# Secure Paired Domination in Graphs

by

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## ABSTRACT

This thesis introduces a new strategy of defending the vertices of a graph - secure paired domination, where guards are required to be paired and, when a vertex is attacked, one or two guards move to defend the attacked vertex, while keeping the graph dominated and the guards paired after the move. We propose nine possible definitions of secure paired domination, compare and contrast each with the others, and obtain properties and inequalities of the secure paired domination (SPD) numbers associated with the definitions. Based on each of the nine definitions, the SPD numbers of five types of special graphs, namely paths, cycles, spiders, ladders and grid graphs, are studied.

We then compare the SPD number of an arbitrary isolate-free graph to various other parameters such as clique partition number, independence number, vertex-covering number, secure domination number and paired domination number. We establish that, for any graph without isolated vertices, its SPD number does not exceed twice the value of any of its other parameters mentioned above. Also, we give classes of trees for which some of the bounds are achieved. As conclusion, some open problems and directions for further studies regarding secure paired domination are listed.

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# DEDICATION

I dedicate this thesis to my parents, Zhuo Kang and Li Juan Nie, my grand father Xu Long Nie, and my girl friend Piching Lee, and all others who had helped me throughout the writing of the thesis.

In memory of my grand mother, Sheng-Di Zhang.

# Chapter 1

# Background of Graph Domination

In recent decades a great amount of effort has been devoted to the study of graph protection and domination. We begin our discussion of some domination strategies that are relevant to this thesis by stating a few preliminary definitions.

Let $G = (V, E)$ denote a finite, simple graph. The *open neighbourhood* $N(x)$ of a vertex $x$ of $G$ is the set of all vertices adjacent to $x$, while its *closed neighbourhood* $N[x]$ is defined by $N[x] = N(x) \cup \{x\}$. For $X \subseteq V$, the *open* and *closed neighbourhoods* of $X$ are defined by $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = \bigcup_{x \in X} N[x]$, respectively. The *private neighbourhood* $\text{pn}(x, X)$ (respectively *external private neighbourhood* $\text{epn}(x, X)$) of $x \in X$ *relative to* $X$ is defined by $\text{pn}(x, X) = N[x] - N[X - \{x\}]$ (respectively $\text{epn}(x, X) = \text{pn}(x, X) - \{x\}$). The vertices in $\text{pn}(x, X)$ (respectively $\text{epn}(x, X)$) are called the *$X$-pns* or *private neighbours* of $x$ *relative to* $X$ (respectively *$X$-epns* or *external private neighbours* of $x$ *relative to* $X$). We abbreviate $X$-epn as epn whenever the context is clear. Furthermore, a vertex $y \in X$ is an *$X$-ipn* or *internal private neighbour* of $x$ *relative to* $X$ if $N(y) \cap X = \{x\}$. The *internal private neighbourhood* $\text{ipn}(x, X)$ is the set of all $X$-ipns of $x$.

We think of each vertex $s$ of $G$ as a possible location of a guard capable of protecting each vertex in its closed neighbourhood, and "domination" requires every vertex to be protected. Formally, a *dominating set* $D$ of $G$ is a subset of $V$ such that every vertex of $G$ is either in $D$ or adjacent to a vertex in $D$. The *domination number* $\gamma(G)$

is the minimum cardinality of a dominating set.

By placing conditions on the nature of the dominating sets and distinguishing between stationary and mobile guards, several "special" kinds of domination have been introduced. The research topic of this thesis, secure paired domination, describes a protection model in which mobile guards work in pairs to protect the vertices of a graph. The use of mobile guards first occurred in Roman domination, which then gave rise to weak Roman domination and secure domination. The first use of mobile guards where guards are not allowed to be isolated from one another occurred in secure total domination.

To place secure paired domination in perspective, this chapter contains the definitions, background, and a selection of noticeworthy results regarding Roman domination, weak Roman domination, secure domination, total domination, secure total domination, and paired domination.

## 1.1 Roman Domination and Weak Roman Domination

### 1.1.1 Roman Domination

The mathematical concept of Roman domination is the oldest known type of domination in which mobile guards are used, and has its historical roots in the time of the ancient Roman Empire [1, 32, 33]. According to ancient history, Rome was founded by Romulus and Remus in 760 – 750 before the Common Era (BCE) on the banks of the Tiber in central Italy. It was a country town whose power gradually grew until it was the centre of a large empire. The expansion of Roman power was due to a variety of political, military, geographical, and economic factors.

In the third century of the Common Era (CE), Rome dominated not only Europe, but also North Africa and the Near East. The Roman army at that time was strong enough to use a *forward defense* strategy, deploying an adequate number of legions to secure on-site every region throughout the empire. However, the Roman Empire's power was greatly reduced over the following hundred years. By the fourth century CE, only twenty-five legions of the Roman army were available, which made a forward defense strategy no longer feasible.

According to E. N. Luttwak, *The Grand Strategy of the Roman Empire*, as cited in [33], to cope with the reducing power of the Empire, Emperor Constantine (Constantine The Great, 274-337 CE) devised a new strategy called a *defense in depth* strategy, which used local troops to disrupt invasion. He deployed mobile Field Armies (FAs), units of forces consisting of roughly six legions powerful enough to secure any one of the regions of the Roman Empire, to stop and throw back the intruding enemy, or to suppress insurrection. By the fourth century CE there were only four FAs available for deployment, whereas there were eight regions to be defended (Britain, Gaul, Iberia, Rome, North Africa, Constantinople, Egypt and Asia Minor) in the empire.

Symbolically, regions are represented as dots, and movement along a line (edge) between regions represents a "step". An FA is capable of deploying to protect an adjacent region only if it moves from a region where there is at least one other FA to help launch it. We call a region *secured* if it has one or more FAs stationed in it already, and a region is considered *securable* if an FA can reach it in one step. The challenge that Emperor Constantine faced was to place four FAs to positions in the eight regions of the empire. Constantine decided to place two FAs in Rome and another two FAs in Constantinople, making all regions either secured or securable – with the exception of Britain, which could only be secured after at least four movements of FAs.

It is mentioned in [1, 33, 35] that Constantine's "defense in depth" strategy was adopted during World War II by General Douglas MacArthur. When conducting military operations in the Pacific theatre, he pursued a strategy of "island-hopping" – moving troops from one island to a nearby one, but only when he could leave behind a large enough garrison to keep the first island secure. The efficiency of Constantine's strategy under different criteria, and ways in which it can be improved, were also discussed in these three articles.

The Roman domination problem can be formalized as follows. A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The *weight* of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph $G$ is called the *Roman domination number* of $G$, denoted by $\gamma_{\mathrm{R}}(G)$.

Some general graph theoretic properties of this parameter is studied in [6], summarized as follows:

**Proposition 1.1.** [6] *For any graph $G$ with order $n$ and maximum degree $\Delta$,*

(*i*) $\gamma(G) \leq \gamma_{\mathrm{R}}(G) \leq 2\gamma(G)$,

(*ii*) $\gamma(G) = \gamma_{\mathrm{R}}(G)$ *if and only if* $G = \overline{K_n}$,

(*iii*) $2n/(\Delta + 1) \leq \gamma_{\mathrm{R}}(G) \leq n(2 + \ln(\frac{1+\delta(G)}{2}))/(1 + \delta(G))$.

Furthermore, some specific values of Roman domination numbers are given in [6], such as the Roman domination numbers for paths and cycles, complete $n$-partite graphs, and $2 \times n$ grid graphs. Characterizations are also given for graphs with $\gamma_{\mathrm{R}}(G) = \gamma(G) + 2$, trees with $\gamma_{\mathrm{R}}(T) = \gamma(T) + 1$, and $\gamma_{\mathrm{R}}(T) = \gamma(T) + 2$.

For an integer $t$, a *wounded spider* is a star $K_{1,t}$ with at most $t - 1$ of its edges subdivided exactly once. Similarly, for an integer $t \geq 2$, a *healthy spider* is a star $K_{1,t}$ with all of its edges subdivided exactly once. In a wounded spider, the vertex of degree $t$ is called the *head vertex*, and the vertices that are distance two from the head vertex are the *foot vertices*. Both vertices in $P_2$ are considered to be head vertices, and in the case of $P_4$, both end vertices are considered foot vertices whereas the two central vertices are head vertices.

**Proposition 1.2.** [6]

(*i*) *If $G$ is a connected graph of order $n$, then $\gamma_{\mathrm{R}}(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V$ of degree $n - \gamma(G)$.*

(*ii*) *If $T$ is a tree on two or more vertices, then $\gamma_{\mathrm{R}}(T) = \gamma(T) + 1$ if and only if $T$ is a wounded spider.*

**Proposition 1.3.** [6] *If $G$ is a connected graph of order $n$, then $\gamma_{\mathrm{R}}(G) = \gamma(G) + 2$ if and only if*

(*i*) *$G$ does not have a vertex of degree $n - \gamma(G)$, and*

(*ii*) *either $G$ has a vertex of degree $n - \gamma(G) - 1$, or $G$ has two vertices $v$ and $w$ such that $|N[v] \cup N[w]| = n - \gamma(G) + 2$.*

**Proposition 1.4.** [6] *If $T$ is a tree of order $n \geq 2$, then $\gamma_{\mathrm{R}}(T) = \gamma(T) + 2$ if and only if either*

(*i*) *$T$ is a healthy spider or*

(*ii*) *$T$ consists of a pair of wounded spiders $T_1$ and $T_2$, not both isomorphic to $P_2$, with a single edge joining $v \in V(T_1)$ and $w \in V(T_2)$, subject to the following conditions:*

  (a) *if either tree is a $P_2$, then neither vertex in $P_2$ is joined to the head vertex of the other tree,*

  (b) *$v$ and $w$ are not both foot vertices.*

Can Propositions 1.2 and 1.3 be generalized to produce a characterization of graphs for which $\gamma_{\mathrm{R}}(G) = \gamma(G) + k$? This question, which is left as an open problem in [6], is answered in [36]:

**Theorem 1.5.** [36] *Let $G$ be a connected graph of order $n$ and domination number $\gamma(G) \geq 2$. If $k$ is an integer such that $2 \leq k \leq \gamma(G)$, then $\gamma_{\mathrm{R}}(G) = \gamma(G) + k$ if and only if*

(*i*) *for any integer $s$ with $1 \leq s \leq k - 1$, $G$ does not have a set $U_t$ of $t$ $(1 \leq t \leq s)$ vertices such that $|\bigcup_{v \in U_t} N[v]| = n - \gamma(G) - s + 2t$, and* (*ii*) *there exists an integer* $l$ *with* $1 \le l \le k$ *such that* $G$ *has a set* $W_l$ *of* $l$ *vertices such that* $|\bigcup_{v \in W_l} N[v]| = n - \gamma(G) - s + 2l$.

It is stated in Proposition 1.1 that the Roman domination number of any graph $G$ is bounded above by twice its domination number. We call graphs which achieve this bound *Roman graphs*. A characterization of Roman trees is given in [19]: a tree is a Roman tree if and only if it can be obtained from a star $K_{1,r}$, $r \ge 1$, by a finite number of applications of three operations. The detailed statement of the theorem is rather technical, so we omit it for brevity.

Research has also been done on Roman domination in regular graphs. The *circulant* $C_n\langle S\rangle$, where $S \subseteq \{1,\ldots,\lfloor n/2\rfloor\}$, is the graph with vertex set $V(C_n\langle S\rangle) = \{0,1,...,n-1\}$ and edge set $E(C_n\langle S\rangle) = \{ij : 0 \le i,j \le n-1$ and $i-j \equiv \pm s \pmod{n}$ for some $s \in S\}$.

In [14] the authors gave some new classes of Roman graphs.

**Theorem 1.6.** [14] *The circulants* $C_n\langle 1,3\rangle$ *are Roman for* $n \ge 7$ *and* $n \not\equiv 4 \pmod{5}$. *The circulants* $C_n\langle\{1,2,\ldots,k\}\rangle$ *are Roman for* $n \ge 4$ $(n \ne 2k)$, $2 \le k \le \lfloor \frac{n}{2} \rfloor$, *and* $n \not\equiv 1 \pmod{(2k+1)}$.

**Theorem 1.7.** [14] *For* $n \equiv 0 \pmod{4}$ *and* $0 \le k \le \frac{1}{2}\left(\lfloor \frac{n-3}{2} \rfloor\right)$, *the generalized Petersen graph* $P(n,2k+1)$ *is Roman. Also,* $P(n,1)$ *is Roman for* $n \ge 3$ *and* $n \not\equiv 2 \pmod{4}$, *and* $P(n,3)$ *is Roman for* $n = 11$, *or* $n \ge 7$ *and* $n \not\equiv 3 \pmod{4}$.

**Theorem 1.8.** [14] *For* $n \ge 1$, $m \ge 1$, *the Cartesian product* $C_{5m} \,\square\, C_{5n}$ *is Roman.*

### 1.1.2 Weak Roman Domination

Weak Roman domination, an alternative defense strategy with the potential of saving Emperor Constantine the Great substantial costs of maintaining legions while still defending the Roman Empire, is introduced in [22].

In the language of graph theory, let $f : V \to \{0,1,2\}$ be a function defined on a graph $G = (V,E)$. A vertex $v$ with $f(u) = 0$ is said to be *undefended with respect* *to f* if it is not adjacent to a vertex with positive weight. The function $f$ is a *weak Roman dominating function* (*WRDF*) if each vertex $u$ with $f(u) = 0$ is adjacent to a vertex $v$ with $f(v) > 0$ such that the function $f' : V \to \{0, 1, 2\}$, defined by

$$f'(u) = 1, f'(v) = f(v) - 1 \text{ and } f'(w) = f(w) \text{ if } w \in V - \{u, v\}$$

has no undefended vertex. Similar to Roman domination, the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. The *weak Roman domination number*, denoted by $\gamma_r G$, is the minimum weight of a WRDF in $G$.

In [22] the authors compare $\gamma(G)$, $\gamma_r(G)$ and $\gamma_{\mathrm{R}}(G)$, determine weak Roman domination numbers for some special classes of graphs. They characterize graphs with weak Roman domination numbers equal to their domination numbers, and forests that have weak Roman domination numbers equal to twice their domination numbers.

**Theorem 1.9.** [22] *For any graph G,*

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_{\mathrm{R}}(G) \leq 2\gamma(G).$$

**Proposition 1.10.** [22] *For* $n \geq 4$ ($n \geq 1$ *for* $P_n$),

$$\gamma_r(P_n) = \gamma_r(C_n) = \lceil \frac{3n}{7} \rceil.$$

**Theorem 1.11.** [22] *For any graph* $G$, $\gamma(G) = \gamma_r(G)$ *if and only if* $G$ *has a minimum dominating set* $S$ *such that for every vertex* $u \in V(G) - S$ *there exists a vertex* $v \in S$ *such that* $\mathrm{epn}(v, S) \cup \{u, v\}$ *induces a clique.*

## 1.2 Secure Domination, Total Domination and Secure Total Domination

### 1.2.1 Secure Domination

A new concept – *secure domination* – that can be used when it is not possible to station two guards on the same vertex, is brought to attention in [8]. The set $X \subseteq V$ is a *secure dominating set (SDS)* of the graph $G = (V, E)$ if for each $u \in V - X$ there exists $v \in N(u) \cap X$ such that $(X - \{v\}) \cup \{u\}$ is a dominating set. Thus we place a guard on each vertex in $X$, and each unoccupied vertex $u$ is adjacent to an occupied vertex $v$ such that if the guard on $v$ moves to $u$, the resulting set of occupied vertices is a dominating set. We say that $v$ $X$-*defends* $u$, or simply $v$ *defends* $u$ if the set $X$ is unimportant or clear from the context.

The minimum cardinality of an SDS is called the *secure dominating number* of $G$ and denoted by $\gamma_s(G)$. If $X$ is dominating, let $S_X = \{v \in X : X - \{v\} \text{ is dominating}\}$ and for $u \in V - X$, let $A(u, X) = \{v : v\ X\text{-defends } u\}$. Using these definitions, the authors established some properties of SDSs in [8]

**Proposition 1.12.** [8] *Let $X$ be a dominating set of $G$. Vertex $v \in X$ defends $u \in V - X$ if and only if* $\text{epn}(v, X) \cup \{u, v\}$ *induces a clique.*

**Theorem 1.13.** [8] *An SDS $X$ is minimal if and only if for each $s \in S_X$ with $N(s) \cap S_X \neq \varnothing$, there exists $u_s \in V - X$ such that, for each $v \in A(u_s, X) - \{s\}$, either*

*(i) there exists $w \in V - X$ such that $N(w) \cap X = \{v, s\}$ and $u_s \notin N(w)$, or*

*(ii) $N(s) \cap X = \{v\}$ and $u_s \in N(v) - N(s)$.*

Furthermore, bounds and/or special values for $\gamma_s$ for some classes of graphs are obtained in [8]. We summarize them below. Unless otherwise indicated we denote the order of a graph by $n$ and its minimum and maximum degrees by $\delta$ and $\Delta$ respectively.

**Proposition 1.14.** [8] *(i)* $\gamma_s(G) \geq n(2\Delta - 1)/(\Delta^2 + 2\Delta - 1)$ *if G is triangle-free;*

*(ii)* $\gamma_s \geq n(2\Delta - 3)/(\Delta^2 + 2\Delta - 5)$ *if G is $K_4$-free;*

*(iii)* $\gamma_s(P_n) = \gamma_s(C_n) = \lceil \frac{3n}{7} \rceil$*;*

*(iv)* $\gamma_r(P_m \square P_k) \leq \gamma_s(P_m \square P_k) \leq \lceil \frac{mk}{3} \rceil + 2,$ *where* $m, k \geq 1$*;*

*(v)* $\gamma_r(C_m \square C_k) \leq \gamma_s(C_m \square C_k) \leq \lceil \frac{mk}{3} \rceil,$ *where* $m, k \geq 1$*;*

*(vi)* $\gamma_s(C_m \square C_k) \geq \frac{7mk}{23}$.

More bounds of $\gamma_s$ in other classes of graphs are established in [7]. We denote the independence number of a graph $G$ by $\beta(G)$. It is shown in [7] that

**Theorem 1.15.** [7] *If G is claw-free, then*

*(i)* $\gamma_r(G) = \gamma_s(G) \leq 2\gamma(G)$*;*

*(ii)* $\gamma_s(G) \leq \frac{3}{2}\beta(G)$*; further,* $\gamma_s(G) \leq \beta(G)$ *if G is also $C_5$-free;*

*(iii)* $\gamma_s(G) \leq 3n/(\delta(G) + 3)$*; further,* $\gamma_s(G) \leq 2n/(\delta(G) + 2)$ *if G is also $C_5$-free.*

**Theorem 1.16.** [7] *Let G be a $K_t$-free graph. Then*

$$\gamma_s(G) \geq n(2\Delta - 2t + 5)/((\Delta + 1)^2 - (t - 1)(t - 2)),$$

*and the bound is sharp.*

A graph $G$ is said to be $\gamma$*-excellent* if each vertex of $G$ is contained in some minimum dominating set of $G$. In 2005, along with some other interesting properties and characterization of $\gamma$-excellent trees, a constructive characterization of $(\gamma, \gamma_s)$-*trees*, i.e., trees with equal domination and secure domination numbers, is obtained in [28]. Let $l$ be a leaf and $v$ be any vertex of a tree. The path $vx_1...x_kl$ is called an *endpath* if $\deg x_i = 2$ for each $i$. A vertex of a tree of order at least three that is adjacent to a leaf is called a *support vertex*. We denote the set of leaves and support vertices of $T$ by $L(T)$ and $S(T)$, respectively.

**Theorem 1.17.** [28] *A tree $T$ is $\gamma$-excellent if and only if $T \in \{K_1, K_2\} \cup \Upsilon$, where $\Upsilon$ is the class of all trees obtained from $P_4$ by a finite sequence of Operations* **O1** - **O4**, *defined as follows:*

**O1.** *Join a support vertex of $T' \in \Upsilon$ to a vertex of $P_2$.*

**O2.** *Join a vertex $v$ of $T' \in \Upsilon$ that lies on an endpath $vxz$ to a vertex of $P_2$.*

**O3.** *Join a vertex $v$ of $T' \in \Upsilon$ that lies on an endpath $vx_1x_2z$ to a leaf of $P_3$.*

**O4.** *Join a leaf of $T' \in \Upsilon$ to a leaf of $P_3$.*

The next result requires the definition of two new operations on a tree $T$, in addition to the four above.

**O5.** Join a support vertex of $T$ to a vertex of $P_5$.

**O6.** Join a vertex $v$ which lies on an endpath $vx_1 \ldots x_3z$ to a vertex of $P_2$.

Let $\mathcal{S}$ be the class of all trees obtained from $P_4$ or $P_7$ by a finite sequence of Operations **O1-O3**, **O5, O6**.

**Theorem 1.18.** [28] *The tree $T$ is a $(\gamma, \gamma_s)$-tree if and only if $T \in \{K_1, K_2\} \cup \mathcal{S}$.*

### 1.2.2 Total Domination and Secure Total Domination

For a graph $G = (V, E)$, a set $S \subseteq V$ is a *total dominating set* if $\bigcup_{s \in S} N(s) = V(G)$. The *total domination number* is the minimum cardinality of a total dominating set of $G$, and is denoted by $\gamma_t(G)$. A new protection strategy, called secure total domination, is introduced in [3]. A set $S \in V$ is said to be a *secure total dominating set* (*STDS*) if $S$ is both a total dominating set and a secure dominating set. The minimum cardinality of an STDS of $G$ is called *the secure total domination number*, denoted by $\gamma_{\rm st}(G)$.

Properties of secure total domination in general graphs are established in [3]. The value of $\gamma_{\rm st}(P_n)$ and a lower bound for $\gamma_{\rm st}$ for $n$-vertex forests with maximum degree $\Delta$ are also determined. These main results are summarized below.

**Proposition 1.19.** [3] *Let $X$ be a TDS of $G$. The vertex $v$ $X$-defends $u$ if and only if* $\operatorname{epn}(v,X)=\varnothing$ *and* $\operatorname{ipn}(v,X)\subseteq N(u)$.

**Theorem 1.20.** [3] *For any graph $G$, $\gamma_{\mathrm{st}}(G)=n$ if and only if $V-S(T)$ is an independent set.*

**Theorem 1.21.** [3] $\gamma_{\mathrm{st}}(P_n)=\lceil 5(n-2)/7\rceil+2$.

**Theorem 1.22.** [3] *If $G$ is an $n$-vertex forest with $\Delta\geq 3$, then*

$$\gamma_{\mathrm{st}}(G)\geq\frac{4\Delta n+4\Delta-3n-4}{6\Delta-5},$$

*and this bound is sharp.*

Our knowledge of bounding $\gamma_{\mathrm{st}}$ is extended in [27], where it is shown that $\gamma_{\mathrm{st}}(G)$ is at most twice the clique partition number $\theta(G)$ of $G$, and less than three times the independence number. It is also shown that these bounds are sharp, with the exception of the bound involving the independence number. For $n\geq 1$, let $J_{2,n}$ be the graph obtained from $K_{2,n}$ by joining the two vertices of degree $n$ (or two nonadjacent vertices of $C_4$ if $n=2$).

