

Spectral Flow in Semifinite von Neumann Algebras

by

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BMath., University of Waterloo, 2003

MMath., University of Waterloo, 2006

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of
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Abstract

Spectral flow, in its simplest incarnation, counts the net number of eigenvalues which change sign as one traverses a path of self-adjoint Fredholm operators in the set of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space. A generalization of this idea changes the setting to a semifinite von Neumann algebra \mathcal{N} and uses the trace τ to measure the amount of spectrum which changes from negative to positive along a path; the operators are still self-adjoint, but the Fredholm requirement is replaced by its von Neumann algebras counterpart, Breuer-Fredholm.

Our work is ensconced in this semifinite von Neumann algebra setting. We prove a uniqueness result in the case when \mathcal{N} is a factor. In the case when the operators under consideration are bounded perturbations of a fixed unbounded operator with τ -compact resolvents, we give a different proof of a p -summable integral formula which calculates spectral flow, and fill in some of the gaps in the proof that spectral flow can be viewed as an intersection number if $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

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To my parents, Cristian and Nadia Georgescu, with love

Chapter 1

Introduction

1.1 Background: Definitions and Theorems

The context for our work is a von Neumann algebra \mathcal{N} equipped with a normal, faithful, semifinite trace τ . The reader to whom this does not immediately evoke a mental picture is directed to [32] for a comprehensive introduction. As shown in [32] (Chapter I.6.7, Proposition 9) the existence of such a trace is equivalent to \mathcal{N} being a semifinite algebra. One possible definition of a semifinite von Neumann algebra is that, if the algebra is decomposed into a direct sum of type I, type II and type III algebras (see [73], Chapter V.1, Theorem 1.19 and Definition 1.21), the type III component is zero.

Definition 1.1.1 ([32], Chapter I.6.1, Definition 1) Suppose τ is a trace on \mathcal{N} . Say that τ is **faithful** if the conditions $S \in \mathcal{N}^+$ and $\tau(S) = 0$ imply $S = 0$. The trace τ is called **semifinite** if, for $\mathcal{C} = \{T \in \mathcal{N}^+ : T \leq S \text{ and } \tau(T) < \infty\}$, we have $\tau(S) = \sup_{T \in \mathcal{C}} \tau(T)$. Finally, τ is called **normal** if for each bounded, monotone increasing net $\{T_i\}_{i \in I} \subset \mathcal{N}^+$ which converges to S , the net $\{\tau(T_i)\}_{i \in I}$ converges to $\tau(S)$. 

In particular, the semifinite property means that any projection P in \mathcal{N} can be written as a sum of pairwise disjoint finite projections in \mathcal{N} ([32], Chapter III.2.4, Corollary 1). If τ is normal, then τ is semifinite if and only if every non-zero element of \mathcal{N}^+ majorizes a non-zero element of \mathcal{N}^+ which has finite trace. In a type III von Neumann algebra there are no nonzero finite projections, which makes the edifice on which this thesis is built collapse (see for example the definition of τ -compact operators in 1.1.1, later used to describe Breuer-Fredholm operators, which in turn are used in the definition of spectral flow). This is the reason we require our algebra to be semifinite.

Usually, our algebra will be denoted \mathcal{N} , though \mathcal{M} will be used whenever the algebra is known to be finite. We only consider algebras that can be realized as von Neumann algebras of operators on a separable Hilbert space \mathcal{H} . To certain continuous paths of operators, we wish to assign a number (called the spectral flow) which gives us information about how the spectrum of the operators is changing as we move along the path.

Spectral flow for paths of self-adjoint Breuer-Fredholm operators is defined in Section 1.1.3; the intervening material sets up the required background for discussing spectral flow in this context (namely, definitions and results for τ -compact and Breuer-Fredholm operators). The results for von Neumann algebras are usually generalizations of results for $\mathcal{N} = \mathcal{B}(\mathcal{H})$ (where $\mathcal{B}(\mathcal{H})$ is the space of bounded operators on \mathcal{H}). As such, whenever the various concepts are introduced, we briefly refer to their parallels from $\mathcal{B}(\mathcal{H})$.

1.1.1 Ideals of operators

The role of this section is two-fold. The first is to briefly introduce τ -compact operators, which play a role in the description of Breuer-Fredholm operators. The second is to describe the trace

class and p -summable operators. These will play a greater role in Chapter 3, when additional restrictions on our operators will allow us to state an integral formula for spectral flow. In a semifinite von Neumann algebra, the τ -compact operators play a similar role to that of the compact operators in $\mathcal{B}(\mathcal{H})$.

Definition 1.1.2 Denote by $\mathcal{K}_{\mathcal{N}}$ the two-sided norm-closed ideal generated by projections of finite trace. The operators in $\mathcal{K}_{\mathcal{N}}$ are called **τ -compact**. \blacktriangledown

Recall that the singular numbers of a compact operator K are the eigenvalues of $|K|$, with multiplicity. Singular values are intimately connected with the trace on $\mathcal{B}(\mathcal{H})$, and can be used to prove various important trace inequalities. For a very readable introduction to singular values, and their connections to problems in the realm of physics, see [65]. The notion of singular numbers is generalized to (τ -measurable) operators affiliated with a semifinite von Neumann algebra by Fack and Kosaki, in [35]. We highlight below the definitions and results from their paper which are relevant to our presentation. Note that if T is an unbounded operator, we say that it is **affiliated with** our von Neumann algebra \mathcal{N} if for any unitary $V \in \mathcal{N}'$ we have $VTV^{-1} = T$ ([64]). If T is densely defined and closed, and $T = U|T|$ is the polar decomposition of T , then it follows that $U \in \mathcal{N}$ and $|T|$ is likewise affiliated with \mathcal{N} . This in turn means that all the spectral projections of $|T|$ are in \mathcal{N} ([32], Chapter I.1, exercise 10).

Remark 1.1.3 If T is a densely-defined, closed operator affiliated with \mathcal{N} , we say that T is **τ -measurable** if there exists a projection E in \mathcal{N} with $\tau(1 - E) < \infty$ and $\text{ran } E \subset \text{Dom}(T)$. Clearly, all bounded operators are τ -measurable (since their domain is all of \mathcal{H} , so we can choose $E = 1$ in the definition). We will mainly use the results for bounded operators, but we wish to point out some subtleties that creep in due to the possibility of considering τ -measurable operators.

If $|T| = \int_0^\infty \lambda dE_\lambda$ is the spectral resolution of $|T|$, then T is τ -measurable if and only if $\lim_{\lambda \rightarrow \infty} \tau(1 - E_\lambda) = 0$. If $\mathcal{N} = \mathcal{B}(\mathcal{H})$, the set of τ -measurable operators is equal to $\mathcal{B}(\mathcal{H})$; at the other end of the spectrum, if \mathcal{M} is an algebra whose trace is finite, the set of τ -measurable operators consists of all closed, densely-defined operators affiliated with \mathcal{M} . \bullet

Definition 1.1.4 For $A \in \mathcal{N}$ and $t > 0$, the t^{th} **singular s-number** (or, briefly, s-number) is

$$\mu_t(A) = \inf\{\|AP\| : P \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - P) \leq t\}.$$

The same definition can be used in the more general setting when A is instead a τ -measurable operator (see Definition 2.1 of [35]). \blacktriangledown

It should be clear that $t \mapsto \mu_t(A)$ is a non-negative, decreasing function. An equivalent formula for μ_t (see Proposition 2.2 of [35]) is given by $\mu_t(T) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(|T|)) \leq t\}$, where $\chi_{(s,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (s, ∞) . The generalized s -numbers can be used to get a handle on certain classes of operators; for example, T is τ -measurable if and only if $\mu_t(T) < \infty$ for all $t > 0$. We select some of the properties of s -numbers to showcase below, though the reader is directed to [35] for more.

Theorem 1.1.5 ([35], Lemma 2.5) Let T, S be τ -measurable operators.

- (i) The map $t \mapsto \mu_t(T)$ is continuous from the right, and $\|T\| = \lim_{t \rightarrow 0^+} \mu_t(T)$.

- (ii) $\mu_t(T) = \mu_t(|T|) = \mu_t(T^*)$ and, for all $\alpha \in \mathbb{C}$, $\mu_t(\alpha T) = |\alpha| \cdot \mu_t(T)$.
- (iii) If f is a continuous, increasing function on $[0, \infty)$ with $f(0) \geq 0$, then for all $t \geq 0$ we have $f(\mu_t(|T|)) = \mu_t(f(|T|))$.
- (iv) If $0 \leq T \leq S$ then $\mu_t(T) \leq \mu_t(S)$ for all $t > 0$.
- (v) $\mu_{t+s}(T + S) \leq \mu_t(T) + \mu_s(S)$ for all $t, s > 0$.
- (vi) $\mu_t(STR) \leq \|S\| \cdot \mu_t(T) \cdot \|R\|$ for S, R bounded operators in \mathcal{N} . ▀

It is also well-known that $S \in \mathcal{K}_{\mathcal{N}}$ if and only if S is bounded and $\mu_t(S) \rightarrow 0$ as $t \rightarrow \infty$. In fact, many authors use this latter property as the definition for $\mathcal{K}_{\mathcal{N}}$ (though again, the reader should be warned that occasionally the condition that S is bounded is dropped, resulting in a possibly larger set). Note first that $\mathcal{K}_{\mathcal{N}}$ is the norm closure of the trace-class operators, and use this description of $\mathcal{K}_{\mathcal{N}}$ and the properties of μ_t proven by Fack and Kosaki in [35] to conclude that $\mathcal{K}_{\mathcal{N}}$ consists of those operators whose s-numbers tend to zero:

- Suppose first that S is trace class; then $\tau(|S|) < \infty$, which implies $\mu_t(S) \rightarrow 0$ as $t \rightarrow \infty$. This can be seen, for example, from the fact that $\tau(|S|) = \int_0^\infty \mu_t(|S|) dt$, so in order for the integral to converge we must have $\mu_t(|S|) \rightarrow 0$ as $t \rightarrow \infty$; moreover, $\mu_t(S) = \mu_t(|S|)$ so $\mu_t(S) \rightarrow 0$. Now, suppose that $S_n \rightarrow S$ in norm, where $\tau(|S_n|) < \infty$. Given $\varepsilon > 0$, we can find a fixed N such that $\|S_n - S\| < \frac{\varepsilon}{2}$ for $n \geq N$. Since $\mu_t(S_N) \rightarrow 0$, there exists a t_0 such that $\mu_t(S_N) < \frac{\varepsilon}{2}$ for $t \geq t_0$. For $t > 2t_0$ we then have, using the properties of μ_t recapped in Theorem 1.1.5,

$$\begin{aligned} \mu_t(S) = \mu_t((S - S_N) + S_N) &\leq \mu_{t/2}(S - S_N) + \mu_{t/2}(S_N) \\ &\leq \|S - S_N\| + \mu_{t/2}(S_N) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $\mu_t(S) \rightarrow 0$ as $t \rightarrow \infty$.

- Conversely, suppose $\mu_t(S) \rightarrow 0$ as $t \rightarrow \infty$. Then for each $n \in \mathbb{N}^*$ we can find t_0 such that $\mu_{t_0}(S) < \frac{1}{2n}$. From the definition of μ_{t_0} as an infimum it follows that we can then find a projection E_n such that $\|SE_n\| < \frac{1}{n}$ and $\tau(1 - E_n) \leq t_0 < \frac{1}{n}$. Let $S_n = S(1 - E_n)$. Then $\tau(|S_n|) \leq \|S\| \cdot \tau(1 - E_n) < \infty$, so S_n is trace class. Moreover, $\|S_n - S\| = \|SE_n\| \rightarrow 0$ as $n \rightarrow \infty$, so $S \in \mathcal{K}_{\mathcal{N}}$ as a norm-limit of trace class operators.

A different proof of this description of τ -compact operators can be found in Proposition 3.4 of [69]. A consequence of this property of τ -compact operators is that, if P is a τ -compact projection then $\tau(P) < \infty$ (since $\mu_t(P) \in \{0, 1\}$, $\mu_t(P) \rightarrow 0$ as $t \rightarrow \infty$, and, as we will see later, $\tau(P) = \int \mu_t(P) dt$).

We denote by \mathcal{L}^p the ideal $\{T \in \mathcal{N} : \tau(|T|^p)^{\frac{1}{p}} < \infty\}$. Similar to the case of $\mathcal{B}(\mathcal{H})$, there is a relation between \mathcal{L}^p and singular values.

Theorem 1.1.6 ([35], Corollary 2.8) If f is a continuous, increasing function on $[0, \infty)$ with $f(0) = 0$ then

$$\tau(f(|T|)) = \int_0^\infty f(\mu_t(T)) dt.$$

In particular, $\tau(|T|^p)^{\frac{1}{p}} = (\int_0^\infty \mu_t(T)^p dt)^{\frac{1}{p}}$ for $0 < p < \infty$. ☞

Remark 1.1.7 In [35], L^p is defined as the set of all closed, densely-defined operators T affiliated with \mathcal{N} for which $\tau(|T|^p)^{\frac{1}{p}} < \infty$, with $\|T\|_p = \tau(|T|^p)^{\frac{1}{p}}$. Using this definition, L^p is a Banach space (this is a well-known result, but not obvious from this definition; see for example Corollary 11.3 and Theorem 13 of [64] for the fact that $\|\cdot\|_1$ is a norm in which L^1 is complete, and [45] for other values of p). If we wish our space \mathcal{L}^p to consist of only the bounded operators, then we need to put a different norm on it if we want it to be complete. A common choice is $\|T\|_p = \max\{\|T\|, \|T\|_p\}$ (used in, for example, [15]). Since $\|\cdot\|$ and $\|\cdot\|_p$ are themselves norms, it is easy to check that $\|\cdot\|_p$ is a norm; moreover, \mathcal{L}^p is the intersection of \mathcal{N} with L^p , so it is also easy to see that it is complete under the norm $\|\cdot\|_p$ described. •

The following theorem can be used to compare the p -norms of two operators.

Theorem 1.1.8 ([33], **Theorem 1.1**) If h is a non-negative operator monotone function and if $0 \leq A, B$ (with A, B τ -measurable), then for all $\alpha > 0$ we have

$$\int_0^\alpha \mu_t(h(A) - h(B)) dt \leq \int_0^\alpha \mu_t(h(|A - B|)) dt. \quad \text{☞}$$

The most commonly used ideal in the following will be $\mathcal{I} = \mathcal{L}^p$; however, we do state some theorems for more general ideals, so we wish to establish a few properties. The norm-closed ideals in a *factor* are discussed in [7], for example. Proposition III.1.7.11 of [7] tells us that, under our standing assumptions on \mathcal{N} , if \mathcal{N} is a finite factor then it has no non-trivial norm-closed ideals, and if \mathcal{N} is a I_∞ or II_∞ factor then the only norm-closed ideal is the one consisting of τ -compact operators. This is as we might expect from our experience with $\mathcal{B}(\mathcal{H})$, where a two sided ideal contains the finite rank operators and is itself contained in the compact operators. However, this no longer holds for a general von Neumann algebra; an ideal \mathcal{I} need not be contained in the τ -compact operators of \mathcal{N} , not even if \mathcal{I} is essential, as we show next by an example. Recall that an ideal \mathcal{I} is called essential if $\mathcal{I} \cap \mathcal{J}$ is non-zero for any other ideal \mathcal{J} , which is equivalent to the description that there is no $E \in \mathcal{N}$ such that $E\mathcal{I} = 0$. For example, if $\mathcal{N} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$, then $\mathcal{K}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ is an essential ideal which is not contained in the compact operators, $\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$. As we will need our ideals to consist of compact operators, we will eventually have to explicitly assume it for the ideals under consideration (see Remark 3.3.1 for the properties required of our ideals).

Another issue that needs to be addressed is the norm we place on our ideals. In all cases under consideration, the norm will satisfy the properties in the definition below.

Definition 1.1.9 ([15], **Definition A.2**) If \mathcal{I} is a (two-sided) $*$ -ideal in \mathcal{N} which is complete in a norm $\|\cdot\|_{\mathcal{I}}$ then we call \mathcal{I} an **invariant operator ideal** if

- (1) $\|S\|_{\mathcal{I}} \geq \|S\|$ for all $S \in \mathcal{I}$,
- (2) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$, and
- (3) $\|ASB\|_{\mathcal{I}} \leq \|A\| \cdot \|S\|_{\mathcal{I}} \cdot \|B\|$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{N}$.

Note that, using polar decomposition, property (2) follows from (3). ☞

To start with, we note that \mathcal{L}^p for $p \geq 1$ satisfies all the above properties, and this will be our main concern. However, there is a wealth of examples of other invariant operator ideals generated from symmetric Banach function spaces. The original definition of Banach function spaces evolved from the fact that for a symmetric norm on a two-sided ideal of $\mathcal{B}(\mathcal{H})$, the norm of an operator is given by a function on the sequence of s -numbers of that operator ([39], Section III.2). By examining the properties of such a function, one can reverse the process: a function ξ with these properties defined on a subset E of the set of sequences of real numbers can be used to create an invariant operator ideal by taking all operators A whose s -numbers are in E and defining $\|A\|$ to be ξ evaluated on the s -numbers of A (Theorem 4.1 of [39], Section III.4). These ideas were adapted to the von Neumann algebra case; the function given by the s -numbers of an operator is no longer a sequence of real numbers but a measurable function on $[0, \infty)$, bounded if the operator is bounded. From a rearrangement-invariant Banach function space E with a norm $\|\cdot\|_E$ (see [70] for a definition, as well as for example of such spaces) one can obtain a symmetric space on \mathcal{N} by once again considering the operators A for which $(s \mapsto \mu_s(A)) \in E$ with norm $\|A\|_{\mathcal{E}} = \|\mu(A)\|_E$ ([71], Theorem 1). By definition, a **symmetric operator space** \mathcal{E} on \mathcal{N} is a linear subspace of the $*$ -algebra of measurable operators affiliated with \mathcal{N} such that, if $T \in \mathcal{E}$ and S is any measurable operator with $\mu_t(S) \leq \mu_t(T)$ for all $t > 0$, then $S \in \mathcal{E}$ and $\|S\|_{\mathcal{E}} \leq \|T\|_{\mathcal{E}}$. From the properties of s -numbers stated in Theorem 1.1.5 (parts (ii) and (iv)), if $S \in \mathcal{E}$ and A, B are bounded then $\mu_t(ASB) \leq \mu_t(\|A\| \cdot S \cdot \|B\|)$, whence $ASB \in \mathcal{E}$ and $\|ASB\|_{\mathcal{E}} \leq \|A\| \cdot \|S\|_{\mathcal{E}} \cdot \|B\|$. Finally, if we consider $\mathcal{E} \cap \mathcal{N}$ with norm $\max\{\|\cdot\|, \|\cdot\|_{\mathcal{E}}\}$, then it is easy to see that the resulting space is a 2-sided ideal which satisfies both properties (1) and (3) in Definition 1.1.9; that is, $\mathcal{E} \cap \mathcal{N}$ is an invariant operator ideal.

It is not clear whether there are invariant operator ideals which cannot be constructed from a symmetric operator space as described above; all the examples of which I am aware are so constructed. From the definition of invariant operator ideals we can also prove a norm inequality, but it holds only in the case when S and T are comparable positive operators. The first part of the following result is proven in [32]; the norm inequality with which we conclude the result follows immediately from the proof in [32], which we sketch briefly for completeness.

Theorem 1.1.10 ([32], Section I.1.6, part of Proposition 10) *If \mathcal{I} is an ideal in a von Neumann algebra \mathcal{N} , $0 \leq S \leq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$. Moreover, if \mathcal{I} is an invariant operator ideal, then $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.* \blacksquare

PROOF From $0 \leq T - S$, we get that $\|S^{1/2}v\|^2 \leq \|T^{1/2}v\|^2$ for any $v \in \mathcal{H}$. We can thus define an operator A by extending the definition $T^{1/2}v \mapsto S^{1/2}v$ from the range of T to the closure of the range, and setting it to 0 on the complement. Moreover, $A \in \mathcal{N}$ (it can be checked that A commutes with all unitaries in \mathcal{N}'). The definition and properties of A make up the proof of Lemma 2 in Section I.1.6 of [32]. It is easy to see that $S^{1/2} = AT^{1/2}$, whence $S = ATA^*$. Since \mathcal{I} is a two-sided ideal and $T \in \mathcal{I}$, it follows that $S \in \mathcal{I}$.

Note from the definition of A that we have $\|A\| \leq 1$. From property (3) in the definition of an invariant operator ideal (Definition 1.1.9),

$$\|S\|_{\mathcal{I}} = \|ATA^*\|_{\mathcal{I}} \leq \|A\| \cdot \|T\|_{\mathcal{I}} \cdot \|A^*\| \leq \|T\|_{\mathcal{I}},$$

showing the desired norm inequality. \blacksquare

For certain unbounded operators whose inverses are in a specific invariant operator ideal, the following result which allows us to compare inverses can be used in conjunction with Theorem 1.1.10 to obtain norm bounds.

Theorem 1.1.11 ([14], Lemma B.1) Suppose A and B are unbounded self-adjoint operators with $\text{Dom}A = \text{Dom}B$ and $0 < c \cdot 1 \leq A \leq B$ on their common domain. Then $0 \leq B^{-1} \leq A^{-1} \leq \frac{1}{c} \cdot 1$ on all of \mathcal{H} . \blacksquare

Theorem 1.1.12 ([15], Theorem B.8) Let \mathcal{I} be an invariant operator ideal, and $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a continuous increasing function such that $f(T)$ is trace class for any $T \in \mathcal{I}_+$. Then $T \mapsto f(T)$ mapping $\mathcal{I}_+ \rightarrow \mathcal{L}^1$ is continuous. \blacksquare

Next, we need to consider powers of ideals. As seen above for $\mathcal{I} = \mathcal{L}^p$, we will often have a norm defined on our ideals; so, in particular, we need to consider if a suitable norm can be defined on the power of an ideal, and what the relationship is between an ideal and its powers.

Definition 1.1.13 Suppose that \mathcal{I} is an ideal of operators. Define for $q \in \mathbb{R}_+$,

$$\mathcal{I}^q = \{T \in \mathcal{N} \mid |T|^{\frac{1}{q}} \in \mathcal{I}\}.$$

Following the notation of [31], use \mathcal{I}^0 to denote the norm closure of the ideal, and \mathcal{I}^∞ for the two-sided ideal generated by the projections of \mathcal{I} . \blacksquare

In [30], Dixmier shows that \mathcal{I}^q is in turn an ideal (Proposition 1). Note that Dixmier approaches the definition of \mathcal{I}^q slightly differently, by defining it to be the ideal whose positive part consists of $\{A^q : A \in \mathcal{I}\}$, but his description is easily seen to agree with the above. The notation is justified by the fact that $(\mathcal{I}^a)^b = \mathcal{I}^{ab}$ and $\mathcal{I}^a \cdot \mathcal{I}^b = \mathcal{I}^{a+b}$ (Proposition 2 of [30]). Moreover, when n is an integer, \mathcal{I}^n is the usual n^{th} power of \mathcal{I} – namely, the ideal consisting of linear combinations of elements of the form $r_1 r_2 \dots r_n$ with all $r_i \in \mathcal{I}$ (this follows by induction from the Proposition 2 of [30] cited above; see Corollary 1 of [30] for a complete proof).

Remark 1.1.14 There is a cautionary word that needs to be said about powers of ideals: as mentioned, $(\mathcal{I}^a)^b = \mathcal{I}^{ab}$ (and this is easy to check); however, we will have occasion to use $\mathcal{I} = \mathcal{L}^p$, and it is *not* true that $(\mathcal{L}^p)^a = \mathcal{L}^{pa}$. Here's a simple example of this phenomenon: Suppose that $\mathcal{I} = \mathcal{L}^{\frac{p}{2}}$. Then $(\mathcal{L}^{\frac{p}{2}})^{\frac{1}{2}} = \{T \mid |T|^2 \in \mathcal{L}^{\frac{p}{2}}\} = \{T \mid \tau\left((|T|^2)^{\frac{p}{2}}\right) < \infty\} = \mathcal{L}^p$; that is, $(\mathcal{L}^{\frac{p}{2}})^{\frac{1}{2}} = \mathcal{L}^p$. The temptation to cancel or combine powers in this situation might occasionally prove strong, but it must be overcome. Note that it is instead the case that $(\mathcal{L}^p)^a = \mathcal{L}^{\frac{p}{a}}$. \bullet

Remark 1.1.15 If $q < 1$ then $\mathcal{I} \subset \mathcal{I}^q$: Suppose that $T \in \mathcal{I}$; then the polar decomposition of T and the fact that \mathcal{I} is an ideal give us immediately that $|T| \in \mathcal{I}$. Now $\frac{1}{q} > 1$, say $\frac{1}{q} = 1 + \varepsilon$ for some fixed $\varepsilon > 0$. Then $|T|^{\frac{1}{q}} = |T| \cdot |T|^\varepsilon$ is in \mathcal{I} since $|T| \in \mathcal{I}$ and (once again) \mathcal{I} is an ideal. It follows from the definition of \mathcal{I}^q that $T \in \mathcal{I}^q$. Hence $\mathcal{I} \subset \mathcal{I}^q$, as claimed. This ordering will prove of some importance in the following, as we will have to consider powers of ideals in order to be able to prove some of our continuity theorems (see e.g. Section 3.5). The more general result $\mathcal{I}^\alpha \subset \mathcal{I}^\beta$ for $0 \leq \beta \leq \alpha \leq \infty$ can be found in [29]. \bullet

For the ideals \mathcal{I} we will consider, we can define a norm on \mathcal{I}^q (for $q < 1$) via

$$\|T\|_{\mathcal{I}^q} = \left(\left\| |T|^{\frac{1}{q}} \right\|_{\mathcal{I}} \right)^q;$$

moreover, \mathcal{I}^q will be invariant operator ideals under their own steam. This is certainly true for $\mathcal{I} = \mathcal{L}^p$, for example. Note, moreover, that the norm $\|\cdot\|_{\mathcal{I}^q}$ defined above corresponds to the appropriate \mathcal{L}^p norm when \mathcal{I} is an \mathcal{L}^p type ideal. Appendices A and B of [15] include proofs of various results and properties for a specific example of an invariant operator ideal and its powers, and can be used as a model of how such proofs can be constructed.

We observe that, if $q < 1$, the inclusion $\mathcal{I} \hookrightarrow \mathcal{I}^q$ is continuous:

$$\begin{aligned} \|T\|_{\mathcal{I}^q} &= \left(\| |T| \cdot |T|^{\frac{1}{q}-1} \|_{\mathcal{I}} \right)^q \\ &\leq \|T\|_{\mathcal{I}}^q \cdot \| |T|^{\frac{1}{q}-1} \|_{\mathcal{I}}^q \\ &\leq \|T\|_{\mathcal{I}}^q \cdot \|T\|_{\mathcal{I}}^{1-q} \\ &\leq \|T\|_{\mathcal{I}}^q \cdot \|T\|_{\mathcal{I}}^{1-q} \\ &= \|T\|_{\mathcal{I}}. \end{aligned}$$

Note that the above proof relies on the fact that \mathcal{I} is an invariant operator ideal – in the first step, we use property (3) of the definition, and in the second to last step, property (1).

Finally, we discuss Hölder's inequality as it applies to powers of ideals. For $\mathcal{I} = \mathcal{L}^1$, whose powers are \mathcal{L}^p for various values of p , Hölder's inequality is a result of Dixmier; we will use it to figure out whether an operator is in \mathcal{L}^p for some p , and to find upper bounds for \mathcal{L}^p norms. As will be obvious from the statement, this theorem applies when the norm on the ideal \mathcal{I} comes from a trace (e.g. $\mathcal{I} = \mathcal{L}^1$); however, for other types of ideals this theorem might have to be proven separately if needed (see for example the proof of Lemma A.3 in [15], where the Hölder inequality is proven for $\mathcal{I} = Li$, the ideal of bounded operators in \mathcal{N} whose generalized s -numbers μ_s are $O(1/\log s)$).

Theorem 1.1.16 (Hölder's inequality, [31], Corollary 2 and 3 of Theorem 6, and [30]) Let τ be a faithful, normal trace on \mathcal{N} and \mathcal{I} the corresponding trace ideal. Suppose p, q, \dots, r are numbers in the interval $[1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} + \dots + \frac{1}{r} = \frac{1}{s}$. Then $\| \|T\|_{\mathcal{I}^{\frac{1}{p}}} = [\tau(|T|^p)]^{\frac{1}{p}}$ defines a norm on $\mathcal{I}^{\frac{1}{p}}$. If $A \in \mathcal{I}^{\frac{1}{p}}, B \in \mathcal{I}^{\frac{1}{q}}, \dots, C \in \mathcal{I}^{\frac{1}{r}}$, then $AB \dots C \in \mathcal{I}^{\frac{1}{s}}$ and

$$\|AB \dots C\|_s \leq \|A\|_p \cdot \|B\|_q \dots \|C\|_r. \quad \blacktriangleright$$

In a more general setting, if \mathcal{I} is not the trace ideal, one might not need to show that Hölder's inequality holds, and might instead find reason to be satisfied with the Cauchy-Schwarz inequality (below). For example, the statements of Theorem 3.5.1 and Theorem 3.5.8 refer to the Cauchy-Schwarz inequality.

Definition 1.1.17 ([15], Lemma B.12) If \mathcal{I} and $\mathcal{I}^{\frac{1}{2}}$ are invariant operator ideals, we say that they satisfy the **Cauchy-Schwarz inequality** if, for all $X, Y \in \mathcal{I}^{\frac{1}{2}}$,

$$\|XY\|_{\mathcal{I}} \leq \|X\|_{\mathcal{I}^{\frac{1}{2}}} \cdot \|Y\|_{\mathcal{I}^{\frac{1}{2}}}. \quad \blacktriangleright$$

The fact that the Cauchy-Schwarz inequality holds if $\mathcal{I} = \mathcal{L}^{\frac{p}{2}}$ (in which case $\mathcal{I}^{\frac{1}{2}} = \mathcal{L}^p$) is, of course, a consequence of the Hölder inequality.

1.1.2 Breuer-Fredholm operators

Fredholm theory was extended to the case of von Neumann algebras by Manfred Breuer ([11] and [12]). Breuer associates to each algebra \mathcal{N} an index group $I(\mathcal{N})$ built up from Murray-von Neumann equivalence classes of finite projections. To each finite projection one can then associate in a natural manner an element of $I(\mathcal{N})$, leading to a definition of Fredholm operators with index in $I(\mathcal{N})$. For example, if \mathcal{N} is a type II_∞ factor, the index group is isomorphic to \mathbb{R} , but is more complex if \mathcal{N} is a general semifinite von Neumann algebra. Better suited to our purposes is a modification of this theory, where the index associated to a projection is given simply by the trace on \mathcal{N} , as outlined in [56] (Appendix B). The resulting theory is similar in flavor to Breuer's, but dependent on the choice of trace τ . It should be noted that an operator which is classified as Fredholm under this modified theory would also be Fredholm under Breuer's classification; the advantage of the change is that the index of a Fredholm operator is always a real number.

In the following, we denote by $[\mathcal{S}]$ the projection onto the closure of the subspace $\mathcal{S} \subset \mathcal{H}$.

Definition 1.1.18 Say that $T \in \mathcal{N}$ is a **Breuer-Fredholm operator** if the projection onto $\ker T$ has finite trace, and there exists a projection $E \in \mathcal{N}$ such that $\tau(1 - E) < \infty$ and $\text{ran } E \subseteq \text{ran } T$. In this case, define the τ -index to be

$$\text{ind } T = \tau([\ker T]) - \tau([\ker T^*]). \quad \blacktriangledown$$

Remark 1.1.19 If $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and τ is the usual trace, the Breuer-Fredholm operators correspond to the Fredholm operators. An overview of the theory for Fredholm operators can be found in, for example, Chapter 2 of [41]. •

The theory describing the properties of Breuer-Fredholm operators is very similar to that of Fredholm operators in $\mathcal{B}(\mathcal{H})$. Namely, the role of the compact operators is played in this context by $\mathcal{K}_{\mathcal{N}}$, and the usual results like Atkinson's theorem (stated below) and the properties of the index carry over.

Theorem 1.1.20 ([56], Theorem B1) Suppose that \mathcal{N} is a von Neumann algebra with faithful, normal, semifinite trace τ , and $\mathcal{K}_{\mathcal{N}}$ denotes the τ -compact operators. If π is the projection onto the quotient $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$, then the Breuer-Fredholm operators are exactly those whose image under π is invertible. \blacktriangledown

We use this idea to prove the following lemma. The types of operators mentioned in this lemma play a major role in Chapter 2.

Lemma 1.1.21 Let T be a bounded operator and P a finite-trace projection. Suppose that, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, T can be written as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with D invertible. Then T is Breuer-Fredholm. \blacktriangledown

PROOF Note that, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, we have

$$\begin{bmatrix} 1 & -BD^{-1} \\ 0 & D^{-1} \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -D^{-1}C & D^{-1} \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & 1 \end{bmatrix}.$$

Since $A - BD^{-1}C \in P\mathcal{N}P$ and $\tau(P) < \infty$, it follows that the two matrices on the right are equal to the identity mod the τ -compact operators. Therefore, T is invertible mod the compacts, which means that it is Breuer-Fredholm. ■

As a lot of the operators we are going to be working with are self-adjoint, let us say a few words about self-adjoint Breuer-Fredholm operators. The essentially positive self-adjoint operators are those which are mapped to a positive operator by the canonical map from \mathcal{N} onto the generalized Calkin algebra $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$; similarly, we define the essentially negative operators. From the point of view of spectral flow, the essentially positive and essentially negative operators are not that interesting (see Remark 1.1.40). If \mathcal{N} is a I_{∞} factor, the Breuer-Fredholm self-adjoint operators split into three components: essentially positive operators, essentially negative operators, and those that are neither. The first two components are contractible (this was originally proven in [3], though a simple proof is also presented in Proposition 1 of [54]). This fact continues to hold in the same manner if \mathcal{N} is not a factor, as the straight line between any essentially positive operator and the identity consists of essentially positive operators, and similarly for essentially negative operators and the negative of the identity. However, the set of operators which are neither essentially positive nor essentially negative can change its nature – for example, it is empty if \mathcal{N} is finite; it is connected if \mathcal{N} is a II_{∞} factor; and it can be disconnected in general (for example, in $\mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$).

A generalization of Breuer-Fredholm operators is obtained by considering operators in a skew corner of \mathcal{N} . We will encounter such operators in the definition of spectral flow. The impetus of this theory comes from the fact that, in certain circumstances, if P_1 and P_2 are projections in \mathcal{N} , then one can associate an index to P_1P_2 considered as an operator from $P_2\mathcal{H}$ to $P_1\mathcal{H}$. This idea was introduced by Brown, Douglas and Fillmore in the context of a type I algebra for projections whose image in the Calkin algebra is the same (see [13], Remark 4.9); since the index of P_1P_2 is equal to the codimension of P_1 in P_2 when $P_1 \leq P_2$, they called the resulting index the “essential codimension” of the pair of projections P_1, P_2 . Perera, in his PhD thesis [52], summarizes the type I results about essential codimension (see Section 1.2), and then generalizes one of the definitions to type II (Section 2.2). In [55], Phillips develops the idea of Fredholm operators from $P\mathcal{H}$ to $Q\mathcal{H}$ (for P, Q projections in a factor \mathcal{N}), and treats the essential codimension as a specific example of an index of such operators. Moreover, he extends the type of projections for which one can calculate the index, by noting that, if P_1 and P_2 are infinite projections such that $\|\pi(P_1) - \pi(P_2)\| < 1$, then P_1P_2 is Breuer-Fredholm, and $\text{ind}(P_1P_2)$ for $P_1P_2 : P_2\mathcal{H} \rightarrow P_1\mathcal{H}$ exists (Lemma 1.1 of [55]). Finally, [18] drops the requirement that P and Q be either infinite or equivalent, and presents a general theory of $(P-Q)$ -Fredholm operators for P, Q projections in a von Neumann algebra. This allows Lemma 1.1 of [55] to be proven in this more general setting. We summarize the needed definitions and results about $(P-Q)$ -Fredholm operators; we shall mostly use this to find $\text{ind}(PQ)$ in the context of calculating spectral flow.

In the following, P and Q are projections in \mathcal{N} .

Definition 1.1.22 Say that $T \in P\mathcal{N}Q$ is **$(P-Q)$ -Fredholm** if

1. $\tau([\ker T \cap Q\mathcal{H}]) < \infty$,

2. $\tau([\ker T^* \cap P\mathcal{H}]) < \infty$, and
3. there exists a projection $P_1 < P$ such that $P_1\mathcal{H} \subset \text{ran } T$ and $\tau(P - P_1) < \infty$.

If $T \in P\mathcal{N}Q$ is $(P-Q)$ -Fredholm, define

$$\text{ind}_{(P-Q)}(T) = \tau([\ker T \cap Q\mathcal{H}]) - \tau([\ker T^* \cap P\mathcal{H}]). \quad \blacktriangledown$$

While it does not make sense, in the case when $P \neq Q$, to talk about invertible operators, nonetheless the notion of 'invertible mod the τ -compacts' generalizes as well as one might expect to this situation.

Definition 1.1.23 Consider $T \in P\mathcal{N}Q$. A **parametrix** for T is an operator $S \in Q\mathcal{N}P$ such that $ST = Q + K_1$ for some $K_1 \in \mathcal{K}_{Q\mathcal{N}Q}$ and $TS = P + K_2$ for some $K_2 \in \mathcal{K}_{P\mathcal{N}P}$. \blacktriangledown

Theorem 1.1.24 ([18], Lemma 3.4) $T \in P\mathcal{N}Q$ is $(P-Q)$ -Fredholm if and only if T has a parametrix $S \in Q\mathcal{N}P$. \blacktriangledown

A summary of the various results for Breuer-Fredholm operators in a skew corner, with appropriate references, can be found in [5]. As already mentioned, this kind of index will be needed when defining the spectral flow. Instead of using the ' $(P-Q)$ ' prefix, we will simply talk about an operator T being Breuer-Fredholm from $Q\mathcal{H}$ to $P\mathcal{H}$ (or just Breuer-Fredholm, if the domain and range under consideration are clear).

A final note should be made about unbounded operators, which we also wish to consider. A discussion of unbounded Fredholm operators can be found in [44], Section IV.5 (in the larger context of operators on Banach spaces). The generalization for $(P-Q)$ -Fredholm is presented in [18], from where we single out the following theorem, which captures the idea that one way of dealing with unbounded operators is to apply a transform to them that gives us back a bounded operator.

Theorem 1.1.25 ([18], Proposition 3.10) If T is a closed, densely-defined operator affiliated with $P\mathcal{N}Q$, T is $(P-Q)$ -Fredholm if and only if $T(1 + |T|^2)^{-\frac{1}{2}}$ is $(P-Q)$ -Fredholm in $P\mathcal{N}Q$. Moreover,

$$\text{ind}_{(P-Q)}(T) = \text{ind}_{(P-Q)}(T(1 + |T|^2)^{-\frac{1}{2}}). \quad \blacktriangledown$$

This particular transformation, $T \mapsto T(1 + |T|^2)^{-\frac{1}{2}}$, will make further appearances; its properties are discussed when we introduce spectral flow for unbounded operators.

1.1.3 Spectral flow definitions

Spectral flow is a homotopy invariant defined for paths of self-adjoint Breuer-Fredholm operators. Suppose that we are dealing with operators whose spectrum consists of discrete eigenvalues. Informally, the spectral flow counts the net number of eigenvalues which change sign from negative to positive as we move along the path. One way of tracking this movement is to examine the change in the projection onto the positive eigenspace of the operators. This approach can then be generalized to cases in which the spectrum of the operators involved is not discrete.

In order to talk about spectral flow we will usually require a continuous path of self-adjoint Breuer-Fredholm operators. The self-adjoint requirement gives us that the spectrum is a subset of

the real line, and the spectrum changes continuously as we go along the path (see [44], Remark 4.9 of Section V4 for results about the continuity of the spectrum for self-adjoint operators). This allows us to make sense of the idea of the spectrum crossing zero; the Breuer-Fredholm requirement will give us a way to measure how much of the spectrum crosses zero. Note that this is merely an intuitive description; all this will be made rigorous below.

In Physics, spectral flow can be used to calculate the number of solutions to certain differential equations; for this point of view, see Chapter 17 of [9], especially Section 17D. A mathematically-minded overview of the importance of spectral flow is given in Section 1.2 below.

1.1.3.1 Spectral flow for Bounded Operators

There are multiple interpretations of spectral flow; the one presented below is due to Phillips, see [55]. The presentation in the aforementioned paper assumes \mathcal{N} is a factor; however, the approach generalizes to semifinite von Neumann algebras, especially once the appropriate Breuer-Fredholm theory is in place (see [18], Section 3, and in particular the remark following Corollary 3.8).

Recall that π is used for the canonical map from \mathcal{N} onto the generalized Calkin algebra, $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$. Denote by χ the characteristic function of the interval $[0, \infty)$. Given a path of self-adjoint Breuer-Fredholm operators $\{F_t\}$, $\chi(F_t)$ is not continuous, but $\pi(\chi(F_t))$ is.

Definition 1.1.26 Suppose $\{F_t\}$ is a continuous path of self-adjoint Breuer-Fredholm operators. Let $P_t = \chi(F_t)$. Choose finitely many points $0 = r_0 < r_1 < \dots < r_n = 1$ such that for u, v in each subinterval $[r_i, r_{i+1}]$ we have $\|\pi(P_u) - \pi(P_v)\| < 1$. Define the **spectral flow** of the path to be


$$\text{sf}(\{F_t\}) := \sum \text{ind}(P_{r_i} P_{r_{i+1}}),$$

where $\text{ind}(P_{r_i} P_{r_{i+1}})$ is the index of $P_{r_i} P_{r_{i+1}}$ as a Breuer-Fredholm operator in $P_{r_i} \mathcal{N} P_{r_{i+1}}$. 

It has been shown that this definition is independent of the partition P_{r_i} (see Lemma 1.3 and Definition 2.2 of [55]).

In the type I case, the spectrum around 0 consists of a discrete collection of eigenvalues; the dimension function for projections takes values in the non-negative integers, so the spectral flow is an integer. In the type II case we can also have continuous spectrum; the dimension function for projections takes values in the non-negative reals, so the spectral flow is itself a real number.

Remark 1.1.27 Note that if the algebra \mathcal{N} is finite, all projections have finite trace. This means that $\pi(P) = 0$ for all projections P , and the condition that the projections $\pi(P_{r_i})$ should be close is trivially satisfied by any $\{r_1, \dots, r_n\} \subseteq [0, 1]$. In particular, it should be clear that the spectral flow depends only on the endpoints of the path. That is, $\text{sf}(\{F_t\}) = \text{ind}(\chi(F_0)\chi(F_1))$.

If \mathcal{N} is not finite, it is not necessarily the case that the spectral flow of the path depends only on the endpoints, as can be seen from Example 1.1.38. 

Remark 1.1.28 It is pertinent at this point to discuss some of the difficulties which arise when generalizing the spectral flow concept from type I factors to type II factors. The reason for this is two-fold: the type I case is easier to illustrate and understand, and the issues become a

stumbling block when we try to use the same approach for certain paths of unbounded operators. The definition given above works for both type I and type II factors (and for general semifinite von Neumann algebras, of course), but is different than Phillips' original definition for type I factors, which we briefly explain below (see [54] for further details).

If we have a path of self-adjoint Fredholm operators $\{F_t\} \subset \mathcal{B}(\mathcal{H})$, then the spectrum of each operator is discrete near 0. Fix some operator F_r on our path; then there exists an $\varepsilon > 0$ such that $\pm\varepsilon \notin \sigma(F_r)$ and F_r restricted to $\chi_{[-\varepsilon, \varepsilon]}(F_r)$ is a finite rank projection. Moreover, there exists a neighbourhood $[r - s, r + s]$ of r such that, for all $k \in [r - s, r + s]$, $\pm\varepsilon \notin \sigma(F_k)$, the projection $P_k = \chi_{[-\varepsilon, \varepsilon]}(F_k)$ is also finite rank, and $k \mapsto P_k$ is continuous (this is the Lemma of [54], p.462). The difference in dimension between the projection onto the positive spectrum of $F_{r+s}P_{r+s}$ (as an operator on $P_{r+s}\mathcal{H}$) and $F_{r-s}P_{r-s}$ (as an operator on $P_{r-s}\mathcal{H}$) can be used to calculate the spectral flow along the segment of the path from F_{r-s} to F_{r+s} .

When moving to, say, a II_∞ factor, we are faced with the problem that the spectrum of the operators under consideration need no longer be discrete near 0. There is no handy gap at $\pm\varepsilon$, no window $[-\varepsilon, \varepsilon]$ on which we can focus to observe the spectrum pass by. We get around this problem by noticing that there is, in some sense, a gap in the spectrum at infinity. So it becomes a question of measuring the change in the projection onto the positive part of the spectrum, which leads to Definition 1.1.26. •

Remark 1.1.29 We will find that endpoints of the form $2P - 1$ where P is a projection play a special role in our presentation (see, for example, the exposition in Section 3.5). To that end, a remark is in order about paths whose endpoints have this special type. Suppose $F_0 = 2P - 1$ and $F_1 = 2Q - 1$ for two projections P and Q ; it should then be clear that $\chi_{[0, \infty)}(F_0) = P$ and $\chi_{[0, \infty)}(F_1) = Q$.

Suppose that our algebra has *finite* trace; then, as already remarked above (Remark 1.1.27), the spectral flow only depends on the endpoints of the path, and $\text{sf}(\{F_t\}) = \text{ind}(PQ)$ as an operator from $Q\mathcal{H}$ to $P\mathcal{H}$. We will show that $\text{ind}(PQ) = \tau(Q - P)$.

Consider $\text{ran } P \cap \ker Q$ and $\text{ran } Q \cap \ker P$. It is easy to check that these are mutually orthogonal closed subspaces of \mathcal{H} which are invariant under P and Q . Let

$$H_1 = [(\text{ran } P \cap \ker Q) \oplus (\text{ran } Q \cap \ker P)]^\perp.$$

Then H_1 is also invariant under P and Q , so with respect to the decomposition of \mathcal{H} given by $(\text{ran } P \cap \ker Q) \oplus (\text{ran } Q \cap \ker P) \oplus H_1$ we have the following block matrix decompositions:

	$\text{ran } P \cap \ker Q$	\oplus	$\text{ran } Q \cap \ker P$	\oplus	H_1
P	1	\oplus	0	\oplus	\tilde{P}
Q	0	\oplus	1	\oplus	\tilde{Q}
$F_0 := 2P - 1$	1	\oplus	-1	\oplus	$2\tilde{P} - 1$
$F_1 := 2Q - 1$	-1	\oplus	1	\oplus	$2\tilde{Q} - 1$,

where \tilde{P} and \tilde{Q} are appropriate operators in $\mathcal{B}(H_1)$. Due to the invariance of H_1 and H_1^\perp , \tilde{P} and \tilde{Q} are also projections; moreover, it is easy to check that $\ker \tilde{P} \cap \text{ran } \tilde{Q} = \ker \tilde{Q} \cap \text{ran } \tilde{P} = 0$, so it follows that \tilde{P} and \tilde{Q} are unitarily equivalent in $\mathcal{B}(H_1)$ (see Proposition 3.2 of [15] or

Proposition 2.1 of [57]). Hence $\tau(\tilde{P}) = \tau(\tilde{Q})$. Clearly, $\tau(P) = \tau([\text{ran } P \cap \ker Q]) + \tau(\tilde{P})$ and $\tau(Q) = \tau([\text{ran } Q \cap \ker P]) + \tau(\tilde{Q})$. Since

$$\text{sf}(\{F_t\}) = \text{ind}(PQ) = \tau([\text{ran } Q \cap \ker P]) - \tau([\text{ran } P \cap \ker Q]),$$

we get $\text{sf}(\{F_t\}) = \tau(Q - P)$.

This result generalizes to algebras for which the trace is not finite, but in that case we have to put additional restrictions on P and Q to ensure that various operators have finite trace. The above is in fact a special case of Theorem 3.1 in [15], quoted in this thesis as Theorem 3.5.2. •

Other possible ways of defining spectral flow follow from definitions for unbounded operators; discussed below.

1.1.3.2 Spectral flow for Unbounded Operators

The unbounded operators under consideration will always have dense domain, ensuring that we can define the adjoint. Since we will only be concerned with self-adjoint operators, we will also have that all our operators are closed. We will not discuss the basic definitions and results for unbounded operators; the curious reader who is unfamiliar with them is referred to [59] or [44].

In the case of unbounded operators, a decision has to be made about how to deal with continuity. There are three subsets of the unbounded operators that one usually considers: perturbations (either bounded or relatively bounded) of a fixed operator, all operators with a fixed domain, and all unbounded operators. An overview of the usual topologies used on these sets and the relationship between them, with further references, can be found in [47] (see Proposition 2.2 and Proposition 2.4). Another possible simplifying assumption is that the operators involved have compact resolvents, which makes it easier to get a handle on the spectrum of the operators involved. We will only consider two of the possible sets: bounded perturbations of a fixed operator, and all unbounded operators.

Due to domain of definition issues, the easiest choice is to restrict ourselves to bounded perturbations of a fixed operator; then we can use the bounded operator norm. If $\{D_t\}$ is a path of unbounded self-adjoint operators, where $D_t = D_0 + A_t$ and $\{A_t\}$ is a norm-continuous path of bounded operators, we can apply the Riesz transform, $D \mapsto D(1 + D^2)^{-\frac{1}{2}}$, to obtain a path of bounded (self-adjoint) operators.

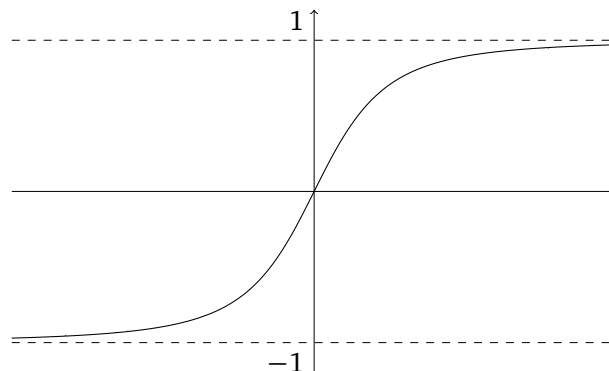


Figure 1.1: The map $x \mapsto x(1 + x^2)^{-\frac{1}{2}}$.

The resulting collection $\{F_t\}$ of operators obtained by applying the Riesz transform is continuous in t (for a proof of this fact, see for example [14], Theorem A.8). Moreover, due to Theorem 1.1.25, we know that if the D_t 's are Breuer-Fredholm, so are the operators $\{F_t\}$. Under certain conditions, we can thus reduce the problem of calculating spectral flow for unbounded operators to the bounded case.

Definition 1.1.30 If $\{D_t\}$ is a path of (unbounded) self-adjoint Breuer-Fredholm operators such that $D_t = D_0 + A_t$ with $\{A_t\}$ a (norm-continuous) path of bounded operators, then

$$\text{sf}(\{D_t\}) = \text{sf}(D_0(1 + D_0^2)^{-\frac{1}{2}}). \quad \blacktriangledown$$

We mention here that restricting ourselves to bounded perturbations gives us that the inverse function is continuous on neighbourhoods of invertible operators; we will need this result later:

Theorem 1.1.31 ([44], Section IV.4, Theorem 1.16 and Remark 1.17) Suppose T is a closed operators for which T^{-1} exists and is bounded, and A is a bounded operator such that $\|A\| \cdot \|T^{-1}\| < 1$. Then $T + A$ has a bounded inverse, with $\|(T + A)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|A\| \cdot \|T^{-1}\|}$. Moreover,

$$\|(T + A)^{-1} - T^{-1}\| \leq \frac{\|T^{-1}\|^2}{1 - \|A\| \cdot \|T^{-1}\|} \cdot \|A\|.$$

Note that if we want to simplify the right hand side of the above inequalities a little bit, we can choose $\|A\| < \frac{1}{2} \cdot \|T^{-1}\|^{-1}$; then $\frac{1}{1 - \|A\| \cdot \|T^{-1}\|} < 2$, so we get $\|(T + A)^{-1}\| < 2\|T^{-1}\|$ and $\|(T + A)^{-1} - T^{-1}\| < 2\|T^{-1}\|^2 \cdot \|A\|$. \blacktriangledown

The most general milieu in which spectral flow can be considered is the set of unbounded operators affiliated with a given von Neumann algebra. The usual topology on the set of unbounded operators is the gap topology, which we now define.

Definition 1.1.32 The **gap distance** between two closed unbounded operators D_1 and D_2 on \mathcal{H} is defined to be $\|P_{D_1} - P_{D_2}\|$, where P_{D_i} is the projection from $\mathcal{H} \times \mathcal{H}$ onto the graph of D_i . The topology generated by the gap distance is called the **gap topology**. \blacktriangledown

It is known that, if D is a self-adjoint unbounded operator, the matrix of P_D relative to the decomposition $H \oplus H$ is

$$\begin{bmatrix} (1 + D^2)^{-1} & D(1 + D^2)^{-1} \\ D(1 + D^2)^{-1} & D^2(1 + D^2)^{-1} \end{bmatrix}$$

(see, for example, [14] Appendix A for how to prove this). In general, however, the Riesz transformation is not continuous with respect to the gap topology, so we cannot treat this case in the same way we did the bounded perturbations one.

Other metrics equivalent to the gap topology on the set of closed unbounded operators are described in Section 3 of [27]. As we do not use these other metrics, we mention instead Theorem 1.1 of [8], which says that the gap metric, on the set of closed self-adjoint operators, is uniformly equivalent to the metric given by $\gamma(T_1, T_2) = \|(T_1 + i)^{-1} - (T_2 + i)^{-1}\|$; occasionally, this is the easier definition to use to show that a path is gap continuous. As part of the Addendum

to [27] (Theorem 1), it is also shown that the topology induced by the gap metric on the space of bounded operators agrees with the topology induced by the norm. It should be noted, however, that the equivalence is not uniform; the set of bounded self-adjoint operators is dense, with respect to the gap metric, in the set of unbounded self-adjoint operators on \mathcal{H} (see, for example, Proposition 1.6 of [8]).

