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Article

A Sustainable Economic Recycle Quantity Model for Imperfect Production System with Shortages

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Abstract: Recycling of products has a great impact on contemporary sustainable business strategies. In this study, a sustainable recycling process in a production-inventory model for an imperfect production system with a fixed ratio of recyclable defective products is introduced. The piecewise constant demand rates of the non-defective items are considered under production run-time, production off-time with positive stock, and production off-time with shortages under varying conditions. Based on the production process, two cases are studied using this model. The first case does not consider recycling processes, while the second case picks up all defective items before sending these items to recycling during the production off-time; the recycled items are added to the main inventory. The aim of this study is to minimize the total cost and identify the optimal order quantity. The manufacturing process with the recycling process provides a better result compared to without recycling in the first case. Some theoretical derivations are developed to enunciate the objective function using the classical optimization technique. To validate the proposed study, sensitivity analysis is performed, and numerical examples are given. Finally, some managerial insights and the scope of future research are provided.

Keywords: economic recycle quantity (ERQ); imperfect production; inventory; sustainable recycling; shortages

1. Introduction and Literature Review

Inventory management control monitors and determines the optimal inventory order to minimize the total inventory cost. In every business, retailers often implement numerous strategies to intensify the profitability and sustainability of the system (Mashud et al. 2021; Popescu and Popescu 2019). The familiar economic order quantity (EOQ) model was first initiated by Harris (1915). Later, Alharkan et al. (2020) worked on a two-echelon supply chain model with lot-size-dependent lead-time, and Taft (1918) extended the EOQ model to consider incremental orders during the production process. Inventory of goods is essential

to the production processes to meet customer demand. The standard economic production quantity (EPQ) models typically assume perfect product quality and perfect production processes. In reality, however, ideal production processes are rare, and defective items will occur. Many industries recycle these faulty items. Some examples of recyclable materials are jute, paper, glass, and cotton and other fabrics.

During the last two decades, many researchers have studied inventory models with imperfect production systems. Defective items are mainly generated due to faulty production process (Rosenblatt and Lee 1986; Hsu and Hsu 2014). Salameh and Jaber (2000) attempted to modify the traditional EPQ model to consider poor-quality items, while Chiu et al. (2014) investigated lot-sizing problems with arbitrary defective rates in finite manufacture systems under rework processes and shortages. Defective items also deteriorate in some cases and emit carbon into the environment (Daryanto and Wee 2019; Mashud et al. 2020a). Despite deterioration, some products are wasted during the production process in the whole supply chain. Oh and Hwang (2006) anticipated an inventory model considering recycled raw materials without shortage. They assumed unequal holding costs for the raw materials and serviceable objects. Benkherouf and Omar (2017) investigated optimal production batch sizing with rework procedures, while Mokhtari (2018) outlined an order lot size combination between producers' and suppliers' defective manufacturing systems considering production rework processes. Al-Salamah (2019) designed an EPQ model for defective manufacturing procedures using synchronous and asynchronous elastic rework rates. A model for a long-run single-stage manufacturing system with imperfect items was developed by Kang et al. (2018) who incorporated rework operation, the inspection process, and planned backordering. In that model, the unfulfilled customer demands during the production phase due to process imperfections were satisfied either at the end of the inspection process or after reworking the imperfect products. Hasan et al. (2020) provided a model for agricultural products where to prevent product deterioration, they used product separation and offered different discount policies, while Nobil et al. (2020) presented a model for imperfect production process with reorder point in an EOQ model. An imperfect multi-stage production system considering defective proportions in the production process and uncertain product demand, and incorporating lean philosophy, was introduced by Tayyab et al. (2019b) who tried to reduce system costs. Onwude et al. (2020) analyzed a food wastage supply chain for fresh agricultural products. However, none of these models reworked defective items at each production stage. This model assumes the reworking of defective items at each production stage to bridge the aforementioned gap, and consequently the defective items (scraped) are perfect after reworking. Very few studies have been carried out with consideration of the recycling process with shortages, so the mentioned model also tries to incorporate shortages in the proposed ERQ model.

Demand is a fundamental key factor in inventory models because it determines the perplexity of optimal results. Sometimes demand becomes uncertain (Mashud et al. 2020b) and sometimes it depends on the price of the products (Mashud et al. 2021a, 2021c). Fuzzy demand is also very popular among the practitioners of inventory research (Tayyab et al. 2019a). Bai and Varanasi (1996) formulated an optimal control problem with indefinite piecewise constant demand and developed an algorithm to determine the optimal production strategy using Pontryagin's minimum principle. Hsu and Hsu (2016) considered a model for defective items with stock-reliant demand with inflation for a Weibull distribution deterioration. Palanivel and Uthayakumar (2016) modified their model by adding payment delays. Viji and Karthikeyan (2016) developed a model for a three-level production process for Weibull distribution deterioration. Recently, Manna et al. (2017) offered an EPQ model for a production rate-reliant defective rate of items under advertisement with demand dependent on a defective production system.

In the collected works of imperfect production systems, most models were developed considering reworks, repair, remanufacture, and different production and supply policies. Some researchers considered two-level and three-level production rates, but production inventory models of three-level constant demand patterns are not available in the literature survey. The main contributions of the proposed model are:

- (i) The proposed model is developed focusing on the recycling of defective items collected from regular production after the proper screening with 100% recovery of raw materials used for production in the corresponding item category.
- (ii) An inventory model is developed for three-level piecewise constant demand, which varies under three different production time parts, from production run-time to production off-time with positive stock and production off-time with shortages.

The remainder of the paper is organized as follows: Notation and assumptions of the model are described in Section 2. Section 3 presents the mathematical formulation of the models. Theoretical results along with optimal solution uniqueness are shown in Section 4. Section 5 comprises numerical examples and convexity graphs. Sensitivity with cost–benefit is analyzed in Section 6. Managerial implementations are delivered in Section 7. Lastly, the conclusion and future research are discussed in Section 8.

2. Assumptions and Notation

In this section, some assumptions and notation necessary to the development of the paper are presented.

2.1. Assumptions

The following assumptions are worthy of mention in formulating the models discussed.

- (i) The production rate is finite and fixed.
- (ii) The defective rate is known, and constant and defective items are produced randomly alongside the perfect product.
- (iii) The demand rate of the perfect product is a piecewise constant function (motivated from [Bai and Varanasi \(1996\)](#)): The demand during production-run and production off-times and stock-out period is as follows:

$$\left\{ \begin{array}{ll} D; & \text{during production – run time} \\ xD; & \text{during production – off time with positive stock} \\ yD; & \text{during production – off time with shortages} \end{array} \right.$$

- (iv) The sum of the demand rate and defective rate is less than the rate of production.
- (v) Lead-time is negligible, and the number of shortages is acceptable while the model is considered for a single items production system.
- (vi) Defective items are recyclable with 100% recovery of raw material usable for the production of same product in the next cycle time, and the value of the recycled materials is higher than that of purchased materials.
- (vii) The holding cost per defective item is same as that of fresh items, while the recycle cost per defective item is equal irrespective of its number.

2.2. Notation

The following notations (Table 1) are used in the development of the model:

Table 1. Notation used in the model.