**Theorem 1.23.** [27] *For any graph $G$ with $\delta(G)\geq 2$, $\gamma_s(G)\leq\gamma_{\mathrm{st}}(G)\leq 2\gamma_s(G)$, and both bounds are sharp.*

**Theorem 1.24.** [27] *If $G$ is connected, then $\gamma_{\mathrm{st}}(G)=\gamma_t(G)$ if and only if $\gamma_{\mathrm{st}}(G)=2$, i.e., if and only if $G=K_2$ or $J_{2,n}$ is a spanning subgraph of $G$ for some $n\geq 1$.*

**Theorem 1.25.** [27] *For all graphs $G$ without isolated vertices, $\gamma_{\mathrm{st}}(G)\leq 2\theta(G)$ and the bound is sharp.*

**Theorem 1.26.** [27] *For all graphs $G$ without isolated vertices, $\gamma_{\mathrm{st}}(G)\leq 3\beta(G)-1$; further, if $\beta(G)=2$, then $\gamma_{\mathrm{st}}(G)\leq 4$.*

## 1.3 Paired Domination

The domination strategies in the previous sections all involve mobile guards. We now discuss a strategy where only stationary guards are used, and where the guards work in pairs. This concept is called paired domination and was introduced in [18]. For a graph $G$, a *matching* $M$ of $G$ is a set of non-adjacent edges of $G$, and called a *perfect matching* if it contains every vertex of $G$ as an end-vertex. For paired domination, each guard is adjacent to another one and the two guards act as backups for each other. Thus, a *paired dominating set* of a graph $G$ is a set $S$ such that the subgraph of $G$ induced by $S$ contains a perfect matching. The minimum cardinality of a paired dominating set is the *paired domination number* $\gamma_{\mathrm{pr}}(G)$.

For a graph $G$, we define the *independent domination number* $i(G)$ as the minimum number of vertices in a maximal independent set. For the corresponding edge parameters, let $i_1(G)$ and $\beta_1(G)$ denote the minimum and maximum cardinalities of maximal independent edge sets.

Some basic results relevant to paired dominations are given in [17]. We summarize them as follows.

**Theorem 1.27.** [17] *If $G$ has no isolated vertices, then*

*(i)* $\gamma(G) \leq \gamma_t(G) \leq \gamma_{\mathrm{pr}}(G) \leq 2\gamma(G) \leq 2i(G)$,

*(ii)* $\gamma(G) \leq 2i_1(G) \leq 2\beta_1(G)$,

*(iii)* $\gamma_{\mathrm{pr}}(G) \leq 2\gamma_t(G) - 2$, *and*

*(iv) given positive integers* $a \leq b \leq c$ *such that* $c$ *is even,* $c \leq 2a$, *and* $c \leq 2b - 2$, *there exists a graph* $G$ *having* $\gamma(G) = a$, $\gamma_t(G) = b$, *and* $\gamma_{\mathrm{pr}}(G) = c$.

**Theorem 1.28.** [17] *If $G$ has no isolated vertices and $|V(G)| = n$, then*

*(i)* $\gamma_{\mathrm{pr}}(G) \geq n/\Delta(G)$, *and*

*(ii)* $2 \leq \gamma_{\mathrm{pr}}(G) \leq n$ *with* $\gamma_{\mathrm{pr}}(G) = n$ *if and only if* $G = mK_2$.

**Theorem 1.29.** [17] *If $G$ is a connected graph on $n \geq 6$ vertices and $\delta(G) \geq 2$, then $\gamma_{\mathrm{pr}}(G) \leq 2n/3$.*

Motivated by a model different from "assigning guards so that each one has a back-up", the subject of paired domination is considered by Haynes and Slater [18]. Further, Fitzpatrick and Hartnell [13] focus on graphs that have a maximal matching whose end vertices form a minimum paired-dominating set, and give a complete characterization of graphs with this property that do not contain leaves (vertices of degree one) and with girth no less than seven.

Let $\mathcal{G}$ denote the leafless graphs of this type that have girth at least seven. Define the infinite family of graphs $\mathcal{F}$ to be the set of those graphs $H$ that can be obtained from three nonempty sets of parallel edges, $\{u_r v_r : r = 1, \ldots, k\}$, $\{w_s x_s : s = 1, \ldots, l\}$, and $\{y_t z_t : t = 1, \ldots, m\}$, by connecting each of the pairs of vertices $(v_r, w_s)$, $(x_s, y_t)$, $(z_t, u_r)$ with a path of length two. The desired characterization follows.

**Theorem 1.30.** [13] *A graph $G$ is in $\mathcal{G}$ if and only if $G$ is also in $\mathcal{F}$.*

In [30], a linear-time algorithm for computing the paired domination number of trees is presented. Furthermore, trees with equal domination and paired domination numbers are characterized, as stated in the following theorem. If $D$ is a paired dominating set of a graph such that $|D| = \gamma_{\mathrm{pr}}(G)$, we also call $D$ a $\gamma_{\mathrm{pr}}$-*set*.

**Theorem 1.31.** [30] *For a tree $T$, $\gamma(T) = \gamma_{\mathrm{pr}}(T)$ if and only if $T \in \mathcal{T}$, where $\mathcal{T}$ is the class of all trees that can be obtained from $P_4$ by a finite sequence of the following four types of operations:*

- *Type* 1*: Attach a path $P_1$ to a vertex of $T' \in \mathcal{T}$ that is in a $\gamma_{\mathrm{pr}}$-set of $T'$.*
- *Type* 2*: Attach a $P_5$ to a vertex $v$ of $T' \in \mathcal{T}$, where $v$ is in a $\gamma_{\mathrm{pr}}$-set of $T'$ and for every minimum dominating set $X$ of $T'$, there is no vertex $u$ such that* $\mathrm{pn}(u, X) = v$ *in $T$.*

- *Type* 3*: Attach a support vertex of* $P_4$ *to a vertex* $v$ *of* $T' \in \mathcal{T}$*, where* $v$ *is a vertex such that for every minimum dominating set* $X$ *of* $T'$*, there is no vertex* $u$ *such that* $pn(u, X) = v$ *in* $T$*.*

- *Type* 4*: Let* $T_1$ *be a tree with* $V(T_1) = \{u_0, u_1, u_2, u_3, u_4\}$ *and* $E(T_1) = \{u_0u_1, u_1u_2, u_1u_3, u_2u_4\}$*. Attach a vertex* $u_0$ *of* $T_1$ *to a vertex of* $T' \in \mathcal{T}$*.*

A series of characterizations of trees whose paired domination numbers satisfy various properties are then established, with some of them improved, over the following years. In [16], trees with equal domination and paired domination numbers are characterized by using simpler labelings than those used in [30].

Constructive characterizations of trees with equal total domination and paired domination numbers, and of trees for which the paired domination number is equal to twice the matching number, are given in [34]. Like the characterization for $(\gamma_{\rm pr}, \gamma)$-trees, a simpler characterization for $(\gamma_{\rm pr}, \gamma_t)$-trees, which categorizes the vertices into four classes, is given in [21].

It is known that the paired domination number of $G$ is bounded above by twice the domination number of $G$. In [15],a constructive characterization of the trees attaining this bound is established. Other characterizations of the same family of trees are given in [24] and [25], the former using the fact that the trees with paired-domination number twice their domination number are precisely the trees with 2-packing number equal to their 3-packing number, where the $k$-packing number $\rho_k$ of $G$ is the maximum cardinality of a set of vertices that are pairwise at distance greater than $k$ apart (*i.e.* a $k$-packing of $G$).

In addition to the results mentioned above, various results regarding the paired domination numbers of graphs (special graphs) are also established. These results can be found in [11], [31], [20], [5], [2]:

**Theorem 1.32.** [11] *Let* $G$ *be a connected cubic graph of order* $n$*. If*

1. $G$ *is* $(K_{1,3}, K_4 - e, C_4)$*-free, then* $\gamma_{\rm pr}(G) \leq 3n/8$*;*

*2. G is* $(K_{1,3}, K_4 - e)$*-free, then* $\gamma_{\text{pr}}(G) \leq 2n/5$*;*

*3. G is* $K_{1,3}$*-free, then* $\gamma_{\text{pr}}(G) \leq n/2$*,*

*and all these bounds are sharp.*

Moreover, the extremal graphs in all these three cases are characterized in [11].

**Theorem 1.33.** [31] *Let* $T$ *be a tree with order* $n$ *and* $l$ *leaves. Then*

$$\gamma_{\text{pr}}(T) \geq (n + 2 - l)/2,$$

*and the bound is sharp.*

Trees which attain this bound are characterized.

**Theorem 1.34.** *Let* $G$ *be a connected graph with order* $n \geq 14$*, then*

$$\gamma_{\text{pr}}(G) \leq \frac{2}{3}(n - 1),$$

*and the bound is sharp.*

Graphs attaining this upper bound are characterized. This result is an improvement to the upper bound of $\frac{2}{3}n$, which is given in [17].

**Theorem 1.35.** [5] *For a cubic graph* $G$ *of order* $n$*,*

$$\gamma_{\text{pr}} \leq 3n/5,$$

**Theorem 1.36.** [2] *For a graph* $G$*,* $\gamma_{\text{pr}}(G) \geq 2\rho_3(G)$*.*

The authors prove, in [2], by induction that the inequality holds for nontrivial trees. They also show that

**Theorem 1.37.** [2] *For graphs* $G$*,* $H$*,*

$$\gamma_{\text{pr}}(G \,\square\, H) \geq \max\{\gamma_{\text{pr}}(G)\rho_3(H), \gamma_{\text{pr}}(H)\rho_3(G)\}.$$

**Corollary 1.38.** [2] *If a graph $G$ satisfies $\gamma_{\mathrm{pr}}(G) = 2\rho_3(G)$, then for any other graph $H$,*

$$\gamma_{\mathrm{pr}}(G \,\square\, H) \geq 2\gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H).$$

For a graph $G$, the *girth* of $G$, denoted by $g(G)$, is the length of the shortest cycle in $G$. Paired domination number of graphs are further studied in [4], where several upper bounds of the parameter are presented in terms of the maximum degree, minimum degree, girth and order.

**Theorem 1.39.** [4] *If $G$ is a connected graph of order $n$ with minimum degree $\delta \geq 2$ and girth $g(G) \geq 6$, then*

$$\gamma_{\mathrm{pr}}(G) \leq \frac{2}{3}(n - (\delta - 1)(\delta - 2)/2).$$

**Theorem 1.40.** [4] *If $G$ is a connected graph of order $n$ with minimum degree $\delta \geq 3$ and girth $g(G) \geq 6$, then*

$$\gamma_{\mathrm{pr}}(G) \leq \frac{2}{3}(n + 1 - \Delta).$$

**Theorem 1.41.** [4] *If $G$ is a connected graph of order $n$ with minimum degree $\delta \geq 3$, then*

$$\gamma_{\mathrm{pr}}(G) \leq \frac{2n}{3} - \frac{g(G)}{6} + \frac{5}{6}.$$

A graph $G$ with no isolated vertex is *paired domination vertex (edge) critical* if for any vertex $v$ of $G$ (for any edge $e$ of $G$) that is not adjacent to a vertex of degree one, $\gamma_{\mathrm{pr}}(G - v) < \gamma_{\mathrm{pr}}(G)$ $(\gamma_{\mathrm{pr}}(G - e) < \gamma_{\mathrm{pr}}(G))$. We call these graphs $\gamma_{\mathrm{pr}}$-vertex (edge)-critical. Criticality of graphs with respect to paired domination was studied, for example, in [23] and [26].

In the rest of this thesis we discuss strategies where the pairs of guards in a PDS become mobile to protect unoccupied vertices. We shall see that there are several possible types of guard movements that lead to slightly different protection strategies and different secure paired domination numbers.

# Chapter 2

# Definitions and Existence of Secure Paired Dominating Sets

## 2.1 Introduction

We introduce a new strategy of domination – secure paired domination, which combines the advantages of both secure domination and paired domination. We propose nine possible definitions of this concept and compare the definitions pairwise, obtaining properties of and inequalities between the secure paired domination numbers associated with the definitions.

## 2.2 Definitions

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $M$ be a matching. We denote the set of end-vertices of all the edges in $M$ by $V(M)$. If $u, v$ are the end-vertices of an edge in $M$, then $u$ is called the *$M$-partner* of $v$. Let $D$ be a paired dominating set of $G$, then for a vertex $v \in D$, we define the $D$-partner of $u$ (or $\mathrm{p}(u)$ whenever the context is obvious) to be the $M$-partner of $u$, denoted by $\mathrm{p}(u, D)$.

Recall that one can think of secure domination in the following way: Place guards on each vertex of an SDS $D$ of $G$. An intruder attacks $G$ at a vertex. If the vertex is in $D$, then the guard stationed at the vertex defends the graph against the attack. If the vertex is not in $D$, then there is a guard at an adjacent vertex and this guard runs along the edge to the vertex to defend the graph against the attack.

**Question 2.1.** *How do we define secure paired domination?*

Note that we cannot just require that a single guard runs to an adjacent vertex to guard against the attack, because in this case not every graph (without isolated vertices) has a secure paired dominating set. Stars have only one type of PDS, and if only one guard moves along an edge, we do not get a PDS. But if both guards move, we do get a PDS of the same type. The graph in Figure 2.1 has two types of PDS's: one set with one pair of guards and the other with two. In the first set, if one or both guards move, the resulting set is not a PDS. In the second case, if one guard moves, the resulting set is not a PDS, but if both guards from the same pair move, another PDS is formed.

Figure 2.1: A graph with two types of PDS's. Only the set on the right is an SPDS.

We consider nine possible definitions of secure paired dominating sets, make some basic observations and remarks, and compare and contrast each with the others.

In Definition 2.1 only a single guard moves. In each even numbered definition, two guards move, and in each subsequent odd numbered definition there is the option of the previous two-guard move or the single guard move of Definition 2.1.

**Definition 2.1.** A PDS $D$ of a graph $G$ is a 1-*secure paired dominating set* (1-*SPDS*) if for each $v \in V(G) - D$, there exist a perfect matching $M$ of $\langle D \rangle$ and an edge $uw \in M$ such that

- $v$ is adjacent to $u$ and

- $D - \{w\} \cup \{v\}$ is a PDS of $G$.

**Definition 2.2.** A PDS $D$ of a graph $G$ is a 2-*secure paired dominating set* (2-*SPDS*) if for each $v \in V(G) - D$, there exist a perfect matching $M$ of $\langle D \rangle$ and an edge $uw \in M$ such that

- $v$ is adjacent to $w$ and
- $D' = (D - \{u\}) \cup \{v\}$ is a PDS of $G$.

According to this definition, the guard on $w$ moves along the edge $wv$ to $v$, and the guard on $u$ moves along $uw$ to $w$ to form a new PDS $D'$, where $M' = (M - uv) \cup \{vw\}$ is a matching of $\langle D' \rangle$.

**Definition 2.3.** A PDS $D$ of a graph $G$ is a 3-*secure paired dominating set* (3-*SPDS*) if for each $v \in V(G) - D$,

(a) there exist a perfect matching $M$ of $\langle D \rangle$ and an edge $uw \in M$ such that

- $v$ is adjacent to $w$ and
- $(D - \{u\}) \cup \{v\}$ is a PDS of $G$, or

(b) there exists $u \in D$ such that

- $v$ is adjacent to $u$ and
- $(D - \{u\}) \cup \{v\}$ is a PDS of $G$.

In the second case, the guard on the vertex $u$ moves to $v$ and the resulting set is a PDS – for example, this is possible in $C_5$ but not in $C_6$. Thus Definition 2.3 is a combination of Definitions 2.1 and 2.2.

**Definition 2.4.** A PDS $D$ of a graph $G$ is a 4-*secure paired dominating set* (4-*SPDS*) if for each $v \in V(G) - D$, there exist a perfect matching $M$ of $\langle D \rangle$, a vertex $x \neq v$, and an edge $uw \in M$ such that

- $\{vw, vx, ux\} \subseteq E(G)$ and

- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$.

Note that if $x = w$, then we have the same move as in Definition 2.2. Here the guard on $u$ moves along $ux$ to $x$, the guard on $w$ moves along $wv$ to $v$ to form $D'$, and $M' = (M - uw) \cup \{vx\}$ is a matching of $D'$.

**Definition 2.5.** A PDS $D$ of a graph $G$ is a 5-*secure paired dominating set* (5-*SPDS*) if for each $v \in V(G) - D$,

(a) there exist a perfect matching $M$ of $\langle D \rangle$, a vertex $x \neq v$, and an edge $uw \in M$ such that

- $vw, vx, ux \in E(G)$ and
- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$, or

(b) there exists $u \in D$ such that

- $v$ is adjacent to $u$ and
- $(D - \{u\}) \cup \{v\}$ is a PDS of $G$.

Definition 2.5 is a combination of Definitions 2.1 and 2.4.

**Definition 2.6.** A PDS $D$ of a graph $G$ is a 6-*secure paired dominating set* (6-*SPDS*) if for each $v \in V(G) - D$, there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that

- $uw, vw, vx, ux \in E(G)$ and
- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$.

In this move the two moving guards do not need to be on matched vertices of the PDS before or after the move, but we do require the locations of the guards to be adjacent.

**Definition 2.7.** A PDS $D$ of a graph $G$ is a 7-*secure paired dominating set* (7-*SPDS*) if for each $v \in V(G) - D$,

(a) there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that

- $uw, vw, vx, ux \in E(G)$ and
- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$, or

(b) there exists $u \in D$ such that

- $v$ is adjacent to $u$ and
- $(D - \{u\}) \cup \{v\}$ is a PDS of $G$.

Again, Definition 2.7 is a combination of Definitions 2.1 and 2.6.

**Definition 2.8.** A PDS $D$ of a graph $G$ is an 8-*secure paired dominating set* (8-*SPDS*) if for each $v \in V(G) - D$, there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that

- $vw, ux \in E(G)$ and
- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$.

For this strategy the moving guards do not need to be on adjacent vertices before or after the move.

**Definition 2.9.** A PDS $D$ of a graph $G$ is a 9-*secure paired dominating set* (9-*SPDS*) if for each $v \in V(G) - D$,

(a) there exist a vertex $x \neq v$ and vertices $u, w \in D$ such that

- $vw, ux \in E(G)$ and
- $(D - \{u, w\}) \cup \{v, x\}$ is a PDS of $G$, or

(b) there exists $u \in D$ such that

- $v$ is adjacent to $u$ and
- $(D - \{u\}) \cup \{v\}$ is a PDS of $G$.

Like the other odd-numbered definitions, Definition 2.9 is a combination of Definition 2.1 and the preceding even-numbered definition.

**Definition 2.10.** The *$i$-th secure paired domination number* $\gamma_{\mathrm{spr}}^{(i)}(G)$ is the smallest cardinality of an $i$-SPDS of $G$, $i = 1, \ldots, 9$.

We now investigate the following questions.

**Question 2.2.** *Which of these definitions – if any – give equivalent moves?*

**Question 2.3.** *Which of these definitions give different moves, but equal secure paired domination numbers?*

**Question 2.4.** *For which of these definitions does $\gamma_{\mathrm{spr}}^{(i)}(G)$ exist for all graphs without isolated vertices?*

## 2.3 Existence of Secure Paired Domination Numbers

We consider Definitions 2.2 to 2.9. They have a trivial hierarchical relationship which we state below.

**Observation 2.1.** *Definition* 2.2 *is a special case of Definition* 2.4*, which is a special case of Definition* 2.6*, which is a special case of Definition* 2.8*. Definition* 2.3 *is a special case of Definition* 2.5*, which is a special case of Definition* 2.7*, which is a special case of Definition* 2.9*.*

Note that $\gamma_{\mathrm{spr}}^{(1)}$ does not exist for all graphs without isolated vertices – for example, $P_3$ does not have a 1-SPDS. We now show that $\gamma_{\mathrm{spr}}^{(i)}(G)$ exists for all isolate-free graphs for $2 \leq i \leq 9$. This result answers Question 2.4.

**Proposition 2.2.** *For each integer $2 \leq i \leq 9$, $\gamma_{\mathrm{spr}}^{(i)}(G)$ exists for all graphs $G$ without isolated vertices.*

*Proof.* It suffices to show that $\gamma_{spr}^{(2)}(G)$ exists for any graph without isolated vertices, because Definition 2 is a special case of all the other definitions.

Let $G$ be a graph without isolated vertices. Let $M$ be a maximum matching of $G$, and let $D$ denote the set of vertices that are matched by $M$. Say $D = \{v_1, u_1, v_2, u_2, \ldots, v_i, u_i, ..., v_m, u_m\}$, where $v_i$ and $u_i$ are the two end-vertices of the same edge in $M$. Obviously $D$ pairwise dominates $V(G)$. We show that $D$ is secure.

Note that the two end-vertices of an edge $u_iv_i \in M$ cannot have different neighbours in $V(G) - D$. Otherwise, suppose $v_i$ is adjacent to $w$, and $u_i$ to $x$, respectively. Then $M' = (M - \{u_iv_i\}) \cup \{v_iw, u_ix\}$ is a matching of $G$ with $|M'| > |M|$; a contradiction.

To show that $D$ is a secure paired dominating set (SPDS) of $G$, suppose there is an attack at a vertex $v \notin D$, where $v$ is adjacent to $v_i$. Then $D' = \{D - u_i\} \cup \{v\}$ is a paired dominating set (DS) of $G$, since $u_i$ has no neighbours outside $D'$ (when both $u_i$ and $v_i$ are adjacent to $v$, $v$ is the only neighbour of $u_i$ or $v_i$, so $D'$ is still a PDS).

It also follows that $\gamma^{(2)}_{\text{spr}}(G) \leq |D| = 2|M|$, and so $\gamma^{(i)}_{\text{spr}}(G) \leq \gamma^{(2)}_{\text{spr}}(G) \leq 2|M|$. ■

We give a necessary and sufficient condition for the existence of $\gamma^{(1)}_{\text{spr}}(G)$, where $G$ is a graph without isolated vertices.