On the set of densely-defined self-adjoint operators on \mathcal{H} , the identity map is continuous from the Riesz topology into the gap topology, but not the other way around. A proof of this fact, and slightly more context, can be found in [47]. Namely, even on bounded perturbations of a fixed self-adjoint unbounded operator the two topologies do not agree. In fact, the norm topology is strictly finer than the Riesz topology, which in turn is strictly finer than the gap topology (Proposition 2.4(1) of [47]). There is a well-known example due to Fuglede of a sequence which converges in gap topology but not in Riesz topology. As this example can be easily found in many other places, such as [47] (equations (2.31) to (2.33)) or [8] (Example 2.14), we do not include it here.

One way of dealing with the gap topology is to use the Cayley transform, which changes the context of the problem from unbounded self-adjoint operators to a subset of the unitary operators (with the norm topology). The Cayley transform is given by $T \mapsto (T - i)(T + i)^{-1}$, and we discuss some of its properties here.

Remark 1.1.33 The function $\lambda \mapsto \frac{\lambda - i}{\lambda + i}$ maps the real axis onto the complex unit circle as shown in the diagram below:

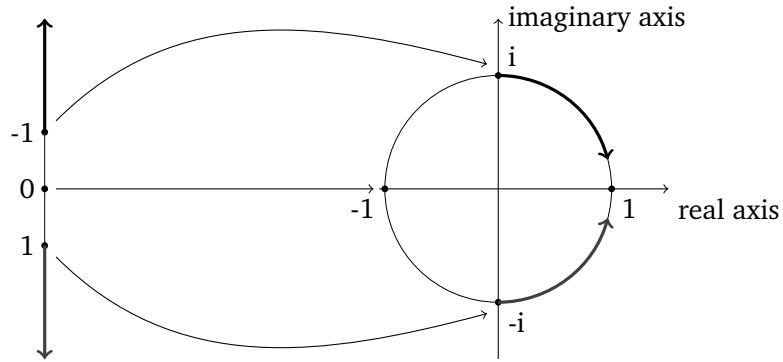


Figure 1.2: The Cayley transform $\lambda \mapsto (\lambda - i)(\lambda + i)^{-1}$.

The inverse of the Cayley transform is given by $\mu \mapsto \frac{1+\mu}{1-\mu} \cdot i$. One possible reference for properties of the Cayley transform when applied to operators is [59]. The fact that, if A is a self-adjoint (possibly unbounded) operator then $V = (A - i)(A + i)^{-1}$ is unitary, is found in Section 121 (Chapter VIII); moreover, with this definition of V , we also have $A = i(1 + V)(1 - V)^{-1}$. The spectral decompositions of A and V are related. If $V = \int_0^{2\pi} e^{i\varphi} dF_\varphi$ is the spectral decomposition of V (with $F_0 = 0$, $F_{2\pi} = 1$), then $E_\lambda = F_{-2\cot^{-1}(\lambda)}$ is a spectral family over $(-\infty, \infty)$ for which $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. •

The Cayley transform allows us to look at spectral flow in a different light; applying the Cayley transform to a gap-continuous path of unbounded self-adjoint operators results in a (continuous)

path in a subset of the unitary operators (namely, those operators which do not have 1 as an eigenvalue). In this new picture, the spectrum is constrained to the unit circle, and we talk about 'spectral flow across -1'. We will use this approach in Section 2.4.

The gap topology presents us with some challenges when it comes to calculating spectral flow; the approach described in Section 1.1.3.1 fails to take into account that, in some sense, the gap topology allows movement of the spectrum through infinity. This is where the problems discussed in Remark 1.1.28 return to haunt us, as in the general case there is no spectral projection we can use that would be guaranteed to change due to spectral flow only. If \mathcal{N} is of type I , then the original approach to spectral flow (explained in Remark 1.1.28) still works, as we can use the gaps in the spectrum close to 0 to zoom in on eigenvalues which are truly switching sign, and ignore what is happening at infinity. This was implemented in [8]. However, if \mathcal{N} is of type II , then we're stuck, as there is no guaranteed gap in the spectrum. The projection onto the positive part of the spectrum can change due to flow through infinity, not just flow through 0; see Section 1.4.2 for an example exhibiting this phenomenon.

Wahl's definition of spectral flow for gap-continuous paths of unbounded operators: A way of getting around this problem is due to Wahl [76]; she applies a series of transformations to the original path, in order to obtain a path of unitaries on which a winding number can be calculated. As part of the process, she introduces a topology weaker than the gap topology and defines the spectral flow in this new context. The details of this topology can be found in [75] and [76]; however, we will just concentrate on the gap topology.

Assume given a path $\{D_t\}$ of self-adjoint Breuer-Fredholm operators, continuous in the gap topology and with invertible endpoints. We claim that there exists a positive integer n such that $\sigma(D_0), \sigma(D_1)$ do not intersect $[-\frac{1}{n}, \frac{1}{n}]$ and for any $t \in [0, 1]$, if $P_t = \chi_{[-\frac{1}{n}, \frac{1}{n}]}(D_t)$, then $D_t|_{P_t \mathcal{H}}$ is τ -compact. To obtain such an n we could proceed as follows:

- since D_0 is invertible, $0 \notin \sigma(D_0)$. But $\sigma(D_0)$ is closed, so there exists a neighbourhood \mathcal{V} of 0 which does not intersect $\sigma(D_0)$; by choosing $m \in \mathbb{N}$ large enough, we can ensure that $[-\frac{1}{m}, \frac{1}{m}] \subseteq \mathcal{V}$, and hence $\sigma(D_0) \cap [-\frac{1}{m}, \frac{1}{m}] = \emptyset$. Since D_1 is also invertible, we can by a similar argument ensure that m is large enough to simultaneously guarantee that $\sigma(D_1) \cap [-\frac{1}{m}, \frac{1}{m}] = \emptyset$.
- for each t , D_t is Breuer-Fredholm. In order to avoid getting all tangled in unbounded operator issues, we use the Cayley transform κ to change the path D_t to a path of unitaries $\{U_t\}$. Then $\{1 + U_t\}$ is Breuer-Fredholm for each t , and the spectrum of each $1 + U_t$ is contained in the circle of radius 1 centered at 1. Denote by $[\alpha \rightarrow \beta]$ the arc of this circle starting at angle α and ending at angle β . If $\chi_{[\alpha \rightarrow \beta]}(1 + U_t)$ has finite trace, then there is a corresponding spectral projection of D_t which has finite trace (see Remark 1.1.33 for a description of the relationship between the spectral projections of U_t and those of D_t).

Fix $s \in [0, 1]$, and denote by N_s the operator $1 + U_s$. Since N_s is Breuer-Fredholm, by Theorem 1.1.20 $\pi(N_s)$ is invertible (where π is the projection onto the generalized Calkin algebra $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$). Thus the spectrum of $\pi(N_s)$ contains a gap at zero; in other words, there is an $r > 0$ such that \bar{B}_r , the closed ball of radius r centered at 0, does not intersect $\sigma(\pi(N_s))$. Moreover, by the upper semicontinuity of the spectrum for operators on a Banach space (see e.g. [44], Chapter IV, Remark 3.3) there is a neighbourhood \mathcal{V} of $\pi(N_s)$ such that

for all $A \in \mathcal{V}$, the spectrum of A also does not intersect \overline{B}_r . The map π is continuous, so there is a neighbourhood \mathcal{U}_s of N_s such that $\pi(\mathcal{U}_s) \subset \mathcal{V}$. Denote by $[\alpha_s \rightarrow \beta_s]$ an arc of the circle of radius 1 centered at 1 which contains zero and is itself contained in \overline{B}_r . By construction, if $N_t \in \mathcal{U}_s$, then $\sigma(\pi(N_t)) \cap [\alpha_s \rightarrow \beta_s] = \emptyset$, whence $\chi_{[\alpha_s \rightarrow \beta_s]}(N_t)$ has finite-trace. Since the path $\{N_t : t \in [0, 1]\}$ is a compact set, finitely many of the \mathcal{U}_s 's cover it, resulting in finitely many arcs $[\alpha_s \rightarrow \beta_s]$. The intersection of these arcs, $[\alpha \rightarrow \beta]$, has the property that $\chi_{[\alpha \rightarrow \beta]}(N_t)$ has finite trace for all $t \in [0, 1]$; note that 0 is a point contained in $[\alpha \rightarrow \beta]$. Use the relationship between the spectrum of N_t and that of D_t (see Remark 1.1.33) to find a segment $J = [-\frac{1}{n}, \frac{1}{n}]$ such that the trace of $\chi_J(D_t)$ is finite for all $t \in [0, 1]$.

Let $n = \min\{m, n\}$. Then n satisfies the conditions set out at the beginning of this paragraph – i.e. $\sigma(D_0), \sigma(D_1)$ do not intersect $[-\frac{1}{n}, \frac{1}{n}]$ and for any $t \in [0, 1]$, if $P_t = \chi_{[-\frac{1}{n}, \frac{1}{n}]}(D_t)$, then $D_t|_{P_t^\perp}$ is τ -compact.

Next, find a continuous function Ξ on \mathbb{R} such that Ξ is odd and non-decreasing, and $\lim_{x \rightarrow \infty} \Xi(x) = 1$, $\Xi^{-1}(0) = \{0\}$ and $\text{supp}(\Xi^2 - 1) \subseteq (-\frac{1}{n}, \frac{1}{n})$ (where n is the integer found in the previous paragraph); Wahl calls such a function a normalizing function for $\{D_t\}$. The restrictions on Ξ mean that it might look something like

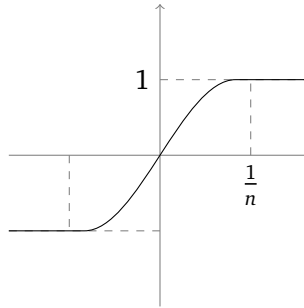


Figure 1.3: A normalizing function Ξ for a gap continuous path D_t (the choice of n depends on the spectrum of the operators).

It is important to note at this point that $\Xi(D_t)$ is not continuous in the norm topology. However, $\{e^{\pi i(\Xi(D_t)+1)}\}$ is a path of unitaries (i.e. norm continuous) which is of the form $1 + K_t$ with $K_t \in \mathcal{K}_{\mathcal{N}}$; so, one can calculate its winding number. The properties of the winding number are covered in Section 3 of [76], but here is the definition. If U_t is a path of unitaries of the form $1 + K_t$ such that $t \mapsto K_t$ is a differentiable map into \mathcal{L}^1 , one can define its winding number

$$w(U_t) = \frac{1}{2\pi i} \int_0^1 \tau \left((1 + K_t)^{-1} \cdot \frac{d}{dt}(K_t) \right) dt.$$

Since $C^1(\mathbb{T}, \mathcal{L}^1)$ is dense in $C^1(\mathbb{T}, \mathcal{K}_{\mathcal{N}})$, the winding number can be extended to all unitaries of the form $1 + K_t$ with $K_t \in \mathcal{K}_{\mathcal{N}}$ (see Proposition 3.2 of [76]).

Definition 1.1.34 For a gap-continuous path of self-adjoint Breuer-Fredholm operators $\{D_t\}$ with invertible endpoints, one defines **spectral flow** by

$$\text{sf}(\{D_t\}) = w(e^{\pi i(\Xi(D_t)+1)}),$$

where w is the winding number described above. ☛

The properties of winding number, and the fact that the spectral flow definition above agrees with the Phillips definition in the case when both can be calculated, can be found in [76]. In order to extend the definition to paths with non-invertible endpoints, Wahl uses the technique explained in the following remark (Remark 1.1.35).

Remark 1.1.35 Suppose $\{D_t\}$ is a path for which D_0 is not invertible, but D_1 is. Since D_0 is Breuer-Fredholm, there is a τ -compact spectral projection P around 0 such that $D_0 + \varepsilon P$ is invertible for some $\varepsilon > 0$. Denote by ρ the straight-line path from $D_0 + \varepsilon P$ to D_0 . Concatenate ρ with our original path $\{D_t\}$; we now have a path with both endpoints invertible, so we can calculate its spectral flow. Since the spectral flow from D_0 to $D_0 + \varepsilon P$ is most likely not 0, we have to adjust our result. All the operators along ρ are bounded perturbations of D_0 , so we can use Phillips' definition for Riesz-continuous paths of unbounded operators (Definition 1.1.30) to calculate the spectral flow along ρ . We can thus adjust our answer and write

$$\text{sf}(\{D_t\}) = \text{sf}(\rho * \{D_t\}) - \text{sf}(\rho),$$

where on the right hand side $\text{sf}(\rho * \{D_t\})$ is calculated using the Wahl definition, and $\text{sf}(\rho)$ is calculated using the Phillips definition. A similar extension and adjustment can be performed if D_1 is not invertible. •

As previously mentioned, there are also standard topologies defined on the set of all unbounded operators with the same (fixed) domain, and the set of operators $\{D + A\}$ where D is a fixed unbounded operator and A is relatively bounded with respect to D . As we will not be using these topologies, we do not define them or go into too many details about the issues involved. Suffice it to say that an overview of the topologies and the relationship between them can be found in [47], Section 2. Moreover, the identity map from each of these spaces into the set of unbounded operators with the gap topology is continuous (Proposition 2.2 of [47]), so Wahl's definition of spectral flow can be used. Wahl, in [76], also tackles the subject of analytic formulas for spectral flow of paths with operators with same domain and compact resolvent (Theorem 6.5); we will also discuss analytic formulas for spectral flow in Section 1.1.3.4, but we will concentrate on the case of bounded perturbations of a fixed operator.

1.1.3.3 Properties of spectral flow

The purpose of this section is to outline some of the basic properties of spectral flow.

Lemma 1.1.36 Regardless of which definition we adopt, spectral flow satisfies the following two properties:

- (a) Spectral flow is additive under concatenation. That is, if ρ and ξ are two paths such that $\rho(1) = \xi(0)$ then $\text{sf}(\rho * \xi) = \text{sf}(\rho) + \text{sf}(\xi)$.
- (b) Spectral flow is invariant under homotopy. That is, if ρ and ξ are homotopic with endpoints fixed then $\text{sf}(\rho) = \text{sf}(\xi)$. 📌

In the case of Phillips' definition (Definition 1.1.26), the first property is obvious since the definition relies on splitting up the path into subpaths and adding up the spectral flows along the subpaths; so, for two concatenated paths, we can choose the point of concatenation to be one of the division points. The second property is the content of Proposition 2.5 of [55]. In the case of

Wahl's definition, the homotopy property follows from the properties of the winding number (see Proposition 3.2 of [76]; the properties of spectral flow are discussed in Section 4 of [76]). In Chapter 2, we will see that these two properties, along with a normalization condition, are sufficient to ensure that a function on a certain set of paths actually calculates the spectral flow.

Lemma 1.1.37 If $\{D_t\}$ is a (continuous) path of invertible operators (either bounded or unbounded) then $\text{sf}(\{D_t\}) = 0$. \blacksquare

PROOF Suppose $\{D_t\}$ is a path which is continuous in the gap topology; in this case, use Wahl's definition of spectral flow. Since the set of invertible operators is open in the gap topology (see [44], for example), we know we can find an n such that the interval $[-1/n, 1/n]$ does not intersect $\sigma(D_t)$ for any t . Choose our normalization function χ such that it is -1 for values $x < -\frac{1}{n}$ and 1 for values of $x > \frac{1}{n}$. Then $e^{\pi i(\chi(D_t)+1)} = 1$ for all t , so $K_t = 0$. It follows that

$$\text{sf}(\{D_t\}) = \frac{1}{2\pi i} \int_0^1 \tau((1 + K_t)^{-1} \cdot \frac{d}{dt}(K_t)) dt = \frac{1}{2\pi i} \int_0^1 \tau(0) dt = 0.$$

The above is sufficient to cover all the kinds of continuity in which we are interested (since the gap topology agrees with the norm topology on the set of bounded operators, and is coarser than the Riesz topology on the set of closed unbounded self-adjoint Breuer-Fredholm operators). \blacksquare

Example 1.1.38 In [54], Phillips showed that if $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and \mathcal{H} is infinite-dimensional, it is possible to build loops whose spectral flow is any integer. We include here his construction of a loop with spectral flow 1.

In order to make the construction explicit, choose $\mathcal{H} = \mathcal{L}^2(\mathbb{T})$ with the usual orthonormal basis $\{h_n = e^{in\theta}\}_{n \in \mathbb{Z}}$ (here, the measure on the circle is assumed to be normalized). Let $P = [\text{span}\{h_0\}]$, $Q = [\text{span}\{h_n\}_{n < 0}]$ and $R = [\text{span}\{h_n\}_{n > 0}]$; so P , Q and R are mutually orthogonal projections, with P rank one and Q and R infinite. Let F_0 and F_1 be the invertible self-adjoint operators which, with respect to the decomposition $P\mathcal{H} \oplus Q\mathcal{H} \oplus R\mathcal{H}$ look like

$$F_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We will define two paths from F_0 to F_1 whose spectral flows are different, allowing us to concatenate them into a loop at F_0 whose spectral flow is non-zero.

Let $f(t) = (1-t)F_0 + tF_1$, the straight line path from F_0 to F_1 . Then $f(1/2)$, with respect to the decomposition $P\mathcal{H} \oplus Q\mathcal{H} \oplus R\mathcal{H}$, looks like

$$f(1/2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so is an operator with one-dimensional kernel. Let $P_t = \chi_{[0, \infty)}(f(t))$. Then $\pi(P_t)$ is constant, so the spectral flow of the path is equal to $\text{ind}(P_0 P_1) = 1$. That is, $\text{sf}(f(t)) = 1$.

On the other hand, if U is the (unitary) operator which takes h_n to h_{n-1} for all $n \in \mathbb{Z}$ (i.e. U is the bilateral shift operator), it is not hard to verify that $F_1 = UF_0U^*$. As the unitary

operators are connected, we can find a path U_t of unitaries such that $U_0 = 1$ and $U_1 = U$. Explicitly, we can think of U as multiplication by the function $f(e^{i\theta}) = e^{-i\theta}$; for each $t \in [0, 1]$ define the functions $f_t(e^{i\theta}) = e^{-it\theta}$ from \mathbb{T} to \mathbb{T} , and let U_t be the unitary operator on $\mathcal{L}^2(\mathbb{T})$ given by multiplication by f_t . It is not hard to check that $\{U_t\}$ is continuous in t , and so defines a path of unitary operators. As a side note, using the Fourier transform we can get that $U_t(h_n) = \sum_{k \geq 0} \frac{(-1)^{n-k+1}}{(n-k-t)\pi} \cdot \sin(t\pi) \cdot h_k$, but this representation is not particularly useful for our purposes. Using our path of unitaries $\{U_t\}$ we can construct $g(t) = U_t F_0 U_t^*$, a new path from F_0 to F_1 . However, all the $g(t)$'s are clearly invertible (since F_0 is invertible), so it follows that $\text{sf}(\{g(t)\}) = 0$ (using Lemma 1.1.37).

Finally, using the additive property of spectral flow, $f(t) * -g(t)$ is a loop at F_0 which has spectral flow $\text{sf}(\{f(t)\}) - \text{sf}(\{g(t)\}) = 1$. The moral of this story is that if \mathcal{N} is not finite, the spectral flow usually depends on the path as well as the endpoints. •

Remark 1.1.39 Let us consider what happens if we join a self-adjoint Breuer-Fredholm operator D to $2\chi_{[0,\infty)}(D) - 1$ via a straight line path. Intuitively, the positive part of the operator is joined to 1, and the negative part is joined to -1; we would expect no spectral flow along this path.

If we are dealing with a case to which Definition 1.1.26 applies, a slightly more general observation is true. Namely, if we have a path $\{D_t\}$ for which $\chi_{[0,\infty)}(D_t)$ is constant, then the spectral flow of $\{D_t\}$ is 0: Let $P_t = \chi_{[0,\infty)}(D_t) =: P$. Then $\pi(P_t) = \pi(P)$ is constant for all t , so the spectral flow is equal to $\text{ind}(P_0 P_1) = \text{ind}(P)$ as an operator from $P\mathcal{H}$ to $P\mathcal{H}$. As P is the identity on $P\mathcal{H}$, $\text{ind}(P) = 0$, so the spectral flow is 0.

Since the positive operators form a convex set, it should be clear that by going from D to $2\chi_{[0,\infty)}(D) - 1$ via a straight line the projection onto the positive part of the spectrum remains unchanged. So the spectral flow is 0, as expected.

One might wonder if a similar result holds for unbounded operators with the gap topology. However, for the straight line $\{D_t\}$ defined as above, we will see that $t \mapsto D_t$ is not in general continuous at 1 in the gap topology (although it is continuous at every $t < 1$). Fix $t \in [0, 1]$ and consider any s in a neighbourhood of t . Denote by U the operator $2P - 1$, recall that we had $D_t = (1-t)D + tU$ (the straight line path from D to U), and note that, since D and U commute, so do the D_t 's. We can thus write

$$\begin{aligned} & (1 + D_t^2)^{-1} - (1 + D_s^2)^{-1} \\ &= (1 + D_t^2)^{-1} [(1 + D_s^2) - (1 + D_t^2)] (1 + D_s^2)^{-1} \\ &= (1 + D_t^2)^{-1} [(D_s - D_t)(D_t + D_s)] (1 + D_s^2)^{-1} \\ &= (1 + D_t^2)^{-1} [(s-t)UD_t + (t-s)D_0] (1 + D_s^2)^{-1} \\ &= (s-t)UD_t(1 + D_t^2)^{-1}(1 + D_s^2)^{-1} + (s-t)(1 + D_t^2)^{-1}UD_s(1 + D_s^2)^{-1} \\ &\quad + (t-s)D_t(1 + D_t^2)^{-1}D_0(1 + D_s^2)^{-1} + (t-s)D_0(1 + D_t^2)^{-1}D_s(1 + D_s^2)^{-1}. \end{aligned}$$

If $s \neq 1$, we can write $D_0 = \frac{1}{1-s}D_s - \frac{s}{1-s}U$. It follows that

$$D_0(1 + D_s^2)^{-1} = \frac{1}{1-s}D_s(1 + D_s^2)^{-1} - \frac{s}{1-s}U(1 + D_s^2)^{-1},$$

and

$$D_0D_s(1 + D_s^2)^{-1} = \frac{1}{1-s}D_s^2(1 + D_s^2)^{-1} - \frac{s}{1-s}UD_s(1 + D_s^2)^{-1},$$

whence $\|D_0(1 + D_s^2)^{-1}\|$ and $\|D_0D_s(1 + D_s^2)^{-1}\|$ are both bounded above by $\frac{1}{1-s} + \frac{s}{1-s}$. Now the functions $\frac{1}{1-s}$ and $\frac{s}{1-s}$ are both undefined at 1 but continuous on $[0, 1)$, so they are uniformly bounded on any closed subinterval of $[0, 1]$ which does not contain 1. This is sufficient to show that $\{D_t\}$ is continuous at any $t < 1$, since $s - t$ can be made arbitrarily small, and the operators which appear in the four terms can be uniformly bounded in norm. On the other hand, if $t = 1$ we have $D_t = D_1 = U$; since $U^2 = 1$ the expression simplifies to

$$\frac{1}{2} - (1 + D_s^2)^{-1} = 1 - (1 + D_s^2)^{-1} - \frac{1}{2} = D_s^2(1 + D_s^2)^{-1} - \frac{1}{2}$$

By the reverse triangle inequality, the norm of the operator on the right-hand side is greater than or equal to $|\|D_s^2(1 + D_s^2)^{-1}\| - \frac{1}{2}|$. If the spectrum of D is unbounded, then $\|D_s^2(1 + D_s^2)^{-1}\| = 1$ for $s \neq 1$, so in particular we cannot have $\frac{1}{2} - (1 + D_s^2)^{-1} \rightarrow 0$ as $s \rightarrow 1$. •

Remark 1.1.40 The sets of essentially positive and essentially negative operators are referred to as the 'trivial components' of \mathcal{BF}_{sa} ; on these sets, spectral flow only depends on the endpoints. To wit, the spectral flow along a constant path is 0, and so the spectral flow of any loop homotopic to a point is 0. Since the trivial components of \mathcal{BF}_{sa} are contractible, it follows that the spectral flow along any two paths with the same endpoints must be the same (as the two can be concatenated to form a loop). •

1.1.3.4 Calculating spectral flow

While the definition using projections promotes an intuitive description of spectral flow, it is not always easy to use; figuring out the projection onto the positive spectrum of an operator is a non-trivial proposition, as is calculating the Breuer-Fredholm index of an operator. As a consequence of this, a certain amount of effort has gone into finding analytical formulas for spectral flow. Further, such analytic formulas connect spectral flow with cohomological concepts; this connection is described in more detail in Section 1.2.

We describe in this section a few of the results in this direction. Our purpose is not at all to have a comprehensive list of such results, but merely to point out recurring features of such theorems. For example, in [15], Carey and Phillips show that if $\{D_t\}$ is a C^1 path of bounded perturbations of D_0 then, under suitable conditions on D_0 ,

$$\text{sf}(\{D_t\}) = \frac{1}{C_q} \int_a^b \tau \left(\frac{d}{dt}(D_t) \cdot e^{-(1+D_t^2)^{1/q}} \right) dt + \gamma_q(D_1) - \gamma_q(D_0).$$

Here, γ_q is a real-valued function defined on $D_0 + \mathcal{N}_{sa}$ and dependent on q (and γ_q evaluates to the same thing for unitarily equivalent operators), C_q is a constant, and the 'suitable conditions' are chosen to ensure that the expression in brackets is in fact trace-class (see Theorem 7.4 in [15] for further details).

We also single out one of the results in [74]. Here $\{D_t\}$ is a path of self-adjoint operators with common domain (C^1 with respect to the usual norm used on the set of operators with common domain). It is additionally required that the D_t 's have compact resolvents (which automatically forces them to be Fredholm operators). For a function ξ satisfying various technical conditions,

in particular ξ is an odd function for which $\xi(x) \rightarrow 1$ as $x \rightarrow \infty$,

$$\text{sf}(\{D_t\}) = \frac{1}{2} \cdot \int_0^1 \text{Tr} \left(\frac{d}{dt}(D_t) \cdot \xi'(D_t) \right) dt + \frac{1}{2} \cdot \tau(2\chi(D_1) - 1 - \xi(D_1)) - \frac{1}{2} \cdot \tau(2\chi(D_0) - 1 - \xi(D_0)).$$

Note that a common feature of these formulas is the use of the trace to calculate the spectral flow; as such, additional summability conditions have to be placed on the operators under consideration. Also, we need our path to be C^1 with respect to an appropriate norm, so $\frac{d}{dt}(D_t)$ can be calculated. We will consider one such formula for spectral flow in Chapter 3. For specific examples, showing both how to apply the definition and the various formulas for spectral flow, we refer to Section 1.4.

1.2 Context

Spectral flow calculations have a place in the grander picture of noncommutative geometry. In the study of the algebraic topology of C^* -algebras, the index of certain operators plays a role, and this in turn can be connected to spectral flow. The main result that we want to get to is the Local Index Theorem; once again, in order to be able to state it, we need to briefly introduce some new concepts. K -theory associates to a C^* -algebra a sequence of groups K_* ; the dual to K -theory is K -homology, K^* . Additionally, de Rham cohomology generalizes to the non-commutative setting as cyclic homology and cohomology. The relationship between K -theory, K -homology, cyclic homology and cohomology is given by the following diagram:

$$\begin{array}{ccc} K_*(\mathfrak{A}) & \xrightarrow{\text{Ch}} & HC_*(\mathfrak{A}) \\ \downarrow & & \downarrow \\ \text{index pairing} & \equiv & \text{duality pairing} \\ \uparrow & & \uparrow \\ K^*(\mathfrak{A}) & \xrightarrow{\text{Ch}} & HC^*(\mathfrak{A}) \end{array}$$

Figure 1.4: Relationship between K -theory/homology and cyclic theory.

The purpose of this section is to explain the above diagram, and to introduce enough background to be able to state the Local Index Theorem, which is connected to the work in Chapter 3. For our purposes, we will assume that our algebra \mathfrak{A} is unital, which will simplify the presentation somewhat.

1.2.1 K -theory and K -homology

The K -theory of a *unital* C^* -algebra is given by a pair of abelian groups $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$. We describe the construction of each of these groups below, but stay away from the deeper aspects of K -theory.

$K_0(\mathfrak{A})$ generalizes the Murray-von Neumann dimension function to non-factors. One way of constructing the group is to start from $\bigcup_{n \in \mathbb{N}} \mathbb{M}_n(\mathfrak{A})$ (that is, matrices with entries in \mathfrak{A} with finitely many non-zero entries), and consider equivalence classes of projections; here, a

projection $p \in \mathbb{M}_n(\mathfrak{A})$ is identified with $(p \oplus 0) \in \mathbb{M}_{n+k}(\mathfrak{A})$ for any $k \in \mathbb{N}$. The resulting object is a semigroup under the operation $[p] + [q] = [p \oplus q]$, and a Grothendieck construction yields a group from it; this is the K_0 group associated to \mathfrak{A} . Note that (we emphasize again that this is the unital case) every element of $K_0(\mathfrak{A})$ can be written as a formal difference $[p] - [q]$, where two elements $[p] - [q]$ and $[p'] - [q']$ are the same if there is some projection r such that $p \oplus q' \oplus r$ is homotopic to $p' \oplus q \oplus r$ (see, for example, [6], the comment after Definition 5.3.1 on p.29).

To get K_1 , we consider homotopy classes of unitaries in $\cup \mathbb{M}_n(\mathfrak{A})$, where $u \in \mathbb{M}_n(\mathfrak{A})$ is identified with $u \oplus 1 \in \mathbb{M}_{n+k}(\mathfrak{A})$ for any $k \in \mathbb{N}$. K-theory can also be defined for non-unital algebras, in which case we make use of the unitization of \mathfrak{A} and construct the K_0 and K_1 groups in this setting; however, the details are a bit more complicated, and as we will concentrate on unital algebras, we refer the interested reader to a resource on K-theory, such as Chapter 3 of [6].

Example 1.2.1 An easy example to derive from the definition is $\mathfrak{A} = \mathbb{C}$. As two finite-rank projections are equivalent if and only if they have the same rank, it should be clear that the constructed semigroup is \mathbb{N} , and $K_0 = \mathbb{Z}$. On the other hand, any two unitaries in $\cup \mathbb{M}_n$ are homotopic (as the unitary group in \mathbb{M}_n for any fixed n is connected), so $K_1 = 0$. Generally speaking, calculations of the K-groups of an algebra are not straight forward, and are more likely to follow from properties of K-theory than from an application of the definition. For example, $K_0(\mathcal{C}(\mathbb{T})) = \mathbb{Z}$ can be deduced from the equality, in this case, with the topological K-theory group of \mathbb{T} (which is defined using complex vector bundles on \mathbb{T} – see [6], Section 5). $K_1(\mathcal{C}(\mathbb{T})) = \mathbb{Z}$ as well, with a 1×1 unitary function $f \in \mathcal{C}(\mathbb{T})$ being mapped to its winding number; by Proposition 8.1.4 of [6], this can be used to construct an isomorphism from K_1 to \mathbb{Z} . In fact, if $f \in \mathcal{C}(\mathbb{T}, \mathbb{M}_n)$ then this isomorphism maps f to the winding number of $\det(f)$. •

One can think of K_0 as being built up from equivalence classes of projections in $\mathfrak{A} \otimes \mathcal{K}(\mathcal{H})$ (see [60], Section 6.4). This idea leads naturally to the semifinite extension of the concept, investigated, for example, in [53]. We will briefly discuss the semifinite versions of K-theory and K-homology at the end of the section; first, we concentrate on the classical definitions and the relationship between them.

There are multiple ways of constructing K-homology; the one discussed below arises from considering equivalence classes of Fredholm modules. Fredholm modules were defined by Atiyah, whose goal was to obtain an abstraction of elliptic operators on a compact manifold.

Definition 1.2.2 ([25], Definition IV.1) A **Fredholm module** over a $*$ -algebra \mathfrak{A} is a pair (\mathcal{H}, F) such that \mathcal{H} is a Hilbert space with a representation of \mathfrak{A} on \mathcal{H} , and F is an operator on \mathcal{H} such that $F = F^*$, $F^2 = 1$ and $[F, a]$ is compact for any $a \in \mathfrak{A}$. A Fredholm module is called **even** if there exists a self-adjoint unitary Γ which anticommutes with F and commutes with all the operators in \mathfrak{A} ; otherwise, it is called **odd**. 🖱

While it might seem that an even spectral triple could be treated as a special case of an odd spectral triple, the existence of the operator Γ forces a specific structure on F and on the operators in \mathfrak{A} , causing the theories for the two kinds of spectral triples to be quite different.

A Fredholm module (\mathcal{H}, F) is called **degenerate** if $[F, a] = 0$ for all $a \in \mathfrak{A}$. The zero element in K-homology is the equivalence class of degenerate modules. The direct sum of two Fredholm modules can be defined in an obvious way, and provides an operation on equivalence classes

of Fredholm modules. With this structure, the equivalence classes of even modules generate a group, denoted K^0 , and similarly the odd modules can be used to construct K^1 .

Note that terminology is not quite standardized when it comes to Fredholm modules. For example in Higson and Roe [41] the requirement that $F^2 = 1$ and $F = F^*$ is relaxed to $F - F^*$ and $1 - F^*F$ both being compact, and modules satisfying the above stricter definition are called 'involutive Fredholm modules'; in other places, the terminology 'normalized Fredholm modules' is used instead. It is nonetheless true that, even starting from the less strict definition of Fredholm module, every equivalence class has an involutive Fredholm module as a representative, and the same K-homology is obtained by restricting to involutive Fredholm modules (Lemma 8.3.5 of [41]). We will occasionally need to consider pairs (\mathcal{H}, F) which are somewhere between our definition of Fredholm modules and the definition used by Higson and Roe; if (\mathcal{H}, F) satisfies all the conditions of a Fredholm module from definition Definition 1.2.2 except the condition that $F^2 = 1$ (note that we still require F to be self-adjoint), we will call (\mathcal{H}, F) a **pre-Fredholm module**. This terminology agrees with the one used in the semifinite case (see e.g. [21]).

Remark 1.2.3 As described in [14], if (\mathcal{H}, F) is a pre-Fredholm module, and $\tilde{F} = 2\chi(F) - 1$, then (\mathcal{H}, \tilde{F}) is a Fredholm module (see Section 1 of [14], discussion immediately following Definition 1.1). The same construction will work in the semifinite case (which is in fact the case considered in the reference given). Moreover, if F has certain desirable properties, so does \tilde{F} (we will address this issue a bit later, as we need some extra definitions). •

Example 1.2.4 As a quick example, we discuss the K-homology of \mathbb{C} . Note that a unital representation of \mathbb{C} into some $\mathcal{B}(\mathcal{H})$ has to take 1 to the identity operator, and hence \mathbb{C} is mapped to the constant multiples of the identity. Any $F \in \mathcal{B}(\mathcal{H})$ which satisfies $F^2 = 1$, $F = F^*$ gives us a Fredholm module (\mathcal{H}, F) for \mathbb{C} ; if, in addition, there is a decomposition of \mathcal{H} with respect to which F looks like $F = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}$, then the module is even. Since all elements of $\mathcal{B}(\mathcal{H})$ commute with the scalar multiples of the identity, all Fredholm modules are degenerate, so the K-homology of \mathbb{C} is trivial. More examples of Fredholm modules will be constructed in Section 1.4. •

As already mentioned, K-theory and K-homology are dual to each other, which means that an element of K-homology induces a homomorphism from K-theory to \mathbb{Z} ; or, phrased another way, one can pair up an element of K-theory with an element of K-homology to get an integer. This is described in detail in Section 8.7 of [41]. It is sufficient to describe how Fredholm modules pair with projections or unitaries in $\mathbb{M}_k(\mathfrak{A})$. To make operators easier to present note that $\mathbb{M}_k(\mathfrak{A})$ acts on $\mathcal{H}^k \cong \mathbb{C}^k \otimes \mathcal{H}$. If (\mathcal{H}, F) is an odd Fredholm module then it pairs with a unitary $U \in \mathbb{M}_k(\mathfrak{A})$. If we let $P = \frac{1+F}{2}$, then P is a projection. Let $P_k = 1 \otimes P$ (where 1 is the identity operator on \mathbb{C}^k); then $P_k U P_k : P_k \mathcal{H}^k \rightarrow P_k \mathcal{H}^k$ is Fredholm, and its index gives the pairing $\langle [U], [(\mathcal{H}, F)] \rangle$. We point out that the pairing described in [41] (for example) is slightly different, as $F^2 = 1$, $F^* = F$ is not assumed as part of the definition of Fredholm module. Note that if $k = 1$ then U is in fact a unitary in \mathfrak{A} , and we are simply calculating $\text{ind}(PUP)$ where, recall, $P = \frac{1+F}{2}$. This index is connected to spectral flow, as described in the beginning of Section 3 of [55]. Namely, if U is such that $[U, F] \in \mathcal{K}(\mathcal{H})$, consider the path $F_t = F + t(UFU^* - F) = F + t[U, F]U^*$. Then each F_t is a compact perturbation of F , so by Definition 1.1.26 the spectral flow of $\{F_t\}$ is $\text{ind}_{(P,Q)}(PQ)$, where $P = \frac{1+F}{2}$ as already established and $Q = \frac{1+UFU^*}{2}$. Since $Q = UPU^*$, we

have that $PQ = PUPU^* = (PUP)(PU^*)$. Using $PU^* = U^*Q$ (and the fact that U is unitary), it is straight forward to show that PU^* is a bijection from $Q\mathcal{H}$ to $P\mathcal{H}$; hence, $\text{ind}_{(P-Q)}(PU^*) = 0$. Finally, since PUP is Fredholm as an operator from $P\mathcal{H}$ to $P\mathcal{H}$, we can use the additivity property of the Fredholm index in a skew corner to write

$$\text{sf}(\{F_t\}) = \text{ind}_{(P-Q)}(PQ) = \text{ind}_{(P-P)}(PUP) + \text{ind}_{(P-Q)}(PU^*) = \text{ind}_{(P-P)}(PUP).$$

Therefore, the pairing between (\mathcal{H}, F) and U (which, recall, is equal to $\text{ind}(PUP)$) is given by the spectral flow along the straight line path from F to UFU^* . Because of the connection to spectral flow, the odd index pairing is the only one that is mentioned in the following; however, for completeness, we also mention the even pairing. So suppose that (\mathcal{H}, F) is an even Fredholm module. The grading operator Γ determines a decomposition of \mathcal{H} into $\mathcal{H}^+ \oplus \mathcal{H}^-$, and with respect to this decomposition we can write $F = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}$, with $UU^* = 1$ on \mathcal{H}^+ and $U^*U = 1$ on \mathcal{H}^- . For $P \in \mathbb{M}_k(\mathfrak{A})$ the pairing $\langle [P], [(\mathcal{H}, F)] \rangle$ is given by the index of the Fredholm operator $P(1 \otimes U^*)P$, as an operator from $P(\mathbb{C}^n \otimes \mathcal{H}^+)$ to $P(\mathbb{C}^n \otimes \mathcal{H}^-)$.

Let us make an observation about the operators F which are part of a Fredholm module; that is, operators for which $F^* = F$ and $F^2 = 1$. One place where such F appear is in the polar decomposition. Suppose D is densely-defined and closed. For such an unbounded operator D we can consider $|D| = (D^*D)^{\frac{1}{2}}$, and obtain a polar decomposition $D = U|D|$, where U is a partial isometry from the closure of the range of D^* to the closure of the range of D . If in addition D is self-adjoint, then \mathcal{H} can be decomposed into three subspaces such that U is -1 on the first, 0 on the second and 1 on the third (corresponding to the parts on which D is negative, zero and positive respectively); see [44] for more details. It follows that when D is invertible, the operator $D|D|^{-1} = U$ is self-adjoint and is equal to the identity when squared. This suggests that one could define an unbounded counterpart to Fredholm modules; especially as, in the study of certain problems, unbounded operators make an appearance in a natural manner (e.g. when considering differential operators on a manifold). We give the definition of unbounded Fredholm modules, and then discuss their relationships to bounded Fredholm modules.

Definition 1.2.5 ([25], Definition IV.2.γ.11) An **unbounded Fredholm module** over a unital $*$ -algebra \mathfrak{A} is a pair (\mathcal{H}, D) such that there exists a unital $*$ -representation of \mathfrak{A} in \mathcal{H} , and D is an (unbounded) self-adjoint operator with compact resolvents such that $[D, a]$ is bounded for any $a \in \mathfrak{A}$. ♣

The only difference between the definition given above and the definition in Connes' book is that Connes calls such an object a K -cycle. It is straight forward to get a bounded Fredholm module from an unbounded one. If D is invertible, then we've already seen we can let $F = D|D|^{-1}$. If D is not invertible, one possible approach is to use the Riesz transform, $F = D(1 + D^2)^{-\frac{1}{2}}$; then (\mathcal{H}, F) is a pre-Fredholm module, from which we can then get a Fredholm module as described in Remark 1.2.3. Alternatively, we can get a Fredholm module by replacing \mathcal{H} by a bigger space and adjusting D so it is invertible on the larger space. One possible way of doing this can be found in [25] (Section IV.2.γ) or, for a slightly different construction, see [16] (Lemma 4).

Our goal, the Local Index Theorem, gives a method for calculating the pairing between K_* (K -theory) and K^* (K -homology) without needing to calculate the index of an operator; instead, the formula is given by residues of (generalized) zeta functions. As part of the proof,

one obtains a representative of the Chern character in cyclic cohomology (see Figure 1.4). The Chern character we are considering is only defined for modules for which F respects additional summability properties. A Fredholm module (\mathcal{H}, F) is called **p -summable** if it satisfies the summability property that $[F, a] \in \mathcal{L}^p$ for $a \in \mathfrak{A}$. Similarly, (\mathcal{H}, D) is said to be a **p -summable unbounded Fredholm module**, if the resolvents of D are in the p^{th} Schatten ideal, instead of being merely compact. Recall that following Definition 1.2.5 we discussed how we can get a bounded Fredholm module from an unbounded one. Optimally, we would like that if (\mathcal{H}, D) has some nice summability properties, then so does the resulting bounded (\mathcal{H}, F) module. This is indeed the case (see Proposition 6.1 in [23] for the p -summable even case, and the later discussion on semifinite modules for the generalization); however, the converse is not always true. For example, the reduced C^* -algebra of the free group on two generators is known to have finitely summable Fredholm modules, but no finitely summable spectral triples (see [24], Theorem 17 and Remark 18). A necessary (but not sufficient) condition for a unital C^* -algebra \mathfrak{A} to have a p -summable unbounded Fredholm module is the existence of a finite trace on \mathfrak{A} . This was proven by Connes in [24], Theorem 8; and it is not hard to see that the proof can be adapted to the semifinite case (we will return to this once we introduce semifinite Fredholm modules). Without going into too many details, we want to point out that if we depart from p -summable modules, then there is a closer correspondence between bounded and unbounded versions. For example, if one replaces \mathcal{L}^p in the definition of p -summable by a different ideal contained in the compact operators, one gets what are called θ -summable modules. We avoid any description of this ideal for now (though we will return to it in Chapter 3); however, we mention the result that if (\mathcal{H}, F) is a θ -summable Fredholm module, one can find an unbounded operator D such that $\text{sign}(D) = F$ and D is θ -summable (see Theorem IV.8.4 in [25]).

The other comment we should make about unbounded Fredholm modules is that they are in practice the same as **spectral triples**. A spectral triple is presented as $(\mathfrak{A}, \mathcal{H}, D)$, where \mathfrak{A} is represented on \mathcal{H} and D is a (usually unbounded) self-adjoint operator. In practice, D often turns out to have compact resolvents; the difference between the two terms is more one of context and meaning. Fredholm modules are used in the description of K -homology (so the algebra \mathfrak{A} is fixed); spectral triples are used when discussing the geometry of a non-commutative space (the algebra and the operator both arise from the underlying space). In the original commutative geometry case, given a space X , one can study the properties of X by examining the commutative C^* -algebra $C(X)$. Noncommutative geometry was born from attempts to extend this theory to topological spaces with relations. The functions on the space are replaced by (not necessarily commutative) algebras. Properties of the algebras are related to the topological properties of the original space, and Riemannian manifolds are replaced by spectral triples $(\mathfrak{A}, \mathcal{H}, D)$. As in the commutative case, one can recover geometric properties of the space from the spectral triple (see Chapter VI of [25]).

For more information about the various equivalent pictures of K -theory and homology, see [41] or [6]. Our next goal is to describe the extensions of some of the concepts described above to the semifinite case.


1.2.1.1 Semifinite K -homology

The idea that concerns us most closely is the introduction of semifinite Fredholm modules, and the related concept of semifinite spectral triples. The definition is a modification of the Fredholm

module one, replacing the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ by a semifinite von Neumann algebra \mathcal{N} with trace τ , and the compact operators of $\mathcal{B}(\mathcal{H})$ by the τ -compact operators in \mathcal{N} ; as a consequence, they are usually denoted 'Breuer-Fredholm modules'.

Definition 1.2.6 ([19], **Definition 3.1**) A **pre-Breuer-Fredholm module** for a unital Banach $*$ -algebra \mathfrak{A} is a pair (\mathcal{N}, F_0) where \mathfrak{A} is represented in the semifinite von Neumann algebra \mathcal{N} via a $*$ -homomorphism π (which can be assumed without loss of generality to be faithful) such that, if τ is a faithful, normal, semifinite trace on \mathcal{N} , then


- $F_0 \in \mathcal{N}$ is self-adjoint and $1 - F_0^2 \in \mathcal{K}_{\mathcal{N}}$, and
- $\{a \in \mathfrak{A} \mid [F_0, a] \in \mathcal{K}_{\mathcal{N}}\}$ is a dense $*$ -subalgebra of \mathfrak{A} .

If, in addition to the above, $F_0^2 = 1$, we refer to (\mathcal{N}, F_0) as a **Breuer-Fredholm module** (i.e. we drop the 'pre-'). 

As in the $\mathcal{B}(\mathcal{H})$ case, additional summability properties on the module allow us to perform certain operations, such as defining a Chern character, which in more general situations either cannot be done or cannot be done in the same way. Using \mathcal{L}^p for the set of p -summable τ -compact operators, we say that (\mathcal{N}, F_0) is **p -summable** if $|1 - F_0^2|^{\frac{1}{2}} \in \mathcal{L}^p$ and we can replace the τ -compact operators in the second condition by \mathcal{L}^p . The **even** and **odd** versions of Breuer-Fredholm modules are defined as in the $\mathcal{B}(\mathcal{H})$ case (see Definition 1.2.2). And, once again, we can define an unbounded version of Fredholm modules.

Definition 1.2.7 ([15], **Definition 2.4**) An **unbounded Breuer-Fredholm module** for \mathfrak{A} is a pair (\mathcal{N}, D_0) where, with \mathfrak{A} and \mathcal{N} as in the previous definition, D_0 is an unbounded self-adjoint operator affiliated with \mathcal{N} such that

- $(1 + D_0^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$, and
- $\{a \in \mathfrak{A} \mid a(\text{Dom } D_0) \subset \text{Dom } D_0 \text{ and } [D_0, a] \in \mathcal{N}\}$ is a dense $*$ -subalgebra of \mathfrak{A} .


If we can replace the first condition by $(1 + D_0^2)^{-\frac{1}{2}} \in \mathcal{L}^p$, we say that (\mathcal{N}, D_0) is a **p -summable unbounded Breuer-Fredholm module**. 

We take a moment to address the seeming differences between the definition above and the definition in the $\mathcal{B}(\mathcal{H})$ case. A discussion of these differences and their overall insignificance will be tackled in Remark 1.2.9, but first some notes on the condition $(1 + D_0^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ or \mathcal{L}^p (as appropriate).

Proposition 1.2.8 Suppose that D is an unbounded self-adjoint operator affiliated with \mathcal{N} .

(a) The following are equivalent:

- (i) $(1 + D^2)^{-1}$ is τ -compact,
- (ii) $(i - D)^{-1}$ is τ -compact,
- (iii) D has τ -compact resolvents.

(b) If \mathcal{I} is a two-sided ideal of \mathcal{N} then $(1 + D^2)^{-\frac{1}{2}} \in \mathcal{I}$ if and only if $(i - D)^{-1} \in \mathcal{I}$. 

PROOF (a) The crux of the proof is the fact that the τ -compact operators form an ideal. We will show that (i) \Leftrightarrow (ii), and then discuss (ii) \Leftrightarrow (iii).

Show that $(1 + D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ implies $(i - D)^{-1} \in \mathcal{K}_{\mathcal{N}}$. To start with, recall that an operator A is τ -compact if and only if A is bounded and $\mu_t(A) \rightarrow 0$ as $t \rightarrow \infty$. Using the fact that $f(x) = x^{\frac{1}{2}}$ is a continuous increasing function on $[0, \infty)$ with $f(0) = 0$ and the properties of s -numbers (Theorem 1.1.5, part (iii)), we can conclude that if A is a positive operator then $\mu_t(A^{1/2}) \rightarrow 0$ as $t \rightarrow \infty$, and so $A^{1/2}$ is also τ -compact. In particular, for our operators, $(1 + D^2)^{-\frac{1}{2}} \in \mathcal{K}_{\mathcal{N}}$. Finally, we have

$$(i - D)^{-1} = \underbrace{-i(1 + D^2)^{-1}}_{\tau\text{-compact}} + \underbrace{[D(1 + D^2)^{-\frac{1}{2}}]}_{\text{bounded}} \cdot \underbrace{(1 + D^2)^{-\frac{1}{2}}}_{\tau\text{-compact}},$$

whence $(i - D)^{-1}$ is τ -compact.

Next, show $(i - D)^{-1} \in \mathcal{K}_{\mathcal{N}}$ implies $(1 + D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$. As $(1 + D^2)^{-1} = -(i - D)^{-1} \cdot (i + D)^{-1}$, this is quite easy to see.

Finally, note that if D has τ -compact resolvents, then in particular $(i - D)^{-1}$ must be τ -compact. Conversely, if an operator has one resolvent which is τ -compact then all its resolvents are τ -compact, using a resolvent expansion. For example, if $(D - i)^{-1}$ is τ -compact, then for any $\lambda \notin \sigma(D_0)$ we have

$$(D - \lambda)^{-1} = \underbrace{(D - i)^{-1}}_{\tau\text{-compact}} \cdot \underbrace{[1 + (\lambda - i)(D - \lambda)^{-1}]}_{\text{bounded}},$$

and so $(D - \lambda)^{-1}$ is also τ -compact.

(b) Suppose first that $(1 + D^2)^{-\frac{1}{2}} \in \mathcal{S}$. Then $(1 + D^2)^{-1} = (1 + D^2)^{-\frac{1}{2}} \cdot (1 + D^2)^{-\frac{1}{2}}$ is also in \mathcal{S} . Using the same expansion for $(i - D)^{-1}$ as above, except now $(1 + D^2)^{-1}$ and $(1 + D^2)^{-\frac{1}{2}}$ are both in \mathcal{S} instead of being merely τ -compact, we can conclude that $(i - D)^{-1} \in \mathcal{S}$.

Conversely, suppose that $(i - D)^{-1} \in \mathcal{S}$. Then, using the resolvent formula for a fixed unbounded operator (as in the previous part of the proof), we can write $(i + D)^{-1}$ in terms of $(i - D)^{-1}$ and conclude $(i + D)^{-1} \in \mathcal{S}$ as well. It follows that $(1 + D^2)^{-1} = -(i + D)^{-1}(i - D)^{-1} \in \mathcal{S}^2$, whence $(1 + D^2)^{-\frac{1}{2}} \in \mathcal{S}$. \blacksquare

Remark 1.2.9 In [17] a **semifinite spectral triple** $(\mathfrak{A}, \mathcal{H}, D)$ is said to consist of a $*$ -algebra $\mathfrak{A} \subset \mathcal{N}$ and an unbounded self-adjoint operator D affiliated with \mathcal{N} such that $(\lambda - D)^{-1}$ is τ -compact for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $[D, a]$ is densely defined and extends to an operator in \mathcal{N} for all $a \in \mathfrak{A}$ (Definition 2.1 of [17]). This might look closer to the $\mathcal{B}(\mathcal{H})$ definition (compare to Definition 1.2.5) than the definition of unbounded Breuer-Fredholm module stated above, but on closer examination the differences turn out to be superficial. To start with, $(1 + D_0^2)^{-1}$ is τ -compact if and only if $(i - D_0)^{-1}$ is τ -compact, which in turn is equivalent to the condition that D_0 has τ -compact resolvents (see Proposition 1.2.8). We further note that the two conditions given for an unbounded (Breuer-)Fredholm module to be p -summable ($(i - D_0)^{-1} \in \mathcal{L}^p$ versus

$(1 + D_0^2)^{-\frac{1}{2}} \in \mathcal{L}^p$) are equivalent (also a consequence of Proposition 1.2.8, since \mathcal{L}^p is a two-sided ideal).

