

SOME PROPERTIES OF CONVOLUTION
OPERATORS IN THE CLASS $\mathcal{P}_\alpha(\beta)$

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Making use of several families of convolution operators, we introduce and study a certain general class $\mathcal{P}_\alpha(\beta)$ ($0 \leq \alpha < 1$; $\beta \geq 0$) of analytic functions in the open unit disk \mathcal{U} . We also investigate the relationships between the class $\mathcal{P}_\alpha(\beta)$ and the Hardy space \mathcal{H}^∞ (of bounded analytic functions in \mathcal{U}). Finally, we consider some interesting applications of the results presented here to a class of generalized hypergeometric functions.

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the class of (*normalized*) functions of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : |z| < 1\}.$$

We also denote by \mathcal{S} the subclass of \mathcal{A} consisting of functions which are univalent in \mathcal{U} .

A function $f(z) \in \mathcal{A}$ is said to be in the class \mathcal{P}_α ($0 \leq \alpha < 1$) if and only if it satisfies the inequality:

$$(1.2) \quad \operatorname{Re}\{f'(z)\} > \alpha \quad (0 \leq \alpha < 1; \quad z \in \mathcal{U}).$$

The class \mathcal{P}_0 was investigated systematically by MacGregor [8] who did refer to numerous earlier studies involving functions whose derivative has a positive real part. Indeed, as readily implied by the Noshiro–Warschawski theorem (*cf.*, *e.g.*, Duren [3, p. 47, Theorem 2.16]), \mathcal{P}_α is a subclass of the class \mathcal{S} .

Let f and g be in the class \mathcal{A} , with $f(z)$ given by (1.1), and $g(z)$ by

$$(1.3) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

For a given $f \in \mathcal{A}$, we define the convolution operator

$$\Omega_f: \mathcal{A} \rightarrow \mathcal{A}$$

by

$$(1.4) \quad \Omega_f(g) = f \star g,$$

where, as usual, $f \star g$ denotes the Hadamard product of f and g :

$$(1.5) \quad (f \star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For a function $f \in \mathcal{A}$ given by (1.1), Owa and Srivastava ([11]; see also [12, p. 338]) defined the generalized Libera integral operator \mathcal{J}_c by

$$(1.6) \quad \begin{aligned} \mathcal{J}_c(f) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \\ &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n. \end{aligned}$$

The operator \mathcal{J}_c , when $c \in \mathbb{N} = \{1, 2, 3, \dots\}$, was introduced by Bernardi [1]. In particular, the operator \mathcal{J}_1 was studied earlier by Libera [6] and Livingston [7].

Clearly, (1.6) yields

$$(1.7) \quad f(z) \in \mathcal{A} \Rightarrow \mathcal{J}_c(f) \in \mathcal{A} \quad (c > -1).$$

Thus, by applying the operator \mathcal{J}_c successively, we can obtain

$$(1.8) \quad \mathcal{J}_c^n(f) = \begin{cases} \mathcal{J}_c \mathcal{J}_c^{n-1}(f) & (n \in \mathbb{N}), \\ f(z) & (n = 0). \end{cases}$$

With a view to introducing an interesting generalization of the class \mathcal{P}_α , we now recall the following definition of a *multiplier transformation* (or *fractional integral* and *fractional derivative*):

DEFINITION 1 (Flett [4, p. 748]). Let the function

$$(1.9) \quad \phi(z) = \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathcal{U})$$

be analytic in \mathcal{U} and let λ be a real number. Then the *multiplier transformation* $I^\lambda \phi$ is defined by

$$(1.10) \quad I^\lambda \phi(z) = \sum_{n=0}^{\infty} (n+1)^{-\lambda} c_n z^n \quad (z \in \mathcal{U}).$$

The function $I^\lambda \phi$ is clearly analytic in \mathcal{U} . It may be regarded as a *fractional integral* (for $\lambda > 0$) or *fractional derivative* (for $\lambda < 0$) of ϕ , and it is readily seen that

$$(1.11) \quad I^\lambda I^\mu \phi = I^{\lambda+\mu} \phi$$

for all real numbers λ and μ . Furthermore, in terms of the Gamma function, we have

$$(1.12) \quad \begin{aligned} I^\lambda \phi(z) &= \frac{1}{z \Gamma(\lambda)} \int_0^z \left[\log \frac{z}{t} \right]^{\lambda-1} \phi(t) dt \quad (\lambda > 0) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} \phi(zt) dt \quad (\lambda > 0), \end{aligned}$$

which can be verified fairly easily by term-by-term integration, using some well-known Γ -function integrals.