**Proposition 2.3.** *For a graph $G$ without isolated vertices, $\gamma^{(1)}_{\text{spr}}(G)$ exists, with $D$ a 1-SPDS of $G$, if and only if $G$ has a matching $M$ such that $D = V(M)$ and any $v \in V(G) - D$ is contained in a cycle $C = v, u_1, ..., u_{2r}, v$, where $\{u_1u_2, u_3u_4, ..., u_{2r-1}u_{2r}\} \subseteq M$.*

*Proof.* Suppose there exists a matching $M$ of $G$ satisfying the given conditions. We show that $D = V(M)$ is a 1-SPDS of $G$. If $M$ is a perfect matching, we are done; otherwise, consider any $v \in V(G) - D$, which is contained in an odd cycle $C$ as described. It is obvious that $v$ is dominated. If there is an attack on $v$, consider $D' = (D - \{u_{2r}\}) \cup \{v\}$ with $M' = (M - E(C)) \cup \{vu_1, u_2u_3, ..., u_{2r-2}u_{2r-1}\}$. Then $D'$ is a PDS of $G$. Consequently, $D$ is a 1-SPDS of $G$, and thus $\gamma^{(1)}_{spr}(G)$ exists.

Conversely, suppose $\gamma^{(1)}_{spr}(G)$ exists and let $D$ be a 1-SPDS of $G$ with $M$ any associated matching. If $|D| = |V(G)|$, then $M$ is a perfect matching of $G$ and there is nothing more to prove.

If $|D| < |V(G)|$, then let $v$ be an arbitrary vertex in $V(G) - D$. Let $u$ be a vertex in $D$ that $D$-defends $v$. Let $D' = (D - \{u\}) \cup \{v\}$, which is a PDS of $G$ with matching $M'$, formed after the guard on $u$ has moved to $v$. Let $u_2$ be the $M$-partner of $u = u_1$ in $D$. Since $u \notin D'$, $u_2$ has a new partner in $D'$, say $u_3$. If $u_3 = v$, we are done because then $v, u_1, u_2$ forms a triangle that satisfies the statement of the proposition. If not, then by definition of $D'$, $u_3 \in D$. Consider the $M$-partner $u_4$ of $u_3$; it has an $M'$-partner, which we call $u_5$. If $u_5 = v$, we are done because then $v, u_1, u_2, u_3, u_4$ is a 5-cycle such that $\{u_1u_2, u_3u_4\} \subseteq M$; otherwise, repeat the procedure to find $u_5, \ldots, u_{2r+1}$, where $u_{2r+1} = v$; such an index $2r+1$ exists because $G$ is finite. Then $C = v, u_1, u_2, \ldots, u_{2r}, v$ is a cycle of $G$ such that $\{u_1u_2, u_3u_4, ..., u_{2r-1}u_{2r}\} \subseteq M$ and the conditions are satisfied. ■

Note that if $\gamma^{(1)}_{spr}(G)$ exists, then $G$ may have minimal 1-SPDS's $D$ and $D'$ such that $|D| \neq |D'|$. See Figure 2.2 for an example.

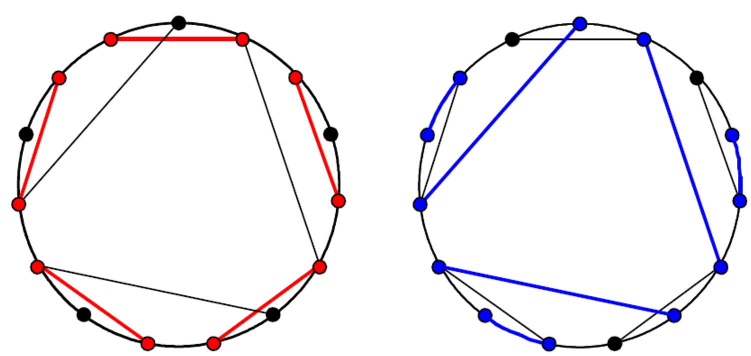


Figure 2.2: A graph $G$ with minimal 1-SPDS's of different cardinalities

Since the class of SPDS's defined by Definitions 2.3 (2.5, 2.7, 2.9) is the union of the classes defined by Definition 2.2 (2.4, 2.6, 2.8, respectively) and Definition 2.1, we conclude the following:

**Corollary 2.4.** *For any bipartite graph* $G$, $\gamma^{(i)}_{\text{spr}}(G) = \gamma^{(i+1)}_{\text{spr}}(G)$ *for each* $i = 2, 4, 6, 8$.

*Proof.* We prove $\gamma^{(2)}_{\text{spr}} = \gamma^{(3)}_{\text{spr}}$ for any bipartite graph $G$, and the remaining cases can be proved similarly.

Suppose there exists a bipartite graph $G$ such that $\gamma^{(3)}_{\text{spr}}(G) < \gamma^{(2)}_{\text{spr}}(G)$. Let $D$ be a minimum 3-SPDS of $G$. If $D$ securely pairwise defends $G$ with only the moves defined in Definition 2.2, then $D$ is a 2-SPDS of $G$, which leads to a contradiction. Otherwise, there exists $v \in V(G) - D$ that is defended with the move defined in Definition 2.1, by some $u \in D$. It follows from Proposition 2.3 that $G$ contains an odd cycle, which again is a contradiction. ■

## 2.4 Comparison of Different Secure Paired Domination Numbers

Next, our attention shifts to the comparison of $\gamma^{(2)}_{\text{spr}}(G)$ to $\gamma^{(9)}_{\text{spr}}(G)$. Observation 2.1 immediately implies the following result.

**Corollary 2.5.** *For any graph $G$ without isolated vertices,*
$\gamma^{(2)}_{\text{spr}}(G) \geq \gamma^{(4)}_{\text{spr}}(G) \geq \gamma^{(6)}_{\text{spr}}(G) \geq \gamma^{(8)}_{\text{spr}}(G)$*, and*
$\gamma^{(3)}_{\text{spr}}(G) \geq \gamma^{(5)}_{\text{spr}}(G) \geq \gamma^{(7)}_{\text{spr}}(G) \geq \gamma^{(9)}_{\text{spr}}(G)$.

We compare each pair of the eight different definitions, giving proofs if two definitions give the same secure paired domination (SPD) number for all graphs, otherwise providing graphs which have different SPD numbers. We start with Definitions 2.2 and 2.3.

**Proposition 2.6.** *For any graph $G$ without isolated vertices, $\gamma^{(2)}_{\text{spr}}(G) \geq \gamma^{(3)}_{\text{spr}}(G)$, where strict inequality holds for some graphs.*

*Proof.* The first part of the statement is obvious. To see the second part, consider a graph $G$ (the *horned pentagon*) as drawn in Figure 2.3.

It is easy to see that $\gamma^{(3)}_{\text{spr}}(G) = 4$, because $D = \{e, d, b, c\}$ is a 3-SPDS, with $e$ matched with $d$: if the attack is at $a$, then $D' = \{D - \{e\}\} \cup \{a\}$ is a PDS, with $a$ matched with $b$ and $d$ with $c$. If $g$ is attacked, then $D' = \{D - \{e\}\} \cup \{g\}$ forms a new PDS of $G$. These are all the cases we have to consider up to isomorphism.

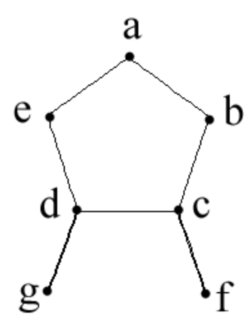



Figure 2.3: The horned pentagon

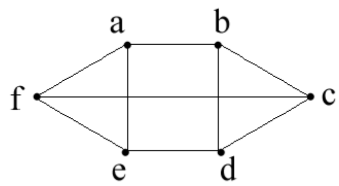



Figure 2.4: The three-prism

However, using only Definition 2, we cannot securely pairwise dominate the graph with $|D| = 4$. Up to isomorphism, there are three PDS's of $G$:

1. $D = \{d, c, e, a\}$, $M = \{ea, dc\}$: if we attack $f$, then $D' = \{e, a, c, f\}$ is the only set that can be formed by the movement of guards according to Definition 2.2. But $D'$ does not dominate $g$.

2. $D = \{d, g, c, b\}$, $M = \{dg, bc\}$: if we attack $f$, then $D' = \{d, g, c, f\}$ is the only set that can be formed by the movement of guards according to Definition 2.2. But $D'$ does not dominate $a$.

3. $D = \{e, d, c, b\}$, $M = \{de, bc\}$: if we attack $a$, then $D' = \{a, e, b, c\}$ and $D'' = \{a, b, e, d\}$) are the only sets that can be formed by the movement of guards according to Definition 2.2. But $D'$ does not dominate $g$, and $D''$ does not dominate $f$.

So $\gamma^{(2)}_{\text{spr}}(G)6$. Since it is easy to show that $G$ can be securely pairwise dominated with 6 vertices, $\gamma^{(2)}_{\text{spr}}(G) = 6$. ■

We next compare Definitions 2.2 and 2.4, then Definitions 2.4 and 2.6, and then 2.6 and 2.8.

**Proposition 2.7.** *For any graph $G$ without isolated vertices, $\gamma^{(2)}_{\text{spr}}(G) \geq \gamma^{(4)}_{\text{spr}}(G)$, and strict inequality holds for some graphs.*

*Proof.* Consider graph $G$ in Figure 2.4, which we call the *three-prism*, with $D = \{a, b\}$.

Obviously $\gamma_{\text{spr}}^{(4)}(G) = 2$ as $D$ securely pairwise dominates the graph. On the other hand, it is impossible to securely pairwise dominate $G$ with $|D| = 2$ using only Definition 2, as one can easily see. ■

**Corollary 2.8.** *For any graph $G$ without isolated vertices, $\gamma_{\text{spr}}^{(3)}(G) \geq \gamma_{\text{spr}}^{(5)}(G)$, and strict inequality holds for some graphs.*

*Proof.* The first part of the statement follows from Corollary 2.5. For the second part, notice that although the graph in Proposition 2.7 (the three-prism) contains $C_3$ and $C_5$, there does not exist a 3-SPDS of size 2. Therefore

$$\gamma_{\text{spr}}^{(3)}(G) = \gamma_{\text{spr}}^{(2)}(G) = 4 > \gamma_{\text{spr}}^{(5)}(G) = 2.$$

■

**Corollary 2.9.** *There is no inequality between $\gamma_{\text{spr}}^{(3)}(G)$ and $\gamma_{\text{spr}}^{(4)}(G)$ that holds for all graphs.*

*Proof.* Let $H_1$ be the horned pentagon and $H_2$ be the three-prism. Note that since $H_1$ does not contain $C_4$, Definition 2.2 and Definition 2.4 give the same move on $H_1$. It then follows from previous propositions and corollaries that

$$\gamma_{\text{spr}}^{(3)}(H_1) = 4 < 6 = \gamma_{\text{spr}}^{(4)}(H_1) \text{ and } \gamma_{\text{spr}}^{(3)}(H_2) = 4 > 2 = \gamma_{\text{spr}}^{(4)}(H_2).$$

Thus the statement follows. ■

**Proposition 2.10.** *For any graph $G$ without isolated vertices, $\gamma_{\text{spr}}^{(4)}(G) \geq \gamma_{\text{spr}}^{(6)}(G)$, and strict inequality holds for some graphs.*

*Proof.* The first part of the proposition follows from Corollary 2.5.

Let $G$ be the graph shown in Figure 2.5, which we call the *octagonal grid*.

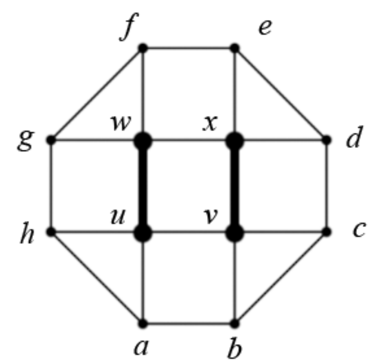



Figure 2.5: The octagonal grid

Consider $D = \{w, u, x, v\}$ with $M = \{wu, xv\}$. Obviously $D$ is an PDS of $G$. We now show that $D$ is 6-secure. If $f$ is attacked, then $D_f = \{f, e, u, v\}$ with $M_f = \{fe, uv\}$ is a new PDS; if $g$ is attacked, then $D_g = \{g, h, x, v\}$ with $M_g = \{gh, xv\}$ is another PDS of $G$. Due to the symmetric nature of $G$, it follows that $D$ is a 6-SPDS of $G$, so $\gamma_{\text{spr}}^{(6)}(G) = 4$.

Now notice that with Definition 2.4, $D$ does not securely pairwise dominate $G$, for if $f$ is attacked, $D' = \{f, w, x, v\}$ with $M' = \{fw, xv\}$ is the set formed by the only move allowed, but it does not guard $a$. Up to isomorphism, there are two other PDS's of $G$ of size 4: $D_1 = \{f, e, u, v\}$ with $M_1 = \{fe, uv\}$, and $D_2 = \{f, e, a, b\}$ with $M_2 = \{fe, ab\}$. For $i = 1, 2$, if $g$ is attacked, $D_i' = \{D_i - \{e\}\} \cup \{g\}$ with $M_i' = \{M - \{fe\}\} \cup \{fg\}$ is the only set formed by the only move allowed, but $d$ is not guarded in either case. Thus $\gamma_{\text{spr}}^{(4)}(G) > 4 = \gamma_{\text{spr}}^{(6)}(G)$. ■

**Corollary 2.11.** *For any graph $G$ without isolated vertices, $\gamma_{\text{spr}}^{(5)}(G) \geq \gamma_{\text{spr}}^{(7)}(G)$, and strict inequality holds for some graphs.*

*Proof.* Similar to the proof to Proposition 2.10. ■

**Corollary 2.12.** *There is no inequality between $\gamma_{\text{spr}}^{(5)}(G)$ and $\gamma_{\text{spr}}^{(6)}(G)$ that holds for all graphs.*

*Proof.* Let $H_1$ be the horned pentagon and $H_2$ be the gridded octagon. By an argument similar to the proof of Corollary 2.9, one can show that $\gamma_{\text{spr}}^{(5)}(H_1) = 4 < 6 = \gamma_{\text{spr}}^{(6)}(H_1)$.

The only three PDS's of $G$ of size 4 in the proof of Proposition2.10 are $D$, $D_1$ and $D_2$, and since none of them is contained in a $C_5$ in $H_2$, it follows from Proposition 2.3 that $\gamma_{\text{spr}}^{(5)}(H_2) = \gamma_{\text{spr}}^{(4)}(H_2) > 4 = \gamma_{\text{spr}}^{(6)}(H_2)$. This completes the proof. ■

**Proposition 2.13.** *For any graph $G$ without isolated vertices, $\gamma_{\text{spr}}^{(6)}(G) \geq \gamma_{\text{spr}}^{(8)}(G)$, and strict inequality holds for some graphs.*

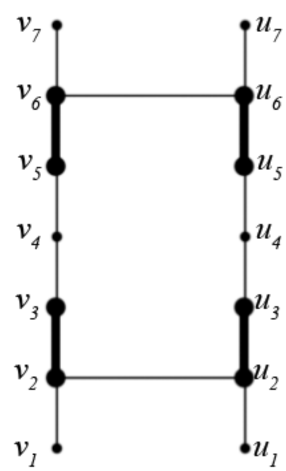



Figure 2.6: The legged rectangle

*Proof.* The first part of the statement follows from Corollary 2.5.

Consider the *legged rectangle* $G$ as illustrated in Figure 2.6.

Notice that $D = \{v_2, v_3, u_2, u_3, v_5, v_6, u_5, u_6\}$ with $M = \{v_2v_3, v_5v_6, u_2u_3, u_5u_6\}$ is an 8-SPDS: if $v_4$ is attacked, then $D' = \{D - \{v_3, u_3\}\} \cup \{v_4, u_4\}$ is a new PDS of $G$ with $M' = \{v_2u_2, v_4v_5, u_4u_5, v_6u_6\}$. If there is an attack against, say, $v_1$, then $D'' = (D - \{v_3\}) \cup \{v_1\}$ dominates $G$ with the obvious matching. Hence $\gamma_{\rm spr}^{(8)}(G) = 8$.

To see that $G$ does not have a 6-SPDS of $G$ of cardinality 8, observe that since $v_6, u_6, v_2, u_2$ are support vertices, they are in any PDS of $G$; furthermore, $v_6u_6, v_2u_2 \notin M$, for otherwise if $v_1$ ($v_7$) is attacked, $u_1$ ($u_7$) is not guarded by the set formed by replacing $u_2$ ($u_6$) by $v_1$ ($v_7$). Thus any SPDS of $G$ has at least four pairs of guards, each containing exactly one of $v_6, u_6, v_2, u_2$. But then it is easy to see that by the moves defined in Definition 2.6, an attack against either $v_4$ or $u_4$ forces one vertex from $\{v_1, u_1, v_7, u_7\}$ out of protection, yielding $\gamma_{\rm spr}^{(6)}(G) \geq 10$ (in fact, one can show that $\gamma_{\rm spr}^{(6)}(G) = 10$). ■

**Corollary 2.14.** *For any graph $G$ without isolated vertices, $\gamma_{\rm spr}^{(7)}(G) \geq \gamma_{\rm spr}^{(9)}(G)$, and strict inequality holds for some graphs.*

*Proof.* Observe that the legged rectangle does not contain odd cycles, hence the corollary is an easy consequence of Proposition 2.3, Corollary 2.5 and Proposition 2.13. ■

**Corollary 2.15.** *There is no inequality between $\gamma_{\rm spr}^{(7)}(G)$ and $\gamma_{\rm spr}^{(8)}(G)$ that holds for all graphs.*

*Proof.* Let $H_1$ be the horned pentagon and $H_2$ be the legged rectangle. It is a trivial fact that $\gamma_{\rm spr}^{(7)}(H_1) = 4 < 6 = \gamma_{\rm spr}^{(8)}(H_1)$. Since $H_2$ contains no odd cycles, $\gamma_{\rm spr}^{(7)}(H_2) = \gamma_{\rm spr}^{(6)}(H_2) > 8 = \gamma_{\rm spr}^{(8)}(H_2)$. Hence the result follows. ■

Note that the legged rectangle reveals a general property about the advantage that Definition 2.8 has over the preceding ones.

**Proposition 2.16.** *For a graph $G$ without isolated vertices, if $\gamma_{\rm spr}^{(8)}(G) < \gamma_{\rm spr}^{(6)}(G)$ and $D$ is a minimum 8-SPDS of $G$, then one (or both) of the following holds:*

- *$G$ contains an even cycle $C$ of which all but exactly two vertices are in $D$; or*
- *$G$ contains two odd cycles $C_1$, $C_2$, such that all except one vertex of $C_i$ are in $D$, $i = 1, 2$.*

*Proof.* Similar to the proof to Proposition 2.3. ■

**Corollary 2.17.** *Let $T$ be a tree. Then $\gamma_{\rm spr}^{(2)}(T) = \ldots, = \gamma_{\rm spr}^{(9)}$.*

*Proof.* Since there are no cycles in a tree, it follows from Proposition 2.3 and 2.16 that, for a tree $T$, all the definitions give the same moves. ■

If $G$ is a tree, we abbreviate $\gamma_{\rm spr}^{(2)}(G) = \cdots = \gamma_{\rm spr}^{(9)}(G)$ as $\gamma_{\rm spr}(G)$.

Now we have completed the pairwise comparisons between all nine definitions.

## 2.5 Basic Properties of 2-SPDS

We finish the chapter by introducing some basic properties of 2-SPDS.

According to Definition 2.2, a PDS $D$ of a graph $G$ is a 2-SPDS if for each $v \in V(G) - D$, there exist a perfect matching $M$ of $\langle D \rangle$ and an edge $uw \in M$ such that $v$ is adjacent to $w$ and $D' = (D - \{u\}) \cup \{v\}$ is a PDS of $G$. In this case we say that $\{u, w\}$ 2-*defends* $v$.

Recall that for any sets $X \subseteq D \subseteq V(G)$, the external private neighbourhood of $X$ relative to $D$ is

$$\text{epn}(X, D) = \{v \in V(G) - D : v \text{ is adjacent to a vertex in } X \text{ but to no vertex in } D - X\}.$$

In our next result we give necessary and sufficient conditions for $\{u, w\}$ to 2-defend $v$. The proof follows immediately from Definition 2.2.

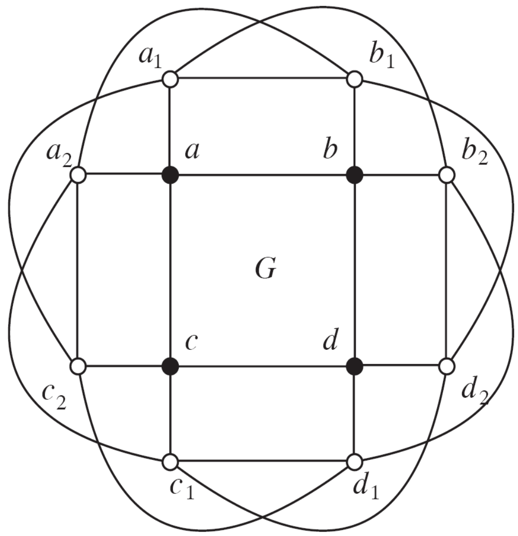



Figure 2.7: $\{a, b, c, d\}$ is a 2-SPDS of $G$

| | | | | |
|---|---|---|---|---|
| $\{a, b\}$ | defends | $a_1$ | but not | $a_2$ |
| $\{a, c\}$ | defends | $a_2$ | but not | $a_1$ |
| $\{a, b\}$ | defends | $b_1$ | but not | $b_2$ |
| $\{b, d\}$ | defends | $b_2$ | but not | $b_1$ |
| $\{c, d\}$ | defends | $c_1$ | but not | $c_2$ |
| $\{a, c\}$ | defends | $c_2$ | but not | $c_1$ |
| $\{c, d\}$ | defends | $d_1$ | but not | $d_2$ |
| $\{b, d\}$ | defends | $d_2$ | but not | $d_1$ |

Table 2.1: $\{a, b, c, d\}$ is a 2-SPDS of $G$

**Proposition 2.18.** *Let $D$ be a PDS of a graph $G$ with associated perfect matching $M$. Then $\{u, w\} \subseteq D$ with $uw \in M$ 2-defends $v \in V(G) - D$ if and only if, without loss of generality, $v$ is adjacent to every vertex in* $\text{epn}(u, D) \cup \{w\}$.

One may think that for a PDS $D$ to be a 2-SPDS, $\langle D \rangle$ must have a matching $M$ such that for each edge $uw \in M$, each vertex of $\text{epn}(u, D)$ is adjacent to each vertex of $\text{epn}(w, D)$, but this is not the case. For the graph $G$ in Figure 2.7, $D = \{a, b, c, d\}$ is a 2-SPDS (see Table 2.1) but $\langle D \rangle$ has no matching with the above-mentioned property.

# Chapter 3

# SPD Numbers for Classes of Graphs

## 3.1 Introduction

We consider five types of special graphs: paths, cycles, spiders, ladders and grid graphs, comparing their secure paired domination numbers according to moves defined in each of Definitions 2.2 to 2.9.