The other difference seems to be that in the definition of unbounded Breuer-Fredholm modules given in Definition 1.2.7, the second condition ($[D_0, a]$ extends to a bounded operator in \mathcal{N}) is required to hold for a is a dense $*$ -subalgebra of \mathfrak{A} instead of for all $a \in \mathfrak{A}$. However, this is because in this definition \mathfrak{A} is required to be complete, whereas the semifinite spectral triples definition just requires \mathfrak{A} to be a $*$ -algebra. Given a semifinite spectral triple $(\mathfrak{A}, \mathcal{H}, D)$ we can get an unbounded Breuer-Fredholm module by considering $(\overline{\mathfrak{A}}, D)$, where $\overline{\mathfrak{A}}$ is the closure of \mathfrak{A} . Then the second condition holds on \mathfrak{A} , which is of course a dense $*$ -subalgebra of $\overline{\mathfrak{A}}$. So to all intents and purposes the definition given is the same as Definition 1.2.5, except with $\mathcal{B}(\mathcal{H})$ and the compact operators $\mathcal{K}(\mathcal{H})$ replaced by a von Neumann algebra \mathcal{N} and its τ -compact operators. •

Recall that when we discussed p -summable Fredholm modules we referred to a result of Connes, which states that the existence of a p -summable Fredholm module for a unital C^* -algebra implies the existence of a finite trace on the algebra ([24]). It is not hard to see that one can replace $\mathcal{B}(\mathcal{H})$ and the trace Tr by a von Neumann algebra \mathcal{N} with faithful, normal, semifinite trace τ , and reach the same conclusion in the semifinite case. The proof relies on the fact that, if (\mathcal{N}, D) is an unbounded Breuer-Fredholm module, then so is (\mathcal{N}, kD) for any constant $k > 0$. The idea is to fix an integer $n > p$ and define a family of operators $D_\varepsilon = \frac{1}{\tau((1+\varepsilon D^2)^{-n})} \cdot (1 + \varepsilon D^2)^{-n}$. Then by design $\tau(D_\varepsilon) = 1$, and one can show that, for any $T \in \mathfrak{A}$, $|\tau(TD_\varepsilon - D_\varepsilon T)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. If one then defines a family of states on \mathcal{N} via $\varphi_\varepsilon(T) = \tau(TD_\varepsilon)$ and denotes by φ a weak- $*$ limit point of the family, it can be shown that $\varphi(1) = 1$ and φ is a trace on \mathfrak{A} , concluding the proof.

It should also be mentioned that the relationship between unbounded Breuer-Fredholm modules and bounded Breuer-Fredholm modules is similar to the one described for Fredholm modules. Namely, one can obtain bounded modules from unbounded ones. Moreover, the fact that if (\mathcal{N}, D_0) is an odd unbounded p -summable Breuer-Fredholm module and $1 < p < \infty$ then $(\mathcal{N}, 2\chi(D_0) - 1)$ is an odd bounded p -summable Breuer-Fredholm module is one of the results of [21] (Corollary 0.4). If $p = 1$, the relationship is more complicated; it is currently known that, if (\mathcal{N}, D_0) is 1-summable then $(\mathcal{N}, 2\chi(D_0) - 1)$ is $1 + \varepsilon$ summable for any $\varepsilon > 0$ (see for example [15], which implies this result for $p \geq 1$). In fact, as shown in [20], part (2) of the Remark on p. 105, an odd unbounded 1-summable module has trivial pairing with K_1 , which seems to imply that there is no bounded 1-summable module associated to it. If one had an example of an odd bounded 1-summable Fredholm module (\mathcal{H}, F) which has non-trivial pairing with K -theory, then F cannot be equal to $2\chi(D_0) - 1$ for any (\mathcal{N}, D_0) an unbounded 1-summable Breuer-Fredholm module.

Unbounded Breuer-Fredholm modules pair up with K -theory classes. Suppose D is an unbounded, densely-defined self-adjoint Breuer-Fredholm operator on \mathcal{H} and $u \in \mathcal{N}$ is such that $[D, u]$ is bounded. If P denotes the projection onto the non-negative spectral subspace of D then the Breuer-Fredholm index $\text{ind}(PuP)$ is equal to the spectral flow along the straight line path from D to $uD u^*$ (Theorem 2.17 of [14]).

Semifinite Fredholm modules seem to occur naturally in some applications; for example, in the investigation of spectral triples for graph C^* -algebras (see [50]), which are however beyond

the scope of this thesis. We mention instead a construction of Connes and Cuntz which requires semifinite Fredholm modules, and which sheds more light on the connection between cyclic cohomology and K-homology. We discuss cyclic cohomology in the next section, and later we will show how to obtain an element of cyclic cohomology from a (semifinite) Fredholm module. Article [26] shows how to partially reverse the process; more precisely, Connes and Cuntz find a description of cyclic cohomology that is based on traces of certain ideals of the free product $\mathfrak{A} * \mathfrak{A}$ (Corollaire 7), and define positivity and parity for such traces. They then use a positive trace on a specific ideal to construct an even summable semifinite Fredholm module (Théorème 15).

Of course, semifinite Fredholm modules are not Fredholm modules in the classical sense. The extent of the machinery available for classical Fredholm modules does not seem to be in place for the semifinite case, in that the group of equivalence classes of semifinite Fredholm modules (which could presumably be defined similarly to the classical theory) has not been systematically investigated, and does not have all the diverse descriptions available for K-homology. In the following, when we talk about the K-homology class of a semifinite spectral triple, we mean the equivalence class obtained if one applies the classical construction of K-homology to the set of semifinite Breuer-Fredholm modules. Note that we can still pair semifinite Fredholm modules with classes in K-theory. Here, the K-theory being used is the classical K-theory. This is worth mentioning because there is some development of K-theory relative to a semifinite factor. To obtain this theory, one looks at equivalence classes of projections in $\mathfrak{A} \otimes \mathcal{K}_{\mathcal{N}}$ (see [53]).

This thesis contains no discussion of what kind of theory one could get from semifinite spectral triples, but this seems like a good place to indicate possible avenues of further investigation. The original approach to K-homology is due to Brown, Douglas and Fillmore; $\text{Ext}(\mathfrak{A})$ is the set of *-homomorphisms $\varphi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ up to unitary equivalence. Under certain assumptions on \mathfrak{A} , this is a group; but if it is not, the group is taken to consist of the invertible elements of Ext . The relationship between the Ext picture of K-homology and the Fredholm operator picture we chose to go with can be found in, for example, Sections 8.4, and 5.1-5.2 of [41] or Section 17.6.4 of [6]. We mention this because there are results regarding the extension group obtained by considering *-homomorphisms into $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$ when \mathcal{N} is a II_{∞} factor. The reader interested in exploring this avenue of thought is directed to [36] and [66] as good starting places – the former describes the construction of Ext in the context of a II_{∞} factor, and the latter explores how this construction fits in with KK -theory. KK -theory is a generalization of Ext due to Kasparov; we do not attempt to explain it here (see, for example, [6]), though it does have a further connection with some of the ideas described earlier. As proven in [42], a semifinite spectral triple represents a class in $KK^1(\mathfrak{A}, \mathcal{I})$, where \mathcal{I} is the norm-closed ideal of compact operators in \mathcal{N} generated by the resolvent of D and the commutators $[F, a]$ (where $F = D(1 + D^2)^{-\frac{1}{2}}$). However, as the ideal \mathcal{I} depends on the operator D , this does not address the question of how different semifinite spectral triples interact with each other.

1.2.2 Cyclic homology and cohomology

The reason for cyclic homology and cohomology is that one can use tools from algebraic topology to handle them. The main references for this section are [48] and [25]. Note that the cyclic (co)homology of a C^* -algebra is trivial; one way of obtaining useful information is to work with a smooth subalgebra. In the case when the algebra is part of a spectral triple $(\mathfrak{A}, \mathcal{H}, D)$, one

uses the derivation to construct this subalgebra – for a brief introduction to this, see Definition 2.2 and Section 3 of [19]; more details can be found in [58].

Define $C_n^\lambda = \mathfrak{A} \otimes \overline{\mathfrak{A}}^{\otimes n} / 1-t$, where $\overline{\mathfrak{A}} = \mathfrak{A}/(\mathbb{C} \times 1)$ and

$$t(a_0 \otimes a_1 \otimes \dots \otimes a_n) = (-1)^n (a_1 \otimes \dots \otimes a_n \otimes a_0).$$

Note that, if \mathfrak{A} has a topology on it, then in most cases the projective tensor product leads to a suitable choice of topology on C_n^λ . The operator $b : C_n^\lambda \rightarrow C_{n-1}^\lambda$ is defined by

$$b(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^k (a_0 \otimes \dots \otimes a_k a_{k+1} \otimes \dots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}).$$

Then $\dots \xrightarrow{b} C_n^\lambda \xrightarrow{b} \dots \xrightarrow{b} C_0^\lambda$ is a complex, and one can define its n^{th} homology group $H_n^\lambda(\mathfrak{A}) = \frac{\ker b}{\text{im } b}$. To facilitate calculations, it is easier to obtain the cyclic homology from a two-dimensional complex called the (B, b) -bicomplex. The map $B : C_n^\lambda \rightarrow C_{n+1}^\lambda$ is defined by

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^n (-1)^{ni} (1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1}).$$

Then one obtains the bicomplex

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \dots \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A} \otimes \overline{\mathfrak{A}}^{\otimes 2} & \xleftarrow{B} & \mathfrak{A} \otimes \overline{\mathfrak{A}} & \xleftarrow{B} & \mathfrak{A} \\ \downarrow b & & \downarrow b & & \\ \mathfrak{A} \otimes \overline{\mathfrak{A}} & \xleftarrow{B} & \mathfrak{A} & & \\ \downarrow b & & & & \\ \mathfrak{A} & & & & \end{array}$$

One can verify that $b^2 = B^2 = bB + Bb = 0$, as befits a bicomplex, so one can define its homology groups as described in [48] (one considers the complex with modules given by direct sums of the off-diagonals of the above complex and module map $b + B$). It follows that a sequence $\{c_n\}_{n \in \mathbb{N}}$ with $c_n \in C_n^\lambda$ is called a **cycle** if $bc_{n+1} + Bc_{n-1} = 0$, and a **boundary** if there exists some chain $\{e_n\}_{n \in \mathbb{N}}$ for which $c_n = be_{n+1} + Be_{n-1}$. The quotient of cycles by boundaries gives the homology $HC(\mathfrak{A})$. Corollary 2.1.10 of [48] tells us that the resulting groups are isomorphic to the cyclic homology.

For cyclic cohomology, consider finite sequences of n -multilinear functionals; in other words, $C^m = \text{Hom}(C_m, \mathbb{C})$. In the cohomology case, it is easier to accommodate a topology on \mathfrak{A} , as we can simply require our functionals to be continuous. The **cyclic cochains** are those $\varphi \in C^m$ which satisfy

$$\varphi(a_0 \otimes \dots \otimes a_m) = (-1)^m \varphi(a_m \otimes a_0 \otimes \dots \otimes a_{m-1}).$$

Define the boundary operator b from n -multilinear forms to $(n + 1)$ -multilinear forms by

$$(b\varphi)(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{j=0}^n (-1)^j \varphi(a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0 \otimes a_1 \otimes \dots \otimes a_n).$$

The m^{th} cyclic cohomology group is $H_\lambda^m(\mathfrak{A})$, the m^{th} cohomology group of the chain complex (C_λ^m, b) . Once again, there is an easier way of obtaining this group from a two dimensional complex instead. Define B from $(n + 2)$ -multilinear forms to $(n + 1)$ -multilinear forms by

$$(B\varphi)(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{j=0}^{n+1} (-1)^{(n-1)j} \varphi(1 \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_{n+1} \otimes a_0 \otimes \dots \otimes a_{j-1})$$

Note that b and B satisfy $b^2 = B^2 = bB + Bb = 0$. A finite sequence of multilinear functionals $\{\varphi_n\}_{n=0}^M$ (where n takes only odd values or only even values) is called a **$(b-B)$ -cocycle** if $b\varphi_n + B\varphi_{n+2} = 0$ for all n . A finite sequence $\{\psi_n\}$ is a **coboundary** if $\psi_n = b\rho_{n-1} + B\rho_{n+1}$ for some sequence $\{\rho_n\}$. The cyclic cohomology of \mathfrak{A} is defined to be $HC^{ev}(\mathfrak{A}) = \frac{\text{even } (b-B) \text{ cocycles}}{\text{even } (b-B) \text{ coboundaries}}$, and similarly for odd.

There is a periodicity operator $S : H_\lambda^m(\mathfrak{A}) \rightarrow H_\lambda^{m+2}(\mathfrak{A})$ defined by $S = n(n + 1)bB^{-1}$, in the following sense. Given $\varphi \in Z_\lambda^{n-1}(\mathfrak{A})$, one has $\varphi \in \text{im } B$, so there exists a Ψ such that $\varphi = B\Psi$. But then $b\Psi \in \frac{\ker b \cap \ker B}{b(\ker B)}$, and it can be shown that $\frac{\ker b \cap \ker B}{b(\ker B)} = \frac{\text{im } B \cap \ker b}{b(\text{im } B)}$ (Lemma 25 on p. 202 of [25]), so $b\Psi$ is an element of $HC^{n+1}(\mathfrak{A})$. One then defines $S\varphi = n(n + 1)b\Psi$. The periodic cyclic cohomology of \mathfrak{A} is the direct limit $H_{\text{per}}^*(\mathfrak{A}) = \varinjlim (H_\lambda^m(\mathfrak{A}), S)$, where m has the same parity as $*$.

As cohomology is dual to homology, cochains in cyclic cohomology are by definition linear functionals defined on chains in cyclic homology. The pairing between homology and cohomology is thus given by evaluation, $\langle \{\varphi_n\}, \{c_n\} \rangle = \sum \varphi_n(c_n)$ for $\{\varphi_n\}$ a cochain and $\{c_n\}$ a chain.

While on the cyclic theory side we have the usual tools of algebraic topology available, calculating the cyclic theory groups is not always particularly easy. Our involvement with cyclic homology and cohomology consists mainly of exhibiting individual elements which are cocycles (via the Chern character), and finding other elements which are in the same cyclic cohomology class.

Before moving away from the subject, we should mention that there are other flavours of cyclic homology and cohomology. We have, here and there, brought up θ -summable Fredholm modules; the Chern character construction we present in the next section only works for p -summable modules, and a modified theory named entire cyclic homology was developed to allow the construction of a Chern character for θ -summable modules. Just to give a feel of how the above theory can be generalized, we note that the elements of (even) entire cyclic cohomology are infinite sequences of maps $\{\varphi_{2n}\}_{n \in \mathbb{N}}$, with $\varphi_{2n} \in C_\lambda^{2n}$, with the additional restriction that the radius of convergence of the series $\sum \|\varphi_{2n}\| \frac{z^n}{n!}$ is infinite. Further details about entire cyclic cohomology can be found in [25].

1.2.3 Local Index Theorem

As already mentioned, using cyclic theory to calculate the pairing between K_* and K^* requires extra summability conditions put on the Fredholm module. For a p -summable Fredholm module, the Chern character is given by a collection of maps defined for integers $n \geq p$ by

$$\omega_n(a_0 \otimes \dots \otimes a_n) = \tau(\Lambda a_0[F, a_1] \dots [F, a_n])$$

where $\Lambda = \Gamma$ if the module is even and $\Lambda = 1$ if the module is odd. Then ω_n is a cyclic cocycle, and so corresponds to an element of $HC^*(\mathfrak{A})$ ([25], p. 292); moreover, $\omega_{n+2} = S\omega_n$ (Theorem 4.1 of [23]), so the class for different n 's is in fact the same. It should be clear why p -summable is required: the expression needs to be trace-class. As an aside, we mention here that there is a Chern character for θ -summable modules ([25], Section IV.8), and even for general modules with no summability assumptions ([49]); however, they live in modified versions of cyclic cohomology, and they consist of infinite collections of maps on \mathfrak{A} .

We next introduce the Chern character from K-theory to cyclic homology. The Chern character of a unitary is given by a sequence of chains

$$\text{Ch}_{2k+1}(u) = (-1)^k \cdot k! \cdot u^* \otimes u \otimes u^* \otimes \dots \otimes u$$


(where there are $2k + 2$ entries in the tensor product). The Chern character of a projection is given by $\text{Ch}_0(p) = p$ and

$$\text{Ch}_{2k}(p) = (-1)^k \frac{(2k)!}{2 \cdot k!} \cdot (2p - 1) \otimes p^{\otimes 2k}.$$

As already mentioned, the Chern character of a module (a representative of a K-homology class) pairs up with the Chern character of a projection or unitary (a representative of a K-theory class) by evaluation.

The Connes-Moscovici theorem gives a formula for figuring out the value of the index pairing from local data [22]; that is, the goal is to obtain a formula which contains residues of zeta-functions of D . A semifinite version was proven by Carey, Phillips, Rennie and Sukochev in [17]; we state this latter result below.

In order to be able to state the Local Index Theorem, we need just a bit more terminology. One defines a partial derivation δ on \mathcal{N} by $\delta(a) = [|D|, a]$; the domain of δ is understood to be all the elements of \mathfrak{A} for which $\delta(a)$ can be extended to a bounded operator. The concept of C^∞ is denoted QC^∞ in the non-commutative setting, to avoid confusion, and is defined below.

Definition 1.2.10 We say that the algebra \mathfrak{A} is QC^k if for all $a \in \mathfrak{A}$, both a and $[D, a]$ are in the domain of δ^k . If \mathfrak{A} is QC^k for all integers $k \geq 1$, say that \mathfrak{A} is QC^∞ . 

In order to make sense of various expressions which appear in the proof, we need summability conditions on D .

Definition 1.2.11 Given a spectral triple $(\mathfrak{A}, \mathcal{H}, D)$, say that it has **spectral dimension** p if

$$p = \inf \{r : \tau((1 + D^2)^{-\frac{r}{2}}) < \infty\}.$$

We now state part of the Local Index Theorem, though we follow it up with more definitions and a quick summary of the rest of the theorem.

Theorem 1.2.12 ([17], **Theorem 4.2 part (1)**) If $(\mathfrak{A}, \mathcal{H}, D)$ is an odd finitely summable QC^∞ semifinite spectral triple of dimension p , $N = \lfloor \frac{p}{2} \rfloor + 1$ and $u \in \mathfrak{A}$ is unitary, then

$$\text{ind}(PuP) = \frac{1}{\sqrt{2\pi i}} \text{res}_{r=\frac{1-p}{2}} \sum_{m=1}^{2N-1} \varphi_m^r(\text{Ch}_m(u))$$

where

$$\begin{aligned} \varphi_m^r(a_0, a_1, \dots, a_m) \\ = \frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \tau \left(\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-1} a_0 R_s(\lambda) [D, a_1] R_s(\lambda) \dots [D, a_m] R_s(\lambda) d\lambda \right) ds \end{aligned}$$

for l a vertical line in \mathbb{C} with real part between 0 and $\frac{1}{2}$ and $R_s(\lambda) = (\lambda - (1 + D^2 + s^2))^{-1}$. Note that the $2N - 1$ appearing in the summation formula is equal to p when p is odd, and to $p + 1$ when p is even. \blacktriangleright

The $\{\varphi_m^r\}$ are shown to form almost a $(b-B)$ -cocycle, modulo holomorphic functions. Additional assumptions allow one to delve further into the residue calculation. As the expressions are quite involved, we introduce a few space-saving notations. For a multi-index $k = (k_1, \dots, k_m)$ (whose size m depends on context) write $|k| = k_1 + k_2 + \dots + k_m$. For $T \in \mathcal{N}$, write $T^{(n)}$ for the iterated commutator with D^2 n times, i.e. $T^{(1)} = [D^2, T]$, $T^{(2)} = [D^2, [D^2, T]]$, and so on. The notation $T^{(0)}$ will be interpreted as the operator T .

Definition 1.2.13 Say that a finitely summable QC^∞ spectral triple $(\mathfrak{A}, \mathcal{H}, D)$ with spectral dimension p has **isolated spectral dimension** if, for any fixed $m \geq 0$, m -multi-index k , and collection of elements a_0, a_1, \dots, a_m of \mathfrak{A} the zeta function

$$\zeta \left(z - \frac{1-p}{2} \right) = \tau \left(a_0 \cdot [D, a_1]^{(k_1)} \dots [D, a_m]^{(k_m)} \cdot (1 + D^2)^{-z + \frac{1-p}{2} - (\frac{m}{2} + |k|)} \right)$$

has an analytic continuation to a deleted neighbourhood of $z = \frac{1-p}{2}$. \blacktriangleright

The isolated spectral dimension assumption is needed to be able to re-write the above formula as a sum of residues, and identify a $(b-B)$ -cocycle $\{\varphi_m\}$ which represents the Chern character of the spectral triple. If, as before, p is the spectral dimension of the triple, and $N = \lfloor \frac{p}{2} \rfloor + 1$, one can let

$$R_j(a_0, \dots, a_m) = \text{res}_{z=\frac{1-p}{2}} \left(z - \frac{1-p}{2} \right)^j \cdot \tau \left(a_0 [D, a_1]^{(k_1)} \dots [D, a_m]^{(k_m)} (1 + D^2)^{-\left(|k| + \frac{m}{2} + z - \frac{1-p}{2}\right)} \right).$$

For an m -tuple $k = (k_1, \dots, k_m)$ let $\alpha(k) = [k_1! k_2! \dots k_m! (k_1 + 1)(k_1 + k_2 + 2) \dots (|k| + m)]^{-1}$ and denote by $\sigma_{n,j}$ the coefficient of z^j in the product $\prod_{j=0}^{n-1} (z + j + \frac{1}{2})$; then one can define the function

$$\varphi_m(a_0 \otimes \dots \otimes a_m) = \sqrt{2\pi i} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|} \cdot \alpha(k) \cdot \sum_{j=0}^{|k| + \frac{m-1}{2}} \sigma_{(|k| + \frac{m-1}{2}), j} \cdot R_j(a_0, \dots, a_m).$$

As stated in [17], Theorem 4.2, part (3), with the additional isolated spectral dimension assumption, $(\varphi_m)_{m=1, \text{odd}}^{2N-1}$ is a $(b-B)$ -cocycle for \mathfrak{A} , and

$$\text{ind}(PuP) = \frac{1}{\sqrt{2\pi i}} \cdot \sum_{m=1}^{2N-1} \varphi_m(\text{Ch}_m(u)).$$

We very briefly outline the proof of the Local Index Formula in Chapter 3. In practice, for a spectral triple in a specific situation, one massages the above formula to get something more manageable; see, for example, [50] for an application of the theorem to a 1-dimensional module. An example which uses the Local Index Formula is included in Section 1.4.

1.3 Summary of Results

As already mentioned, our setting is a von Neumann algebra \mathcal{N} equipped with a faithful, normal, semifinite trace τ , and our main concern is calculating spectral flow for paths of self-adjoint Breuer-Fredholm operators (though in Chapter 2, we have occasion to use the Cayley picture and discuss spectral flow across -1 for certain paths of unitary operators). We weave three main threads: recognizing when a map calculates spectral flow, obtaining spectral flow as the result of integral formulas, and interpreting spectral flow geometrically.

In Chapter 2, we give an axiomatic description of spectral flow in the case when \mathcal{N} is a type II factor. Our goal is to claim that a map satisfying certain properties must calculate the spectral flow; however, special care needs to be taken when dealing with unbounded operators. For bounded operators, different proofs are needed in the type II_1 case (Theorem 2.2.2) and the type II_∞ case (Theorem 2.3.6), but the list of properties required is the same, so we reproduce it below to capture the flavour of these results:

Theorem Let \mathcal{N} be a type II factor and denote by $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ the set of paths of bounded self-adjoint Breuer-Fredholm operators with invertible endpoints. Suppose we have a map $\mu : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times) \rightarrow \mathbb{R}$ which satisfies the following properties:

- (concatenation) For $\rho, \xi \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ with $\rho(1) = \xi(0)$,

$$\mu(\rho * \xi) = \mu(\rho) + \mu(\xi).$$

- (homotopy) For ρ and ξ homotopic paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ (with endpoints not necessarily fixed, but remaining in the invertible operators),

$$\mu(\rho) = \mu(\xi).$$

- (normalization) There exists a finite trace projection P_0 such that, for any projection $Q_0 \leq P_0$ and any projection $R \leq 1 - Q_0$,

$$\mu(\{tQ_0 + (2R - 1 + Q_0)\}_{t \in [-1, 1]}) = \tau(Q_0).$$

Then $\mu = \text{sf}$.



In order to deal with gap-continuous paths of unbounded operators, we apply the Cayley transform to obtain paths of unitaries. The normalization property is then stated for unitary operators, but is nonetheless similar – see Section 2.4, Theorem 2.4.1.

In Chapters 3 and 4, we concentrate on bounded perturbations of a fixed unbounded operator. In Chapter 3, we present a proof of the p-summable integral formula: if (\mathcal{H}, D) is a p-summable unbounded Breuer-Fredholm module, and $\{D_t\} \subset D + \mathcal{N}_{sa}$ is a C^1 path with unitarily equivalent endpoints, then

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_{p/2}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{p}{2}} \right) dt.$$

There are three main steps to this proof: first show that $\alpha_T(X) = \tau(X(1 + D_t^2)^{-\frac{q}{2}})$ is a closed one-form for q much larger than p , second that the formula holds if p is replaced by any such q which is large enough, and as the third and final step, use analytic continuation to conclude that the formula works at $q = p$. The second step especially requires us to evolve a formula that works for a certain set of bounded operators, but we leave the details to its respective chapter. We state the results which allow us to implement each of these steps in the slightly generalized form in which they are proved; these results are then applied towards the desired formula in the proof of Theorem 3.7.9.

Theorem (Theorem 3.4.1) If T_0 is self-adjoint (bounded or unbounded, depending on circumstances), $\mathcal{S} \subset \mathcal{N}_{sa}$ is a Banach space such that $\|X\|_{\mathcal{S}} \geq \|X\|$ for $X \in \mathcal{S}$ and g is a function which satisfies a number of restrictions (see the statement of the actual theorem for details), then $\alpha_T(X) = \text{const} \cdot \tau(Xg(T)^q)$ is a closed one-form on $T_0 + \mathcal{S}$. \blacktriangledown

Theorem (Corollary 3.5.18) Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ such that $(1 + D_0^2)^{-1} \in \mathcal{I}$ for some small power invariant (see Remark 3.3.1 for definition) operator ideal \mathcal{I} and that k is a continuous function on $\mathbb{R} \setminus \{0\}$ which is non-zero on $(0, 1]$ and for which $\lim_{x \rightarrow 0} \frac{k(x)}{x^{3/2}} = 0$. We can define

$$h(x) = \begin{cases} x^{-\frac{3}{2}}k(x) & \text{for } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which is in turn continuous on \mathbb{R} and non-zero on $(0, 1]$. Let $F_0 = D_0(1 + D_0^2)^{-\frac{1}{2}}$, $\mathcal{I} = \mathcal{I}^{1-\varepsilon}$, and suppose that $h(T)$ is trace class for all $T \in \mathcal{I}_{sa}$ and $X \mapsto \tau(Xh(1 - F^2))$ is an exact one-form on $F_0 + \mathcal{I}_{F_0}$. Then

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t) \cdot k((1 + D_t^2)^{-1}) \right) dt + \beta(D_0) - \beta(D_1),$$

where \tilde{C} is a constant and β is a map from $D_0 + \mathcal{N}_{sa} \rightarrow \mathbb{R}$. \blacktriangledown

If the endpoints of the path are unitarily equivalent, then the β correction terms cancel. In specific cases (when k is the power of a function) we can use analytic continuation to improve the formula, as indicated by the following result.

Theorem (Corollary 3.6.7) Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ for which the equality

$$C(m) \cdot \text{sf}(\{D_t\}) = \int_0^1 \tau \left(\frac{d}{dt}(D_t)g(D_t)^m \right) dt$$

holds for all $m \geq N$. Let $q_0 = \inf\{m : \tau(g(D_t)^m) < \infty\}$. Suppose that $C(m)$ has an analytic continuation $C(z)$ to the half-plane $\{\Re(z) > q_0\}$. Suppose further that

- $g : \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous
- $\|g(D_t)\| \leq 1$ for all t , and
- for any $m > q_0$, $t \mapsto g(D_t)^m$ is continuous in trace norm.

Then the formula

$$C(m) \cdot \text{sf}(\{D_t\}) = \int_0^1 \tau \left(\frac{d}{dt}(D_t)g(D_t)^m \right) dt$$

also holds for any m between q_0 and N (not including q_0). ☛

Finally, in Chapter 4 we address the geometric interpretation of spectral flow; namely, spectral flow can be obtained as the intersection number of the path of operators with a certain submanifold Φ_1 of the manifold Φ of bounded perturbations of D_0 . This intersection number has a meaning in singular cohomology, allowing it to be related to de Rham cohomology and integral formulas for spectral flow. In the *type I case*, and when D_0 has *compact resolvents*, the interpretation of spectral flow as an intersection number is due to Getzler ([38]), who then uses it to prove an integral formula in the case when D_0 is θ -summable. For some of the required results, Getzler's proof was simply an outline, omitting some of the subtle operator theoretic details; we flesh out these details, allowing us to glimpse a possible interpretation in the type II case. However, we should underline that such a type II description remains still out of reach. The main results of Chapter 4 do not go beyond those of Getzler, and consist of finding a class in $H^1(\Phi, \Phi_0)$ which calculates spectral flow (Corollary 4.1.20), and then exhibiting a representative of a corresponding class in $H_{dR}^1(\Phi, \Phi_0)$ when D_0 is θ -summable, allowing us to get the following integral formula for spectral flow:

Theorem (Proposition 4.2.5 and Theorem 4.2.18) Suppose D_0 is a self-adjoint unbounded operator for which $\text{Tr}(e^{-tD_0^2}) < \infty$ for all $t > 0$. Denote by Φ the manifold $D_0 + \mathcal{N}_{sa}$, and by Φ_0 the submanifold of Φ consisting of invertible operators. Define a one-form on Φ and a zero-form on Φ_0 by $\alpha_{D,\varepsilon}(X) = \left(\frac{\varepsilon}{\pi}\right)^{1/2} \text{Tr}(Xe^{-\varepsilon D^2})$ for $X \in \mathcal{B}(\mathcal{H})_{sa}$, and $\eta_\varepsilon(D) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \text{Tr}(De^{-tD^2})t^{-\frac{1}{2}} dt$ respectively.

Then $\alpha_\varepsilon \oplus -\frac{1}{2}\eta_\varepsilon \in H_{dR}^1(\Phi, \Phi_0)$. Moreover, for $\rho = \{D_t\}_{t \in [0,1]} \subset \Phi$ a (differentiable) path for which D_0 and D_1 are invertible,

$$\text{sf}(\rho) = \int_\rho \alpha_\varepsilon + \frac{1}{2}\eta_\varepsilon(D_1) - \frac{1}{2}\eta_\varepsilon(D_0). \quad \text{☛}$$

1.4 Examples

Below, we single out type I and type II factors in which we can discuss spectral flow, and we explain how the various constructions in this dissertation apply to each case.

1.4.1 Example in a Type I Factor

In this section, we concentrate on a path of self-adjoint unbounded operators affiliated with a type I von Neumann algebra. We calculate the spectral flow of the path both from the definition and from the integral formula. Next, we explain the interpretation of spectral flow as an intersection number. Finally, we describe the interpretation of the Local Index Theorem in this context. The example presented in this section is a standard example; the calculation of spectral flow from the integral formula can also be found in [8]. It is a particularly nice example to work with, because it is easy to understand, one can show the spectral flow using a diagram, and the calculations are straight-forward.

This example is constructed in terms of functions on the unit circle, \mathbb{T} ; if we need to integrate such functions, the assumption is that the Lebesgue measure is used, normalized so that the measure of \mathbb{T} is 1. Let $\mathcal{H} = L^2(\mathbb{T})$, and fix the orthonormal basis $\{h_n := e^{int}\}_{n \in \mathbb{Z}}$. Consider the algebra $\mathcal{N} = \mathcal{B}(L^2(\mathbb{T}))$, a type I_∞ factor. Let D be the differentiation operator $\frac{1}{i} \frac{d}{dt}$; that is, D maps the basis vector h_n to $n \cdot h_n$. Note that $(1 + D^2)^{-1}$ is compact, and

$$\mathrm{Tr}((1 + D^2)^{-\frac{q}{2}}) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{1 + n^2} \right)^{\frac{q}{2}}.$$

In particular, if $q > 1$, this sum is finite; that is, $(1 + D^2)^{-1}$ is q -summable for any $q > 1$.

Let $D_0 = D$ and $D_t = D_0 + t \cdot 1$. Note that $D_1 = u^* D u$, where u is the unitary operator which takes h_n to h_{n+1} for $n \in \mathbb{Z}$ (so u^* takes h_n to h_{n-1}). Moreover, D_t takes h_n to $(n + t)h_n$. The picture below shows the spectrum of D_t for all $t \in [0, 1]$, with the spectrum of D_0 (which consists of all integers) on the vertical line to the left and the spectrum of D_1 on the vertical line to the right. The dotted line shows how each eigenvalue of D_0 changes with t until it becomes an eigenvalue of D_1 .

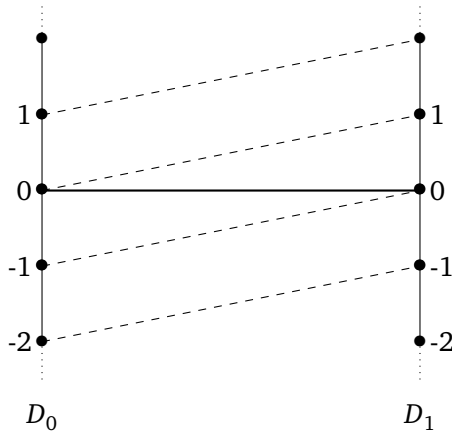


Figure 1.5: Spectral image of the path D_t

By definition, $D_1 = u^* D u$ takes h_n to $(n + 1)h_n$. The picture makes it clear that the spectral flow from D to $u^* D u$ is 1, as the eigenvalue which is -1 at D_0 becomes 0 at D_1 , whereas the other eigenvalues either remain negative or non-negative all along the path.

As an exercise, let us calculate the spectral flow using Definition 1.1.26. It is easy to see that

$$P_t := \chi_{[0,\infty)}(D_t) = \begin{cases} \text{span}\{h_n\}_{n \geq 0} & \text{for } 0 \leq t < 1, \text{ and} \\ \text{span}\{h_n\}_{n \geq -1} & \text{for } t = 1. \end{cases}$$

Clearly $\pi(P_t)$ is constant (since P_1 and P_0 differ by a finite-rank projection), so the spectral flow is $\text{ind}(P_0 P_1)$ as an operator from $P_1 \mathcal{H}$ to $P_0 \mathcal{H}$. But

$$\text{ind}(P_0 P_1) = \tau([\text{ran } P_1 \cap \ker P_0]) - \tau([\text{ran } P_0 \cap \ker P_1]) = \tau([\text{span}\{h_{-1}\}]) - 0 = 1.$$

Therefore, by using the definition, we get $\text{sf}\{D_t\} = 1$, as expected.

In Chapter 3, we prove that the spectral flow can be calculated by integrating a specific one-form on $D + \mathcal{N}_{sa}$. Namely, we will show that, if D is q -summable, then

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_q} \int_0^\infty \tau \left(\frac{d}{dt} (D_t) (1 + D_t^2)^{-\frac{q}{2}} \right) dt,$$

where $\tilde{C}_q = \int_{-\infty}^\infty (1 + x^2)^{-\frac{q}{2}}$. As already observed, D is q -summable for any $q > 1$; for such q , we have

$$\begin{aligned} \int_0^1 \tau \left(\frac{d}{dt} D_t (1 + D_t^2)^{-\frac{q}{2}} \right) dt &= \int_0^1 \tau((-D + u^* D u) (1 + D_t^2)^{-\frac{q}{2}}) dt \\ &= \int_0^1 \sum_{n \in \mathbb{Z}} (1 + n^2 + t^2 + 2nt)^{-\frac{q}{2}} dt \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 (1 + (n+t)^2)^{-\frac{q}{2}} dt \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} (1 + x^2)^{-\frac{q}{2}} dx \quad (\text{where } x = n + t) \\ &= \int_{-\infty}^\infty (1 + x^2)^{-\frac{q}{2}} = \tilde{C}_q. \end{aligned}$$

Dividing by \tilde{C}_q , $\frac{1}{\tilde{C}_q} \int_0^1 \tau \left(\frac{d}{dt} D_t (1 + D_t^2)^{-\frac{q}{2}} \right) dt = 1$; hence, from the integral formula we also get $\text{sf}(\{D_t\}) = 1$.

Spectral flow as an intersection number

In order to use Getzler's interpretation of spectral flow as the intersection number of the path with a specific submanifold of $D + \mathcal{N}_{sa}$, we need to modify the path so that its endpoints are invertible. Since we do not wish to change the spectral flow along the path, we will modify each endpoint by adding to it the projection onto its kernel. Note that, for example, the spectral flow from D_0 to $\hat{D}_0 := D_0 + P_{\ker D_0}$ is zero, since adding the projection changes the eigenvalue at 0 to a 1 and leaves the other eigenvalues unchanged. This ensures that \hat{D}_0 is invertible (since 0 is not an eigenvalue of \hat{D}_0), and also that the spectral flow along a path from \hat{D}_0 to D_0 is zero. So we extend our path by connecting D_0 to \hat{D}_0 and D_1 to \hat{D}_1 with straight lines.

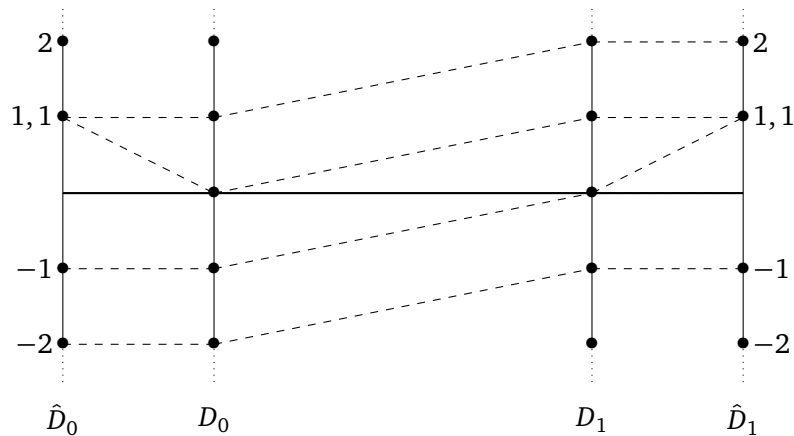


Figure 1.6: Spectral image of the path obtained by extending D_t

Getzler's idea reduces to the fact that the path can be modified so that each operator along it is invertible except for possibly finitely many whose zero eigenspace has dimension one. Using a cohomology argument, he shows that there is a well-defined intersection number of the resulting path with the submanifold consisting of operators whose zero eigenspace has dimension one. This intersection number is the spectral flow.

We apply this idea to our example. The only non-invertible operators on the path from \hat{D}_0 to \hat{D}_1 are D_0 and D_1 , and 0 is an eigenvalue with multiplicity 1 for each of them. As there are only two operators along the path which are not invertible, and zero is an eigenvalue of multiplicity one for each of them, we do not have to do any further modification of our path; we just need to figure out the intersection number at each point. The intersection number is calculated by writing the operators close to the intersection point with respect to some suitably chosen decomposition of \mathcal{H} into a one-dimensional subspace and its complement, and examining the real numbers

$$\Psi(A) = a - A_{12}A_{22}^{-1}A_{21} \text{ for } A = \begin{bmatrix} a & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If the sign changes, the intersection number is assigned as 1 or -1 appropriately; if the sign doesn't change the intersection number is zero. Note that the value of $\Psi(A)$ depends on the decomposition used for A (more details on this will be found in Chapter 4).

Denote by P the projection onto $\ker(D_0) = \text{span}\{h_0\}$, and by $L_s = D_0 + (1-s)P$ the straight line path from \hat{D}_0 to D_0 , for $s \in [0, 1]$. Denote by Q the projection onto $\ker(D_1) = \text{span}\{h_{-1}\}$, and by $R_s = D_1 + sQ$ for $s \in [0, 1]$ (so $\{R_s\}$ is the straight line path from D_1 to \hat{D}_1). The path with invertible endpoints that we are looking at is the concatenation of $\{L_s\}$, $\{D_t\}$ and $\{R_s\}$; we do not bother changing the indices. We want to calculate the intersection number of this path with the submanifold of operators whose kernel has dimension one.

Let us calculate the intersection number at D_0 . Recall that P denotes the projection onto the kernel of D_0 . With respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, $D_0 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_0 \end{bmatrix}$ with \tilde{D}_0 invertible. Moreover, for any $t \in [0, 1]$ we can write $D_t = \begin{bmatrix} t & 0 \\ 0 & \tilde{D}_t \end{bmatrix}$, where \tilde{D}_t is also

invertible if $t < 1$, and $L_s = \begin{bmatrix} 1-s & 0 \\ 0 & \tilde{D}_0 \end{bmatrix}$. Note that $\Psi(D_0) = 0$; to the left of D_0 we have that $\Psi(L_s) = 1 - s > 0$ for $s < 1$ and to the right, $\Psi(D_t) = t > 0$ for $t > 0$. Note that if we calculate Ψ with respect to this decomposition, then $\Psi(D_1)$ is not in fact defined (since \tilde{D}_1 is not invertible); however, we only care about being able to calculate Ψ (or, more precisely, figure out the change in sign of Ψ) in a small enough neighbourhood of D_0 . Hence, since the sign of Ψ does not change, the intersection number at D_0 is zero. This reflects the fact that no eigenvalue actually changes sign at this point of the path.

Now calculate the intersection number at D_1 . Recall that we denoted the projection onto the kernel of D_1 by Q . With respect to the decomposition $Q\mathcal{H} \oplus Q^\perp\mathcal{H}$, $D_1 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_1 \end{bmatrix}$ with \tilde{D}_1 invertible. Moreover, for $t \in [0, 1]$ we have $D_t = \begin{bmatrix} -1+t & 0 \\ 0 & \tilde{D}_t \end{bmatrix}$ with respect to the same decomposition, and $R_s = \begin{bmatrix} s & 0 \\ 0 & \tilde{D}_1 \end{bmatrix}$. Once again, $\Psi(D_1) = 0$; to the left of D_1 we have $\Psi(D_t) = -1 + t < 0$ for $t < 1$, and to the right we have $\Psi(R_s) = s > 0$ for $s > 0$. Hence, the sign of Ψ changes from negative to positive at D_1 , so the intersection number at D_1 is one.

Adding the two values obtained above we get that the intersection number of the path with the submanifold is 1. Hence, viewed as an intersection number, the spectral flow is also calculated to be 1.

Fredholm modules and the local index formula

We point out that, with $\mathfrak{A} = \mathcal{C}^\infty(\mathbb{T})$, acting by multiplication on $\mathcal{H} = \mathcal{L}^2(\mathbb{T})$, and D defined as above, we can show that $(\mathfrak{A}, D, \mathcal{H})$ is an odd spectral triple of spectral dimension 1. Note that if $g \in \mathfrak{A}$ and $f \in \text{Dom}(D_0) \subset \mathcal{L}^2(\mathbb{T})$ then $gf \in \text{Dom}(D_0)$ and, writing M_g for the operator given by multiplication by the function g , we have $[DM_g - M_gD](f) = f \cdot Dg = M_{Dg}(f)$. Since Dg is continuous, M_{Dg} is a bounded operator on $\mathcal{L}^2(\mathbb{T})$. Finally, $(1 + D_0^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ follows from the fact that the s -numbers of this operator go to zero. Recall that spectral dimension one means that $\tau((1 + D^2)^{-\frac{r}{2}}) < \infty$ for all $r > 1$ (this trace was discussed when the example was first introduced); note, however, that $\tau((1 + D^2)^{-\frac{1}{2}})$ is not finite, so $(\mathfrak{A}, \mathcal{H}, D)$ is *not* 1-summable. To get a bounded Breuer-Fredholm module, let $F = 2\chi_{[0, \infty)}(D) - 1$. Then (\mathcal{H}, F) is a bounded Fredholm module for \mathfrak{A} ; as discussed in Section 1.2.1.1 we already know that (\mathcal{H}, F) is r -summable for any $r > 1$ (since $(\mathfrak{A}, \mathcal{H}, D)$ is r -summable).

To see how the Local Index Formula applies in this context we refer to the final formula after Theorem 1.2.12. As the spectral dimension p is 1, we get $N = \lfloor \frac{1}{2} \rfloor + 1 = 1$, and most of the summations only have one term; the expression simplifies to

$$\begin{aligned} \text{sf}(D, u^*Du) &= \frac{1}{\sqrt{2\pi i}} \cdot \varphi_1(\text{Ch}_1(u)) \\ &= \frac{1}{\sqrt{2\pi i}} \cdot \sqrt{2\pi i} \cdot R_0(u^*, u) \\ &= \text{res}_{z=0} \tau(u^*[D, u]^{(0)}(1 + D^2)^{-(z+\frac{1}{2})}) \\ &= \text{res}_{z=0} \tau((u^*Du - D)(1 + D^2)^{-(z+\frac{1}{2})}). \end{aligned}$$

For the unitary u which we used in the earlier example (that is, u takes the basis vector h_n to h_{n+1}) we had $u^*Du - D = D_1 - D_0 = 1$, resulting in the formula

$$\text{sf}(D, u^*Du) = \text{res}_{z=0} \tau((1 + D^2)^{-(z+\frac{1}{2})}).$$

Suppose $F(z)$ is a meromorphic continuation of $\tau((1 + D^2)^{-(z+\frac{1}{2})})$ to a neighbourhood of zero; we know then that $F(s) = \sum_{n \in \mathbb{Z}} (1 + n^2)^{-(s+\frac{1}{2})}$ for all $s > 0$. We want to calculate $\text{res}_{z=0} F(z)$. The formula for $F(s)$ is similar to the definition of the Riemann zeta function, so to calculate the desired residue we can take the same approach as when calculating the residue of the Riemann zeta function at 1 – namely, find a formula for $F(s)$ which agrees with the above sum for $s > 0$ but which remains valid in some neighbourhood of zero, and use this new formula to calculate the residue.

To start with, we assume $s > 0$, and manipulate the sum formula for $F(s)$. It is easy to check that $\sum_{n \geq 1} (1 + n^2)^{-s-\frac{1}{2}} = \sum_{n \geq 1} n[(1 + n^2)^{-s-\frac{1}{2}} - (1 + (n+1)^2)^{-s-\frac{1}{2}}]$ (the difference between the partial sums of these two series goes to zero). Therefore

$$\begin{aligned} F(s) &= \sum_{n \in \mathbb{Z}} (1 + n^2)^{-s-\frac{1}{2}} = 1 + 2 \cdot \sum_{n \geq 1} (1 + n^2)^{-s-\frac{1}{2}} \\ &= 1 + 2 \cdot \sum_{n \geq 1} n[(1 + n^2)^{-s-\frac{1}{2}} - (1 + (n+1)^2)^{-s-\frac{1}{2}}] \\ &= 1 + 2 \cdot \sum_{n \geq 1} n \cdot [(s + \frac{1}{2}) \cdot \int_{1+n^2}^{1+(n+1)^2} x^{-s-\frac{3}{2}} dx] \\ &= 1 + 2 \cdot (s + \frac{1}{2}) \cdot \sum_{n \geq 1} \int_{1+n^2}^{1+(n+1)^2} \lfloor (x-1)^{1/2} \rfloor x^{-s-\frac{3}{2}} dx \\ &= 1 + 2 \cdot (s + \frac{1}{2}) \cdot \int_2^\infty \lfloor (x-1)^{1/2} \rfloor x^{-s-\frac{3}{2}} dx. \end{aligned}$$

In the penultimate line of the calculations above we used the fact that if x is between $1 + n^2$ and $1 + (n+1)^2$ then $n < (x-1)^{1/2} < n+1$, so $\lfloor (x-1)^{1/2} \rfloor = n$ (where $\lfloor \cdot \rfloor$ denotes the integer part of the number). Using $\{ \cdot \}$ for the fractional part of the number, we note that $\lfloor (x-1)^{1/2} \rfloor = (x-1)^{1/2} - \{(x-1)^{1/2}\}$, leading to the formula

$$F(s) = 1 + 2 \cdot \left(s + \frac{1}{2}\right) \cdot \left[\int_2^\infty (x-1)^{1/2} x^{-s-\frac{3}{2}} dx - \int_2^\infty \{(x-1)^{1/2}\} x^{-s-\frac{3}{2}} dx \right].$$

From the binomial expansion, we get $(x-1)^{\frac{1}{2}} \cdot x^{-(s+\frac{3}{2})} = \sum_{k \geq 0} \binom{1/2}{k} x^{-s-k-1} (-1)^k$. Integrate this formula term by term to get a function $G_s(x) = \sum_{k \geq 0} \binom{1/2}{k} (-1)^k \cdot \frac{1}{-s-k} \cdot x^{-s-k}$, defined for $x > 1$ (when the sum converges by the ratio test). Note that $G_s(x) = x^{-s} \cdot \sum_{k \geq 0} \binom{1/2}{k} (-1)^k \cdot \frac{1}{-s-k} \cdot x^{-k}$, and that the value of the sum decreases as x increases (the only part depending on x is the x^{-k} factor), and so the sum is uniformly bounded for, say, $x \geq 2$. For $s > 0$ we have $\lim_{x \rightarrow \infty} x^{-s} = 0$, whence $\lim_{x \rightarrow \infty} G_s(x) = 0$. It follows that, for all $s > 0$,

$$\begin{aligned} F(s) &= 1 + 2 \cdot \left(s + \frac{1}{2}\right) \cdot \left[\lim_{x \rightarrow \infty} G_s(x) - G_s(2) \right] - 2 \cdot \left(s + \frac{1}{2}\right) \cdot \int_2^\infty \{(x-1)^{1/2}\} x^{-s-\frac{3}{2}} dx \\ &= 1 + 2 \cdot \left(s + \frac{1}{2}\right) \cdot (-1) \cdot G_s(2) - 2 \cdot \left(s + \frac{1}{2}\right) \cdot \int_2^\infty \{(x-1)^{1/2}\} x^{-s-\frac{3}{2}} dx. \end{aligned}$$

This is the definition that we want to extend to a (complex) neighbourhood of zero. Recall that $G_s(2) = 2^{-s} \cdot \sum_{k \geq 0} \binom{1/2}{k} (-1)^{k+1} \cdot \frac{1}{s+k} \cdot \frac{1}{2^k}$. For $z \in \mathbb{C}$ with $|z| < 1$, we can thus define $G_z(2)$ by replacing all s 's by z in this expression; as $z + k$ would then be non-zero for all $k \geq 0$, and the sum converges absolutely, there is no problem with this definition. It is also easy to check that the integral still remaining in the formula converges, giving us the function

$$\begin{aligned} F(z) &= 1 + 2 \cdot \left(z + \frac{1}{2}\right) \cdot (-1) \cdot G_z(2) - 2 \cdot \left(z + \frac{1}{2}\right) \cdot \int_2^\infty \{(x-1)^{1/2}\} x^{-z-\frac{3}{2}} dx \\ &= 1 + 2 \cdot \left(z + \frac{1}{2}\right) \cdot (-1) \cdot \left[(-1) \frac{1}{z} \frac{1}{2^z} + \frac{1}{2} \cdot \frac{1}{z+1} \frac{1}{2^{z+1}} + \frac{\frac{1}{2} \cdot (\frac{1}{2}-1)}{2!} \frac{1}{z+2} \cdot \frac{1}{2^{z+2}} + \dots \right] \\ &\quad - 2 \cdot \left(z + \frac{1}{2}\right) \cdot \int_2^\infty \{(x-1)^{1/2}\} x^{-z-\frac{3}{2}} dx. \end{aligned}$$

$F(z)$ agrees with $\tau((1+D^2)^{-(z+\frac{1}{2})})$ on the set $z \in \mathbb{R}_+^*$, so is a meromorphic continuation to a neighbourhood which includes zero. From here it is easy to see that

$$\text{res}_{z=0} F(z) = \lim_{z \rightarrow 0} z F(z) = 2 \cdot \frac{1}{2} \cdot (-1) \cdot (-1) \frac{1}{2^0} = 1,$$

and so $\text{sf}(D, u^* Du) = \text{res}_{z=0} \tau((1+D^2)^{-(z+\frac{1}{2})}) = 1$. Therefore, from the Local Index Formula, we get $\text{sf}(D, u^* Du) = 1$; our previous calculations had $\text{sf}(D, u^* Du) = 1$ as well, so all is in agreement. Notice that, even for a relatively simple example, the residue is not necessarily that easy to calculate.

1.4.2 Example in a Type II Factor

Some of the standard examples of type II factors (especially type II_∞) are obtained via a crossed product construction; we summarize the approach we are going to use, though it should be pointed out that the construction can be performed in a more general setting if desired. For X a finite measure space, $\mathcal{H} = \mathcal{L}^2(X)$ is a Hilbert space. The algebra $\mathfrak{A} = \mathcal{L}^\infty(X)$ can be considered as a commutative subalgebra of $\mathcal{B}(\mathcal{L}^2(X))$, where \mathfrak{A} acts on $\mathcal{L}^2(X)$ by multiplication operators (i.e. $a \in \mathcal{L}^\infty(X)$ defines the operator $v \mapsto av$ for all $v \in \mathcal{L}^2(X)$ – we will occasionally write M_a for this operator, but only if we need to refer to it outside of a calculation). In addition, we have an *abelian* topological group G (in fact, G will be either \mathbb{Z} or \mathbb{R}) and an action $\alpha : G \rightarrow \text{Aut}(\mathfrak{A})$ which is continuous with respect to the strong operator topology on \mathfrak{A} . In order to reduce the number of brackets, we will denote $\alpha(r)$ by α_r .

We point out here that many of the standard references which describe the von Neumann crossed algebra construction (e.g. [32] and [73]) concentrate on the case when G is discrete and countable, which is more tractable than the general case, but clearly does not cover $G = \mathbb{R}$. The full construction for G a locally compact topological group was first described in [72], and some of the results for the case when G is abelian can be found in [51].

1.4.2.1 Crossed Product Construction

There are multiple ways of presenting the von Neumann cross product $\mathfrak{A} \rtimes_\alpha G$. The most common is to introduce the Hilbert space $\tilde{\mathcal{H}} = \mathcal{L}^2(G, \mathcal{H})$ (that is, measurable functions $\xi : G \rightarrow \mathcal{H}$ such

that $\int_G \|\xi(s)\|^2 ds < \infty$), and construct representations of both \mathfrak{A} and G in $\mathcal{B}(\tilde{H})$. Namely, for $\xi \in \tilde{H}$, define the operators

$$\begin{aligned} (V_a \xi)(s) &= \alpha_{-s}(a) \cdot \xi(s) & \text{for } a \in \mathfrak{A}, \text{ and} \\ (U_r \xi)(s) &= \xi(s - r) & \text{for } r \in G. \end{aligned}$$

The von Neumann algebra generated by the V_a 's and U_r 's, i.e. $\{V_a, U_r : a \in \mathfrak{A}, r \in G\}''$, is called the **von Neumann cross product** of \mathfrak{A} by α , and is denoted $\mathfrak{A} \rtimes_\alpha G$. This is the same notation as the one often used in the literature for the C^* crossed product, which is a different mathematical object (see e.g. [51], Section 7.6); however, as we never need to use the C^* crossed product in the following, we do not need to differentiate between the two.

