Developments in The Cubing Problem

Noah Janssen & Dr. Ulrike Stege

University of Victoria, Department of Computer Science

Motivation

- Rectangular partition of polyhedrons has been shown to be an NP-complete problem when the polyhedron is 3-dimensions [1].

- The problem of partitioning polyhedrons shows up in various fields and has various applications in those fields [3].

- An example of an application would be to reduce large objects down to a small number of simple pieces to facilitate easier 3D printing of large objects.

- We focus on a special case of the polyhedron partitioning problem, specifically that of orthogonal polyhedrons, or as we know them, cornerhedrons and are trying to develop a fast algorithm to solve the polyhedron partitioning problem in this case.

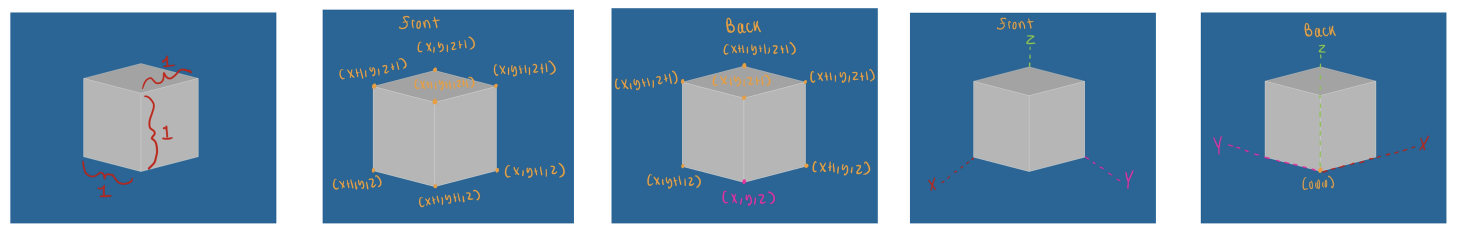
Terminology

- **Unit Cube (UC):** A 3D cube where each edge is of length 1. See Figure 1 (a).

- The coordinates of a UC are assigned to the back corner of the UC that faces away from the viewer see Figure 1 (b)-(c).

- **Peak:** The outward-facing corner of a UC that faces the viewer. See Figure 3 (a).

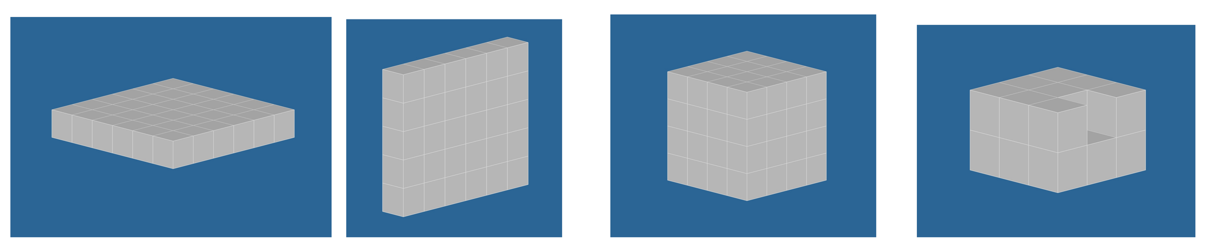
- **Origin Cube (OC):** The UC whose back corner that faces away from the viewer has the coordinates that act as the origin for the rest of the shape it is contained in. See 1 (d)-(e).



(a) (b) (c) (d) (e)

Figure 1. (a): A UC with 3 of its edges labelled with their length. (b): A UC with the coordinates of each of its corners labelled from the front. (c): A UC with the coordinates of each of its corners labelled from the back. (d)-(e): An OC from the front and the back with its back corner facing away from the viewer labelled as the origin.

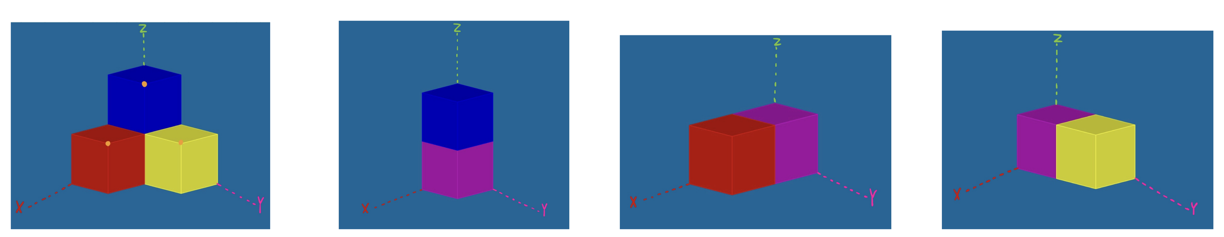
- **Box:** A 2D or 3D rectangle or square made of UCs that contains no gaps. See Figure 2.