## 3.2 Paths and Cycles

We start with the secure domination number of paths and cycles, as stated in the following proposition.

**Proposition 3.1.** [8] *For any integer $n$, $\gamma_s(P_n) = \lceil \frac{3n}{7} \rceil$. If $n = 3$, then $\gamma_s(C_3) = 1$; otherwise $\gamma_s(C_n) = \gamma_s(P_n)$.*

Observe that since a path does not contain cycles, Definitions 2.3 to 2.9 give the same moves as Definition 2.2. Consequently it suffices to consider only $\gamma^{(2)}_{\text{spr}}(P_n)$, for which our strategy is to contract each pair of guards in an SPDS of $P_n$ to one guard of an SDS of the resulting path, apply Proposition 3.1, then extend the contracted guards back to pairs again. For all trees $T$ we abbreviate $\gamma_{\text{spr}}^{(2)}(T)$ to $\gamma_{\text{spr}}(T)$.

**Proposition 3.2.** *For the graphs $P_n$, $\gamma_{\text{spr}}(P_n) = 2\left\lceil \frac{3n}{10} \right\rceil$.*

*Proof.* Let $D$ be a minimum 2-SPDS of $P_n$, and $|D| = 2m, m \geq 1$. Now we replace each of the $m$ pairs of guards in $D$ by a single guard on a single vertex, resulting in a path $P_{n-m}$ with a set of $m$ guards, which we denote by $D'$. It is obvious that $D'$ is an SDS of $P_{n-m}$. On the other hand, we can begin with a minimum SDS $D'$ with $|D'| = m$ of $P_{n-m}$ and replace each guard in $D'$ by a pair of guards. This will form an SPDS $D$ of $P_n$ with $|D| = 2m$. Hence $m = \gamma_{\text{s}}(P_{n-m})$. The desired result is obtained from solving the equality, $m = \left\lceil \frac{3(n-m)}{7} \right\rceil$ and rounding up to ensure that we obtain an even integer.

For each integer $n \in [2, 10]$, a minimum SPDS of $P_n$ is shown in Figure 3.1. ■

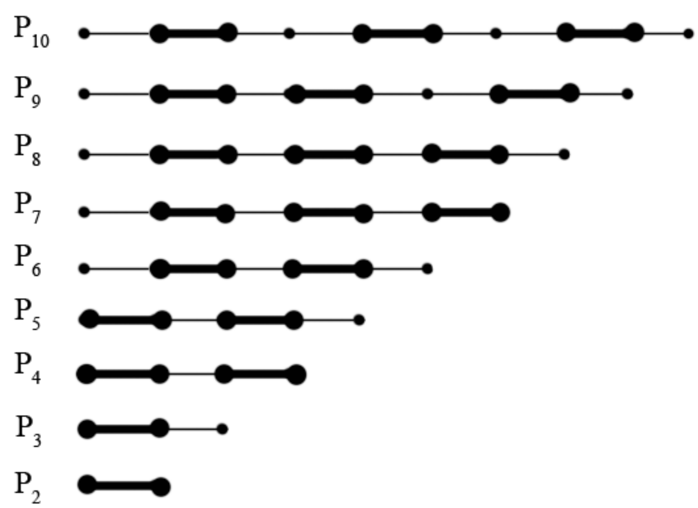



Figure 3.1: SPDSs for $P_2$ to $P_{10}$

With some simple modification, the strategy we used to obtain $\gamma_{\text{spr}}(P_n)$ can also be applied to finding $\gamma_{\text{spr}}(C_n)$, the secure paired domination number of cycles of length n.

**Proposition 3.3.** *For the graphs $C_n$ and for $i = 2, \ldots, 9$,*

$$\gamma_{\text{spr}}^{(i)}(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \\ \gamma_{\text{spr}}^{(i)}(P_n) & \text{otherwise} \end{cases}$$

*Proof.* It is clear that $\gamma_{\text{spr}}^{(i)}(C_4) = 2$ for $i = 2, \ldots, 9$.

If $n \neq 4$ but is even, $C_n$ contains no odd cycles nor $C_4$; if $n$ is odd, Proposition 2.3 yields that $\gamma_{\text{spr}}^{(1)}(C_n) = n - 1$, hence $\gamma_{\text{spr}}^{(i)}(C_n) = \gamma_{\text{spr}}^{(i+1)}(C_n)$ for $i = 2, 4, 6, 8$. Since $C_n$ does not contain $C_4$, $\gamma_{\text{spr}}^{(2)}(C_n) = \gamma_{\text{spr}}^{(3)}(C_n) = \cdots = \gamma_{\text{spr}}^{(9)}(C_n)$. Therefore in each case, $\gamma_{\text{spr}}^{(i)}(C_n) = \gamma_{\text{spr}}^{(j)}(C_n)$ for all $i, j \in \{2, 3, \ldots, 9\}$. By a similar argument as in Proposition 3.2, one can deduce that for $n \neq 4$, $\gamma_{\text{spr}}^{(2)}(C_n) = 2\left\lceil \frac{3n}{10} \right\rceil$, which completes the proof. ■

## 3.3 Spiders

A *branch vertex* of a tree is a vertex of degree at least three and a *support vertex* is a vertex adjacent to a leaf. For $v \in V(T)$ and a leaf $l$ of $T$, a $(v, l)$-*endpath*, or $v$-*endpath* if the leaf is unimportant, or *endpath* if neither $v$ nor $l$ is important, is a path $P$ from $v$ to $l$ such that each internal vertex of $P$ has degree two in $T$. A $v$-$L$ path is any path from $v$ to a leaf. A *spider* $S(q_1, ..., q_p)$ is a tree with exactly one branch vertex $v$, which has degree $p$ and is called the *body vertex*, and $p$ $v$-endpaths, called the *legs*, of lengths $1 \leq q_1 \leq \cdots \leq q_p$.

For positive integers $p$, $q$, we define a $(p, q)$-*spider* $S(p; q)$ to be a spider with $p$ legs, each of length $q$. Let $P$ be some leg in $S(p; q)$. We label the vertices in $V(P) - \{v\}$ by $u_{1,i}, i = 1, 2, ...q$, where $d(u_{1,i}, v) = i$. We similarly label the corresponding vertices of the other legs $u_{j,1}, \ldots, u_{j,q}$, $j = 2, \ldots, p$. We call the path $u_{i,1}, u_{i,2}, \ldots, u_{i,q}$ ($i = 1, 2, \ldots, p$) the $i^{\text{th}}$ *proper leg*, and denote it by $L_i$. In this section we first determine the SPD number for $(p, q)$ - spiders, and then generalize the result to all spiders.

**Proposition 3.4.** *For $p \geq 3$, we have*

$$\gamma_{\text{spr}}(S(p;q)) = p\gamma_{\text{spr}}(P_{q-1}) + \eta(q),$$

*where $\eta(q) = 2$ when $q \equiv 0,\ 1,\ 3,\ 4,\ 7 \pmod{10}$, and $\eta(q) = 0$ otherwise.*

*Proof.* Let $D$ be an SPDS of $G = S(p;q)$, and let $v$ be the body vertex. Let $\{s,t\}$ be a pair of vertices in $D$ that $D$-defends $v$, and $L_1$ be the proper leg that contains $t$ (notice that $\{s,t\} = \{v, u_{1,1}\}$ or $\{u_{1,1}, u_{1,2}\}$).

There are two cases to consider: either the pair $\{s,t\}$ $D$-defends some vertex $u_{j,1}$ from another proper leg, or it does not.

If $\{s,t\}$ defends some vertex $u_{j,1}$, $j \geq 2$, then we can assume that it defends $u_{j,1}$ for all $j = 2, \ldots, p$. Decompose $G$ into $p$ subgraphs as follows. Let $H_1 = \langle V(L_1) \cup \{v, u_{2,1}, u_{3,1}, ..., u_{p,1}\}\rangle$ and $H_i = \langle V(L_i) - \{u_{i,1}\}\rangle$ for $i = 2, ..., p$. Let $D_i$ be a minimum SPDS of $H_i$, for $i = 1, 2, ..., p$.

Observe that if we securely pairwise dominate each component, then the whole graph is securely pairwise dominated. This leads to the inequality

$$|D| \leq \sum_{i=1}^{p} |D_i| = \gamma_{\text{spr}}(P_{q+2}) + (p-1)\gamma_{\text{spr}}(P_{q-1}).$$

On the other hand, if $\{s,t\}$ $D$-defends no vertex from any proper leg other than $L_1$, then we establish the following inequality by decomposing $G$ into $H_1 = \langle L_1 \cup \{v\}\rangle$ and $H_i = L_i, i = 2, 3, ..., p$:

$$|D| \leq \sum_{i=1}^{p} |D_i| = \gamma_{\text{spr}}(P_{q+1}) + (p-1)\gamma_{\text{spr}}(P_q),$$

where $D_i$ is a minimum SPDS of $H_i$, for $i = 1, 2, \ldots, p$. Thus

$$\gamma_{\text{spr}}(G) \leq \min\{\gamma_{\text{spr}}(P_{q+1}) + (p-1)\gamma_{\text{spr}}(P_q), \gamma_{\text{spr}}(P_{q+2}) + (p-1)\gamma_{\text{spr}}(P_{q-1})\}. \quad (3.1)$$

Notice that when $q \equiv 6 \pmod{10}$, we can place the guards on $G$ so that there is a guard at $u_{i,1}$ for all $i = 1, \ldots, p$, all protecting the body vertex $v$. In this case we only need to put $\gamma_{\text{spr}}(P_q) = \gamma_{\text{spr}}(P_{q+1}) - 2$ guards on $H_1$. It can be shown by exhaustively examining all congruence classes modulo 10 that this is the only case when strict inequality in 3.1 is achieved.

Hence we conclude:

- $\gamma_{\text{spr}}(S(p;q)) = \gamma_{\text{spr}}(P_{q+2}) + (p-1)\gamma_{\text{spr}}(P_{q-1})$, if $q \equiv 0, 1, 3, 4, 7 \pmod{10}$.
- $\gamma_{\text{spr}}(S(p;q)) = p\gamma_{\text{spr}}(P_q)$, if $q \equiv 6 \pmod{10}$,
- $\gamma_{\text{spr}}(S(p;q)) = \gamma_{\text{spr}}(P_{q+1}) + (p-1)\gamma_{\text{spr}}(P_q)$, if $q \equiv 2, 5, 8, 9 \pmod{10}$,

which simplifies to the desired result. ■

Figure 3.2 illustrates minimum SPDSs for $S(p;q)$ when $q = 1, \ldots, 10$.

It is worth noticing that, when $p = 1$ or 2, $S(p;q)$ is in fact a path of length $q$ and $2q$, respectively. The following remark shows that in each case, the SPD numbers obtained by regarding the graph as a spider, or a path, agree.

**Remark 3.5.** *For $q \geq 2$,*

- $\gamma_{\text{spr}}(S(1;q)) = \gamma_{\text{spr}}(P_{q+1})$,
- $\gamma_{\text{spr}}(S(2;q)) = \gamma_{\text{spr}}(P_{2q+1})$.

The idea of decomposing spiders, used in the proof to Proposition 3.4, provides a good strategy to find the SPD numbers for any general spider graph.

**Proposition 3.6.** *For positive integers $q_1, \ldots, q_p$,*

$$\gamma_{\text{spr}}(S(q_1, \ldots, q_p) = \sum_{j=1}^{p} \gamma_{\text{spr}}(P_{q_j - 1}) + \eta,$$

*where $\eta = 0$ if*

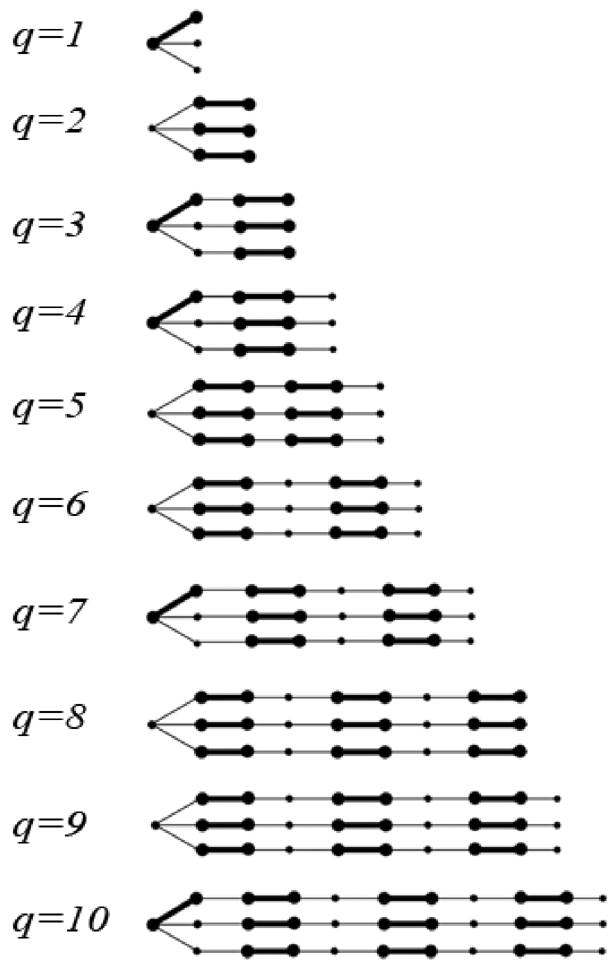



Figure 3.2: Minimum SPDSs for $S(p;q)$ when $q = 1, \ldots, 10$.

- *a) there exists* $t$ *such that* $q_t \equiv 8 \pmod{10}$*, or*
- *b) there does not exist* $j$ *such that* $q_j \equiv 1, 4, 7, 8$*, and there exists* $t$ *such that* $q_t \equiv 2, 5, 6,$ *or* $9 \pmod{10}$*,*

*and* $\eta = 2$ *otherwise.*

*Proof.* Let $G = S(q_1, \ldots, q_p)$. By following the proof method of Proposition 3.4, we obtain that $\gamma_{\text{spr}}(G) = \sum_{j=1}^{p} \gamma_{\text{spr}}(P_{q_j-1}) + \eta$, where $\eta = 0$ or 2, depending on the value of $q_j$, $j = 1, \ldots, p$. To find the exact value of $\eta$, the following cases are considered.

- **Case 1** There exists a $t$ such that $q_t \equiv 8 \pmod{10}$.

  Then we can securely pairwise dominate $\langle V(L_t) \cup \{v, u_{1,1}, \ldots, u_{p,1}\}\rangle$ by $\gamma_{\text{spr}}(P_{q_t-1})$ guards, as Proposition 3.2 yields $\gamma_{\text{spr}}(P_n) = \gamma_{\text{spr}}(P_{n+3})$ if and only if $n \equiv 7 \pmod{10}$. Hence we only need $\gamma_{\text{spr}}(P_{q_j-1})$ guards for $L_j$, $j \neq t$, from which it follows that in this case, $\eta = 0$.

- **Case 2** There does not exist $j$ such that $q_j \equiv 8 \pmod{10}$, and there exists a $t$ such that $q_t \equiv 1, 4, 7 \pmod{10}$.

  We only consider $q_t \equiv 1 \pmod{10}$, as the other two cases follow similarly. Let $D_t$ be a minimum SPD of $\langle V(L_t) - \{v_{t,1}\}\rangle$, and consider vertex $v_{t,1}$ in $L_t$; it is defended by a pair of guards from $L_t$ or $\langle \{v_{t,1}, v\} \cup V(L_r)\rangle$ for some $r \neq t$, which is a path of length $q_r + 1$. Note that since $q_t - 1 \equiv 0 \pmod{10}$, we know $v_{t,2} \notin D_t$. So we need $\gamma_{\text{spr}}(P_{q_t-1}) + 2$ guards in the former subcase. In the latter, we still need $\gamma_{\text{spr}}(P_{q_t-1}) + 2$ guards because $\gamma_{\text{spr}}(P_{n+3}) = \gamma_{\text{spr}}(P_n) + 2$ unless $n \equiv 7 \pmod{10}$. Therefore $\eta = 2$ in this case.

- **Case 3** There does not exist $j$ such that $q_j \equiv 1, 4, 7, 8$, and there exists $t$ such that $q_t \equiv 2, 5, 6,$ or $9 \pmod{10}$.

  We claim $\eta = 0$. Consider the following two subcases:

– If $q_j \equiv 2, 5,\ 6$ or $9 \pmod{10}$ for all $j = 1, \ldots, p$, it is easy to see that we need only $\gamma_{\rm spr}(P_{q_j - 1})$ guards to securely pairwise dominate $PL_j \cup \{v\}$, $j = 1, \ldots, p$.

– If there exists $t$ such that $t \equiv 0, 3$, the result follows from the fact that $\gamma_{\rm spr}(P_n) = \gamma_{\rm spr}(P_{n-1})$ for $n \equiv 0, 3 \pmod{10}$.

- **Case 4** For all $j = 1, \ldots, p$, $q_j \equiv 0, 3 \pmod{10}$,

  Then it is an easy consequence of Proposition 3.4 that $\eta = 2$.

Thus the desired result holds. ■

## 3.4 Ladders

A *ladder* of length $n-1$, which is denoted by $L_n$, is the Cartesian product $K_2 \square P_n$. We call the two $n$-paths $P_U$ and $P_V$, and let $V(L_n) = \{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$, where $u_i$ ($v_i$) is the $i^{\rm th}$ vertex of $P_U$ ($P_V$) and $u_i$ is adjacent to $v_i$ for all $i = 1, \ldots, n$. A *section* (of length $k$) of $L_n$ is the subgraph of $L_n$ induced by $\{u_i, v_i, \ldots, u_{i+k}, v_{i+k}\}$ for some $i$.

Now we give the SPD numbers for $L_n$.

**Theorem 3.7.** *For $i = 2, \ldots, 9$, $\gamma^{(i)}_{\rm spr}(L_n) = 2\gamma_{\rm s}(P_n)$, where $n = 2, 3, \ldots$.*

*Proof.* We use induction on $n$ to show that (a) $2\gamma_{\rm s}(P_n)$ guards can 2-defend $L_n$, and (b) fewer than $2\gamma_{\rm s}(P_n)$ guards cannot 9-defend $L_n$. It will follow that $\gamma^{(i)}_{\rm spr} = 2\gamma_{\rm s}(P_n)$, $i = 2, \ldots, 9$.

Figure 3.3 gives minimum 2-SPDs of $L_1$ to $L_7$. Using the same patterns as in this figure, one can easily show that statement (a) is true for $n = 1, 2, \ldots$.

Now we show that (b) holds for $n = 2, \ldots, 7$. We label the vertices of $L_n$ by $u_1, v_1, \ldots, u_n, v_n$ as in Figure 3.3. For $L_3$, $D = \{u_2, v_2\}$ is the unique PDS with two guards, and it is easy to see that $D$ is not a 9-PDS. It follows that $L_4$ cannot be 9-defended by fewer than four vertices either. Consider $L_5$. Let $D_1 = \{u_2, u_4, v_2, v_4\}$

with $M_1 = \{u_2v_2, u_4v_4\}$, and $D_2 = \{u_1, u_2, v_4, v_5\}$ with $M_2 = \{u_1u_2, v_4v_5\}$. $D_1$ and $D_2$ are the only two minimum PDS's of $L_5$ up to isomorphism; however, in either case, if there is an attack against $v_1$, $u_3$ will be left undominated. This implies that neither $D_1$ nor $D_2$ 9-defends $L_5$, and thus (b) holds for $n = 5$. It follows immediately that (b) holds for $L_6$ and $L_7$. Note that for $L_7$, $D = \{u_2v_2, u_4v_4, u_6v_6\}$ is the only 9-SPDS of size six.

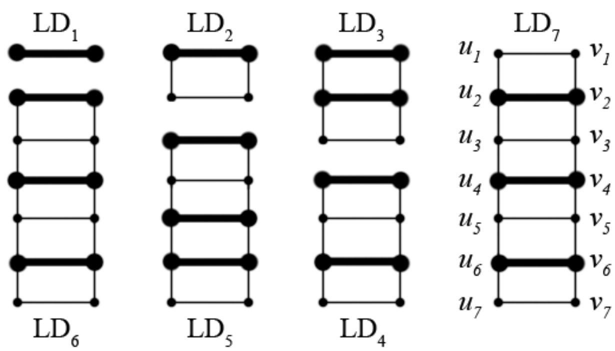



Figure 3.3: Minimum SPDSs of ladders of length 1 to 7

Now assume: $\gamma^{(9)}_{\mathrm{spr}}(L_n) = 2\gamma_{\mathrm{s}}(P_n)$ for $n = 1, \ldots, t$, where $t \geq 8$.

Let $n = t + 1$. Let $G = L_{t+1}$ and $D$ be a minimum 9-SPDS of $G$. Suppose to the contrary that $|D| \leq 2\gamma_{\mathrm{s}}(P_n) - 2$. Consider the $L_5$ induced by $\{u_i, v_i \mid i = n-4, n-3, n-2, n-1, n\}$, which, as shown in Figure 3.3, requires at least six guards to be securely pairwise dominated. Denote the set of these guards by $D^*$. Now

$$|D - D^*| \;\leq\; 2\gamma_{\mathrm{s}}(P_n) - 8 = 2\left(\gamma_{\mathrm{s}}(P_{n-7}) + 3\right) - 8 = 2\gamma_{\mathrm{s}}(P_{n-7}) - 2. \qquad (3.2)$$

Let $G'$ denote the $L_{n-7}$ induced by $\{u_i, v_i \mid i = 1, \ldots, n-7\}$. If $D' = D - D^*$ securely pairwise dominates $G'$, then (3.2) leads to a contradiction against our assumption. So $D'$ does not securely dominates $G'$, yielding $\{u_{n-6}, v_{n-6}\} \cap D \neq \varnothing$.

However, this is not possible because there is only one minimum 2-SPDS of $L_7$, and it does not intersect the two vertices with the smallest index ($u_{n-6}$ and $v_{n-6}$ in our case). ■

## 3.5 Grid Graphs

Consider the grid graph $G = P_m \,\Box\, P_k$ embedded in the $X - Y$ plane such that the vertices of $G$ have integer coordinates and occur in $m$ rows and $k$ columns. For $i \in \{1, ..., m\}$, $j \in \{1, ..., k\}$, let $v_{i,j}$ denote the vertex in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column. Let $S$ denote the rectangular region in the plane whose corners correspond to the coordinates of the vertices $v_{1,1}, v_{m,1}, v_{1,k}$ and $v_{m,k}$. A square in the plane whose corners correspond to the vertices $v_{i,j}, v_{i+1,j}, v_{i,j+1}, v_{i+1,j+1}$ is called a *tiling square* and is considered to have an area of one unit. A triangle in the plane that is bounded by two sides and a diagonal of a tiling square is called a *tiling triangle*.