Notation	Unit	Description
P	Constant	Production rate
D	Constant	Rate of demand of non-defective items during production run-time
d	Constant	Production rate defective items
C _o	\$/Cycle	Setup cost
C _h	\$/Units	Inventory holding cost
C _p	\$/Units	Per unit production cost
C _s	\$/Units	Shortage cost
C _R	\$/Units	Raw material cost per unit item
C _r	\$/Units	Recycle cost per unit item of defective items
TCD	\$/Unit time	Total cost of inventory (for case 1)
TCR	\$/Unit time	Total cost of inventory (for case 2)
t ₁	Time units	Production run-time with positive stock
t ₂	Time units	Production off-time with positive stock
t ₃	Time units	Production off-time with negative stock
t ₄	Time units	Production run-time with negative stock
x	Constant	Ratio of demand rates of production off-time with positive stock and production run-time
y	Constant	Ratio of demand rates of production off-time with negative stock and production run-time
m	Constant	Shape parameter of the accurate average demand
<i>Decision variables</i>		
W	Units	Quantity of defective items produced per cycle time
Q	Units	Volume of lot
Q _s	Units	Maximum shortage level
Q _d	Units	Maximum on-hand stock of non-defective items
T	Time units	Production cycle time. Therefore, $T = t_1 + t_2 + t_3 + t_4$

3. Model Formulation

During the production run-time of cycle time T , a lot size of Q units is produced at a production rate P , and with it, W units of defective items are produced at a defective rate d . The production process runs during $[0, K]$, and during the production-off time, the stock depletes due to customer demand. At time C , the stock becomes zero, and shortages start until G . Current demand is expressed at the demand rate “ D ” during production run-time, which is calculated again using a ratio coefficient to obtain a rate “ xD ” during production off-time while the stock is positive. Shortages also occur at the demand rate “ yD ” during the time t_3 while the stock is negative. Thus, several inventory flow situations exist for different values of the ratios “ x ” and “ y ”, for which two examples are depicted in the following Figures 1 and 2.

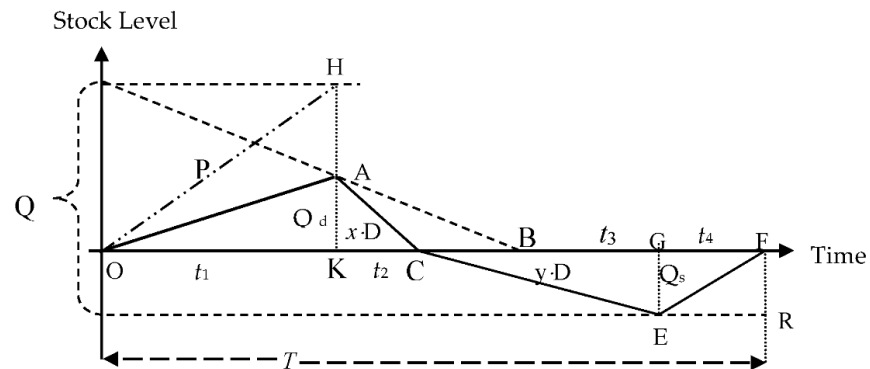


Figure 1. On-hand stock flow for $x > 1$ and $y < x$.

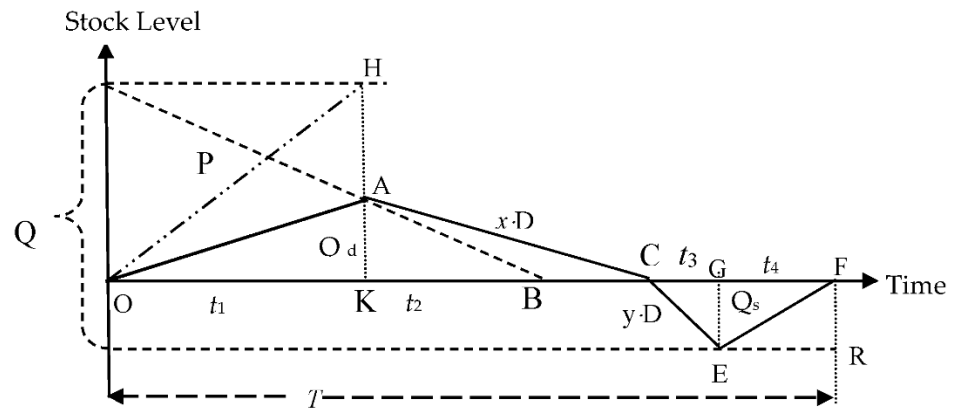


Figure 2. On-hand stock flow for $x < 1$ and $y > x$.

The stock is positive during the replenishment period, which increases by “ $(P - D - d)$ ” units of non-defective items over time through the line OA of Figure 2 during the time period t_1 . Replenishment ends at point A with a maximum number of Q_d (non-defective) units. Therefore,

$$t_1 = \frac{Q_d}{P - D - d} \tag{1}$$

Production stops at the end of the time period t_1 at point K, and the inventory decreases at another demand rate of $x D$ units per unit time, expressed in Figure 2 as line AC during t_2 . At the end of time period t_2 , the inventory reaches point C with zero on-hand stock. In this case, if the demand rate remains the same as “ D ”, then the inventory decreases as indicated by the dashed line AB. Therefore,

$$t_2 = \frac{Q_d}{x D} \tag{2}$$

Therefore,

$$t_1 + t_2 = \frac{\{P - (1 - x)D - d\} Q_d}{x D (P - D - d)} \tag{3}$$

At the end of time t_2 , the stock is 0 and then decreases at the demand rate $y D$ units per unit time. During time t_3 , shortages reach a total of Q_s units.

$$t_3 = \frac{Q_s}{y D} \tag{4}$$

As soon as the shortage level reaches Q_s at point E, production starts. The shortage amount Q_s is served at the rate of $(P - D - d)$ units with the current demand, which is fulfilled at the rate of D during time t_4 . After this period, the stock becomes zero again at the point F. Therefore,

$$t_4 = \frac{Q_s}{P - D - d} \tag{5}$$

Hence,

$$t_3 + t_4 = \frac{\{P - (1 - y)D - d\} Q_s}{y D (P - D - d)} \tag{6}$$

A total of Q units is produced in $t_1 + t_4$ time where W units are defective. Therefore,

$$t_1 + t_4 = \frac{W}{d} = \frac{Q}{P} \tag{7}$$

The following relation is obtained using Equation (7) in addition to Equations (1) and (5).

$$Q_d + Q_s = (P - D - d) \frac{W}{d} \tag{8}$$

If there are no defective items (i.e., $d = 0$) and no demand (i.e., $D = 0$), then the inventory is replenished through the line OH.

Given that the ratios of demand rates x and y modify the total demand D during production off-time with positive and negative stock respectively. The average demand rate in a cycle becomes $\left(\frac{2+x+y}{4m}D\right)$ and requires a total of $(Q - W)$ units of fresh items for the cycle time T . Therefore,

$$T = \frac{4m(Q - W)}{(2 + x + y)D} = \frac{4m(P - d)W}{(2 + x + y)dD} \tag{9}$$

Thus

$$\frac{t_1 + t_2}{T} = \frac{(2 + x + y)d\{P - (1 - x)D - d\}Q_d}{4mx(P - D - d)(P - d)W} \tag{10}$$

and

$$\frac{t_3 + t_4}{T} = \frac{(2 + x + y)d\{P - (1 - y)D - d\}Q_s}{4my(P - D - d)(P - d)W} \tag{11}$$

“ m ” is an arbitrary constant that has a single value for every set of system parameter values. The theoretical value “ m ” is explained in Section 4.3.