We now concentrate on getting a handle on the elements of $\mathfrak{A} \rtimes_\alpha G$ and the interaction between them. Since (as can be checked by a simple calculation) $U_r V_a = V_{\alpha_r(a)} U_r$, we can view some of the elements of $\mathfrak{A} \rtimes_\alpha G$ as functions from G to \mathfrak{A} . If G is discrete, the formal sum $\sum_{r \in G} V_{a_r} U_r$ corresponds to the function $r \mapsto a_r$. Since $(V_a U_r \xi)(s) = \alpha_{-s}(a) \xi(s - r)$, each function $\varphi \in C_c(G, \mathfrak{A})$ defines an operator S_φ on $L^2(G, \mathfrak{A})$ by

$$(S_\varphi \xi)(s) = \int_G \alpha_{-s}(\varphi(r)) \xi(s - r) dr.$$

By considering what happens to a formal sum $\sum V_{a_r} U_r$ when we wish to take involutions, and what happens when two such sums are multiplied, we are led to the definition of involution and multiplication on $\mathcal{C}_c(G, \mathfrak{A})$ by

$$\begin{aligned} (\varphi^*)(s) &= \alpha_s(\varphi(-s)^*) \\ (\varphi \diamond \psi)(s) &= \int_G \varphi(r) \alpha_r(\psi(s - r)) dr. \end{aligned}$$

The non-standard notation \diamond is being used for the operation convolution twisted by α , to clearly differentiate it from pointwise multiplication.

It is possible to view the crossed product $\mathfrak{A} \rtimes_\alpha G$ as a Hilbert algebra, as explained below; the machinery of Hilbert algebras can thus be used to define the trace on the crossed product, and to identify trace class operators. Since X has finite measure, $\mathfrak{A} = \mathcal{L}^\infty(X) \subset \mathcal{L}^2(X) = \mathcal{H}$, so $\mathcal{C}_c(G, \mathfrak{A}) \subseteq \mathcal{C}_c(G, \mathcal{H}) \subseteq \mathcal{L}^2(G, \mathcal{H}) = \tilde{\mathcal{H}}$. One can define a trace τ on \mathfrak{A} via $\tau(a) = \int_X a d\mu$; then, for $\varphi, \psi \in \mathcal{C}_c(G, \mathfrak{A})$, the scalar product

$$(\varphi | \psi) = \int_G \tau(\varphi(r) \psi(r)^*) dr$$

turns $\mathcal{C}_c(G, \mathfrak{A})$ into a **Hilbert algebra**; that is, considering the fact that $L^2(G, \mathcal{H})$ is the completion of $C_c(G, \mathfrak{A})$ with respect to this scalar product, the algebra $C_c(G, \mathfrak{A})$ equipped with the earlier defined multiplication \diamond and involution $*$ satisfies

1. The subalgebra generated by $\{\varphi \diamond \psi\}$ is dense in $L^2(G, \mathcal{H})$.
2. For each $\varphi \in C_c(G, \mathfrak{A})$, the map defined by $\xi \mapsto \varphi \diamond \xi$ extends to a bounded operator L_φ on $L^2(G, \mathfrak{A})$.

3. The involution on $C_c(G, \mathfrak{A})$ is compatible with the adjoint operator on L_φ ; in other words, $(\varphi \diamond \xi | \psi) = (\xi | \varphi^* \diamond \psi)$.

4. $(\varphi | \psi) = (\psi^* | \varphi^*)$.

Verifying that the above conditions hold for the given involution and product can be modelled on the proofs of Lemma 5.3 to 5.9 of [72]. We will also denote by L_φ the left multiplication by φ for functions $\varphi \in \mathcal{L}^2(G, \mathfrak{A})$ which are not necessarily in $C_c(G, \mathfrak{A})$; the domain of such L_φ contains $C_c(G, \mathfrak{A})$, but for some functions φ it might not be possible to extend the domain of definition to all of $\mathcal{L}^2(G, \mathfrak{H})$ (i.e. L_φ might not be a bounded operator). The **bounded elements** of $\mathcal{L}^2(G, \mathfrak{H})$ are the functions $\varphi \in L^2(G, \mathfrak{A})$ for which L_φ (defined as above) is a bounded operator. The left von Neumann algebra of $C_c(G, \mathfrak{A})$ is the von Neumann algebra $\tilde{\mathcal{N}}$ generated by the $\{L_\varphi : \varphi \text{ is a bounded element of } \mathcal{L}^2(G, \mathfrak{A})\}$. For the specific crossed product algebra in which we are constructing our example, we will show that $\mathcal{N} = \mathfrak{A} \rtimes_\alpha G$ is unitarily equivalent to $\tilde{\mathcal{N}}$ (see the section entitled “A type II_∞ factor” below). Hilbert algebras are well studied, so for example appealing to Theorem 1 in Section 6.2 of [32], we know that one can define a semifinite normal trace on $\tilde{\mathcal{N}}$ such that the trace ideal of $\tilde{\mathcal{N}}$ consists of elements $C_{\psi^* \diamond \varphi}$ where φ, ψ are bounded elements of $\mathcal{L}^2(G, \mathfrak{A})$; moreover, the trace on this ideal is given by

$$\widetilde{Tr}(C_{\psi^* \diamond \varphi}) = (\varphi | \psi).$$

This will allow us to use the unitary equivalence between \mathcal{N} and $\tilde{\mathcal{N}}$ to define a trace on \mathcal{N} . The trace for a crossed product obtained via an action of \mathbb{R} is also discussed in some detail in Section 2 of [56].

We will also need to classify $\mathfrak{A} \rtimes_\alpha G$ as a von Neumann algebra. To that end, we need a little bit more terminology. The action of G on \mathfrak{A} is called **ergodic** if the fixed point algebra $\mathfrak{A}^\alpha = \{a \in \mathfrak{A} : \alpha_s(a) = a \text{ for all } s \in G\}$ consists of the scalars in \mathfrak{A} . The other concept we need is that of Connes spectrum. Given an action α of a group G on an algebra \mathfrak{A} , we can define an action of $L^1(G)$ on \mathfrak{A} by $\alpha_f(a) = \int_G f(r) \alpha_r(a) dr$ for $f \in L^1(G)$. The spectrum of an action α , denoted $\text{sp}(\alpha)$, consists of the set $\cap \{\gamma \in \hat{G} \mid \hat{f}(\gamma) = 0\}$, where \hat{f} denotes the Fourier transform of f , \hat{G} denotes the dual group of G , and the intersection is taken over all $f \in L^1(G)$ for which $\alpha_f = 0$. Finally, the *Connes spectrum* of α , denoted $\Gamma(\alpha)$, is defined to be $\cap_p \text{sp}(\alpha_p)$ where the intersection is taken over the non-zero projections $p \in \mathfrak{A}^\alpha$, and α_p denotes the restriction of α to the algebra $p\mathfrak{A}p$. A brief overview of the Connes spectrum can be found in [32], in the notes added to the second edition. Theorem 8.11.15 of [51] states that, for G abelian, $\mathfrak{A} \rtimes_\alpha G$ is a factor if and only if the action of G on \mathfrak{A} is ergodic and $\Gamma(\alpha) = \hat{G}$.

1.4.2.2 A type II_∞ factor

We are going to apply the above cross product construction for $\mathfrak{A} = \mathcal{L}^\infty(\mathbb{T}^2)$. Fix θ irrational and define an action of $G = \mathbb{R}$ on \mathfrak{A} by $\alpha_s(a)(z_1, z_2) = a(e^{-2\pi i s} z_1, e^{-2\pi i \theta s} z_2)$. Construct the von Neumann cross product $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ as an algebra on $L^2(\mathbb{R}, \mathfrak{H})$, as described above. Recall that $\tilde{\mathfrak{H}} = \mathcal{L}^2(\mathbb{R}, \mathfrak{H})$ is isomorphic to $\mathcal{L}^2(\mathbb{R}) \otimes \mathfrak{H}$ (where the pure tensor $v \otimes g$, with $v \in \mathcal{L}^2(G)$ and $g \in \mathfrak{H} = \mathcal{L}^2(\mathbb{T}^2)$, corresponds to the function $g \mapsto v(r)g$ in $\mathcal{L}^2(G, \mathfrak{H})$ – a proof of this well known fact can be found in Proposition 13.2.1 of [43]); this picture of $\tilde{\mathfrak{H}}$ will occasionally facilitate calculations, and will be occasionally used in the following.

Our path will consist of bounded perturbations of the operator $D = \frac{1}{2\pi i} \frac{d}{ds}$ on $\mathcal{L}^2(\mathbb{R}, \mathcal{H})$ (which is equal to $\frac{1}{2\pi i} \frac{d}{ds} \otimes 1$ when viewed as an operator on $L^2(\mathbb{R}) \otimes \mathcal{H}$), and we need to show that D is p -summable for some p . The Fourier transform on $\mathcal{L}^2(\mathbb{R})$ will help us get a nicer picture of the action of D .

We quickly review the properties of the Fourier transform that we will need. Recall that, if $f \in \mathcal{L}^1(\mathbb{R})$, the Fourier transform F maps f to \hat{f} , where $\hat{f}(r) = \int_{\mathbb{R}} f(x) e^{-2\pi i x r} dx$ for every $r \in \mathbb{R}$; this definition can be extended by continuity from $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ to all of $\mathcal{L}^2(\mathbb{R})$, making F into a unitary operator (see e.g. [37], Theorem 8.29). If $f \in L^1$ is such that \hat{f} is also in L^1 , then the inverse Fourier transform is given by $F^{-1}(f)(s) = \hat{f}(-s)$ (the Fourier Inversion Theorem; see e.g. Theorem 8.26 of [37]). Working with the Fourier transform will require us to occasionally refer to the convolution of functions, which in keeping with standard notation we denote by $*$. We will use the following properties without further reference:

- if $f, g \in \mathcal{L}^2(\mathbb{R})$ then $F^{-1}(Ff \cdot Fg) = f * g$ ([37], Theorem 8.4); note that this implies $F(f * g) = (Ff) \cdot (Fg)$
- for $f \in C^1 \cap C_0$ with $f' \in L^1$, $F(\frac{d}{ds}f(s)) = 2\pi i s \hat{f}(s)$ ([37], Theorem 8.22e)

Recall that we used λ for the left regular representation of \mathbb{R} on $\mathcal{L}^2(\mathbb{R})$, $(\lambda_r f)(s) = f(r - s)$. We will use Λ for the integrated form of λ , which gives us a representation of $\mathcal{L}^1(\mathbb{R})$ on $\mathcal{L}^2(\mathbb{R})$ by convolution operators; that is, $(\Lambda_f g)(s) = \int_{\mathbb{R}} f(r) \lambda_r(g(s)) dr = f * g$.

Denote by U the unitary operator $F \otimes 1$; then one can calculate that $U^{-1}DU = M_s \otimes 1$, where M_s is the operator on $\mathcal{L}^2(\mathbb{R})$ defined by multiplication by the variable s . Hence for g a suitable function we have $g(U^{-1}DU) = M_g \otimes 1$, and in particular if $g \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ then $g(D) = U \cdot g(U^{-1}DU) \cdot U^{-1} = \Lambda_{\hat{g}} \otimes 1$. We will show that under these conditions $g(D)$ is a trace class operator in \mathcal{N} .

We now describe a unitarily equivalent picture of the crossed product algebra $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ described in the the section entitled ‘‘Crossed product construction’’. It is called the **implemented crossed product**, and requires a representation of the group on the same Hilbert space on which \mathfrak{A} is acting. In our case, start by defining, for each $r \in \mathbb{R}$, a unitary W_r on $\mathcal{H} = \mathcal{L}^2(\mathbb{T}^2)$ via $(W_r v)(z_1, z_2) = v(e^{-2\pi i s} z_1, e^{-2\pi i \theta s} z_2)$ (note the similarity to the definition of α_r ; in fact, $W_r(v) = \alpha_r(v)$ for $v \in \mathfrak{A} = \mathcal{L}^\infty(\mathbb{T}^2) \subset \mathcal{L}^2(\mathbb{T}^2)$). Then $r \mapsto W_r$ is a representation of \mathbb{R} on $\mathcal{B}(\mathcal{H})$. Moreover, one can define a unitary W on $\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}, \mathcal{H})$ via $(W \xi)(s) = W_s(\xi(s))$. It can then be checked that $(WV_a W^* \xi)(s) = a \cdot \xi(s)$ (so, as an operator on $\mathcal{L}^2(\mathbb{R}) \otimes \mathcal{H}$, $WV_a W^*$ is $1 \otimes M_a$), and $(WU_r W^* \xi)(s) = W_r \xi(s - r)$ (so, again as an operator on $\mathcal{L}^2(\mathbb{R}) \otimes \mathcal{H}$, $WU_r W^*$ is $\lambda_r \otimes W_r$, where λ_r is the left regular representation of \mathbb{R} on $\mathcal{L}^2(\mathbb{R})$). In other words, the unitary W can be used to untwist the representation of \mathfrak{A} on $\tilde{\mathcal{H}}$ by getting rid of the action of α , at the cost of making the representation of \mathbb{R} on $\tilde{\mathcal{H}}$ more complicated. Moreover, for $\varphi \in \mathcal{C}_c(\mathbb{R}, \mathfrak{A})$, we get that

$$(WS_{\varphi} W^* \xi)(s) = \int_G \varphi(r) \alpha_r(\xi(s - r)) dr = (L_{\varphi} \xi)(s).$$

Recall that we denoted by $\tilde{\mathcal{N}}$ the left von Neumann algebra of $C_c(\mathbb{R}, \mathfrak{A})$, which is generated by the L_{φ} 's; it follows that $\tilde{\mathcal{N}} = W \mathcal{N} W^*$. Hence, we can use the trace defined on $\tilde{\mathcal{N}}$ to define a trace on \mathcal{N} ; namely, $\text{Tr}(S_{\varphi}) = \tilde{\text{Tr}}(L_{\varphi})$ for every L_{φ} which is trace class in $\tilde{\mathcal{N}}$.

1.4.2.3 Some trace class operators in $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$

Recall from the previous section that, for the unbounded operator D in which we are interested and functions g for which $\Lambda_{\widehat{g}}$ is a bounded operator on $\mathcal{L}^2(\mathbb{R})$, we had $g(D) = \Lambda_{\widehat{g}} \otimes 1$. We want to show that this operator is trace class in \mathcal{N} . For this section only, we will occasionally use the notation $1_{\mathfrak{A}}$ to differentiate the unit in the algebra \mathfrak{A} (i.e. $1_{\mathfrak{A}}$ stands for the function $\mathbb{T}^2 \rightarrow \mathbb{C}$ which is everywhere one) from the real number 1.

Suppose that $f \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \subset L^2(\mathbb{R})$ is a positive function. We will show that, for $\varphi(s) = \widehat{f}(s) \cdot 1_{\mathfrak{A}}$, $S_{\varphi} = \Lambda_{\widehat{f}} \otimes 1$, and that, moreover, S_{φ} is trace class in \mathcal{N} . In order to calculate the trace of S_{φ} , we will define $\psi(s) = \widehat{f^{1/2}}(s) \cdot 1_{\mathfrak{A}}$ and show that ψ is a bounded element of $\mathcal{L}^2(\mathbb{R}, \mathcal{H})$ for which $\varphi = \psi^* \diamond \psi$; thus, we will be able to conclude that $\text{Tr}(S_{\varphi}) = (\psi | \psi)$.

- Since $f \in \mathcal{L}^1(\mathbb{R})$ we certainly have a definition of \widehat{f} ; the fact that $f \in \mathcal{L}^{\infty}(\mathbb{R})$ means that multiplication by f is a bounded operator on $\mathcal{L}^2(\mathbb{R})$, and thus, by appealing to the Fourier transform, so is $\Lambda_{\widehat{f}}$. Therefore, $\Lambda_{\widehat{f}} \otimes 1$ is a bounded operator on $\mathcal{L}^2(\mathbb{R}) \otimes \mathcal{H}$. Showing that $\Lambda_{\widehat{f}} \otimes 1 = S_{\varphi}$ will allow us to conclude that $S_{\varphi} = \Lambda_{\widehat{f}} \otimes 1 \in \mathcal{N}$.

Suppose $\xi(s) = g(s)u$ for some $g \in C_c(\mathbb{R})$ and $u \in \mathfrak{A}$. Then

$$\begin{aligned} (S_{\varphi}\xi)(s) &= \int_{\mathbb{R}} \alpha_{-s}(\varphi(r))\xi(s-r) dr \\ &= \int_{\mathbb{R}} \alpha_{-s}(\widehat{f}(r) \cdot 1_{\mathfrak{A}}) \cdot g(s-r)u dr \\ &= \int_{\mathbb{R}} \widehat{f}(r)g(s-r)u dr \\ &= (\widehat{f} * g)u \\ &= (\Lambda_{\widehat{f}} \otimes 1_{\mathfrak{A}})\xi. \end{aligned}$$

As such functions are dense in $\mathcal{L}^2(\mathbb{R}, \mathcal{H})$, the desired equality follows. Moreover, since $f^{1/2} \in \mathcal{L}^2(\mathbb{R}) \cap \mathcal{L}^{\infty}(\mathbb{R})$, the exact same proof shows that $S_{\psi} = \Lambda_{\widehat{f^{1/2}}} \otimes 1_{\mathfrak{A}}$, so S_{ψ} is also an element of \mathcal{N} .

- Now show that $\varphi = \psi^* \diamond \psi$. We will need the fact that $\widehat{f^{1/2}(-s)} = \widehat{f^{1/2}}(s)$, which can be easily verified from the definition of \widehat{f} using the fact that f is a positive function.

$$\begin{aligned} (\psi^* \diamond \psi)(s) &= \int_{\mathbb{R}} \psi^*(r)\alpha_{-r}(\psi(s-r)) dr \\ &= \int_{\mathbb{R}} \alpha_r(\psi(-r)^*) \cdot \alpha_{-r}(\psi(s-r)) dr \\ &= \int_{\mathbb{R}} \alpha_r(\widehat{f^{1/2}(-r)} \cdot 1_{\mathfrak{A}}) \cdot \alpha_{-r}(\widehat{f^{1/2}}(s-r) \cdot 1_{\mathfrak{A}}) dr \\ &= \int_{\mathbb{R}} \widehat{f^{1/2}}(r)\alpha_r(1_{\mathfrak{A}}) \cdot \widehat{f^{1/2}}(s-r)\alpha_{-r}(1_{\mathfrak{A}}) dr \\ &= 1_{\mathfrak{A}} \cdot \int_{\mathbb{R}} \widehat{f^{1/2}}(r) \cdot \widehat{f^{1/2}}(s-r) dr \\ &= 1_{\mathfrak{A}} \cdot (\widehat{f^{1/2}} * \widehat{f^{1/2}})(s) \\ &= 1_{\mathfrak{A}} \cdot \widehat{f}(s), \end{aligned}$$

which is equal to $\varphi(s)$, as claimed.

Hence, from the definition of the trace on \mathcal{N} , we get

$$\begin{aligned}
\text{Tr}(S_\varphi) &= (\psi|\psi) = \int_{\mathbb{R}} \tau(\psi(r) \cdot \psi(r)^*) dr \\
&= \int_{\mathbb{R}} \tau(\widehat{f^{1/2}}(r) \cdot \widehat{f^{1/2}}(r) \cdot 1_{\mathfrak{A}}) dr \\
&= \int_{\mathbb{R}} |\widehat{f^{1/2}}(r)|^2 \cdot \tau(1_{\mathfrak{A}}) dr \\
&= \|\widehat{f^{1/2}}\|_2^2 \\
&= \|f^{1/2}\|_2^2 \\
&= \int_{\mathbb{R}} f(x) dx.
\end{aligned}$$

A slightly more general result is proven in Lemma 8.1 of [20] (namely, in the definition of φ , $1_{\mathfrak{A}}$ can be replaced by any element $a \in \mathfrak{A}$).

We now connect this up again to our unbounded operator D . From the above work, it follows that if g is a positive function in $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$, then $g(D) = \Lambda_g \otimes 1$ is a trace class operator in \mathcal{N} and

$$\text{Tr}(g(D)) = \int_{\mathbb{R}} g(r) dr.$$

1.4.2.4 The algebra $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ is a II_{∞} factor

First show that $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ is a factor. It is well-known that the action defined by α , the irrational rotation on the torus, is ergodic (one way to see this is to start out with a function f fixed by α , write out its Fourier coefficients, and conclude that the only way f remains unchanged is if all the coefficients corresponding to non-constant terms are zero).

Since \mathfrak{A}^{α} consists of the scalars only, the only projections in \mathfrak{A}^{α} are 0 and 1, so the intersection in the definition of $\Gamma(\alpha)$ is taken over only $p = 1$; that is, $\Gamma(\alpha) = \text{sp}(\alpha)$. We claim that if $f \in L^1$ is such that $\alpha_f = 0$, then $\hat{f} = 0$ (which in turn implies f is itself 0 almost everywhere). Since the zero function is the only function f with $\alpha_f = 0$, we can then conclude from the definition that $\text{sp}(\alpha) = \mathbb{R}$.

So suppose that $f \in L^1$ is such that $\alpha_f = 0$. For each fixed $s \in \mathbb{R}$, $\hat{f}(s) = \int_{\mathbb{R}} f(r) e^{-2\pi i s r} dr$ by definition. For each $n, m \in \mathbb{Z}$, define a function $g_{n,m}(z_1, z_2) = z_1^n \cdot z_2^m$ for $z_1, z_2 \in \mathbb{T}$. By assumption, $\alpha_f(g_{n,m}) = 0$, whence $(\alpha_f(g_{n,m}))(z_1, z_2) = 0$ (as $\mathcal{L}^2(\mathbb{T}^2)$ functions, so almost everywhere), i.e. $\int_{\mathbb{R}} f(r) \cdot \alpha_r(g_{n,m})(z_1, z_2) dr = 0$ a.e. But, by definition,

$$\alpha_r(g_{n,m})(z_1, z_2) = (e^{-2\pi i r} z_1)^n \cdot (e^{-2\pi i r \theta} z_2)^m = e^{-2\pi i r(n+\theta m)} \cdot z_1^n \cdot z_2^m;$$

note that the power function is multiplicative for complex numbers if the power is an integer. We can thus conclude that $\int_{\mathbb{R}} f(r) e^{2\pi i r s} dr = 0$ whenever $s = n + \theta m$ for $n, m \in \mathbb{Z}$. However, $\{n + \theta m : n, m \in \mathbb{Z}\}$ is dense in \mathbb{R} , and since \hat{f} is continuous for $f \in \mathcal{L}^1$, we must have that $\hat{f} = 0$.

This concludes the proof that the action of α on \mathfrak{A} is ergodic and $\Gamma(\alpha) = \hat{G}$, so by the aforementioned Theorem 8.11.15 of [51], $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ is a factor.

We can now proceed to follow the same exposition as paragraph 17.3 of [28] to show that the trace on $\mathcal{N} = \mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ takes values in $[0, \infty]$, so $\mathfrak{A} \rtimes_{\alpha} G$ is a II_{∞} factor. For any $a \in [0, \infty)$, consider the characteristic function $f := \chi_{[-a, a]}$. Then, as shown earlier, $\Lambda(\hat{f}) \otimes 1$ is a trace class operator in \mathcal{N} , with $\text{Tr}(\Lambda(\hat{f}) \otimes 1) = \int f dx = \int \chi_{[-a, a]} = 2a$. As a can be any positive real number, this concludes the proof.

1.4.2.5 Spectral flow calculations

Recall we had $D = \frac{1}{2\pi i} \frac{d}{ds} \otimes 1$ on $L^2(\mathbb{R}) \otimes \mathcal{H}$, and we showed that for $g \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ we have $g(D) = \Lambda_{\hat{g}} \otimes 1$ (beginning of section “A type II_{∞} factor”), which is trace class in \mathcal{N} (section entitled “Some trace class operators in $\mathfrak{A} \rtimes_{\alpha} \mathbb{R}$ ”), with

$$\text{Tr}(g(D)) = \int_{\mathbb{R}} g(r) dr.$$

Let $D_0 = D$ and $D_t = D + \theta \cdot t$. Note here that this is the same θ as the θ used in the definition of the action of \mathbb{R} on \mathbb{T}^2 . The reason for this will become apparent from the proof that D_1 is unitarily equivalent to D_0 ; however, the same proof could be adapted for other perturbations of D by a scalar.

It seems intuitive that we should have $\text{sf}(D_0, D_1) = \theta$. After all, via the unitary $U = F \otimes 1$, D_0 is unitarily equivalent to $M_s \otimes 1$ and D_t is unitarily equivalent to $M_{s+t\theta} \otimes 1$, so the spectrum shifts by θ , as shown in the diagram below:

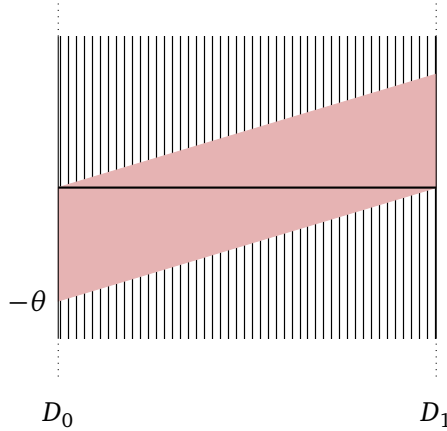


Figure 1.7: Spectral image of the path D_t

For any $t \in [0, 1]$ the spectrum of D_t consists of the whole real line, and D_t has no eigenvalues. Note that it is no longer the case that the value θ can be measured by an intersection number between the spectral diagram of the path and a line, as in this context the intersection number does not make sense. We can, however, still calculate spectral flow using Definition 1.1.30. Let $P_t = \chi_{[0, \infty)}(D_t) = V_{\chi_{[-t\theta, \infty)}}$; since the projections all have the same image in the generalized Calkin algebra, the spectral flow is equal to $\text{ind}_{(P_0, P_1)}(P_0 P_1)$, which, since $P_0 \leq P_1$, simplifies to $\text{Tr}(P_1 - P_0)$. Therefore,

$$\text{sf}(\{D_t\}) = \text{ind}_{(P_0, P_1)}(P_0 P_1) = \text{Tr}(P_1 - P_0) = \text{Tr}(V_{\chi_{[-\theta, 0]}}) = \int_{\mathbb{R}} \chi_{[-\theta, 0]}(r) dr = \theta.$$

Now we want to show that we can also apply Theorem 3.0.8. Define the function $a : \mathbb{T}^2 \rightarrow \mathbb{C}$ by $a(z_1, z_2) = z_2$. As the range of a is contained in \mathbb{T} , M_a is a unitary operator in \mathfrak{A} . Moreover, as we show from the calculations below, $D_1 = V_a^* D_0 V_a$, so D_1 and D_0 are unitarily equivalent. First note that

$$\alpha_{-s}(a)(z_1, z_2) = a(e^{2\pi i s} z_1, a^{2\pi i s \theta} z_2) = e^{2\pi i s \theta} z_2 = e^{2\pi i s \theta} a(z_1, z_2).$$

Thus, for $\xi \in \mathcal{L}^2(\mathbb{R}, \mathcal{H})$ such that $\xi \in \text{Dom}(D)$

$$\begin{aligned} (V_a \xi)(s) &= \alpha_{-s}(a) \xi(s) = e^{2\pi i s \theta} a(z_1, z_2) \xi(s), \\ (D(V_a \xi))(s) &= \frac{1}{2\pi i} \frac{d}{ds} [e^{2\pi i s \theta} a \xi(s)] \\ &= \frac{1}{2\pi i} [2\pi i \theta e^{2\pi i s \theta} a \xi(s) + e^{2\pi i s \theta} a \frac{d}{ds} (\xi(s))] \\ &= \theta e^{2\pi i s \theta} a \xi(s) + e^{2\pi i s \theta} a \cdot \frac{1}{2\pi i} \frac{d}{ds} (\xi(s)) \\ &= \theta \alpha_{-s}(a) \xi(s) + (\alpha_{-s}(a) D \xi)(s) \\ &= \theta (V_a \xi)(s) + (V_a D \xi)(s) \\ (V_a^* (D V_a \xi))(s) &= \theta (V_a^* V_a \xi)(s) + (V_a^* V_a D \xi)(s) \\ &= \theta \xi(s) + D \xi(s) \\ &= ((\theta + D) \xi)(s). \end{aligned}$$

Therefore, $V_a^* D V_a \xi = D_1 \xi$, as claimed.

We now calculate the spectral flow using the formula in Theorem 3.0.8. Note that with $g(s) = (1+s^2)^{-1}$ we have $g \in \mathcal{L}^1 \cap \mathcal{L}^2$, so $g(D)$ is trace class, which means that D is 2-summable. Applying the formula with $p = 2$, we have the constant $\tilde{C}_1 = \int_{-\infty}^{\infty} (1+x^2)^{-1} dx$, and

$$\begin{aligned} \text{sf}(\{D_t\}) &= \frac{1}{\tilde{C}_1} \cdot \int_0^1 \text{Tr} \left(\frac{d}{dt} (D_t) (1 + D_t^2)^{-1} \right) dt = \frac{1}{\tilde{C}_1} \cdot \int_0^1 \text{Tr}(\theta \cdot (1 + D_t^2)^{-1}) dt \\ &= \frac{1}{\tilde{C}_1} \cdot \int_0^1 \int_{-\infty}^{\infty} (\theta \cdot (1 + (s + \theta t)^2)^{-1}) ds dt \\ &= \frac{1}{\tilde{C}_1} \cdot \theta \cdot \int_0^1 \int_{-\infty}^{\infty} (1 + x^2)^{-1} dx dt \\ &= \frac{1}{\tilde{C}_1} \cdot \theta \cdot \int_{-\infty}^{\infty} (1 + x^2)^{-1} dx \cdot \int_0^1 1 dt \\ &= \frac{1}{\tilde{C}_1} \cdot \theta \cdot \tilde{C}_1 \cdot 1 \\ &= \theta. \end{aligned}$$

1.4.2.6 A gap continuous path in a type II_1 factor

We will use once again the cross product construction to obtain a factor. Start with $\mathcal{L}^\infty(\mathbb{T})$ acting by multiplication on $\mathcal{L}^2(\mathbb{T})$. Fix θ irrational and define an action α of \mathbb{Z} on $\mathcal{L}^\infty(\mathbb{T})$ by $(\alpha_n f)(z) = f(e^{in\theta} z)$ for $n \in \mathbb{Z}$. It is well known that $\mathcal{N} = \mathcal{L}^\infty(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is a II_1 factor (see e.g. [32], Theorem 1 in Section I.9.4). We will construct our example in the copy of $\mathcal{L}^\infty(\mathbb{T})$ in \mathcal{N} .

The purpose of this example is to exhibit a gap continuous path where the spectrum flows “through infinity”, an expression which will hopefully be clearer by the end of the section. The example below is an explicit instance of the main idea of the proof of Theorem 1.10 in [8], where Booss-Bavnbek, Lesch and Phillips show that the unbounded self-adjoint Fredholm operators are path connected. The idea is to construct a path of unitary operators $\{f_t\}$ such that their spectrum moves from the top half of the circle to the bottom half of the circle, passing through

1, and then apply the inverse Cayley transform to obtain a gap-topology continuous path of unbounded operators $\{D_t\}$.

To make descriptions easier, we will use the notation $[\alpha \rightarrow \beta]$ for the arc along the unit circle measured counterclockwise starting at angle α , and ending at angle β (see Figure 2.2 for a diagram if needed). For fixed $t \in [0, 1]$, f_t is a function on the circle which is the cumulative effect of the following operations, in the given order: map the circle to itself by keeping the top half fixed and mapping the bottom half to the top, map the arc $[0 \rightarrow \pi]$ to $[\pi/4 \rightarrow 3\pi/4]$, and finally rotate the circle clockwise by $t\pi$; we will work out the exact formula for f_t anon. We will identify f_t with the operator on $\mathcal{L}^2(\mathbb{T})$ given by multiplication by f_t , thus obtaining our desired path of unitary operators.

The range of f_t is shown in the figure below (as a subset of the unit circle) for a few select values of t :

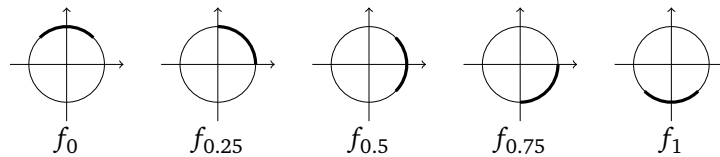


Figure 1.8: The range of the function f_t for a few select values of t .

Let us now work out the exact formula for f_t (where $t \in [0, 1]$ is fixed) following the steps given in our earlier description:

- fix the top half of the circle and map the bottom half of the circle to the top half; that is, start with the function

$$g(e^{\pi i \theta}) = \begin{cases} e^{i\theta} & \text{if } \theta \in [0, \pi) \\ e^{-i\theta} & \text{otherwise,} \end{cases}$$

- apply the function which squeezes the arc $[0 \rightarrow \pi]$ to $[\pi/4 \rightarrow 3\pi/4]$ to g to get the function

$$f(e^{\pi i \theta}) = \begin{cases} e^{i(\frac{\theta}{2} + \frac{\pi}{4})} & \text{if } \theta \in [0, \pi) \\ e^{i(\frac{2\pi - \theta}{2} + \frac{\pi}{4})} & \text{otherwise,} \end{cases}$$

- apply a clockwise rotation by $t\pi$ to f to get the final formula for f_t :

$$f_t(e^{i\theta}) = \begin{cases} e^{i(\frac{\theta}{2} + \frac{\pi}{4} - t\pi)} & \text{if } \theta \in [0, \pi) \\ e^{i(\frac{2\pi - \theta}{2} + \frac{\pi}{4} - t\pi)} & \text{otherwise.} \end{cases}$$

Note that $f_t \in \mathcal{L}^\infty(\mathbb{T})$, and $\sigma(f_t) = \text{essential range of } f_t = [(\pi/4 - t\pi) \rightarrow (3\pi/4 - t\pi)]$. Moreover, since $\text{ran } f_t \subseteq \mathbb{T}$, f_t is unitary when considered as a multiplication operator on $\mathcal{L}^2(\mathbb{T})$. Note that 1 is not an eigenvalue of f_t , so f_t is in the range of the Cayley transform κ .

We apply the inverse Cayley transform to our f_t 's in order to figure out what the unbounded operators D_t should be. For each $t \in [0, 1]$ define

$$\alpha_t(\theta) = \begin{cases} \frac{\theta}{2} + \frac{\pi}{4} - t\pi & \text{if } \theta \in [0, \pi) \\ \frac{2\pi - \theta}{2} + \frac{\pi}{4} - t\pi & \end{cases},$$

and D_t a function on the circle by

$$D_t(e^{i\theta}) = \frac{\sin(\alpha_t(\theta))}{\cos(\alpha_t(\theta)) - 1}.$$

Since $\kappa(D_t) \in \mathcal{L}^\infty(\mathbb{T})$, D_t is affiliated with $\mathcal{L}^\infty(\mathbb{T})$. Examining the definition of D_t (and the pictures in Figure 1.9, which show some of the D_t 's as functions of θ), we see that $D_t \in \mathcal{L}^\infty$ for $t < \frac{1}{4}$ and $t > \frac{3}{4}$, but D_t is an unbounded function for other values of t . Since $\{f_t\}$ is continuous as a path of operators in \mathcal{L}^∞ , $\{D_t\}$ is gap continuous; however, again from the definition, for some values of t the differences $D_t - D_0$ are not bounded, let alone norm-continuous.

Since D_t is invertible for each t (note that $\chi_{[-\frac{1}{2}, \frac{1}{2}]}(D_t) = 0$ for all $t \in [0, 1]$), the spectral flow along $\{D_t\}$ should be zero. However, if we look at $\chi_{[0, \infty)}(D_t)$ we get

- $\chi_{[0, \infty)}(D_t) = 0$ if $t \in [0, \frac{1}{4}]$
- $\chi_{[0, \infty)}(D_t) = \chi_{[0, 2\pi t - \frac{\pi}{2}] \cup [\frac{5\pi}{2} - 2\pi t, 2\pi]}$ if $t \in [\frac{1}{4}, \frac{3}{4}]$
- $\chi_{[0, \infty)}(D_t) = 1$ if $t \in [\frac{3}{4}, 1]$

This is what we mean when we refer to “spectral flow through infinity” when talking about gap continuous paths – the projection onto the positive spectrum changes even though there is no spectral flow through zero. Of course, using Wahl’s definition of spectral flow for gap-continuous paths, we do get that the spectral flow of $\{D_t\}$ is zero; the calculations are the same as those in Lemma 1.1.37 (where we show that the spectral flow of a path of invertible operators is zero), so we do not repeat them here.

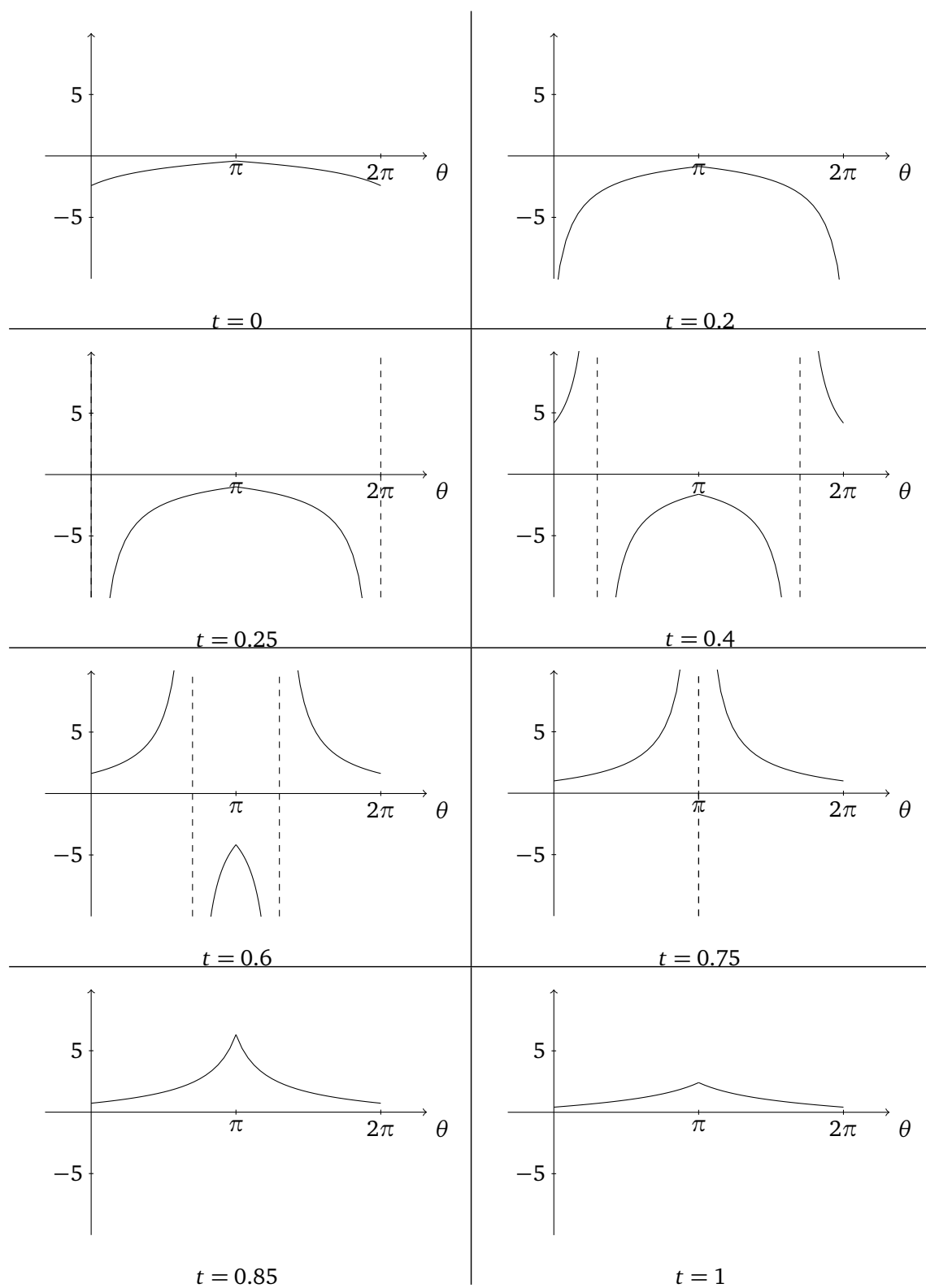


Figure 1.9: The family of functions D_t , shown for a few values of t , as functions of $\theta \in [0, 2\pi]$.

Chapter 2

Uniqueness of Spectral Flow

In this chapter, \mathcal{N} will denote a semifinite factor. As usual, denote a fixed trace on \mathcal{N} by τ , the map into the generalized Calkin algebra $\mathcal{N}/\mathcal{K}_{\mathcal{N}}$ by π and the self-adjoint Breuer-Fredholm operators in \mathcal{N} by \mathcal{BF}_{sa} ; in addition, let $\mathcal{BF}_{sa}^{\times}$ be the invertible operators in \mathcal{BF}_{sa} . The topology on \mathcal{BF}_{sa} is the usual norm topology.

Denote by $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$ the set of paths in \mathcal{BF}_{sa} with endpoints in $\mathcal{BF}_{sa}^{\times}$ (that is, continuous functions $\rho : [0, 1] \rightarrow \mathcal{BF}_{sa}$ for which $\rho(0), \rho(1) \in \mathcal{BF}_{sa}^{\times}$). It is possible to calculate the spectral flow of such paths using Definition 1.1.26; we are interested in investigating a set of conditions on a real-valued function defined on paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^{\times})$ which are sufficient to ensure that the function calculates the spectral flow. This was done in the case when $\mathcal{N} = \mathcal{B}(\mathcal{H})$ by Lesch (see [47]); the proof below follows the same main steps, but the details are by necessity different. Recall (Lemma 1.1.36) that spectral flow is invariant under homotopy with endpoints fixed and is additive under concatenation; these will be two of the required conditions for our function, the picture being completed by a normalization requirement. In Section 2.4 the setting will be modified in order to allow us to consider paths of unbounded operators.

In the following, two paths ρ and ξ will be called homotopic if there exists a continuous function $H : [0, 1] \times [0, 1] \rightarrow \mathcal{BF}_{sa}$ such that

1. $H(0, t) = \rho(t)$ and $H(1, t) = \xi(t)$ for all $t \in [0, 1]$
2. $H(s, 0), H(s, 1) \in \mathcal{BF}_{sa}^{\times}$ for all $s \in [0, 1]$

Note that, unlike the standard definition of homotopy, we do not require the endpoints to stay fixed, simply that they remain in the set of invertible operators. $H(s, t)$ will sometimes be written as the collection of functions $H_s(t)$.

Remark 2.0.1 Note that one could consider spectral flow as a function on $\Omega(\mathcal{BF}_{sa})$ instead; however, some changes need to be made to the setup in this case. In particular, homotopy would have to be restricted to homotopy with fixed endpoints. If $H : [0, 1] \times [0, 1] \rightarrow \mathcal{BF}_{sa}$ is a homotopy which does not fix the endpoints, then $\rho \sim H(0, t) * \xi * -H(1, t)$ with endpoints fixed, and the properties of spectral flow (Lemma 1.1.36) tell us that $\text{sf}(\rho) = \text{sf}(H_0) + \text{sf}(\xi) + \text{sf}(H_1)$. So if there is any spectral flow along H_0 or H_1 , we might very well get that $\text{sf}(\rho) \neq \text{sf}(\xi)$. Recall however that spectral flow along a path of invertible operators is 0 (Lemma 1.1.37), so we can choose to consider homotopies which allow the endpoints to move, as long as we set the additional condition that they remain in the set of invertible operators. Due to the nature of the following proof, it is easier to follow this latter course. In particular, we will want to apply a homotopy to some of our paths which block diagonalizes the operators (with respect to a fixed decomposition of \mathcal{H}) – while it is possible to ensure that the endpoints stay invertible, it is not at all obvious how to change the construction so that the endpoints are already block diagonal, allowing them to remain unchanged by the homotopy.

However, calculating spectral flow for paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ is not unrelated to calculating spectral flow for paths in $\Omega(\mathcal{BF}_{sa})$. Since $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times) \subseteq \Omega(\mathcal{BF}_{sa})$, a function on \mathcal{BF}_{sa} which calculates spectral flow can be restricted to a function on $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. On the other hand, a function on $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ which calculates spectral flow can be used to calculate spectral flow on $\Omega(\mathcal{BF}_{sa})$ by extending the path so the endpoints are invertible and adjusting the answer accordingly (for example, as explained in Remark 1.1.35). •

Say that a map $\mu : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times) \rightarrow \mathbb{R}$ has

♣ the **concatenation property** if, for $\rho, \xi \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ with $\rho(1) = \xi(0)$,

$$\mu(\rho * \xi) = \mu(\rho) + \mu(\xi);$$

♡ the **homotopy property** if, for ρ and ξ homotopic paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$,

$$\mu(\rho) = \mu(\xi).$$

In addition, in order for μ to calculate the spectral flow, we will require it to satisfy a normalization property.

Any interval $[a, b]$ (with $a < b$) can be mapped onto $[0, 1]$ by the linear function which maps a to 0 and b to 1. This mapping is a homeomorphism; given a continuous function $\rho : [a, b] \rightarrow \mathcal{BF}_{sa}$, the composition $[0, 1] \rightarrow [a, b] \xrightarrow{\rho} \mathcal{BF}_{sa}$ is a path as defined earlier. If we talk about paths $[a, b] \rightarrow \mathcal{BF}_{sa}$, it should be assumed that this composition is to be performed first, in order for all our paths to be standardized.

Remark 2.0.2 If ρ is a path consisting entirely of invertible operators, the following argument, identical to the one used in [47], shows that $\mu(\rho) = 0$: ρ is homotopic with endpoints in the invertibles to the constant path $\xi(t) = \rho(0)$. The homotopy property of μ gives us that $\mu(\rho) = \mu(\xi)$. However, since ξ is constant, $\xi \sim \xi * \xi$; hence $\mu(\xi) = 2\mu(\xi)$, which means $\mu(\xi) = 0$. Therefore, $\mu(\rho) = 0$. •

2.1 Normalization Property

If \mathcal{N} is a semifinite factor, the **normalization property** will take the following form:

♠ There exists a finite trace non-zero projection $P_0 \in \mathcal{N}$ such that for any projection $Q \leq P_0$, the following condition holds:

whenever $R \leq 1 - Q$ is a projection,

$$\mu(\{tQ + (2R - 1 + Q)\}_{t \in [-1, 1]}) = \tau(Q) \quad (2.1)$$

An explanation of this normalization property is forthcoming (Remark 2.1.2 below), but first we address possible simplifications, and the relation of the above to the type I case.

Remark 2.1.1 If \mathcal{N} is in fact a finite factor, then the finite trace requirement is redundant, as all projections have finite trace. In particular, if desired, one could choose $P_0 = 1$.

The normalization requirement can also be simplified if \mathcal{N} is type I . In such a case, we could instead choose P_0 to be minimal, which means that the condition $Q \leq P_0$ is satisfied only by $Q = 0$ and $Q = P_0$. When $Q = 0$ the normalization condition holds trivially (see Remark 2.0.2 for a proof that μ evaluated on a constant path will be 0), so the normalization condition given above reduces to the Lesch normalization condition. The normalization condition used by Lesch in [47] is in fact:

There is a rank one orthogonal projection $P \in \mathcal{B}(\mathcal{H})_{sa}$ such that, for all operators $A \in \mathcal{B}(\mathcal{H})_{sa}$,

$$\mu(\{tP + (I - P)A(I - P)\}_{t \in [-\frac{1}{2}, \frac{1}{2}]}) = 1.$$

Note, however, that there is a slight mistake in the statement of the above normalization condition, as $(I - P)A(I - P)$ might not be invertible, which contradicts the requirement that the endpoints of the path should be invertible. This could be fixed by changing the requirement to 'all $A \in \mathcal{B}(\mathcal{H})_{sa}$ for which $(1 - P)A(1 - P)$ is invertible', but the proof only uses those A which on $P^\perp \mathcal{H}$ look like $1 \oplus -1$ for some decomposition of $P^\perp \mathcal{H}$. If stated only for these kind of operators, the condition becomes,

There is a rank one projection $P \in \mathcal{B}(\mathcal{H})_{sa}$ such that, for any projection $R < P^\perp$,

$$\mu(\{tP + (2R - 1 + P)\}_{t \in [-\frac{1}{2}, \frac{1}{2}]}) = 1.$$

This is what the normalization property 2.1 simplifies to as well if we choose P_0 to be minimal. •

Remark 2.1.2 Note that, by design, Q and R are mutually orthogonal projections. With respect to the decomposition $Q\mathcal{H} \oplus R\mathcal{H} \oplus (1 - (Q + R))\mathcal{H}$, the matrix

$$tQ + (2R - 1 + Q) = tQ + R + [-(1 - (Q + R))]$$

which appears in the normalization property looks like

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

As t goes from -1 to 1 it is only the $Q\mathcal{H}$ corner which changes.

Recall that a self-adjoint bounded operator T can be connected to $2\chi_{[0, \infty)}(T) - 1$ by a straight line path, and the spectral flow is zero along this path (see Remark 1.1.39). This kind of construction will appear in the proof, and perhaps suggests both why on $Q^\perp \mathcal{H}$ the matrices have the form they do, and why we are content to only look at these kind of matrices instead of considering all invertible operators on the $Q^\perp \mathcal{H}$ corner. •

Lemma 2.1.3 Suppose the normalization property 2.1 holds. Then, for any finite projection $P \in \mathcal{N}$ and any projection $R \leq 1 - P$ we have

$$\mu(\{tP + (2R - 1 + P)\}_{t \in [-1, 1]}) = \tau(P)$$

(i.e. we do not need the fact that $P \leq P_0$). ☛

PROOF Consider the following cases:

(i) P is a projection with $\tau(P) \leq \tau(P_0)$ (where P_0 is the specific projection appearing in property 2.1). Suppose that $S \leq 1 - P$ is a projection; for the path defined by $B_t = tP + (2S - 1 + P)$, we want to calculate $\mu(\{B_t\})$.

Since P is a finite projection, there exists a unitary U and a $Q \leq P_0$ such that $UPU^* = Q$. Note that, since $S \leq 1 - P$, $R := USU^* \leq 1 - Q$. The group of unitaries is connected, so there exists a path U_t such that $U_0 = 1$ and $U_1 = U$. Let $H_s(t) = U_s B_t U_s^*$. This defines a homotopy from $\{B_t\}$ to $\{tQ + (2R - 1 + Q)\}$. The latter is a path for which the normalization property holds by assumption; that is, μ evaluates to $\tau(Q)$ on it. It follows, by the homotopy property of μ , that $\mu(\{B_t\}) = \tau(P)$ (which is equal to $\tau(Q)$), as desired.

(ii) Suppose P and S are mutually orthogonal finite projections, and that Equation (2.1) holds for each of P and S . We would like to show that it holds for $P + S$; that is, if $R \leq 1 - (P + S)$ is a fixed projection, and $\rho(t)$ is the path $\{t(P + S) + (2R - 1 + (P + S))\}_{t \in [-1, 1]}$ then

$$\mu(\rho) = \tau(P) + \tau(S).$$

Let $\rho_1(t) = tP + ((2R - 1) + P)$ and $\rho_2(t) = tS + (2(R + P) - (1 - S))$. That is, with respect to the decomposition $P\mathcal{H} \oplus S\mathcal{H} \oplus R\mathcal{H} \oplus (P + S + R)^\perp \mathcal{H}$ we have

$$\begin{aligned} \rho(t) &= t \oplus t \oplus 1 \oplus -1 \\ \rho_1(t) &= t \oplus -1 \oplus 1 \oplus -1 \\ \rho_2(t) &= 1 \oplus t \oplus 1 \oplus -1. \end{aligned}$$

Note that, since $R \leq 1 - (P + S) \leq 1 - P$ and the normalization property holds for P , we know that $\mu(\rho_1) = \tau(P)$. Similarly, since $(R + P) \leq 1 - S$, we have $\mu(\rho_2) = \tau(S)$. By the concatenation property, $\mu(\rho_1 * \rho_2) = \tau(P) + \tau(S)$. To finish, we claim that $\rho \sim \rho_1 * \rho_2$ via

$$H_s(t) = \begin{cases} (1-s)\rho(t) + s\rho_1(2t+1) & \text{if } t \in [-1, 0] \\ (1-s)\rho(t) + s\rho_2(2t-1) & \text{if } t \in [0, 1] \end{cases}$$

That is, once again with respect to the decomposition $P\mathcal{H} \oplus S\mathcal{H} \oplus R\mathcal{H} \oplus (P + S + R)^\perp \mathcal{H}$,

$$H_s(t) = \begin{cases} [(1-s)t + s(2t+1)] \oplus [(1-s)t + (-s)] \oplus 1 \oplus -1 & \text{if } t \in [-1, 0] \\ [(1-s)t + s(1)] \oplus [(1-s)t + s(2t-1)] \oplus 1 \oplus -1 & \text{if } t \in [0, 1]. \end{cases}$$

Since P and S have finite trace, $H_s(t)$ is Breuer-Fredholm; moreover, it should be clear that $H_s(t)$ is continuous in s and t (in particular, note that for t close to 0, $2t + 1$ is close to 1 and $2t - 1$ is close to -1).