Let $U$ be a subset of $V(G)$ such that $\langle U \rangle$ has a perfect matching. We call $U$ a *tiling set* of $G$. The *domination region* $R(U)$ is the region in the plane covered by those tiling squares and tiling triangles, all of whose corners are at distance at most one from some vertex in $U$. We say that $R(U)$ *covers* a vertex $v_{i,j}$ of $G$ if $v_{i,j}$ lies on the boundary or in the interior of $R(U)$.

We aim to place copies $U_1, ..., U_t$ of $U$ on (the representation of) $G$ in such a way that each vertex of $G$ is covered by at least one domination region $R(U_i)$. The set $D = \bigcup_{i=1}^{t} U_i$ will then correspond to a PDS of $G$. To ensure that $D$ is an SPDS, several vertices of $G$ will need to be covered by at least two domination regions. This means that fewer vertices in $D$ will have private neighbours, thus enabling guards placed on the vertices in $D$ to move to defend attacks at neighbouring vertices. However, to minimize $|D|$, we need to avoid overlapping domination regions, if possible.

We begin our description by placing infinitely many copies $U_1, U_2, ...$ of $U$ on the $X - Y$ plane, in such a way that all points with integer coordinates are covered by at least one domination region. See Figure 3.4 for an example of placements of the set $U$ with $|U| = 2$. There may exist regions in the plane that are not part of the domination regions $R(U_i)$. These regions are shown as white regions in Figures 3.4 and 3.5 , and we also refer to them as *white regions*. In Figure 3.4 the regularity of the placement of the copies of $U$ implies that there is a $1-1$ correspondence between the $R(U_i)$ and the white regions, while in Figure 3.5 , there is a $1-1$ correspondence between groups of three white regions (one tiling square and two tiling triangles) and three domination regions. Thus, if we define $\sigma$ to denote the ratio of the number of guards used to the area of the plane covered by the domination regions plus the area of the white regions, we see that $\sigma = \frac{2}{5}$ for the placement in Figure 3.4, and $\sigma = \frac{6}{14} = \frac{3}{7}$ for the placement in Figure 3.5.

The significance of these values is that if we use the placement in Figure 3.4 , together with some additional guards along the first two and last two rows and columns, to obtain an upper bound $\mu_{m,n}$ for $\gamma_{\text{spr}}(P_m \Box P_k)$, then $\mu_{m,n} \geq \frac{2}{5}(m-1)(k-1)$. Similarly, if we use the placement in Figure 3.5 , we obtain that $\mu_{m,n} \geq \frac{3}{7}(m-1)(k-1)$. Since $\frac{2}{5} < \frac{3}{7}$, we use the placement in Figure 3.4 . (Other placements also exist, but give worse bounds than those mentioned here.)

We now give an upper bound of the SPD numbers for grid graph $G = P_m \Box P_k$ for positive integers $m, k$.

**Theorem 3.8.** *For $i = 2, \ldots, 9$, and $1 \leq k \leq m$,*

$$\gamma_{\text{spr}}^{(i)}(P_m \Box P_k) \leq \gamma_{\text{spr}}^{(2)}(P_m \Box P_k) \leq 2\left(\left\lceil \frac{3k-2}{10} \right\rceil + \left\lceil \frac{m-3}{5} \right\rceil + k\left\lfloor \frac{m-2}{5} \right\rfloor + R(k)\right),$$

*where*

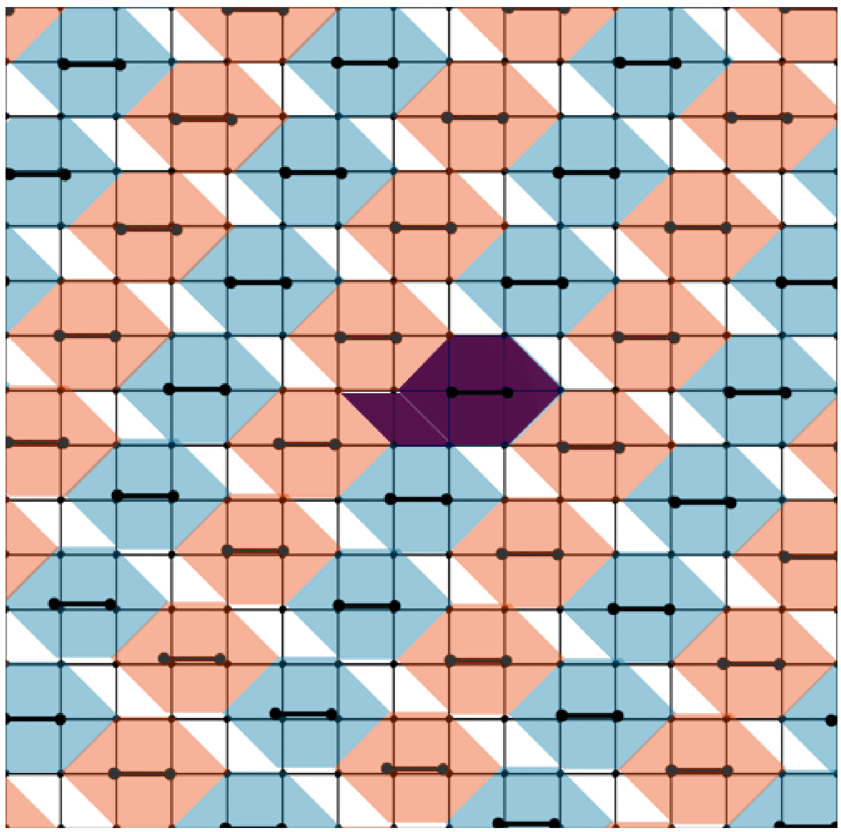


Figure 3.4: $\sigma = \frac{2}{5}$

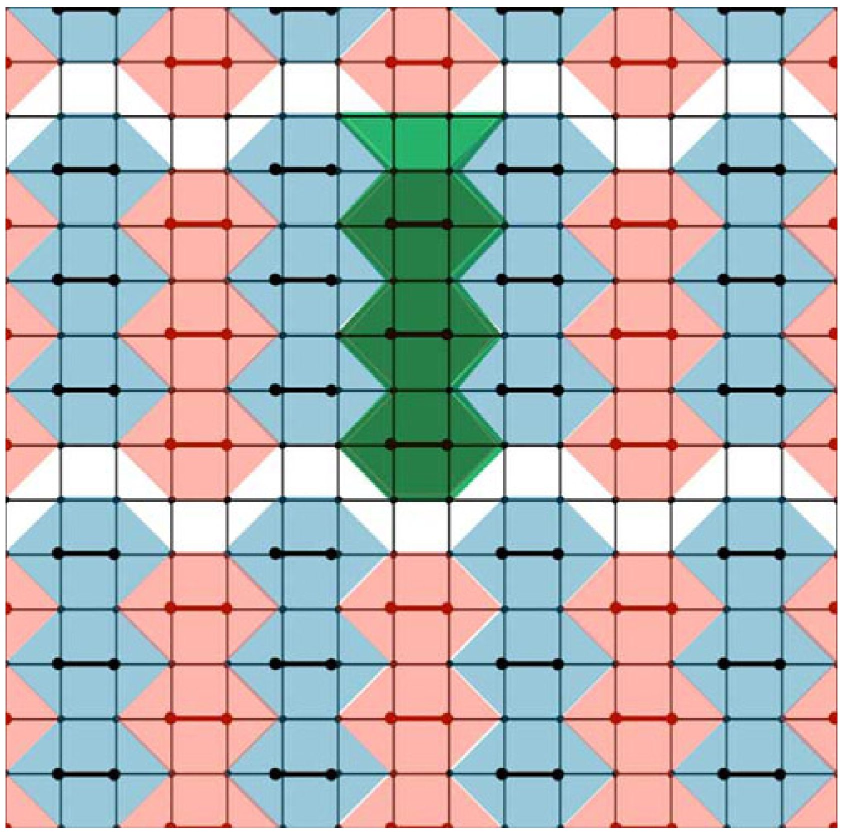


Figure 3.5: $\sigma = \frac{3}{7}$

$$R(k) = \begin{cases} \left\lceil \frac{3k+4}{10} \right\rceil + \left\lceil \frac{k-1}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \textit{if} \quad m \equiv 0 \pmod{5} \\ \left\lceil \frac{3k}{10} \right\rceil + \left\lceil \frac{k-1}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-3}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \textit{if} \quad m \equiv 1 \pmod{5} \\ \left\lceil \frac{3k-1}{10} \right\rceil & \textit{if} \quad m \equiv 2 \pmod{5} \\ \left\lceil \frac{3k+3}{10} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil & \textit{if} \quad m \equiv 3 \pmod{5} \\ \left\lceil \frac{3k-3}{10} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \textit{if} \quad m \equiv 4 \pmod{5} \end{cases}$$

*Proof.* Let $G = P_m \,\square\, P_k$. The first inequality is obvious.

To show the second inequality holds, we construct a 2-SPDS of $G$ of size at most $2\left(\left\lceil \frac{3k-2}{10} \right\rceil + \left\lceil \frac{m-3}{5} \right\rceil + k\left\lfloor \frac{m-2}{5} \right\rfloor + R_1(k) + R_2(k)\right)$, where

$$R_1(k) = \begin{cases} \left\lceil \frac{3k+4}{10} \right\rceil & \text{if} \quad m \equiv 0 \pmod{5} \\ \left\lceil \frac{3k}{10} \right\rceil & \text{if} \quad m \equiv 1 \pmod{5} \\ \left\lceil \frac{3k-1}{10} \right\rceil & \text{if} \quad m \equiv 2 \pmod{5} \\ \left\lceil \frac{3k+3}{10} \right\rceil & \text{if} \quad m \equiv 3 \pmod{5} \\ \left\lceil \frac{3k-3}{10} \right\rceil & \text{if} \quad m \equiv 4 \pmod{5} \end{cases}$$

and

$$R_2(k) = \begin{cases} \left\lceil \frac{k-1}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \text{if} \quad m \equiv 0 \pmod{5} \\ \left\lceil \frac{k-1}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-3}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \text{if} \quad m \equiv 1 \pmod{5} \\ 0 & \text{if} \quad m \equiv 2 \pmod{5} \\ \left\lceil \frac{k-2}{5} \right\rceil & \text{if} \quad m \equiv 3 \pmod{5} \\ \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \text{if} \quad m \equiv 4 \pmod{5}. \end{cases}$$

Consider the standard plane embedding of $G$ depicted in Figure 3.6, where $G$ has $m$ rows and $k$ columns.

Place a pair of guards at the end vertices of each boldfaced edge shown in Figure 3.6. Call the set of guards in the first row $S_1$, and let the set of pairs of guards $\{v_{i,j}, v_{i,j+1} \mid 2 \leq i \leq m-1, 2 \leq j \leq k-1\}$ be $S_2$. We also place guards on pairs of large black vertices of form $v_{i,1}, v_{i+1,1}$ $(2 \leq i \leq m-1)$, and denote the set of these guards by $S_3$.

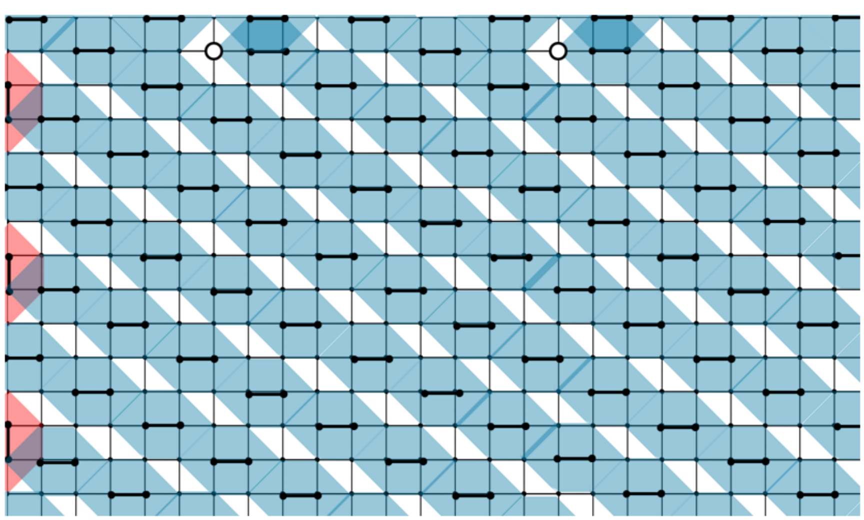


Figure 3.6: Guards in $S_1, S_2, S_3$.

Depending on $r_k$, the least residue of $k$ modulo 10, there are ten cases regarding the domination of the end of row 1, as shown in Figure 3.7. Similarly as in $S_3$, we place guards at each pair of large black vertices of form $v_{i,k}, v_{i+1,k}$ $(2 \leq i \leq m-1)$ shown in Figure 3.7. The set consisting of these guards is denoted by $S_4$. It is easy to see that in each case except when $r_k = 4, 7$, every vertex in row 1 either belongs to $S_1$ or is dominated by two guards; when $r_k = 4, 7$, the vertices which are dominated by one guard are shown as white circled vertices, and it is obvious that these vertices are defended. Therefore, all the vertices in row 1 are securely paired dominated.

Finally, place guards in row $m$ as shown in Figure 3.8, where there are five cases depending on $r_m$, the least residue of $m$ modulo 5, and each case is divided into ten sub-cases, depending on $r_k$. It is shown in Figure 3.8 that all vertices in the last row are securely pairwise dominated when $r_k = 2$. Denote the guards in row $m$ by $S_5$. By following a similar argument used for row 1, it can be shown that every vertex is securely pairwise dominated in all other cases.

Let $D = \bigcup_{i=1,\ldots,5} S_i$. Labelled by white vertices in Figures 3.6 and 3.7, all the vertices except in row $m$ which are dominated by one guard are $D$-defended (vertices in row $m$ are discussed previously); further, there is no vertex $v$ such that all the guards dominating $v$ have external private neighbours (white vertices). Hence $D$ is a 2-SPDS of $G$. Now we prove that $|D| = M$ by calculating the size of $S_1, \ldots, S_5$. We begin with $S_1$ and $S_3$, of which the size is easy to obtain:

$$|S_1| = 2\left\lceil \frac{3k-2}{10} \right\rceil, |S_3| = 2\left\lceil \frac{k-3}{5} \right\rceil.$$

In the last row, we have

$$|S_5| = 2\left\lceil \frac{3k+4}{10} \right\rceil, 2\left\lceil \frac{3k}{10} \right\rceil, 2\left\lceil \frac{3k-1}{10} \right\rceil, 2\left\lceil \frac{3k+3}{10} \right\rceil, 2\left\lceil \frac{3k-3}{10} \right\rceil,$$

when $m \equiv 0, 1, 2, 3, 4 (\mathrm{mod}\ 5)$, respectively.

Now consider $S_2 \cup S_4$. On the $i$th row $(2 \leq i \leq m-1)$ we can put $\lceil \frac{k-3}{5} \rceil$, $\lceil \frac{k}{5} \rceil$, $\lceil \frac{k-2}{5} \rceil$, $\lceil \frac{k-4}{5} \rceil$, $\lceil \frac{k-1}{5} \rceil$ pairs of guards, when $i \equiv 0, 1, 2, 3, 4 \pmod 5$ respectively. Also

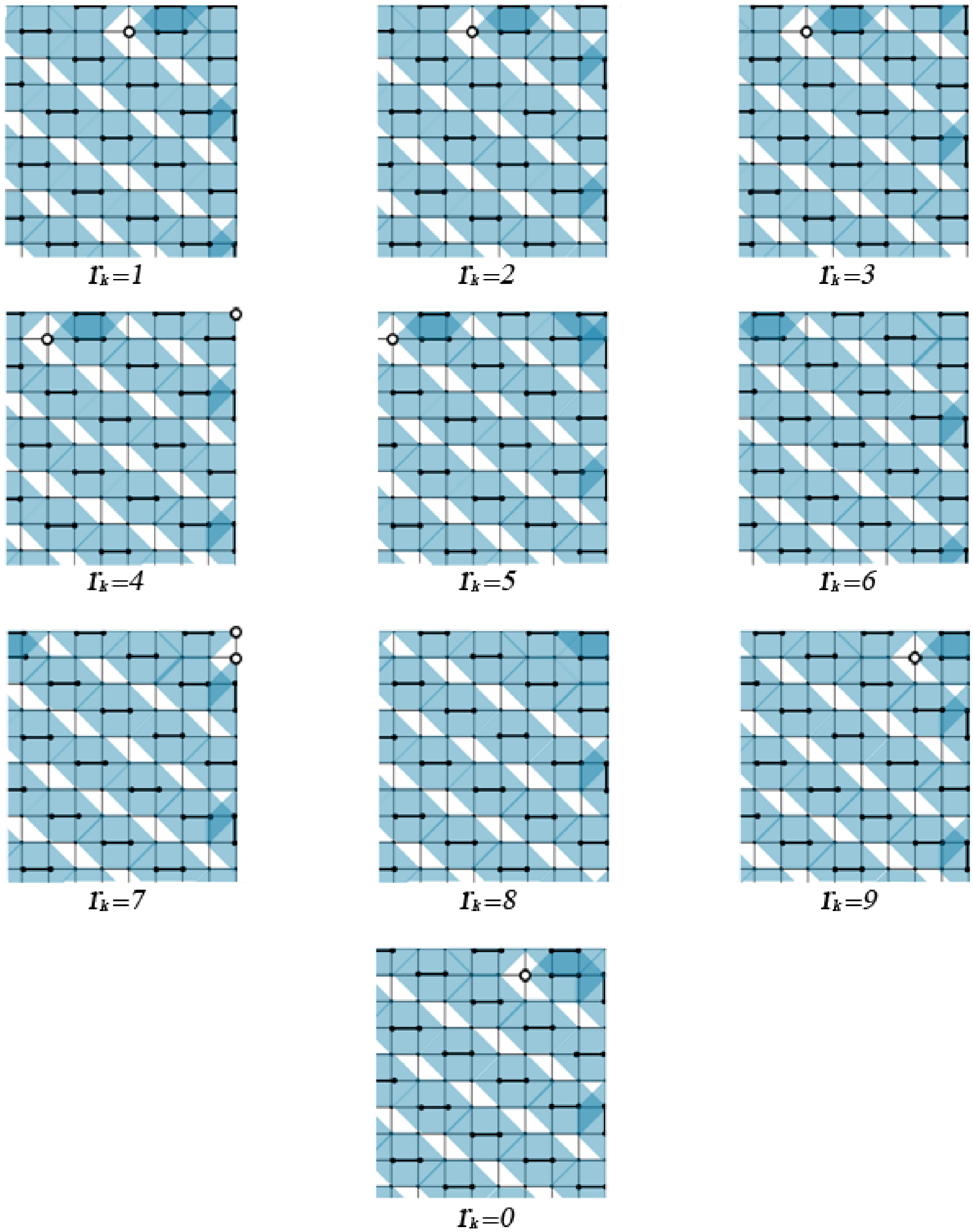
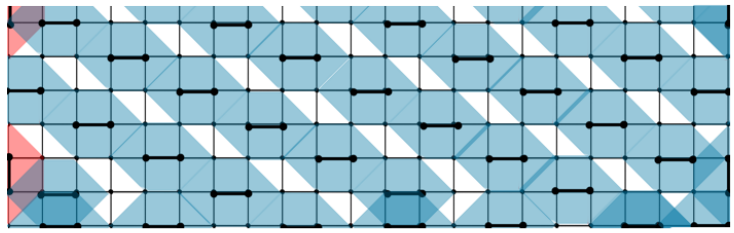
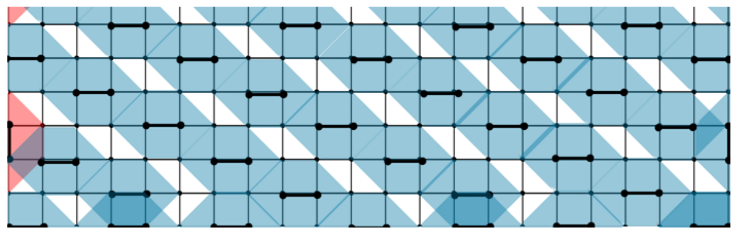


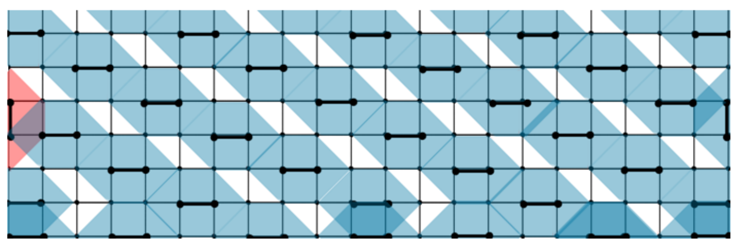

Figure 3.7: Guards in $S_4$, when $r_k = 0, \ldots, 9$.



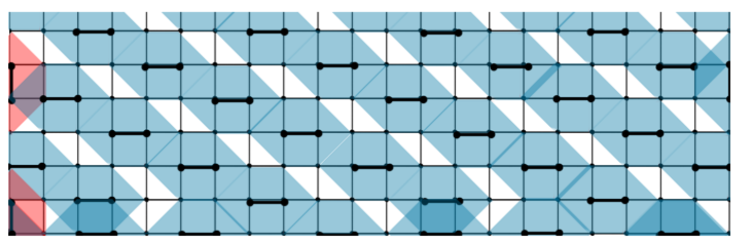
$r_m = 0$



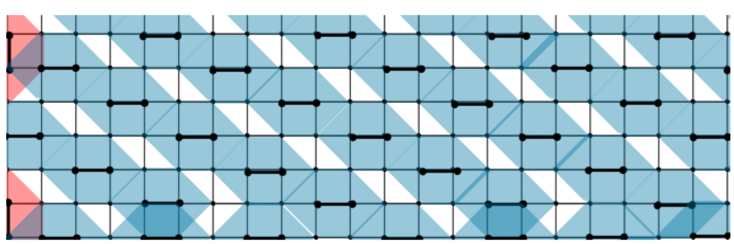
$r_m = 1$



$r_m = 2$



$r_m = 3$



$r_m = 4$

Figure 3.8: guards in $S_5$, when $r_m = 0, \ldots, 4$.

observe that there are exactly $k$ pairs of guards in every adjacent five rows in $S_2 \cup S_4$, and that the pattern repeats after five rows. Therefore

$$|S_2| + |S_4| = 2k \left\lfloor \frac{m-2}{5} \right\rfloor + 2R_2(k),$$

where $R_2(k)$ counts the number of pairs of guards in $V_1$ in the bottom $r_m$ rows of $G$.

The desired upper bound is obtained from $|D| = \sum_{i=1}^{5} |S_i|$. ∎

We now adapt the tiling pattern used in $P_m \square P_k$ to find an upper bound of $\gamma_{\mathrm{spr}}^{(i)}(C_m \square C_k)$, where $i = 2, \ldots, 9$.