(a) (b) (c) (d)

Figure 2. (a): An example of a 2D box. (b) An example of another variety of a 2D box. (c) An example of a 3D box. (d) An example of a shape that is not a box since it contains a gap.

Cornerhedron: A shape *C* composed of UCs with an OC as its back corner facing away from the viewer, such that every one of *C*'s UCs can be contained, with the OC, in a box made of the UCs in C without changing their positions. See Figure 3.

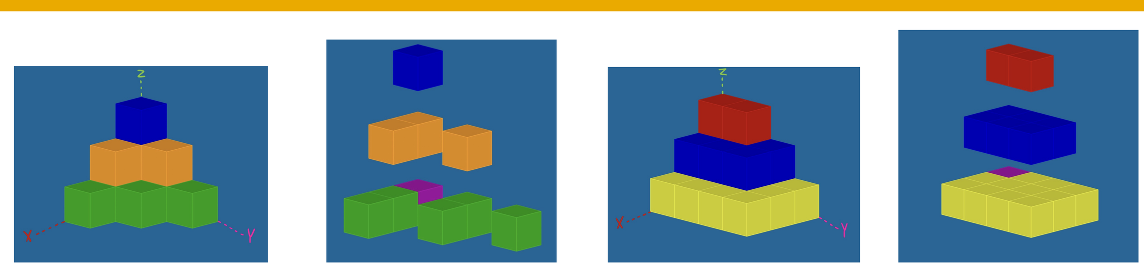


(a) (b) (c) (d)

Figure 3. (a): A cornerhedron with the axes drawn in and its peaks dotted in orange. (b-d) UCs of the cornerhedron contained in a box with the OC (purple) made of UCs of the cornerhedron without changing their positions.

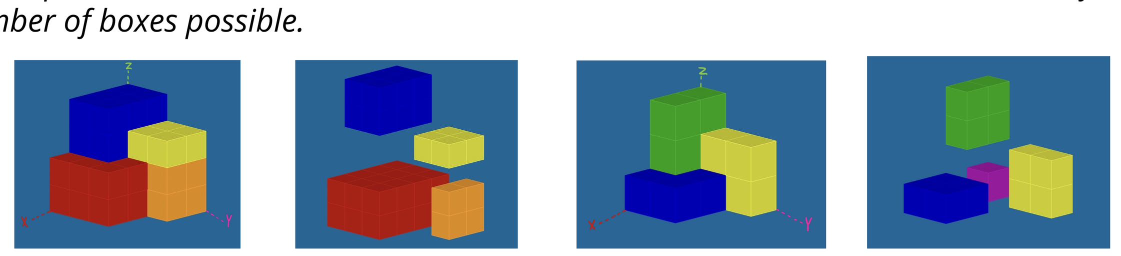
The Cubing Problem

Partition any given cornerhedron into the minimum number of boxes possible. See Figures 4 and 5.



(a) (b) (c) (d)

Figure 4. (a): A cornerhedron with the axes drawn in. (b): The cornerhedron from (a) split into the minimum number of boxes possible. (c): A cornerhedron with the axes drawn in. (d): The cornerhedron from (c) split into the minimum number of boxes possible.



(a) (b) (c) (d)

Figure 5. (a): A cornerhedron with axes. (b): The cornerhedron from (a) split into a minimum number of boxes. (c): A cornerhedron with axes. (d): The cornerhedron from (c) split into a minimum number of boxes.