Definition 1 leads us naturally to

DEFINITION 2. The *fractional derivative* $D^\lambda \phi$ of order $\lambda \geq 0$, for an analytic function ϕ given by (1.9), is defined by

$$(1.13) \quad D^\lambda \phi(z) = I^{-\lambda} \phi(z) = \sum_{n=0}^{\infty} (n+1)^\lambda c_n z^n \quad (\lambda \geq 0; z \in \mathcal{U}).$$

It follows from Definition 2 that

$$(1.14) \quad D^m \phi(z) = \left[\frac{d}{dz} z \right]^m \phi(z) \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

More importantly, making use of Definition 2, we now introduce an interesting generalization of the class \mathcal{P}_α of functions in \mathcal{A} which satisfy the inequality (1.2).

DEFINITION 3. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\alpha(\beta)$ ($0 \leq \alpha < 1; \beta \geq 0$) if and only if

$$2^{-\beta} D^\beta f \in \mathcal{P}_\alpha \quad (0 \leq \alpha < 1; \beta \geq 0).$$

Observe that $\mathcal{P}_\alpha(0) = \mathcal{P}_\alpha$. Furthermore, since $f \in \mathcal{A}$, it follows from (1.1) and (1.13) that

$$(1.15) \quad 2^{-\beta} D^\beta f(z) = z + \sum_{n=2}^{\infty} \left[\frac{n+1}{2} \right]^\beta a_n z^n \quad (z \in \mathcal{U}),$$

which shows that $2^{-\beta} D^\beta f \in \mathcal{A}$ if $f \in \mathcal{A}$.

The object of the present paper is to investigate various useful properties of the general class $\mathcal{P}_\alpha(\beta)$ by using such families of convolution operators as those

mentioned above. We also relate the class $\mathcal{P}_\alpha(\beta)$ with the Hardy space \mathcal{H}^∞ of bounded analytic functions in \mathcal{U} , and consider several applications of our results to a class of generalized hypergeometric functions.

2. A PRELIMINARY LEMMA

In our present investigation of the general class $\mathcal{P}_\alpha(\beta)$ ($0 \leq \alpha < 1$; $\beta \geq 0$), we shall require the following

LEMMA (Miller and Mocanu [9, p. 301, Theorem 10]). *Let $M(z)$ and $N(z)$ be analytic in \mathcal{U} with*

$$(2.1) \quad M(0) = N(0) = 0,$$

and let γ be a real number. If $N(z)$ maps \mathcal{U} onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

$$(2.2) \quad \operatorname{Re} \left\{ \frac{M'(z)}{N'(z)} \right\} > \gamma \quad (z \in \mathcal{U}) \implies \operatorname{Re} \left\{ \frac{M(z)}{N(z)} \right\} > \gamma \quad (z \in \mathcal{U})$$

and

$$(2.3) \quad \operatorname{Re} \left\{ \frac{M'(z)}{N'(z)} \right\} < \gamma \quad (z \in \mathcal{U}) \implies \operatorname{Re} \left\{ \frac{M(z)}{N(z)} \right\} < \gamma \quad (z \in \mathcal{U}).$$

3. EXAMPLES OF CONVOLUTION OPERATORS WITH INTEGRAL REPRESENTATIONS

Throughout this section, let $f(z) \in \mathcal{A}$ be given by (1.1). Suppose also that [cf. Equations (1.6) and (1.8)]

$$\begin{aligned}
(3.1) \quad \mathcal{F}_p(f) &= \mathcal{J}_{c_1} \cdots \mathcal{J}_{c_p}(f) \\
&= z + \sum_{n=2}^{\infty} \frac{(c_1+1)\cdots(c_p+1)}{(c_1+n)\cdots(c_p+n)} a_n z^n \\
&\quad (c_j > -1 \quad (j = 1, \dots, p); \quad p \in \mathbb{N}).
\end{aligned}$$

Then, in view of the definitions (1.5) and (1.6), it is not difficult to express the functional \mathcal{F}_p as a convolution operator given by

$$(3.2) \quad \mathcal{F}_p(f) = \mathcal{J}_{c_1} \left[\frac{z}{1-z} \right] \star \cdots \star \mathcal{J}_{c_p} \left[\frac{z}{1-z} \right] \star f.$$

For various special choices for the parameters c_j ($j = 1, \dots, p$), the function $\mathcal{F}_p(f)$ can be simplified considerably, giving us some (single) integral representations which are contained in the following examples.

