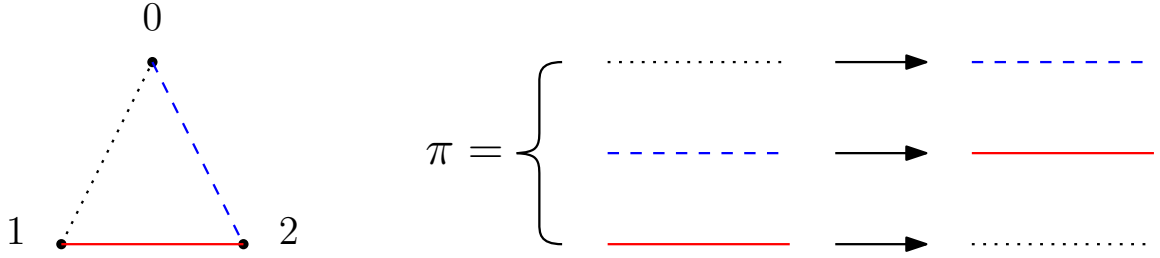


Figure 1.7: A 3-edge coloured graph H and a group $\Gamma = \langle \pi \rangle$.

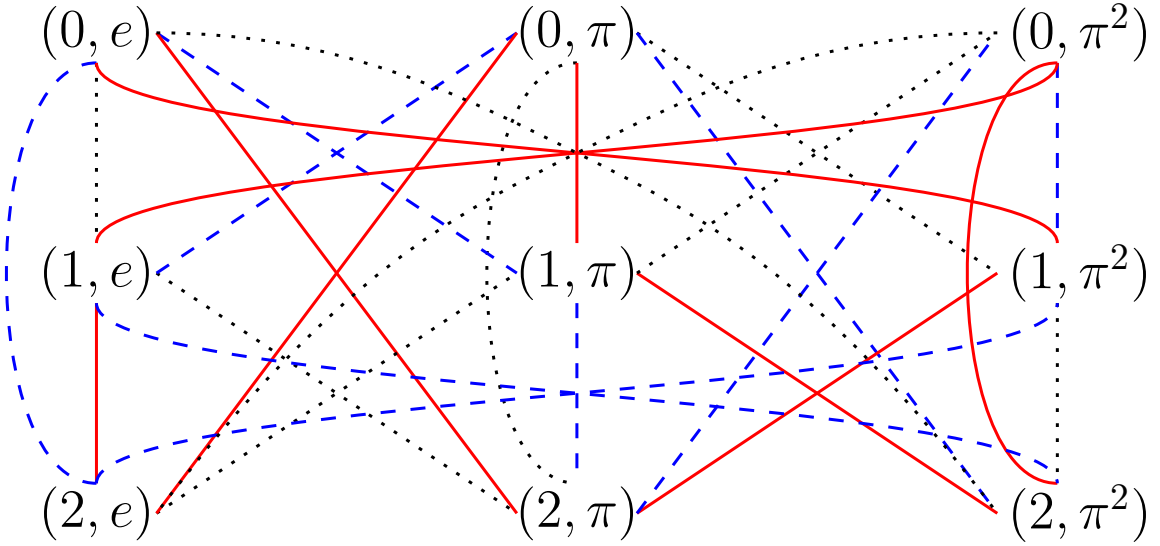


Figure 1.8: The switch graph $P_\Gamma(H)$ using H and Γ from Figure 1.7.

We make note of the following theorem.

Theorem 1.13.2. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 and Γ be an Abelian group acting on \mathbb{Z}_m . Suppose $G \rightarrow_\Gamma H$, then the sequence Σ which makes G monochromatic has length less than or equal to $|V(G)|$. That is, if $G \rightarrow_\Gamma H$, then we only need to switch at each vertex once.*

Proof. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 and Γ be an Abelian group acting on \mathbb{Z}_m . Suppose $G \rightarrow_\Gamma H$. Since Γ is a group, $G^{(v,\pi)(v,\pi')} = G^{(v,\pi\pi')}$. And since Γ is Abelian, $G^{(u,\sigma)(v,\pi)} = G^{(v,\pi)(u,\sigma)}$. Therefore, $G^{(v,\pi)(u,\sigma)(v,\pi')} = G^{(v,\pi\pi')(u,\sigma)}$. The result follows. \square

The following results describe the role of the switch graph in reducing switch

homomorphisms to homomorphisms.

Lemma 1.13.3. [4, 21] Let H be a m -edge coloured graph and Γ be a group acting on \mathbb{Z}_m . Then, $H \rightarrow P_\Gamma(H)$

Proof. Let $X = \{(u, e) | u \in V(H)\}$ be a subset of $V(P_\Gamma(H))$. Then the subgraph induced by X in $P_\Gamma(H)$ is isomorphic to H . So by the inclusion homomorphism, $H \rightarrow P_\Gamma(H)$. \square

Lemma 1.13.4. [4, 21] Let H be a m -edge coloured graph and Γ be a group acting on \mathbb{Z}_m . Then, $P_\Gamma(H) \rightarrow_\Gamma H$

Proof. Let H be a m -edge coloured graph and Γ be a group acting on \mathbb{Z}_m . For each $(u, \alpha) \in V(P_\Gamma(H))$ we perform the switch $P_\Gamma(H)^{((u, \alpha), \alpha^{-1})}$. We denote the resulting graph as $P'_\Gamma(H)$. Let h be the projection of $P'_\Gamma(H) \rightarrow H$ defined by $h(u, \alpha) = u$. We claim h is a homomorphism.

Let $(u, \alpha)(v, \beta)$ be an edge of $P'_\Gamma(H)$. Then uv is an edge of colour say ϕ in H . By definition, $(u, \alpha)(v, \beta)$ is of colour $\alpha\beta(\phi)$ in $P'_\Gamma(H)$. Thus after switching, $(u, \alpha)(v, \beta)$ is of colour $\beta^{-1}\alpha^{-1}\alpha\beta(\phi) = \phi$ in $P'_\Gamma(H)$, establishing the claim. Hence $P_\Gamma(H) \cong_\Gamma P'_\Gamma(H) \rightarrow H$, and thus $P_\Gamma(H) \rightarrow_\Gamma H$. \square

Theorem 1.13.5. [4, 21] Let G and H be m -edge coloured graphs and Γ be an Abelian group acting on \mathbb{Z}_m . Then, $G \rightarrow_\Gamma H$ if and only if $G \rightarrow P_\Gamma(H)$

Proof. Let G and H be m -edge coloured graphs and Γ be an Abelian group acting on \mathbb{Z}_m .

Suppose $G \rightarrow P_\Gamma(H)$. Then $G \rightarrow P_\Gamma(H) \rightarrow_\Gamma H$ by Lemma 1.13.4.

Now suppose $G \rightarrow_\Gamma H$. Let $h : G \rightarrow_\Gamma H$ be the Γ -switchable homomorphism mapping G to H . Then there exists a graph G' that is switch isomorphic to G and

such that $G' \rightarrow H$. Let uv be an edge of colour ϕ in G' . Then $h(u)h(v)$ is an edge of colour ϕ in H .

Let Σ be a sequence of switches on G such that $G^\Sigma = G'$. Since Γ is Abelian we need only switch once at each vertex. Let σ_x be the element of Γ performed by Σ at vertex x . We define a new function $g : G \rightarrow P_\Gamma(H)$ such that $g(v) = (h(v), \sigma_v^{-1})$. We claim that g is a homomorphism. Let uv be an edge of colour ϕ' in G . Then uv is of colour $\sigma_u \sigma_v(\phi') = \phi$ in G' . Thus, $h(u)h(v)$ is of colour ϕ in H . This gives, $g(u)g(v) = (h(u), \sigma_u^{-1})(h(v), \sigma_v^{-1})$ is an edge of colour $\sigma_u^{-1} \sigma_v^{-1}(\phi) = \phi'$ in $P_\Gamma(H)$. Therefore g is a homomorphism as desired. \square

Lemma 1.13.6. *Let Γ be an Abelian group and H be a monochromatic K_2 . Then the graph underlying $P_\Gamma(H)$ is isomorphic to $K_{|\Gamma|, |\Gamma|}$.*

Proof. Let Γ be an Abelian group. Suppose $H = K_2$ consists of the two vertices u and v joined by an edge of colour ϕ . Since H is loop-free, vertices (u, π) and (u, π') are not adjacent for all $\pi, \pi' \in \Gamma$. Therefore if $A = \{(v, \alpha) | \alpha \in \Gamma\}$ and $B = \{(u, \beta) | \beta \in \Gamma\}$, then (A, B) is a partition $V(P_\Gamma(H))$ into independent sets. By definition, the vertices $(u, \alpha) \in A$ and $(v, \beta) \in B$ are joined by an edge of colour $\alpha\beta(\phi)$. The result follows. \square

The following is an important observation about the switch graph. We recall that the action of a group Γ on a set X is called regular if for every $\phi, \phi' \in X$, there exists precisely one $\pi \in \Gamma$ such that $\pi(\phi) = \phi'$. In particular only the identity has a fixed point. We recall that if an Abelian group is transitive, then it is regular.