Conjectures Proven

Lower Bound on the Number of Boxes a Cornerhedron Can be Partitioned Into

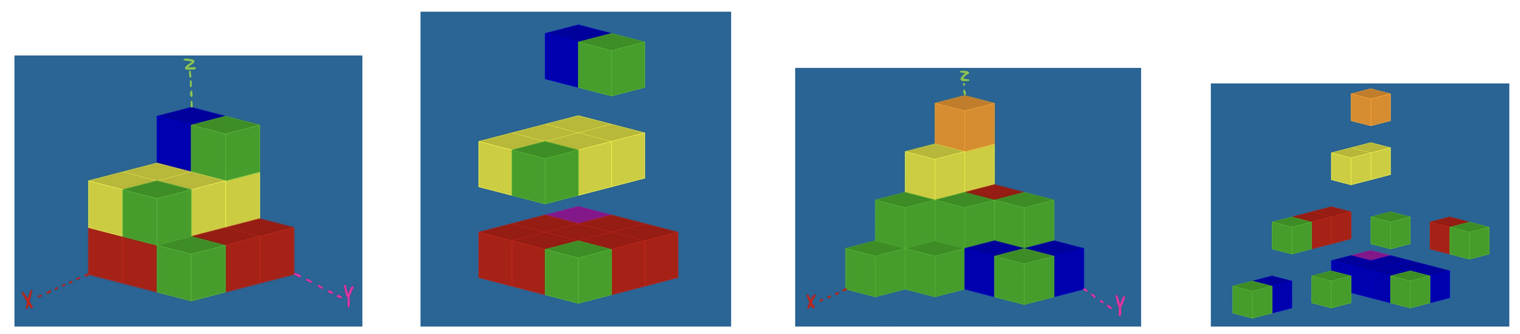
The number of boxes a cornerhedron can be partitioned into is greater than or equal to the number of peaks in the cornerhedron. Proposed in [1] but I proved it.

Proof Idea:

A box contains 1 peak. Since this box would be made of only UCs of the cornerhedron without changing their positions, then a peak in a cornerhedron must remain a peak in the box it ends up in when the cornerhedron is partitioned into boxes.

Thus if more than 1 peak ended up in the same box then a box would have 2 peaks, which is not possible.

Therefore the number of boxes in a partition of a cornerhedron must be at least equal to the number of peaks in the cornerhedron. See Figure 6.

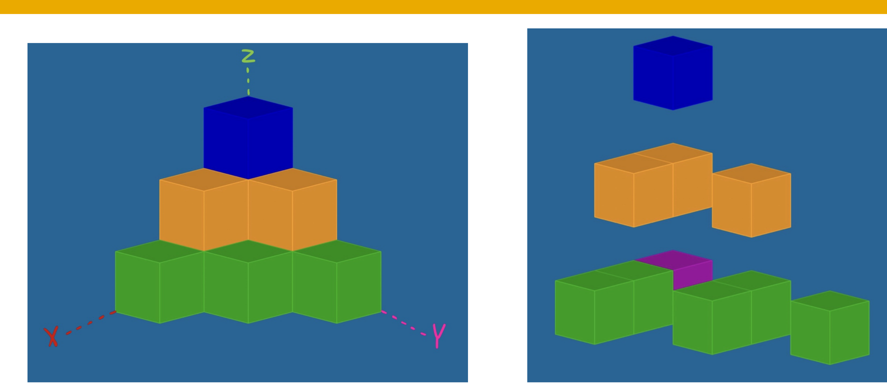


(a) (b) (c) (d)

Figure 6. The peaks of all cornerhedra are green. (a): A cornerhedron with axes. (b): The cornerhedron from (a) split into a minimum number of boxes. (c): A cornerhedron with axes. (d): The cornerhedron from (c) split into a minimum number of boxes.

Not Every Cornerhedron Can be Partitioned Into Boxes Without at Least 1 Cut Splitting 2 Layers

There exists at least 1 cornerhedron that cannot be partitioned into boxes without at least 1 cut splitting 2 vertical or horizontal layers of the cornerhedron. See Figure 7.



(a) (b)

Figure 7. (a): A cornerhedron that cannot be partitioned into boxes without at least 1 cut splitting 2 layers. (b): The cornerhedron from (a) partitioned into boxes.

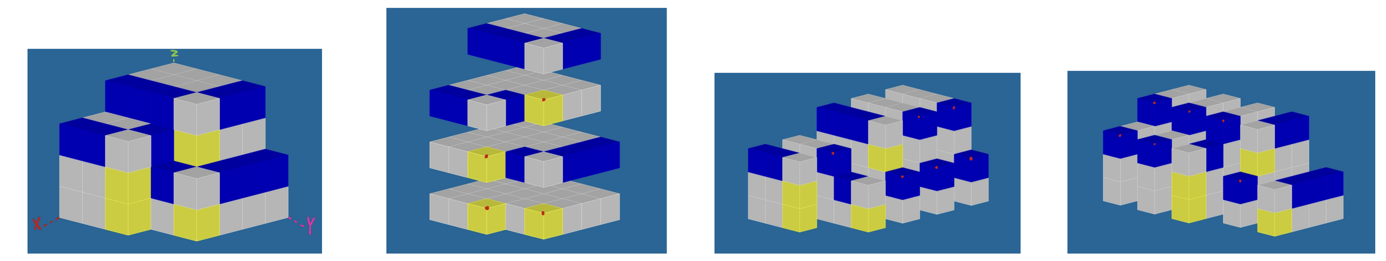
There Exist Cornerhedra That Cannot be Partioned Into a Number of Boxes Equal to Their Number of Peaks

The minimum number of boxes some cornerhedra can be partitioned into is greater than the number of peaks in that cornerhedron. See Figure 5.