EXAMPLE 1. Setting

$$c_j = j + \gamma \quad (\gamma > -2; \quad j = 1, \dots, p)$$

in (3.1), we obtain

$$\begin{aligned}
(3.3) \quad \mathcal{F}_p(f) &= \{B(p, \gamma+2)\}^{-1} \int_0^1 t^\gamma (1-t)^{p-1} f(zt) dt \\
&\quad (\gamma > -2; \quad p \in \mathbb{N})
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
(3.4) \quad \mathcal{F}_p(f) &= \{z^{\gamma+1} B(p, \gamma+2)\}^{-1} \int_0^z t^\gamma \left[1 - \frac{t}{z}\right]^{p-1} f(t) dt \\
&\quad (\gamma > -2; \quad p \in \mathbb{N}),
\end{aligned}$$

where $B(\alpha, \beta)$ denotes the Beta function defined by

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

EXAMPLE 2. For $\gamma = r$ ($r \in \mathbb{N}_0$), the last integral representation (3.4) can be rewritten in the form:

$$(3.5) \quad \mathcal{F}_p(f) = \frac{(p+r+1)!}{(p-1)!(r+1)!} z^{-r-1} \int_0^z t^r \left[1 - \frac{t}{z}\right]^{p-1} f(t) dt$$

$$(r \in \mathbb{N}_0; \quad p \in \mathbb{N}),$$

which, for $r = 0$, was given by Bernardi [1, p. 438, Example 3].

EXAMPLE 3. Setting $c_j = 1$ ($j = 1, \dots, p$) in (3.1), and making use of (1.12), we have

$$(3.6) \quad \mathcal{F}_p(f) = \mathcal{I}_1^p(f) = 2^p I^p f(z)$$

$$= \frac{2^p}{(p-1)!} z^{-1} \int_0^z \left[\log \frac{z}{t}\right]^{p-1} f(t) dt \quad (p \in \mathbb{N}).$$

4. INCLUSION PROPERTIES OF THE GENERAL CLASS $\mathcal{P}_\alpha(\beta)$

We begin by stating a generalization of an interesting result due to Bernardi [1, p. 432, Theorem 4] as

THEOREM 1. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then $\mathcal{I}_c(f)$ defined by (1.6) is also in the class $\mathcal{P}_\alpha(\beta)$.*

Proof. A simple calculation shows that

$$(4.1) \quad \frac{d}{dz} D^\beta(\mathcal{J}_c(f)) = \frac{c+1}{z^{c+1}} \int_0^z t^c \left\{ \frac{d}{dt} D^\beta f(t) \right\} dt,$$

where the operators \mathcal{J}_c ($c > -1$) and D^λ ($\lambda \geq 0$) are defined by (1.6) and (1.13), respectively. In view of (4.1), we set

$$(4.2) \quad M(z) = \frac{c+1}{2^\beta} \int_0^z t^c \left\{ \frac{d}{dt} D^\beta f(t) \right\} dt \quad \text{and} \quad N(z) = z^{c+1},$$

so that

$$(4.3) \quad \operatorname{Re} \left\{ \frac{M'(z)}{N'(z)} \right\} = \operatorname{Re} \left\{ 2^{-\beta} \frac{d}{dz} (D^\beta f(z)) \right\}.$$

Since, by hypothesis, $f \in \mathcal{P}_\alpha(\beta)$, the second member of (4.3) is greater than α ($z \in \mathcal{U}$), and hence

$$(4.4) \quad \operatorname{Re} \left\{ \frac{M'(z)}{N'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; \quad z \in \mathcal{U}).$$

Thus, applying the lemma of Section 2, we have

$$(4.5) \quad \operatorname{Re} \left\{ \frac{M(z)}{N(z)} \right\} = \operatorname{Re} \left\{ 2^{-\beta} \frac{d}{dz} D^\beta(\mathcal{J}_c(f)) \right\} > \alpha$$

$$(0 \leq \alpha < 1; \quad \beta \geq 0; \quad z \in \mathcal{U}),$$

which evidently completes the proof of Theorem 1.