The foremost target of this work is to minimize the probable total cost of inventory where defective items are formed with perfect goods, but defective items are recyclable, and recycled raw materials are reusable for the production of the same category of new product. Firstly, the manufacturer must define all costs, production features, all abilities of the production procedure, and their position on recycling. The inventory holding cost of the inventory depends on average inventory. Average inventory also increases if defective items are stored properly during production run-time before they are sent to recycling. Recycling processes also incur a cost per defective item.

The following two cases are discussed regarding recycling of defective items.

Case 1: EPQ model for defective items with three levels of piecewise constant demand under shortages

Case 2: ERQ model for defective items with three levels of piecewise constant demand under shortages

3.1. Case 1 (EPQ Model for Defective Items with Three Levels of Piecewise Constant Demand under Shortages)

The total cost function of the inventory has the following components.

(1) Average setup cost or fixed cost (FC):

$$FC = \frac{1}{T}C_o = \frac{(2 + x + y)dD}{4m(P - d)W} \times C_o$$

(2) Average production cost (PC):

$$PC = \frac{1}{T}QC_p = \frac{(2 + x + y)PD}{4m(P - d)}C_p$$

(3) Average raw material cost (RMC): Each production lot size is a quantity for which raw materials are purchased for exact production in each time cycle. Therefore,

$$RMC = \frac{1}{T}QC_R = \frac{(2 + x + y)PD}{4m(P - d)}C_R$$

(4) Average holding cost (HC):

Defective items have no holding cost in this case. Therefore,
Total inventory = $\frac{1}{2}Q_d(t_1 + t_2)$

And average inventory = $\frac{1}{2}Q_d \left(\frac{t_1+t_2}{T} \right) = \frac{d(2+x+y)\{P-(1-x)D-d\}}{4mx(P-D-d)(P-d)} \cdot \frac{Q_s^2}{2W}$
 Therefore,

$$HC = \frac{d(2+x+y)\{P-(1-x)D-d\}}{8xm(P-d)(P-D-d)W} \left\{ (P-D-d) \frac{W}{d} - Q_s \right\}^2 C_h$$

(5) Average shortage cost (SC):

$$SC = \frac{Q_s}{2} \times C_s \times \frac{t_3+t_4}{T} = \frac{(2+x+y)d\{P-(1-y)D-d\}}{4my(P-D-d)(P-d)} \left(\frac{Q_s^2}{2W} \right) \times C_s$$

Hereafter, the total cost function is $TCD = FC + PC + RMC + HC + SC$

$$TCD = \frac{(2+x+y)}{4m(P-d)} \left[\frac{dD}{W} C_o + PDC_P + PDC_R + \frac{d\{P-(1-y)D-d\}}{2y(P-D-d)W} Q_s^2 C_s + \frac{d\{P-(1-x)D-d\}}{2x(P-D-d)W} \left\{ (P-D-d) \frac{W}{d} - Q_s \right\}^2 C_h \right] \tag{12}$$

The objective is to discover the optimal number of defective items and the shortage level in order to minimize total cost of the inventory per unit time. The corresponding optimal production lot size (EPQ) is also consequently determined.

3.2. Case 2 (ERQ Model for Defective Items with Three Levels of Piecewise Constant Demand under Shortages)

The producer receives a fixed portion of recyclable defective items during regular production, and defective items are sent to be recycled during production off-time. Recyclable materials become raw materials again for the new production cycle, yet to meet the demand, supplementary raw materials must be purchased from the suppliers. The target of the study is to control raw material procurement and the recycle size and production procedure to fulfill demand while minimizing the total inventory cost. The recycling system is shown in Figure 3.

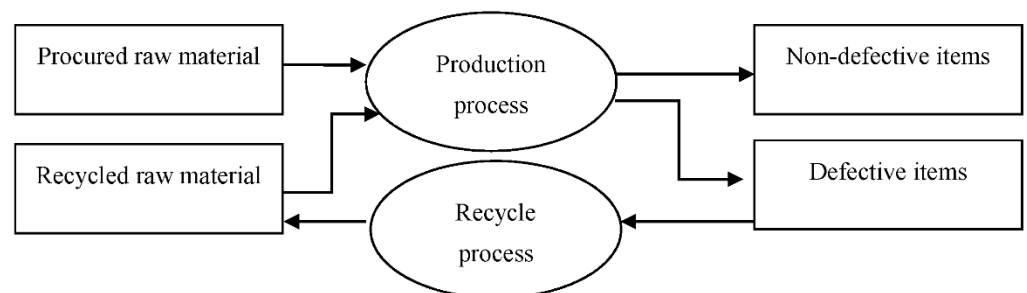


Figure 3. Framework of recycling.

In this case, the production cycle starts with a shortage of amount Q_s in the inventory, which accumulates at the rate $(P - D - d)$ over the time t_4 . During this time, the current demand is also fulfilled at the demand rate D . The inventory accumulates at the rate $(P - D - d)$ over time t_1 , and production stops at the end of t_1 . After that, the inventory level starts to decrease due to the demand rate xD over the time t_2 and due to the third level of demand rate yD , while shortages reach Q_s overtime t_3 . In the production-runtime $(t_1 + t_4)$, W units of defective items are produced and are recycled during production off-time $(t_2 + t_3)$ and added to the raw material of the next production cycle. Consequently, the inventory of the ERQ model procures raw materials for $(Q - W)$ units of items in each cycle time, and the process repeats.

The total cost function of the inventory system comprises the following cost components:

- (1) Average production cost (PC) is:

$$PC = \frac{1}{T}QC_p = \frac{(2+x+y)PD}{4m(P-d)}C_p$$

- (2) Average raw material cost (RMC): Each production lot size is Q , but raw materials for $(Q - W)$ units of items in each time cycle are purchased. Therefore,

$$RMC = \frac{1}{T}(Q - W)C_R = \frac{(2+x+y)D}{4m}C_R$$

- (3) Average holding cost (HC): Holding cost of defective items is included during time $(t_1 + t_4)$ before the items are sent for recycling. Therefore, Total inventory

$$\begin{aligned} &= \frac{1}{2}Q_d \times t_1 + \frac{1}{2}Q_d \times t_2 + \frac{1}{2}d(t_1 + t_4) \times (t_1 + t_4) \\ &= \frac{1}{2}Q_d(t_1 + t_2) + \frac{1}{2}d(t_1 + t_4)^2 \end{aligned}$$

Average inventory

$$\begin{aligned} &= \frac{1}{2}Q_d \left(\frac{t_1 + t_2}{T} \right) + \frac{1}{2}d \frac{(t_1 + t_4)^2}{T} \\ &= \frac{(2+x+y)}{4m(P-d)} \cdot \left[\frac{d\{P - (1-x)D - d\}}{2x(P-D-d)W} \left\{ (P-D-d) \frac{W}{d} - Q_s \right\}^2 + \frac{DW}{2} \right] \end{aligned}$$

Hence,

$$HC = \frac{(2+x+y)}{4m(P-d)} \left[\frac{d\{P - (1-x)D - d\}}{2x(P-D-d)W} \left\{ (P-D-d) \frac{W}{d} - Q_s \right\}^2 + \frac{DW}{2} \right] C_h$$