(iii) Finally, suppose $P \in \mathcal{N}$ is a projection with $\tau(P_0) < \tau(P) < \infty$. Then we can write P as a sum of mutually orthogonal projections $P_1 + \dots + P_n + R$, where $\tau(P_i) = \tau(P_0)$ for all $1 \leq i \leq n$ and $\tau(R) < \tau(P_0)$. Repeated applications of the previous two cases can be used to show that (2.1) holds for P . ■

The above proof should indicate why the statement of the normalization property is more complicated for a general semifinite factor than it was in the type I case. Suppose we require

2.1 to hold for a projection P_0 with finite trace k , instead of for all projections $Q \leq P_0$. Using the proof technique above, we can show that property 2.1 holds for all projections whose traces are rational multiples of k , but it is not possible to extend the result for projections whose traces are the remaining real numbers in $[0, 1]$ (for a II_1 factor) or $[0, \infty]$ (for a II_∞ factor). However, we have already discussed that, if \mathcal{N} is type I , a projection of trace 1 can be used to generate all possible traces (whether $\{0, \dots, n\}$ or $\{0, 1, \dots, \infty\}$).

A final note should be made about why we restrict ourselves to factors instead of considering general von Neumann algebras. To start with, note that the above proof makes repeated use of the Comparison Theorem for projections, and the fact that two finite projections whose traces are equal are unitarily equivalent. Neither of these holds in a general von Neumann algebra. In fact, the result does not work in general; the normalization condition, as stated, is not strong enough. As a simple example, consider $\mathcal{N} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$, and the function $\mu(\{A_t \oplus B_t\}) = \text{sf}(A_t)$ defined on suitable paths. It is not hard to see that μ satisfies the homotopy, concatenation and normalization conditions stated above, but does not in general calculate spectral flow of a path in \mathcal{N} . One could replace in the normalization condition the phrase “there exists a finite projection P_0 ” by “for every projection J in the centre of the algebra $Z(\mathcal{N})$, there exists a finite trace projection $P_0 \leq J$ ”. Then, if μ satisfies this stronger normalization condition, and J is a central projection of \mathcal{N} for which $J\mathcal{N}J$ is a factor, it is not hard to see that for any appropriate path $\{JA_tJ\}$ the function $\tilde{\mu}(\{JA_tJ\}) = \mu(\{JA_tJ + J^\perp\})$ must be equal to the spectral flow defined on appropriate paths in $J\mathcal{N}J$. However, this would only be helpful if the von Neumann algebra is a countable direct sum of factors; it is unclear how to reasonably generalize the normalization condition to von Neumann algebras which are not factors.

2.2 Uniqueness in a II_1 factor

In this case, all projections are finite, so all operators are Breuer-Fredholm.

Remark 2.2.1 Suppose $\rho(t)$ and $\xi(t)$ are two paths in $\Omega(\mathcal{B}\mathcal{F}_{sa}, \mathcal{B}\mathcal{F}_{sa}^\times)$ with the same endpoints. Let $H_s(t) = s\rho(t) + (1-s)\xi(t)$. Then $H_s(t)$ is self-adjoint and Breuer-Fredholm for each t . Moreover, the endpoints remain unchanged; hence, H_s is a homotopy between ρ and ξ . Note that this would not necessarily be true if \mathcal{M} was a II_∞ factor (say), since the set of Breuer-Fredholm operators is not convex (for example, the straight line path from -1 to 1 passes through 0, which is not Breuer-Fredholm if 1 is not a finite projection). •

Theorem 2.2.2 Suppose \mathcal{M} is a II_1 factor, and $\mu : \Omega(\mathcal{B}\mathcal{F}_{sa}, \mathcal{B}\mathcal{F}_{sa}^\times) \rightarrow \mathbb{R}$ satisfies the concatenation and homotopy properties stated earlier. In addition, suppose μ satisfies the following normalization property:

- ♠ There exists a (necessarily finite trace) projection $P_0 \in \mathcal{M}$ such that for any projection $Q \leq P_0$, the following condition holds:

whenever $R \leq 1 - Q$ is a projection,

$$\mu(\{tQ + (2R - 1 + Q)\}_{t \in [-1, 1]}) = \tau(Q) \quad (2.2)$$

Then $\mu = \text{sf}$. ☞

PROOF Suppose that $\{B_t\}$ is a path in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$; we want to show that $\mu(\{B_t\}) = \text{sf}(\{B_t\})$. We will prove first that this holds for a subset of all possible paths, and then show that the general case reduces to this subset:

(i) Suppose first that $B_0 = 2P - 1$ and $B_1 = 2Q - 1$ for some projections P and Q . Then $\text{sf}(\{B_t\}) = \tau(Q - P)$ (see Remark 1.1.29).

(a) Suppose $P \leq Q$. Since \mathcal{M} is a type II_1 factor, $\{B_t\}$ is homotopic to the straight line path from B_0 to B_1 (see Remark 2.2.1). Using Lemma 2.1.3 (with $Q - P$ as the projection and $R = 0$), we can dispense with the P_0 in the normalization condition, and we get that μ evaluated on the straight line path is the trace of $Q - P$; that is, $\mu(\{B_t\}) = \text{sf}(\{B_t\})$.

(b) Now suppose P and Q are any two projections. Since \mathcal{M} is a factor, either $P \prec Q$ or $P \succ Q$; assume without loss of generality that $P \prec Q$. Hence, there exists a unitary U such that $UPU^* = Q_0 \leq Q$. Let U_t be a path connecting 1 to U (so $U_0 = 1$ and $U_1 = U$). Let $H_s(t) = U_s B_t U_s^*$. Then H_0 is our original path, and H_1 is a path between $2Q_0 - 1$ and $2Q - 1$. Using the previous case, $\mu(H_1) = \tau(Q - Q_0)$. From the homotopy property, and since $\tau(Q_0) = \tau(P)$, it follows that $\mu(H_0) = \tau(Q - P)$; so, once again, $\mu(\{B_t\}) = \text{sf}(\{B_t\})$.

(ii) For any operator T , denote by \tilde{T} the symmetry $2\chi_{[0,\infty)}(T) - 1$. Note that, if T is invertible, the straight line path from T to \tilde{T} consists of invertible operators; by Remark 2.0.2, μ evaluates to 0 on this path. Extend the path B_t by connecting \tilde{B}_0 to B_0 , and B_1 to \tilde{B}_1 by straight lines; call the resulting (normalized) path $\{C_t\}$. The value of μ along this new path is the same as $\mu(\{B_t\})$, and by case (i) we know that μ is equal to the spectral flow along the new path. ■

2.3 Uniqueness in a II_∞ factor

If we had a path $\begin{bmatrix} A_t & 0 \\ 0 & T \end{bmatrix}$ where T is fixed and invertible, and the corner on which A_t acts is finite trace, then we'd be, to all intents and purposes, back in the II_1 case. Our goal is to show that any path can be manipulated via homotopy and concatenation into paths of this special type. Start with a path $\rho \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. The proof will follow the following steps:

1. find paths ρ_1, \dots, ρ_n such that $\rho \sim \rho_1 * \rho_2 * \dots * \rho_n$ and additionally, for each ρ_i , there is a finite projection P_i such that:
 - P_i commutes with the endpoints of ρ_i , and
 - $(1 - P_i)\rho_i(t)(1 - P_i)$ is invertible for all t (as an operator on $(1 - P_i)\mathcal{H}$).

Note that, by the concatenation property, $\mu(\rho) = \sum \mu(\rho_i)$.

2. for any path ρ_i as created after the first step, there is a homotopic path $\tilde{\rho}_i$ such that all the operators along the path $\tilde{\rho}_i$ commute with P_i . Moreover, $(1 - P_i)\tilde{\rho}_i(t)(1 - P_i)$ is still invertible (as an operator on $(1 - P_i)\mathcal{H}$).
3. for any path as created after the second step, there is a homotopy which fixes the lower right hand corner.

For paths of the kind obtained after step 3, we can apply the result from the II_1 case to show that our function calculates spectral flow.

We will first state the necessary lemmas, which implement the steps outlined above; the main result, which brings them all together, is stated at the end.

Implementation of Step 1

We will want to cut up our path into smaller pathlings. However, we cannot guarantee that the operators at the subdivision points are invertible. The following lemma will allow us to manufacture paths with invertible endpoints; moreover, we can ensure that the difference between the invertible endpoint path and the original pathling is in some sense small.

Lemma 2.3.1 Assume given a path $\rho \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$, a fixed $c \in [0, 1]$ and an $\varepsilon > 0$. Then there exists a path $\xi \subset \mathcal{BF}_{sa}$ such that $\xi(0) = \rho(c)$, $\xi(1)$ is invertible, and $\|\xi(t) - \rho(c)\| \leq \varepsilon$ for all $t \in [0, 1]$. Moreover, with $\rho_1 = \rho|_{[0,c]} * \xi$ and $\rho_2 = (-\xi) * \rho|_{[c,1]}$ we have that $\rho_1, \rho_2 \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ and $\rho \sim \rho_1 * \rho_2$. \blacksquare

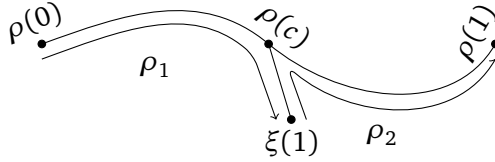


Figure 2.1: Given the path ρ , construct ξ such that with ρ_1 and ρ_2 as defined in the theorem (and indicated in this figure by arrows) we have $\rho \sim \rho_1 * \rho_2$.

PROOF Choose a $\delta < \varepsilon/2$ such that $P = \chi_{[-\delta, \delta]}(\rho(c))$ has finite trace (such a δ exists since $\rho(c)$ is Breuer-Fredholm). Let $A = \rho(c) + \varepsilon P$. Then A is invertible and $\|A - \rho(c)\| = \varepsilon\|P\| \leq \varepsilon$. Let $\xi(t)$ be the straight line path from $\rho(c)$ to A ; that is, $\xi(t) = (1-t)\rho(c) + tA = \rho(c) + t\varepsilon P$. Clearly, $\xi(t)$ is self-adjoint for all t ; moreover, since P is a finite trace projection, if we apply the map π into the generalized Calkin algebra to $\xi(t)$ we get $\pi(\xi(t)) = \pi(\rho(c))$; so $\xi(t)$ is also Breuer-Fredholm for all t .

Denote by ρ_1 the path obtained by following ρ from $\rho(0)$ to $\rho(c)$ and ξ from $\rho(c)$ to A ; parametrically, we could write

$$\rho_1(t) = \begin{cases} \rho(2tc) & \text{for } t \in [0, \frac{1}{2}] \\ \xi(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases} .$$

Denote by ρ_2 the path obtained by following ξ (in reverse) from A to $\rho(c)$, and ρ from $\rho(c)$ to $\rho(1)$; parametrically, we could write

$$\rho_2(t) = \begin{cases} \xi(1-2t) & \text{for } t \in [0, \frac{1}{2}] \\ \rho((2-2t)c + (2t-1)) & \text{for } t \in [\frac{1}{2}, 1] \end{cases} .$$

Note that $\rho_1, \rho_2 \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. We claim that $\rho \sim \rho_1 * \rho_2$. To this end, note that ρ is homotopic to the path

$$\tilde{\rho}(t) = \begin{cases} \rho(4tc) & \text{for } t \in [0, \frac{1}{4}] \\ \rho(c) & \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \\ \rho(4(1-t)c + (4t-3)) & \text{for } t \in [\frac{3}{4}, 1] \end{cases}$$

obtained by traversing ρ at different speeds (namely, between $[0, \frac{1}{4}]$ it covers ρ from 0 to c ; between $[\frac{1}{4}, \frac{3}{4}]$ it is the constant $\rho(c)$, and between $[\frac{3}{4}, 1]$ it covers ρ from c to 1). On the other hand,

$$(\rho_1 * \rho_2)(t) = \begin{cases} \rho_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \rho_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} \rho(4tc) & \text{if } t \in [0, \frac{1}{4}] \\ \xi(4t-1) & \text{if } t \in [\frac{1}{4}, \frac{1}{2}] \\ \xi(3-4t) & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ \rho((4-4t)c + (4t-3)) & \text{if } t \in [\frac{3}{4}, 1]. \end{cases}$$

Finally, the definition of ξ makes it quite obvious that $\tilde{\rho} \sim \rho_1 * \rho_2$ via

$$H_s(t) = \begin{cases} \tilde{\rho}(t) & \text{for } t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \\ \rho(c) + s(4t-1)\varepsilon P & \text{for } t \in [\frac{1}{4}, \frac{1}{2}] \\ \rho(c) + s(-4t+3)\varepsilon P & \text{for } t \in [\frac{1}{2}, \frac{3}{4}]. \end{cases}$$

By transitivity of homotopy, $\rho \sim \rho_1 * \rho_2$, concluding the proof. \blacksquare

Lemma 2.3.2 Assume given a path $\rho \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. Then there exist paths ρ_1, \dots, ρ_n and finite projections P_1, \dots, P_n such that $\rho \sim \rho_1 * \rho_2 * \dots * \rho_n$ and, for each fixed i , we have that $(1 - P_i)\rho_i(t)(1 - P_i)$ is invertible as an operator on $(1 - P_i)\mathcal{H}$ for all t along the path ρ_i . \blacktriangleright

PROOF At every fixed $t \in [0, 1]$, $\pi(\rho(t))$ is invertible; that is, $0 \notin \sigma(\pi(\rho(t)))$. Hence there exists $\delta_t > 0$ such that $[-\delta_t, \delta_t] \notin \sigma(\pi(\rho(t)))$. Let $P_t = \chi_{[-\delta_t, \delta_t]}(\rho(t))$. Then P_t is a finite projection; moreover, $(1 - P_t)\rho(t)(1 - P_t)$ is invertible as an operator on $(1 - P_t)\mathcal{H}$.

Invertible operators in a von Neumann algebra form an open set in the norm topology. Hence, there exists an open ball \mathcal{V}_t around $\rho(t)$ such that for every $S \in \mathcal{V}_t$ we have $(1 - P_t)S(1 - P_t)$ is invertible (again, as an operator on $(1 - P_t)\mathcal{H}$).

The neighbourhoods \mathcal{V}_t cover $\{\rho(t)\}_{t \in [0, 1]}$, so, by compactness, there is a finite subcover. This means that we can choose $0 = s_0 < s_1 < s_2 < \dots < s_n = 1$ points in $[0, 1]$ such that $\rho([s_i, s_{i+1}]) \subset \mathcal{V}_{s_i}$; so, in particular, $(1 - P_{s_i})(\rho(t))(1 - P_{s_i})$ is invertible for all $t \in [s_i, s_{i+1}]$ (where the P_{s_i} are the finite projections established above).

It is certainly true that ρ is obtained from the concatenation of $\rho|_{[s_0, s_1]}$, \dots , $\rho|_{[s_{n-1}, s_n]}$. However, there is no reason that the $\rho(s_i)$'s should be invertible, and we would like our paths to have invertible endpoints (as required by the definition of $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$, where we would like our paths to reside). To this effect, we employ Lemma 2.3.1 to fix the problem.

We don't have to worry about $s_0 = 0$ and $s_n = 1$, as our path ρ had invertible endpoints by assumption; but at all other division points s_i , we might have to take some sort of action. Let us first consider the division point at s_1 . Using Lemma 2.3.1, we would like to connect $f(s_1)$ via a path ξ to an invertible operator, in such a way that ξ is contained in $\mathcal{V}_{s_0} \cap \mathcal{V}_{s_1}$. This will ensure that the decomposition of the operators along ξ with respect to either P_0 or P_1 has the desired form. We know that $(1 - P_{s_0})\rho(s_1)(1 - P_{s_0})$ and $(1 - P_{s_1})\rho(s_1)(1 - P_{s_1})$ are both invertible, by construction; moreover, there exists an $\varepsilon > 0$ such that, if M is any operator for which the distance from M to $\rho(s_i)$ is less than ε , then $(1 - P_{s_0})M(1 - P_{s_0})$ and $(1 - P_{s_1})M(1 - P_{s_1})$ are both invertible. Apply Lemma 2.3.1 with this ε to find paths ρ_1 and ρ_2 with invertible endpoints and such that $\rho \sim \rho_1 * \rho_2$. By induction, apply the lemma to the remaining division points of the original path to complete the proof. ■

For each of the paths ρ_i in the above lemma, there exists a finite projection P such that $(1 - P)\rho_i(t)(1 - P)$ is invertible for all t . We want to modify ρ_i such that each operator along the path commutes with P . To this end, we will first modify the endpoints, using the next two lemmas, and then the rest of the path.

Lemma 2.3.3 Suppose T is a bounded operator which, with respect to some decomposition $P\mathcal{H} \oplus (1 - P)\mathcal{H}$ looks like $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Suppose T is invertible, and D is also invertible as an operator on the corner $(1 - P)\mathcal{H}$. Then $A - BD^{-1}C$ is invertible and the inverse of T is, with respect to the same decomposition $P\mathcal{H} \oplus (1 - P)\mathcal{H}$,

$$\begin{bmatrix} R & -RBD^{-1} \\ -D^{-1}CR & D^{-1}(1 + CRBD^{-1}) \end{bmatrix},$$

where $R = (A - BD^{-1}C)^{-1}$.

Moreover, for any $t \in [0, 1]$, the operator $\begin{bmatrix} A - (2t - t^2)BD^{-1}C & (1 - t)B \\ (1 - t)C & D \end{bmatrix}$ is also invertible, with inverse

$$\begin{bmatrix} R & -(1 - t)RBD^{-1} \\ -(1 - t)D^{-1}CR & D^{-1}(1 + (1 - t)^2CRBD^{-1}) \end{bmatrix}. \quad \blacktriangleright$$

PROOF The fact that T is invertible means that there exists an $S = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$ such that $ST = TS = 1$. Multiplication of the appropriate block submatrices for $TS = 1$ gives us

$$\begin{cases} AR_1 + BR_3 = 1 \\ CR_1 + DR_3 = 0 \end{cases}$$

Since D is invertible we must have $R_3 = -D^{-1}CR_1$; this formula, substituted in the first equation, gives us $(A - BD^{-1}C)R_1 = 1$. From $ST = 1$ get $R_1(A - BD^{-1}C) = 1$, which tells us that $A - BD^{-1}C$ is invertible on $P\mathcal{H}$, and its inverse is exactly R_1 .

Let $R = (A - BD^{-1}C)^{-1}$ (that is, dispense with the index in R_1 , for ease of notation). If we multiply T on the right by the matrix $\begin{bmatrix} R & -RBD^{-1} \\ -D^{-1}CR & D^{-1}(1 + CRBD^{-1}) \end{bmatrix}$ we get the identity matrix, as the following calculations show:

$$\begin{aligned} AR - BD^{-1}CR &= (A - BD^{-1}C)R \\ &= (A - BD^{-1}C)(A - BD^{-1}C)^{-1} = 1 \\ \\ -ARB D^{-1} + BD^{-1}(1 + CRBD^{-1}) &= -ARB D^{-1} + BD^{-1} + BD^{-1}CRBD^{-1} \\ &= BD^{-1} + (-A + BD^{-1}C)RBD^{-1} \\ &= BD^{-1} + (-1)[(A - BD^{-1}C)R]BD^{-1} \\ &= BD^{-1} - BD^{-1} = 0 \\ \\ CR - DD^{-1}CR &= 0 \\ \\ -CRBD^{-1} + DD^{-1}(1 + CRBD^{-1}) &= 1. \end{aligned}$$

Since we know that T is invertible, the right inverse must be the inverse.

Similarly, the fact that $\begin{bmatrix} A - (2t - t^2)BD^{-1}C & (1 - t)B \\ (1 - t)C & D \end{bmatrix}$ is invertible for $t \in [0, 1]$, with the purported inverse having the expression given, is verified by multiplying the two matrices, and noticing that we get the identity. ■

Lemma 2.3.4 Suppose ρ is a path in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ and P is a projection operator such that $(1 - P)\rho(t)(1 - P)$ is invertible for all $t \in [0, 1]$. Then there exists a path ξ homotopic to ρ such that $(1 - P)\xi(t)(1 - P)$ is invertible for all $t \in [0, 1]$ still holds, and in addition P commutes with the endpoints. ■

PROOF We are going to extend the path by connecting $\rho(0)$, through invertible elements, to an operator which satisfies the requirements. Suppose that, with respect to the decomposition

$P\mathcal{H} \oplus (1 - P)\mathcal{H}$, $\rho(0)$ looks like $\begin{bmatrix} A & C \\ C^* & D \end{bmatrix}$, where A and D are self-adjoint and D is invertible.

Connect this matrix to $\begin{bmatrix} A - CD^{-1}C^* & 0 \\ 0 & D \end{bmatrix}$ via the path

$$\rho_0(t) = \begin{bmatrix} A - (2t - t^2)CD^{-1}C^* & (1 - t)C \\ (1 - t)C^* & D \end{bmatrix} \text{ for } t \in [0, 1].$$

Note that, certainly, $\rho_0(t)$ is self-adjoint for each $t \in [0, 1]$. By Lemma 2.3.3, the path ρ_0 consists of invertible operators, whence $\mu(\rho_0) = 0$ (see Remark 2.0.2) and $\rho_0 \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. Moreover, since the projection onto the $1 - P$ corner is D all along the path, $(1 - P)\rho_0(t)(1 - P)$ is invertible for all t .

Similarly, define a path $\rho_1(t)$ which connects $\rho(1)$ to an operator which commutes with P . Let $\xi = (-\rho_0) * \rho * \rho_1$. Then ξ has the desired properties. ■

Implementation of step 2

Lemma 2.3.5 Suppose ρ is a path for which there exists a finite projection P which commutes with the endpoints and such that $(1 - P)\rho(t)(1 - P)$ is invertible for all t . Then ρ is homotopic to a path ξ such that $\xi(t)$ commutes with P for all t and $(1 - P)\xi(t)(1 - P)$ is invertible for all t . \blacksquare

PROOF Suppose $\rho(t) = \begin{bmatrix} A_t & C_t \\ C_t^* & D_t \end{bmatrix}$. By hypothesis, $C_0 = C_1 = 0$ and D_t is invertible for all t .

Let $H(s, t) = \begin{bmatrix} A_t & (1-s)C_t \\ (1-s)C_t^* & D_t \end{bmatrix}$; this defines a homotopy between the original path and

$$\xi(t) = \left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_t \end{bmatrix} \right\}_{t \in [0,1]}.$$

The fact that $C_0 = C_1 = 0$ means that the $H_s(0)$ and $H_s(1)$ remain constant (and hence invertible) for all s , so H_s is indeed a homotopy of paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. Finally, note that $\xi(t)$ satisfies the requirements of the theorem. \blacksquare

Implementation of step 3

Suppose we have a path which looks like $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_t \end{bmatrix} \right\}$ with respect to some decomposition $P\mathcal{H} \oplus (1 - P)\mathcal{H}$ with $\tau(P)$ finite and D_t invertible for all t . By the end of this step, we want to end up with a path where the bottom right corner is constant; to this end, consider the function $[0, 1] \times [0, 1] \rightarrow \mathcal{N}$

$$H(s, t) = \begin{bmatrix} A_t & 0 \\ 0 & D_{st} \end{bmatrix}.$$

We claim that this defines a homotopy of the original path to a path in which only the upper left corner changes, while the lower right hand corner remains fixed at D_0 . We need to show that this is indeed a homotopy of paths in $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$.

First note that the endpoints of H_s are invertible for each $s \in [0, 1]$. Since the endpoints of the original path are invertible, we know that A_0, A_1 are invertible; moreover, by assumption D_t is invertible for all t . The endpoint of H_s at $t = 0$ is $\begin{bmatrix} A_0 & 0 \\ 0 & D_0 \end{bmatrix}$, which is invertible. The endpoint of H_s at $t = 1$ is $\begin{bmatrix} A_1 & 0 \\ 0 & D_s \end{bmatrix}$, which is invertible since both A_1 and D_s are invertible.

Moreover, $H(s, t)$ is Breuer-Fredholm for each s, t , since $\ker H(s, t) = \ker A_t$, and by assumption $A_t : P\mathcal{H} \rightarrow P\mathcal{H}$, with $\tau(P) < \infty$.

Finally, we have to show $H(s, t)$ is continuous. Assume given $\varepsilon > 0$; we want to find a $\delta > 0$ such that, if $|s_1 - s_2| < \delta$ and $|t_1 - t_2| < \delta$ then $\|H(s_1, t_1) - H(s_2, t_2)\| < \varepsilon$, or, in other words,

$$\left\| \begin{bmatrix} A_{t_1} & 0 \\ 0 & D_{s_1 t_1} \end{bmatrix} - \begin{bmatrix} A_{t_2} & 0 \\ 0 & D_{s_2 t_2} \end{bmatrix} \right\| < \varepsilon.$$

Our choice of δ is going to be dictated by existing continuity conditions:

1. The path $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_t \end{bmatrix} \right\}$ is (uniformly) continuous. This means that we can find a $\delta_1 > 0$ such that if $|t_1 - t_2| < \delta_1$ then both $\|A_{t_1} - A_{t_2}\| < \varepsilon$ and $\|D_{t_1} - D_{t_2}\| < \varepsilon$.
2. The function $(s, t) \mapsto st$ (from $[0, 1] \times [0, 1] \rightarrow [0, 1]$) is likewise (uniformly) continuous; hence, we can find a $\delta_2 > 0$ such that if both $|s_1 - s_2| < \delta_2$ and $|t_1 - t_2| < \delta_2$ then $|s_1 t_1 - s_2 t_2| < \delta_1$.

Let $\delta = \min\{\delta_1, \delta_2\}$ (and note $\delta > 0$). Suppose that $|s_1 - s_2| < \delta$ and $|t_1 - t_2| < \delta$. Then, since $\delta \leq \delta_2$, it follows from item 2 above that $|s_1 t_1 - s_2 t_2| < \delta_1$, which in turn means (from item 1) that $\|D_{s_1 t_1} - D_{s_2 t_2}\| < \varepsilon$. However, since $|t_1 - t_2| < \delta \leq \delta_1$, we also have (from item 1) $\|A_{t_1} - A_{t_2}\| < \varepsilon$; and so

$$\left\| \begin{bmatrix} A_{t_1} & 0 \\ 0 & D_{s_1 t_1} \end{bmatrix} - \begin{bmatrix} A_{t_2} & 0 \\ 0 & D_{s_2 t_2} \end{bmatrix} \right\| < \varepsilon,$$

completing the proof that $H(s, t)$ is continuous.

Therefore, $H(s, t)$ is a continuous function from $[0, 1] \times [0, 1]$ to \mathcal{BF}_{sa} , so it is a homotopy from the original path $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_t \end{bmatrix} \right\}$ to the path $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_0 \end{bmatrix} \right\}$.

Let $Q = \chi_{[0, \infty)}(D_0)$; we would like to change the lower right hand corner of operators in our current path from D_0 to $2Q - 1$. To this end, denote by Q_s the straight line path from D_0 to $2Q - 1$; that is, $Q_s = sD_0 + (1 - s)(2Q - 1)$. Since D_0 is invertible, Q_s is also invertible for all s . Let

$$G(s, t) = \begin{bmatrix} A_t & 0 \\ 0 & Q_s \end{bmatrix}.$$

This defines a homotopy from $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & D_0 \end{bmatrix} \right\}$ to $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & 2Q - 1 \end{bmatrix} \right\}$.

Main Theorem

Theorem 2.3.6 Suppose \mathcal{N} is a II_∞ factor and $\mu : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times) \rightarrow \mathbb{R}$ satisfies the earlier homotopy and concatenation properties, and in addition the normalization property:

- ♠ There exists a finite projection P_0 such that for any projection $Q_0 \leq P_0$ and any projection $R \leq 1 - Q_0$

$$\mu(\{tQ_0 + (2R - 1 + Q_0)\}_{t \in [-1, 1]}) = \tau(Q_0).$$

Then $\mu = \text{sf}$. ☛

PROOF Apply the various lemmas in steps 1 to 3 to ensure that we only have to prove μ calculates spectral flow along paths which can be written as $\left\{ \begin{bmatrix} A_t & 0 \\ 0 & 2S - 1 \end{bmatrix} \right\}$ with respect to some decomposition $P\mathcal{H} \oplus (1 - P)\mathcal{H}$ with $\tau(P)$ finite and S a projection.

Define $\nu : \Omega(P\mathcal{N}P_{sa}, P\mathcal{N}P_{sa}^\times) \rightarrow \mathbb{R}$ by

$$\nu(\rho(t)) = \mu \left(\left\{ \left[\begin{array}{cc} \rho(t) & 0 \\ 0 & 2S - 1 \end{array} \right] \right\}_{t \in [0,1]} \right).$$

$P\mathcal{N}P$ is a II_1 factor. Concatenation and homotopy hold for ν since they hold for μ , so we just have to check the normalization property. We will show that the normalization property holds for the finite projection P . Suppose $Q \leq P$ and $R \leq P - Q$ are both projections in $\mathcal{B}(P\mathcal{H})$. By definition,

$$\nu(tQ + (2R - 1 + Q)) = \mu \left(\left\{ \left[\begin{array}{cc} tQ + (2R - 1 + Q) & 0 \\ 0 & 2S - 1 \end{array} \right] \right\} \right).$$

By Lemma 2.1.3, the normalization property holds with P_0 replaced by any finite trace projection; in particular, it holds for P . Since $Q \leq P$ and $R \oplus S \leq 1 - Q$,

$$\mu \left(\left\{ \left[\begin{array}{cc} tQ + (2R - 1 + Q) & 0 \\ 0 & 2S - 1 \end{array} \right] \right\}_{t \in [-1,1]} \right) = \tau(Q).$$

Therefore, the normalization property holds for ν , which means that ν calculates spectral flow on $P\mathcal{N}P$. It follows from the definition of ν that

$$\mu \left(\left[\begin{array}{cc} A_t & 0 \\ 0 & 2S - 1 \end{array} \right] \right) = \text{sf}(\{A_t\}) = \text{sf} \left(\left[\begin{array}{cc} A_t & 0 \\ 0 & 2S - 1 \end{array} \right] \right),$$

concluding the proof. ■

As already mentioned in Section 1.1.2, the self-adjoint Breuer-Fredholm operators in a type II_∞ factor split into three components, two of which are contractible. As such, the interesting calculations of spectral flow only involve the component consisting of self-adjoint Breuer-Fredholm operators which are neither essentially positive nor essentially negative; denote this component by $\mathcal{BF}_{sa,*}$. In this case, the normalization condition can be slightly simplified, so we state the uniqueness theorem for this component as a corollary:

Corollary 2.3.7 Suppose \mathcal{N} is a II_∞ factor and $\mu : \Omega(\mathcal{BF}_{sa,*}, (\mathcal{BF}_{sa,*})^\times) \rightarrow \mathbb{R}$ satisfies the earlier homotopy and concatenation properties, and in addition this new normalization property:

- ◇ There exists a finite projection P_0 such that for any projection $Q_0 \leq P_0$ there exists an infinite and co-infinite projection $R \leq 1 - Q_0$ for which

$$\mu(\{tQ_0 + (2R - 1 + Q_0)\}_{t \in [-1,1]}) = \tau(Q_0).$$

Then $\mu = \text{sf}$. ☯

PROOF The normalization condition implies that for any finite projection Q and any infinite and co-infinite projection $R \leq 1 - Q$ we have

$$\mu(\{tQ + (2R - 1)\}) = \tau(Q).$$

This follows from first using unitary equivalence of infinite and co-infinite projections to replace the 'there exists' in the condition by 'for any', and then going from $Q_0 \leq P_0$ to any Q just as in the proof of Lemma 2.1.3 (the only change needed is to mention $R \leq 1 - Q$ is infinite and coinfinite at every step, and URU^* is infinite and co-infinite if R is).

Define $\tilde{\mu} : \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times) \rightarrow \mathbb{R}$ as follows: Given $\rho \in \Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$, ρ must be contained in one of the three connected components. Define

$$\tilde{\mu}(\rho) = \begin{cases} \text{sf}(\rho) & \text{if } \rho \subset \mathcal{BF}_{sa,+} \text{ or } \rho \subset \mathcal{BF}_{sa,-} \\ \mu(\rho) & \text{otherwise} \end{cases}$$

Then $\tilde{\mu}$ satisfies the homotopy and concatenation property because each of sf and μ do, and we only have to check that the normalization property for $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ holds. Suppose Q is any finite projection and $R \leq 1 - Q$ is any projection. We want to show that

$$\tilde{\mu}(\{tQ + (2R - 1 + Q)\}_{t \in [-1,1]}) = \tau(Q)$$

If R is infinite and co-infinite, then

$$\tilde{\mu}(\{tQ + (2R - 1 + Q)\}_{t \in [-1,1]}) = \mu(\{tQ + (2R - 1 + Q)\}_{t \in [-1,1]}) = \tau(Q)$$

by the remark at the beginning of the proof regarding the stronger normalization property.

If R is finite (and co-infinite), then $Q + R$ is a finite projection; as, on $Q\mathcal{H} \oplus R\mathcal{H} \oplus (Q + R)^\perp \mathcal{H}$ the path $\{tQ + (2R - 1 + Q)\}$ looks like $t \oplus 1 \oplus -1$, it should be clear that the path lies in the essentially negative component of \mathcal{BF}_{sa} . Similarly, if R is infinite but co-finite, the path $\{tQ + (2R - 1 + Q)\}$ lies in the essentially positive component of \mathcal{BF}_{sa} . In either case,

$$\tilde{\mu}(\{tQ + (2R - 1 + Q)\}_{t \in [-1,1]}) = \text{sf}(\{tQ + (2R - 1 + Q)\}_{t \in [-1,1]}) = \tau(Q).$$

Thus the function $\tilde{\mu}$ satisfies the normalization property, and hence calculates spectral flow on $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$. Therefore, we can conclude that μ , which is a restriction of $\tilde{\mu}$, calculates spectral flow on $\Omega(\mathcal{BF}_{sa,*}, (\mathcal{BF}_{sa,*})^\times)$. ■

2.4 Unbounded Operators

Now, suppose we have a map from $[0, 1]$ into the self-adjoint unbounded operators which is continuous with respect to the gap topology. The definition used for spectral flow in this case is the one provided by Wahl in [76], and given here as Definition 1.1.34. The purpose of this section is to show that spectral flow is a unique function satisfying certain constraints similar to those from the bounded case.

In order to avoid some of the problems inherent in working with unbounded operators (such as issues having to do with the domain not being all of \mathcal{H}), we use the Cayley transform to translate the question into one about bounded operators. As already discussed in Section 1.1.3.2 of Chapter 1, applying the Cayley transform to a gap continuous path results in a norm-continuous path of unitary operators. The image of the Cayley transform is in fact all unitary operators which do not have 1 as an eigenvalue (see [8], Theorem 1.1b). Moreover, the

spectrum is transformed as shown in Figure 1.2; hence, the spectral flow of the original path can be thought of, in this new picture, as spectral flow across -1 from the upper half of the circle to the lower half of the circle. We write sf for the spectral flow across -1 for appropriate paths of unitary operators, and rely on context to clarify the meaning.

Denote by \mathcal{U}_κ the unitary operators which are in the image of the Cayley transform, and by \mathcal{U}_κ^{+1} those unitaries in \mathcal{U}_κ which do not have -1 in their spectrum (these correspond to the invertible operators in our usual self-adjoint picture). The paths $\Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$ will thus have the condition that the endpoints should be in \mathcal{U}_κ^{+1} (note that it is not a problem for any of the operators to have 1 in the spectrum, subject to the earlier caveat that 1 should not be an eigenvalue).

Theorem 2.4.1 Suppose that there is a function $\mu : \Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1}) \rightarrow \mathbb{R}$ which satisfies the homotopy and concatenation property (see page 55, except replace $\Omega(\mathcal{BF}_{sa}, \mathcal{BF}_{sa}^\times)$ by $\Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$), and in addition the normalization property below:

♡ There exists a finite trace non-zero projection $P_0 \in \mathcal{N}$ such that for any projection $Q \leq P_0$, the following condition holds:

whenever $R \leq 1 - Q$ is a projection,

$$\mu \left(\left\{ \left(\frac{t^2 - 1}{t^2 + 1} - \frac{2t}{t^2 + 1} \cdot i \right) \oplus i \oplus -i \right\}_{t \in [-1, 1]} \right) = \tau(Q) \quad (2.3)$$

(where the operator is written with respect to the decomposition $Q\mathcal{H} \oplus R\mathcal{H} \oplus (1 - (Q + R))$).

Then, $\mu = \text{sf}$. ☛

Using a slight variation on Lemma 2.1.3, we can obtain that the above normalization condition holds for any projection Q , not just $Q \leq P_0$. The only change in the proof are the entries in the matrices involved: the corner which is changing goes from i to $-i$ using the path $\frac{t^2-1}{t^2+1} - \frac{2t}{t^2+1} \cdot i$, instead of linearly from -1 to 1 , and the fixed parts stay at $(2R - 1)i$ instead of $2R - 1$. As the proof follows almost identically, we do not show it again.

For paths of (self-adjoint Breuer-Fredholm) unbounded operators continuous in the gap topology, the only definition we have for spectral flow is the one due to Wahl (see Section 1.1.3.2). We know that it satisfies the concatenation and homotopy properties (see Lemma 1.1.36), so we only have to check the normalization condition. Suppose that $\{\rho(t)\}_{t \in [-1, 1]}$ is a path for which each $\rho(t)$ has a matrix decomposition as described in the normalization condition. In order to use Wahl's definition, we apply the inverse Cayley transform κ^{-1} to obtain a path of self-adjoint operators which looks like

$$\{t \oplus -1 \oplus 1\}$$

with respect to the same decomposition. If Ξ is a normalization function for $\{\rho(t)\}$ (as described in Section 1.1.3.2), then the unitaries $U_t := e^{\pi i(\Xi(\rho(t))+1)}$ look like $\begin{bmatrix} -e^{\pi i \Xi(t)} & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the decomposition $Q\mathcal{H} \oplus Q^\perp\mathcal{H}$ (both -1 and 1 get mapped to 1 under this transformation).

Calculate the winding number of the path U_t (note that the limits of integration are -1 and 1 since our path is indexed on the interval $[-1, 1]$ instead of the more usual $[0, 1]$):

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-1}^1 \tau \left((U_t)^{-1} \frac{d}{dt} (U_t - 1) \right) dt \\
&= \frac{1}{2\pi i} \int_{-1}^1 \tau \left(\begin{bmatrix} -e^{-\pi i \Xi(t)} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -e^{\pi i \Xi(t)} \cdot \pi i \Xi'(t) & 0 \\ 0 & 0 \end{bmatrix} \right) dt \\
&= \frac{1}{2\pi i} \int_{-1}^1 \tau \left(\begin{bmatrix} \pi i \Xi'(t) & 0 \\ 0 & 0 \end{bmatrix} \right) dt \\
&= \frac{1}{2\pi i} \cdot \pi i \tau(Q) \int_{-1}^1 \Xi'(t) dt \\
&= \frac{1}{2} \cdot \tau(Q) \cdot \Xi(t) \Big|_{-1}^1 \\
&= \tau(Q).
\end{aligned}$$

To get the last line, recall that any normalization function Ξ evaluates to 1 at 1 and -1 at -1, by design. Since the spectral flow of the path is the winding number just calculated, we conclude that $\text{sf}(\{\rho(t)\})$ is equal to $\tau(Q)$, as desired.

So certainly Wahl's definition of spectral flow satisfies the conditions imposed above. Suppose μ is another function which satisfies the homotopy, concatenation and normalization conditions; we want to show that $\mu = \text{sf}$.

We will denote by $[\alpha \rightarrow \beta]$ the arc of the unit circle from angle α to angle β in the counter-clockwise direction (regardless of whether α or β is larger).

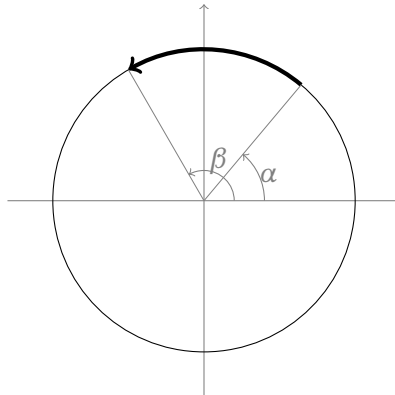


Figure 2.2: The arc highlighted in this figure will be denoted $[\alpha \rightarrow \beta]$.

The idea of the proof is to split up our path, and apply a homotopy to each piece in order to introduce holes in the spectrum at $\pm i$. Note that $\pm i$ are arbitrarily chosen; any two other points could be used, as long as one is in the upper half circle and one in the lower half circle. Once we have such holes in the spectrum, we can reduce our problem to the bounded case, as demonstrated by the proof of the following lemma.

Lemma 2.4.2 Suppose that $\rho \in \Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$ is such that $\{-i, i\} \notin \sigma(\rho(t))$ for any $t \in [0, 1]$. Then $\mu(\rho) = \text{sf}(\rho)$. \blacksquare

PROOF The fact that there are gaps in the spectrum allows us to follow the proof used by Lesch in the type I case (see [47]), a proof that was inspired by Proposition 2.10 of [8].

Let $P_t = \chi_{[\frac{\pi}{2} \rightarrow \frac{3\pi}{2}]}(\rho(t))$. The path of projections $\{P_t\}$ is continuous; this follows from the holomorphic functional calculus, which for a suitable curve Γ chosen to go around the 'left side' of the circle (using the holes at i and $-i$), tells us that

$$\chi_{[\frac{\pi}{2} \rightarrow \frac{3\pi}{2}]}(\rho(t)) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \rho(t))^{-1} d\lambda.$$

Since $\{P_t\}$ is continuous, there exists a path of unitaries U_t such that $P_t = U_t P_0 U_t^*$ (see [6], Propositions 4.3.3 and 4.6.5).

Let $P = P_0$. Define the homotopy

$$H(s, t) = U_{st} \rho(t) (U_{st})^*.$$

This allows us to get a homotopy to a path which, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, looks like $\begin{bmatrix} A_t & 0 \\ 0 & B_t \end{bmatrix}$, where A_t and B_t have to be unitary. Because of the way the original decomposition was chosen, $\sigma(B_t) \subseteq [\frac{3\pi}{2} \rightarrow \frac{\pi}{2}]$, so in particular $-1 \notin \sigma(B_t)$. On the other hand, A_t is in the image of the Cayley transform (to start with, 1 cannot be an eigenvalue of A_t , since $\sigma(A_t) \subseteq [\frac{\pi}{2} \rightarrow \frac{3\pi}{2}]$, and, further, $1 + H(1, t)$ is Breuer-Fredholm implies that so is $1 + A_t$). We will use these facts momentarily.

As in the bounded case, we can replace B_t by B_0 via the homotopy

$$G(s, t) = \begin{bmatrix} A_t & 0 \\ 0 & B_{(1-s)t} \end{bmatrix}.$$

Finally, since B_0 is unitary and $-1 \notin \sigma(B_0)$, we can replace B_0 by $i \oplus -i$ for some appropriate decomposition of \mathcal{H} (namely, let $Q = \chi_{[0 \rightarrow \pi]}(B)$, write B_0 according to its decomposition with respect to $Q\mathcal{H} \oplus Q^\perp\mathcal{H}$, and squeeze the upper arc to i and the lower arc to $-i$).

Define $\nu : \Omega(P\mathcal{N}P_{sa}, P\mathcal{N}P_{sa}^\times)$ by

$$\nu(\{\rho(t)\}) = \mu(\{\kappa(\rho(t)) \oplus i \oplus (-i)\})$$

where κ is the Cayley transform. Note that, if $0 \notin \sigma(\rho(t))$ for some t , then $-1 \notin \sigma(\kappa(\rho(t)))$; so the resulting path is in the domain of definition of μ . Since μ has the concatenation and homotopy properties, so does ν . Moreover, if $Q \leq P$ is any finite trace projection and $R \leq P - Q$, then with respect to the decomposition $P = Q + R + (P - (Q + R))$ we have

$$\nu(\{t \oplus 1 \oplus -1\}) = \mu\left(\left\{\left(\frac{t-i}{t+i} \oplus i \oplus -i\right) \oplus i \oplus -i\right\}\right);$$

the right hand side is equal to $\tau(Q)$ by the normalization property of μ .

Therefore ν satisfies the normalization property as well, so it must be the spectral flow on $P\mathcal{N}P$. Apply the same process to $\tilde{\nu}(\{\rho(t)\}) = \text{sf}(\{\kappa(\rho(t)) \oplus i \oplus (-i)\})$. By the uniqueness of spectral flow on $P\mathcal{N}P$, we can conclude that $\tilde{\nu} = \nu$. Since $A_t \in \kappa(P\mathcal{N}P)$, it follows that $\mu(\{A_t \oplus i \oplus -i\}) = \text{sf}(\{A_t \oplus i \oplus -i\})$.

Since $\{A_t \oplus i \oplus -i\}$ is homotopic to our original path $\rho(t)$, we can now conclude that $\mu(\{\rho(t)\}) = \text{sf}(\{\rho(t)\})$. ■

Therefore, in order to conclude that the spectral flow function is unique (subject to the outlined constraints), it is sufficient to show that we can homotope any path ρ to a concatenation of paths of the type covered in the lemma. The homotopy we use will be reminiscent of the one used in the case of bounded operators.

Lemma 2.4.3 Suppose that U is an unitary operator which, with respect to some decomposition of \mathcal{H} , can be written as $\begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ in such a way that $-1 \notin \sigma(Y)$. Then, for any fixed $t \in [0, 1]$, the matrix

$$Z = \begin{bmatrix} X - tV(tY + 1)^{-1}W & \sqrt{1 - t^2} \cdot V(tY + 1)^{-1} \\ \sqrt{1 - t^2} \cdot (tY + 1)^{-1}W & (Y + t)(tY + 1)^{-1} \end{bmatrix}$$

is also unitary. Moreover,

- (a) Suppose λ is either 1 or -1 and that $\lambda \notin \sigma(U)$; so $U - \lambda$ is invertible, say with inverse $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then $Z - \lambda$ is also invertible (except in the case when λ and t are both 1), with inverse

$$\begin{bmatrix} A & \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} B \\ \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} C & \frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t} D \end{bmatrix};$$

that is, $\lambda \notin \sigma(Z)$. Note that when $t = 1$ we get $Z = \begin{bmatrix} X - V(Y + 1)^{-1}W & 0 \\ 0 & 1 \end{bmatrix}$; this should make it clear why the result fails when both $\lambda = 1$ and $t = 1$.

- (b) Suppose $t \neq 1$ and 1 is an eigenvalue of Z . If $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of Z corresponding to 1, then $\begin{bmatrix} \sqrt{1+t} \cdot x \\ \sqrt{1-t} \cdot y \end{bmatrix}$ is an eigenvector of U corresponding to 1, so 1 is an eigenvalue of U .

Note in particular that this implies that, if 1 is not an eigenvalue of U , then 1 cannot be an eigenvalue of Z except in the case when $t = 1$.

- (c) $1 + Z$ is Breuer-Fredholm. ☛

PROOF The requirement that $-1 \notin \sigma(Y)$ is needed in order for the expressions used to define the entries of Z to make sense. If $t = 0$ then $tY + 1 = 1$ is invertible; on the other hand, if $t \neq 0$, then $tY + 1 = t(Y + \frac{1}{t})$ is invertible if and only if $-\frac{1}{t} \notin \sigma(Y)$ for $t \in (0, 1]$, or, equivalently, $\sigma(Y) \cap (-\infty, -1] = \emptyset$. Indeed, since Y is a cut-down of a unitary operator, $\sigma(Y) \subseteq \mathbb{D}$. The only point which is in both $(-\infty, -1]$ and \mathbb{D} is -1 , which is not in $\sigma(Y)$ by hypothesis.

We need to check that Z is unitary. Since U is unitary, we have $UU^* = 1$; we use this to verify that when we multiply Z by Z^* we get the identity operator. In order to not clutter up the presentation with tedious and not particularly illuminating calculations, we relegate this verification to the appendices (see Lemma C.0.8). By symmetry, we can use the fact that $U^*U = 1$ to show that also $Z^*Z = 1$. This concludes the proof that Z is unitary.

Let us now check that Z has the properties claimed in the statement of the lemma.

(a) We want to show that $-1 \notin \sigma(U)$ implies $-1 \notin \sigma(Z)$, and that $1 \notin \sigma(U)$ and $t \neq 1$ implies $1 \notin \sigma(Z)$. The proof is (once again) purely algebraic. From the fact that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the inverse of $U - \lambda$ we derive certain equalities, which we then use to show that when we multiply $Z - \lambda$ by the purported inverse we indeed get the identity. Once again, we relegate details to the appendix, so as to avoid a long and unenlightening calculation (see Lemma C.0.9). As we are able to exhibit an explicit inverse of $Z - \lambda$, we can conclude that $\lambda \notin \sigma(Z)$.

(b) Suppose that

$$\begin{bmatrix} X - tV(tY + 1)^{-1}W - 1 & \sqrt{1 - t^2}V(tY + 1)^{-1} \\ \sqrt{1 - t^2}(tY + 1)^{-1}W & (Y + t)(tY + 1)^{-1} - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Multiplying the second row by the vector we get

$$\sqrt{1 - t^2}(tY + 1)^{-1}Wx + ((Y + t)(tY + 1)^{-1} - 1)y = 0.$$

The left side of this equality simplifies to

$$\begin{aligned} & \sqrt{1 - t^2}(tY + 1)^{-1}Wx + ((Y + t) - (tY + 1))(tY + 1)^{-1}y = \\ & (tY + 1)^{-1}[\sqrt{1 - t^2}Wx + (1 - t)(Y - 1)y] = \\ & (tY + 1)^{-1}\sqrt{1 - t}[W(\sqrt{1 + t}x) + (Y - 1)(\sqrt{1 - t}y)]. \end{aligned}$$

We know this expression evaluates to 0; moreover, $(tY + 1)^{-1}$ is invertible, and $\sqrt{1 - t} \neq 0$ (since $t \neq 1$). Hence, we must have that $W(\sqrt{1 + t}x) + (Y - 1)(\sqrt{1 - t}y) = 0$ (that is, $\begin{bmatrix} W & Y - 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{1 + t}x \\ \sqrt{1 - t}y \end{bmatrix} = 0$). We can also rearrange this equation to get $Wx = -\sqrt{\frac{1-t}{1+t}}(Y - 1)y$, which we will use later.

Multiplying the first row by the vector we get

$$(X - 1 - tV(tY + 1)^{-1}W)x + \sqrt{1 - t^2}V(tY + 1)^{-1}y = 0.$$

Again, we simplify the left hand side of the equation

$$\begin{aligned} & (X - 1)x - tV(tY + 1)^{-1}Wx + \sqrt{1 - t^2}V(tY + 1)^{-1}y = \\ & (X - 1)x + V(tY + 1)^{-1}[-tWx + \sqrt{1 - t^2}y]. \end{aligned}$$

Using the fact that $Wx = -\sqrt{\frac{1-t}{1+t}}(Y - 1)y$, we can simplify this further

$$\begin{aligned} & (X - 1)x + V(tY + 1)^{-1}\left[\frac{t(\sqrt{1-t})}{\sqrt{1+t}}(Y - 1)y + \sqrt{1 - t^2}y\right] = \\ & (X - 1)x + V(tY + 1)^{-1} \cdot \frac{\sqrt{1-t}}{\sqrt{1+t}}(tY)y + V(tY + 1)^{-1} \cdot \sqrt{1 - t} \cdot \left(\sqrt{1 + t} - \frac{t}{\sqrt{1+t}}\right) \cdot y = \\ & (X - 1)x + \frac{\sqrt{1-t}}{\sqrt{1+t}} \cdot V(1 - (tY + 1)^{-1})y + \frac{\sqrt{1-t}}{\sqrt{1+t}}V(tY + 1)^{-1}y = \\ & (X - 1)x + \frac{\sqrt{1-t}}{\sqrt{1+t}} \cdot Vy - \frac{\sqrt{1-t}}{\sqrt{1+t}} \cdot V(tY + 1)^{-1}y + \frac{\sqrt{1-t}}{\sqrt{1+t}}V(tY + 1)^{-1}y = \\ & (X - 1)x + \frac{\sqrt{1-t}}{\sqrt{1+t}} \cdot Vy. \end{aligned}$$

We know that this expression is 0. Multiply by $\sqrt{1+t}$ and rearrange to get

$$(X-1)(\sqrt{1+t} \cdot x) + V(\sqrt{1-t} \cdot y) = 0$$

Therefore, from the above calculations, we can conclude that

$$\begin{bmatrix} X-1 & V \\ W & Y-1 \end{bmatrix} \begin{bmatrix} \sqrt{1+t} \cdot x \\ \sqrt{1-t} \cdot y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As $t \neq 1$, both $\sqrt{1+t}$ and $\sqrt{1-t}$ are non-zero. Therefore, since $\begin{bmatrix} x \\ y \end{bmatrix}$ is not the zero vector, neither is $\begin{bmatrix} \sqrt{1+t} \cdot x \\ \sqrt{1-t} \cdot y \end{bmatrix}$. Hence, the latter is an eigenvector of $\begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ corresponding to eigenvalue 1.

(c) Note that $(Y+t)(tY+1)^{-1} + 1 = (Y+t+(tY+1))(tY+1)^{-1} = (t+1)(Y+1)(tY+1)^{-1}$. Since $t \in [0, 1]$ and $-1 \notin \sigma(Y)$, this expression is invertible. By Lemma 1.1.21, $Z+1$ is Breuer-Fredholm (recall P has finite trace by hypothesis). ■

We use this kind of transformation to create gaps at i and $-i$ along the path. What we want is to split up the path in portions, so that on each portion we can write the operators as $\begin{bmatrix} X & V \\ W & Y \end{bmatrix}$, with $\sigma(X)$ contained in the left side of the unit circle, away from $\pm i$. Since we want to apply the homotopy suggested by Lemma 2.4.3, we also need to ensure that $-1 \notin \sigma(Y)$, and in the operators obtained after the homotopy, $\sigma(X - V(Y+1)^{-1}W)$ is also kept away from $\pm i$.

In the following, the arc $[\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}]$ is going to make an appearance. For quick reference, we show this arc below:

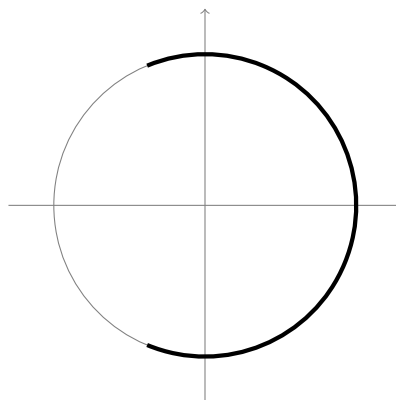


Figure 2.3: The arc $[\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}]$.