REMARK 1. It follows from the definitions (1.6) and (1.13) that

$$(4.6) \quad 2^{-\beta} D^\beta(\mathcal{J}_c(f)) = \mathcal{J}_c(2^{-\beta} D^\beta f) \quad (c > -1; \quad \beta \geq 0; \quad f \in \mathcal{A}),$$

which can be used to give an alternative proof of Theorem 1 along the lines of Bernardi [1, p. 432].

In conjunction with the first part of the definition (3.1), Theorem 1 readily yields

COROLLARY 1. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then the function $\mathcal{F}_p(f)$ defined by (3.1) is also in the class $\mathcal{P}_\alpha(\beta)$.*

The next inclusion property of the class $\mathcal{P}_\alpha(\beta)$, contained in Theorem 2 below, would involve the operator \mathcal{J}_1^λ ($\lambda > 0$) defined by

$$(4.7) \quad \mathcal{J}_1^\lambda(f) = 2^\lambda I^\lambda f(z) \quad (\lambda > 0; f \in \mathcal{A}),$$

which, for $\lambda = p \in \mathbb{N}$, was considered already in (3.6). Clearly, we have

$$(4.8) \quad f(z) \in \mathcal{A} \Rightarrow \mathcal{J}_1^\lambda(f) \in \mathcal{A} \quad (\lambda > 0).$$

THEOREM 2. *Let the function $f(z)$ be in the class $\mathcal{P}_\alpha(\beta)$. Then the function $\mathcal{J}_1^\lambda(f)$ ($\lambda > 0$) defined by (4.7) is also in the class $\mathcal{P}_\alpha(\beta)$.*

Proof. Making use of (1.10) and (1.13), the definition (4.7) immediately yields [cf. Equation (4.6)]

$$(4.9) \quad 2^{-\beta} D^\beta(\mathcal{J}_1^\lambda(f)) = \mathcal{J}_1^\lambda(2^{-\beta} D^\beta f) \quad (\beta \geq 0; \lambda > 0; f \in \mathcal{A}).$$

Therefore, setting

$$(4.10) \quad g(z) = 2^{-\beta} D^\beta f \quad \text{and} \quad G(z) = \mathcal{J}_1^\lambda(g),$$

we must show that

$$(4.11) \quad \operatorname{Re}\{G'(z)\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathcal{U})$$

whenever $f \in \mathcal{P}_\alpha(\beta)$.

From the second integral representation in (1.12), we obtain

$$(4.12) \quad G'(z) = \frac{2^\lambda}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} t g'(zt) dt \quad (\lambda > 0),$$

so that

$$(4.13) \quad \operatorname{Re}\{G'(z)\} = \frac{2^\lambda}{\Gamma(\lambda)} \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} t \operatorname{Re}\{g'(zt)\} dt \quad (\lambda > 0).$$

Since $f \in \mathcal{P}_\alpha(\beta)$, we have

$$(4.14) \quad \operatorname{Re}\{g'(zt)\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathcal{U}; 0 \leq t \leq 1),$$

and hence (4.13) yields

$$(4.15) \quad \operatorname{Re}\{G'(z)\} > \frac{2^\lambda}{\Gamma(\lambda)} \alpha \int_0^1 \left[\log \frac{1}{t} \right]^{\lambda-1} t dt = \alpha \quad (0 \leq \alpha < 1; \lambda > 0),$$

which completes the proof of Theorem 2.