Proposition 1.13.7. *Let Γ be an Abelian group acting transitively on \mathbb{Z}_m and H be a m -edge coloured graph such that H is a monochromatic K_2 . Then $P_\Gamma(H)$ is properly edge coloured. Further, each colour class is a perfect matching of $P_\Gamma(H)$.*

Proof. Let Γ be an Abelian group. If two edges incident to a vertex have the same colour, then there exists distinct $\pi, \pi' \in \Gamma$ such that $\pi(\alpha) = \pi'(\alpha)$. Therefore $\alpha = \pi^{-1}\pi'(\alpha)$. Since Γ is regular, $\pi^{-1}\pi' = e_\Gamma$. Therefore $\pi = \pi'$, a contradiction. Since the group is transitive, each vertex is incident with exactly one edge of each colour. \square

Corollary 1.13.8. [21] Let Γ be an Abelian group acting on \mathbb{Z}_m , G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Then, $G \rightarrow_\Gamma H$ if and only if $G \rightarrow P_\Gamma(H)$. \square

Theorem 1.13.9. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 and Γ be an Abelian group acting on \mathbb{Z}_m . Then Γ -HOM- H is solvable in polynomial time.

Proof. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 and Γ be an Abelian group acting on \mathbb{Z}_m .

Using the switch graph $P_\Gamma(H)$ and Proposition 1.13.7 we can either map G to $P_\Gamma(H)$ or we arrive at a contradiction using the following algorithm: Let x, y be the vertices of H and suppose xy is of colour ϕ . Let v be a vertex of G . Map v to (x, e_Γ) in $P_\Gamma(H)$. Let u be a neighbour of v in G with uv of colour ϕ' . Then there is a unique $\alpha \in \Gamma$ such that $\alpha(\phi) = \phi'$. Map u to (y, α) . Continue in this fashion until either a homomorphism is constructed or a contradiction is reached. \square

Corollary 1.13.10. Let Γ be an Abelian group acting on \mathbb{Z}_m . Then Γ -SWITCHABLE 2-COL is solvable in polynomial time. \square

Proposition 1.13.11. Let H be a m -edge coloured graph and Γ be an Abelian group acting transitively on \mathbb{Z}_m . If the underlying graph of H contains an odd cycle, then Γ -HOM- H is NP-complete.

Proof. Let H be a m -edge coloured graph and Γ be an Abelian group acting transitively on \mathbb{Z}_m . Let C be an odd cycle in the underlying graph of H . Then H

can be switched with respect to Γ so that C is monochromatic of some colour ϕ . Therefore $P_\Gamma(H)$ contains an odd cycle which is monochromatic of colour ϕ . Let H' be the underlying graph of the subgraph of $P_\Gamma(H)$ induced by the edges of colour ϕ . The reduction is from $\text{HOM-}H'$. Since H' has an odd cycle, by [14] $\text{HOM-}H'$ is NP-Complete.

Let the graph G' be an instance of $\text{HOM-}H'$. The instance of $\Gamma\text{-HOM-}H$ is the m -edge coloured graph G which is monochromatic of colour ϕ and has underlying graph G' . The reduction can be accomplished in polynomial time. By Theorem 1.13.5, $G \rightarrow_\Gamma H$ if and only if $G \rightarrow P_\Gamma(H)$. Since G only has edges of colour ϕ , $G \rightarrow P_\Gamma(H)$ if and only if there is a homomorphism of G' to H' . This completes the proof. \square

Thus when Γ is an Abelian group acting transitively on \mathbb{Z}_m , finding a dichotomy theorem for $\Gamma\text{-HOM-}H$ comes down to determining the complexity of $\Gamma\text{-HOM-}H$ when H is 2-colourable. We note that such problems can be NP-complete. For example, if H is a 4-cycle with three edges of colour 0 and one edge of colour 1, then $S_2\text{-HOM-}H$ is NP-complete [3].

1.13.2 Γ has Property \mathcal{T}_j : Symmetric & Alternating Groups

In this section we summarize the results from [9] that pertain to switching with respect to symmetric, alternating, and related groups.

Definition 1.13.12. *Let Γ be a group acting on \mathbb{Z}_m . We say Γ has property $\mathcal{T}_{i,j}$ if there exist $i, j, k \in \mathbb{Z}_m$ and $\pi \in \Gamma$ such that $\pi(k) = k$ and $\pi(i) = j$.*

Theorem 1.13.13. [9] *Let G be a m -edge coloured graph and Γ be a group acting transitively on \mathbb{Z}_m . If Γ has property $\mathcal{T}_{i,j}$ and is transitive, then for an edge uv of colour i , there exists a sequence Σ such that uv is of colour j in G^Σ and $G \setminus uv = G^\Sigma \setminus uv$.*

Proof. Let G be a m -edge coloured graph and Γ be a group acting on \mathbb{Z}_m . Suppose Γ has property $\mathcal{T}_{i,j}$. Let uv be an edge of colour i in G . Since Γ has property $\mathcal{T}_{i,j}$, there exists $\pi_1, \pi_2 \in \Gamma$ such that $\pi_1(k) = k$, $\pi_1(i) = j$, and $\pi_2(j) = k$. Consider the sequence $\Sigma = (v, \pi_2^{-1}), (u, \pi_1^{-1}), (v, \pi_2), (u, \pi_1)$. Then edge uv in G^Σ is of colour

$$\pi_2^{-1}\pi_1^{-1}\pi_2\pi_1(i) = \pi_2^{-1}\pi_1^{-1}\pi_2(j) = \pi_2^{-1}\pi_1^{-1}(k) = \pi_2^{-1}(k) = j$$

Further, every edge of $G^\Sigma - uv$ is the same colour as in G . □

Definition 1.13.14. Let Γ be a group acting on \mathbb{Z}_m . We say Γ has property \mathcal{T}_j if Γ has property $\mathcal{T}_{i,j}$ for all $i \in \mathbb{Z}_m$.

Corollary 1.13.15. [9] Let G and H be m -edge coloured graphs and Γ be a group acting transitively on \mathbb{Z}_m . Suppose Γ has property \mathcal{T}_j for some $j \in \mathbb{Z}_m$. Then $G \cong_\Gamma H$ if and only if G and H have the same underlying graph. □

Corollary 1.13.16. [9] Let G and H be m -edge coloured graphs with underlying graphs G' and H' . Let Γ be a transitive group acting on \mathbb{Z}_m . If Γ has property \mathcal{T}_j for some $j \in \mathbb{Z}_m$, then $G \rightarrow_\Gamma H$ if and only if $G' \rightarrow H'$. □

Corollary 1.13.17. [9] Let G and H be m -edge coloured graphs where H and Γ be a group acting transitively on \mathbb{Z}_m with property \mathcal{T}_j for some $j \in \mathbb{Z}_m$. If H is bipartite, then Γ -HOM- H is solvable in polynomial time. If H is not bipartite, then Γ -HOM- H is NP-Complete. □

Proposition 1.13.18. For each $m \geq 3$ the symmetric group S_m and for $m \geq 4$ the alternating group A_m acting on \mathbb{Z}_m has property \mathcal{T}_j . □

We note that Corollary 1.13.17 applies to the symmetric and alternating groups. This is intuitive as for $m = 3$, S_3 contains all transpositions, and so property \mathcal{T}_j

holds. For $m \geq 4$, both A_m and S_m contain all cyclic permutations of 3 elements, and because $m \geq 4$ some element is always stabilized and so \mathcal{T}_j holds.

We also note that we do not have a complete characterization of which groups have property \mathcal{T}_j .

1.13.3 Dihedral Groups

For this section we assume; without loss of generality, that for $m \geq 3$, D_m is the group acting on \mathbb{Z}_m and is generated by the following two permutations:

$$r = (012 \dots (m-1))$$

$$s = \begin{cases} (1(m-1))(2(m-2)) \dots ((\frac{m}{2}-1)(\frac{m}{2}+1)) & \text{if } m \text{ is even} \\ (1(m-1))(2(m-2)) \dots ((\frac{m-1}{2})(\frac{m-1}{2}+1)) & \text{if } m \text{ is odd} \end{cases}$$

Theorem 1.13.19. [9] Let $m \geq 3$ be an odd integer. The dihedral group acting on \mathbb{Z}_m has property \mathcal{T}_j for all $j \in \mathbb{Z}_m$.

Proof. Let $m \geq 3$ be an odd integer and Γ be permutably isomorphic to D_m acting on \mathbb{Z}_m .

Let i, j be distinct elements of \mathbb{Z}_m . Then either the least residue of $(i-j) \pmod m$ or the least residue of $(j-i) \pmod m$ is an even integer. Let k be the midpoint of the even interval between i and j . Then $r^k s r^{-k}$ is a permutation where $r^k s r^{-k}(k) = k$ and $r^k s r^{-k}(i) = j$. Therefore $\mathcal{T}_{i,j}$ hold for all i, j . \square

The following is a consequence of Corollary 1.13.17.

Corollary 1.13.20. [9] Let $m \geq 3$ be an odd integer, k be a positive integer, and G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Then determining $G \rightarrow_{D_m} H$ is solvable in polynomial time. \square

Theorem 1.13.21. [9] Let $m \geq 3$ be an even integer, and i be an element of \mathbb{Z}_m . Then D_m has property $\mathcal{T}_{i,j}$ for all j such that $i - j \equiv 0 \pmod{2}$.

Proof. Let m be an even integer.