**Corollary 3.9.** *For* $i = 2, \ldots, 9$, $1 \leq k \leq m$,

$$\gamma_{\mathrm{spr}}^{(i)}(C_m \square C_k) \leq \gamma_{\mathrm{spr}}^{(2)}(C_m \square C_k) \leq \gamma_{\mathrm{spr}}^{(2)}(P_m \square P_k); \tag{3.3}$$

*further, if* $m \equiv 0, 2, 3 \pmod 5$, *then*

$$\gamma_{\mathrm{spr}}^{(2)}(C_m \square C_k) \leq 2\left(k \left\lfloor \frac{m}{5} \right\rfloor + R_1(k) + R_2(m)\right), \tag{3.4}$$

*where*

$$R_1(k) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 5 \\ \left\lceil \frac{k}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil & \text{if } m \equiv 2 \pmod 5 \\ \left\lceil \frac{k}{5} \right\rceil + \left\lceil \frac{k-2}{5} \right\rceil + \left\lceil \frac{k-4}{5} \right\rceil & \text{if } m \equiv 3 \pmod 5 \end{cases}$$

*and*

$$R_2(m) = \begin{cases} 0 & \text{if } k \equiv 0 \pmod 5 \\ \left\lceil \frac{m-1}{5} \right\rceil + \left\lceil \frac{m-4}{5} \right\rceil & \text{if } k \equiv 1 \pmod 5 \\ \left\lceil \frac{m-2}{5} \right\rceil & \text{if } m \equiv 2, 3, 4 \pmod 5 \end{cases}$$

*Proof.* $P_m \square P_k$ is a spanning subgraph of $C_m \square C_k$, so (3.3) is true.

When $m \equiv 0, 2, 3 \pmod 5$, consider $D$, the set consisting of the large black vertices in Figure 3.9, where private neighbours of guards are labeled by white circles, each of which is obviously protected by the unique adjacent pair of guards. It is thus easy to verify that $D$ is a 2-SPDS of $C_m \square C_k$. Now, following a similar counting argument as in the proof of Theorem 3.8, it is routine to show that $|D| = 2\left(k\left\lfloor \frac{m}{5} \right\rfloor + R_1(k) + R_2(m)\right)$.

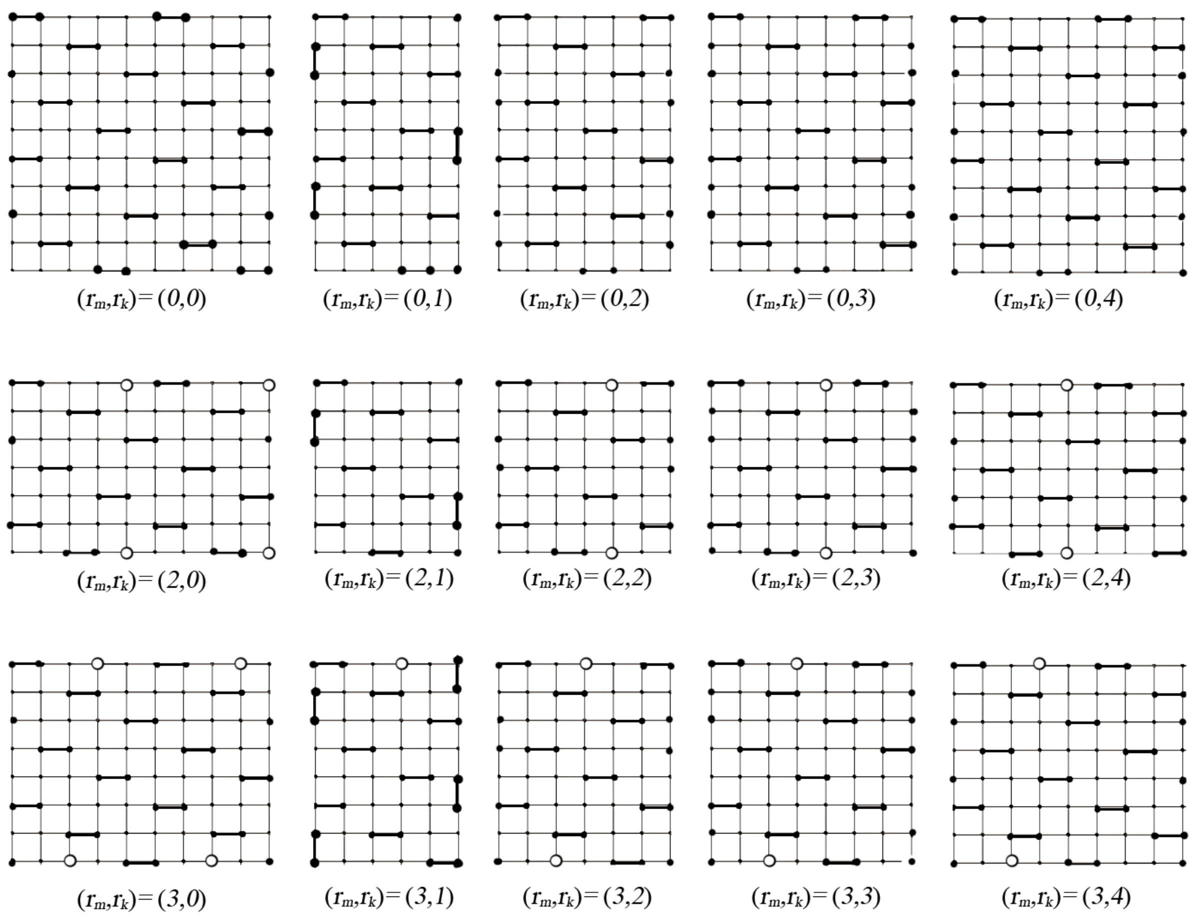
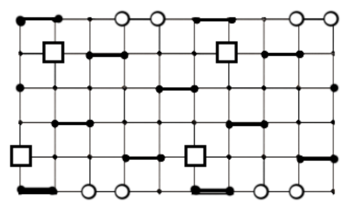


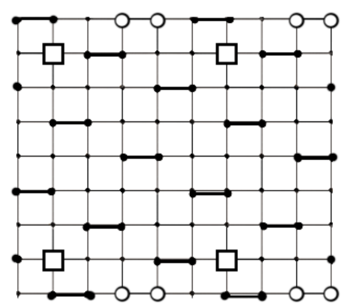

Figure 3.9: 2-SPDSs of $C_m \square C_k$ when $m \equiv 0, 2, 3 \pmod 5$

.

To see (3.4) is not an upper bound of $C_m \square C_k$ when $m \equiv 1, 4 \pmod 5$, notice that if we place guards similar as before, some vertices are not protected, as shown in Figure 3.10. Thus we place guards in the first and the last row in the same fashion



$(r_m, r_k) = (1,0)$



$(r_m, r_k) = (4,0)$

Figure 3.10: Vertices labeled by white squares are not protected.

as in $P_m \square P_k$, which leads to the same upper bound as in Theorem 3.8:

$$\gamma_{\mathrm{spr}}^{(i)}(P_m \square P_k) \leq 2\left(\left\lceil \frac{3k-2}{10} \right\rceil + \left\lceil \frac{m-3}{5} \right\rceil + k\left\lfloor \frac{m-2}{5} \right\rfloor + R(k)\right),$$

where

$$R(k) = \begin{cases} \lceil \frac{3k}{10} \rceil + \lceil \frac{k-1}{5} \rceil + \lceil \frac{k-2}{5} \rceil + \lceil \frac{k-3}{5} \rceil + \lceil \frac{k-4}{5} \rceil & \text{if} \quad m \equiv 1 \pmod{5} \\ \\ \lceil \frac{3k-3}{10} \rceil + \lceil \frac{k-2}{5} \rceil + \lceil \frac{k-4}{5} \rceil & \text{if} \quad m \equiv 4 \pmod{5} \end{cases}$$

■

Now we give the 2-SPD numbers for $P_m \square P_m$ when $m$ is small.

It is easy to see that, when $m = 2, 3$, $\gamma_{\mathrm{spr}}^{(2)}(P_m \square P_m) = 2, 6$, respectively. When $m = 4$, we consider the standard embedding of $P_4 \square P_4$ into $X$-$Y$ plane with the vertices labeled $v_{i,j}$, $i, j \in \{1, \ldots, 4\}$ as before. It is easy to show that $D = \{v_{1,2}, v_{1,3}, v_{2,1}, v_{3,1}, v_{4,2}, v_{4,3}, v_{2,4}, v_{3,4}\}$ (with the obvious matching) is a minimum 2-SPDS of the graph, and thus $\gamma_{\mathrm{spr}}^{(2)}(P_4 \square P_4) = 8$. The case when $m = 5$ are discussed in the following result.

**Proposition 3.10.** $\gamma_{\mathrm{spr}}^{(2)}(P_5 \square P_5) = 12$

*Proof.* Let $G = P_5 \square P_5$. First we show $\gamma_{\mathrm{spr}}^{(2)}(G) \leq 12$ by giving a 2-SPDS of $G$ of size 12.

Consider $D$, the set of guards shown as end vertices of the boldfaced edges in Figure 3.11. In the figure, each vertex marked by a yellow dot is dominated by one guard, while the other vertices are dominated by at least two guards. One can easily see that every vertex with a yellow dot is defended by its unique neighbouring guard. Further, observe that there is no vertex with more than one neighbouring guard, each of which has a private external neighbour. Therefore $D$ is indeed a 2-SPDS of $G$, and $|D| = 12$.

Now we show that there does not exist a 2-SPDS of $G$ that has a smaller cardinality

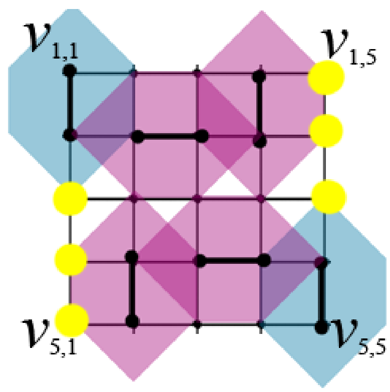



Figure 3.11: a minimum 2-SPDS of $P_5 \square P_5$

than $D$. We call the cycle induced by $\{v \in V(G) \mid \deg v \leq 3\}$ the *edge* of $G$. We embed $G$ into $X$-$Y$ plane and label its vertices as before.

First observe that, to dominate the four corner squares, there are four guards on the edge of $G$, and no two of these four guards are adjacent. Thus each guard is partnered by another guard either on the edge of $G$, or in the rows and columns next to the edge. Note that none of these eight guards dominates $v_{3,3}$.

Recall that $\gamma^{(2)}_{\text{spr}}(P_4 \square P_4) = 8$, but from the above argument it easily follows that it is impossible to securely pairwise dominate $G = P_5 \square P_5$ (in particular, both its four corners and $v_{3,3}$ with four pairs of guards), using only the movements defined in Definition 2.2. Hence, it suffices to show that there is no 2-SPDS of $G$ with ten vertices either.

Suppose to the contrary that there exists a 2-SPDS $D'$ of $G$ such that $|D'| = 10$. Then, as argued above, $D'$ contains four pairs of guards such that no pair is adjacent to another, and each contains a guard on the edge of $G$. Now, we let $v_{3,3}$ be $D'$-defended by guard $w \in D'$, and depending on whether or not $w$ is on $v_{3,3}$, there are two cases:

- **Case 1** $w$ lies on $v_{3,3}$

  Assume without loss of generality that the $D'$ - partners of $w$ lies on $v_{4,3}$. We claim that $v_{2,3}$ is $D'$-defended only by $w$. To see this, suppose $v_{2,3}$ is defended by a guard $x \neq w$. Then $x$ lies on $v_{2,3}$ or $x$ moves to $v_{2,3}$ to defend it against an attack, and we are forced to use only three pairs of guards to dominate the four corners $v_{1,1}, v_{1,5}, v_{5,1}$ and $v_{5,5}$, which is clearly not possible.

  Since $w$ defends $v_{2,3}$, $v_{5,3}$ is dominated by a guard other than $s$. This guard lies on $v_{5,2}$, $v_{5,3}$, or $v_{5,4}$, and we may assume that with out loss of generality that there is a guard $t$ on $v_{5,2}$ whose $D'$-partner $t'$ lies on $v_{5,1}, v_{4,2}$, or $v_{5,3}$. In each case neither $t$ nor $t'$ dominates $v_{3,1}$. Let $y$ be the guard that $D'$-defends $v_{3,1}$ and let $y'$ be its $D'$-partner. If neither $y$ nor $y'$ protects $v_{1,1}$, then at least six additional guards are required to dominate $v_{1,1}$, $v_{1,5}$ and $v_{5,5}$, which is impossible. Hence

$y$ or $y'$ lies on $v_{2,1}$. Similarly, it can be shown that there are guards on both $v_{1,4}$ and $v_{4,5}$.

Now, let $v \in \{y, y'\}$ be the guard on $v_{2,1}$, and let $u \in \{y, y'\} - \{v\}$. Then either $u$ defends $v_{1,2}$ or it does not, as discussed in the following two subcases.

- **Subcase 1** $u$ defends $v_{1,2}$.

  In this case, $u$ lies on $v_{1,1}$ or $v_{2,2}$. In either case, after the pair $u, v$ move along the path $v_{2,1}$, $v_{1,1}$, $v_{1,2}$ or the path $v_{2,1}, v_{2,2}, v_{1,2}$ to defend an attack against $v_{1,2}$, $v_{3,1}$ is no longer dominated by $\{u, v\}$. However, this is impossible since the $D'$-partner of $t$ does not dominate $v_{3,1}$ in any of its three possible positions, nor does any other guard.

- **Subcase 2** $u$ does not defend $v_{1,2}$.

  Then in order to defend $v_{1,2}$, the $D'$-partner of the guard on $v_{1,4}$ lies on $v_{1,3}$, and the guards on $v_{1,4}$ and $v_{1,3}$ move along the path $v_{1,4}, v_{1,3}, v_{1,2}$ to defend $v_{1,2}$. It is routine to show that regardless of the placements of other guards, $v_{1,5}$ is left undominated, which is a contradiction.

- **Case 2** There is no guard in $D'$ that lies on $v_{3,3}$

  Let $p$ be the guard defending $v_{3,3}$ and without loss of generality, assume $p$ lies on $v_{3,2}$. Let $q$ be the $D'$-partner of $p$. Then, up to isomorphism, there are two possible situations: $q$ lies on $v_{3,1}$ or $v_{4,2}$. In either case, since both $v_{1,1}$ and $v_{5,1}$ need to be defended, which requires at least four additional guards, we are forced to use the remaining two pairs of guards to form a 2-SPDS of the subgraph induced by $\{v_{1,5}, v_{2,4}, v_{2,5}, v_{3,4}, v_{3,5}, v_{4,4}, v_{4,5}, v_{5,5}\}$, which can be easily shown to be impossible.

Therefore, we conclude that there is no 2-SPDS of $G$ of size 10, which leads to the desired result. ∎

Finally, we can use Figure 3.12 and follow a similar argument used in the proof of Proposition 3.10 to obtain the 2-SPD number for $P_6 \,\square\, P_6$.

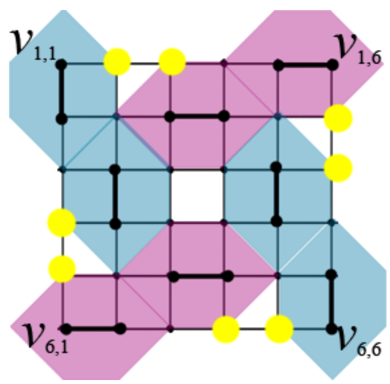



Figure 3.12: a minimum 2-SPDS of $P_6 \Box P_6$

**Proposition 3.11.** $\gamma_{\rm spr}^{(2)}(P_6 \,\square\, P_6) = 16$

*Proof.* Similar to the proof of Proposition 3.10. ■

**Corollary 3.12.** *For integers $m, n \geq 1$ and $i = 2, \ldots, 9$,*

- $\gamma_{\rm spr}^{(i)}(P_{5m} \,\square\, P_{5n}) \leq 12mn$,
- $\gamma_{\rm spr}^{(i)}(P_{6m} \,\square\, P_{6n}) \leq 16mn$.

## 3.6 Trees and Forests

Our main aim is to estimate SPD numbers for trees by establishing an upper and a lower bound for $\gamma_{\rm spr}(T)$, where $T$ is a tree.

Notice that, different from secure domination, secure paired domination in a tree requires moving a pair of vertices to defend another vertex. This property suggests that lower bounds for $\gamma_{\rm spr}(T)$ are not in direct proportion to $n = |V(T)|$ or $\Delta(T)$. An example is $K_{1,t}$, where $t$ can be big (and thus so can $n = t + 1$), whereas obviously $\gamma_{\rm spr}(K_{1,t}) = 2$. The next observation shows we can bound the SPD bumber of a connected graph by its diameter (the proof is obvious and omitted).

**Observation 3.13.** *Let $G$ be a connected graph with diam$(G) = d$. Then, for $i = 2, \ldots, 9$,*

$$\gamma_{\rm spr}^{(i)}(G) \geq \gamma_{\rm spr}^{(i)}(P_{d+1}). \tag{3.5}$$

**Remark 3.14.** *Equality in (3.5) may be obtained, for example, for stars and paths.*

Our next results relate to line graphs. We start with an algorithm, which takes a minimum SDS of the line graph of a graph and produces an SPDS of a supergraph of the original graph.

**Algorithm 3.15.** *Let $G$ be a graph without isolated vertices, and let $L(G)$ be the line graph of $G$ with $V(L(G)) = \{uv \mid uv \in E(G)\}$. Let $D_L$ be a minimum SDS of $L(G)$.*

*Inputs of the algorithm: $G$, $L(G)$, $D_L$.*

*Steps:*

1. *Create sets $X = D_L$, $D_H = \varnothing$, $M_H = \varnothing$, and graph $H = G$.*

2. *If $X = \varnothing$, the algorithm terminates; otherwise, choose a vertex $uv \in X$;*

3. 
   - *if $\{u, v\} \cap D_H = \varnothing$, add $u, v$ to $D_H$, $uv$ to $M_H$, and delete $uv$ from $X$,*
   - *if (without loss of generality) $\{u, v\} \cap D_H = \{u\}$, add a new vertex $v'$ to $H$ such that $v'$ is adjacent to only $v$, and add $v, v'$ to $H$, $vv'$ to $M_H$, and delete $uv$ from $X$,*
   - *otherwise, delete $uv$ from $X$;*

4. *Return to Step* 1.

**Remark 3.16.** *Algorithm 3.15 returns a graph $H$ of which $G$ is an induced subgraph, and $D_H$ of $H$ with associated matching $M_H$. Since $D_L$ protects every vertex of $L(G)$, a set $D_H$ protects every edge and thus every vertex of $G$. Hence $D_H$ is an SPDS of $H$.*

**Proposition 3.17.** *Let $G$ be a graph and $L(G)$ be the line graph of $G$. Then*

$$\gamma_{\mathrm{spr}}(G) \leq 2\gamma_{\mathrm{s}}(L(G)). \tag{3.6}$$

*Proof.* For $G$, Algorithm 3.15 provides a graph $H$ such that $H - G$ contains only leaves of $H$, and such that $\gamma_{\mathrm{spr}}(H) \leq 2\gamma_{\mathrm{s}}(L(G))$. If $v \in V(H) - V(G)$, then $v \in D_H$. Let $v'$ be the partner of $v$. If $v'$ has a neighbour $u$ such that $u \notin D_H$, let $D' = (D_H - \{v\}) \cup \{u\}$. Suppose each neighbour of $v'$ is in $D_H$. If $u \in N(v')$ and $u, u' \in D_H$, then $uu' \in M_H$, and there exists $w \in \mathrm{epn}(u', D_H)$. Let $D' = (D_H - \{v\}) \cup \{w\}$, and $M_H = (M - \{vv', uu'\}) \cup \{v'u, u'w\}$. If there is no such $u \in N(v')$, let $D' = D_H - \{v, v'\}$.

In each case $D'$ is an SPDS of $G$ and the result follows. ■

Next we present a class of graphs for which (3.6) achieves equality.

**Proposition 3.18.** *Let $T$ be a tree with line graph $L(T)$. Then $\gamma_{\rm spr}(T) = 2\gamma_{\rm s}(L(T))$ if there is a minimum SPDS $D$ of $T$ such that every path of order six in $T$ contains at least five vertices in $D$.*

*Proof.* For convention, for a tree $T$ and $D$ an SPDS of $T$, we call a path a *$D$-bad path* (*bad path* hereafter whenever the context is clear) if $|V(P)| = 6$ and $|V(P) - D| \geq 2$. Now let $T$ be a tree such that there exists a minimum SPDS $D$ of $T$, such that $T$ contains no $D$-bad paths. Let $M$ be the matching associated with $D$. By Proposition 3.17 it suffices to show:

$$\gamma_{\rm spr}(T) \geq 2\gamma_{\rm s}(L(T)).$$

In $L(T)$, consider $D' = \{v_1v_2 \in V(L(T)) \mid v_1 \in D, v_2 \in D, v_1v_2 \in M\}$, which has size $\frac{1}{2}\gamma_{\rm spr}(T)$. We show that $D'$ is an SDS of $L(T)$. Suppose to the contrary that there exists a vertex $uv \in V(L(T))$ such that $uv$ is not $D'$-defended.

If $u, v \notin D$, there exist four vertices $u_1, u_2, v_1, v_2 \in D$ with $u_1u_2, v_1v_2 \in M$, such that $u_1$, $v_1$ is adjacent to $u$, $v$, respectively, as $D$ is an SPDS of $T$; but this is not the case, as $u, v \notin D$ are contained in a bad path, contradicting our assumption.

If $u, v \in D$, then $uv \notin M$. Let $u_1$ and $v_1$ be the $D$-partner of $u$, $v$, respectively. In $L(T)$, we have $uu_1, vv_1 \in D'$, and since $uv$ is not defended, each of $uu_1$ and $vv_1$ has an $D'$-epn. This implies that, in $T$, $u_1$, $v_1$ have neighbours $u_2$, $v_2$, respectively, and that $u_2, v_2 \notin D$. We then have a contradiction, as now $u_2, u_1, u, v, v_1, v_2$ form a bad path in which $u_2, v_2 \notin D$.

Hence, exactly one of $u, v$ is in $D$. Without loss of generality we assume that $u \in D$ and $v \notin D$, and let $u_1$ be the $D$-partner of $u$. Then there are two cases, depending on whether or not $v$ is a leaf in $T$:

- **Case 1** If $v$ is a leaf.