- (4) Average shortage cost (SC):

$$SC = \frac{Q_s}{2} \times C_s \times \frac{t_3 + t_4}{T} = \frac{(2+x+y)d\{P - (1-y)D - d\}}{4my(P-D-d)(P-d)} \frac{Q_s^2}{2W} \times C_s$$

- (5) Average recycle cost (RC):

$$RC = \frac{1}{T}WC_r = \frac{(2+x+y)dD}{4m(P-d)}C_r$$

Hence, the total cost function is

$$TCR = FC + PC + RMC + HC + SC + RC$$

$$\begin{aligned} TCR = \frac{(2+x+y)}{4m(P-d)} \left[\frac{dD}{W}C_o + PDC_p + (P-d)DC_R + \frac{d\{P - (1-x)D - d\}}{2x(P-D-d)W} \left\{ (P-D-d) \frac{W}{d} - Q_s \right\}^2 C_h + \right. \\ \left. \frac{DW}{2}C_h + \frac{d\{P - (1-y)D - d\}}{y(P-D-d)} \frac{Q_s^2}{2W} \times C_s + dDC_r \right] \end{aligned} \tag{13}$$

The objective is to find out the optimal number of defective items (ERQ) to be recycled and the optimal shortage level so as to minimize the total cost of the inventory per unit time. The corresponding optimal production lot size (EPQ) is also consequently determined.

In the next section, the uniqueness of optimal solution of the total cost functions TCD and TCR are explained and justified.

4. Theoretical Derivations

Convexity of total cost functions TCD and TCR relating to the decision variables was established using the Hessian matrix, allowing for the unique optimal solution to be derived. Special cases of the optimal values are also deliberated.

4.1. Case 1 (EPQ Model for Defective Items with Three Levels of Piecewise Constant Demand under Shortages)

The convexity of TCD and uniqueness of optimal solution regarding the decision variables are established to prove the following theorem.

Theorem 1. *The total cost function TCD gives minimum value regarding W and Q_s simultaneously when (1 - x)D + d > 0, P - (1 - y)D - d > 0, and hence TCD emits a unique optimal solution W* and Q_s*.*

Proof. TCD is a function of W and Q_s, and therefore the first order partial derivatives of TCD with regard to W and Q_s are

$$\frac{\partial(TCD)}{\partial W} = \frac{(2+x+y)}{4m(P-d)} \left[-\frac{dDC_o}{W^2} - \frac{(P-D-d)\{P-(1-x)D-d\}C_h}{2xd} + \frac{d\{P-(1-x)D-d\}C_hQ_s^2}{2x(P-D-d)W^2} - \frac{d\{P-(1-y)D-d\}C_sQ_s^2}{2y(P-D-d)W^2} \right] \tag{14}$$

And

$$\frac{\partial(TCD)}{\partial Q_s} = \frac{(2+x+y)}{4m(P-d)} \left[-\frac{d\{P-(1-x)D-d\}}{x(P-D-d)W} \left\{ (P-D-d)\frac{W}{d} - Q_s \right\} C_h + \frac{d\{P-(1-y)D-d\}C_s}{y(P-D-d)W} Q_s \right] \tag{15}$$

The second order partial derivatives of TCD are

$$\frac{\partial^2(TCD)}{\partial W^2} = \frac{dD(2+x+y)}{2m(P-d)W^3} C_o + \frac{d(2+x+y)[y\{P-(1-x)D-d\}C_h + x\{P-(1-y)D-d\}C_s]Q_s^2}{4mxy(P-d)(P-D-d)W^3} \tag{16}$$

$$\frac{\partial^2(TCD)}{\partial Q_s^2} = \frac{d(2+x+y)[y\{P-(1-x)D-d\}C_h + x\{P-(1-y)D-d\}C_s]}{4mxy(P-d)(P-D-d)W} \tag{17}$$

Also

$$\frac{\partial^2(TCD)}{\partial W\partial Q_s} = -\frac{d(2+x+y)[y\{P-(1-x)D-d\}C_h + x\{P-(1-y)D-d\}C_s]Q_s}{4mxy(P-d)(P-D-d)W^2} \tag{18}$$

And

$$\frac{\partial^2(TCD)}{\partial Q_s\partial W} = -\frac{d(2+x+y)[y\{P-(1-x)D-d\}C_h + x\{P-(1-y)D-d\}C_s]Q_s}{4mxy(P-d)(P-D-d)W^2} \tag{19}$$

The Hessian matrix of TCD is

$$H_{ij} = \begin{bmatrix} \frac{\partial^2(TCD)}{\partial W^2} & \frac{\partial^2(TCD)}{\partial W\partial Q_s} \\ \frac{\partial^2(TCD)}{\partial Q_s\partial W} & \frac{\partial^2(TCD)}{\partial Q_s^2} \end{bmatrix}$$

Hence, the first principal minor

$$|H_{11}| = \frac{\partial^2(TCD)}{\partial W^2} = \frac{(2+x+y)}{4m(P-d)} \left[\frac{2dD}{W^3} \times C_o + \frac{d\{P-(1-x)D-d\}C_h}{x(P-D-d)W^3} Q_s^2 + \frac{d\{P-(1-y)D-d\}C_s}{y(P-D-d)W^3} Q_s^2 \right]$$

It is assumed that, $P - D - d > 0$. Therefore,

$$P - (1 - x)D - d > 0, P - (1 - y)D - d > 0 \text{ and } P - d > 0. \text{ Consequently, } |H_{11}| > 0.$$

Once more the second principal minor

$$\begin{aligned} |H_{22}| &= \frac{\partial^2(TCD)}{\partial W^2} \times \frac{\partial^2(TCD)}{\partial Q_s^2} - \frac{\partial^2(TCD)}{\partial Q_s \partial W} \times \frac{\partial^2(TCD)}{\partial W \partial Q_s} \\ &= \frac{d^2 D (2 + x + y)^2 [y\{P - (1 - x)D - d\}C_h + x\{P - (1 - y)D - d\}C_s]C_o}{8m^2 xy(P - d)^2(P - D - d)W^4} > 0. \end{aligned}$$

Since the first and second principal minors of the Hessian matrix for TCD are positive, the Hessian matrix is positive definite. Hence TCD is a nonnegative, differentiable, and strictly convex function regarding the decision variables W and Q_s simultaneously, and TCD is minimum at the unique optimal values W^* and Q_s^* . Solving the necessary conditions $\frac{\partial(TCD)}{\partial W} = 0$ and $\frac{\partial(TCD)}{\partial Q_s} = 0$ from Equations (14) and (15) the following results are obtained.

$$\frac{Q_s}{W} = \frac{y(P - D - d)\{P - (1 - x)D - d\}C_h}{d[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]} \tag{20}$$

And

$$W = d \sqrt{\frac{2DC_o}{C_h}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s}} \tag{21}$$

Therefore, the optimal solution is given by

$$W^* = d \sqrt{\frac{2DC_o}{C_h}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s}} \tag{22}$$

And

$$Q_s^* = y \sqrt{2DC_o} \sqrt{\frac{(P - D - d)\{P - (1 - x)D - d\}}{\{P - (1 - y)D - d\}}} \sqrt{\frac{C_h}{[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]C_s}} \tag{23}$$

Then the corresponding optimal production lot size is

$$Q^* = P \sqrt{2DC_o} \sqrt{\frac{1}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{C_h C_s}} \tag{24}$$

And the optimal cycle time is

$$T^* = \sqrt{\frac{2C_o}{D}} \cdot \sqrt{\frac{\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}}{(P - D - d)[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \cdot \frac{C_s + C_h}{\sqrt{C_h C_s}} \tag{25}$$

Also the maximum on-hand stock is

$$Q_d^* = x \sqrt{2DC_o} \sqrt{\frac{(P - D - d)\{P - (1 - y)D - d\}}{\{P - (1 - x)D - d\}}} \sqrt{\frac{C_s}{[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]C_h}} \tag{26}$$

□

Special Cases

(i) If $d = 0$, then $W^* = 0$.