COROLLARY 2. *If $0 \leq \alpha < 1$ and $0 \leq \beta < \gamma$, then $\mathcal{P}_\alpha(\gamma) \subset \mathcal{P}_\alpha(\beta)$.*

Proof. Setting $\lambda = \gamma - \beta > 0$ in Theorem 2, we observe that

$$(4.16) \quad \begin{aligned} f(z) \in \mathcal{P}_\alpha(\gamma) &\Rightarrow \mathcal{J}_1^{\gamma-\beta}(f) \in \mathcal{P}_\alpha(\gamma) \\ &\Leftrightarrow \{2^{-\gamma} D^\gamma(\mathcal{J}_1^{\gamma-\beta}(f))\} \in \mathcal{P}_\alpha \\ &\Leftrightarrow 2^{-\beta} D^\beta f \in \mathcal{P}_\alpha \\ &\Leftrightarrow f \in \mathcal{P}_\alpha(\beta), \end{aligned}$$

and the proof of Corollary 2 is completed.

Next we define a function $h(z) \in \mathcal{A}$ by

$$(4.17) \quad h(z) = \sum_{n=1}^{\infty} \left[\frac{n+1}{2} \right] z^n = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \quad (z \in \mathcal{U}).$$

Then, in terms of the convolution operator Ω_f defined by (1.4), we have

$$(4.18) \quad \Omega_h(f) = (h \star f)(z) = \frac{1}{2} \{ f(z) + z f'(z) \} \quad (f \in \mathcal{A}),$$

which, when compared with (1.14) with $m = 1$, yields

$$(4.19) \quad \Omega_h(f) = \frac{1}{2} D^1 f \quad (f \in \mathcal{A}).$$

We now state and prove yet another inclusion property of the class $\mathcal{P}_\alpha(\beta)$, which is given by

THEOREM 3. *If $0 \leq \alpha < 1$ and $\beta \geq 0$, then*

$$(4.20) \quad \mathcal{P}_\alpha(\beta+1) \subset \mathcal{P}_\mu(\beta) \quad \left[\mu = \frac{4\alpha+1}{5} \right].$$

Proof. In view of (4.19) and Theorem 2 of Owa and Nunokawa [10, p. 580], we have

$$(4.21) \quad \begin{aligned} f \in \mathcal{P}_\alpha(\beta+1) &\Leftrightarrow 2^{-\beta-1} D^{\beta+1} f \in \mathcal{P}_\alpha \\ &\Rightarrow \Omega_h(2^{-\beta} D^\beta f) \in \mathcal{P}_\alpha \\ &\Rightarrow 2^{-\beta} D^\beta f \in \mathcal{P}_\mu \quad \left[\mu = \frac{4\alpha+1}{5} \right] \\ &\Leftrightarrow f \in \mathcal{P}_\mu(\beta) \quad \left[\mu = \frac{4\alpha+1}{5} \right], \end{aligned}$$

which evidently proves Theorem 3.

REMARK 2. Since $0 \leq \alpha < 1$, we have

$$\mu = \frac{4\alpha + 1}{5} > \alpha,$$

and hence $\mathcal{P}_\mu(\beta) \subset \mathcal{P}_\alpha(\beta)$.

REMARK 3. Since $\Omega_h(\mathcal{P}_0) \subset \mathcal{P}_0$, as observed by Livingston [7, p. 356], we can apply the relationship (4.19) to conclude that $2^{-1}D^1 f$ need not be contained in \mathcal{P}_0 whenever $f \in \mathcal{P}_0$. Thus, by Definition 3,

$$\mathcal{P}_0(0) \subset \mathcal{P}_0(1).$$

5. RELATIONSHIPS WITH THE HARDY SPACE

For a function f analytic in \mathcal{U} , we define the integral means by

$$(5.1) \quad M_p(r, f) = \begin{cases} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{1/p} & (0 < p < \infty) \\ \max_{|z| \leq r} |f(z)| & (p = \infty). \end{cases}$$

The Hardy space \mathcal{H}^p ($0 < p \leq \infty$) is the class of all functions f analytic in \mathcal{U} for which

$$(5.2) \quad \lim_{r \rightarrow 1} \{M_p(r, f)\} < \infty \quad (0 < p \leq \infty).$$

For the general theory of \mathcal{H}^p spaces, see (for example) Duren [2] and Koosis [5].

A simple relationship between the class $\mathcal{P}_\alpha(\beta)$ and the Hardy space \mathcal{H}^p is given by

THEOREM 4. $\mathcal{P}_0(1) \subset \mathcal{H}^\omega$.