Let i, j be distinct elements of \mathbb{Z}_m such that $i - j \equiv 0 \pmod{m}$. Then either $(i - j)$ or $(j - i)$ is a positive even integer. Let k be the midpoint of the even interval between i and j . Then $r^k s r^{-k}$ is a permutation where $r^k s r^{-k}(k) = k$ and $r^k s r^{-k}(i) = j$. Therefore $\mathcal{T}_{i,j}$ holds for all i, j such that $i - j \equiv 0 \pmod{m}$. \square

Let $m \geq 2$ be an even integer. Let F be a m -edge coloured graph. Let F_2 be the 2-edge coloured graph obtained by replacing each edge of colour ϕ by an edge of colour ϕ' where $\phi' \in \mathbb{Z}_2$ and $\phi \equiv \phi' \pmod{2}$.

Theorem 1.13.22. [9] Let $m \geq 3$ be an integer and G and H be m -edge coloured graphs. Then $G \rightarrow_{D_m} H$ if and only if $G_2 \rightarrow_{S_2} H_2$.

Proof. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 .

Suppose $G \rightarrow_{\Gamma} H$. Then there exists a sequence Σ such that G^{Σ} is monochromatic. We extend the sequence Σ to G_2 such that for a switch (v, π) where π sends even elements of \mathbb{Z}_m to odd elements of \mathbb{Z}_m (and vice versa) in Σ , we apply the switch $(v, (01))$ in Σ' . Then $G_2^{\Sigma'}$ is monochromatic in \mathbb{Z}_2 .

Suppose $G_2 \rightarrow_{S_2} H_2$. Then there exists a sequence Σ such that G_2^{Σ} is monochromatic in \mathbb{Z}_2 . Since Γ has property $\mathcal{T}_{i,j}$ for all i, j such that $i - j \equiv 0 \pmod{2}$, there exists a sequence Σ' on G such that $G^{\Sigma'}$ has only edges of colours 0 and 1 and the parity of each edge colour is the same in G . We alter the sequence Σ such that for a switch $(v, (01))$ in Σ , we instead apply the switch (v, rs) . This switch swaps 0 with 1 and vice versa. Therefore $(G^{\Sigma'})^{\Sigma}$ is monochromatic in \mathbb{Z}_m . \square

The results of Brewster, Foucaud, Hell, and Naserasr [3] then imply the following.

Corollary 1.13.23. [9] Let $m \geq 2$ be an even integer. Let F and H be a m -edge coloured graphs, such that F is a monochromatic K_2 . If $H_2 \rightarrow_{S_2} F_2$, then D_m -HOM- H is solvable in polynomial time. If $H_2 \not\rightarrow_{D_m} F_2$, then D_m -HOM- H is NP-complete. \square

The condition $H_2 \rightarrow_{S_2} F_2$ is checkable in polynomial time by the results in [3]. We sketch the algorithm.

By Theorem 1.13.5 we know that $H_2 \rightarrow_{S_2} F_2$ if and only if $H_2 \rightarrow P_\Gamma(F_2)$. Here, $P_\Gamma(F_2)$ is a 2-edge-coloured 4-cycle whose edges alternate in the colours 0, 1. Since $P_\Gamma(F_2)$ is vertex-transitive and each vertex is incident with exactly one edge of each colour, choosing an image for some vertex of H_2 determines the image of each vertex of H_2 . If a unique image is determined for each vertex of H_2 , then $H_2 \rightarrow P_\Gamma(F_2)$, otherwise a contradiction is reached and $H_2 \not\rightarrow P(F_2)$.

Corollary 1.13.24. *Let $m \geq 2$ be an even integer, and let k be a positive integer. If $k \leq 2$, then D_m -switchable k -colouring is polynomial time decidable. If $k \geq 3$, then D_m -switchable k -colouring is NP-complete.* \square

Chapter 2

Switchable 2-Colouring

2.1 Γ is the Quaternions: A Tutorial by Way of Example

In this section we talk about the Γ -switchable 2-colouring problem for the quaternions. In a sense, this example foreshadows the development of the main results of the thesis.

Let $m = 8$ and $Q_8 \leq S_8$ be the permutation group defined by

$$\begin{aligned}
 Q_8 &= \{\pi_1 = (), \\
 &\pi_{-1} = (02)(13)(46)(57), \\
 &\pi_i = (0123)(4765), \\
 &\pi_{-i} = (0321)(4567), \\
 &\pi_j = (0426)(1537), \\
 &\pi_{-j} = (0624)(1735), \\
 &\pi_k = (0527)(1634), \\
 &\pi_{-k} = (0725)(1436)\}
 \end{aligned}$$

The group Q_8 is called the *quaternions*. This is an interesting group because it is the smallest Hamiltonian group; it is not Abelian and every subgroup is normal. The quaternions and D_4 are the smallest nilpotent groups. These are the two extraspecial groups of order $2^{1+2(1)}$.

An important difference between Q_8 and D_4 is that Q_8 admits no non-trivial stable elements on \mathbb{Z}_8 , so the technique used for the symmetric, alternating, and dihedral groups does not apply.

We now outline our strategy for solving the Q_8 -switchable 2-colouring problem. The first step is to show that the partition of the edge colours determined by the cosets of a special normal subgroup of Q_8 allows the problem to be reduced. We then consider the restriction of the problem to 8-edge-coloured even cycles, and characterize the situations where they are switch equivalent to a monochromatic even cycle with respect to Q_8 . The next step is to show that an 8-edge coloured bipartite graph G is switch equivalent to a monochromatic bipartite graph with respect to Q_8 if and only if every cycle in G can be made monochromatic by switching with respect to Q_8 , and stronger, if and only if every fundamental cycle with respect to some spanning tree can be made monochromatic by switching with respect to Q_8 . The last of these equivalences implies a polynomial time algorithm for the Q_8 -switchable 2-colouring problem, and also a polynomial time algorithm that either switches a given 8-edge coloured bipartite graph so that it is monochromatic (and hence a the Q_8 -switchable 2-colouring exists), or returns an easily checkable certificate that is not possible (and hence that no the Q_8 -switchable 2-colouring exists).

We will describe a solution to the 2-colouring problem which is similar to the solution for 2-edge coloured graphs given by Zaslavsky [30].

Theorem 2.1.1. [30] Let G and H be 2-edge coloured graphs. Suppose H is a monochromatic K_2 . Then, $G \rightarrow_{S_2} H$ if and only if every cycle C in G has an even

number of edges of colour 0 and an even number of edges of colour 1. \square

The concept of a block system for a group will allow us to reduce the problem.

Definition 2.1.2. *Let $\Gamma \leq S_m$ be a group. A block system $\Sigma = \{\Delta_1, \Delta_2, \dots, \Delta_k\}$ is a partition of \mathbb{Z}_m such that for $1 \leq i \leq k$ and every $\pi \in \Gamma$, either $\pi(\Delta_i) = \Delta_i$ or $\pi(\Delta_i) \cap \Delta_i = \emptyset$.*

Before we continue, we note that Q_8 is regular on \mathbb{Z}_8 . Therefore, $|Q_8| = |\mathbb{Z}_8| = 8$, and so there exists a bijection $b : \mathbb{Z}_8 \rightarrow Q_8$. Fixing such a bijection allows Q_8 to act on itself. In what follows, we use the following bijection.

$$b(\phi) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \pi_1 & \pi_i & \pi_{-1} & \pi_{-i} & \pi_j & \pi_k & \pi_{-j} & \pi_{-k} \end{pmatrix}$$

Theorem 2.1.3. *Let $\Gamma \leq S_m$ be a regular group and N be a normal subgroup of Γ . Then the cosets of N are a block system over Γ . \square*

For example, for $N = \{\pi_1, \pi_{-1}, \pi_i, \pi_{-i}\}$ the block system admitted on Γ is $\{\Delta_1 = \{\pi_1, \pi_{-1}, \pi_i, \pi_{-i}\}, \Delta_2 = \{\pi_j, \pi_{-j}, \pi_k, \pi_{-k}\}\}$, whereas for $N' = \{\pi_1, \pi_{-1}\}$ the block system admitted on Γ is $\{\Delta_1 = \{\pi_1, \pi_{-1}\}, \Delta_2 = \{\pi_i, \pi_{-i}\}, \Delta_3 = \{\pi_j, \pi_{-j}\}, \Delta_4 = \{\pi_k, \pi_{-k}\}\}$.

Since every subgroup of Q_8 is normal, there is a choice as to which subgroup we should attempt to use to reduce the problem.

We briefly state a definition and an observation that is critical.

Definition 2.1.4. *Let Γ be a group and $\alpha, \beta \in \Gamma$. The commutator of α and β is $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$.*

Definition 2.1.5. *Let Γ be a group. The commutator subgroup of Γ is the subgroup of Γ generated by the set of all the commutators of Γ and is denoted as $[\Gamma, \Gamma]$.*

We state a well known theorem about the commutator subgroup.

Theorem 2.1.6. *Let Γ be a group and N be a normal subgroup of Γ . Then Γ/N is Abelian if and only if N contains the commutator subgroup of Γ . \square*

The commutator subgroup of Q_8 is $\{\pi_1, \pi_{-1}\}$. We will discuss the commutator subgroup in more detail in Section 2.5.5.

In an effort to determine when a bipartite 8-edge coloured graph is switch equivalent with respect to Q_8 to a monochromatic graph, we begin by looking at cycles.