Therefore, no defective items will be produced in the inventory if the defective rate is zero.

(ii) If $x = y = 1$ and $d = 0$, then we get

$$Q^* = \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{P}{(P-D)}} \cdot \sqrt{\frac{C_s + C_h}{C_s}}$$

This is the lot size in the standard EPQ model. Therefore, the optimal solution of the total cost function of this model conforms to the standard EPQ model.

(iii) If $x = y$, then,

$$W^* = d \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{x(C_s + C_h)}{(P-D-d)\{P-(1-x)D-d\}C_s}}$$

$$\text{and } Q^* = P \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{x(C_s + C_h)}{(P-D-d)\{P-(1-x)D-d\}C_s}}$$

These are the results of the inventory model for two levels of piecewise constant demand.

(iv) If $x = y = 1$, then,

$$W^* = d \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{(C_s + C_h)}{(P-D-d)(P-d)C_s}}$$

$$\text{And } Q^* = P \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{(C_s + C_h)}{(P-D-d)(P-d)C_s}}$$

These are the results of the inventory model of defective items for constant demand.

4.2. Case 2 (ERQ Model for Defective Items with Three Levels of Piecewise Constant Demand under Shortages)

The convexity of TCR and hence the uniqueness of optimal values of the decision variables are determined to prove the following theorem.

Theorem 2. *The total cost function TCR gives a minimum value regarding the decision variables W and Q_s and hence a unique solution exists for W* and Q_s*.*

Proof. TCR is a function of W and Q_s, and the relation between TCR and TCD is obtained as

$$TCR = TCD - \frac{(2 + x + y)D}{4m(P - d)} \left[d(C_R - C_r) - \frac{W}{2}C_h \right] \tag{27}$$

From the first order and second order partial derivatives, the following results are obtained from Equation (27).

$$\frac{\partial^2(TCR)}{\partial W^2} = \frac{\partial^2(TCD)}{\partial W^2}; \frac{\partial^2(TCR)}{\partial Q_s^2} = \frac{\partial^2(TCD)}{\partial Q_s^2}; \frac{\partial^2(TCR)}{\partial W \partial Q_s} = \frac{\partial^2(TCD)}{\partial W \partial Q_s} \text{ and } \frac{\partial^2(TCR)}{\partial Q_s \partial W} = \frac{\partial^2(TCD)}{\partial Q_s \partial W}$$

The Hessian matrix of TCR is

$$H_{ij} = \begin{bmatrix} \frac{\partial^2(TCR)}{\partial W^2} & \frac{\partial^2(TCR)}{\partial W \partial Q_s} \\ \frac{\partial^2(TCR)}{\partial Q_s \partial W} & \frac{\partial^2(TCR)}{\partial Q_s^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2(TCD)}{\partial W^2} & \frac{\partial^2(TCD)}{\partial W \partial Q_s} \\ \frac{\partial^2(TCD)}{\partial Q_s \partial W} & \frac{\partial^2(TCD)}{\partial Q_s^2} \end{bmatrix}$$

Therefore, from Theorem 1, it can be concluded that the first principal minor

$$|H_{11}| = \frac{\partial^2(TCR)}{\partial W^2} = \frac{\partial^2(TCD)}{\partial W^2} > 0$$

And the second principal minor

$$|H_{22}| = \frac{\partial^2(TCR)}{\partial W^2} \times \frac{\partial^2(TCR)}{\partial Q_s^2} - \frac{\partial^2(TCR)}{\partial Q_s \partial W} \times \frac{\partial^2(TCR)}{\partial W \partial Q_s} = \frac{\partial^2(TCD)}{\partial W^2} \times \frac{\partial^2(TCD)}{\partial Q_s^2} - \frac{\partial^2(TCD)}{\partial Q_s \partial W} \times \frac{\partial^2(TCD)}{\partial W \partial Q_s} > 0.$$

Since the first and second principal minor of Hessian matrix for TCR are positive, the Hessian matrix is positive definite. Hence TCR is a nonnegative, differentiable, and strictly convex function regarding W and Q_s concurrently and then TCR is minimum at the unique optimal solution W^* and Q_s^* .

The necessary conditions of minimization of TCR are the first order partial derivatives of TCR with respect to W and Q_s are equivalent to zero, and hence the following results are obtained.

$$\frac{Q_s}{W} = \frac{y(P - D - d)\{P - (1 - x)D - d\}C_h}{d[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]} \tag{28}$$

And

$$W = d\sqrt{\frac{2DC_O}{C_h}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \tag{29}$$

Therefore, the required economic recycle quantity (ERQ) is

$$W^* = d\sqrt{\frac{2DC_O}{C_h}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \tag{30}$$

And shortage level is

$$Q_s^* = \frac{y(P - D - d)\{P - (1 - x)D - d\}C_h}{\sqrt{\left[\begin{matrix} x\{P - (1 - y)D - d\}C_s \\ + y\{P - (1 - x)D - d\}C_h \end{matrix} \right]} C_h} \sqrt{\frac{2DC_O}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \tag{31}$$

The corresponding production lot size of the inventory is

$$Q^* = P\sqrt{\frac{2DC_O}{C_h}} \sqrt{\frac{x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \tag{32}$$

And the optimal cycle time is

$$T^* = \sqrt{\frac{2C_O}{D}} \cdot \frac{\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}}{\sqrt{\left[\begin{matrix} (P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s \\ + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h \end{matrix} \right]}} \times \frac{C_h + C_s}{\sqrt{\left[\begin{matrix} x\{P - (1 - y)D - d\}C_s \\ + y\{P - (1 - x)D - d\}C_h \end{matrix} \right]} C_h} \tag{33}$$

Also, the maximum on-hand inventory is

$$Q_d^* = \frac{x(P - D - d)\{P - (1 - y)D - d\}C_s}{\sqrt{\left[\begin{matrix} x\{P - (1 - y)D - d\}C_s \\ + y\{P - (1 - x)D - d\}C_h \end{matrix} \right]} C_h} \sqrt{\frac{2DC_0}{(P - D - d)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}C_s + dD[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]}} \quad (34)$$

□

Special Cases

(i) If $d = 0$, then $W^* = 0$.

Therefore, no defective items will be produced in the inventory if the defective rate is zero.

(ii) If $x = y = 1$ and $d = 0$, then,

$$Q^* = \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{P}{(P - D)}} \cdot \sqrt{\frac{C_s + C_h}{C_s}}$$

This is the lot size in the standard EPQ model.

Therefore, the optimal outcomes are conformable with the standard EPQ model.