Proof. Suppose that $f \in \mathcal{P}_0(1)$. Then, by Definition 3, we have

$$(5.3) \quad \operatorname{Re}\{(2^{-1} D^1 f)'\} > 0 \quad (z \in \mathcal{U}),$$

which, in view of a known result [2, p. 34, Theorem 3.2], implies that

$$(5.4) \quad (2^{-1} D^1 f)' \in \mathcal{H}^p \quad (p < 1).$$

By the Hardy–Littlewood theorem [2, p. 88, Theorem 5.12], (5.4) shows that $D^1 f \in \mathcal{H}^p$ for all $p < \omega$. Also, by Corollary 2, we have

$$(5.5) \quad f \in \mathcal{P}_0(1) \subset \mathcal{P}_0(0) = \mathcal{P}_0,$$

which yields the inequality:

$$(5.6) \quad \operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U}).$$

Therefore, by using the same arguments as above, we find from (5.6) that $f \in \mathcal{H}^p$ for all $p < \omega$. Thus, in particular, $f \in \mathcal{H}^1$ and $D^1 f \in \mathcal{H}^1$.

Next, by comparing (4.18) and (4.19), we obtain

$$(5.7) \quad f'(z) = \frac{1}{z} \{D^1 f(z) - f(z)\},$$

which readily yields

$$(5.8) \quad \int_0^{2\pi} |f'(re^{i\theta})| d\theta \leq \frac{1}{r} \left\{ \int_0^{2\pi} |D^1 f(re^{i\theta})| d\theta + \int_0^{2\pi} |f(re^{i\theta})| d\theta \right\} \quad (r = |z|)$$

or, equivalently,

$$(5.9) \quad M_1(r, f') \leq \frac{1}{r} \left\{ M_1(r, D^1 f) + M_1(r, f) \right\}.$$

Proceeding to the limit as $r \rightarrow 1$, we find from this last inequality (5.9) that

$$(5.10) \quad \lim_{r \rightarrow 1} \{M_1(r, f')\} < \infty,$$

showing that $f' \in \mathcal{H}^1$. Thus, by applying another known result [2, p. 42, Theorem 3.11], we conclude that f is continuous in

$$\bar{\mathcal{U}} = \mathcal{U} \cup \partial \mathcal{U} = \{z : |z| \leq 1\}.$$

Finally, since $\bar{\mathcal{U}}$ is compact, f is bounded in $\bar{\mathcal{U}}$. Hence f is a bounded analytic function in \mathcal{U} , which completes the proof of Theorem 4.

As an interesting consequence of Theorem 4 and Corollary 2, we shall now prove

COROLLARY 3. *If*

$$(5.11) \quad f \in \mathcal{P}_\alpha(\beta) \quad (0 \leq \alpha < 1; \beta \geq 1),$$

then f is a bounded univalent function in \mathcal{U} .

Proof. By Corollary 2, we have

$$(5.12) \quad \mathcal{P}_\alpha(\beta) \subset \mathcal{P}_\alpha(0) = \mathcal{P}_\alpha \quad (0 \leq \alpha < 1; \beta > 0).$$

Furthermore, by an observation of Bernardi [1, p. 429], we get

$$(5.13) \quad \mathcal{P}_\alpha \subseteq \mathcal{P}_0 \subset \mathcal{S} \quad (0 \leq \alpha < 1).$$

Combining (5.11) and (5.12), we conclude that f is a *univalent* function in \mathcal{U} under the hypothesis (5.11). This, in view of Theorem 4, implies the assertion of Corollary 3 that f is a *bounded univalent* function in \mathcal{U} .

6. APPLICATIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

Let ρ_j ($j = 1, \dots, r$) and σ_j ($j = 1, \dots, s$) be complex numbers with

$$(6.1) \quad \sigma_j \neq 0, -1, -2, \dots \quad (j = 1, \dots, s).$$

Then the generalized hypergeometric function ${}_rF_s(z)$ is defined by (cf., e.g., [12, p. 333])

$$(6.2) \quad \begin{aligned} {}_rF_s(z) &\equiv {}_rF_s(\rho_1, \dots, \rho_r; \sigma_1, \dots, \sigma_s; z) \\ &= \sum_{n=0}^{\infty} \frac{(\rho_1)_n \cdots (\rho_r)_n}{(\sigma_1)_n \cdots (\sigma_s)_n} \frac{z^n}{n!} \quad (r \leq s + 1), \end{aligned}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(6.3) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

We note that the ${}_rF_s(z)$ series in (6.2) converges absolutely for $|z| < \infty$ if $r < s + 1$, and for $z \in \mathcal{U}$ if $r = s + 1$.