Definition 2.1.7. *Let $C = v_1, v_2, \dots, v_{2k}, v_1$ be an 8-edge coloured even cycle. If xy is an edge of C and b is the bijection from above, for simplicity of notation we write $b(xy)$ rather than $b(c(xy))$ to denote the colour of xy . We define the forward product of C with respect to v_i as*

$$\vec{\Pi}(C, v_i) = b(v_i v_{i+1}) b(v_{i+1} v_{i+2}) \cdots b(v_{i-1} v_i),$$

and the backward product of C with respect to v_i as

$$\overleftarrow{\Pi}(C, v_i) = b(v_i v_{i-1}) b(v_{i-1} v_{i-2}) \cdots b(v_{i+1} v_i),$$

where subscripts are modulo $2k$.

Let $\pi \in Q_8$. If there exists some $\pi' \in Q_8$ such that $\pi' = \pi_1 \pi$ or $\pi' = \pi_{-1} \pi$, then we write $\pi' = \pi_{\pm 1} \pi$. We note that $\pi' = \pi_{\pm 1} \pi$ if and only if π, π' belong to the same coset of the subgroup $\{\pi_1, \pi_{-1}\}$. We also note that for any elements $\pi, \pi' \in Q_8$, $\pi \pi' = \pi' \pi$ or $\pi \pi' = \pi_{-1} \pi' \pi$. Using the notation described above, $\pi \pi' = \pi_{\pm 1} \pi' \pi$.

Lemma 2.1.8. *Let $C = v_1, v_2, \dots, v_{2k}, v_1$ be an 8-edge coloured even cycle. Let b be the bijection from above. Then $\vec{\Pi}(C, v_1) = \pi_{\pm 1} \overleftarrow{\Pi}(C, v_1)$.*

Proof. Let G be a 8-edge coloured graph, C be a cycle of length $2k$ in G . Then,

$$\begin{aligned}
\vec{\Pi}(C, v_1) &= b(v_1v_2)b(v_2v_3) \cdots b(v_{2k}v_1) \\
&= \pi_{\pm 1}b(v_2v_1)\pi_{\pm 1}b(v_3v_2) \cdots \pi_{\pm 1}b(v_1v_{2k}) \\
&= \pi_{\pm 1}b(v_2v_1)b(v_3v_2) \cdots b(v_1v_{2k}) \\
&= \pi_{\pm 1}(\pi_{\pm 1}b(v_1v_{2k}))b(v_2v_1)b(v_3v_2) \cdots b(v_{2k}v_{2k-1}) \\
&= \pi_{\pm 1}b(v_1v_{2k})b(v_2v_1)b(v_3v_2) \cdots b(v_{2k}v_{2k-1}) \\
&= \pi_{\pm 1}b(v_1v_{2k})(\pi_{\pm 1}b(v_{2k}v_{2k-1}))b(v_2v_1)b(v_3v_2) \cdots b(v_{2k-1}v_{2k-2}) \\
&= \pi_{\pm 1}b(v_1v_{2k})b(v_{2k}v_{2k-1})b(v_2v_1)b(v_3v_2) \cdots b(v_{2k-1}v_{2k-2}) \\
&= \dots \\
&= \pi_{\pm 1}(b(v_1v_{2k})b(v_{2k}v_{2k-1}) \cdots b(v_2v_1)) \\
&= \pi_{\pm 1} \overleftarrow{\Pi}(C, v_1)
\end{aligned}$$

□

Lemma 2.1.9. *Let $C = v_1, v_2, \dots, v_{2k}, v_1$ be an 8-edge coloured even cycle. Let b be the bijection from above. Then $\vec{\Pi}(C, v_i) = \pi_{\pm 1} \vec{\Pi}(C, v_j)$.*

Proof. The proof is very similar to the the proof of Lemma 2.1.8. □

The consequence of Lemma 2.1.8 and Lemma 2.1.9 is that the product of a cycle is invariant with respect to $\pi_{\pm 1}$ regardless of the starting vertex or cycle direction. Further, the product of a cycle is invariant with respect to $\pi_{\pm 1}$ with respect to switching. This is obvious as switching u with respect to π changes the weight in the product by $\pi^2 = \pi_{\pm 1}$. Since the product of cycles regardless of the starting vertex is invariant with respect to $\pi_{\pm 1}$, we write $\vec{\Pi}(C, \cdot)$ when the starting vertex is not specified.

The following theorems are similar to Theorem 2.1.1.

Theorem 2.1.10. *Let G and H be 8-edge coloured graphs. Suppose H is a monochromatic K_2 . Then, $G \rightarrow_{Q_8} H$ if and only if G is bipartite and for every cycle C in G , $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$.*

Proof. Suppose $G \rightarrow_{Q_8} H$. Then there exists a sequence Σ such that G^Σ is monochromatic of colour ϕ . Then for a cycle C in G^Σ , we have $\vec{\Pi}(C, \cdot) = \phi^{2k} = \pi_{\pm 1}$.

Now suppose for all cycles C in G , $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$. Let T be a spanning tree of G . We can switch T to be monochromatic of colour ϕ using a sequence of switches Σ . Consider edge $uv \in \bar{T}$ in G^Σ . Since the product of the cycle is invariant with $\pi_{\pm 1}$, the edge uv must be of colour $\pi_{\pm \phi}$. Let $\Sigma = (u, \pi_{-j})(v, \pi_{-i})(u, \pi_j)(v, \pi_i)$ be a sequence of switches. Then in G^Σ , the edge uv is of colour $\pi_{-1}\pi_{\pm \phi}$ and every other edge remains the same. \square

Theorem 2.1.11. *Let G and H be 8-edge coloured graphs and $\mathcal{C}(G)$ be the set of all cycles of G . Suppose H is a monochromatic K_2 . Then, $G \rightarrow_{Q_8} H$ if and only if for every cycle $C \in \mathcal{C}(G)$, there are an even number of edges from each coset of $\{\pi_1, \pi_{-1}\}$.*

Proof. By Theorem 2.1.10 we need only verify that $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$ for all cycles in G . As the product of two elements from a coset is $\pi_{\pm 1}$, and $\pi\pi' = \pi'\pi\pi_{\pm 1}$, the result follows. \square

Theorem 2.1.12. *Let G and H be 8-edge coloured graphs where H is a monochromatic K_2 . Let T be a spanning tree of G and let b be the bijection from above. Then, every cycle C in G has $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$ if and only if when G is switched such that T is monochromatic of colour π_ϕ , the colour of every edge in \bar{T} is $\pi_{\pm \phi}$.*

Proof. Let G and H be 8-edge coloured graphs where H is a monochromatic K_2 , T be a spanning tree of G , and b be the bijection from above.

Suppose when G is switched such that T is monochromatic of colour π_ϕ , the colour of every edge in the cotree of T is of colour $\pi_{\pm \phi}$. Let C be an even cycle in G . Then

$$\vec{\Pi}(C, \cdot) = \pi_{\pm\phi}^{2k} = \pi_{\pm 1}^k = \pi_{\pm 1}.$$

Suppose every cycle $C = v_1, v_2, \dots, v_{2k}$ in G has $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$. Let T be a spanning tree of G , and Σ be the sequence of switches making T monochromatic π_ϕ . From the proof of Theorem 2.1.10 we know that the product of any cycle is invariant with respect to $\pi_{\pm 1}$ with respect to switching. Therefore the cotree edge corresponding to C is of colour $\pi_{\pm\phi}$. \square

It is of import to note that Theorem 2.1.12 gives a polynomial time algorithm for Q_8 -SWITCHABLE 2-COL. Further, the proof gives a polynomial time algorithm to find a sequence of switches required to switch a graph to be monochromatic.

The following theorem summarizes our work in this section.

Theorem 2.1.13. *Let G and H be 8-edge coloured graphs such that H is a monochromatic K_2 , T be a spanning tree of G , and Q_8 be a group acting on itself. Then the following are equivalent*

1. $G \rightarrow_{Q_8} H$;
2. For every cycle C in G , $\vec{\Pi}(C, \cdot) = \pi_{\pm 1}$;
3. For every cycle C in G , there are an even number of edges of colours from each coset of $\{\pi_1, \pi_{-1}\}$;
4. When G is switched so that T is monochromatic of colour π_ϕ , the colour every edge in the cotree of T is of colour $\pi_{\pm\phi}$.

\square

2.2 Overview of Main Result

We now outline our strategy for solving the Γ -switchable 2-colouring problem. We assume in what follows that the set of edge colours is \mathbb{Z}_m .

The first step is to consider the restriction of the problem to m -edge coloured even cycles, and characterize the situations where they are switch equivalent to a monochromatic even cycle with respect to Γ . Since any even cycle of length $2k$ can be switched so that it is *nearly monochromatic of colours ϕ and ϕ'* (has $2k - 1$ edges of colour ϕ and 1 edge of colour ϕ'), we are led to the study of which nearly monochromatic even cycles can be transformed into a monochromatic even cycle by switching.

An edge colour ϕ' is said to *agree* with an edge of colour ϕ if a nearly monochromatic even cycle with all edges of colour ϕ , except for one edge of colour ϕ' can be transformed to be monochromatic of colour ϕ by switching. Agreeance in the sense just described is an equivalence relation on the set of edge colours. We then show the so-called agreeance classes are independent of cycle length.

The next step is to show that a m -edge coloured bipartite graph G is switch equivalent to a monochromatic bipartite graph with respect to Γ if and only if every cycle in G can be made monochromatic by switching with respect to Γ , and stronger if and only if every fundamental cycle with respect to some spanning tree can be made monochromatic by switching with respect to Γ . The proof of the last of these equivalences implies a polynomial time algorithm for the Γ -switchable 2-colouring problem, and also a polynomial time algorithm that either switches a given m -edge coloured bipartite graph so that it is monochromatic (and hence a the Γ -switchable 2-colouring exists), or returns an easily checkable certificate that is not possible (and hence no Γ -switchable 2-colouring exists).