(iii) If $x = y$ then,

$$W^* = d \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{x(C_s + C_h)}{(P - D - d)\{P - (1 - x)D - d\}C_s + xdD(C_s + C_h)}}$$

$$Q^* = P \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{x(C_s + C_h)}{(P - D - d)\{P - (1 - x)D - d\}C_s + xdD(C_s + C_h)}}$$

These are the results of the ERQ model for two levels of piecewise constant demand.

(iv) If $x = y = 1$, then,

$$W^* = d \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{(C_s + C_h)}{(P - D - d)(P - d)C_s + dD(C_s + C_h)}}$$

$$Q^* = P \sqrt{\frac{2DC_0}{C_h}} \sqrt{\frac{(C_s + C_h)}{(P - D - d)(P - d)C_s + dD(C_s + C_h)}}$$

These are the results of the ERQ model of defective items for constant demand.

4.3. Value of “m”

It is assumed that the arbitrary constant “m” determines the accurate average demand as well as total inventory cost but has no significance on the optimal solution. It is introduced to avoid the complexity of the theoretical results. The assumption is justified by the obtained optimal solutions of TCD and TCR and their special cases. The value of “m” is calculated in the following manner.

Addition of Equations (3) and (6) gives the cycle time

$$T = \frac{x\{P - (1 - y)D - d\}Q_s + y\{P - (1 - x)D - d\}Q_d}{xyD(P - D - d)} \quad (35)$$

Then, Equation (9) implies

$$m = \frac{d(2 + x + y)[x\{P - (1 - y)D - d\}Q_s + y\{P - (1 - x)D - d\}Q_d]}{4xy(P - d)(P - D - d)W} \quad (36)$$

Putting the optimal values of Q_s and Q_d of both of the total cost function TCD and TCR, the same value of “ m ” is obtained as follows:

$$m = \frac{(2 + x + y)\{P - (1 - x)D - d\}\{P - (1 - y)D - d\}(C_s + C_h)}{4(P - d)[x\{P - (1 - y)D - d\}C_s + y\{P - (1 - x)D - d\}C_h]} \tag{37}$$

5. Numerical Illustrations and Case Study

The inventory model is analyzed considering a piecewise constant demand rate that may vary from one level to another. Certain products, for instance, such as bricks, mustard oil, and shopping mall products, have different demands during their production-run periods than during production-off shortage periods. The practical implications of the proposed model can be observed in a brick production company. For the production time of brick raw materials, some imperfect bricks are made before firing, for instance, during the setting and drying periods. These defective bricks are collected and stored properly in stock before they are sent for recycling during production off-time and added to the inventory for the next production cycle. Furthermore, the demand for bricks during the production off-time and shortage time may increase or decrease from the demand during their production run-time.

In order to study the applicability of the model, numerical illustrations are provided below considering the following static input values of parameters taken from MJB brick production company in Bangladesh where: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, x = 0.75,$ and $y = 0.5$.

5.1. For Case 1

Optimal decision variable values:

$$W^* = 136, Q^* = 6822, Q_s^* = 414, Q_d^* = 131, \text{ and } T^* = 1.5877$$

Optimal cycle time level values:

$$t_1^* = 0.3275, t_2^* = 0.0388, t_3^* = 0.1843, t_4^* = 1.0370$$

Optimal costs and the total inventory cost:

$$FC = 630, PC = 214,859, SC = 478, RMC = 214,859, HC = 151, \text{ and } TCD = 430,978$$

Cycle time verification for TCD:

$$t_1^* + t_2^* + t_3^* + t_4^* = 0.3275 + 0.0388 + 0.1843 + 1.0370 = 1.5876 = T^*$$

Hence, the model is consistent.

For the values of independent variables in the range $120 < W < 150$ and $400 < Q_s < 430$, Figure 4 depicts the convexity of the total cost function TCD with respect to W and Q_s .

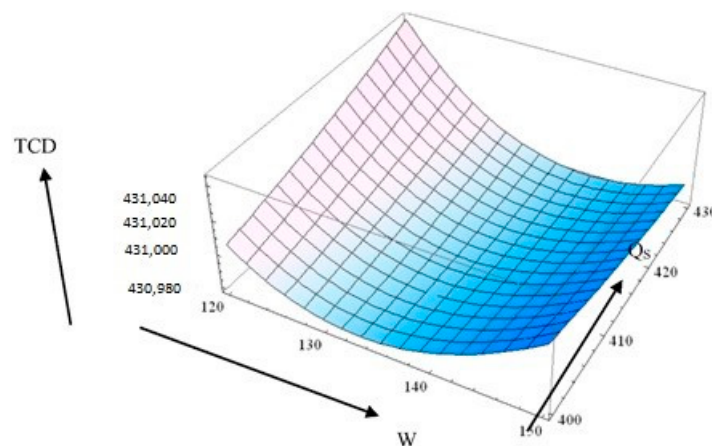


Figure 4. Convexity of TCD with respect to W and Q_s .

For $Q_s = 414$ and $120 < W < 150$, Figure 5 depicts the convexity of the total cost function TCD vs. W .

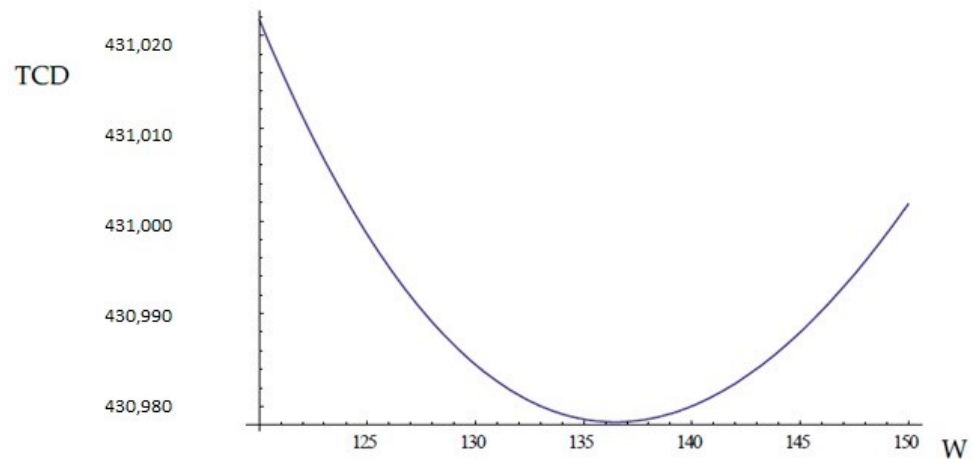


Figure 5. TCD vs. W graph.

Fixing $W = 136$ and $250 < Q_s < 350$, the following Figure 6 shows that TCD is convex with respect to Q_s , and the total inventory cost reaches minimum at “ $Q_s = 414$ ”.