Applying Theorem 3 to the generalized hypergeometric function defined by (6.2), we can derive an interesting (presumably new) property of this important class of functions involving the space $\mathcal{P}_\alpha(\beta)$. More generally, we shall prove

THEOREM 5. *Let the function*

$$z {}_{r+1}F_{s+1}(\rho_1, \dots, \rho_r, 1+\lambda^{-1}; \sigma_1, \dots, \sigma_s, \lambda^{-1}; z) \quad (r \leq s+1; \lambda > 0)$$

be in the class $\mathcal{P}_\alpha(\beta)$.

Then the function

$$z {}_r F_s(\rho_1, \dots, \rho_r; \sigma_1, \dots, \sigma_s; z)$$

is in the class $\mathcal{P}_\delta(\beta)$ for δ given by

$$(6.4) \quad \delta = \frac{\lambda + 2\alpha}{\lambda + 2} \quad (\lambda > 0; \quad 0 \leq \alpha < 1).$$

Proof. From (1.15), (6.2), and (6.3), we have

$$(6.5) \quad \begin{aligned} & 2^{-\beta} D^\beta(z {}_{r+1} F_{s+1}(\rho_1, \dots, \rho_r, 1+\lambda^{-1}; \sigma_1, \dots, \sigma_s, \lambda^{-1}; z)) \\ &= z + \sum_{n=1}^{\infty} \left[\frac{n+1}{2} \right]^\beta \frac{(\rho_1)_n \cdots (\rho_r)_n}{(\sigma_1)_n \cdots (\sigma_s)_n} \frac{(1+\lambda^{-1})_n}{(\lambda^{-1})_n} \frac{z^{n+1}}{n!} \\ &= z + \sum_{n=1}^{\infty} \left[\frac{n+1}{2} \right]^\beta (\lambda_{n+1}) \frac{(\rho_1)_n \cdots (\rho_r)_n}{(\sigma_1)_n \cdots (\sigma_s)_n} \frac{z^{n+1}}{n!} \\ &= (1-\lambda) w(z) + \lambda z w'(z), \end{aligned}$$

where, for convenience,

$$(6.6) \quad \begin{aligned} w(z) &= z + \sum_{n=1}^{\infty} \left[\frac{n+1}{2} \right]^\beta \frac{(\rho_1)_n \cdots (\rho_r)_n}{(\sigma_1)_n \cdots (\sigma_s)_n} \frac{z^{n+1}}{n!} \\ &= 2^{-\beta} D^\beta(z {}_r F_s(\rho_1, \dots, \rho_r; \sigma_1, \dots, \sigma_s; z)). \end{aligned}$$

Now, in view of Definition 3 and a known result [10, p. 580, Theorem 2], the assertion of Theorem 5 follows immediately from (6.5) and (6.6).

A special case of Theorem 5 when $\lambda = \frac{1}{2}$ (so that $\delta = \mu$, where δ and μ are given by (6.4) and (4.20), respectively) can indeed be derived *directly* from Theorem 3.

Finally, by applying Theorem 1 and Corollary 1, we obtain

THEOREM 6. *Let the function*

$$z {}_r F_s(\rho_1, \dots, \rho_r; \sigma_1, \dots, \sigma_s; z) \quad (r \leq s+1)$$

be in the class $\mathcal{P}_\alpha(\beta)$.

Then the function

$$z {}_{r+p} F_{s+p}(\rho_1, \dots, \rho_r, c_1+1, \dots, c_p+1; \sigma_1, \dots, \sigma_s, c_1+2, \dots, c_p+2; z)$$

is also in the class $\mathcal{P}_\alpha(\beta)$ for $c_j > -1$ ($j = 1, \dots, p$).

The proof of Theorem 6 is much akin to that of Theorem 4 (and Corollary 3) of Owa and Srivastava [11, p. 128]. The details may be omitted.

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