  Since $uv$ is not $D'$-defended, $uu_1$ has a $D'$-epn. Consequently, $u_1$ has a neighbour $u_2$ in $T$, $u_2 \neq u$, $u_2 \notin D$. Observe that $u_2$ is not a leaf, for otherwise either $v$ or $u_2$ is not defended in $T$. Thus $u_2$ has a neighbour $u_3 \in D$ whose $D$-partner is denoted by $u_4$. Now $v, u, u_1, u_2, u_3, u_4$ forms a bad path with $v, u_2$ not in $D$, which is not the case.

- **Case 2** If $v$ is not a leaf.

  Then let $w$ be a neighbour of $v$. It is obvious that $w \in D$, as otherwise $w$ is dominated by another pair of guards, resulting in a bad path. We denote the $D$-partner of $w$ by $w_1$. Since $uv$ is not defended by $D'$ in $L(T)$, $uu_1$ has a $D'$-epn in $L(T)$, so that $u_1$ has a neighbour $u_2 \in T$ that does not belong to $D$. Again, a bad path is formed, leading to a contradiction.

Consequently, $D'$ is an SDS of $L(T)$, and $\gamma_{\mathrm{spr}}(T) = 2\gamma_{\mathrm{s}}(L(T))$. ■

**Remark 3.19.** *In fact, for a tree $T$, $D$ an SPDS of $T$, and $D' \subseteq V(L(T))$ formed as in the proof of Proposition 3.18, $D'$ is not an SDS of $L(T)$ whenever $T$ contains a $D$-bad path.*

**Remark 3.20.** $\gamma_{\mathrm{spr}}(P_n) = 2\gamma_{\mathrm{s}}(L(P_n))$ *if and only if* $n = 2, 4, 5, 7, 8, 11, 14, 17$.

*Proof.* $L(P_n) = P_{n-1}$ for all $n \geq 2$. The results follows directly from Proposition 3.1 and Proposition 3.2. ■

# Chapter 4

# Bounds of Secure Paired Domination Numbers

## 4.1 Introduction

We compare the secure paired domination number of any graph $G$ with various other parameters such as clique partition number, independence number, vertex-covering number, secure domination number and paired domination number. As a result we establish that $\gamma_{\text{spr}}(G)$ does not exceed $2x$, where $x$ is any of the parameters mentioned above. Finally, we give classes of trees for which some of the bounds are achieved.

## 4.2 Upper Bounds of Secure Paired Domination Numbers Relating to Other Parameters

**Observation 4.1.** *Let $G$ be a graph with no isolated vertices. Then $\gamma^{(i)}_{\text{spr}}(G) \leq 2\gamma_{\text{s}}(G)$ for $2 \leq i \leq 9$.*

Let $G = (V, E)$ be a graph. A *clique covering* of $G$ is a partition of $V$ where each subset induces a clique, and the *clique partition number* of $G$, denoted by $\theta(G)$ (or $\theta$ if the context is clear), is the minimum number of sets in a clique cover of $G$. In a clique covering, a set $X \subseteq V(G)$ such that $|X| = 1$ is called a *singleton*.

Recall that the independence number of a graph, $\beta(G)$, is the cardinality of the largest independent set of $G$. A set $S \subseteq G$ is called a *vertex-covering* of $G$ if every edge of $G$ has at least one endpoint contained in $S$. The *vertex-covering number* of $G$ is the cardinality of a minimum vertex-covering of $G$, and is denoted by $\alpha(G)$. By definition, $\alpha(G) + \beta(G) = |V(G)|$ for any graph $G$.

**Theorem 4.2.** *For a graph $G$ without isolated vertices and $2 \le i \le 9$, $\gamma_{\text{spr}}^{(i)}(G) \le 2\theta(G)$, where $\theta(G)$ is the clique partition number of $G$.*

*Proof.* Let $\mathcal{P} = V_1 \cup \cdots \cup V_\theta$ be a minimum clique partition of $G$ that contains as few singletons as possible. Assume that $r \le \theta$ is an integer such that $|V_i| = 1$ for $i < r$ and $|V_i| \ge 2$ for $i \ge r$.

Suppose $V_1$ and $V_2$ are singletons; say $V_1 = \{v_1\}$ and $V_2 = \{v_2\}$. Then $v_1v_2 \notin E(G)$, for otherwise $V_1 \cup V_2$ is a clique, and $\mathcal{P}' = (V_1 \cup V_2) \cup \cdots \cup V_\theta$ is a clique cover of $G$ in $\theta - 1$ cliques, a contradiction. Hence

$$\text{if } V_1 = \{v_1\}, \text{ then } v_1 \text{ is adjacent to a vertex in } V_k, \text{ where } k \ge r. \tag{4.1}$$

Suppose $x, y \in V_k$ for some $k \ge r$ such that $v_1x,\ v_2y \in E(G)$, and assume firstly that $x \ne y$. Define $V_1' = V_1 \cup \{x\}$, $V_2' = V_2 \cup \{y\}$. If $|V_k| = 2$, then $\mathcal{P}' = V_1' \cup V_2' \cup \cdots \cup V_{k-1} \cup V_{k+1} \cup \cdots \cup V_\theta$ is a clique cover of $G$ in $\theta - 1$ cliques, a contradiction, and if $|V_k| \ge 3$, then $\mathcal{P}' = V_1' \cup V_2' \cup \cdots \cup V_k' \cup \cdots \cup V_\theta$, where $V_k' = V_k - \{x, y\}$, is a clique cover of $G$ in $\theta$ cliques that contains fewer singletons than $\mathcal{P}$. This is also a contradiction. Hence $x = y$.

Now, if $|V_k| \ge 3$, let $V_1'$ be as defined above and let $V_k' = V_k - \{x\}$. Then $\mathcal{P}'' = V_1' \cup V_2 \cup \cdots \cup V_k' \cup \cdots \cup V_\theta$ is a clique cover of $G$ in $\theta$ cliques that contains fewer singletons than $\mathcal{P}$, again a contradiction. Hence we have proved that

$$\begin{array}{l}\text{if } V_1 = \{v_1\} \text{ and } V_2 = \{v_2\}, \text{ and } v_1 \text{ and } v_2 \text{ are adjacent to vertices in } V_k, \\ \text{then } |V_k| = 2 \text{ and } v_1 \text{ and } v_2 \text{ are adjacent to the same vertex in } V_k.\end{array} \tag{4.2}$$

Together with (4.1), the last paragraph also shows that

$$\text{if } V_1 = \{v_1\}, \text{ then } v_1 \text{ is adjacent to a vertex in } V_k, \text{ where } |V_k| = 2. \tag{4.3}$$

For each $i \geq r$, let $u_i, v_i$ be arbitrary vertices in $V_i$; note that $u_i v_i \in E(G)$. Let $D = \bigcup_{i=r}^{\theta} \{u_i, v_i\}$ and consider $x \in \bigcup_{i=1}^{r-1} V_i$. By (4.3), $x$ is adjacent to $u_k$ (say) for some $k$ such that $V_k = \{u_k, v_k\}$, and by (4.2), $v_k$ is not adjacent to any vertex in $\bigcup_{i=1}^{r-1} V_i - \{x\}$. Hence $\{u_k, v_k\}$ defends $x$ (the guard on $u_k$ moves to $x$ while the guard on $v_k$ moves to $u_k$; each vertex in $G$ remains dominated). Obviously, $\{u_k, v_k\}$ defends $V_k$. If no vertex in $V_j$, $j \geq r$, is adjacent to a vertex in $\bigcup_{i=1}^{r-1} V_i$, then clearly $\{u_j, v_j\}$ is an SPDS of $G[V_j]$. Therefore $D$ is a 2-SPDS of $G$, from which the desired result follows. ■

**Proposition 4.3.** *Let $G$ be a graph with no isolated vertices. Then for $2 \leq i \leq 9$*

$$\gamma_{\text{spr}}^{(i)}(G) \leq 2\min\{\alpha(G), \beta(G)\},$$

*where $\alpha(G), \beta(G)$ are the vertex-covering number and the independence number of $G$, respectively.*

*Proof.* Let $G$ be a graph of order $n$, and $S = \{v_1, v_2, \ldots, v_{\beta(G)}\}$ be a maximum independent set of $G$. Since $G$ does not contain isolated vertices, $v_i$ has neighbours in $V(G) - S$ for all $1 \leq i \leq \beta(G)$.

Let $M$ be a maximum matching from $S$ to $V(G) - S$ and let $D$ be the set of end-vertices of the edges in $M$. It is easy to see that for a vertex $v \in S$, $v \notin D$ implies $N(v) \subseteq D$. Further, since the complement of an $\alpha$-set is a $\beta$-set, $|D| \leq 2\min\{\alpha(G), \beta(G)\}$.

Now we show that $D$ is a 2-SPDS of $G$. Obviously $D$ is dominating, so it suffices to show that all vertices are $D$-defended. Suppose to the contrary that $w \in V(G)$ is not $D$-defended.

If $w \in S$, let $x \in N(w)$ and note that $x \in D - S$. Let $y \in S$ be the $D$-partner of $x$. If $p \in \text{epn}(y, D)$, then $M' = (M - \{xy\}) \cup \{xw, yp\}$ is a matching from $S$ to $V(G) - S$ with $|M'| > |M|$, a contradiction. Hence $\text{epn}(y, D) = \varnothing$, and so $\{x, y\}$ defends $w$.

If $w \notin S$, let $x \in S$ be a neighbour of $w$. Note that $x \in D$, otherwise we can extend $M$. Let $y \in V(G)-S$ be the $D$-partner of $x$. Since $w$ is not $D$-defended, $y$ has a $D$-epn $p$ such that $p \notin N[w]$. If $p \in S$, then $(M - \{xy\}) \cup \{xw, py\}$ forms a larger matching than $M$, leading to a contradiction. So $p \in V(G) - S$. Since $S$ is a maximum independent set, $p$ has a neighbour $q \in S - D$. But again, $M \cup \{pq\}$ is a larger matching than $M$, which is not the case. Therefore $\{x, y\}$ defends $w$ in this case also. It follows that $\gamma_{\text{spr}}^{(2)}(G) \leq 2\min\{\alpha(G), \beta(G)\}$, and thus $\gamma_{\text{spr}}^{(i)}(G) \leq 2\min\{\alpha(G), \beta(G)\}$ for $i = 2, \ldots, 9$. ■

**Proposition 4.4.** *Let $G = (V, E)$ be a graph with no isolated vertices. Then $\gamma_{\text{spr}}^{(i)}(G) \leq 2\gamma_{\text{pr}}(G)$ for $2 \leq i \leq 9$.*

*Proof.* It suffices to prove the inequality for $i = 2$.

Let $D = \{u_1, ..., u_k, v_1, ..., v_k\}$ be a $\gamma_{\text{pr}}$-set of $G$ with associated matching $M = \{u_1v_1, ..., u_kv_k\}$. We prove the desired inequality by constructing a 2-SPDS $D'$ with associated matching $M'$ such that $|D'| \leq 2|D|$.

**Construction D1.** Initially, let $D_0 = D$ and $M_0 = \varnothing$, and construct $D' = D_k$ and $M' = M_k$ recursively as follows. For $i = 1, ..., k$,

(a) if there exist $x \in N(u_i) - D_{i-1}$ and $y \in N(v_i) - D_{i-1}$ such that $x \neq y$, then choose arbitrary $x_i \in N(u_i) - D_{i-1}$ and $y_i \in N(v_i) - D_{i-1}$ such that $x_i \neq y_i$, and let $D_i = D_{i-1} \cup \{x_i, y_i\}$ and $M_i = M_{i-1} \cup \{u_ix_i, v_iy_i\}$;

(b) otherwise, let $D_i = D_{i-1}$ and $M_i = M_{i-1} \cup \{u_iv_i\}$.

If the condition in Construction D1(a) holds, we say that $\{u_i, v_i\}$ *is doubled to* $\{u_i, x_i\} \cup \{v_i, y_i\}$. Clearly, $D \subseteq D'$ and $|D'| \leq 2|D|$. It remains to show that $D'$ is a 2-SPDS of $G$.

Suppose to the contrary that there exists $w \in V(G)$ such that $w$ is not $D'$-defended. Since $D$ is a PDS, $w$ has a neighbour $u \in D$ whose $D$-partner is $v$. If $\{u, v\}$ is not doubled in the construction of $D'$, then either $N(u) - D' = N(v) - D' = \{w\}$, or $N[v] \subseteq D'$. But in either case $\{u, v\}$ defends $w$, which is not the case, hence assume that $\{u, v\}$ is doubled to $\{u, x\} \cup \{v, y\}$. Since $\{u, x\}$ does not defend $w$, there exists $s \in \text{epn}(x, D')$. However, by the construction of $D'$, $x \notin D$ and so $D$ does not dominate $s$, a contradiction. ■

**Corollary 4.5.** *If the graph $G$ has a $\gamma_{\text{pr}}$-set that contains a vertex of degree one, then $\gamma_{\text{spr}}(G) < 2\gamma_{\text{pr}}(G)$.*

*Proof.* Suppose $D$ is a $\gamma_{\text{pr}}$-set of $G$ such that $\deg u = 1$ for some $u \in D$. Let $v$ be the $D$-partner of $u$ and construct the SPDS $D'$ according to Construction D1. Then $\{u, v\}$ is not doubled and the result follows. ■

**Corollary 4.6.** *Let $G$ be a graph with no isolated vertices. Then $\gamma_{\text{spr}}^{(i)}(G) \leq 4\gamma(G)$ for $2 \leq i \leq 9$.*

*Proof.* The result follows from Proposition 4.4 and the obvious fact that $\gamma_{\text{pr}}(G) \leq 2\gamma(G)$ for all graphs without isolated vertices. ■

The bound in Corollary 4.6 is weak. For the special classes of graphs mentioned in Chapter 3, $\gamma_{\text{spr}}^{(2)} \leq 2\gamma$ in all cases. For the ladders $L_4$, $L_6$ and $L_7$ we in fact see that

$$\gamma_{\text{spr}}^{(2)}(L_4) = 4 < 2\gamma(L_4) = 6 \text{ and } \gamma_{\text{spr}}^{(2)}(L_6) = \gamma_{\text{spr}}^{(2)}(L_7) = 6 < 2\gamma(L_6) = 2\gamma(L_7) = 8,$$

hence strict inequality holds. We now improve this bound, and also the bounds in Theorem 4.2 and Proposition 4.3, by proving that $\gamma_{\text{spr}}^{(2)}(G) \leq 2\gamma(G)$ for all graphs $G$ without isolated vertices.

**Theorem 4.7.** *For any graph $G$ without isolated vertices, $\gamma(G) \leq \gamma_{\text{spr}}^{(2)}(G) \leq 2\gamma(G)$.*

**Proof.** The first inequality is trivial, so we only prove the second one. Among all $\gamma$-sets of $G$, let $D$ be one such that the number of edges in the subgraph $\langle D \rangle$ of $G$

induced by $D$ is as large as possible. Define

$$D_{\text{epn}} = \{v \in D : \text{epn}(v, D) \neq \varnothing\}.$$

We prove that $D_{\text{epn}} = D$. Suppose this is not the case. Then there exists $x \in D$ such that $\text{epn}(x, D) = \varnothing$. If $x$ is adjacent to $u \in D$, then $D - \{x\}$ dominates $G$, which is impossible. Thus $x$ is isolated in $\langle D \rangle$. Since $G$ has no isolated vertices, $x$ is therefore adjacent to $y \in V(G) - D$. Since $y \notin \text{epn}(x, D) = \varnothing$, $y$ is adjacent to $w \in D - \{x\}$. Let $D' = (D - \{x\}) \cup \{y\}$. Then $D'$ dominates $G$. Moreover, $y$ is adjacent to $w \in D'$, hence $\langle D' \rangle$ contains more edges than $\langle D \rangle$ does, contradicting the choice of $D$.

For each $u_i \in D$, let $v_i \in \text{epn}(u_i, D)$ and define $S = D \cup \{v_i : u_i \in D\}$ with associated matching $M = \{u_i v_i : u_i \in D\}$. Clearly, $S$ is a PDS of $G$ and $|S| = 2\gamma(G)$. Note that $\text{pn}(v_i, S) = \varnothing$ for each $v_i \in S$. We show that $S$ is also an SPDS of $G$.

Consider arbitrary $w \in V(G) - S$ and let $u_i$ be any vertex in $D$ adjacent to $w$. Let $S_w = (S - \{v_i\}) \cup \{w\}$ and $M_w = (M - \{u_i v_i\}) \cup \{u_i w\}$. Since $D \subseteq S_w$, $S_w$ is a PDS of $G$ and hence $w$ is $S$-protected. ■

The cycle $C_4$ is an example of a graph for which the first inequality in Theorem 4.7 is an equality. We now construct an infinite class $\mathcal{X}$ of graphs for which $\gamma_{\text{spr}} = \gamma$. For any integer $k \geq 1$, we first construct the graph $H_k$ with $k$ components. We begin with the set $D_k = \{u_1, w_1, ..., u_k, w_k\}$ and the matching $M = \{u_1 w_1, ..., u_k w_k\}$. Let $U_1, W_1, ..., U_k, W_k$ be disjoint sets of vertices such that $|U_i|, |W_i| \geq 2$ for each $i$. Join $u_i$ to each vertex in $U_i$, and $w_i$ to each vertex in $W_i$, $i = 1, ..., k$. Add all edges between $U_i$ and $W_i$, so that $H_k[U_i \cup W_i] \cong K_{|U_i|,|W_i|}$. This is the graph $H_k$. Now form the class $\mathcal{X}_k$ by adding any edges (including none, or all) between vertices in $D_k$. Finally, let $\mathcal{X} = \bigcup_{i=1}^{\infty} \mathcal{X}_k$. The graph in Figure 4.1 is the graph in $\mathcal{X}_2$ obtained by adding all possible edges between vertices in $D_2$, and where $|U_1| = |U_2| = |W_1| = |W_2| = 2$.

It is clear that $D_k$ dominates any graph $G \in \mathcal{X}_k$. Suppose $X$ is a dominating set of $G$ of cardinality $q \leq 2k - 1$. By the pigeonhole principle, there is at least one index $i$

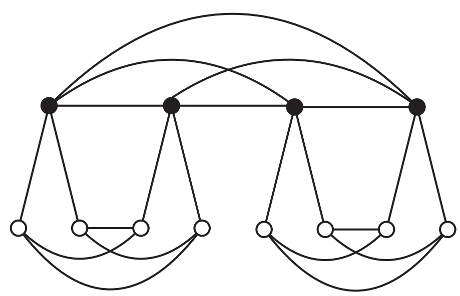


Figure 4.1: A graph $G$ with $\gamma(G) = \gamma_{\mathrm{spr}}(G) = 4$

such that $|X \cap (U_i \cup W_i \cup \{u_i, w_i\})| \leq 1$. Since no vertex in $V(G) - (U_i \cup W_i \cup \{u_i, w_i\})$ dominates a vertex in $U_i \cup W_i$, and no single vertex in $U_i \cup W_i \cup \{u_i, w_i\}$ dominates $U_i \cup W_i$, this is clearly impossible. Hence $\gamma(G) = 2k$. Clearly, $D_k$ is also a PDS of $G$. Moreover, since any vertex in $U_i$ dominates all vertices in $W_i$, and any vertex in $W_i$ dominates all vertices in $U_i$, each vertex in $U_i \cup W_i$ is $D_k$-protected and thus $D_k$ is an SPDS of $G$.

## 4.3 Extremal Trees

We now consider trees that achieve equality in Proposition 4.4. As before, for a tree $T$, we abbreviate $\gamma_{\text{spr}}^{(i)}(T)$ $(i = 2, \ldots, 9)$ to $\gamma_{\text{spr}}(T)$, and we call $T$ a $(\gamma_{\text{spr}}, 2\gamma_{\text{pr}})$-*tree* if $\gamma_{\text{spr}}(T) = 2\gamma_{\text{pr}}(T)$. A *double star* is a tree obtained by joining the central vertices of two nontrivial stars.

We begin with the following proposition.

**Proposition 4.8.** *Let $T$ be a tree with a $\gamma_{\text{pr}}$-set $D$ such that each vertex in $D$ has an epn that is a leaf. Then*

*(i) $D$ is the unique $\gamma_{\text{pr}}$-set of $T$,*

*(ii) $\gamma_{\text{spr}}(T) = 2\gamma_{\text{pr}}(T)$, and*

*(iii) $T$ has a spanning subforest consisting of double stars.*

*Proof.* By the hypothesis, $D \subseteq S(T)$. However, since $D$ is a PDS, each leaf of $T$ is dominated by a vertex that has a $D$-partner, and so $S(T) \subseteq D$. Hence $D = S(T)$.

$(i)$ Suppose to the contrary that $X \neq D$ is also a $\gamma_{\text{pr}}$-set of $T$ and let $v \in D - X$. Let $u \in \text{epn}(v, D) \cap L(T)$. Since $v \notin X$ and $X$ dominates $u$, $u \in X$. But then $u$ has no partner in $X$, a contradiction.

$(ii)$ For each $v \in D$, let $v' \in \text{epn}(v, D) \cap L(T)$ and define $D' = D \cup \{v' : v \in D\}$ and $M' = \{vv' : v \in D\}$. Then $|D'| = 2|D|$ and $D'$ is a PDS of $T$ with matching $M'$. To show that $D'$ is an SPDS, consider $w \in V(T) - D'$. Since $D$ is a PDS, $w \in N(v)$ for some $v \in D$, and $(D' - \{v'\}) \cup \{w\}$ is a PDS of $T$. Hence $\{v, v'\}$ protects $w$.

Suppose $X$ is an SPDS of $T$ with $|X| < |D'|$ and matching $M_X$. Since $X$ is a PDS, $D = S(T) \subseteq X$. Since $|X| \leq |D'| - 2$, there exist $u, v \in D$ such that $uv \in M_X$ and $\text{epn}(u, D) \cap X = \text{epn}(v, D) \cap X = \varnothing$. Consider $u' \in \text{epn}(u, D) \cap L(T)$, $v' \in \text{epn}(v, D) \cap L(T)$. Then $u' \in \text{epn}(u, X)$ and $v' \in \text{epn}(v, X)$. But then $\{u, v\}$ does not protect either $u'$ or $v'$, and so $X$ does not protect these vertices, a contradiction. Therefore $\gamma_{\text{spr}}(T) = |D'| = 2|D| = 2\gamma_{\text{pr}}(T)$.

*(iii)* Assume that $M$ is the matching associated with $D$ and delete all edges of $T[D]$ except those in $M$. Let $Y = \bigcup_{v \in D} \text{epn}(v, D)$ and $Z = V(T) - D - Y$. Note that each vertex in $Z$ is adjacent to at least two vertices in $D$. For each $z \in Z$, choose a vertex $d_z \in D \cap N(z)$ arbitrarily, and delete all edges incident with $z$ except the edge $zd_z$. Also delete all edges in $T[Y]$. Since each vertex in $D$ has a private neighbour in $L(T)$, the resulting graph is a forest consisting of double stars whose centres are the vertices in $D$. ■

The path $P_8 = v_1, ..., v_8$ is an example of a tree with a unique $\gamma_{\text{pr}}$-set $D = \{v_2, v_3, v_6, v_7\}$ that has the forest $2P_4$ as spanning subgraph, and $P_4$ is a double star. Also, each vertex in $D$ has an epn. But $\gamma_{\text{spr}}(P_8) = 6 < 2\gamma_{\text{pr}}(P_8)$; note that neither $v_3$ nor $v_6$ has a leaf private neighbour.