The spectrum of X is going to be constrained away from the highlighted area. The endpoints of the arc are chosen somewhat arbitrarily; the only requirement was that the arc should not contain -1 , but should have $\pm i$ in its interior. The following theorem will be used to argue about spectra of operators which are close to each other:

Theorem 2.4.4 ([44], p. 208, Theorem 3.1 and Remark 3.2) Suppose T is a closed linear operator, A is a bounded linear operator and Γ is a compact subset of the resolvent set of T . Then, if $\|A\| < \min_{z \in \Gamma} \|(T - z)^{-1}\|^{-1}$, Γ is contained in the resolvent set of $T + A$. \blacktriangleright

We can now establish the result below.

Lemma 2.4.5 Given a path ρ we can find ρ_1, \dots, ρ_n such that $\rho \sim \rho_1 * \dots * \rho_n$ and, for each ρ_i there exists a finite trace projection P_i such that, with respect to the decomposition $P_i \mathcal{H} \oplus P_i^\perp \mathcal{H}$,

each $\rho_i(t)$ can be written as a matrix $\begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ satisfying

- $\sigma(X) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$,
- $-1 \notin \sigma(Y)$, and
- $\sigma(X - V(Y + 1)^{-1}W) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$. \blacktriangleright

PROOF Since $\rho(t) + 1$ is Breuer-Fredholm, there exists a $0 < \theta < \frac{\pi}{4}$ such that the projection $P := \chi_{[\pi - \theta \rightarrow \pi + \theta]}(\rho(t))$ has finite trace. Then, with respect to the decomposition $P \mathcal{H} \oplus P^\perp \mathcal{H}$, $\rho(t)$ looks like $U_0 = \begin{bmatrix} X_0 & 0 \\ 0 & Y_0 \end{bmatrix}$; moreover, $\sigma(X_0) \subseteq [\pi - \theta \rightarrow \pi + \theta] \subseteq [\frac{3\pi}{4} \rightarrow \frac{5\pi}{4}]$ and $\sigma(Y_0) \subseteq [\pi + \theta \rightarrow \pi - \theta]$. In particular, $\sigma(X_0) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$ and $-1 \notin \sigma(Y_0)$. Any other unitary U can be written as $\begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ with respect to the same decomposition; we want to find a neighbourhood of U_0 in which U satisfies the conditions stated in the theorem.

Let $\delta_1 = \min_{z \in [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}]} \|(X_0 - z)^{-1}\|^{-1}$, $\delta_2 = \|(Y_0 + 1)^{-1}\|^{-1}$. The following restrictions on U would help us achieve our requirements:

- Suppose $\|X - X_0\| < \delta_1$. Then, by Theorem 2.4.4, $\sigma(X) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$.
- If $\|Y - Y_0\| < \delta_2$, by the same theorem, $-1 \notin \sigma(Y)$.
- Suppose $\|V\|, \|W\| < \sqrt{\frac{\delta_1 \cdot \delta_2}{4}}$, $\|X - X_0\| < \frac{\delta_1}{2}$ and $\|Y - Y_0\| < \frac{\delta_2}{2}$. Then

$$\|Y - Y_0\| \cdot \|(Y_0 + 1)^{-1}\| \leq \frac{\delta_2}{2} \cdot \frac{1}{\delta_2} = \frac{1}{2} < 1,$$

so by Theorem 1.1.31 we have that

$$\|(Y + 1)^{-1}\| = \|(Y_0 + 1 + (Y - Y_0))^{-1}\| \leq \frac{\|(Y_0 + 1)^{-1}\|}{1 - \frac{1}{2}} = \frac{2}{\delta_2}.$$

In turn, this estimate allows us to say that $\|V(Y + 1)^{-1}W\| \leq \frac{\delta_1 \cdot \delta_2}{4} \cdot \frac{2}{\delta_2} = \frac{\delta_1}{2}$, which means that

$$\|(X - V(Y + 1)^{-1}W) - X_0\| \leq \|X - X_0\| + \|V(Y + 1)^{-1}W\| < \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1.$$

Hence, applying Theorem 2.4.4 once again, we get $\sigma(X - V(Y + 1)^{-1}W) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$ as well.

Combining all these requirements, if we let $\delta = \min \left\{ \frac{\delta_1}{2}, \frac{\delta_2}{2} \right\} > 0$, which implies in turn that $\delta < \sqrt{\frac{\delta_1 \cdot \delta_2}{4}}$, we can get an open ball around U_0 (of radius δ) such that, if U is in this open ball and $U = \begin{bmatrix} X & V \\ W & Y \end{bmatrix}$ with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, then X, V, W and Y satisfy the three conditions laid out in the theorem.

Since at each $\rho(t)$ we can place an open ball as described above, by compactness of $\{\rho(t)\}$ there is a finite subcover. The proof continues as in the bounded case: we have to make sure that -1 is not in the spectrum of the division points, which we can do by taking a small projection around -1 and rotating it up, in such a way that we don't go out of the open cover. ■

Lemma 2.4.6 Suppose ρ is a path in $\Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$ for which there exists a finite trace projection P with respect to which $\rho(t)$ can be written as $\begin{bmatrix} X_t & V_t \\ W_t & Y_t \end{bmatrix}$ such that

- $\sigma(X_t) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$
- $-1 \notin \sigma(Y_t)$, and
- $\sigma(X_t - V_t(Y_t + 1)^{-1}W_t) \cap [\frac{11\pi}{8} \rightarrow \frac{5\pi}{8}] = \emptyset$.

Then ρ is homotopic to a path ξ such that $\sigma(\xi(t)) \cap \{i, -i\} = \emptyset$. ☞

PROOF Define the function $F : [0, 1] \times [0, 1] \rightarrow \mathcal{U}$ by

$$F(s, t) = \begin{bmatrix} X_t - sV_t(sY_t + 1)^{-1}W_t & \sqrt{1-s^2} \cdot V_t(sY_t + 1)^{-1} \\ \sqrt{1-s^2} \cdot (sY_t + 1)^{-1}W_t & (Y_t + s)(sY_t + 1)^{-1} \end{bmatrix}.$$

Then $F(s, t)$ is continuous. The problem with this function is that (unless $P = 1$), it is not a homotopy of paths in $\Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$, since $F_1(t)$ has 1 as an eigenvalue (with eigenspace $P^\perp\mathcal{H}$) for all t .

However, we do know that 1 is not an eigenvalue of $F(s, t)$ for any $t \in [0, 1]$ as long as $s < 1$ (by Lemma 2.4.3). So all we need to do is find a place to stop before we get to $s = 1$. We know that $\sigma(F_1(t)) \subset [\frac{5\pi}{8} \rightarrow \frac{11\pi}{8}] \cup \{1\}$; so in particular $\sigma(F_1(t))$ does not contain either i or $-i$. For each t , there exists a $\gamma_t > 0$ such that if T is a unitary matrix with $\|T - F_1(t)\| < \delta_t$, then $\sigma(T) \cap \{i, -i\} = \emptyset$. By compactness, finitely many of the open balls $B(F_1(t); \gamma_t)$ cover $\{F_1(t)\}$.

We want to find an r , $0 \leq r < 1$, such that $F_r(t)$ is contained in the union of these finitely many open balls. Denote the open balls by $B(F_1(t_i); \gamma_{t_i})$ with $t_0 < t_1 < \dots < t_n$. The path $\{F_1(t)\}$ is a compact set contained in the open set $\cup B(F_1(t_i); \gamma_{t_i})$, so there exists an ε -neighbourhood of $\{F_1(t)\}$ which is also contained in the set. By uniform continuity of the map F , there is a δ such that if $|s_1 - 1| < \delta$ then $|F_{s_1}(t) - F_1(t)| < \varepsilon$ for all t . Fix $r < 1$ such that $|r - 1| < \delta$. Then F_r is contained in the aforementioned ε -neighbourhood of F_1 , which in turn is a subset of $\cup B(F_1(t_i); \gamma_{t_i})$.

By design, the unitaries in $B(F_1(t_i); \gamma_{t_i})$ do not contain $\pm i$ in their spectrum. It follows that $\sigma(F_r(t)) \cap \{-i, i\} = \emptyset$. Then $F|_{[0, r] \times [0, 1]}$ is a homotopy of paths in $\Omega(\mathcal{U}_\kappa, \mathcal{U}_\kappa^{+1})$ (as follows mainly from Lemma 2.4.3):

- $F(s, t)$ is unitary; in addition, if $s < 1$, 1 is not an eigenvalue of $F(s, t)$ since it is not an eigenvalue of $F(0, t)$ (part (b) of aforementioned lemma)
- $F(s, 0)$ and $F(s, 1)$ do not have -1 in the spectrum for any $s \in [0, r]$, as $F(0, 0)$ and $F(0, 1)$ are known to not have -1 in the spectrum (part (a) of aforementioned lemma)

Therefore $\xi(t) = F_r(t)$ satisfies the requirements. ■

Remark 2.4.7 Consider the homotopy F_s from the proof of Lemma 2.4.6. While it is true that F_1 is not in the image of the Cayley transform, we could define spectral flow for paths of unitary operators which have a gap in the spectrum at $\pm i$ similarly to the definition in the type I gap continuous case (see Section 1.1.3.2 for an overview of this type of definition for spectral flow). Using the Wahl definition of spectral flow, it would follow that F_0 is homotopic to F_r (where the r is the one found in the proof of the theorem), so $\text{sf}\{F_0\} = \text{sf}\{F_r\}$; then, using the spectral flow theory for operators with a gap at $\pm i$, we could conclude that F_r is homotopic to F_1 and so $\text{sf}\{F_r\} = \text{sf}\{F_1\}$.

It follows that, theoretically speaking, one can use the construction described above to actually calculate the spectral flow of a path (namely, break up the path, modify the break points to be invertible, apply our homotopy to each pathling, and add up the spectral flows of the resulting pathlings). Unfortunately, the construction is not particularly easy to perform. Moreover, trying to define spectral flow this way and then derive the properties of spectral flow (or even show that it is well defined) has not led anywhere fruitful, and, since it is not clear what the homotopy actually does to the spectrum (besides squishing some of it towards 1), provides no additional insight into the spectral flow. ●

Chapter 3

Integral Formulas for Spectral Flow

In this chapter, we present a different proof of the following result from [15].

Theorem 3.0.8 (Spectral Flow [15], Corollary 9.4) Let (\mathcal{N}, D_0) be an odd p -summable Breuer-Fredholm module for the unital Banach $*$ -algebra \mathfrak{A} , and let $P = \chi_{[0, \infty)}(D_0)$. Then for each $u \in \mathcal{U}(\mathfrak{A})$ for which the domain of D_0 is invariant and $[D_0, u]$ is bounded, PuP is a Breuer-Fredholm operator in $P\mathcal{N}P$.

Define the constant $\tilde{C}_{\frac{p}{2}} = \int_{-\infty}^{\infty} (1+x^2)^{-\frac{p}{2}} dx$, which can also be shown to be equal to $\frac{\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p}{2})}$. Then $\alpha_D(X) = \frac{1}{\tilde{C}_{\frac{p}{2}}} \cdot \tau(X(1+D^2)^{-\frac{p}{2}})$ is an exact one-form on the manifold $D_0 + \mathcal{N}_{sa}$.

Moreover, if $\{D_t^u\}$ is any piecewise C^1 path in $D_0 + \mathcal{N}_{sa}$ from D_0 to uD_0u^* (for example, the linear path connecting the two operators), then

$$\text{ind}(PuP) = \text{sf}(\{D_t^u\}) = \frac{1}{\tilde{C}_{\frac{p}{2}}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t^u) \cdot (1+(D_t^u)^2)^{-\frac{p}{2}} \right) dt,$$

the integral of the above-mentioned exact one-form along the path $\{D_t^u\}$. ✎

This result is used in the proof in [17] of the Local Index Theorem, which is stated and explained in Section 1.2.3 of this thesis (see Theorem 1.2.12). Because of this connection, we very briefly explain the approach to the proof of the Local Index Theorem in the next section. Then, returning to Theorem 3.0.8, in Section 3.2 we describe the proof given by Carey and Phillips in [15]. The rest of the chapter is concerned with our new proof, the steps of which are summarized in Section 3.3.

3.1 Application of Formula to Local Index Theorem

The proof described below can be found, in detail, in [17]. Recall that the idea of the Local Index formula (Theorem 1.2.12) is to start with a p -summable semifinite spectral triple $(\mathfrak{A}, \mathcal{H}, D)$, with $\mathfrak{A} \subset \mathcal{N}$ for a semifinite von Neumann algebra \mathcal{N} with trace τ , and, for each u for which $[D, u]$ is bounded, to end up with a formula which calculates spectral flow from D to uD_0u^* as a sum of residues of zeta functions of D .

To allow for more maneuvering room, one doubles up the data, replacing \mathcal{H} by $\mathbb{C}^2 \otimes \mathcal{H}^2$, and \mathcal{N} by $\tilde{\mathcal{N}} = \mathbb{M}_2 \otimes \mathbb{M}_2(\mathcal{N})$. In this new setting, the operators D and u are replaced by

$$\tilde{D} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -iu^* \\ iu & 0 \end{bmatrix}.$$

One then defines a grading on $\mathbb{C}^2 \otimes \mathcal{H}^2$ via $\Gamma = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and a graded trace on \mathcal{N} by $S\tau(a) = \frac{1}{2}\tau(\Gamma a)$ (where $\tau(\Gamma a)$ is used to indicate that we add up the values of the trace τ on the diagonal entries of Γa , considered as an element of $\mathbb{M}_4(\mathcal{N})$).

The next step is to use Theorem 3.0.8 to get a formula for spectral flow in this new setting (i.e. using the supertrace). The goal is to show that on the manifold

$$\Phi = \{\tilde{D} + X \mid X \in \mathcal{N}_{sa} \text{ and } [X, \Gamma] = 0\},$$

at any $D_1 \in \Phi$ the map $\alpha_{\tilde{D}_1}(X) = S\tau\left(X(1 + \tilde{D}_1^2)^{-\frac{n}{2}}\right)$ defines an exact form (this is Lemma 5.6 of [17]). Moreover, if one defines

$$\tilde{D}_r = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} (1-r)D + ru^*Du & 0 \\ 0 & (1-r)D + ruDu^* \end{bmatrix},$$

the spectral flow from D to u^*Du is given by $\frac{1}{2} \cdot \frac{1}{c_{n/2}} \cdot \int_0^1 S\tau\left(\frac{d}{dr}(\tilde{D}_r) \cdot (1 + \tilde{D}_r^2)^{-\frac{n}{2}}\right) dr$ (Lemma 5.7 of [17]). As this is the part of the proof that uses Theorem 3.0.8, we include a sketch of the proof of these lemmas. In order to change the setting to an odd module, one replaces D by $\hat{D} = \Gamma\tilde{D}$ and considers the space $\Gamma\Phi = \{\Gamma\tilde{D} + \Gamma X \mid X \in \mathcal{N}_{sa} \text{ and } [X, \Gamma] = 0\}$, which can be verified to be a subspace of $\hat{D} + \mathcal{N}_{sa}$. Then (\tilde{N}, \hat{D}) is an odd p-summable Breuer-Fredholm module, allowing us to define an exact one-form as in Theorem 3.0.8. The correspondence of paths in $\Gamma\Phi$ with paths in Φ is sufficient to prove that $\alpha_{\tilde{D}_1}$ is also an exact one-form. To prove Lemma 5.7 of [17] (the supertrace formula for spectral flow), one simply evaluates the supertrace and relies on the formula in the original setting, for the Fredholm module (\mathcal{N}, D) . Let $A_r = (1-r)D + ru^*Du$ (the straight line path from D to u^*Du) and $B_r = (1-r)D + ruDu^*$ (the straight line path from D to $uD u^*$). Then $\text{sf}(D, uDu^*) = \frac{1}{c_{\frac{n}{2}}} \int_0^1 \tau\left(\frac{d}{dr}A_r(1 + A_r^2)^{-\frac{n}{2}}\right) dr$ and $\text{sf}(D, u^*Du)$ is given by a similar formula, except with A_r replaced by B_r . On the other hand, $\tilde{D}_r = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} A_r & 0 \\ 0 & B_r \end{bmatrix}$, whence it is easily verified that

$$\frac{d}{dr}(\tilde{D}_r) \cdot (1 + \tilde{D}_r^2)^{-\frac{n}{2}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} \frac{d}{dr}(A_r)(1 + A_r^2)^{-\frac{n}{2}} & 0 \\ 0 & \frac{d}{dr}(B_r)(1 + B_r^2)^{-\frac{n}{2}} \end{bmatrix}.$$

The supertrace of $\frac{d}{dr}(\tilde{D}_r) \cdot (1 + \tilde{D}_r^2)^{-\frac{n}{2}}$ is thus, by definition,

$$\frac{1}{2} \cdot \tau\left(1 \otimes \begin{bmatrix} \frac{d}{dr}(A_r)(1 + A_r^2)^{-\frac{n}{2}} & 0 \\ 0 & -\frac{d}{dr}(B_r)(1 + B_r^2)^{-\frac{n}{2}} \end{bmatrix}\right),$$

which evaluates to $\frac{1}{2} \cdot 2 \cdot \tau\left(\frac{d}{dr}(A_r)(1 + A_r^2)^{-\frac{n}{2}}\right) - \tau\left(\frac{d}{dr}(B_r)(1 + B_r^2)^{-\frac{n}{2}}\right)$. In particular,

$$\begin{aligned} & \frac{1}{2} \cdot \frac{1}{c_{n/2}} \cdot \int_0^1 S\tau\left(\frac{d}{dr}(\tilde{D}_r) \cdot (1 + \tilde{D}_r^2)^{-\frac{n}{2}}\right) dr \\ &= \frac{1}{2} \cdot \frac{1}{c_{n/2}} \cdot \int_0^1 \tau\left(\frac{d}{dr}(A_r)(1 + A_r^2)^{-\frac{n}{2}}\right) - \tau\left(\frac{d}{dr}(B_r)(1 + B_r^2)^{-\frac{n}{2}}\right) dr \\ &= \frac{1}{2} \cdot [\text{sf}(D, u^*Du) - \text{sf}(D, uDu^*)]. \end{aligned}$$

Since $\text{sf}(D, uDu^*) = -\text{sf}(D, u^*Du)$, this shows that the supertrace formula, which is the integral of the exact one-form $\alpha_{\tilde{D}_1}$, calculates the spectral flow from D to u^*Du (which is the negative of the spectral flow from D to uDu^*).

The fact that $\alpha_{\tilde{D}_1}$ is an exact one-form allows one to change the path of integration. Start with $D_{r,0} = \tilde{D}_r$, and gradually change the off-diagonal entries (for each s_0 , D_{r,s_0} is a straight line):

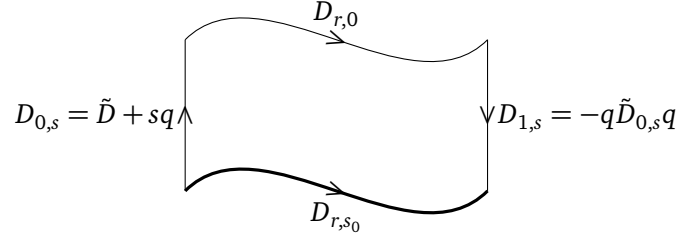


Figure 3.1: Changing the path of integration to obtain a new formula for spectral flow – the integral along the thickened line is equal to the sum of the integrals along the remaining three lines, in the direction indicated.

Fix s_0 . The integral along the line from $D_{0,0}$ to D_{0,s_0} is equal to the negative of the integral from $D_{1,0}$ to D_{1,s_0} (this follows from $S\tau(\frac{d}{ds}(D_{1,s})(1+D_{1,s}^2)^{-\frac{n}{2}}) = -S\tau(\frac{d}{ds}(D_{0,s})(1+D_{0,s}^2)^{-\frac{n}{2}})$, Lemma 5.9 of [17]). The integral along $D_{r,0}$ gives a multiple of the spectral flow. Finally, taking the limit as $s_0 \rightarrow \infty$ and using the fact ([17], Lemma 5.8) that $\lim_{s_0 \rightarrow \infty} \int_0^1 S\tau\left(\frac{d}{dr}D_{r,s_0} \cdot (1+D_{r,s_0}^2)^{-\frac{n}{2}}\right) dr = 0$, one gets

$$\begin{aligned} \text{sf}(D, uDu^*) &= \frac{1}{C_{n/2}} \int_0^\infty S\tau\left(\frac{d}{ds}D_{0,s}(1+D_{0,s}^2)^{-\frac{n}{2}}\right) ds \\ &= \frac{1}{C_{(\frac{p}{2}+r)}} \int_0^\infty S\tau(q(1+s^2+\tilde{D}^2+s\{\tilde{D},q\})^{-(\frac{p}{2}+r)}) ds, \end{aligned}$$

where in the second line we used the definition of $D_{0,s}$ and replaced n by $p+2r$, where p is the spectral dimension of the spectral triple.

Since the spectrum of $1+s^2+\tilde{D}^2+s\{\tilde{D},q\} = 1+D_{0,s}^2$ is contained in $[1, \infty)$, one can find a vertical line l in the complex plane such that any complex number $\lambda \in l$ is not in the spectrum of $1+s^2+\tilde{D}^2+s\{\tilde{D},q\}$. One then uses the Cauchy integral formula to rewrite the integrand

$$(1+s^2+\tilde{D}^2+s\{\tilde{D},q\})^{-(\frac{p}{2}+r)} = \frac{1}{2\pi i} \int_l \lambda^{-(\frac{p}{2}+r)} \cdot (\lambda - (1+\tilde{D}^2+s^2+s\{\tilde{D},q\}))^{-1} d\lambda.$$

The resolvent in the integrand can then be expanded, allowing us to (eventually) get back to resolvents of \tilde{D} instead. It is easy to check that, algebraically speaking, for any $M \geq 0$ we have the equality

$$(\lambda - (A+B))^{-1} = \sum_{m=0}^M ((\lambda - A)^{-1}B)^m (\lambda - A)^{-1} + ((\lambda - A)^{-1}B)^{M+1} (\lambda - (A+B))^{-1}.$$

Taking into account spectra and domains, one verifies that this formula can be used with $A = 1 + \tilde{D}^2 + s^2$, $B = \{\tilde{D}, q\}$ and λ a complex number on the line l . That is, using the resolvent

formula and rearranging the last expression, one gets, with $R_s(\lambda) = (\lambda - (1 + \tilde{D}^2 + s^2))^{-1}$ and $\tilde{R}_s(\lambda) = (\lambda - (1 + \tilde{D}^2 + s^2 + s\{\tilde{D}, q\}))^{-1}$, that for any $M > 0$

$$C_{\frac{p}{2}+r} \text{sf}(D, uDu^*) = \int_0^\infty S\tau \left(\sum_{m=0}^M \frac{1}{2\pi i} \int_l \lambda^{-(\frac{p}{2}+r)} s^m q \cdot (R_s(\lambda)\{\tilde{D}, q\})^m \cdot R_s(\lambda) d\lambda + \frac{1}{2\pi i} \int_l \lambda^{-(\frac{p}{2}+r)} q \cdot (R_s(\lambda)\{\tilde{D}, q\})^{M+1} \cdot \tilde{R}_s(\lambda) d\lambda \right).$$

One then shows that each term in the sum appearing in the supertrace is trace-class, allowing the exchange of the supertrace and summation symbol. It is at this point that terms with m odd can be shown to have supertrace zero (see the proof of Lemma 7.3 of [17]). Moreover, if M is chosen large enough (say $M = 2\lfloor \frac{p}{2} \rfloor + 1$), then the remainder term (i.e. the term containing $\tilde{R}_s(\lambda)$), considered as a function of $r \in \mathbb{C}$, can be shown to be holomorphic on some half-plane $\Re(r) > -\frac{p-\delta'}{2}$ with $0 < \delta' < 1$. This is the content of Lemmas 7.3 and 7.4 of [17].

We have avoided so far mentioning the order of an operator, but now we come to a result in which order plays an important role. So as to accomodate unbounded operators D which are not invertible, one replaces the usual modulus and derivation by $|D|_1 = (1 + D^2)^{\frac{1}{2}}$ and $\delta_1(T) = [|D|_1, T]$ for $T \in \text{Dom}(\delta)$. An operator is said to have order at most r (with $r \in \mathbb{R}$) if it belongs to the space $|D|_1^r \cdot (\cap_{n \geq 0} \delta_1^n)$. A more in-depth discussion of order can be found in Section 6.1 of [17]; we simply mention that operators of order zero are bounded, and operators of order $-n$ are $\frac{p}{n}$ -summable (Remark after Definition 6.6 in [17]), which suggests that operators with a smaller order are better behaved. Lemma 6.9 of [17] establishes the means for permuting powers of $R_s(\lambda)$ past an operator of order n , at the price of commutators with D^2 and an error term of suitably small order. Repeated application of this result leads to Lemma 6.11 of [17], which can be used to expand the $(R_s(\lambda)\{\tilde{D}, q\})^m$ appearing in the last stated formula for spectral flow. There is one more thing to worry about before we state the formula, namely, the order of $\{\tilde{D}, q\}$. A simple calculation using the definitions of \tilde{D} and q shows that

$$\{\tilde{D}, q\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & [D, u^{-1}] \\ -[D, u] & 0 \end{bmatrix}.$$

The notation $T^{(j)}$ denotes the iterated commutator of an operator T with D^2 j times; be warned that usually such iterated commutators will not be bounded operators. Corollary 6.8 of [17] shows that if $(\mathfrak{A}, \mathscr{H}, D)$ is a QC^∞ spectral triple and $a \in \mathfrak{A}$ then, for any $n \geq 0$, $a^{(n)}$ and $[D, a]^{(n)}$ both have order at most n . In particular, $\{\tilde{D}, q\}$ has order at most 0, and the $\{\tilde{D}, q\}^{(j)}$ factors appearing below have order at most j . Finally, here's the formula implied by Lemma 6.11 of [17]: for $M \geq 0$ we have

$$(R_s(\lambda)\{\tilde{D}, q\})^m R_s(\lambda) = \sum_{|k|=0}^M C(k) \{\tilde{D}, q\}^{(k_1)} \{\tilde{D}, q\}^{(k_2)} \dots \{\tilde{D}, q\}^{(k_m)} R_s(\lambda)^{m+|k|+1} + E_{M,m}(\lambda),$$

where the summation is taken over m -tuples $k = (k_1, k_2, \dots, k_m)$, the notation $|k|$ denotes $k_1 + k_2 + \dots + k_m$, and $C(k)$ is a constant which depends on k and m . Moreover, $E_{M,m}$ is an error term whose formula is known and whose order is at most $-2m - M - 3$ (Lemma 6.9 of [17]). In other words, the order of $E_{M,m}$ can be made as small as desired by increasing the value of M (the number of terms in the summation).

Grouping the resolvents together allows us to take out of the integral the operators $\{\tilde{D}, q\}^{(k)}$ – which do not depend on λ – and apply the Cauchy formula in the opposite direction to evaluate the line integrals (Proposition 8.1 of [17]). As the goal is to eventually evaluate the residue of the right hand side at $r = -\frac{p-1}{2}$, we do not attempt to simplify the error term, as it is holomorphic in a neighbourhood of the point of interest. With the choice of $N = \lfloor \frac{p}{2} \rfloor + 1$, this gives the formula (from Proposition 8.1 of [17])

$$C_{\frac{p}{2}+r} \text{sf}(D, u^* Du) = \sum_{m=1, \text{ odd}}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \cdot C(k) \cdot \frac{\Gamma(\frac{p}{2}+r+m+|k|)}{\Gamma(\frac{p}{2}+r)(|k|+m)!} \times \\ \times \int_0^\infty s^m \cdot S\tau \left(q\{\tilde{D}, q\}^{(k_1)} \dots \{\tilde{D}, q\}^{(k_m)} (1 + \tilde{D}^2 + s^2)^{-(\frac{p}{2}+r+m+|k|)} \right) ds + \\ + \text{a function in } r \text{ which is holomorphic for } \Re(r) > -\frac{p-\delta'}{2}.$$

Note that the power of $R_s(\lambda)$ appearing in the integral means we have to use the Cauchy formula for the derivative of a function instead, which accounts for the appearance of the constant involving gamma functions.

The proof of Proposition 8.2 of [17] appeals to the Laplace transform to evaluate the s integrals and show that

$$C_{\frac{p}{2}+r} \text{sf}(D, u^* Du) = \sum_{m=1, \text{ odd}}^{2N-1} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|+m} \cdot C(k) \cdot \frac{\Gamma(\frac{p}{2}+r+|k|+\frac{m-1}{2})\Gamma(\frac{m+1}{2})}{2\cdot\Gamma(\frac{p}{2}+r)(|k|+m)!} \times \\ \times S\tau \left(q\{\tilde{D}, q\}^{(k_1)} \dots \{\tilde{D}, q\}^{(k_m)} (1 + \tilde{D}^2)^{-(\frac{p}{2}+r+|k|+\frac{m-1}{2})} \right) + \\ + \text{a function in } r \text{ which is holomorphic for } \Re(r) > -\frac{p-\delta'}{2}.$$

Finally, one examines the formula from the point of view of residues. The left hand side has a simple pole at $r = -\frac{p-1}{2}$, and the residue there is exactly $\text{sf}(D, uDu^*)$. The right hand side thus has an analytic continuation to a neighbourhood of $r = -\frac{p-1}{2}$. The isolated spectral dimension assumption is now needed to be able to re-write the expression as a sum of residues, since the whole point of this assumption is that each of the term analytically continues to a neighbourhood of $-\frac{p-1}{2}$.

At various points in the proof (i.e. at points where the argument of the supertrace can be explicitly computed), one can evaluate the supertrace and obtain a formula which depends on the original setup. In particular, we mention this description: define

$$\varphi_m^r(a_0, \dots, a_r) = \frac{-2\sqrt{2\pi}i}{\Gamma(\frac{m+1}{2})} \cdot \int_0^\infty s^m \tau \left(\frac{1}{2\pi i} \int_l \lambda^{-(\frac{p}{2}+r)} a_0 R_s(\lambda) [D, a_1] R_s(\lambda) \dots [D, a_m] R_s(\lambda) d\lambda \right) \\ \text{for } \Re(r) > -\frac{m-1}{2}, \text{ and}$$

$$\varphi_m(a_0, \dots, a_m) = \text{res}_{r=-\frac{p-1}{2}} \varphi_m^r(a_0, \dots, a_m).$$

The collection $(\varphi_m)_{m=1, \text{ odd}}^{2N-1}$ is a (b, B) -cocycle (called the **residue cocycle**) which represents the class of the Chern character of $(\mathfrak{A}, \mathcal{H}, D)$. This is itself a non-trivial result, proven in [19].

3.2 Outline of the Carey-Phillips proof of p -summable formula

In Section 7 of [24], Connes introduced a generalization of p -summable modules called θ -summable, the distinguishing feature of which being that (\mathcal{H}, D) is θ -summable if e^{-tD^2}

is trace class for any $t > 0$. A different and useful description of θ -summable is that $(1 + D^2)^{-1}$ belongs to a certain ideal of operators, a description of which can be given via s -numbers (see Section 2.2, Definition 2.4 and Appendix B of [15]). In [15], Carey and Phillips proved integral formulas for spectral flow in the context of θ -summable modules, and used these formulas to derive the p -summable version stated at the beginning of the chapter. The approach relies on the fact that, if (\mathcal{H}, D_0) is p -summable, then it is θ -summable, and one can use Laplace transforms and the knowledge from the θ -summable case to obtain the desired result as sketched below.

Recall that the Laplace transform of a function $f(t)$ with domain the non-negative real numbers is given by $F(s) = \lim_{k \rightarrow 0} \int_{-k}^{\infty} e^{-st} f(t) dt$ for $s \in \mathbb{C}$ and $\Re(s) > a$ (where the value of $a \in \mathbb{R}$ depends on f and is chosen such that the integral is finite). The method used in the above-mentioned paper is to prove that the Laplace transform gives the expected value if we replace s by an operator:

Theorem 3.2.1 (Spectral Flow [15], Lemma 9.1) If $n > 0$ (not necessarily an integer) and D is an unbounded self-adjoint operator affiliated with \mathcal{N} such that $(1 + D^2)^{-n}$ is trace-class, then the integral $\int_0^{\infty} t^{n-1} e^{-t(1+D^2)} dt$ converges in both trace and operator norm to $\Gamma(n)(1 + D^2)^{-n}$. \blacksquare

If D_0 and D_1 are unitarily equivalent then using the formula for θ -summable operators we have that for any $\varepsilon > 0$, $\text{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \cdot \int_0^1 \tau(D'_t \cdot e^{-\varepsilon D_t^2}) dt$ (Corollary 7.10 of [15]). The Laplace transform for the power function gives us that $\frac{1}{\Gamma(n-\frac{1}{2})} \int_0^{\infty} \varepsilon^{n-\frac{3}{2}} e^{-\varepsilon} d\varepsilon = 1$. Hence, the following calculations can be performed:

$$\begin{aligned}
\text{sf}(D_0, D_1) &= \frac{1}{\Gamma(n-\frac{1}{2})} \int_0^{\infty} \varepsilon^{n-\frac{3}{2}} \cdot e^{-\varepsilon} \text{sf}(D_0, D_1) d\varepsilon \\
&= \frac{1}{\Gamma(n-\frac{1}{2})} \int_0^{\infty} \varepsilon^{n-\frac{3}{2}} \cdot e^{-\varepsilon} \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau(D'_t e^{-\varepsilon D_t^2}) dt d\varepsilon \\
&= \frac{1}{\sqrt{\pi} \Gamma(n-\frac{1}{2})} \int_0^{\infty} \int_0^1 \tau(D'_t \varepsilon^{n-1} e^{-\varepsilon(1+D_t^2)}) dt d\varepsilon \\
&= \frac{1}{\sqrt{\pi} \Gamma(n-\frac{1}{2})} \int_0^1 \tau(D'_t \int_0^{\infty} \varepsilon^{n-1} e^{-\varepsilon(1+D_t^2)} d\varepsilon) dt \\
&= \frac{1}{\sqrt{\pi} \Gamma(n-\frac{1}{2})} \int_0^1 \tau(D'_t \Gamma(n)(1 + D_t^2)^{-n}) dt \quad (\text{using the operator Laplace transform}) \\
&= \frac{\Gamma(n)}{\sqrt{\pi} \Gamma(n-\frac{1}{2})} \int_0^1 \tau(D'_t (1 + D_t^2)^{-n}) dt.
\end{aligned}$$

Finally, algebraic manipulation shows that the constant is the same as calculated earlier. Of course, we have omitted the details; for example, one has to check that the order of the integrals can be switched. The proof of the result for θ -summable operators is quite involved (see [15] for details), so a direct approach to the problem simplifies the presentation greatly.

If D_0 and D_1 are not unitarily equivalent, then we need correction terms at the endpoints; the formula is given by:

$$\text{sf}(D_a, D_b) = \frac{1}{\sqrt{\pi}} \int_a^b \tau(D'_t e^{-D_t^2}) dt + \frac{1}{2} [\eta_1(D_b) - \eta_1(D_a) + \tau(P_{\ker(D_b)}) - \tau(P_{\ker(D_a)})].$$

where P_M is the projection onto the subspace M and $\eta_1(D) = \frac{1}{\pi} \int_1^{\infty} \tau(D e^{-tD^2}) t^{-\frac{1}{2}} dt$ ([15], Corollary 8.11). As before, we can use $\text{sf}(D_a, D_b) = \text{sf}(\sqrt{\varepsilon} D_a, \sqrt{\varepsilon} D_b)$ to get the result for the

finitely-summable case. However, when we apply the Laplace transform trick to one of the $\eta_1(D)$ correction terms, starting from the integral $\int_0^\infty \varepsilon^{k-1} e^{-\varepsilon} d\varepsilon = \Gamma(k)$, we get:

$$\begin{aligned} \frac{1}{\pi} \int_1^\infty \int_0^\infty \varepsilon^{k-1} e^{-\varepsilon} \tau(\sqrt{\varepsilon} e^{-t k \varepsilon D^2}) t^{-\frac{1}{2}} dt &= \frac{1}{2\pi} \int_0^\infty \int_1^\infty \tau(\varepsilon^{k-\frac{1}{2}} t^{-\frac{1}{2}} D e^{-\varepsilon(tD^2+1)}) dt d\varepsilon \\ &= \frac{1}{2\pi} \int_1^\infty \tau(t^{-\frac{1}{2}} D \int_0^\infty \varepsilon^{k-\frac{1}{2}} e^{-\varepsilon(tD^2+1)} d\varepsilon) dt \\ &= \frac{1}{2\pi} \int_1^\infty \tau(t^{-\frac{1}{2}} D \Gamma(k + \frac{1}{2}) (1 + tD^2)^{-(k+\frac{1}{2})}) dt \quad (\dagger) \\ &= \frac{\Gamma(k+\frac{1}{2})}{2\pi} \int_1^\infty \tau(t^{-\frac{1}{2}} D (1 + tD^2)^{-(k+\frac{1}{2})}) dt. \end{aligned}$$

In order to ensure that the trace of the operator at (\dagger) is finite, we need $k \geq \frac{p+1}{2}$.

It was hoped, especially in view of its application to the proof of the Local Index formula, that the p -summable case could be proved directly, without having to rely on the θ -summable results. In [14], a formula similar to the one we are seeking was shown, except the power of $(1 + D_t^2)^{-1}$ was an integer $m > \frac{p}{2} + 1$ (Theorem 2.17 of [14]). As a consequence of this assumption the proof has features which do not generalize to the case when $\frac{p}{2}$ is a real number; nonetheless, the goal was to find a proof similar to the one in [14].

3.3 Outline of analytic continuation proof of p -summable formula

The idea is to show that the integral formula for spectral flow works with $\frac{p}{2}$ replaced by any large enough q ; the larger power gives us extra maneuvering room, as we can split up the $(1 + D^2)^{-q}$ factor and still have part of it be trace class (we do this, for example, during the proof that the one-form is closed). We then use analytic continuation of complex functions to show that the formula works for all $q \geq \frac{p}{2}$.

To show that the integral formula calculates the spectral flow, the proof follows the same steps as for the finitely-summable case when the power is an integer (see [14]), but changes the proof so it works for all (sufficiently large) real powers. Note that the proof is split into a 'bounded case' and an 'unbounded case'. One way of showing that a formula calculates spectral flow is to start with an integral formula which works for special linear paths of bounded operators and build up the general formula from there. The 'bounded case' and the 'unbounded case' are connected by the Riesz transform, $D \mapsto D(1 + D^2)^{-\frac{1}{2}}$, which maps unbounded operators to bounded operators – see Section 1.1.3.2 for a review of the Riesz transform. The following two points explain how we decide what the corresponding 'bounded case' should be:

- given a summability condition on D_0 , say $(1 + D_0^2)^{-1} \in \mathcal{S}$ for some operator ideal \mathcal{S} , the Riesz transform maps paths in $D_0 + \mathcal{N}_{s_a}$ to paths in $F_0 + \mathcal{S}$, where $F_0 = D_0(1 + D_0^2)^{-\frac{1}{2}}$ and \mathcal{S} is a Banach space whose exact nature will be revealed in Section 3.5.
- for appropriate choices of function k , if $\{F_t\} \subset F_0 + \mathcal{S}$ is the image of the path $\{D_t\}$ under the Riesz transform, then

$$\tau \left(\frac{d}{dt} (D_t) k((1 + D_t^2)^{-1}) \right) = \tau \left(\frac{d}{dt} (F_t) (1 - F_t^2)^{-\frac{3}{2}} k(1 - F_t^2) \right)$$

Suppose $k(x)$ is a suitable function and we want to show that the integral

$$\int_0^1 \tau\left(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1})\right) dt$$

can be used to calculate spectral flow. The course indicated by the above observations is clear:

(a) Modify the desired formula, replacing $(1+D_t^2)^{-1}$ by $1-F_t^2$, and add a factor of $(1-F_t^2)^{-\frac{3}{2}}$, to obtain the corresponding integral formula in the bounded case; that is, consider the integral $\int_0^1 \tau\left(\frac{d}{dt}(F_t)h(1-F_t^2)\right) dt$, where $h(x) = x^{-\frac{3}{2}}k(x)$. Some care might be required in doing this:

- The operator $(1+D_t^2)^{-1}$ is always positive; in order for $h(1-F_t^2)$ to make sense for all F_t in our manifold $F_0 + \mathcal{S}$, we might have to replace $1-F_t^2$ by $|1-F_t^2|$ (so consider $h(|x|)$ instead of $h(x)$). This is indeed the case for $h(x) = x^p$ (the function we are ultimately interested in), as x^p might not make sense if $x < 0$ and p is not an integer. Note that this is one of the reasons the proof for general real powers p differs from the one for integer powers presented in [14].
- The extra factor of $|1-F_t^2|^{-\frac{3}{2}}$ might very well cause problems, as there is no reason to suppose that $1-F_t^2$ is invertible. However, for suitable functions k , say if $\lim_{x \rightarrow 0} x^{-\frac{3}{2}}k(x) = 0$, we can still make sense of the expression $(1-F_t^2)^{-\frac{3}{2}}k(1-F_t^2)$: namely, define

$$l(x) = \begin{cases} x^{-\frac{3}{2}}k(x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0; \end{cases}$$

then l is a continuous function on \mathbb{R} , and whenever we write $h(F)$ we really mean $l(F)$. This is, for example, the approach taken in [15], where $k(x) = e^{-\frac{1}{x}}$.

(b) Show that the one form $\tau(Xh(1-F_t^2))$ is exact, so its integral is independent of the path over which it is calculated (one way to do this is to show that the one-form is closed, which is the approach we take).

(c) Show that the integral formula holds in the bounded case. It is during this step that we find the suitable normalizing factor for the one-form, and also where we have to introduce correction terms for a path whose endpoints are not unitarily equivalent.

(d) Reduce the unbounded case to the bounded case. The vital point here is that, if $D_t = D_0 + A_t$ describes a C^1 path in $D_0 + \mathcal{N}_{sa}$, then its Riesz image $\{F_t\}$ should be C^1 in $F_0 + \mathcal{S}$ for an appropriately chosen Banach space \mathcal{S} . Then the equality of traces mentioned earlier allows us to rewrite the bounded formula into a formula depending on $\{D_t\}$.

The goal is to apply this program for $h(x) = x^q$, for any q large enough. Along the way, we try to present results in some generality:

- We start by presenting, in Section 3.4, a set of conditions which are sufficient to prove that $\alpha_T : X \mapsto \tau(Xg(T)^k)$ is a closed one-form on an affine operator space $T_0 + \mathcal{S}$. The choice of function g will be different for the bounded case and the unbounded case.

- We consider under what conditions the integral formula

$$\int \tau \left(\frac{d}{dt}(F_t) \cdot h(1 - F_t^2) \right) dt$$

can be suitably modified to calculate spectral flow. As part of this process, both a constant normalization factor and correction terms emerge. We then use this to obtain, for unbounded operators, an integral formula of the type

$$\text{const} \cdot \int \tau \left(\frac{d}{dt}(D_t) \cdot k((1 + D_t^2)^{-1}) \right) dt + \text{correction terms.}$$

This is the content of Section 3.5.

- We show in Section 3.6 that, with some conditions on the function g , the complex function

$$z \mapsto \int_0^1 \tau(A \cdot g(D_t)^z) dt$$

is analytic, and explain the idea behind the analytic continuation step.

Finally, in Section 3.7, we put these various facts together to prove the formula stated at the beginning of the chapter.

Remark 3.3.1 Our manifolds will be modelled on ideals of operators from which we need to demand certain features in order for our proofs to work. Say that \mathcal{I} is a **small power invariant operator ideal** if $\mathcal{I} \subset \mathcal{K}_{\mathcal{N}}$ is an invariant operator ideal (as defined in Definition 1.1.9) whose powers \mathcal{I}^q for $0 < q < 1$ are also invariant operator ideals for the norm $\|A\|_{\mathcal{I}^q} = (\|A^{\frac{1}{q}}\|_{\mathcal{I}})^q$; moreover, for $0 < q \leq 1$, $\mathcal{I}^q \subset \mathcal{K}_{\mathcal{N}}$, and the ideals \mathcal{I}^q and $\mathcal{I}^{q/2}$ satisfy the Cauchy-Schwarz inequality (Definition 1.1.17). •

3.4 Closed forms of the type $\tau(Xg(T)^q)$

In the following, we will have reason to consider two one-forms of the same type, so this section's purpose is to prove that these one-forms are closed and exact in a more general context. As always, we have a von Neumann algebra \mathcal{N} equipped with a faithful, normal semifinite trace τ . The one-forms are defined on a manifold of the type $T_0 + \mathcal{S}$ for a fixed self-adjoint operator T_0 (where either $T_0 \in \mathcal{N}_{sa}$ or, if T_0 is unbounded, then T_0 is affiliated with \mathcal{N}) and a real Banach space $\mathcal{S} \subset \mathcal{N}_{sa}$, and are of the type $\alpha_T(X) = C \cdot \tau(Xg(T)^q)$, where C is a constant, $q \geq 3$ is a fixed real number, $T \in T_0 + \mathcal{S}$, X is in the tangent space at T of the manifold (namely \mathcal{S}), and $g : T_0 + \mathcal{S} \rightarrow \mathbb{R}$ is a suitable function.

In order to define the one-form and show that the one-form is closed using the technique below, we will need to place some restrictions on the function g . In all the cases we are interested in, we have that $\|X\|_{\mathcal{S}} \geq \|X\|$ for $X \in \mathcal{S}$, a condition which we need for some of our continuity proofs; for example, the identity function from \mathcal{S} equipped with $\|\cdot\|_{\mathcal{S}}$ norm to \mathcal{S} equipped with operator norm is then continuous, so if $X \mapsto f(X)$ is continuous from \mathcal{N} to \mathcal{N} , then $X \mapsto f(X)$ is continuous from \mathcal{S} to \mathcal{N} . We will write $\mathcal{S}_{\|\cdot\|}$ to stand for \mathcal{S} equipped with operator norm. We first state the conditions we need g to satisfy, and immediately afterwards explain the reasoning behind some of the choices:

1. $g(T)$ is a positive operator in \mathcal{N} for all $T \in T_0 + \mathcal{S}$ (so $g(T)^q$ is defined and in \mathcal{N}).
2. $\tau(g(T)^t) < \infty$ for $t \geq \frac{q-3}{2}$ for all $T \in T_0 + \mathcal{S}$.
3. For each fixed $t \geq \frac{q-3}{2}$ and $T \in T_0 + \mathcal{S}$, the function $X \mapsto g(T+X)^t$ from \mathcal{S} to \mathcal{L}^1 is continuous. Note that it is sufficient to prove continuity at $X = 0$ for each T , as continuity at some other X_0 then follows by replacing T with $T + X_0$ (which is also an element of $T_0 + \mathcal{S}$).
4. For each $T \in T_0 + \mathcal{S}$ the function $X \mapsto g(T+X)$ is differentiable as a function from \mathcal{S} to \mathcal{N} . As above, it is sufficient to show differentiability at $X = 0$ for each T .

For fixed $X \in \mathcal{S}$ and $T \in T_0 + \mathcal{S}$ we can define the function $s \mapsto g(T + sX)$ from \mathbb{R} to \mathcal{N} . This function is differentiable by 4 above; its derivative at $s = 0$ plays an important role, and it is useful to introduce a notation for the difference quotient. Namely, define the function

$$d_{g,s}(T, X) = \begin{cases} \frac{g(T+sX) - g(T)}{s} & s \neq 0 \\ \lim_{s \rightarrow 0} \frac{g(T+sX) - g(T)}{s} & s = 0. \end{cases}$$

Note that $s \mapsto d_{g,s}(T, X)$ is a continuous function (since $s \mapsto g(T + sX)$ is itself continuous, and differentiable at $s = 0$). Further restrictions on g are in fact placed on this function.

5. $d_{g,0}(T, X)$ (the derivative of $s \mapsto g(T + sX)$ at $s = 0$) can be written as a sum, where each term is of the following type:
 - $g_1(T)Xg_1(T)$ with $T \mapsto g_1(T)$ continuous from $T_0 + \mathcal{S}$ to \mathcal{N} , or
 - $g_1(T)Xg_2(T)$ with $T \mapsto g_1(T)$ and $T \mapsto g_2(T)$ both continuous from $T_0 + \mathcal{S}$ to \mathcal{N} ; each such term (when $g_1 \neq g_2$) can be paired up with a corresponding term of the form $g_2(T)Xg_1(T)$

First, a few remarks about these various conditions. Note the repeated appearance of $\frac{q-3}{2}$, instead of the (perhaps) expected q . This is a consequence of the method of proof; we will want to split up $g(T)^q$ into a product $g(T)^a \cdot g(T)^b$, and we need to ensure that at least one of these two factors is trace class (see the proof of Lemma 3.4.7). Property 5 might also look a bit strange; it is needed so that, when we use the trace property on some of the resulting expressions, we get a permutation of the same terms (see the calculations following Lemma 3.4.8). Examples of spaces $T_0 + \mathcal{S}$ and functions g which satisfy these conditions can be found in Sections 3.7.1 and 3.7.2.

Note that, from the definition of α and conditions on g , it is easy to check that for each fixed T , α_T is a bounded linear functional on the tangent space at T . It might be possible to show that α is C^1 when considered as a map from the manifold $T_0 + \mathcal{S}$ to the cotangent space; however, this is in fact irrelevant to our purposes. Our goal is to calculate the exterior derivative and hence conclude that the integral of α is independent of path, for which we really only need directional derivatives. In fact, in order to show that $d\alpha = 0$, we could relax properties 3 and 4 to $s \mapsto g(T + sX)^t$ is continuous as a function from \mathbb{R} to \mathcal{L}^1 for fixed t , T and X , and respectively

$s \mapsto g(T + sX)$ is differentiable from \mathbb{R} to \mathcal{N} for fixed T and X , allowing us to a large extent to ignore the norm on \mathcal{S} ; this would be sufficient for the following proof to go through. The main result of this section is then:

Theorem 3.4.1 Suppose T_0 is a fixed self-adjoint operator affiliated with \mathcal{N} , \mathcal{S} is a real Banach space such that $\mathcal{S} \subset \mathcal{N}_{sa}$ and $\|X\|_{\mathcal{S}} \geq \|X\|$ for $X \in \mathcal{S}$, and g is a function satisfying conditions 1 - 5 above. Then for any $C > 0$, $\alpha_T(X) = C \cdot \tau(Xg(T)^q)$ is a closed one-form on the manifold $T_0 + \mathcal{S}$ (that is, $d\alpha = 0$). \blacktriangleright

The rest of the section is dedicated to proving this result. Before we proceed, note that if $\alpha_T(X)$ is closed then so is $C\alpha_T(X)$ for any constant C , so in the following we assume without loss of generality that $C = 1$.

The definition of exterior derivative applied to the one-form α gives us that

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]),$$

where $[X, Y]$ is the bracket product of X and Y viewed as constant vector fields. It is easy to check that $\Phi_t(F) = F + tX$ and $\Psi_s(F) = F + sY$ are flows for the constant vector fields X and Y respectively, and that $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all s, t ; so it follows that $[X, Y] = 0$ (for a proof that the bracket product of two vector fields is zero if and only if their flows commute see [68], Chapter 5, Lemma 13). Hence, showing that the one-form is closed reduces to showing that the derivatives in the direction X of $\alpha_T(Y)$ and in the direction Y of $\alpha_T(X)$ are equal; that is,

$$\left. \frac{d}{ds} \right|_{s=0} \tau(Yg(T + sX)^q) = \left. \frac{d}{ds} \right|_{s=0} \tau(Xg(T + sY)^q).$$

We start with the left hand side and manipulate it until we can conclude that it is the same as the right hand side. By definition,

$$\left. \frac{d}{ds} \right|_{s=0} \tau(Yg(T + sX)^q) = \lim_{s \rightarrow 0} \tau \left(Y \cdot \frac{1}{s} \cdot [g(T + sX)^q - g(T)^q] \right).$$

Let $n = [q]$ and $r = \frac{q}{n+1}$ (note that $0 < r < 1$). It is easy to check (by expanding) that

$$g(T + sX)^q - g(T)^q = \sum_{i=0}^n g(T + sX)^{r(n-i)} [g(T + sX)^r - g(T)^r] g(T)^{ri}.$$

We will use this formula more than once, so for the purpose of future references we will summarize this well-known result in a lemma:

Lemma 3.4.2 Given a real number $t > 0$, let $n = [t]$ and $r = \frac{t}{n+1}$ (note that $0 < r < 1$). If A and B are positive bounded operators, then

$$A^t - B^t = \sum_{i=0}^n A^{r(n-i)} (A^r - B^r) B^{ri}. \quad \blacktriangleright$$

The condition that A and B be positive is necessary to ensure that A^r and B^r are defined (since r is not an integer). The point of using this expansion is that, since $0 < r < 1$, we can apply the following theorem.