Conjectures and Open Questions

Janssen's Hidden Peak Conjecture

A cornerhedron that gains at least 1 peak, if you split any 2 of its vertical or horizontal layers from each other must have a minimum number of boxes it can be partitioned into, which is greater than the number of peaks of that cornerhedron. See Figure 8.

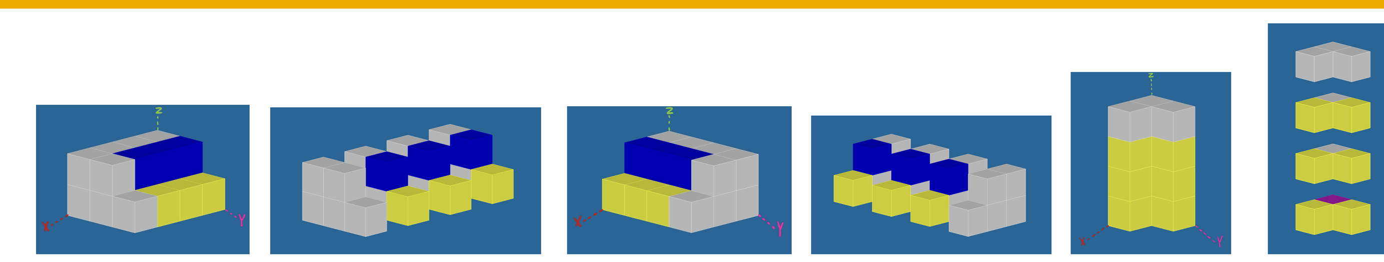


(a) (b) (c) (d)

Figure 8. (a): A cornerhedron that gains at least 1 peak if any 2 layers contained in it are split. All UCs that would gain peaks if vertical layers are split are blue and all UCs that would gain peaks if horizontal layers are split are yellow. (b): The cornerhedron from (a) with all horizontal layers split. All UCs that gain peaks are yellow and dotted in red. (c): The cornerhedron from (a) with all pairs of vertical layers going from the front left to the back right split. All UCs that gain peaks are blue and dotted in red. (d): The cornerhedron from (a) with all pairs of vertical layers going from the front right to the back left split. All UCs that gain peaks are blue and dotted in red.

No 2 Hidden Peaks Conjecture

There cannot exist a cornerhedron such that the splitting of any 2 layers, vertical or horizontal, results in the creation of at least 2 new peaks. See Figure 9.



(a) (b) (c) (d) (e) (f)

Figure 9. (a): A cornerhedron where at least 2 peaks are created when vertical layers are split from the front left to the back right. (b): The cornerhedron from (a) with all vertical layers from the front left to the back right split. (c): A cornerhedron where at least 2 peaks are created when vertical layers are split from the front right to the back left. (d): The cornerhedron from (c) with all vertical layers from the front right to the back left split. (e): A cornerhedron that gains at least 2 peaks when any of its horizontal layers are split. (f): The cornerhedron from (e) with all of its horizontal layers split.

Splitting Cut Required?

Is a splitting cut required to partition a cornerhedron into a number of boxes equal to the number of peaks of that cornerhedron? Right now it looks to be so, but this has not been proven.

Acknowledgements

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References

[1] V. J. Dielissen and A. Kaldewaij, "Rectangular partition is polynomial in two dimensions but NP-complete in three," Information processing letters, vol. 38, no. 1, pp. 1–6, 1991, doi: 10.1016/0020-0190(91)90207-X.

[2] Therese Biedl, Martin Derka, Veronika Irvine, Anna Lubiw, Debajyoti Mondal, Alexi Turcotte, "Partitioning Orthogonal Histograms into Rectangular Boxes," in *Proceedings 13th Latin Americal Symposium (Latin 2018)*., 2018, pp. 146-160.

[3] Vladislav A. Chekanin and Alexander V. Chekanin, "Solving the Problem of Decomposition of an Orthogonal Polyhedron of Arbitrary Dimension," in *Proceedings 9th Modern Engineering: Science and Education*., June 2020, pp. 52-59.