The condition stated in Proposition 4.8 is not necessary for a tree to satisfy equality in the bound of Proposition 4.4. Let $T_1$ be the tree obtained by joining a new leaf to the vertex $v_3$ of $P_8$ labelled as above. Then $v_6$ has no leaf private neighbour, but $\gamma_{\text{spr}}(T_1) = 8 = 2\gamma_{\text{pr}}(T_1)$. In fact, if $T_n$ is any tree constructed from $n$ copies of $T_1$ by joining vertices corresponding to the vertex $v_3$ of $T_1$ in any way (providing the graph is acyclic and connected), then $T_n$ has $n$ vertices without leaf private neighbours, but $\gamma_{\text{spr}}(T_n) = 8n = 2\gamma_{\text{pr}}(T_n)$.

We next show that if the tree $T$ has a PDS $D$ such that there are two vertices of $D$ "close" to each other without leaf private neighbours, then $\gamma_{\text{spr}}(T) < 2\gamma_{\text{pr}}(T)$.

**Theorem 4.9.** *If a tree $T$ has a $\gamma_{\mathrm{pr}}$-set $D$ such that there exist $u, v \in D$, with $uv \in E(G)$, and neither $u$ nor $v$ has a leaf private neighbour, then $\gamma_{\mathrm{spr}}(T) < 2\gamma_{\mathrm{pr}}(T)$.*

*Proof.* By Corollary 4.5 we may assume that $D$ contains no leaves of $T$, for otherwise we are done. Since $u$ and $v$ have no leaf private neighbours, and no leaf adjacent to $u$ or $v$ is in $D$, $u$ and $v$ are not support vertices.

Assume firstly that $u$ and $v$ are paired in $D$. Then for each $w \in N(\{u, v\}) - D$ there exists a vertex $z \in D - \{u, v\}$ such that $1 \leq d(w, z) \leq 2$; let $Z_w$ be the set of all vertices in $D - \{u, v\}$ satisfying this inequality.

Say $D = \{u_1, ..., u_k, v_1, ..., v_k\}$, where $u = u_k$, $v = v_k$, and construct the set $D'$ and its associated matching $M'$ as in Construction D1, with the following additional requirement.

**(R)** If $\{u_i, v_i\}$ is doubled to $\{u_i, x_i\} \cup \{v_i, y_i\}$ and $u_i \in Z_w$ for some $w \in N(\{u, v\}) - D$, whenever possible let $x_i$ be the vertex adjacent to $u_i$ on the $u_i - w$ path in $T$. Choose $y_i$ similarly if $v_i \in Z_{w'}$ for some $w' \in N(\{u, v\}) - D$.

If $|D'| < 2|D|$, we are done, so assume $|D'| = 2|D|$. Then for each $i$, $\{u_i, v_i\}$ is doubled to $\{u_i, x_i\} \cup \{v_i, y_i\}$; in particular, $\{u_k, v_k\}$ is doubled to $\{u_k, x_k\} \cup \{v_k, y_k\}$. Let $D'' = D' - \{x_k, y_k\}$ and $M'' = (M' - \{u_k x_k, v_k y_k\}) \cup \{u_k v_k\}$. We show that $D''$ is an SPDS of $T$.

Suppose to the contrary that there exists $w \in V(G)$ such that $w$ is not $D''$-defended. Since $D$ is a PDS, $w$ has a neighbour $a \in D$ whose $D$-partner is $b$. If $\{a, b\} \neq \{u, v\}$, then we obtain a contradiction as in the proof of Proposition 4.4, so assume $a = u$, $b = v$. Since $\{u, v\}$ does not $D''$-defend $w$, there exists $s \in \mathrm{epn}(v, D'')$. Recall that $Z_s \neq \varnothing$. Since $s \in \mathrm{epn}(v, D'')$, $d(s, z) = 2$ for each $z \in Z_s$. Let $i$ be the smallest index such that $\{u_i, v_i\} \cap Z_s \neq \varnothing$; without loss of generality say $u_i \in Z_s$. By assumption, $\{u_i, v_i\}$ is doubled to $\{u_i, x_i\} \cup \{v_i, y_i\}$, but $x_i$ is not adjacent to $s$ because $s \in \mathrm{epn}(v, D'')$. This violates requirement **(R)** and we conclude that $D''$ is an SPDS of $T$.

Assume therefore that $u$ and $v$ are not partners in $D$.

Let $D = \{u_1, ..., u_k, v_1, ..., v_k\}$, where $u = u_{k-1}$ and $v = v_k$, and $M = \{u_iv_i : i = 1, ..., k\}$. We define $Z_w$ for each $w \in N(\{u, v\}) - D$ , and construct $D'$, $M'$ as in the case where $u, v$ are paired. Again, assume that $\{u_i, v_i\}$ is doubled to $\{u_i, x_i\} \cup \{v_i, y_i\}$ for all $i = 1, \ldots, k$; in particular, $\{u_{k-1}, v_{k-1}\}, \{u_k, v_k\}$ are doubled to $\{u_{k-1}, x_{k-1}\} \cup \{v_{k-1}, y_{k-1}\}$ and $\{u_k, x_k\} \cup \{v_k, y_k\}$, respectively.

Note that if $v_{k-1}$ is not a support vertex, then we are done by setting pair $\{u, v\} = \{v_{k-1}, u_{k-1}\}$ and applying the same argument as in the previous case. Thus $v_{k-1}$ has a leaf neighbour $l_1$. Similarly, $u_k$ has a leaf neighbour $l_2$.

Let $D'' = (D' - \{y_{k-1}, x_{k-1}, y_k, x_k\}) \cup \{l_1, l_2\}$, and

$$M'' = (M' - \{v_{k-1}y_{k-1}, u_{k-1}x_{k-1}, v_ky_k, u_kx_k\}) \cup \{l_1v_{k-1}, uv, u_kl_2\}.$$

We claim that $D''$ is an SPDS of $T$. Suppose this is not the case, then there exists a vertex $w \in V(G)$ that is not $D''$-defended. It is clear that since $l_1$ is a leaf, each neighbour of $v_{k-1}$ is $D''$-defended by $\{l_1, v_{k-1}\}$. It follows that $w \notin N(v_{k-1})$. Similarly, $w \notin N(u_k)$. We conclude therefore that $w \in N(\{u, v\})$, where, as in the previous case, we obtain a contradiction.

In either case we constructed an SPDS $D'$ of $T$ with $|D'| < 2|D|$. The proof is thus complete. ■

**Proposition 4.10.** *The conclusion of Theorem* 4.9 *does not hold if* $d(u, v) > 1$.

*Proof.* Consider the tree $T$ by joining two $P_5$'s $x_1, \ldots, x_5$ and $y_1, \ldots, y_5$ to a $P_7 = v_1, \ldots, v_7$ so that $x_1$ is adjacent to $v_3$ and $y_1$ to $v_5$, and attaching leaves $x_6, y_6$ to $x_3$ and $y_3$, respectively.

Let $D = \{v_2, v_3, v_5, v_6, x_3, x_4, y_3, y_4\}$ and $M = \{v_2v_3, v_5v_6, x_3x_4, y_3y_4\}$. It is easy to see that $D$ is a minimum PDS of $T$ and thus $\gamma_{\text{pr}}(T) = 8$. Note that $u = v_3$ and $v = v_5$ are the only vertices in $D$ without leaf private neighbours, and $d(u, v) = 2$.

We show that $\gamma_{\text{spr}}(T) = 16$ by arguing that it is impossible to securely pairwise dominate $T$ with only seven pairs of guards. Suppose to the contrary that there exists an SPDS $D'$ of $T$ such that $|D'| = 14$. Let $M'$ be the associated matching of $D'$. Observe that $x_3, x_4, y_3, y_4 \in D'$; however, $x_3x_4, y_3y_4 \notin M'$, as otherwise $x_5, x_6, y_5, y_6$ are not defended. Therefore $x_4, x_5, y_4, y_5 \in D'$ and $x_4x_5, y_4y_5 \in M'$. To defend $x_6$, $\{x_2, x_6\} \cap D' \neq \varnothing$. Similarly, $\{y_2, y_6\} \cap D' \neq \varnothing$. Hence $|D' \cap \{x_2, x_3, x_4, x_5, x_6, y_2, y_3, y_4, y_5, y_6\}| \geq 8$. Also note that $v_2, v_6 \in D'$. Hence $|D' \cap \{v_1, v_3, v_4, v_5, v_7, x_1, y_1\}| \leq 4$.

Now if $v_3, v_5 \in D'$, then $\{v_2v_3, v_5v_6\} \nsubseteq M'$, for otherwise $\{v_1, v_4, v_7\} \cap D' = \varnothing$, and $v_4$ is not defended. Thus the remaining two vertices $u, w \in D'$ must be chosen in such a way that $\{v_2, v_3, v_5, v_6, u, w\}$ has a perfect matching that does not include both edges $v_2v_3$ and $v_5v_6$. Up to isomorphism, there are only two ways to do this: $\{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq D'$ and $\{v_1v_2, v_3v_4, v_5v_6\} \subseteq M'$, and $\{v_1, v_2, v_3, x_1, v_5, v_6\} \subseteq D'$ and $\{v_1v_2, v_3x_1, v_5v_6\} \subseteq M'$. In each case, $v_7 \in \mathrm{epn}(v_6, D')$, so $v_5$ does not protect $y_1$. Therefore $y_2 \in D'$. But then $y_6 \in \mathrm{epn}(y_3, D')$ and so $y_2$ does not defend $y_1$ either. Thus $\{v_3, v_5\} \nsubseteq D'$.

Obviously, if $\{v_3, v_5\} \cap D' = \varnothing$, then $v_4$ is not pairwise dominated, so assume without loss of generality that $v_3 \in D'$ and $v_5 \notin D'$. Then $v_6, v_7 \in D'$ and $v_6v_7 \in M'$. If $v_2v_3 \in M'$, then $\{v_1, v_4\} \cap D' = \varnothing$ and so neither $v_1$ nor $v_4$ is protected. Hence $v_2v_3 \notin M'$, so that $v_1 \in D'$ and $\mathrm{p}(v_3) \in \{v_4, x_1\}$. In either case all vertices in $D'$ have been determined, hence $y_1 \notin D'$. Thus $y_2 \in D'$ to protect $y_1$. This is impossible because $y_6 \in \mathrm{epn}(y_3, D')$.

Thus $T$ satisfies the conditions of Theorem 4.9 with the only exception that $d(u, v) = 2$, but $\gamma_{\mathrm{spr}}(T) = 2\gamma_{\mathrm{pr}}(T)$. ■

Next we consider the trees $T$ which $\gamma_{\mathrm{spr}}(T) = \gamma_{\mathrm{pr}}(T)$, starting with a property of general graphs.

**Proposition 4.11.** *Let $G$ be a graph with PDS $D$. Then for $i = 2, \ldots, 9$, $D$ is an $i$-SPDS of $G$ if for each $v \in V(G) - D$ there exists a vertex $u \in N(v) \cap D$ such that* $\mathrm{epn}(\mathrm{p}(u), D) = \varnothing$.

*Proof.* It is sufficient to prove the case for $i = 2$.

Suppose to the contrary that $D$ is not a 2-SPDS of $G$. Then there exists a vertex $v \in V(G)$ that is not $D$-defended. Let $u \in N(v) \cap D$ and $u' = p(u, D)$. If $u'$ has no $D$-epn, then $v$ is 2-defended by $u, u'$. So $\text{epn}(u', D) \neq \varnothing$. A contradiction is then obtained by applying the same argument to each $u \in N(v) \cap D$. ■

It is worth noting that the converse of Proposition 4.11 does not hold for graphs in general – for a counter-example, consider $G = C_4$, where $\gamma_{\text{spr}}(G) = \gamma_{\text{pr}}(G) = 2$ but the given condition is clearly not satisfied. The converse of the proposition, however, does hold for all trees.

**Proposition 4.12.** *Let $T$ be a tree with PDS $D$. Then $D$ is an SPDS of $T$ if and only if for each $v \in V(T) - D$ there exists a vertex $u \in N(v) \cap D$ such that* $\text{epn}(\text{p}(u), D) = \varnothing$.

*Proof.* If $T$ has a minimum PDS $D$ for which the given condition is satisfied, then $D$ is also an SPDS and thus $\gamma_{\text{spr}}(T) = \gamma_{\text{pr}}(T)$.

Suppose now that $\gamma_{\text{spr}}(T) = \gamma_{\text{pr}}(T)$. Let $D$ be an SPDS of $T$. Suppose to the contrary that there exists $v \in V(T)$ such that $\text{epn}(\text{p}(u)) \neq \varnothing$ for all $u \in N(v) \cap D$. Let $w \in N(v) \cap D$ such that $w$ $D$-defends $v$ when $v$ is attacked, and let $w' = \text{p}(u)$. It follows that $D' = (D - \{w'\}) \cup \{v\}$ is a PDS of $T$. By assumption, $w'$ has an $D$-epn $z$. However, since $T$ is a tree, $N(z) \cap N(v) = \varnothing$. Thus $z$ is not dominated by $D'$, leading to a contradiction. The proof is therefore complete. ■

We show that a result similar to Proposition 4.8 $(ii)$ holds for the domination number instead of the paired domination number (Note that Proposition 4.8 $(i)$ is not true in the case of dominating sets, while $(iii)$ is trivial).

**Proposition 4.13.** *If the tree $T$ has a $\gamma$-set $D$ such that each vertex in $D$ has an epn that is a leaf, then $\gamma_{\text{spr}}(T) = 2\gamma(T)$.*

*Proof.* The result is trivial if $T = K_2$, so assume $T$ has order at least three. Then $D$ contains no leaf of $T$ and so $S(T) \subseteq D$. On the other hand, since each vertex in $D$

has a leaf neighbour, $D = S(T)$. Let $D = \{v_1, \ldots, v_k\}$, and let $u_i$ be an arbitrary leaf neighbour of $v_i$. Consider $D' = \{v_1, \ldots, v_k, u_1, \ldots, u_k\}$ and $M' = \{v_1u_1, \ldots, v_ku_k\}$.

It is easy to see that $D'$ is a PDS of $T$. We show that $D'$ is secure. Let $x$ be any vertex in $T - D'$ and let $v \in D \cap N(x)$, $u = \mathrm{p}(v)$. When $x$ is attacked, $D'' = (D' - \{u\}) \cup \{x\}$ forms a new PDS of $T$, and this implies $D'$ is an SPDS of $T$.

We now show that $2|S(T)|$ is the minimum number of guards one can use to securely pairwise dominate $T$. Clearly, $S(T)$ is a subset of any SPDS of $T$, so it suffices to show that each vertex in $S(T)$ corresponds to a unique pair of guards in any SPDS $D'$ of $T$. Suppose to the contrary that there exist $u, v \in D'$ such that $uv \in M'$, the associated matching of $D'$, and that $u, v \in S(T)$. Let $u', v'$ be a leaf partner of $u, v$, respectively. Then it is easy to see that If there is an attack against any vertex in $\{u', v'\}$, the other one will be left undominated, which is not the case. Therefore the proof is complete. ■

# Chapter 5

# Open Problems

## 5.1 Introduction

We conclude the thesis with a list of some open problems and ideas for further research.

## 5.2 List of Problems

**Problem 5.1.** *Characterize graphs (or trees, in particular) whose SPD numbers equal twice their clique partition numbers.*

**Problem 5.2.** *Characterize graphs (or trees, in particular) whose SPD numbers equal twice their PD numbers.*

**Problem 5.3.** *Characterize graphs (or trees, in particular) with equal SPD and PD numbers.*

**Problem 5.4.** *Characterize graphs (or trees, in particular) whose SPD numbers equal twice their secure domination numbers.*

**Problem 5.5.** *Find a (good) lower bound for SPD numbers of grid graphs.*

It may also be interesting to study criticality and stability aspects of secure paired domination: how do the SPD numbers of a graph change when a vertex or edge is deleted, or when an edge is added? In particular, critical graphs - that is, graphs whose SPD numbers change whenever the graph is altered by one of these operations - may exhibit interesting properties.

A future direction is to study defending the vertices of a graph against an infinite sequence of attacks, using paired domination and guards moving as defined in the nine definitions, according to two further models: Only one or two guards move to defend against each attack, or any number of guards may move to defend against each attack.

# Bibliography

[1] J. Arquilla and H. Fredricksen, "Graphing" an Optimal Grand Strategy. *Military Operations Research* **1**(3) (1995), 3–17.

[2] B. Bresar, M. A. Henning and D. F. Rall, Paired-domination of Cartesian Products of Graphs. *Utilitas Math.* **73** (2007), 255–265.

[3] S. Benecke, E. J. Cockayne and C. M. Mynhardt, Secure Total Domination in Graphs. *Utilitas Math.* **74** (2007), 247–259.

[4] X. G. Chen, W. C. Shiu and W. H. Chan, Upper Bounds on the Paired-domination Number. *Appl. Math. Lett.* **21** (2008), no. 11, 1194–1198.

[5] X. G. Chen, L. Sun and H. M. Xing, Paired-domination Numbers of Cubic Graphs. *Acta Math. Sci. Ser. A Chin. Ed.* **27** (2007), no. 1, 166–170.

[6] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, Roman Domination in Graphs. *Discrete Math.* **278** (2004), 11–22.

[7] E. J. Cockayne, O. Favaron and C. M. Mynhardt, Secure Domination, Weak Roman Domination and Forbidden Subgraphs. *Bulletin of the ICA* **39** (2003), 87–100.

[8] E. J. Cockayne, P. J. P. Grobler, W. R. Grundlingh, J. Munganga and J. H. van Vuuren, Protection of a Graph. *Utilitas Math.* **67** (2005), 19–32.

[9] P. Dorbec, S. Gravier and M. A. Henning, Paired-domination in Generalized Claw-free Graphs. *J. Comb. Optim.* **14** (2007), no. 1, 1–7.

[10] M. Edwards, R. G. Gibson, M. A. Henning and C. M. Mynhardt, Diameter of Paired Domination Edge-cirtical Graphs. *Australas. J. Combin.* **40** (2008), 279–291.

[11] O. Favaron and M. A. Henning, Paired-domination in Claw-free Cubic Graphs. *Graphs Combin.* **20** (2004), no. 4, 447–456.

[12] O. Favaron, H. Karami and S. M. Sheikholeslami, Paired-domination Number of a Graph and Its Complement. *Discrete Math.* **308** (2008), no. 24, 6601–6605.

[13] S. Fitzpatrick and B. Hartnell, Paired-domination. *Discuss. Math. Graph Theory.* **18** (1998), no. 1, 63–72.

[14] X. Fu, Y. Yang and B. Jiang, Roman Domination in Regular Graphs. *Discrete Math.* **309** (2009), no. 6, 1528–1537.

[15] T. W. Haynes and M. A. Henning, Trees with Large Paired-domination Number. *Utilitas Math.* **71** (2006), 3–12.

[16] T. W. Haynes, M. A. Henning and P. J. Slater, Trees with Equal Domination and Paired-domination Numbers. *Ars Combin.* **76** (2005), 169–175.

[17] T. W. Haynes and P. J. Slater, Paired-Domination in Graphs. *Networks* **32** (1998), no. 3, 199–206.

[18] T. W. Haynes and P. J. Slater, Paired-domination and the Paired-domatic Number. *Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)* **109** (1995), 65–72.

[19] M. A. Henning, A Characterization of Roman Trees. *Discuss. Math. Graph Theory* **22** (2002), no. 2, 325–334.

[20] M. A. Henning, Graphs with Large Paired-domination Number. *J. Comb. Optim.* **13** (2007), no. 1, 61–78.

[21] M. A. Henning, Trees with Equal Total Domination and Paired-domination Numbers. *Utilitas Math.* **69** (2006), 207–218.

[22] M. A. Henning and S. T. Hedetniemi, Defending the Roman Empire—a New Strategy. *The 18th British Combinatorial Conference (Brighton, 2001). Discrete Math.* **266** (2003), no. 1-3, 239–251.

[23] M. A. Henning and C. M. Mynhardt, The Diameter of Paired-domination Vertex Critical Graphs. *Czechoslovak Math. J.* **58** (133) (2008), no. 4, 887–897.

[24] M. A. Henning and P. D. Vestergaard, Trees with Paired-domination Number Twice Their Domination Number. *Utilitas Math.* **74** (2007), 187–197.

[25] X. Hou, A Characterization of $(2\gamma, \gamma_{\mathrm{p}})$-trees. *Discrete Math.* **308** (2008), no. 15, 3420–3426.

[26] X. Hou and M. Edwards, Paired Domination Vertex Critical Graphs. *Graphs Combin.* **24** (2008), no. 5, 453–459.

[27] W. F. Klostermeyer and C. M. Mynhardt, Secure Domination and Secure Total Domination in Graphs. *Discuss. Math. Graph Theory* **28** (2008), no. 2, 267–284.

[28] C. M. Mynhardt, H. C. Swart and E. Ungerer, Excellent Trees and Secure Domination. *Utilitas Math.* **67** (2005), 255–267.

[29] K. E. Proffitt, T. W. Haynes and P. J. Slater, Paired-domination in Grid Graphs. *Proceedings of the Thirty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, LA, 2001)* **150** (2001), 161–172.

[30] H. Qiao, L. Kang, M. Cardei and D. Du, Paired-domination of Trees, *J. Global Optim.* **25** (2003), no. 1, 43–54.

[31] J. Raczek, Lower Bound on the Paired Domination Number of a Tree. *Australas. J. Combin.* **34** (2006), 343–347.

[32] C. S. ReVelle, Can you protect the Roman Empire? *Johns Hopkins Magazine* **50**(2), April 1997.

[33] C. S. ReVelle and K. E. Rosing, Defendens Imperium Romanum: A Classical Problem in Military Strategy. *The American Mathematical Monthly* **107**, No. 7 (Aug. -Sep., 2000), 585–594.

[34] E. Shan, L. Kang and M. A. Henning, A Characterization of Trees with Equal Total Domination and Paired-domination Numbers. *Australas. J. Combin.* **30** (2004), 31–39.

[35] I. Stewart, Defend the Roman Empire! *Scientific American* December 1999, 136–138.

[36] H. Xing, X. Chen and X. G. Chen, A Note on Roman Domination in Graphs. *Discrete Math.* **306** (2006), no. 24, 3338–3340.