Theorem 3.4.3 ([51], p. 8) If B is a bounded positive operator and $0 < t < 1$ then

$$B^t = \frac{\sin(t\pi)}{\pi} \int_0^\infty \lambda^{-t} (1 + \lambda B)^{-1} B \, d\lambda. \quad \clubsuit$$

Hence $g(T + sX)^r - g(T)^r$ can be written as

$$\frac{\sin(r\pi)}{\pi} \cdot \int_0^\infty \lambda^{-r} \left[(1 + \lambda g(T + sX))^{-1} g(T + sX) - (1 + \lambda g(T))^{-1} g(T) \right] d\lambda.$$

Moreover, using the resolvent formula, it is easy to see that, for any A and B positive bounded operators,

$$\begin{aligned} & (1 + \lambda A)^{-1} A - (1 + \lambda B)^{-1} B \\ &= (1 + \lambda A)^{-1} \cdot (A(1 + \lambda B) - (1 + \lambda A)B) \cdot (1 + \lambda B)^{-1} \\ &= (1 + \lambda A)^{-1} \cdot (A - B) \cdot (1 + \lambda B)^{-1}; \end{aligned}$$

so, with $A = g(T + sX)$ and $B = g(T)$, we get

$$\begin{aligned} & (1 + \lambda g(T + sX))^{-1} g(T + sX) - (1 + \lambda g(T))^{-1} g(T) \\ &= (1 + \lambda g(T + sX))^{-1} \cdot (g(T + sX) - g(T)) \cdot (1 + \lambda g(T))^{-1}. \end{aligned}$$

Substitute these formulas into the derivative calculation, to get

$$\begin{aligned} & \left. \frac{d}{ds} \right|_{s=0} \tau(Y g(T + sX)^q) \\ &= \lim_{s \rightarrow 0} \tau \left(Y \cdot \frac{1}{s} \cdot [g(T + sX)^q - g(T)^q] \right) \\ &= \lim_{s \rightarrow 0} \tau \left(Y \cdot \frac{1}{s} \cdot \left[\sum_{i=0}^n g(T + sX)^{r(n-i)} \left[\frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{-r} \left[(1 + \lambda g(T + sX))^{-1} \cdot (g(T + sX) - g(T)) \cdot (1 + \lambda g(T))^{-1} \right] d\lambda \right] g(T)^{ri} \right] \right). \end{aligned}$$

We combine the difference $g(T + sX) - g(T)$ with the factor of $\frac{1}{s}$ appearing in the limit, thereby obtaining the difference quotient at 0 of the function $s \mapsto g(T + sX)$. Recall that this difference quotient was denoted by $d_{g,s}(T, X)$, and the derivative at 0 by $d_{g,0}(T, X)$. In order to make the formulas easier to read, we will also use $R_\lambda(A)$ for $(1 + \lambda A)^{-1}$. We remark here that if A is a positive operator then $\|R_\lambda(A)\| \leq 1$, an inequality which will be used often. Using these notations in our derivative calculation we get

$$\begin{aligned} & \lim_{s \rightarrow 0} \tau \left(Y \cdot \frac{g(T + sX)^q - g(T)^q}{s} \right) = \\ & \lim_{s \rightarrow 0} \tau \left(Y \sum_{i=0}^n g(T + sX)^{r(n-i)} \cdot \frac{\sin(r\pi)}{\pi} \left[\int_0^\infty \lambda^{-r} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \, d\lambda \right] g(T)^{ri} \right). \end{aligned}$$

Note that, in each term, either $g(T + sX)^{r(n-i)}$ or $g(T)^{ri}$ must be trace class; this is because

$$r(n-i) + ri = rn = q - r > q - 1,$$

so at least one of $r(n-i)$ and ri must be greater than or equal to $\frac{q-3}{2}$. This will be important in the proofs below. In order to avoid having to distinguish which of $g(T + sX)^{r(n-i)}$ and $g(T)^{ri}$ is trace class (which will not be relevant beyond the fact that one of them is), we introduce a

special notation to deal with both cases at once. Namely, for an operator A denote by $\|A^t\|_*$ the \mathcal{L}^1 norm if t is large enough that $A^t \in \mathcal{L}^1$, and the operator norm otherwise.

To start with, since one of $g(T + sX)^{r(n-i)}$ and $g(T)^{ri}$ is trace class, we can pull the trace into the sum (each term is also trace class). Moreover, $g(T + sX)^{r(n-i)}$ and $g(T)^{ri}$ are bounded operators, so we can pull them into the integral without changing the value of the expression. We get

$$\lim_{s \rightarrow 0} \tau \left(Y \cdot \frac{g(T+sX)^q - g(T)^q}{s} \right) = \lim_{s \rightarrow 0} \sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \cdot \tau \left(\int_0^\infty \lambda^{-r} Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri} d\lambda \right).$$

We would like to conclude that this limit evaluates to

$$\sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \cdot \int_0^\infty \lambda^{-r} \tau \left(Y \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot d_{g,0}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri} \right) d\lambda,$$

the expression obtained by first exchanging the integral and trace and then plugging in $s = 0$.

Consider one term at a time; that is, in most of the following (until we return to the derivative calculation), i is fixed, and we recall that r and n are determined by the exponent q in the formula for α , and hence are also fixed. For each $s \in [-1, 1]$, define

$$J_s(\lambda) = \lambda^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}.$$

$J_s(\lambda)$ is the integrand in our last limit expression, so in order to accomplish our goal we need to show that $\lim_{s \rightarrow 0} \tau \left(\int_0^\infty J_s(\lambda) d\lambda \right)$ converges to $\int_0^\infty \tau(J_0(\lambda)) d\lambda$. The proof will follow the following steps:

1. Show that $\lambda \mapsto J_s(\lambda)$ is continuous in trace norm.
2. For $s \in [-1, 1]$, $|\tau(J_s(\lambda))|$ is bounded above by a non-negative function $h(\lambda)$ for which $\int_0^\infty h(\lambda) d\lambda$ is finite (the function h is the same for all s).
3. Show that $\tau(J_s(\lambda)) \rightarrow \tau(J_0(\lambda))$ pointwise as $s \rightarrow 0$.

From (2) we can conclude that $\int_0^\infty \tau(J_s(\lambda)) d\lambda$ converges, which along with (1) is sufficient to allow us to use Fubini's Theorem to switch the order of the trace and the integral. The rest we need in order to apply the Lebesgue Dominated Convergence Theorem and pull the limit in the integral, and finally to evaluate the limit. We can then use the trace property and the restrictions on $d_{g,0}(T, X)$ to show that each term which appears in the formula for $\left. \frac{d}{ds} \right|_{s=0} \tau(X g(T + sY)^q)$ also appears in the formula for $\left. \frac{d}{ds} \right|_{s=0} \tau(Y g(T + sX)^q)$.

Step 1 of the proof is the continuity of the integrand as a function of λ , shown in the following lemma.

Lemma 3.4.4 Fix $s \in [-1, 1]$. Recall that, with fixed $0 < r < 1$ a real number and n, i non-negative integers, we defined

$$J_s(\lambda) = \lambda^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}.$$

The map $\lambda \mapsto J_s(\lambda)$ is continuous (in $\|\cdot\|_1$ norm) on the interval $(0, \infty)$. ☛

PROOF Fix $\eta \in (0, \infty)$ and show that, for λ close enough to η , $\|J_s(\eta) - J_s(\lambda)\|$ can be made as small as we want. Using the fact that either $g(T + sX)^{r(n-i)}$ or $g(T)^{ri}$ has to be trace class, we can re-write the difference and use the triangle inequality to get

$$\begin{aligned} & \|J_s(\eta) - J_s(\lambda)\|_1 \\ & \leq \|\eta^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\eta g(T + sX) \cdot d_{g,s}(T, X) \cdot [R_\eta g(T) - R_\lambda g(T)] \cdot g(T)^{ri}\|_1 \\ & \quad + \|\eta^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot [R_\eta g(T + sX) - R_\lambda g(T + sX)] \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \quad + \|(\eta^{-r} - \lambda^{-r}) \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1. \end{aligned}$$

Next, we can use Hölder's inequality to show that each term can be made arbitrarily small. The argument relies on the fact that $\lambda \mapsto \lambda^{-r}$ is a continuous function, as are the functions $\lambda \mapsto R_\lambda g(T)$ and $\lambda \mapsto R_\lambda g(T + sX)$. For example, for the first term,

$$\begin{aligned} & \|\eta^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\eta g(T + sX) \cdot d_{g,s}(T, X) \cdot [R_\eta g(T) - R_\lambda g(T)] \cdot g(T)^{ri}\|_1 \\ & \leq \eta^{-r} \|Y\| \cdot \|g(T + sX)^{r(n-i)}\|_* \cdot \|R_\eta g(T + sX)\| \cdot \|d_{g,s}(T, X)\| \cdot \|R_\eta g(T) - R_\lambda g(T)\| \cdot \|g(T)^{ri}\|_*. \end{aligned}$$

Note that Y , $g(T + sX)^{r(n-i)}$, $d_{g,s}(T, X)$ and $g(T)^{ri}$ are constants (as s and i are fixed); hence, their norms are constant. So is $\|R_\eta g(T + sX)\|$, as η is fixed. It follows that, since $\lambda \mapsto R_\lambda g(T)$ is continuous, the right hand side of the inequality can be made arbitrarily small.

Similarly, the remaining two terms can be compared to the product of the norms of their factors; the fact that they can be made arbitrarily small relies on the continuity of the functions $\lambda \mapsto \lambda^{-r}$ and $\lambda \mapsto R_\lambda g(T + sX)$, and the fact that $\|R_\lambda g(T + sX)\|$ and $\|R_\lambda g(T)\|$ are both at most one regardless of the value of $\lambda > 0$. Hence, the remaining two terms can in turn be made arbitrarily small. This allows us to conclude that

$$\lambda \mapsto \lambda^{-\frac{r}{2}} Y g(T + sX)^{r(n-i)} (1 + \lambda g(T + sX))^{-1} d_{g,s}(T, X) (1 + \lambda g(T))^{-1} g(T)^{ri}$$

is continuous in trace norm. ■

The continuity of the function $s \mapsto g(T + sX)^t$ in operator norm follows from the continuity of the spectral calculus; for the sake of completeness, we state below the spectral calculus continuity theorem and prove that it implies $s \mapsto g(T + sX)^k$ is continuous in operator norm.

Theorem 3.4.5 ([73], **Proposition 4.10**) Suppose \mathfrak{A} is a C^* -algebra and $K \subset \mathbb{C}$ is compact. Denote by \mathfrak{A}_K the set of normal elements of \mathfrak{A} whose spectrum is contained in K . If f is a continuous function on K , then the functional calculus $x \mapsto f(x)$ (from \mathfrak{A}_K to \mathfrak{A}) is continuous. ☛

Lemma 3.4.6 Consider $t \in \mathbb{R}_+$, and fix $T \in T_0 + \mathcal{S}$ and $X \in \mathcal{S}$. The function from $[-1, 1]$ to \mathcal{N} defined by $s \mapsto g(T + sX)^t$ is continuous. ☛

PROOF First note that it is sufficient to prove that the given function is continuous at $s = 0$, since then continuity at any $s_0 \in [-1, 1]$ follows by replacing T with $T + s_0 X \in T_0 + \mathcal{S}$, and using continuity at zero of the resulting function.

We will use the continuity of the spectral calculus (see Theorem 3.4.5). In detail, let $m = \sup_{s \in [-1, 1]} \|g(T + sX)\|$ (note $m < \infty$ since $s \mapsto g(T + sX)$ is continuous in norm as a consequence of property 4 of g , and $[-1, 1]$ is compact). The spectrum of $g(T + sX)$ is contained in

$[0, \|g(T + sX)\|] \subset [0, m]$ for each s ; moreover, the function $x \mapsto x^t$ is continuous on $[0, m]$. So we can use the above-mentioned theorem to conclude that, since $g(T + sX) \rightarrow g(T)$ in operator norm (continuity at $s = 0$ of $s \mapsto g(T + sX)$), we must also have $g(T + sX)^t \rightarrow g(T)^t$ in operator norm. ■

The following lemma accomplishes step 2 in our proof (finding a function $h(\lambda)$ which can be used as a bound for $J_s(\lambda)$ for all $s \in [-1, 1]$ simultaneously).

Lemma 3.4.7 Assume $s \in [-1, 1]$. Recall that, with fixed $0 < r < 1$ a real number and n, i non-negative integers, we defined

$$J_s(\lambda) = \lambda^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}.$$

There exists a constant k which does not depend on either s or λ such that, for each $\lambda \in (0, \infty)$,

$$\|J_s(\lambda)\|_1 \leq \min\{\lambda^{-r}, \lambda^{-r-1}\} \cdot k.$$

It follows that

$$\int_0^\infty \|J_s(\lambda)\|_1 d\lambda < k \cdot \int_0^\infty \min\{\lambda^{-r}, \lambda^{-r-1}\} d\lambda < \infty. \quad \text{☞}$$

PROOF Recall that $s \mapsto d_{g,s}(T, X)$ is continuous, so since $[-1, 1]$ is compact, we can define the constant $k_0 = \sup_{s \in [-1, 1]} \|d_{g,s}(T, X)\|$. As the value of $\min\{\lambda^{-r}, \lambda^{-r-1}\}$ depends on how λ compares to 1, we will divide the proof into cases according to the value of λ .

Consider first $\lambda \in (0, 1)$; then $\lambda^{-r} < \lambda^{-r-1}$. We have

$$\begin{aligned} & \|\lambda^{-r} Y g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \leq \lambda^{-r} (\|Y\| \cdot \|g(T + sX)^{r(n-i)}\|_* \cdot \|R_\lambda g(T + sX)\| \cdot \|d_{g,s}(T, X)\| \cdot \|R_\lambda g(T)\| \cdot \|g(T)^{ri}\|_*). \end{aligned}$$

Note that $\|Y\|$ and $\|g(T)^{ri}\|_*$ are constants, while $\|d_{g,s}(T, X)\|$ is bounded by k_0 ; moreover, $\|R_\lambda g(T)\|$ and $\|R_\lambda g(T + sX)\|$ are both at most one (since $g(T)$ and $g(T + sX)$ are both positive operators by assumption 1 on g). To prove the desired inequality, we still need to show that $\|g(T + sX)^{r(n-i)}\|$ is uniformly bounded for $s \in [-1, 1]$. We know that $s \mapsto g(T + sX)^{r(n-i)}$ is continuous in operator norm (by Lemma 3.4.6), and if $r(n-i) \geq \frac{q-3}{2}$ then it is also continuous in trace norm (Property 3 of function g), so $\sup_{s \in [-1, 1]} \|g(T + sX)^{r(n-i)}\|_*$ exists. Let

$$k_1 = k_0 \cdot \|Y\| \cdot \|g(T)^{ri}\|_* \sup_{s \in [-1, 1]} (\|g(T + sX)^{r(n-i)}\|_*);$$

then k_1 is a constant (independent of s and λ) satisfying

$$\|\lambda^{-r} Y g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \leq \lambda^{-r} \cdot k_1.$$

This shows the desired result for $\lambda \in (0, 1)$.

Now consider $\lambda \in [1, \infty)$. In this case $\lambda^{-r-1} \leq \lambda^{-r}$. Recall that r, n and i are fixed, and since $r(n-i) + ri = rn = q - r > q - 1$ it follows that at least one of $r(n-i)$ and ri must be greater than or equal to $\frac{q-1}{2} = \frac{q-3}{2} + 1$. We will use this information to define our constant;

namely, consider the two cases $r(n-i) \geq \frac{q-3}{2} + 1$ and $r(n-i) < \frac{q-3}{2}$, exactly one of which holds for our fixed r, n and i , and in each case define a constant k_2 such that $\|J_s(\lambda)\| \leq \lambda^{-r-1} \cdot k_2$.

Suppose first that $r(n-i) \geq \frac{q-3}{2} + 1$; so $r(n-i) - 1 \geq \frac{q-3}{2}$, whence $g(T+sX)^{r(n-i)-1}$ is still in \mathcal{L}^1 by property 2 of function g and, using Hölder's inequality, we can write:

$$\begin{aligned} & \|\lambda^{-r} \cdot Y \cdot g(T+sX)^{r(n-i)} \cdot R_\lambda g(T+sX) \cdot d_{g,s}(T,X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \leq \lambda^{-r} \|Y\| \cdot \|g(T+sX)^{r(n-i)-1}\|_1 \cdot \|g(T+sX) \cdot R_\lambda g(T+sX)\| \cdot \|d_{g,s}(T,X)\| \\ & \quad \cdot \|R_\lambda g(T)\| \cdot \|g(T)^{ri}\|. \end{aligned}$$

Recall that $g(T+sX)$ is a positive operator; since the function $\frac{x}{1+\lambda x}$ can be easily shown to be less than $\frac{1}{\lambda}$ for x non-negative and $\lambda > 0$, the Spectral Theorem implies that $\|g(T+sX) \cdot R_\lambda g(T+sX)\| \leq \frac{1}{\lambda}$. Moreover, $\|R_\lambda g(T)\| \leq 1$ and $\|d_{g,s}(T,X)\|$ is bounded above by the constant k_0 defined at the beginning of the proof. As $\|g(T)^{ri}\|$ does not depend on s , we just have to show that $\|g(T+sX)^{r(n-i)-1}\|_1$ is uniformly bounded for $s \in [-1, 1]$. However, this follows immediately from Property 3 of the function g , so define the constant

$$k_2 = k_0 \cdot \|Y\| \cdot \|g(T)^{ri}\| \cdot (\sup_{s \in [-1,1]} \|g(T+sX)^{r(n-i)-1}\|_1)$$

independent of s and λ for which

$$\|\lambda^{-r} Y g(T+sX)^{r(n-i)} \cdot R_\lambda g(T+sX) \cdot d_{g,s}(T,X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \leq \lambda^{-r-1} \cdot k_2.$$

This gives us the desired inequality in the case when $r(n-i) \geq \frac{q-1}{2}$. On the other hand, if $r(n-i) < \frac{q-1}{2}$ then we must have $ri \geq \frac{q-1}{2} = \frac{q-3}{2} + 1$, so we can perform a similar calculation by writing $(1 + \lambda g(T))^{-1} g(T)^{ri} = g(T)(1 + \lambda g(T))^{-1} g(T)^{ri-1}$. Namely,

$$\begin{aligned} & \|\lambda^{-r} Y g(T+sX)^{r(n-i)} \cdot R_\lambda g(T+sX) \cdot d_{g,s}(T,X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \leq \lambda^{-r} \cdot \|Y\| \cdot \|g(T+sX)^{r(n-i)}\| \cdot \|R_\lambda g(T+sX)\| \cdot \|d_{g,s}(T,X)\| \\ & \quad \cdot \|g(T) \cdot R_\lambda g(T)\| \cdot \|g(T)^{ri-1}\|_1 \\ & \leq \lambda^{-r} \cdot \|Y\| \cdot \|g(T+sX)^{r(n-i)}\| \cdot 1 \cdot k_0 \cdot \frac{1}{\lambda} \cdot \|g(T)^{ri-1}\|_1. \end{aligned}$$

Once again, we used the fact that $\|g(T) \cdot R_\lambda g(T)\| \leq \frac{1}{\lambda}$, which holds because $g(T)$ is a positive operator. Hence, in this case we need to let

$$k_2 = k_0 \cdot \|Y\| \cdot \|g(T)^{ri-1}\|_1 \cdot (\sup_{s \in [-1,1]} \|g(T+sX)^{r(n-i)}\|),$$

to obtain

$$\|\lambda^{-r} Y g(T+sX)^{r(n-i)} \cdot R_\lambda g(T+sX) \cdot d_{g,s}(T,X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \leq \lambda^{-r-1} \cdot k_2.$$

Therefore, for $k = \max\{k_1, k_2\}$, we obtain the desired result

$$\|J_s(\lambda)\|_1 \leq k \cdot \min\{\lambda^{-r}, \lambda^{-r-1}\}.$$

As $\int_0^1 \lambda^{-r} d\lambda$ and $\int_1^\infty \lambda^{-r-1} d\lambda$ are both finite (recall $0 < r < 1$), $\int_0^\infty \min\{\lambda^{-r}, \lambda^{-r-1}\} d\lambda$ converges. Using the inequality shown above, we can thus conclude that $\int_0^\infty \|J_s(\lambda)\|_1 d\lambda < \infty$. ■

Finally, show that $J_s(\lambda)$ converges pointwise to $J_0(\lambda)$ (step 3 of the proof).

Lemma 3.4.8 Fix $\lambda \in (0, \infty)$. Recall that, with fixed $0 < r < 1$ a real number and n, i non-negative integers, we defined

$$J_s(\lambda) = \lambda^{-r} \cdot Y \cdot g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}.$$

Then $\tau(J_s(\lambda)) \rightarrow \tau(J_0(\lambda))$ as $s \rightarrow 0$. ▀

PROOF This is similar to the proof of Lemma 3.4.4, in that it relies heavily on the triangle inequality and Hölder's inequality, except in this case we are interested in the continuity in trace norm of $s \mapsto J_s(\lambda)$ at zero instead of the continuity of $\lambda \mapsto J_s(\lambda)$ when s is fixed. First, the triangle inequality gives us

$$\begin{aligned} & \|\lambda^{-r} \cdot Y g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot d_{g,s}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri} - \\ & \quad \lambda^{-r} \cdot Y g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot d_{g,0}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \leq \|\lambda^{-r} \cdot Y g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot [d_{g,s}(T, X) - d_{g,0}(T, X)] \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \quad + \|\lambda^{-r} \cdot Y g(T + sX)^{r(n-i)} \cdot [R_\lambda g(T + sX) - R_\lambda g(T)] \cdot d_{g,0}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \quad + \|\lambda^{-r} \cdot Y [g(T + sX)^{r(n-i)} - g(T)^{r(n-i)}] \cdot R_\lambda g(T) \cdot d_{g,0}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1. \end{aligned}$$

From Hölder's inequality, the first term,

$$\begin{aligned} & \|\lambda^{-r} \cdot Y g(T + sX)^{r(n-i)} \cdot R_\lambda g(T + sX) \cdot [d_{g,s}(T, X) - d_{g,0}(T, X)] \cdot R_\lambda g(T) \cdot g(T)^{ri}\|_1 \\ & \leq \lambda^{-r} \|Y\| \|g(T + sX)^{r(n-i)}\|_* \|R_\lambda g(T + sX)\| \|d_{g,s}(T, X) - d_{g,0}(T, X)\| \|R_\lambda g(T)\| \|g(T)^{ri}\|_*. \end{aligned}$$

Since λ is fixed, λ^{-r} and $\|R_\lambda g(T)\|$ are both constants, as are $\|Y\|$ and $\|g(T)^{ri}\|$; also, $\|R_\lambda g(T + sX)\| \leq 1$ regardless of the value of s (since $g(T + sX)$ is a positive operator). In order to conclude that the right hand side of the inequality can be made arbitrarily small by choosing s suitably, we need that $\|g(T + sX)^{r(n-i)}\|$ is uniformly bounded for s small, and that $\|d_{g,s}(T, X) - d_{g,0}(T, X)\| \rightarrow 0$ as $s \rightarrow 0$. However, from Lemma 3.4.6 we know that $g(T + sX)^{r(n-i)} \rightarrow g(T)^{r(n-i)}$ in operator norm, and if $ri > \frac{q-3}{2}$ then $g(T + sX)^{r(n-i)} \rightarrow g(T)^{r(n-i)}$ in trace norm as well (property 3 of function g). Hence, $\|g(T + sX)^{r(n-i)}\|_*$ can be ensured to be at most $\|g(T)^{r(n-i)}\|_* + 1$ by choosing s small enough. Finally, recall that $d_{g,s}(T, X) \rightarrow d_{g,0}(T, X)$ in operator norm (by the definition of $d_{g,s}$); this concludes the proof that the first term can be made arbitrarily small.

Similarly, the other two terms are bounded above by the product of the norms of their factors. From the resolvent formula, since $g(T + sX) \rightarrow g(T)$, it follows easily that $R_\lambda g(T + sX) \rightarrow R_\lambda g(T)$ in operator norm; and, as already pointed out above, $g(T + sX)^{r(n-i)} \rightarrow g(T)^{r(n-i)}$ in either operator norm or (if suitable) trace norm. The only additional observation we need is that $\|d_{g,0}(T, X)\|$ is constant; using the properties already mentioned in the above paragraph, it follows that the remaining two terms can be made arbitrarily small. Therefore, $\tau(J_s(\lambda)) \rightarrow \tau(J_0(\lambda))$ pointwise. ■

From Lemma 3.4.7 we know that $|\tau(J_s(\lambda))| \leq k \cdot \min\{\lambda^{-r}, \lambda^{-r-1}\}$ (where k is a constant which does not depend on s), which is integrable on $(0, \infty)$, and $\tau(J_s(\lambda))$ converges pointwise to $\tau(J_0(\lambda))$ as $s \rightarrow 0$. We can thus use the Lebesgue Dominated Convergence theorem to conclude that for any sequence $\{r_n\}$ converging to 0 we have $\int_0^\infty \tau(J_{r_n}(\lambda)) d\lambda \rightarrow \int_0^\infty \tau(J_0(\lambda)) d\lambda$. Since

\mathbb{R} is first-countable, this is sufficient to ensure that $\int_0^\infty \tau(J_s(\lambda)) d\lambda$ converges to $\int_0^\infty \tau(J_0(\lambda)) d\lambda$ as $s \rightarrow 0$. This concludes the proof that

$$\begin{aligned} & \left. \frac{d}{ds} \right|_{s=0} \tau(Y g(T + sX)^q) \\ &= \lim_{s \rightarrow 0} \tau\left(Y \cdot \frac{g(T+sX)^q - g(T)^q}{s}\right) \\ &= \sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \cdot \int_0^\infty \lambda^{-r} \tau\left(Y \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot d_{g,0}(T, X) \cdot R_\lambda g(T) \cdot g(T)^{ri}\right) d\lambda. \end{aligned}$$

By assumption, $d_{g,0}(T, X) = \lim_{s \rightarrow 0} \frac{1}{s} \cdot (g(T + sX) - g(T))$ can be written as sums of the form $g_1(T)Xg_2(T)$ (property 5 of function g), so we can break up the trace to get sums of terms which look like

$$\tau(Y \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot g_1(T)Xg_2(T) \cdot R_\lambda g(T) \cdot g(T)^{ri}).$$

The trace property then allows us to write

$$\begin{aligned} & \tau(Y \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot g_1(T) \cdot X \cdot g_2(T) \cdot R_\lambda g(T) \cdot g(T)^{ri}) \\ &= \tau(X \cdot g_2(T) \cdot R_\lambda g(T) \cdot g(T)^{ri} \cdot Y \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot g_1(T)). \end{aligned}$$

Since the various functions of T commute with each other, the right hand side can be written as

$$\tau(X \cdot g(T)^{ri} \cdot R_\lambda g(T) \cdot g_2(T) \cdot Y \cdot g_1(T) \cdot R_\lambda g(T) \cdot g(T)^{r(n-i)}).$$

However, by symmetry, $\left. \frac{d}{ds} \right|_{s=0} \tau(X g(T + sY)^q)$ is equal to

$$\sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \cdot \int_0^\infty \lambda^{-r} \tau(X \cdot g(T)^{r(n-i)} \cdot R_\lambda g(T) \cdot d_{g,0}(T, Y) \cdot R_\lambda g(T) \cdot g(T)^{ri}) d\lambda,$$

so it should be clear that changing the index from i to $n - i$ and expanding $d_{g,0}(T, Y)$ will show that the expression we obtained for the derivative in the X direction appears in the derivative for the Y direction. Here we rely on Property 5 of the function g , which ensures that either $g_1 = g_2$ or, if $g_1 \neq g_2$ then the term $g_2(T)Yg_1(T)$ also appears in $d_{g,0}(T, Y)$. This concludes the proof that the two limits are equal, and hence that α is closed.

We will use this method for both $\tau(X|1 - F^2|^q)$ in the bounded case (which we can re-write as $\tau(X((1 - F^2)^2)^{\frac{q}{2}})$ to get rid of the absolute value sign, giving us $g(F) = (1 - F^2)^2$) and $\tau(X(1 + D^2)^{-m})$ in the unbounded case (giving us $g(D) = (1 + D^2)^{-1}$). The fact that these two functions satisfy the desired properties will be shown in Section 3.5.

Since \mathcal{S} is a Banach space, and the manifold under consideration is simply $T_0 + \mathcal{S}$, the fact that α is exact follows as in the Poincaré Lemma (see [46], Theorem V.4.1 for details). Namely, the zero-form $\theta(T) = C \int_0^1 \tau((T - T_0)g(T_t)^q) dt$ has the property that $d\theta = \alpha$. Note that the Poincaré Lemma in [46] is stated for a Banach space \mathcal{S} ; we have to adjust the formula for $T_0 + \mathcal{S}$ by adding and subtracting T_0 depending on whether we want a tangent vector or a point on the manifold – however, the formula is the same as in the Poincaré Lemma up to differences of notation. Using properties of integrals, it follows that α is independent of the path over which it is integrated. In Appendix D we have a proof of the fact that $d\theta = \alpha$ and that the integral of α is independent of path for a specific manifold $T_0 + \mathcal{S}$ and function g ; this could serve as a model for how the general proof should work.

3.4.1 Other descriptions of the function g

The purpose of this section is to give an alternate description of function g used in the definition of our one-form; see page 85 for a list of the current restrictions placed on g . Recall in particular the definition of $d_{g,s}$:

$$d_{g,s}(T, X) = \begin{cases} \frac{g(T+sX) - g(T)}{s} & s \neq 0 \\ \lim_{s \rightarrow 0} \frac{g(T+sX) - g(T)}{s} & s = 0, \end{cases}$$

where the limit is calculated with respect to the operator norm. Property 3 of the function g , which requires that $X \mapsto g(T + X)^t$ is continuous from \mathcal{S} to \mathcal{L}^1 for $t \geq \frac{q-3}{2}$ and fixed $T \in T_0 + \mathcal{S}$, can be replaced by

- 3.(a) For each fixed $t \geq \frac{q-3}{2}$, and $T \in T_0 + \mathcal{S}$, $\{\|g(T + X)\|_t : X \in \mathcal{S} \text{ with } \|X\|_{\mathcal{S}} \leq 1\}$ is bounded.
- 3.(b) For each fixed $t \geq \frac{q-3}{2}$, $T \in T_0 + \mathcal{S}$, and $s \in [-1, 1]$ we have $d_{g,s}(T, X) \in \mathcal{L}^t$. Moreover, for t and T fixed the set $\{\|d_{g,s}(T, X)\|_t : s \in [-1, 1], X \in \mathcal{S} \text{ with } \|X\|_{\mathcal{S}} \leq 1\}$ is bounded.

These two conditions imply property 3. Denote by $\|X\|_t$ the norm given by $\tau(|X|^t)^{\frac{1}{t}}$; recall that we replaced this norm by the maximum between it and the operator norm on \mathcal{L}^t in order to ensure that \mathcal{L}^t is complete, and that L^t endowed with $\|\cdot\|_t$ is a subset of the densely-defined closed operators affiliated with \mathcal{N} . In order to prove our desired result, we need to know that, in restricted situations, $\|A^r - B^r\|_p$ is bounded by $\||A - B|^r\|_p$. This is Lemma 3.4.9 below; for the needed theorems on the behaviour of generalized s -numbers, refer back to Section 1.1.1. We note that this is a well-known result, the history of which can be found at the end of [21] (page 161).

Lemma 3.4.9 Suppose $0 < r < 1$ and A, B are positive operators. If $p \geq 1$ is such that $A, B \in \mathcal{L}^{pr}$, then

$$\|A^r - B^r\|_p \leq \||A - B|^r\|_p. \quad \blacksquare$$

PROOF We use generalized s -numbers to reduce the problem to a question about functions. With the use of Theorem 1.1.6 (which tells us that $\tau(|T|^p) = \int_0^\infty \mu_t(T)^p dt$ for $T \in \mathcal{L}^p$), we are reduced to proving that

$$\int_0^\infty \mu_t(A^r - B^r)^p dt \leq \int_0^\infty \mu_t(|A - B|^r)^p dt.$$

Since $x \mapsto x^r$ is operator monotone for $0 < r \leq 1$, and it is also clearly a non-negative function for $x > 0$, we have by Theorem 1.1.8 applied with $h(x) = x^r$ that

$$\int_0^s \mu_t(A^r - B^r) dt \leq \int_0^s \mu_t(|A - B|^r) dt \text{ for all } s \geq 0.$$

However, $t \mapsto \mu_t(A^r - B^r)$ and $t \mapsto \mu_t(|A - B|^r)$ are non-negative, decreasing functions, so by Lemma A.0.5 in the appendix, we can conclude that $\int_0^s \mu_t(A^r - B^r)^p dt \leq \int_0^s \mu_t(|A - B|^r)^p dt$ for each $s \in \mathbb{R}_+$. Finally, taking the limit as $s \rightarrow \infty$ of both sides gives us the desired result. \blacksquare

We are now ready to prove that the two conditions are sufficient to ensure that $X \mapsto g(T + X)^t$ is continuous in \mathcal{L}^1 norm.

Lemma 3.4.10 Suppose g is a function which satisfies properties 1 and 2 on 85, and in addition properties 3.(a) and 3.(b) stated at the beginning of this section. Consider $t \in \mathbb{R}_+$ with $t \geq \frac{q-3}{2}$ and fix $T \in T_0 + \mathcal{S}$; then $g(T + X)^t \in \mathcal{L}^1$ by property 2 of g . The function from \mathcal{S} to \mathcal{L}^1 defined by $X \mapsto g(T + X)^t$ is continuous at $X = 0$ (i.e. g also satisfies property 3 on 85). \blacksquare

PROOF By assumptions 3.(a) and 3.(b), we know that $\{\|g(T + X)\|_t : \|X\|_{\mathcal{S}} \leq 1\}$ and $\{\|d_{g,s}(T, X)\|_t : \|X\|_{\mathcal{S}} \leq 1, s \in [-1, 1]\}$ are bounded, say by l and m respectively (chosen such that neither l nor m is zero). Using the formula from Lemma 3.4.2, we can write

$$g(T + X)^t - g(T)^t = \sum_{i=0}^n g(T + X)^{ri} (g(T + X)^r - g(T)^r) g(T)^{r(n-i)},$$

where $n = \lfloor t \rfloor$ and $r = \frac{t}{n+1}$. Apply the triangle and Hölder inequalities respectively (with the understanding that the norm $\|\cdot\|_{\frac{t}{u}}$ represents the operator norm when the denominator u is 0) to get

$$\begin{aligned} \|g(T + X)^t - g(T)^t\|_1 &\leq \sum_{i=0}^n \|g(T + X)^{ri} (g(T + X)^r - g(T)^r) g(T)^{r(n-i)}\|_1 \\ &\leq \sum_{i=0}^n \|g(T + X)^{ri}\|_{\frac{t}{ri}} \cdot \|g(T + sX)^r - g(T)^r\|_{\frac{t}{r}} \cdot \|g(T)^{r(n-i)}\|_{\frac{t}{r(n-i)}}. \end{aligned}$$

In order to justify the use of Hölder's inequality, we have to check that $g(T + X)^{ri} \in L^{\frac{t}{ri}}$ and so on for the other factors; we will only show the first, as the other are similar. That is, $\|g(T + X)^{ri}\|_{\frac{t}{ri}} = \tau(|g(T + X)^{ri}|^{\frac{t}{ri}})^{\frac{ri}{t}} = \tau(g(T + X)^t)^{\frac{ri}{t}}$ (for the last equality we use the fact that $g(T + X)$ is a positive operator). Since $\tau(g(T + X)^t)^{\frac{1}{t}} = \|g(T + X)\|_t$ is finite by property 2 of function g , it follows that $\|g(T + X)^{ri}\|_{\frac{t}{ri}}$ is also finite, so Hölder's inequality can be used. Algebraically manipulate the right-hand side of the above inequality, and if $\|X\|_{\mathcal{S}} \leq 1$ we can use the upper bound on $\|g(T + X)\|_t$ to write

$$\begin{aligned} \|g(T + X)^t - g(T)^t\|_1 &\leq \|g(T + X)^r - g(T)^r\|_{\frac{t}{r}} \left(\sum_{i=0}^n \|g(T + X)^{ri}\|_{\frac{t}{ri}} \|g(T)^{r(n-i)}\|_{\frac{t}{r(n-i)}} \right) \\ &\leq \|g(T + X)^r - g(T)^r\|_{\frac{t}{r}} \left(\sum_{i=0}^n (\|g(T + X)\|_t)^{ri} (\|g(T)\|_t)^{r(n-i)} \right) \\ &\leq \|g(T + X)^r - g(T)^r\|_{\frac{t}{r}} \left(\sum_{i=0}^n (l^{ri} l^{r(n-i)}) \right) \\ &\leq \|g(T + X)^r - g(T)^r\|_{\frac{t}{r}} ((n+1)l^{rn}). \end{aligned}$$

Note $\frac{t}{r} = n+1 = \lfloor t \rfloor + 1 \geq 1$ (since $t \geq 0$), so $\|g(T + X)^r - g(T)^r\|_{\frac{t}{r}} \leq \| |g(T + X) - g(T)|^r \|_{\frac{t}{r}}$ by Lemma 3.4.9. Now $\| |g(T + X) - g(T)|^r \|_{\frac{t}{r}} = \|g(T + X) - g(T)\|_t^r$; so, for any $X \in \mathcal{S}$ with $\|X\|_{\mathcal{S}} \leq 1$, we have

$$\|g(T + X)^t - g(T)^t\|_1 \leq ((n+1)l^{rn}) \cdot \|g(T + X) - g(T)\|_t^r.$$

Given $\varepsilon > 0$, let $s = \min\left\{\left(\frac{\varepsilon}{2(n+1)l^{rn} \cdot m^r}\right)^{\frac{1}{r}}, 1\right\}$ (by design, n and r depend on t only, and l and m depend on T but not on X); then $0 < s \leq 1$. We can write

$$g(T+X) - g(T) = s \cdot \frac{g(T + s \cdot \frac{X}{s}) - g(T)}{s} = s \cdot d_{g,s}(T, X/s).$$

If $\|X\| < s$ then $\left\|\frac{X}{s}\right\| < 1$, so by hypothesis $\|d_{g,s}(T, \frac{X}{s})\|_t \leq m$. So for any X such that $\|X\| < s$ we have

$$\begin{aligned} \|g(T+X)^t - g(T)^t\|_1 &\leq ((n+1)l^{rn}) \cdot \|g(T+X) - g(T)\|_t^r \\ &\leq ((n+1)l^{rn}) \cdot \|s d_{g,s}(T, \frac{X}{s})\|_t^r \\ &\leq ((n+1)l^{rn}) \cdot s^r \cdot \|d_{g,s}(T, \frac{X}{s})\|_t^r \\ &\leq ((n+1)l^{rn}) \cdot s^r \cdot m^r, \end{aligned}$$

which, given our choice of value for s , simplifies to be less than ε . Similar to Lemma 3.4.6 we can show that, since $\|g(T+X)\| \leq \|g(T+X)\|_t$ is uniformly bounded for $\|X\|_{\mathcal{S}} \leq 1$, $X \mapsto g(T+X)^t$ is continuous at $X = 0$ as a function from \mathcal{S} to \mathcal{N} . This allows us to conclude the continuity of $X \mapsto g(T+X)^t$ from \mathcal{S} to \mathcal{L}^1 at $X = 0$. \blacksquare

Therefore, Property 3 of the function g can be replaced by 3.(a) and 3.(b). We will use this alternate description in the bounded case, in Section 3.7.1.

3.5 Spectral flow integral formula

The approach below is the same as the one used in [15] (Theorem 4.1) and [55] (Theorem 3.1). In the unbounded case we consider paths in the manifold $D_0 + \mathcal{N}_{sa}$, where D_0 is self-adjoint Breuer-Fredholm, and satisfies an additional summability condition, usually stated as $(1 + D_0^2)^{-1} \in \mathcal{S}$ for some operator ideal \mathcal{S} (in our case, D_0 is p -summable so $(1 + D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$; and, as a different example, the requirement that D_0 is θ -summable implies $(1 + D_0^2)^{-1}$ belongs to an ideal Li_0 , as proven in Corollary B.6 of [15]). Once we apply the Riesz transform $D \mapsto D(1 + D^2)^{-\frac{1}{2}}$ we end up with paths in some manifold $F_0 + \mathcal{S}$, where \mathcal{S} is a real Banach space, as well as an operator ideal related to \mathcal{S} . The equality that allows us to flip between the two pictures (bounded and unbounded) is $(1 + D^2)^{-1} = 1 - F_D^2$ (where F_D is the image of D under the Riesz transform).

An overview of how the bounded case and unbounded case formulas are related was already presented in Section 3.3. Suffice it to reiterate that in the bounded case we consider one-forms $X \mapsto \frac{1}{C} \cdot \tau(Xk(1 - F^2))$, for some suitable constant C and function k . We show that integrating this one-form gives spectral flow for straight-line paths whose endpoints are of the form $2P - 1$ and $2Q - 1$ respectively, for two projections P and Q . The general formula is then obtained from this case. Note that we could consider a slightly more general case – namely, try to handle one-forms $\alpha_F : X \mapsto \frac{1}{C} \cdot \tau(Xk(F))$ in some affine space $F_0 + \mathcal{S}$ which is not necessarily related to the image of the Riesz transform; however, the current approach would lead to very technical conditions on k and \mathcal{S} , and we did not wish to clutter the presentation. We address these technical conditions in Remark 3.5.20.

We refer to Section 1.1.1 for a review of powers of ideals, if needed; they are about to make an appearance in the description of the manifold $F_0 + \mathcal{S}$ in which we can find the image of the

path $\{D_t\}$ under the Riesz transform. Recall once again that, if $F_D = D(1 + D^2)^{-\frac{1}{2}}$, we have the equality $(1 + D^2)^{-1} = 1 - F_D^2$, so if $(1 + D^2)^{-1} \in \mathcal{I}$ so is $1 - F_D^2$. In fact, if $D = D_0 + A$ for $A \in \mathcal{N}$, then $F_D - F_{D_0}$ cannot be just any bounded operator, as will be shown in Corollary 3.5.12. The spaces that we are compelled to consider in the bounded case are $F_0 + \mathcal{S}\mathcal{G}_{F_0}$, where F_0 is self-adjoint Breuer-Fredholm and

$$\mathcal{S}\mathcal{G}_{F_0} = \{X \in \mathcal{I}_{sa}^{\frac{1}{2}} : 1 - (F_0 + X)^2 \in \mathcal{I}\}.$$

Note that the description of $\mathcal{S}\mathcal{G}_{F_0}$ depends on the choice of F_0 ; however, Theorem 3.5.1 below tells us that the space itself is independent of the choice of base point. The description and properties of $\mathcal{S}\mathcal{G}_{F_0}$ summarized in the next paragraph can be found in [15], p. 143-144 and Appendix B (note that $\mathcal{S}\mathcal{G}_{F_0}$ is denoted in [15] by \mathcal{I}_{F_0} ; be warned that a typo on page 143 of this article suggests that \mathcal{I}_{F_0} denotes the space $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ instead, which was clearly not intended).

As $\mathcal{S}\mathcal{G}_{F_0}$ looks, perhaps, impenetrable, we give a nicer description of the operators contained in this space. Suppose that $F_0 = 2P - 1$ for some projection P . Then, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$ we have $F_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Writing any $X \in \mathcal{S}\mathcal{G}_{F_0}$ with respect to the same decomposition, we would have $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, and since $X \in \mathcal{I}^{\frac{1}{2}}$, we could conclude $X_{11} \in P\mathcal{I}^{\frac{1}{2}}P$, $X_{12} \in P\mathcal{I}^{\frac{1}{2}}P^\perp$, $X_{21} \in P^\perp\mathcal{I}^{\frac{1}{2}}P$, and, finally, $X_{22} \in P^\perp\mathcal{I}^{\frac{1}{2}}P^\perp$. However, we can obtain a stronger restriction on the X_{ij} 's by using the condition $1 - (F_0 + X)^2 \in \mathcal{I}$. Namely, when we calculate $1 - (F_0 + X)^2$ we get $\begin{bmatrix} -2X_{11} - X_{11}^2 & X_{11}X_{12} + X_{12}X_{22} \\ X_{21}X_{11} + X_{22}X_{21} & -X_{22}^2 + 2X_{22} \end{bmatrix}$. As $X_{11}^2 \in P\mathcal{I}P$ and $X_{22}^2 \in P^\perp\mathcal{I}P^\perp$, it follows that in order for $1 - (F_0 + X)^2$ to be in \mathcal{I} we must have $X_{11} \in P\mathcal{I}P$ and $X_{22} \in P^\perp\mathcal{I}P^\perp$ (using the fact that the product of two operators in $\mathcal{I}^{\frac{1}{2}}$ is in \mathcal{I}). This leads to the description of $\mathcal{S}\mathcal{G}_{F_0}$ as all operators in $\mathcal{I}^{\frac{1}{2}}$ that, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, have the form

$$\begin{bmatrix} P\mathcal{I}P & P\mathcal{I}^{\frac{1}{2}}P^\perp \\ P^\perp\mathcal{I}^{\frac{1}{2}}P & P^\perp\mathcal{I}P^\perp \end{bmatrix}.$$

The fact that all operators which have the above decomposition are in $\mathcal{S}\mathcal{G}_{F_0}$ is easy to check using the fact that \mathcal{I} and $\mathcal{I}^{\frac{1}{2}}$ are ideals, $\mathcal{I} \subset \mathcal{I}^{\frac{1}{2}}$ and $\mathcal{I}^{\frac{1}{2}} \cdot \mathcal{I}^{\frac{1}{2}} = \mathcal{I}$ (see Definition 1.1.13 and remarks following it).

If F_0 is not equal to $2P - 1$ for some projection P , then the above still holds but with $P = \chi_{[0, \infty)}(F_0)$. The fact that $2P - 1$ is then in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ and $F_0 + \mathcal{S}\mathcal{G}_{F_0} = (2P - 1) + \mathcal{S}\mathcal{G}_{2P-1}$ is the content of further discussion, so we will not go into details now (see Theorem 3.5.1 below, and the beginning of Step 2 of the proof).

Theorem 3.5.1 ([15], Lemma B.12) With the norm $\|X\|_{\mathcal{S}\mathcal{G}_{F_0}} = \|X\|_{\mathcal{I}^{\frac{1}{2}}} + \|XF_0 + F_0X\|_{\mathcal{I}}$, $\mathcal{S}\mathcal{G}_{F_0}$ is a Banach space. Moreover, if $F \in \mathcal{S}\mathcal{G}_{F_0}$ then $F_0 + \mathcal{S}\mathcal{G}_{F_0} = F + \mathcal{S}\mathcal{G}_F$, and the two norms on $\mathcal{S}\mathcal{G}_{F_0}$ are equivalent. \blacksquare

We will show that, with $\mathcal{J} = \mathcal{J}^{1-\varepsilon}$ for some $0 < \varepsilon < 1$, if $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ with $(1 + D_0^2)^{-1} \in \mathcal{J}$, then $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}_{\mathcal{J}F_0}$. Hence, these are the kind of spaces we consider for the bounded case formula.

The final goal is, for a path $\{D_t\}$ in $D_0 + \mathcal{N}_{sa}$, to figure out conditions we can put on the function k to ensure we get a spectral flow formula of the type

$$\text{sf}(\{D_t\}) = \frac{1}{C} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)k((1 + D_t^2)^{-1}) \right) dt + \beta(D_1) - \beta(D_0),$$

where the appropriate choices for constant C and correction terms $\beta(D)$ will come out of the proof. We have divided the proof into three steps:

1. Suppose first that $\{F_t\}$ is a straight line path from $F_0 = 2P - 1$ to $F_1 = 2Q - 1$, with $\{F_t\} \subset F_0 + \mathcal{S}_{\mathcal{J}F_0}$ for some suitable operator ideal \mathcal{J} . Then Lemma 3.5.3 will show that, for judiciously chosen functions h ,

$$\text{sf}(\{F_t\}) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right) dt.$$

2. Next, consider a general path $\{F_t\}$ in $F_0 + \mathcal{S}_{\mathcal{J}F_0}$. Use concatenation and homotopy to obtain a path of the type considered in the previous step, allowing us to extend the formula. The correction terms enter the proof at this step.
3. If $\{D_t\}$ is a path of unbounded operators, let $F_t = D_t(1 + D_t^2)^{-\frac{1}{2}}$. Then $\text{sf}(\{D_t\}) = \text{sf}(\{F_t\})$ by definition, and $(1 + D_t^2)^{-1} = 1 - F_t^2$. We show that $\{F_t\}$ is a C^1 path in an appropriate affine space, from where it immediately follows that if h is suitably chosen

$$\text{sf}(\{D_t\}) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(F_t) \cdot h((1 + D_t^2)^{-1}) \right) dt + \beta(D_1) - \beta(D_0).$$

Lemma 3.5.17 will allow us to get rid of the $\frac{d}{dt}(F_t)$ in this formula, by showing that $\tau \left(\frac{d}{dt}(F_t)h((1 + D_t^2)^{-1}) \right) = \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{3}{2}}h((1 + D_t^2)^{-1}) \right)$.

Implementation of step 1

We need to show that the integral formula calculates spectral flow in the special case when the endpoints of the path are of the form $2P - 1$ and $2Q - 1$ for P, Q projections. Note that, if P and Q are projections for which $Q - P \in \mathcal{K}_{\mathcal{N}}$ then, since $\pi(P) = \pi(Q)$, the spectral flow of the straight line path from $2P - 1$ to $2Q - 1$ is $\text{ind}(PQ)$ (directly from Definition 1.1.26). The following theorem gives us a formula for calculating this index:

Theorem 3.5.2 ([15], Theorem 3.1) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous odd function with $f(1) \neq 0$. Let P and Q be projections with $Q - P \in \mathcal{K}_{\mathcal{N}}$ and $f(Q - P)$ trace class. Then $\text{ind}(PQ) = \frac{1}{f(1)} \tau(f(Q - P))$, where $\text{ind}(PQ)$ is the index of PQ as an operator from $Q\mathcal{H}$ to $P\mathcal{H}$. ▮

The above result applied to a suitable family of functions will allow us to get the spectral flow as an integral, but only in the case when our paths are linear and have special endpoints.

Lemma 3.5.3 For P and Q projections, let $F_0 = 2P - 1$ and $F_1 = 2Q - 1$. Suppose $F_1 \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$, where \mathcal{S} is a small power invariant operator ideal (see Remark 3.3.1 for the definition). Denote by $\{F_t\}$ the straight-line path from F_0 to F_1 (that is, $F_t = F_0 + t(F_1 - F_0)$ for $t \in [0, 1]$). Suppose, moreover, that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, non-zero on $(0, 1]$ and for which $h(T)$ is trace class for all $T \in \mathcal{S}_{sa}$. Then

$$\text{sf}(\{F_t\}) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(F_t) \cdot h(1 - F_t^2) \right) dt,$$

with $C = \int_{-1}^1 h(1 - s^2) ds$. ☛

PROOF By definition, for every $F \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$ we have $1 - F^2 \in \mathcal{S}$, so in particular $1 - F_t^2 \in \mathcal{S}$. The hypotheses on h thus give us that $h(1 - F_t^2)$ is trace class, whence $\tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right) < \infty$. Since $F_t = F_0 + t(F_1 - F_0) = (2P - 1) + 2t(Q - P)$, we have $\frac{d}{dt}(F_t) = F_1 - F_0 = 2(Q - P)$ and

$$\begin{aligned} 1 - F_t^2 &= 1 - (2P - 1 + 2t(Q - P))^2 \\ &= 4t(1 - t)(Q - P)^2. \end{aligned}$$

Hence,

$$\tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right) = \tau(2(Q - P) \cdot h(4t(1 - t)(Q - P)^2)).$$

For each fixed $t \in (0, 1)$ let $f_t(x) = 2xh(4t(1 - t)x^2)$; we claim that f_t satisfies the requirements of Theorem 3.5.2. It is easy to see that f_t is an odd continuous function of x (recall h is continuous by hypothesis). Also, $f_t(1) = 2h(4t(1 - t))$; since $4t(1 - t) \in (0, 1]$ and h is not 0 on $(0, 1]$, we get that $f_t(1) \neq 0$. Finally, $Q - P \in \mathcal{S}\mathcal{G}_{F_0} \subset \mathcal{S}^{\frac{1}{2}} \subset \mathcal{K}\mathcal{N}$, and $f_t(Q - P) = 2(Q - P)h(4t(1 - t)(Q - P)^2)$, which is trace class by the assumptions on h (since $4t(1 - t)(Q - P)^2 \in \mathcal{S}$). Hence we can apply Theorem 3.5.2 with $f = f_t$ to get that, for any $t \in (0, 1)$,

$$\text{ind}(PQ) = \frac{1}{f_t(1)} \tau(f_t(Q - P)) = \frac{1}{2h(4t(1 - t))} \cdot \tau(2(Q - P)h(4t(1 - t)(Q - P)^2)).$$

Rearranging,

$$\tau(2(Q - P)h(4t(1 - t)(Q - P)^2)) = 2 \text{ind}(PQ) \cdot h(4t(1 - t)).$$

Since the left-hand side of this last equality is equal to $\tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right)$ (see the computation earlier in the proof),

$$\begin{aligned} \int_0^1 \tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right) dt &= \int_0^1 2h(4t(1 - t)) \text{ind}(PQ) dt \\ &= \text{ind}(PQ) \cdot \int_0^1 2h(4t(1 - t)) dt \\ &= \text{sf}(\{F_t\}) \cdot \int_0^1 2h(4t(1 - t)) dt; \end{aligned}$$

in the last line we used the fact that, since $F_t = F_0 + t(F_1 - F_0)$ and $F_t - F_0 \in \mathcal{H}_N$, we have $\pi(F_t) = \pi(F_0)$ for all t and so $\text{sf}(\{F_t\}) = \text{ind}(PQ)$ (by Definition 1.1.26).

Let $C = \int_0^1 2h(4t(1-t)) dt$. Rearranging, we get

$$\text{sf}(\{F_t\}) = \frac{1}{C} \cdot \int_0^1 \tau \left(\frac{d}{dt}(F_t)h(1-F_t^2) \right) dt.$$

To finish the proof, note that the change of variables $s = 2t - 1$ gives us $C = \int_{-1}^1 h(1-s^2) ds$. ■

Implementation of step 2

In this step, we relate the calculation of spectral flow for a general path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ to a straight-line path of the type for which we already have a formula.