The last step for m -edge coloured graphs is to then determine the exact solution to the so-called agreeance classes for all groups. To complete our work on (m, n) -mixed graphs we demonstrate a reduction of the Γ -switchable 2-colouring problem from n -arc coloured graphs to $2n$ -edge coloured graphs, thus solving the Γ -switchable

2-colouring problem for (m, n) -mixed graphs.

In the remainder of this section, we make some observations that are used throughout the remainder of the thesis.

If the group Γ is not transitive on \mathbb{Z}_m , then G is not switchably homomorphic to a monochromatic K_2 when elements from different orbits appear on edges of G . For cases when Γ is not transitive and elements of separate orbits do not appear, we can restrict both the set of colours and Γ to the collection of reachable colours. We therefore need only consider groups that are transitive on the set of colours. Thus, for the remainder of this thesis, we assume all groups Γ are transitive on \mathbb{Z}_m .

The following lemma allows us to restrict our attention to bipartite (m, n) -mixed graphs.

Lemma 2.2.1. *Let Γ be a group acting on \mathbb{Z}_m and G and H be m -edge coloured graphs such that G is an odd cycle and H is a monochromatic K_2 . Then G is not switchably homomorphic to H .*

Proof. The result follows as the underlying graph of G is not homomorphic to the underlying graph of H . □

2.3 m -Edge Coloured Cycles

We begin by characterizing the situations in which a m -edge coloured even cycle is switch equivalent to a monochromatic even cycle with respect to switching by a permutation group $\Gamma \leq S_m$.

Definition 2.3.1. *An m -edge coloured cycle of length $2k$ is called nearly monochromatic of colours ϕ and ϕ' if it has $2k - 1$ edges of colour ϕ and one edge of colour ϕ' .*

Definition 2.3.2. Let G be a m -edge coloured C_{2k} , which is nearly monochromatic of colours ϕ and ϕ' , and Γ be a group acting on \mathbb{Z}_m . We define the relation \sim_{2k} on \mathbb{Z}_m by $\phi \sim_{2k} \phi'$ if and only if G is Γ -switchably equivalent to a monochromatic C_{2k} of colour ϕ .

Proposition 2.3.3. Let G be a m -edge coloured C_{2k} which is nearly monochromatic of colour ϕ and ϕ' . Let H be a monochromatic K_2 of colour ϕ . If $\phi \sim_{2k} \phi'$, then $G \rightarrow_{\Gamma} H$. \square

Lemma 2.3.4. Let G be a m -edge coloured graph and ϕ and ϕ' be elements of \mathbb{Z}_m . If G can be made monochromatic of colour ϕ , then it can be made monochromatic of colour ϕ' .

Proof. Let G be a m -edge coloured graph, ϕ, ϕ' be elements of \mathbb{Z}_m , and Γ be a group acting on the colours of G . Since Γ is transitive, there exists a permutation π such that $\pi(\phi) = \phi'$.

Since G is a bipartite graph, there exists a bipartition (A, B) of G . Then switching A with π acts on each edge in G exactly one time and $\pi(\phi) = \phi'$. Therefore the resulting graph $G^{(A, \pi)}$ is monochromatic ϕ' . \square

Lemma 2.3.5. The relation \sim_{2k} is an equivalence relation.

Proof. To prove \sim_{2k} is reflexive, we observe an empty sequence of switches transforms a monochromatic m -edge coloured graph to a monochromatic m -edge coloured graph.

To prove \sim_{2k} is symmetric, suppose $\phi' \sim_{2k} \phi$. let G be a m -edge coloured C_{2k} which is nearly monochromatic of colours ϕ and ϕ' . Suppose the vertex sequence of the cycle is labelled $v_1, v_2, \dots, v_{2n}, v_1$ with the colour of edge $v_{2n}v_1$ being ϕ' . Let $\pi(\phi') = \phi$ and $\Sigma = (v_{2k-1}, \pi)(v_{2k-3}, \pi) \dots (v_3, \pi)$. Then G^{Σ} has $2k - 1$ edges of colour ϕ and 1 edge, v_1v_2 , coloured ϕ' . By our supposition, there exists a sequence Σ' such

that $(G^\Sigma)^{\Sigma'}$ is monochromatic of colour ϕ . Therefore $G^{\Sigma'\Sigma}$ is a monochromatic cycle and $\phi \sim_{2k} \phi'$.

To prove \sim_{2k} is transitive, suppose $\phi \sim_{2k} \phi'$ and $\phi' \sim_{2k} \phi''$. Let G, G', G'' be m -edge coloured cycles of length $2k$. By the above notation, we let the vertices of each be $v_1, v_2, \dots, v_{2k}, v_1$.

There are $2k - 1$ edges of colour ϕ' in G with edge v_1v_{2k} of colour ϕ in G . Similarly there are $2k - 1$ edges of colour ϕ'' in G' with edge v_1v_{2k} of colour ϕ' in G' and $2k - 1$ edges of colour ϕ'' with edge v_1v_{2k} of colour ϕ in G'' . We shall show G'' can be switched to be monochromatic of colour ϕ'' .

By hypothesis, there is a sequence Σ' such that $G'^{\Sigma'}$ is monochromatic of colour ϕ'' . Applying Σ' to G'' will leave all the edges of colour ϕ'' unchanged (as in $G'^{\Sigma'}$) and the product of those switches of the form (v_1, π) or (v_{2k}, π) will switch v_1v_{2k} from colour ϕ to colour $\sigma(\phi)$ for some $\sigma \in \Gamma$. We observe by the fact that $G'^{\Sigma'}$ is monochromatic $\sigma(\phi') = \phi''$. Let Σ'' be the subsequence of Σ' consisting of the switches at v_1 or v_{2k} .

Now we alter Σ'' and apply to $G''^{\Sigma''}$ as follows. For each switch (v_1, π) in Σ'' , we apply (v_i, π^{-1}) to v_i for each odd i . Similarly, for (v_{2k}, π) in Σ'' , apply (v_i, π^{-1}) to v_i for each even i . The net effect is to apply σ^{-1} to each edge of $G''^{\Sigma''}$. Thus each edge of colour ϕ'' switches to ϕ' and the edge v_1v_{2k} of colour $\sigma(\phi)$ becomes colour ϕ . Thus we can switch G'' to be G . By hypothesis, G can be switched to be monochromatic of colour ϕ' . By Lemma 2.3.4, the resulting graph can be switched to be monochromatic of colour ϕ'' as required. \square

We denote the equivalence class with respect to \sim_{2k} by $[\phi]_{2k} = \{\phi' \mid \phi' \sim_{2k} \phi\}$.

Definition 2.3.6. *If $\phi' \in [\phi]_{2k}$ we say ϕ' agrees with ϕ or that ϕ' belongs to the agreeance class of ϕ .*

If a cycle C cannot be switched with respect to Γ to be monochromatic, we say that this cycle *disagrees with* Γ .

We now seek to show that the agreeance classes are independent of cycle length.

Lemma 2.3.7. *The agreeance class for C_4 is contained in the agreeance class for larger cycles. That is, $[\phi]_4 \subseteq [\phi]_{2k}$ for all $k \in \{2, 3, \dots\}$.*

Proof. Let G and H be m -edge coloured graphs such that G is a C_{2k} which is nearly monochromatic of colours ϕ and ϕ' , and H is a monochromatic K_2 of colour ϕ . Suppose $\phi' \in [\phi]_4$. Let G' be a m -edge coloured C_4 which is nearly monochromatic of colours ϕ and ϕ' . Then G has a homomorphism to G' . Since $\phi' \in [\phi]_4$, G' has a Γ -switchable homomorphism to H . And so $G \rightarrow G' \rightarrow_{\Gamma} H$. Therefore $\phi' \in [\phi]_{2k}$, for all $k \geq 2$. \square

Lemma 2.3.8. *The agreeance class for the large cycles is contained in the agreeance class for C_4 . That is, $[\phi]_4 \supseteq [\phi]_{2k}$ for all $k \in \{2, 3, \dots\}$.*

Proof. We prove Lemma 2.3.8 by way of construction using the following chain: $[\phi]_4 \supseteq [\phi]_6 \supseteq \dots \supseteq [\phi]_{2k}$.

Let G be the m -edge coloured graph with:

$$V(G) = \{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_k\} \text{ and}$$

$$E(G) = \{v_1u_1, v_1w_1, v_ku_k, v_kw_k\} \cup \{u_iu_{i+1}, w_iw_{i+1}, v_iv_{i+1} \mid i \in \{1, 2, \dots, k-1\}\}$$

Then G has 3 cycles, two of length $2k$, and one of length $2k+2$. Let the cycle with edge u_1u_2 of length $2k$ be called C_1 , the other length $2k$ cycle be C_2 , and the length $2k+2$ cycle be C_3 . Finally, let G be nearly monochromatic of colours ϕ and ϕ' , where u_1u_2 is of colour ϕ' . We note the edge u_1u_2 is contained C_1 and C_3 . Assume $\phi' \in [\phi]_{2k+2}$. Then there exists a sequence of switches Σ such that in G^{Σ} , the cycle C_3 is monochromatic ϕ . We note that v_1v_2 and $v_{k-1}v_k$ might not be of colour ϕ in G^{Σ} .