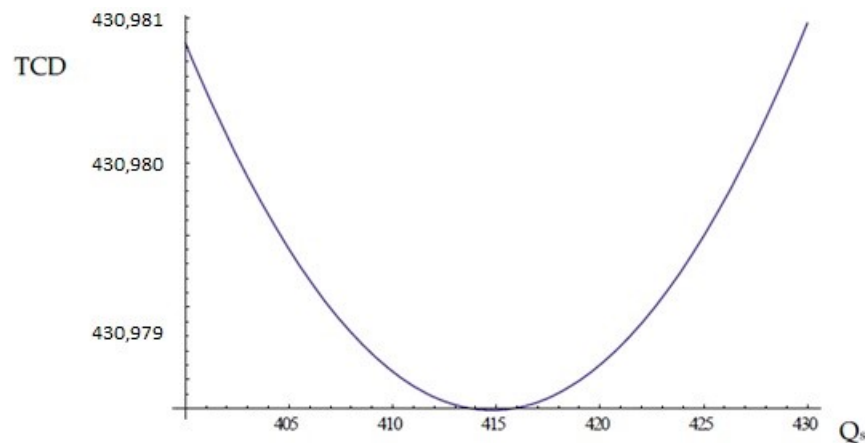


Figure 6. TCD vs. Q_s graph.

5.2. For Case 2

Optimal decision variable values: $W^* = 98, Q^* = 4910, Q_s^* = 298, Q_d^* = 94, T^* = 1.1426$

Optimal cycle time level values: $t_1^* = 0.2357, t_2^* = 0.0279, t_3^* = 0.1326, t_4^* = 0.7462$

Optimal costs and total inventory cost: $FC = 875, PC = 214,859, SC = 344, RC = 430, RMC = 210,562, HC = 530$ and $TCR = 427,602$

Cycle time verification: $t_1^* + t_2^* + t_3^* + t_4^* = 0.2357 + 0.0279 + 0.1326 + 0.7462 = 1.1425 = T^*$. Hence, the model is consistent.

For the values of independent variables in the range $80 < W < 120$ and $250 < Q_s < 350$, the following Figure 7 depicts the convexity of the total cost function TCR for W and Q_s .

For $Q_s = 300$ and $80 < W < 120$, the following figure (Figure 8) depicts the convexity of the total cost function with respect to “ W ”.

For $W = 98$ and $250 < Q_s < 350$, Figure 9 shows that TCR is convex with respect to Q_s and the total cost of the inventory is minimum at “ $Q_s = 300$ ”.

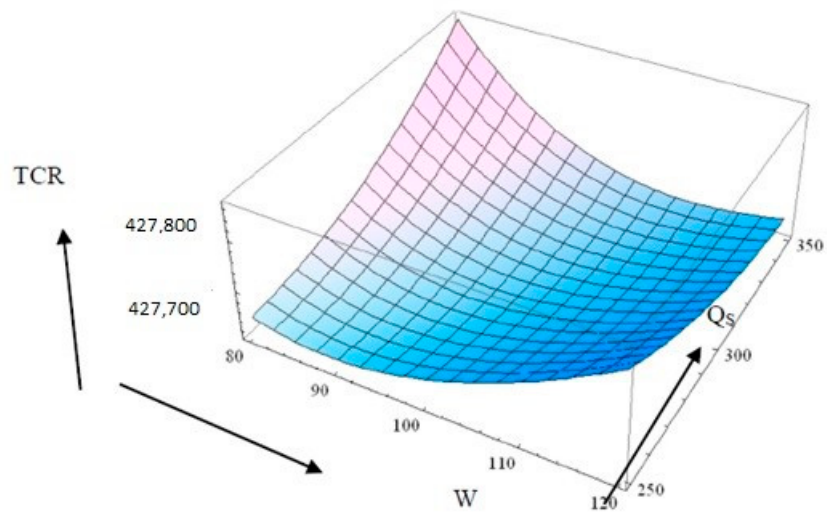


Figure 7. Convexity of TCR relating to W and Q_s .

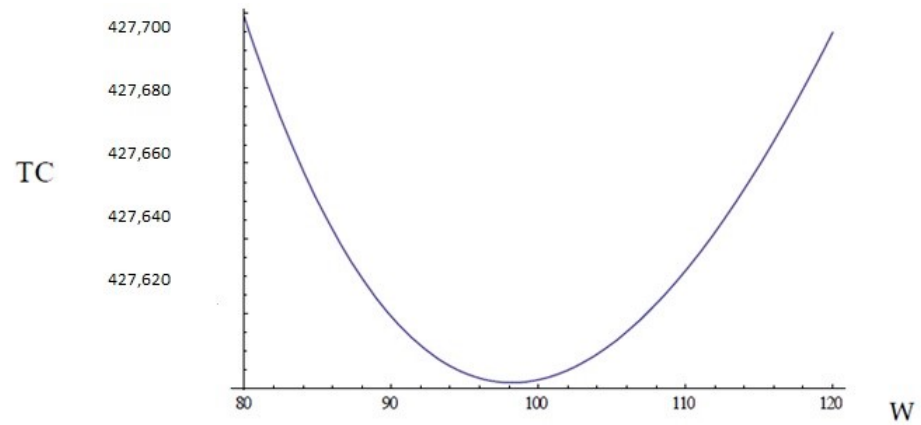


Figure 8. TC vs. W graph.

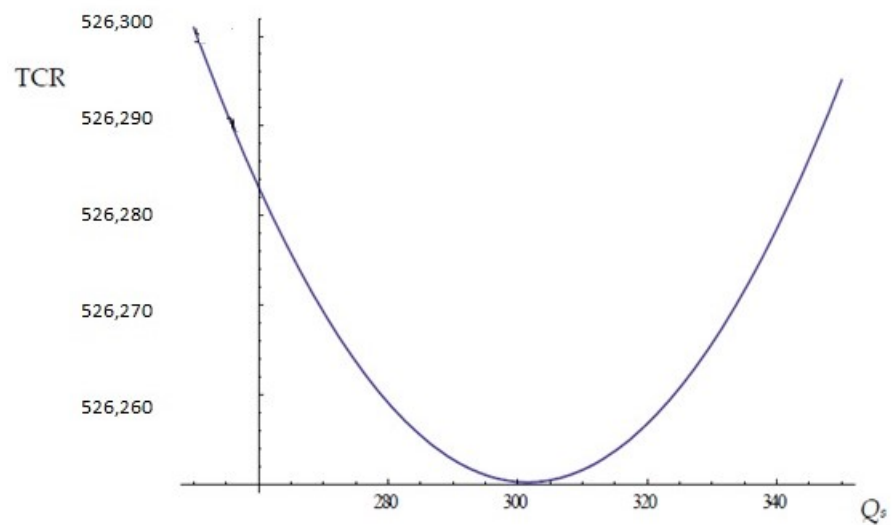


Figure 9. TCR vs. Q_s graph.

6. Sensitivity Analysis

In this section, a sensitivity analysis of the anticipated model of various system parameters is examined. When the sensitivity of a parameter is analyzed, only changes to a parameter are considered keeping other parameters as fixed. The sensitivity analyses of the key parameters are given in Tables 2–9.

Table 2. Sensitivity of “*x*” to different decision variables and costs. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, y = 1$.

<i>x</i>	0.5	0.75	1.75	5	Observations
W^*	98	98	99	99	Increase
Q^*	4894	4909	4929	4940	Increase
Q_s^*	301	299	295	293	Decrease
Q_d^*	90	94	99	102	Increase
t_1^*	0.225	0.235	0.248	0.254	Increase
t_2^*	0.0410	0.027	0.012	0.004	Decrease
t_3^*	0.133	0.132	0.131	0.130	Decrease
t_4^*	0.752	0.746	0.737	0.733	Decrease
T^*	1.152	1.142	1.129	1.122	Decrease
FC	867	875	885	890	Increase
RMC	208,019	210,562	213,820	215,565	Increase
PC	212,264	214,859	218,184	219,965	Increase
HC	519	530	545	552	Increase
SC	347	344	340	338	Increase
RC	424	430	436	440	Increase
TCR	422,442	427,602	434,210	437,755	Increase

Table 3. Sensitivity of “*y*” to different decision variables and costs. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, x = 1$.