Let Σ' be the sequence obtained from Σ by reversing the order of the sequence, replacing each permutation with its inverse permutation and replacing all switches on vertices u_i with switches on w_i and vice versa. Then in $G^{\Sigma'\Sigma}$ we see that C_1 is monochromatic ϕ .

Therefore $[\phi]_{2k} \supseteq [\phi]_{2k+2}$. □

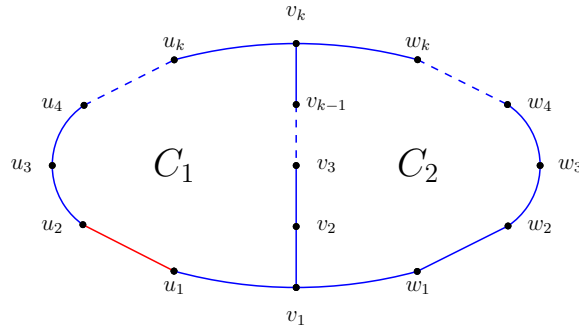


Figure 2.1: Illustration for Lemma 2.3.8.

Theorem 2.3.9. *Let Γ be a group and ϕ be an element of \mathbb{Z}_m , the agreeance class of ϕ is independent of cycle length. That is, $[\phi]_4 = [\phi]_{2k}$ for all $k \in \{2, 3, \dots\}$.*

Proof. The result follows immediately from Lemma 2.3.7 and Lemma 2.3.8. □

As we have collapsed all agreeance classes to a single class which depends only on the group, we henceforth denote this class as $[\phi]_\Gamma$. We note that; for a fixed m and Γ , $[\phi]_\Gamma$ can be computed in constant time as there is a finite number of m -edge coloured 4-cycles.

Theorem 2.3.10. *Let G and H be m -edge coloured graphs where G is C_{2k} and H is a monochromatic K_2 of colour ϕ . Then it can be determined in polynomial time whether there is a Γ -switchable homomorphism of G to H .*

Proof. Let the vertices of G be labelled $v_1, v_2, \dots, v_{2n}, v_1$. By switching at v_2, v_3, \dots, v_{2k} we can make G nearly monochromatic of colours ϕ and ϕ' , for some ϕ' which without

loss of generality is the colour of v_1v_{2k} after the switches. Then $G \rightarrow_\Gamma H$ if and only if $\phi' \in [\phi]_\Gamma$. Testing this condition can be done in constant time. \square

2.4 m -Edge Coloured Graphs

We begin the section by making some observations. Recall that by Lemma 2.2.1, it suffices to consider bipartite graphs.

Lemma 2.4.1. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 . If G is bipartite and has a monochromatic edge set, then $G \rightarrow_\Gamma H$ for every transitive group Γ .*

Proof. Let G be a bipartite m -edge coloured graph that is monochromatic of colour ϕ . Since G is bipartite there exists a bipartition (A, B) of G . Since Γ is transitive, there exists a $\pi \in \Gamma$ such that $\pi(\phi) = \phi'$ for all $\phi' \in \mathbb{Z}_m$. Then $G^{(A, \pi)}$ is monochromatic of colour ϕ' \square

An important class of bipartite graphs that are of vital import are trees.

Lemma 2.4.2. *Let T be a m -edge coloured tree and H be a m -edge coloured graph such that H is a monochromatic K_2 of colour ϕ . Then, for any transitive group Γ , $T \rightarrow_\Gamma H$.*

Proof. Let T be a m -edge coloured tree. Let v_1, v_2, \dots, v_n be a depth first search ordering of T rooted at v_1 . For each $i \in 2, \dots, n$, switch v_i so that the edge from v_i to its parent in the depth first search ordering has colour ϕ .

We observe that if the induced subtree $T[v_1, \dots, v_{i-1}]$ is monochromatic of colour ϕ , then after switching at v_i , so is the induced subtree $T[v_1, \dots, v_i]$. \square

Definition 2.4.3. Let G and H be m -edge coloured graphs such that H is a subgraph of G . A retraction from G to H , is a homomorphism $r : G \rightarrow H$ such that $r(x) = x$ for all $x \in V(H)$. If there exists a retraction of G to H , we call H a retract of G .

We shall use the following result of Hell [13].

Theorem 2.4.4. Let G be a bipartite graph. Suppose P is a shortest path from u to v in G . Then G admits a retraction to P . \square

In our attempt to show that the Γ -switchable 2-colouring problem is solvable in polynomial time, we seek to find a description of similar to Theorem 1.2.3 that guarantees a yes-instance.

Theorem 2.4.5. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Let $\mathcal{C}(G)$ be the set of all cycles of G . Then $G \rightarrow_{\Gamma} H$ if and only if for each cycle $C \in \mathcal{C}(G)$, $C \rightarrow_{\Gamma} H$.

Proof. Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 and $\mathcal{C}(G)$ be the set of all cycles of G .

Suppose $G \rightarrow_{\Gamma} H$, then there exists a sequence of switches Σ that transforms G so it becomes monochromatic, after which each cycle in $\mathcal{C}(G)$ is also monochromatic. Therefore for each $C \in \mathcal{C}(G)$, $C \rightarrow_{\Gamma} H$.

Now suppose that for all $C \in \mathcal{C}(G)$, $C \rightarrow_{\Gamma} H$. Suppose G has no cycles, then by Lemma 2.4.2, there exists a sequence of switches Σ that transforms G so it becomes monochromatic, and so $G \rightarrow_{\Gamma} H$. Now let us suppose the theorem holds for any m -edge coloured graph G' such that $|\mathcal{C}(G')| < k$ for $k \geq 1$.

Consider a m -edge coloured graph G such that $|\mathcal{C}(G)| = k$. Then there exists an edge uv in G that belongs to a cycle. Let $G' = G - uv$. By the inductive hypothesis, then there exists a sequence of switches Σ that transforms G' so it becomes monochromatic of colour ϕ . Let P be a shortest uv path in G' . Then by Theorem 2.4.4, P

is a retract of G' . Thus there is a Γ -switchable retraction $r : G' \rightarrow_\Gamma P$ where P is a monochromatic path of colour ϕ , $r(u) = u$, and $r(v) = v$. By applying Σ to G and the retraction r to the resulting m -edge coloured graph G' , we see G admits a Γ -switchable homomorphism to a cycle $C \in \mathcal{C}(G)$ that is nearly monochromatic of colour ϕ . By supposition, $C \rightarrow_\Gamma H$. Therefore $G \rightarrow_\Gamma C \rightarrow_\Gamma H$, and by composition of homomorphisms, $G \rightarrow_\Gamma H$. \square

Theorem 2.4.5 implies our structural property similar to Theorem 1.2.3.

Corollary 2.4.6. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 , Γ be a transitive group acting on \mathbb{Z}_m , $\mathcal{C}(G)$ be the set of cycles of G , and Θ_Γ be the set of cycles disagreeing with Γ . Then the following are equivalent:*

1. $G \rightarrow_\Gamma H$
2. For all cycles $C \in \mathcal{C}(G)$, $C \rightarrow_\Gamma H$
3. For all cycles $C \in \Theta_\Gamma$, $C \not\rightarrow_\Gamma G$.

\square

While Theorem 2.4.5 provides a description of which m -edge coloured graphs G have a Γ -switchable homomorphism to a monochromatic K_2 of colour ϕ based on cycles, we immediately note the problem that $|\mathcal{C}(G)|$ can be exponential in G .

We therefore seek a description that leads to a polynomial time algorithm.

Theorem 2.4.7. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 , T be a spanning tree of G , and Γ be a transitive group acting on \mathbb{Z}_m . Then $G \rightarrow_\Gamma H$ if and only if after G is switched with respect to Γ such that T is monochromatic of colour ϕ , each cotree edge agrees with ϕ .*

Proof. Let G and H be a m -edge coloured graphs such that H is a monochromatic K_2 , T be a spanning tree of G , and Γ be a group acting on \mathbb{Z}_m . By Lemma 2.4.2, there exists a sequence such that T can be switched to be monochromatic of any colour. Without loss of generality let the colour be ϕ .

Let uv be a cotree edge with respect to T . Then if the colour of uv disagrees with ϕ , we have a no instance and $G \not\rightarrow_{\Gamma} H$. So we assume each cotree edge is in agreeance with ϕ .

Suppose after G is switched such that T is monochromatic of colour ϕ , each cotree edge agrees with ϕ . If G has no cotree edges with respect to T , then $G \rightarrow_{\Gamma} H$. Suppose $G \rightarrow_{\Gamma} H$ for all G with $|\overline{T}| < k$.

Consider G where $|\overline{T}| = k$. Let uv be an edge in \overline{T} . Then by our supposition $G - uv$ can be made monochromatic of colour ϕ by some sequence Σ . Because agreeance classes are closed with respect to switching, the edge uv in G^{Σ} agrees with ϕ . Let P be the shortest uv path in $(G - uv)^{\Sigma}$. Then by Theorem 2.4.4, P is a retract of $(G - uv)^{\Sigma}$. Thus there is a Γ -switchable homomorphism $r : (G - uv) \rightarrow_{\Gamma} P$ where P is a monochromatic path of colour ϕ , $r(u) = u$, and $r(v) = v$. By applying Σ to G and the retraction r to G , we see G admits a Γ -switchable homomorphism to a cycle $C \in \mathcal{C}(G)$ that is nearly monochromatic of colours ϕ and ϕ' for some ϕ' . By supposition, $C \rightarrow_{\Gamma} H$. Therefore $G \rightarrow_{\Gamma} C \rightarrow_{\Gamma} H$, and by composition of homomorphisms, $G \rightarrow_{\Gamma} H$. \square

Corollary 2.4.8. *Let G and H be m -edge coloured graphs where H is a monochromatic K_2 of colour ϕ . Then Γ -HOM- H is solvable in polynomial time.*

2.5 More on Agreeance Classes

In this section we discuss agreeance classes in more detail.