<i>y</i>	0.5	0.8	1.50	2	5	Observations
W^*	98.36	99.36	100.4	100.84	101.2	Increase
Q^*	4918	4968	5020	5042	5058	Increase
Q_s^*	297	305	309	313	316	Increase
Q_d^*	96	91.72	90.72	89.5	88	Decrease
t_1^*	0.24098	0.2293	0.22682	0.22383	0.22157	Decrease
t_2^*	0.02142	0.02547	0.02016	0.01989	0.01969	Decrease
t_3^*	0.13202	0.084992	0.04605	0.02789	0.01404	Decrease
t_4^*	0.74265	0.7643	0.77721	0.78455	0.79016	Increase
T^*	1.13709	1.10406	1.07025	1.05619	1.04548	Decrease
FC	879	905	934	947	956	Increase
RMC	211,938	220,500	229,841	233,914	237,093	Increase
PC	216,263	225,000	234,532	238,688	241,931	Increase
HC	536	552	575	584	591	Increase
SC	342	352	359	362	364	Increase
RC	432	450	469	477	483	Increase
TCR	430,392	447,762	466,711	474,972	481,421	Increase

Table 4. Sensitivity of defective rate to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, x = 1.5, y = 1.5$.

<i>d</i>	100	110	120	130	140
PCB	0.797%	0.878%	0.96%	1.042%	1.125%
Remarks	Cost–benefit increases with the increase in “ <i>d</i> ”.				

Table 5. Sensitivity of “recycle cost” to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_s = 3, x = 1.5, y = 1.5$.

C_r	5	10	15	20	25
PCB	0.797%	0.697%	0.59%	0.497%	0.398%
Remarks	Cost–benefit decreases with increase in recycle cost.				

Table 6. Sensitivity of “holding cost” to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_R = 50, C_r = 5, C_s = 3, x = 1.5, y = 1.5$.

C_h	10	20	30	40	50
PCB	0.797%	0.737%	0.689%	0.648%	0.613%
Remarks	Cost–benefit decreases with increase in holding cost.				

Table 7. Sensitivity of “ x ” to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, y = 1.5$.

x	0.5	1	1.5	2	2.5
PCB	0.7976%	0.7972%	0.7970%	0.7969%	0.7969%
Remarks	“ x ” is not sensitive to cost–benefit.				

Table 8. Sensitivity of “ y ” to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_R = 50, C_r = 5, C_s = 3, x = 1.5$.

y	0.5	1	1.5	2	2.5
Percentage of cost–benefit	0.7990%	0.7975%	0.7970%	0.7968%	0.7966%
Remarks	“ y ” is not sensitive to cost–benefit.				

Table 9. Sensitivity of raw material cost to cost–benefit. The static input values of the parameters are: $P = 5000, D = 4500, d = 100, C_o = 1000, C_p = 50, C_h = 10, C_r = 5, C_s = 3, x = 1.5, y = 1.5$.

C_R	50	55	60	65	70
PCB	0.797%	0.854%	0.906%	0.954%	0.997%
Remarks	Cost–benefit increases with the increase in raw material cost.				

From Table 2, it is observed that total cost increases with increasing values of “ x ”, and as a result, total order quantity upsurges. In contrast, the total cycle length “ T ” decreases. The significant increases in all the associated costs, e.g., PC, HC, RC are also noted. However, the maximum shortage Q_s is notably reduced.

Explanations: From Table 3, it is clear that with the rise of “ y ”, the total cost is amplified, while the total order quantity is also increased. In contrast, the total cycle length “ T ” declines. A noteworthy intensification in all the related costs, e.g., PC, HC, RC, can also be observed. Moreover, the maximum shortage Q_s notably rises.

Cost–Benefit of ERQ Model

Cost–benefit is a tool used to gain insights into the decision-making process. In the cost–benefit analysis, we considered the comparative total cost of two inventory systems: the system without recycling and system with recycling of defective items. Thus,

Average cost–benefit ACB is

$$ACB = TCD - TCR = \frac{(2 + x + y)D}{4m(P - d)} \left[d(C_R - C_r) - \frac{W}{2}C_h \right] \tag{38}$$

Percentage of cost–benefit (PCB) is

$$\text{PCB} = \left(\frac{ACB}{TCD} \times 100 \right) \% \quad (39)$$

Therefore, from Equation (38), we see that the cost–benefit increases when the cost of raw materials increases and the cost–benefit decreases when holding cost, recycling cost, or both costs increase.

While the effect of a parameter is analyzed, only changes to one parameter is considered while other parameters remain static.

7. Managerial Insights

The proposed model helps to control inventories and to determine the economic recycle quantity (ERQ), optimal production lot size, cycle time, total inventory cost, maximum on-hand stock, and maximum backlogs. From the numerical analysis, sensitivity analysis (Tables 2–9), and cost–benefit analysis, the following findings provide managerial insights to inventory managers.

- Recycling larger or smaller quantities of defective items than the ERQ will increase the total inventory cost. Therefore, inventory managers must evaluate the ERQ of the item before recycling.
- When the company decides to recycle, production lot sizes should be set up in favor of the ERQ, and this strategy will provide greater facility to the manager to decline the total cost.
- If raw material costs increase significantly, recycling will benefit the company because it will diminish the load on raw-materials and use the wastage items.
- The total cost is more sensitive to demand increases during shortage time than it is to increases during production time with positive stock.
- If the defective rate is high, companies will receive better cost–benefit from recycling because it produces more defective items, which are usually treated as rejected, but the recycling process converts them to useable, which provides huge revenue for the managers.

8. Conclusions

In this study, we developed an imperfect production model with and without the recycling of defective items. The model is a generalization of models considering recycled raw materials. It is worth mentioning that recycled materials can be an alternative source of raw materials in the event of any uncertainties. Moreover, recycling defective production reduces waste and hence is also environmentally friendly. The study can be applied to bricks, textile, glass, paper, and jute production. It is demonstrated that the total cost function is strictly convex with a unique optimal solution. While this model considers a single item production system, multi-item inventory systems of the same recyclable raw material can be developed for further research. This paper outlines when to stop the production process and when to start it. Proper management of the recycling process is also illustrated in the model with some insights for managers.

Furthermore, the model can be extended to consider time-dependent defective rates and selling prices (Mashud et al. 2020a). Considering the environmental emissions (Mashud et al. 2021b; Mishra et al. 2021) would be another interesting extension of this study. As the recycling process is shown in the model, a proper waste management (Tsai et al. 2021) could be another important discussion that is missing in this study. Some marketing strategies, for instance, discount on imperfect items (Mashud et al. 2020a), trade credit financing (Liao et al. 2018, 2020; Srivastava et al. 2018; Mashud et al. 2021a, 2021d) can be implemented in the proposed model to make it more lucrative to the practitioners. Other imperfect process in both the EOQ and EPQ models (Lin and Srivastava 2015; Srivastava et al. 2021) can also be consider as an interesting extension.

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