Proposition 2.5.1. *Let Γ be a group that acts transitively on \mathbb{Z}_m and let ϕ and ϕ' be elements of \mathbb{Z}_m . Then $|\llbracket\phi\rrbracket_\Gamma| = |\llbracket\phi'\rrbracket_\Gamma|$.*

Proof. Let Γ be a group that acts transitively on \mathbb{Z}_m and let ϕ and ϕ' be colours of \mathbb{Z}_m . Since Γ is transitive, for each $\lambda \in \llbracket\phi\rrbracket_\Gamma$ and $\lambda' \in \llbracket\phi'\rrbracket_\Gamma$, there exists an element $\pi \in \Gamma$ such that $\pi(\lambda) = \lambda'$. Since $\lambda \in \llbracket\phi\rrbracket_\Gamma$, a cycle C that is nearly monochromatic of colours ϕ and λ can be made monochromatic. Further, since C is bipartite, it has a bipartition, say (A, B) . It is clear that $C^{(A, \pi)}$ is a graph that is nearly monochromatic of colours $\pi(\phi)$ and $\pi(\lambda) = \lambda'$, and $C^{(A, \pi)}$ can be made monochromatic of colour $\pi(\phi)$. Therefore $\lambda' \in \llbracket\pi(\phi)\rrbracket_\Gamma$ for all $\lambda \in \llbracket\phi\rrbracket_\Gamma$. This implies $\pi(\llbracket\phi\rrbracket_\Gamma) = \llbracket\phi'\rrbracket_\Gamma$. It directly follows that $|\llbracket\phi\rrbracket_\Gamma| = |\llbracket\phi'\rrbracket_\Gamma|$. \square

2.5.1 Γ is Abelian

We now revisit some of the the classes of groups already studied to examine the agreeance classes. We will begin with the best behaved groups, the Abelian groups. We shall to prove the following.

Theorem 2.5.2. *Let Γ be an Abelian group that acts transitively on \mathbb{Z}_m and ϕ be an element of \mathbb{Z}_m . Then $\llbracket\phi\rrbracket_\Gamma = \{\phi\}$.*

Proof. Let Γ be an Abelian group that acts transitively on \mathbb{Z}_m and ϕ, ϕ' be elements of \mathbb{Z}_m , and H be a m -edge coloured graph such that H is a monochromatic K_2 . We use $P_\Gamma(H)$ (see Definition 1.13.1).

Consider a m -edge coloured even cycle G which is nearly monochromatic of colours ϕ and ϕ' where $\phi' \neq \phi$. Without loss of generality, we can assume that the graph underlying G is isomorphic to C_4 . Suppose there exists a homomorphism $h : G \rightarrow P_\Gamma(H)$. Then by Proposition 1.13.7, the edges of colour ϕ must map to a single edge in $P_\Gamma(H)$ as the edges of colour ϕ . In particular, $h(u)$ and $h(v)$ is an edge of colour

ϕ in $P_\Gamma(H)$, a contradiction. Therefore, by Corollary 1.13.8, G is not switchably homomorphic to a monochromatic K_2 . Therefore $\phi' \notin [\phi]_\Gamma$ for $\phi \neq \phi'$. \square

We have shown that the agreeance class of a colour with respect to any Abelian group is trivial. This is not always the case.

2.5.2 Γ Has Property \mathcal{T}_j

In contrast to the Abelian case, there is a single agreeance class of colours with respect to S_m .

Theorem 2.5.3. *Let $m \geq 3$ be an integer, $\Gamma = S_m$, and let ϕ be an element of \mathbb{Z}_m . Then $[\phi]_\Gamma = \mathbb{Z}_m$.*

Proof. Let S_m be the symmetric group, ϕ, ϕ', ϕ'' be distinct elements of \mathbb{Z}_m , and H be a m -edge coloured graph such that H is a monochromatic K_2 .

Consider a m -edge coloured even cycle G which is nearly monochromatic of colours ϕ and ϕ' where $\phi' \neq \phi$. Let uv be the edge of colour ϕ . Without loss of generality, we can assume that the graph underlying G is isomorphic to C_4 . Since S_m contains all permutations, there exists transpositions $\pi_1 = (\phi\phi')$ and $\pi_2 = (\phi\phi'')$ in S_m . Let uv be the edge of G of colour ϕ' .

Consider $\Sigma = (v, \pi_2^{-1})(u, \pi_1^{-1})(v, \pi_2)(u, \pi_1)$. We can see that the overall action on uv is

$$\pi_2^{-1}\pi_1^{-1}\pi_2\pi_1(\phi') = \pi_2^{-1}\pi_1^{-1}\pi_2(\phi) = \pi_2^{-1}\pi_1^{-1}(\phi'') = \pi_2^{-1}(\phi'') = \phi.$$

For an edge uw ,

$$\pi_1^{-1}\pi_1(\phi) = \pi_1^{-1}(\phi') = \phi.$$

And similarly for an edge vx ,

$$\pi_2^{-1}\pi_2(\phi) = \pi_2^{-1}(\phi') = \phi.$$

This completes the proof. □

Using the sequence of switches from the previous proof is more general than in the case for groups with property \mathcal{T}_j . Using the same sequence in Section 1.13.2, we obtain the following:

Theorem 2.5.4. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 , Γ be a group acting on \mathbb{Z}_m . If Γ has property \mathcal{T}_j , then $[\phi]_\Gamma = \mathbb{Z}_m$. □*

This leads to the following immediate corollaries.

Corollary 2.5.5. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Let $m \geq 4$ be an integer, $\Gamma = A_m$, and ϕ be an element of \mathbb{Z}_m . Then $[\phi]_\Gamma = \mathbb{Z}_m$. □*

Corollary 2.5.6. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Let $m \geq 3$ be an odd integer, Γ be D_m (acting on \mathbb{Z}_m), and ϕ be an element of \mathbb{Z}_m . Then $[\phi]_\Gamma = \mathbb{Z}_m$. □*

2.5.3 Γ is a Dihedral Group of Even Order

In contrast to the previous cases, the even dihedral group has a non-trivial agreeance class.

Corollary 2.5.7. *Let G and H be m -edge coloured graphs such that H is a monochromatic K_2 . Let $m \geq 3$ be an even integer, D_m be as described in Section 1.13.3, and ϕ be an element of \mathbb{Z}_m . Then,*

$$[\phi]_\Gamma = \{\phi' \in \mathbb{Z}_m \mid \phi - \phi' \equiv 0 \pmod{2}\} = [D_m, D_m](\phi).$$

Proof. The proof of the leftmost equality comes from the proof of Theorem 1.13.22.

We prove the rightmost equality.

Let D_m be as described in Section 1.13.3. We say a rotation r' is *even* in D_m if $r' = r^m = e_{D_m}$.

If π and π' are reflections, then $\pi = \pi^{-1}$ and $\pi' = \pi'^{-1}$ and $\pi\pi'$ is a rotation. So,

$$[\pi, \pi'] = \pi^{-1}\pi'^{-1}\pi\pi' = (\pi\pi')^2$$

is an even rotation.

If π is a reflection and π' is a rotation, then $\pi\pi'\pi = \pi'^{-1}$. So,

$$[\pi, \pi'] = \pi^{-1}\pi'^{-1}\pi\pi' = \pi\pi'^{-1}\pi\pi' = \pi'^2$$

is an even rotation.

If π and π' are rotations, then $\pi\pi' = \pi'\pi$. So,

$$[\pi, \pi'] = \pi^{-1}\pi'^{-1}\pi\pi' = \pi^{-1}\pi\pi'^{-1}\pi' = e_{D_m}.$$

That is $[D_m, D_m] = \{e_{D_m}, r^2, r^{-2}, r^4, r^{-4}, \dots, r^{m-2}, r^{2-m}\}$. Therefore, any $\pi \in [D_m, D_m]$ has the property that if $\pi(\phi) = \phi'$, then $\phi - \phi' \equiv 0 \pmod{2}$. The result follows. \square

2.5.4 Γ is the Quaternions

Theorem 2.1.13 gives the following.

Corollary 2.5.8. *Let G and H be 8-edge coloured graphs such that H is a monochromatic K_2 . Let Q_8 be as described in Section 2.1, and ϕ be an element of \mathbb{Z}_m . Then $[\phi]_\Gamma = \{\phi, \pi_{-1}(\phi)\} = [Q_8, Q_8](\phi)$. \square*

Hence we have examples of groups for which the agreeance classes are singletons, the set of all colours, and non-trivial.

2.5.5 Commutator Subgroup: The Whole Story

The operation we applied in the proof of Theorem 2.5.3 is $\pi_2^{-1}\pi_1^{-1}\pi_2\pi_1(\phi') = \phi$. This is the commutator of π_2 and π_1 , and is denoted by $[\pi_2, \pi_1]$. Therefore $[\pi_2, \pi_1](\phi') = \phi$. Thus, if there is a commutator $[\alpha, \beta]$ such that $[\alpha, \beta](\phi') = \phi$, then $\phi' \in [\phi]_\Gamma$. Moreover, this argument works for any product of commutators as shown below. Note also that $[\beta, \alpha] = [\alpha, \beta]^{-1}$, so $[\beta, \alpha](\phi) = \phi'$. Hence $[\Gamma, \Gamma](\phi) \in [\phi]_\Gamma$.

We state the following well known properties of the commutator subgroup.

Let Γ be a group, the fact that the commutator subgroup is a normal subgroup of Γ and the fact the quotient group $\Gamma/[\Gamma, \Gamma]$ is Abelian allow us to prove the following two statements.

Lemma 2.5.9. *Let Γ be a group acting transitively on \mathbb{Z}_m and ϕ be an element of \mathbb{Z}_m . Then the agreeance class $[\phi]_\Gamma$ is closed with respect to the action of an element of $[\Gamma, \Gamma]$.*

Proof. Let Γ be a group acting transitively on \mathbb{Z}_m , π be an element of $[\Gamma, \Gamma]$, and G be a m -edge coloured C_4 such that the vertices of G are $\{u, v, a, b\}$, the edges $\{va, ab, bu\}$ are of colour ϕ , and uv is of colour ϕ' for $\phi' \in [\phi]_\Gamma$. We shall show $\pi(\phi') \in [\phi]_\Gamma$.

Since $\pi \in [\Gamma, \Gamma]$, then $\pi = \pi_k\pi_{k-1}\dots\pi_1$ for $\pi_i = [\alpha_i, \beta_i]$ where $\alpha_i, \beta_i \in \Gamma$.

Let Σ_{π_i} be the sequence $(v, \alpha_i^{-1})(u, \beta_i^{-1})(v, \alpha_i)(u, \beta_i)$. Let us consider the colour of the edges of $G^{\Sigma_{\pi_i}}$.

Suppose uv is of colour ϕ_{i-1} after switching with the sequence $\Sigma_{\pi_{i-1}}\Sigma_{\pi_{i-2}}\dots\Sigma_{\pi_1}$. Then all other edges are still of ϕ . After switching, the edge uv the is of colour $\alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i(\phi_{i-1}) = \pi_i(\phi_{i-1}) = \phi_i$. Similarly, the edge vb is of colour $\alpha_i^{-1}\alpha_i(\phi) = \phi$

and the edge au is of colour $\beta_i^{-1}\beta_i(\phi) = \phi$. Finally, the colour of the edge ab remains colour ϕ .

Let Σ_π be the sequence $\Sigma_\pi = \Sigma_{\pi_k}\Sigma_{\pi_{k-1}}\dots\Sigma_{\pi_1}$. Let us consider the colour of the edges of G^{Σ_π} .

In G^{π_i} only the colour of edge uv changes. The colour of the edge uv in G^{Σ_π} is $\pi_k\pi_{k-1}\dots\pi_1(\phi') = \pi(\phi')$. As $\phi' \in [\phi]_\Gamma$, we have $\pi(\phi') \in [\phi]_\Gamma$. \square

Theorem 2.5.10. *Let Γ be a group acting transitively on \mathbb{Z}_m and ϕ be an element \mathbb{Z}_m . Then $[\phi]_\Gamma = [\Gamma, \Gamma](\phi)$. That is, the agreeance class of ϕ is the orbit of ϕ under the action of $[\Gamma, \Gamma]$.*

Proof. By [8] Theorem 1.6A, the orbits of $[\Gamma, \Gamma]$ form a block system for Γ and the number of orbits divides the index $|\Gamma : [\Gamma, \Gamma]|$.

We claim the agreeance classes are the blocks in this block system. Let $x \in \Gamma$, $\phi \in \mathbb{Z}_m$, and $\psi \in [\phi]_\Gamma$. Then

$$\begin{aligned} x(\psi) &\in x([\psi]_\Gamma) \\ &= x([\phi]_\Gamma) \\ &= [x(\phi)]_\Gamma \end{aligned}$$

Therefore the blocks are unions of agreeance classes. Since the quotient group $\Gamma/[\Gamma, \Gamma]$ acts regularly on the blocks, the number of blocks is equal to $|\Gamma/[\Gamma, \Gamma]|$. \square

2.6 n -Arc Coloured Graphs

In this section, we use previous results to solve the Γ -switchable 2-colouring problem for n -arc coloured graphs. This completes the solution to the Γ -switchable 2-colouring problem for (m, n) -mixed graphs and transitive groups Γ .

Theorem 2.6.1. *Let H be a n -arc coloured graph such that H is a monochromatic T_2 , and Γ be a transitive subgroup of $e \times S_2 \times S_2^n$. Let H' be a $2n$ -edge coloured graph that is a monochromatic K_2 . Then there exists a group Γ' acting transitively on \mathbb{Z}_{2n} , such that Γ -HOM- H polynomially transforms to Γ' -HOM- H' .*

Proof. Let G be a n -arc coloured graph. Then without loss of generality, G is bipartite with bipartition (A, B) . Construct a $2n$ -edge coloured bipartite graph G' as follows.

For each arc $uv \in A(G)$, we add edge uv to $E(G')$; if the arc $uv \in A(G)$ has colour $c(uv)$ in G and $u \in A$, then we assign colour $c(uv)^+$ to edge $uv \in E(G')$. If arc $uv \in A(G)$ has colour $c(uv)$ in G and $u \in B$, then we assign colour $c(uv)^-$ to edge $uv \in E(G')$.

Let $\Gamma \leq e \times S_2 \times S_2^n$. We define the group Γ' as follows. Let $\pi \in \Gamma$. If $\pi(\phi) = \phi'$ and the arc direction is preserved, we define $\pi' \in \Gamma'$ as $\pi'(\phi^+) = \phi'^+$ and $\pi'(\phi^-) = \phi'^-$. Otherwise, if $\pi(\phi) = \phi'$ and the arc direction is reversed, we define $\pi' \in \Gamma$ as $\pi'(\phi^+) = \phi'^-$ and $\pi'(\phi^-) = \phi'^+$. Then $\Gamma \leq S_{2n}$.

It is straightforward to verify that $G' \rightarrow_{\Gamma'} H'$ if and only if $G \rightarrow_{\Gamma} H$.

Since G' can be constructed in polynomial time, and $|\Gamma|$ is constant (because Γ is fixed) the transformation can be carried out in polynomial time. \square

2.7 Reconfiguration is in P

Let G be a m -edge coloured bipartite graph. Suppose G can be switched to be monochromatic. In this section, we show that the number of switches is $O(|E(G)|)$.

Our first result is a consequence of Theorem 2.4.7.

Corollary 2.7.1. *Let G be a m -edge coloured bipartite graph. Let Γ be a group acting transitively on \mathbb{Z}_m . If G is Γ -switch equivalent to a monochromatic graph, then the*

sequence Σ of switches which transforms G to be monochromatic satisfies,

$$|\Sigma| \leq |V(G)| - 1 + c_\Gamma(|E(G)| - |V(G)| + 1)$$

where c_Γ depends only on Γ .

Proof. Let G be a m -edge coloured bipartite graph. Let Γ be a group acting transitively on \mathbb{Z}_m . Suppose G is Γ -switch equivalent to a monochromatic graph. From Lemma 2.4.2 we can switch a spanning tree T to be monochromatic ϕ in $|V(G)| - 1$ switches. From Theorem 2.5.10 we can transform a cotree edge uv to be of colour ϕ by switching its endpoints with an element of $[\Gamma, \Gamma]$. If the element of $[\Gamma, \Gamma]$ is $\pi = \pi_k \pi_{k-1} \cdots \pi_1$, then for each π_i we need to switch u twice and v twice. Therefore we can make the $|E(G)| - |V(G)| + 1$ cotree edges colour ϕ with c_Γ switches where c_Γ is the maximum of $4k$ over all $\pi = \pi_k \pi_{k-1} \cdots \pi_1$ for $\pi \in [\Gamma, \Gamma]$. \square

Corollary 2.7.2. *Let Γ be an Abelian group acting transitively on \mathbb{Z}_m . Then $c_\Gamma = 0$.*

Proof. The result follows from Theorem 1.13.2. \square

Corollary 2.7.3. *Let Γ be a non-Abelian group acting transitively on \mathbb{Z}_m such that $|\Gamma| \leq 95$. Then $c_\Gamma = 4$.*

Proof. By the proof of Corollary 2.7.1, c_Γ is the maximum of $4k$ over all $\pi = \pi_k \pi_{k-1} \cdots \pi_1$ for $\pi \in [\Gamma, \Gamma]$. The smallest group Γ such that $x \in [\Gamma, \Gamma]$ is not a commutator itself is of order 96. Therefore $k = 1$. The result follows. \square

Chapter 3

Conclusion

We have shown that Γ -SWITCHABLE k -COL is solvable in polynomial time if $k \leq 2$. We have thus completed a step towards the dichotomy theorem for Γ -SWITCHABLE k -COL. Equivalently, we have shown that Γ -HOM- H is solvable in polynomial time for all (m, n) -mixed graphs H on at most 2 vertices. This is a step in obtaining a dichotomy theorem for Γ -HOM- H for all transitive permutation groups Γ .

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