Consider the function

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0, \end{cases}$$

and for any F self-adjoint write \tilde{F} for $\text{sign}(F)$. Note that, if $P = \chi_{[0,\infty)}(F)$, then $\tilde{F} = 2P - 1$. It should also be clear that $\tilde{F}^2 = 1$. Suppose that additionally $F \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$ for some appropriate F_0 ; we want to show that $\tilde{F} \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$. Since $F + \mathcal{S}\mathcal{G}_F = F_0 + \mathcal{S}\mathcal{G}_{F_0}$ (Theorem 3.5.1), it is sufficient to check that $\tilde{F} \in F + \mathcal{S}\mathcal{G}_F$. Going back to the definition of $F + \mathcal{S}\mathcal{G}_F$, it is easy to see that $1 - \tilde{F}^2 = 0 \in \mathcal{J}$, so we next have to show that $\tilde{F} - F \in \mathcal{J}^{\frac{1}{2}}$. By assumption, $1 - F^2 \in \mathcal{J}$. However, $1 - F^2 = \tilde{F}^2 - F^2 = (\tilde{F} - F)(\tilde{F} + F)$ (here we used the fact that F and \tilde{F} commute). Since $\tilde{F} + F$ is invertible, we have $\tilde{F} - F = (1 - F^2) \cdot (\tilde{F} + F)^{-1}$, which gives us that $\tilde{F} - F \in \mathcal{J}$ (since $1 - F^2 \in \mathcal{J}$ and \mathcal{J} is an ideal). Finally, since $\mathcal{J} \subset \mathcal{J}^{\frac{1}{2}}$ (see Remark 1.1.15), we have $\tilde{F} - F \in \mathcal{J}^{\frac{1}{2}}$.

If $\{F_t\}$ is any path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$, then extend the path by connecting F_0 to \tilde{F}_0 and F_1 to \tilde{F}_1 via straight lines (see Figure 3.2 below). As there is no spectral flow from F_0 to \tilde{F}_0 or from F_1 to \tilde{F}_1 (see Remark 1.1.39), the additivity property of spectral flow allows us to conclude that the spectral flow along the path $\tilde{F}_0 \dashrightarrow F_0 \text{ --- } F_1 \dashrightarrow \tilde{F}_1$ is the same as the spectral flow of $\{F_t\}$. On the other hand, we can join \tilde{F}_0 to \tilde{F}_1 by a straight line (indicated in the figure by a dotted line), which also lies in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. Under the assumption that $\alpha_F(X) = \tau(Xh(1 - F_t^2))$ defines an exact one-form on $F_0 + \mathcal{S}\mathcal{G}_{F_0}$, integrating it along either path from \tilde{F}_0 to \tilde{F}_1 will give us the same answer. Finally, we know that integrating the one-form along the straight line path from \tilde{F}_0 to \tilde{F}_1 gives us the spectral flow from \tilde{F}_0 to \tilde{F}_1 (this is Lemma 3.5.3).

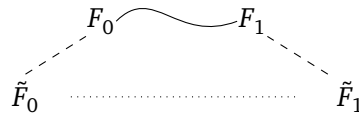


Figure 3.2: Extend the original path F_t as shown by the dashed lines. Integrating our one-form along either path from \tilde{F}_0 to \tilde{F}_1 should give the same value – the spectral flow from F_0 to F_1 .

Hence, in order to get the spectral flow it is not sufficient to integrate the one-form along the original path; we need to adjust our formula to include correction terms, consisting of the integral of the one-form from \tilde{F}_0 to F_0 and from F_1 to \tilde{F}_1 (i.e. along the dashed lines in Figure 3.2). This gives us

Theorem 3.5.4 Let $\{F_t\}$ be a C^1 path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. Suppose that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that h is non-zero on $(0, 1]$ and $h(T)$ is trace-class for all $T \in \mathcal{J}_{sa}$. Moreover, suppose that $\alpha_F : X \mapsto \tau(Xh(1 - F^2))$ is a one-form on $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ whose integral is independent of the path of integration. Define a function $\gamma : F_0 + \mathcal{S}\mathcal{G}_{F_0} \rightarrow \mathbb{R}$ by $\gamma(F) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(G_t)h(1 - G_t^2) \right) dt$, where $\{G_t\}$ is the straight line path from F to \tilde{F} . Then

$$\text{sf}(\{F_t\}) = \frac{1}{C} \cdot \int_0^1 \tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right) dt + \gamma(F_1) - \gamma(F_0),$$

where $C = \int_{-1}^1 h(1 - s^2) ds$. ☛

Remark 3.5.5 Note that, if F_0 is unitarily equivalent to F_1 then $\gamma(F_0) = \gamma(F_1)$ (since the expressions whose traces we are calculating are also unitarily equivalent). •

Implementation of step 3

We would now like to reduce the unbounded case to the bounded case. Suppose that $\{D_t\}$ is a path in $D_0 + \mathcal{N}_{sa}$ and $\{F_t\}$ is its image under the Riesz transform, with $(1 + D_0^2)^{-1} \in \mathcal{J}$ for some small power invariant operator ideal \mathcal{J} (see Remark 3.3.1 for the definition). Recall the notation

$$\mathcal{S}\mathcal{G}_{F_0} = \{X \in \mathcal{J}^{\frac{1}{2}} : 1 - (F_0 + X)^2 \in \mathcal{J}\}.$$

Clearly, $1 - F_t^2 \in \mathcal{J}$; however, in order to use Theorem 3.5.4, we would need $\{F_t\}$ to be a C^1 path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. The various norm inequalities used in the proof are not strong enough to prove this (assuming it is even true); in order for this approach to work, we will have to replace \mathcal{J} by $\mathcal{J} = \mathcal{J}^{1-\varepsilon}$ for some $0 < \varepsilon < 1$ (recall, with the aid of Remark 1.1.15, that $\mathcal{J} \subset \mathcal{J}^{1-\varepsilon}$), and show $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$.

Hence, if h is a function which satisfies the hypotheses of Theorem 3.5.4 for $\mathcal{J} = \mathcal{J}^{1-\varepsilon}$, we could easily conclude that

$$\text{sf}(\{D_t\}) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(F_t) \cdot h((1 + D_t^2)^{-1}) \right) dt + \gamma \left(D_1(1 + D_1^2)^{-\frac{1}{2}} \right) - \gamma \left(D_0(1 + D_0^2)^{-\frac{1}{2}} \right).$$

A second goal of this section is to get rid of $\frac{d}{dt}(F_t)$ and replace it by $\frac{d}{dt}(D_t)$. The price we pay for this is a factor of $(1 + D_t^2)^{-\frac{3}{2}}$; in Lemma 3.5.17, we will show that $\tau \left(\frac{d}{dt}(F_t)h(1 - F_t^2) \right)$ is equal to $\tau \left(\frac{d}{dt}(D_t) \cdot (1 + D_t^2)^{-\frac{3}{2}} \cdot h((1 + D_t^2)^{-1}) \right)$. The extra factor of $(1 + D_t^2)^{-\frac{3}{2}}$ is a consequence of the derivative $\frac{d}{dt}(F_t)$, as should be expected since we already have that $(1 + D_t^2)^{-1} = 1 - F_t^2$.

Remark 3.5.6 This is just an interesting aside. Note that the derivative of the function $x(1 + x^2)^{-\frac{1}{2}}$ is $(1 + x^2)^{-\frac{3}{2}}$. Of course, the derivative of $D_t(1 + D_t^2)^{-\frac{1}{2}}$ is not as easy to calculate, as the D_t 's

are unbounded and anyway the operators involved do not commute. However, once we take the trace we can rearrange some of the operators, and we amusingly enough recover the commutative derivative (note the $(1 + D_t^2)^{-\frac{3}{2}}$ in the trace equality formula). •

Fix $0 < \varepsilon < 1$ and let $\mathcal{G} = \mathcal{G}^{1-\varepsilon}$. We want to show that applying the Riesz transform to $\{D_t\}$ gives us a path in $F_0 + \mathcal{L}\mathcal{G}_{F_0}$. In order to show that $\{F_t\}$ is continuous, it is sufficient (as we will see below) to show that $1 - F_t^2$ is continuous in the \mathcal{G} norm, and that $\{F_t - F_0\}$ is continuous in the $\mathcal{G}^{\frac{1}{2}}$ norm; a similar statement holds if we replace the word continuous by C^1 . Before we proceed to state and prove this result, we make a note of some norm bounds and continuity results that we will need (and that are either known or easily provable from the Spectral Theorem).

Lemma 3.5.7 Let D be a self-adjoint unbounded operator.

- (i) For $0 < r \leq 1$ (with $r \in \mathbb{R}$), and $\lambda \in [0, \infty)$, $\|(1 + D^2 + \lambda)^{-r}\| < (1 + \lambda)^{-r}$.
- (ii) For $r \in \mathbb{R}$ such that $\frac{1}{2} < r \leq 1$, and $\lambda \in [0, \infty)$, $\|D(1 + D^2 + \lambda)^{-r}\| \leq (1 + \lambda)^{\frac{1}{2}-r}$.
- (iii) (Remark A.5 of [14]) $\lambda \mapsto (1 + D^2 + \lambda)^{-1}$ is a continuous function from $[0, \infty)$ into \mathcal{N} .
- (iv) (Remark A.5 of [14]) $\lambda \mapsto D(1 + D^2 + \lambda)^{-1}$ is a continuous function from $[0, \infty)$ into \mathcal{N} .
- (v) (Lemma A.6 of [14]) If $D = D_0 + A$ for some self-adjoint bounded operator A , and $\lambda \in [0, \infty)$ is fixed, then

$$\begin{aligned} \|(1 + D^2 + \lambda)^{-1} - (1 + D_0^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-3/2} \cdot \|A\|, \text{ and} \\ \|D(1 + D^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-1} \cdot \|A\|. \end{aligned}$$

PROOF For $x \in \mathbb{R}$ it is immediately clear that $|(1 + x^2 + \lambda)^{-r}| \leq (1 + \lambda)^{-r}$, so we can use the Spectral Theorem to conclude that

$$\|(1 + D^2 + \lambda)^{-r}\| \leq (1 + \lambda)^{-r}.$$

Note that, for $r < \frac{1}{2}$, $D(1 + D^2 + \lambda)^{-r}$ is not bounded. For $\frac{1}{2} < r \leq 1$, we let $f(x) = \frac{x}{(1 + \lambda + x^2)^r}$, and find that $|f(x)| \leq f(\sqrt{\frac{1 + \lambda}{2r - 1}})$. The right hand side simplifies to $(1 + \lambda)^{\frac{1}{2}-r} \cdot (2r - 1)^{r-\frac{1}{2}} \cdot (2r)^{-r}$. Since $(2r - 1) \leq 1$ and $r - \frac{1}{2} \geq 0$ we have $(2r - 1)^{r-\frac{1}{2}} \leq 1$; likewise, $2r > 1$ and $r > 0$ give us $(2r)^{-r} \leq 1$. Hence $|f(x)| \leq (1 + \lambda)^{\frac{1}{2}-r}$, so finally we apply the Spectral Theorem to get

$$\|D_t(1 + D_t^2 + \lambda)^{-r}\| \leq (1 + \lambda)^{\frac{1}{2}-r}.$$

When $r = \frac{1}{2}$, $D_t(1 + D_t^2 + \lambda)^{-\frac{1}{2}}$ is still a bounded operator, but we can do no better than claim that its norm is less than or equal to 1, which is not strong enough to prove later integral convergences.

We refer to Remark A.5 of [14] for the continuity results (also proven from the Spectral Theorem); namely,

$$\begin{aligned} \|(1 + D^2 + \lambda)^{-1} - (1 + D^2 + \eta)^{-1}\| &\leq (1 + \lambda)^{-1} \cdot |\lambda - \eta| \cdot (1 + \eta)^{-1}, \text{ and} \\ \|D(1 + D^2 + \lambda)^{-1} - D(1 + D^2 + \eta)^{-1}\| &\leq \frac{1}{2\sqrt{1 + \lambda}} \cdot |\lambda - \eta| \cdot (1 + \eta)^{-1}, \end{aligned}$$

allowing us to conclude the continuity of the two functions. We also do not prove part (v). ■

Theorem 3.5.8 ([15], **Lemma B.15**) If \mathcal{S} is an invariant operator ideal which satisfies the Cauchy-Schwarz inequality, then $\{F_t\} \subset F_0 + \mathcal{S}_{F_0}$ is a C^1 path if and only if $\{F_t\}$ is C^1 in $\mathcal{S}^{\frac{1}{2}}$ -norm and $\{1 - F_t^2\}$ is C^1 in \mathcal{S} -norm. \blacksquare

PROOF Write $F_t = F_0 + X_t$; by definition of \mathcal{S}_{F_0} , $X_t \in \mathcal{S}^{\frac{1}{2}}$ is such that $1 - (F_0 + X_t)^2 \in \mathcal{S}$.

Suppose first that $\{X_t\}$ is C^1 in $\mathcal{S}^{\frac{1}{2}}$, and $\{1 - F_t^2\}$ is C^1 in \mathcal{S} . We want to calculate $\|\cdot\|_{\mathcal{S}_{F_0}} \frac{d}{dt}(F_t)$ and show it is continuous.

By hypothesis, $\{F_t\}$ is C^1 in $\mathcal{S}^{\frac{1}{2}}$ -norm; let $G_t = \|\cdot\|_{\mathcal{S}^{\frac{1}{2}}} \frac{d}{dt}(X_t)$. Then $\{G_t\} \subset \mathcal{S}^{\frac{1}{2}}$ is continuous. Note that, since $\mathcal{S}^{\frac{1}{2}}$ is an invariant operator ideal, $\|X\|_{\mathcal{S}^{\frac{1}{2}}} \geq \|X\|$ for any $X \in \mathcal{S}^{\frac{1}{2}}$, which allows us to conclude that $\|\cdot\|_{\mathcal{S}^{\frac{1}{2}}} \frac{d}{dt}(X_t)$ is also equal to G_t for any $t \in [0, 1]$. Fix $t_0 \in [0, 1]$; we will show that $\|\cdot\|_{\mathcal{S}_{F_0}} \frac{d}{dt} \Big|_{t=t_0} (F_t) = G_{t_0}$.

By the definition of the norm on \mathcal{S}_{F_0} ,

$$\left\| \frac{F_t - F_{t_0}}{t - t_0} - G_{t_0} \right\|_{\mathcal{S}_{F_0}} = \left\| \frac{X_t - X_{t_0}}{t - t_0} - G_{t_0} \right\|_{\mathcal{S}^{\frac{1}{2}}} + \left\| F_0 \left(\frac{X_t - X_{t_0}}{t - t_0} - G_{t_0} \right) + \left(\frac{X_t - X_{t_0}}{t - t_0} - G_{t_0} \right) F_0 \right\|_{\mathcal{S}}. \quad (\dagger)$$

The second term on the right hand side can be rearranged to $\left\| \frac{\{F_0 X_t\} - \{F_0 X_{t_0}\}}{t - t_0} - (F_0 G_{t_0} + G_{t_0} F_0) \right\|_{\mathcal{S}}$,

so in order to conclude that $\frac{F_t - F_{t_0}}{t - t_0} \xrightarrow{\|\cdot\|_{\mathcal{S}_{F_0}}} G_{t_0}$ it is sufficient to show $\|\cdot\|_{\mathcal{S}} \frac{d}{dt} \{F_0, X_t\} = \{F_0, G_t\}$.

Since $\|\cdot\|_{\mathcal{S}} \frac{d}{dt}(X_t) = G_t$, it is not hard to check that $\|\cdot\|_{\mathcal{S}} \frac{d}{dt} \{F_0, X_t\} = \{F_0, G_t\}$. As $\|Y\|_{\mathcal{S}} \geq \|Y\|$ for all $Y \in \mathcal{S}$, if $\|\cdot\|_{\mathcal{S}} \frac{d}{dt} (\{F_0, X_t\})$ exists, it must equal $\|\cdot\|_{\mathcal{S}} \frac{d}{dt} (\{F_0, X_t\}) = \{F_0, G_t\}$. It is easy to check that $\{F_0, X_{t_0}\} = (1 - F_0^2) - (1 - (F_0 + X_t)^2) - X_t^2$; now $t \mapsto 1 - (F_0 + X_t)^2$ is C^1 in \mathcal{S} by hypothesis, so we concentrate on the X_t^2 term. As $X_t \in \mathcal{S}^{\frac{1}{2}}$, we certainly have $X_t^2 \in \mathcal{S}$, and the fact that \mathcal{S} has the Cauchy-Schwarz property allows us to conclude that $\|\cdot\|_{\mathcal{S}} \frac{d}{dt}(X_t^2) = X_t G_t + G_t X_t$, which is continuous in \mathcal{S} norm. For example,

$$\begin{aligned} & \left\| \frac{X_t^2 - X_{t_0}^2}{t - t_0} - (X_{t_0} G_{t_0} + G_{t_0} X_{t_0}) \right\|_{\mathcal{S}} \\ & \leq \left\| X_t \cdot \frac{X_t - X_{t_0}}{t - t_0} - X_t G_{t_0} \right\|_{\mathcal{S}} + \|X_t G_{t_0} - X_{t_0} G_{t_0}\|_{\mathcal{S}} + \left\| \frac{X_t - X_{t_0}}{t - t_0} X_{t_0} - G_{t_0} X_{t_0} \right\|_{\mathcal{S}} \\ & \leq \|X_t\|_{\mathcal{S}^{\frac{1}{2}}} \cdot \left\| \frac{X_t - X_{t_0}}{t - t_0} - G_{t_0} \right\|_{\mathcal{S}^{\frac{1}{2}}} + \|X_t - X_{t_0}\|_{\mathcal{S}^{\frac{1}{2}}} \cdot \|G_{t_0}\|_{\mathcal{S}^{\frac{1}{2}}} + \left\| \frac{X_t - X_{t_0}}{t - t_0} - G_{t_0} \right\|_{\mathcal{S}^{\frac{1}{2}}} \cdot \|X_{t_0}\|_{\mathcal{S}^{\frac{1}{2}}}; \end{aligned}$$

this allows us to use the fact that $\{X_t\}$ is C^1 in $\mathcal{S}^{\frac{1}{2}}$ to conclude that the difference quotient of X_t^2 at $t = t_0$ converges to $X_{t_0} G_{t_0} + G_{t_0} X_{t_0}$ in \mathcal{S} . The fact that $\{X_t G_t + G_t X_t\}$ is continuous in \mathcal{S} follows similarly from the fact that $\{X_t\}$ and $\{G_t\}$ are both continuous in $\mathcal{S}^{\frac{1}{2}}$. Thus, we can conclude from this work that $\{F_0, X_{t_0}\}$ is C^1 in \mathcal{S} (as both $1 - (F_0 + X_t)^2$ and X_t^2 are C^1) and its derivative is $\{F_0, G_{t_0}\}$. Going back to (\dagger) , it follows that $\|\cdot\|_{\mathcal{S}_{F_0}} \frac{d}{dt}(F_t) = G_t$, which can now easily be checked to be continuous in \mathcal{S}_{F_0} norm. Therefore, $\{F_t\}$ is C^1 in \mathcal{S}_{F_0} .

Conversely, suppose $\|\cdot\|_{\mathcal{S}_{F_0}} \frac{d}{dt}(F_t)$ exists and is equal to $\{G_t\}$, and that $\{G_t\}$ is continuous in $\|\cdot\|_{\mathcal{S}_{F_0}}$ norm. Since $\|X\|_{\mathcal{S}_{F_0}} \geq \|X\|_{\mathcal{S}^{\frac{1}{2}}}$ for all $X \in \mathcal{S}_{F_0} \subset \mathcal{S}^{\frac{1}{2}}$, it follows immediately that

$\{F_t\}$ is C^1 in $\mathcal{S}^{\frac{1}{2}}$ norm, and $\|\cdot\|_{\mathcal{S}^{\frac{1}{2}}} \cdot \frac{d}{dt}(X_t) = G_t$. From (\dagger) we can now conclude that $\frac{d}{dt}\{F_0, X_t\}$ converges to $\{F_0, G_t\}$ in \mathcal{S} , and similarly that $\{F_0, X_t\}$ is C^1 in \mathcal{S} . Proceed as in the converse direction to relate $\{F_0, X_t\}$ to $\{1 - (F_0 + X_t)^2\}$ and conclude that $\{1 - F_t^2\}$ is C^1 in \mathcal{S} . ■

The above result makes clear what we have to do in order to show that $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}_{F_0}$. The fact that $1 - F_t^2$ is continuous in the \mathcal{S} norm follows almost immediately from the following result (though we point out that this result is itself a direct consequence of Lemma B.6 of [14]), and is shown in Theorem 3.5.10.

Theorem 3.5.9 ([15], Lemma 6.1) For D_0 an unbounded self-adjoint operator and A bounded and self-adjoint, let $D = D_0 + A$. Then

$$-(f(\|A\|) - 1)(1 + D_0^2)^{-1} \leq (1 + D^2)^{-1} - (1 + D_0^2)^{-1} \leq (f(\|A\|) - 1)(1 + D_0^2)^{-1},$$

where $f(a) = 1 + \frac{1}{2} \cdot (a^2 + a\sqrt{a^2 + 4})$. ☛

Theorem 3.5.10 ([15], Proposition 6.4) Suppose D_0 is an unbounded self-adjoint operator affiliated with \mathcal{N} for which $(1 + D_0^2)^{-1} \in \mathcal{S}$ for some invariant operator ideal \mathcal{S} , and $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$. Let $F_t = D_t(1 + D_t^2)^{-\frac{1}{2}}$ for each t ; then $t \mapsto 1 - F_t^2$ is C^1 in the norm on \mathcal{S} . Moreover, if \mathcal{S} is a small power invariant operator ideal, for any $0 < \varepsilon < 1$, $1 - F_t^2$ is C^1 in the norm on $\mathcal{S}^{1-\varepsilon}$. ☛

PROOF We first tackle showing that $\{1 - F_t^2\}$ is continuous at zero in \mathcal{S} -norm. Recall that $1 - F_t^2 = (1 + D_t^2)^{-1}$, and let $A_t = D_t - D_0$. Rearrange the inequality from Theorem 3.5.9 into

$$0 \leq (1 + D_t^2)^{-1} - (1 + D_0^2)^{-1} + (f(\|A_t\|) - 1) \cdot (1 + D_0^2)^{-1} \leq 2(f(\|A_t\|) - 1) \cdot (1 + D_0^2)^{-1}.$$

The norm on an invariant operator ideal respects the order of positive operators (Theorem 1.1.10), so from the reverse triangle inequality, and re-writing the expression to use F_t 's, we can get

$$\|(1 - F_t^2) - (1 - F_0^2)\|_{\mathcal{S}} \leq 3 \cdot (f(\|A_t\|) - 1) \cdot \|1 - F_0^2\|_{\mathcal{S}}.$$

As $t \mapsto A_t$ is continuous in operator norm by assumption and f is a continuous function, this proves the continuity at zero of $t \mapsto 1 - F_t^2$ in \mathcal{S} -norm. The continuity at other values of t can be shown in the same way, replacing F_0 by F_t and F_t by F_s for s close enough to t .

We now attempt to find the derivative of $\{1 - F_t^2\} \subset \mathcal{S}$ and show it is continuous. Recall that $\{A_t\}$ (where $A_t = D_t - D_0$) is C^1 in \mathcal{N}_{sa} by hypothesis, and write A'_t for $\frac{d}{dt}(A_t)$. Fix $t_0 \in [0, 1]$; we have

$$\begin{aligned} \|\cdot\|_{\mathcal{S}} \lim_{t \rightarrow t_0} \frac{(1 - F_t^2) - (1 - F_{t_0}^2)}{t - t_0} &= \|\cdot\|_{\mathcal{S}} \lim_{t \rightarrow t_0} \frac{(1 + D_t^2)^{-1} - (1 + D_{t_0}^2)^{-1}}{t - t_0} \\ &= \|\cdot\|_{\mathcal{S}} \lim_{t \rightarrow t_0} \frac{-D_{t_0}(1 + D_{t_0}^2)^{-1}(A_t - A_{t_0})(1 + D_t^2)^{-1} - (1 + D_{t_0}^2)^{-1}(A_t - A_{t_0})D_t(1 + D_t^2)^{-1}}{t - t_0} \\ &\quad \text{(by Lemma 3.7.5 applied with } x = 1\text{).} \end{aligned}$$

Using the fact that \mathcal{I} is an invariant operator ideal, and the continuity of the various factors, we show that the limit evaluates to $-D_{t_0}(1 + D_{t_0}^2)^{-1}A'_{t_0}(1 + D_{t_0}^2)^{-1} - (1 + D_{t_0}^2)^{-1}A'_{t_0}D_{t_0}(1 + D_{t_0}^2)^{-1}$. The difference between the first term and its purported limit satisfies the following inequality:

$$\begin{aligned} & \left\| -D_{t_0}(1 + D_{t_0}^2)^{-1} \cdot \frac{A_t - A_{t_0}}{t - t_0} \cdot (1 + D_t^2)^{-1} - (-D_{t_0}(1 + D_{t_0}^2)^{-1} \cdot A'_{t_0} \cdot (1 + D_{t_0}^2)^{-1}) \right\|_{\mathcal{I}} \\ & \leq \|D_{t_0}(1 + D_{t_0}^2)^{-1}\| \cdot \left\| \frac{A_t - A_{t_0}}{t - t_0} - A'_{t_0} \right\| \cdot \|(1 + D_t^2)^{-1}\|_{\mathcal{I}} \\ & \quad + \|D_{t_0}(1 + D_{t_0}^2)^{-1}\| \cdot \|A'_{t_0}\| \cdot \|(1 + D_t^2)^{-1} - (1 + D_{t_0}^2)^{-1}\|_{\mathcal{I}}. \end{aligned}$$

But t_0 is fixed, A'_{t_0} denotes $\lim_{t \rightarrow t_0} \frac{A_t - A_{t_0}}{t - t_0}$ by definition, and $t \mapsto (1 + D_t^2)^{-1} = 1 - F_t^2$ is continuous in the norm on \mathcal{I} (as established at the beginning of this proof); hence, it follows that $\left\| -D_{t_0}(1 + D_{t_0}^2)^{-1} \cdot \frac{A_t - A_{t_0}}{t - t_0} \cdot (1 + D_t^2)^{-1} - (-D_{t_0}(1 + D_{t_0}^2)^{-1} \cdot A'_{t_0} \cdot (1 + D_{t_0}^2)^{-1}) \right\|_{\mathcal{I}} \rightarrow 0$ as $t \mapsto t_0$.

For the second term, we have

$$\begin{aligned} & \left\| -(1 + D_{t_0}^2)^{-1} \cdot \frac{A_t - A_{t_0}}{t - t_0} \cdot D_t(1 + D_t^2)^{-1} - (-(1 + D_{t_0}^2)^{-1} \cdot A'_{t_0} \cdot D_{t_0}(1 + D_{t_0}^2)^{-1}) \right\|_{\mathcal{I}} \\ & \leq \|(1 + D_{t_0}^2)^{-1}\|_{\mathcal{I}} \cdot \left\| \frac{A_t - A_{t_0}}{t - t_0} - A'_{t_0} \right\| \cdot \|D_t(1 + D_t^2)^{-1}\| \\ & \quad + \|(1 + D_{t_0}^2)^{-1}\|_{\mathcal{I}} \cdot \|A'_{t_0}\| \cdot \|D_t(1 + D_t^2)^{-1} - D_{t_0}(1 + D_{t_0}^2)^{-1}\|. \end{aligned}$$

Since t_0 is fixed, $A'_{t_0} = \frac{d}{dt}(A_t)$ by definition, and $t \mapsto D_t(1 + D_t^2)^{-1}$ is continuous in operator norm (see Lemma 3.5.7), it follows that

$$\|\cdot\|_{\mathcal{I}} \cdot \lim_{t \rightarrow t_0} -(1 + D_{t_0}^2)^{-1} \cdot \frac{A_t - A_{t_0}}{t - t_0} \cdot D_t(1 + D_t^2)^{-1} = -(1 + D_{t_0}^2)^{-1} \cdot A'_{t_0} \cdot D_{t_0}(1 + D_{t_0}^2)^{-1}.$$

So $\|\cdot\|_{\mathcal{I}} \cdot \lim_{t \rightarrow t_0} \frac{(1 - F_t^2) - (1 - F_{t_0}^2)}{t - t_0} = -D_{t_0}(1 + D_{t_0}^2)^{-1}A'_{t_0}(1 + D_{t_0}^2)^{-1} - (1 + D_{t_0}^2)^{-1}A'_{t_0}D_{t_0}(1 + D_{t_0}^2)^{-1}$ as claimed, and the derivative of $\{1 - F_t^2\}$ exists at every $t = t_0 \in [0, 1]$. The proof that $t \mapsto -D_t(1 + D_t^2)^{-1}A'_t(1 + D_t^2)^{-1} - (1 + D_t^2)^{-1}A'_tD_t(1 + D_t^2)^{-1}$ is continuous as a function from $[0, 1]$ to \mathcal{I} is similar to the above, and relies on the continuity of the functions $t \mapsto (1 + D_t^2)^{-1}$, $t \mapsto D_t(1 + D_t^2)^{-1}$ and $t \mapsto A'_t$ in operator norm, as well as $t \mapsto (1 + D_t^2)^{-1}$ in \mathcal{I} norm; we omit the details.

Finally, let us consider $\{1 - F_t^2\}$ as a path in the invariant operator ideal $\mathcal{I}^{1-\varepsilon}$ for $0 < \varepsilon < 1$. We can use Remark 1.1.15 (which tells us that $\mathcal{I} \subset \mathcal{I}^{1-\varepsilon}$ and $\|A\|_{\mathcal{I}^{1-\varepsilon}} \leq \|A\|_{\mathcal{I}}$ for $A \in \mathcal{I}$) to conclude that $\{1 - F_t^2\}$ is C^1 in the norm on $\mathcal{I}^{1-\varepsilon}$ for any $0 < \varepsilon < 1$. \blacksquare

We will need the following formula for $F_t - F_0$ in a few places, such as to prove continuity of $\{F_t\}$ (see Corollary 3.5.12), and to calculate the derivative of $\{F_t\}$ (see Lemma 3.5.15). We note that the proof relies on Theorem 3.4.3 and careful algebraic manipulation, but as checking the convergence of the various expressions involved is not trivial, we send the interested reader to the source for the proof.

Theorem 3.5.11 ([14], Lemma 2.7) Suppose D_0 is an unbounded self-adjoint operator affiliated with a von Neumann algebra \mathcal{N} and $D = D_0 + A$ for $A \in \mathcal{N}_{sa}$. Denote by F_{D_0} and F_D the images of

the two operators under the Riesz transform; that is, $F_{D_0} = D_0(1 + D_0^2)^{-\frac{1}{2}}$ and $F_D = D(1 + D^2)^{-\frac{1}{2}}$. Then

$$F_D - F_{D_0} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} A_t (1 + D_0^2 + \lambda)^{-1} - D_t (1 + D_t^2 + \lambda)^{-1} A_t D_0 (1 + D_0^2 + \lambda)^{-1}] d\lambda.$$

For any fixed $0 < \sigma < \frac{1}{2}$ one can show that $F_D - F_{D_0} = B_\sigma (1 + D_0^2)^{-\sigma}$, where $B_\sigma \in \mathcal{N}$ and $\|B_\sigma\| \leq C(\sigma)\|A\|$ for some constant $C(\sigma)$ depending only on D_0 and σ . \blacksquare

We note that the formula for $F_D - F_{D_0}$ stated above is found in the proof (and not the statement) of Lemma 2.7 of [14]; the result cited only states the last part, which is obtained by multiplying the integrand by $(1 + D_0^2)^\sigma (1 + D_0^2)^{-\sigma}$ and pulling out the bounded operator $(1 + D_0^2)^{-\sigma}$. We are going to prove later that $\{F_t\}$ is in fact C^1 in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ (see Lemma 3.5.15); for now, we just reassure ourselves that it is indeed a path (i.e. continuous as a map from $[0, 1]$ into $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ with the appropriate norm).

Corollary 3.5.12 (cf [14], Corollary 2.8 and [15], Corollary 6.3) Suppose $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ is a C^1 path and $(1 + D_0^2)^{-1} \in \mathcal{I}$ for some small power invariant operator ideal \mathcal{I} . Fix $0 < \varepsilon < 1$, and let $\mathcal{J} = \mathcal{I}^{1-\varepsilon}$. If $F_t = D_t(1 + D_t^2)^{-1}$, then $\{F_t\}$ is a path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. \blacksquare

PROOF First show that $F_t \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$ for all t ; that is, show that $F_t - F_0 \in \mathcal{J}^{\frac{1}{2}}$ and is such that $1 - (F_0 + (F_t - F_0))^2 \in \mathcal{J}$. Let $\sigma = \frac{1-\varepsilon}{2}$, and note that $0 < \sigma < \frac{1}{2}$. By hypothesis, $(1 + D_0^2)^{-1} \in \mathcal{I}$, which means that $(1 + D_0^2)^{-\sigma} \in \mathcal{I}^{\frac{1-\varepsilon}{2}} = \mathcal{J}^{\frac{1}{2}}$. Using the description in Theorem 3.5.11, we know that we can write $F_t - F_0 = B_t(1 + D_0^2)^{-\sigma}$, where B_t is a bounded operator. But $(1 + D_0^2)^{-\sigma} \in \mathcal{J}^{\frac{1}{2}}$ and $\mathcal{J}^{\frac{1}{2}}$ is an ideal, so $F_t - F_0 \in \mathcal{J}^{\frac{1}{2}}$. Therefore, using the fact that $1 - F_t^2 = (1 + D_t^2)^{-1} \in \mathcal{I} \subset \mathcal{J}$, $F_t - F_0$ satisfies the description of the elements of $\mathcal{S}\mathcal{G}_{F_0}$, and so $F_t \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$.

Recall that $F_t + \mathcal{S}\mathcal{G}_{F_t} = F_0 + \mathcal{S}\mathcal{G}_{F_0}$ (Theorem 3.5.1); since we can reparametrize our line so any point F_t becomes F_0 , it is thus sufficient to show continuity at zero, as continuity at any other point would be similar. The definition of the norm on $\mathcal{S}\mathcal{G}_{F_0}$ gives us

$$\|F_t - F_0\|_{\mathcal{S}\mathcal{G}_{F_0}} = \|F_t - F_0\|_{\mathcal{J}^{\frac{1}{2}}} + \|F_0(F_t - F_0) + (F_t - F_0)F_0\|_{\mathcal{J}}.$$

To tackle the $\mathcal{J}^{\frac{1}{2}}$ -norm, use the expression $F_t - F_0 = B_t(1 + D_0^2)^{-\sigma}$ from Theorem 3.5.11; the norm bound on B_t included in that result tells us that $t \mapsto B_t$ is operator norm continuous at $t = 0$. Using the properties of invariant operator ideals, $\|F_t - F_0\|_{\mathcal{J}^{\frac{1}{2}}} \leq \|B_t\| \cdot \|(1 + D_0^2)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}$; it follows that $\|F_t - F_0\|_{\mathcal{J}^{\frac{1}{2}}} \rightarrow 0$ as $t \rightarrow 0$.

On the other hand, $F_0(F_t - F_0) + (F_t - F_0)F_0 = (F_t^2 - F_0^2) - (F_t - F_0)^2$, and we can show that each of these terms is continuous:

- By Theorem 3.5.10, $t \mapsto \{1 - F_t^2\}$ is continuous in the \mathcal{J} norm; it follows that so is $t \mapsto F_t^2 - F_0^2 = (1 - F_0^2) - (1 - F_t^2)$.
- Since \mathcal{J} has the Cauchy-Schwarz property, $t \mapsto (F_t - F_0)^2$ is continuous in the \mathcal{J} norm (from the inequality $\|(F_t - F_0)^2\|_{\mathcal{J}} \leq \|F_t - F_0\|_{\mathcal{J}^{\frac{1}{2}}} \cdot \|F_t - F_0\|_{\mathcal{J}^{\frac{1}{2}}}$).

Therefore, $\|F_0(F_t - F_0) + (F_t - F_0)F_0\|_{\mathcal{J}} \rightarrow 0$ as $t \rightarrow 0$ as well. It follows that $\{F_t\}$ is continuous at $t = 0$ in the norm of $F_0 + \mathcal{S}\mathcal{G}_{F_0}$, and hence $\{F_t\}$ is a path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. \blacksquare

Thus $\{F_t\} \subset F_0 + \mathcal{S}\mathcal{G}_{F_0}$ is indeed a path, and according to Theorem 3.5.8, in order to check that $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$, we need only check that $\{F_t\}$ is C^1 in $\mathcal{G}^{\frac{1}{2}}$ -norm (we already know that $\{1 - F_t^2\}$ is C^1 in \mathcal{G} norm, by Theorem 3.5.10). Our goal is to use the difference formula for $F_D - F_{D_0}$ from Theorem 3.5.11 to get an integral formula for the derivative of $\{F_t\}$ in terms of the original path $\{D_t\}$, allowing us to use the properties of these operators to argue about continuity (see Lemma 3.5.15 for the eventual formula of $\frac{d}{dt}(F_t)$). To this end, we will need some helping bounds and convergence results. We split up the results into two lemmas, one dealing with the λ -continuity, and one with the t -continuity of various expressions appearing in the integral formula for $F_D - F_{D_0}$.

Lemma 3.5.13 Suppose $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ is continuous, with $(1 + D_0^2)^{-1} \in \mathcal{G}$, and $\{B_t\}$ is a path of bounded operators. Fix $0 < \varepsilon < 1$ and let $\mathcal{G} := \mathcal{G}^{1-\varepsilon}$.

(a) Fix $t \in [0, 1]$. Then

$$\begin{aligned} \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} &\leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}, \text{ and} \\ \|D_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} &\leq (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}. \end{aligned}$$

(b) For fixed $s, t \in [0, 1]$,

$$\begin{aligned} \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1}B_t(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} &\leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|B_t\| \cdot \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}, \\ \|D_t(1 + D_t^2 + \lambda)^{-1}B_tD_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} &\leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|B_t\| \cdot \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}. \end{aligned}$$

(c) Fix $t \in [0, 1]$. The function $\lambda \mapsto (1 + D_t^2 + \lambda)^{-1}$ is uniformly continuous as a function from \mathbb{R}_+ to $\mathcal{G}^{\frac{1}{2}}$.

(d) Fix $t \in [0, 1]$. The function $\lambda \mapsto D_t(1 + D_t^2 + \lambda)^{-1}$ is uniformly continuous as a function from \mathbb{R}_+ to $\mathcal{G}^{\frac{1}{2}}$.

(e) Fix $t \in [0, 1]$. The functions from \mathbb{R}_+ to $\mathcal{G}^{\frac{1}{2}}$ given by

$$\begin{aligned} \lambda &\mapsto (1 + \lambda)(1 + D_t^2 + \lambda)^{-1}B_t(1 + D_t^2 + \lambda)^{-1}, \text{ and} \\ \lambda &\mapsto D_t(1 + D_t^2 + \lambda)^{-1}B_tD_t(1 + D_t^2 + \lambda)^{-1} \end{aligned}$$

are both continuous. ☞

PROOF We will need the fact that $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)} \in \mathcal{G}^{\frac{1}{2}}$. To start with, $(1 + D_0^2)^{-1} \in \mathcal{G}$ by hypothesis. Theorem 3.5.9 tells us that $(1 + D_t^2)^{-1} \leq f(\|A_t\|) \cdot (1 + D_0^2)^{-1}$ (where $A_t = D_t - D_0$ and $f(a) = 1 + \frac{1}{2} \cdot a^2 + \frac{1}{2} \cdot a \cdot \sqrt{a^2 + 4}$); hence, by Theorem 1.1.10, $(1 + D_t^2)^{-1} \in \mathcal{G}$ as well. From the definition of powers of ideals, $(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)} \in \mathcal{G}^{\frac{1}{2}}$ (recall that $\mathcal{G} = \mathcal{G}^{1-\varepsilon}$). Now $x \mapsto x^r$ is an operator monotone function for $0 < r \leq 1$. We know $0 \leq (1 + D_t^2 + \lambda)^{-1} \leq (1 + D_t^2)^{-1} \leq 1$ (see e.g. Theorem 1.1.11), so it follows $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)} \leq (1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}$. Use Theorem 1.1.10 again to conclude that, since $(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}$ is in $\mathcal{G}^{\frac{1}{2}}$, then so is $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}$. Moreover,

$\|(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \leq \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}$. We will use this latter norm inequality quite often in the following.

The idea for most of the proofs will be to split up $(1 + D_t^2 + \lambda)^{-1}$ into the product of $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)}$ and $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}$, the second of which is in $\mathcal{J}^{\frac{1}{2}}$ by the above discussion, and the first of which can be handled using operator norms (see Lemma 3.5.7).

(a) By the above discussion, $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \varepsilon\right)} \in \mathcal{J}^{\frac{1}{2}}$, so we can write

$$\begin{aligned} \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} &\leq \|(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)}\| \cdot \|(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}, \end{aligned}$$

and similarly

$$\|D_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \leq \dots \leq (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}.$$

(b) Use the fact that $\mathcal{J}^{\frac{1}{2}}$ is an invariant operator ideal, along with the estimates from part (a) and Lemma 3.5.7 to write

$$\begin{aligned} &\|(1 + \lambda) \cdot (1 + D_t^2 + \lambda)^{-1} \cdot B_t \cdot (1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq (1 + \lambda) \cdot \|(1 + D_t^2 + \lambda)^{-1}\| \cdot \|B_t\| \cdot \|(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq (1 + \lambda) \cdot \frac{1}{1 + \lambda} \cdot \|B_t\| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &= (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|B_t\| \cdot \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\|D_t(1 + D_t^2 + \lambda)^{-1} B_t D_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq \dots \leq (1 + \lambda)^{-\frac{1}{2}} \cdot \|B_t\| \cdot (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}. \end{aligned}$$

(c) Fix λ and η in \mathbb{R}_+ . Since $(1 + D_t^2 + \lambda)$ and $(1 + D_t^2 + \eta)$ have the same domain, we can use the resolvent formula to write

$$(1 + D_t^2 + \eta)^{-1} - (1 + D_t^2 + \lambda)^{-1} = (1 + D_t^2 + \eta)^{-1}(\lambda - \eta)(1 + D_t^2 + \lambda)^{-1}.$$

Hence,

$$\begin{aligned} &\|(1 + D_t^2 + \eta)^{-1} - (1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq \|(1 + D_t^2 + \eta)^{-1}\| \cdot |\lambda - \eta| \cdot \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq (1 + \eta)^{-1} \cdot |\eta - \lambda| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\ &\leq 1 \cdot |\eta - \lambda| \cdot 1 \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}. \end{aligned}$$

Since t is fixed, $\|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}$ is a constant, so we can control the distance between $(1 + D_t^2 + \eta)^{-1}$ and $(1 + D_t^2 + \lambda)^{-1}$ by making the difference between η and λ sufficiently small. Therefore, $\lambda \mapsto (1 + D_t^2 + \lambda)^{-1}$ is uniformly continuous in $\mathcal{J}^{\frac{1}{2}}$.

(d) Fix λ and η in \mathbb{R}_+ . Since we're dealing with unbounded operators, a priori

$$D_t(1 + D_t^2 + \lambda)^{-1} - D_t(1 + D_t^2 + \eta)^{-1} \subseteq D_t[(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \eta)^{-1}].$$

Since $D_t(1 + D_t^2 + \lambda)^{-1}$ and $D_t(1 + D_t^2 + \eta)^{-1}$ are both bounded operators, equality follows. Again, we can use the resolvent formula to write

$$D_t(1 + D_t^2 + \eta)^{-1} - D_t(1 + D_t^2 + \lambda)^{-1} = D_t(1 + D_t^2 + \eta)^{-1}(\lambda - \eta)(1 + D_t^2 + \lambda)^{-1}.$$

Hence, using the estimates in (a) and the beginning of the proof,

$$\begin{aligned} & \|D_t(1 + D_t^2 + \eta)^{-1} - D_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq \|D_t(1 + D_t^2 + \eta)^{-1}\| \cdot |\lambda - \eta| \cdot \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq (1 + \eta)^{-\frac{1}{2}} \cdot |\lambda - \eta| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq 1 \cdot |\lambda - \eta| \cdot 1 \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}. \end{aligned}$$

As in the previous part, the fact that $\lambda \mapsto D_t(1 + D_t^2 + \lambda)^{-1}$ is uniformly continuous follows.

(e) This follows from the fact that $\lambda \mapsto (1 + D_t^2 + \lambda)^{-1}$ and $\lambda \mapsto D_t(1 + D_t^2 + \lambda)^{-1}$ are both continuous in $\mathcal{J}^{\frac{1}{2}}$ -norm (parts (c) and (d) of this lemma), combined with the fact that, for any $\lambda \in \mathbb{R}_+$, $(1 + D_t^2 + \lambda)^{-1}$ and $D_t(1 + D_t^2 + \lambda)^{-1}$ are bounded operators with norm at most 1 (see Lemma 3.5.7 and note that $(1 + \lambda)^{-1}$ and $(1 + \lambda)^{-\frac{1}{2}}$ are both at most 1 for $\lambda \in \mathbb{R}_+$). We show the calculation for the first function, and omit the second since it is similar. Fix $\lambda \in \mathbb{R}_+$; then for any $\mu \in \mathbb{R}_+$,

$$\begin{aligned} & \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1}B_t(1 + D_t^2 + \lambda)^{-1} - (1 + \mu)(1 + D_t^2 + \mu)^{-1}B_t(1 + D_t^2 + \mu)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq \|[(1 + \lambda) - (1 + \mu)] \cdot (1 + D_t^2 + \lambda)^{-1}B_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \quad + \|(1 + \mu) \cdot [(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \mu)^{-1}] \cdot B_t \cdot (1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \quad + \|(1 + \mu) \cdot (1 + D_t^2 + \mu)^{-1} \cdot B_t \cdot [(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \mu)^{-1}]\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq |\lambda - \mu| \cdot \|(1 + D_t^2 + \lambda)^{-1}\| \cdot \|B_t\| \cdot \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \quad + |(1 + \mu)| \cdot \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \mu)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot \|B_t\| \cdot \|(1 + D_t^2 + \lambda)^{-1}\| \\ & \quad + |(1 + \mu)| \cdot \|(1 + D_t^2 + \mu)^{-1}\| \cdot \|B_t\| \cdot \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \mu)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}. \end{aligned}$$

By choosing μ such that $|\mu - \lambda| < 1$, we can ensure $|1 + \mu| < |\lambda| + 2$; since $\|B_t\|$ is constant and $\|(1 + D_t^2 + \lambda)^{-1}\|$ and $\|(1 + D_t^2 + \mu)^{-1}\|$ both have norms less than 1, the continuity of $\lambda \mapsto (1 + D_t^2 + \lambda)^{-1}$ in $\mathcal{J}^{\frac{1}{2}}$ norm implies that by choosing μ close enough to λ we can make the righthand side of the above inequality as small as we want, concluding the proof of the continuity of $\lambda \mapsto (1 + \lambda)(1 + D_t^2 + \lambda)^{-1}B_t(1 + D_t^2 + \lambda)^{-1}$. ■

We now tackle the continuity of the same type of expressions as in the previous lemma, except from the point of view of continuity as functions in t rather than functions in λ . As we will be using these expressions to prove convergence of various integral expressions, it is not sufficient to prove continuity, but must additionally get λ -dependent bounds for the differences.

Lemma 3.5.14 Suppose $\{D_t = D_0 + A_t\} \subset D_0 + \mathcal{N}_{sa}$ is continuous, with $(1 + D_0^2)^{-1} \in \mathcal{S}$, and $\{B_t\}$ is a path of bounded operators. Fix $0 < \varepsilon < 1$ and let $\mathcal{S} := \mathcal{S}^{1-\varepsilon}$.

(a) There exists a $K \in \mathbb{R}$ such that

$$\|(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}} \leq K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}}$$

for all $t \in [0, 1]$ and $\lambda \in \mathbb{R}_+$.

(b) Suppose $\{A_t\}$ is C^1 , and denote by A'_0 the derivative of $\{A_t\}$ at 0. Then, as $t \rightarrow 0$, $(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} (1 + D_0^2 + \lambda)^{-1}$ converges to $(1 + \lambda)(1 + D_0^2 + \lambda)^{-1} A'_0 (1 + D_0^2 + \lambda)^{-1}$ in $\|\cdot\|_{\mathcal{S}^{\frac{1}{2}}}$ norm. More specifically, for $t \in [0, 1]$,

$$\begin{aligned} & \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} (1 + D_0^2 + \lambda)^{-1} - (1 + \lambda)(1 + D_0^2 + \lambda)^{-1} A'_0 (1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{S}^{\frac{1}{2}}} \\ & \leq (1 + \lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}} \cdot u_t, \end{aligned}$$

where $u_t \rightarrow 0$ as $t \rightarrow 0$ and u_t does not depend on λ .

(c) Suppose $\{A_t\}$ is C^1 , and denote by A'_0 the derivative of $\{A_t\}$ at 0. Then, as $t \rightarrow 0$, $D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} D_0(1 + D_0^2 + \lambda)^{-1}$ converges to $D_0(1 + D_0^2 + \lambda)^{-1} A'_0 D_0(1 + D_0^2 + \lambda)^{-1}$ in $\|\cdot\|_{\mathcal{S}^{\frac{1}{2}}}$ norm. More specifically, for $t \in [0, 1]$,

$$\begin{aligned} & \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} D_0(1 + D_0^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1} A'_0 D_0(1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{S}^{\frac{1}{2}}} \\ & \leq (1 + \lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}} \cdot u_t, \end{aligned}$$

where u_t is as in (b).

(d) The map from $[0, 1]$ to $\mathcal{S}^{\frac{1}{2}}$ given by $t \mapsto (1 + \lambda)(1 + D_t^2 + \lambda)^{-1} B_t (1 + D_t^2 + \lambda)^{-1}$ is continuous. In fact, for $t, s \in [0, 1]$ we have

$$\begin{aligned} & \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} B_t (1 + D_t^2 + \lambda)^{-1} - (1 + \lambda)(1 + D_s^2 + \lambda)^{-1} B_s (1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{S}^{\frac{1}{2}}} \\ & \leq K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}} \cdot (1 + \lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot v_{s,t}, \end{aligned}$$

where K is the same constant as in (a), $v_{s,t}$ do not depend on λ , and $v_{s,t} \rightarrow 0$ as $t \rightarrow s$.

(e) The map from $[0, 1]$ to $\mathcal{S}^{\frac{1}{2}}$ given by $t \mapsto D_t(1 + D_t^2 + \lambda)^{-1} B_t D_t(1 + D_t^2 + \lambda)^{-1}$ is continuous. In fact, for $t, s \in [0, 1]$ we have

$$\begin{aligned} & \|D_t(1 + D_t^2 + \lambda)^{-1} B_t D_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1} B_s D_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{S}^{\frac{1}{2}}} \\ & \leq K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{S}^{\frac{1}{2}}} \cdot (1 + \lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot v_{s,t}, \end{aligned}$$

where K and $v_{s,t}$ are as in (d). ▮

PROOF One can check that $(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)} \in \mathcal{G}^{\frac{1}{2}}$ (this is shown in the beginning of the proof of Lemma 3.5.13). Additionally, from Lemma 3.5.7 we will need the operator norm bounds

$$\|(1 + D_t^2 + \lambda)^{-1}\| \leq (1 + \lambda)^{-1}, \text{ and } \|D_t(1 + D_t^2 + \lambda)^{-1}\| \leq (1 + \lambda)^{-\frac{1}{2}}.$$

We also refer to Lemma 3.5.7 (part (v)) for the additional norm inequalities

$$\begin{aligned} \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_0^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-3/2} \cdot \|A_t\|, \text{ and} \\ \|D_t(1 + D_t^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-1} \cdot \|A_t\|. \end{aligned}$$

(a) As in the beginning of the proof of Lemma 3.5.13, we can show that, for any $\lambda \in \mathbb{R}_+$, $\|(1 + D_t^2 + \lambda)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}} \leq \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}$; so we just need to relate this latter norm to $\|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}$. By Theorem 3.5.9, we have $(1 + D_t^2)^{-1} \leq f(\|A_t\|) \cdot (1 + D_0^2)^{-1}$ for each $t \in [0, 1]$, where $f(x) = 1 + \frac{1}{2} \cdot x^2 + \frac{1}{2} \cdot x \cdot \sqrt{x^2 + 4}$. Since $\{A_t\}$ is continuous on $[0, 1]$, there exists an $M \geq 0$ such that $\|A_t\| \leq M$ for all t , and since f is increasing on $[0, M]$ it follows that $f(\|A_t\|) \leq f(M)$ for all t . So we have $(1 + D_t^2)^{-1} \leq f(M) \cdot (1 + D_0^2)^{-1}$. Since the function $x \mapsto x^r$ is operator monotone for $r \leq 1$, $(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)} \leq f(M)^{\frac{1}{2} - \frac{\varepsilon}{2}} \cdot (1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}$. From Theorem 1.1.10 it follows that

$$\|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}} \leq f(M)^{\frac{1}{2} - \frac{\varepsilon}{2}} \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}.$$

Hence $K = f(M)^{\frac{1}{2} - \frac{\varepsilon}{2}}$ satisfies the requirements; in particular, note that the value of K depends on the path $\{D_t\}$ and on ε , but does not depend on either λ or t .

(b) Use the estimates from Lemma 3.5.13.

$$\begin{aligned} &\|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} (1 + D_0^2 + \lambda)^{-1} - (1 + \lambda)(1 + \lambda + D_0^2)^{-1} A'_0 (1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} \\ &\leq (1 + \lambda) \cdot \|(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - (1 + D_0^2 + \lambda)^{-1} \cdot A'_0\| \cdot \|(1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} \\ &\leq (1 + \lambda) \cdot \|(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - (1 + D_0^2 + \lambda)^{-1} \cdot A'_0\| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}. \end{aligned}$$

By the triangle inequality, and the norm estimates mentioned earlier in the proof,

$$\begin{aligned} &\|(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - (1 + D_0^2 + \lambda)^{-1} \cdot A'_0\| \\ &\leq \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_0^2 + \lambda)^{-1}\| \cdot \left\| \frac{A_t}{t} \right\| + \|(1 + D_0^2 + \lambda)^{-1}\| \cdot \left\| \frac{A_t}{t} - A'_0 \right\| \\ &\leq (1 + \lambda)^{-\frac{3}{2}} \cdot \|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + (1 + \lambda)^{-1} \cdot \left\| \frac{A_t}{t} - A'_0 \right\| \\ &\leq (1 + \lambda)^{-1} \left[\|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + \left\| \frac{A_t}{t} - A'_0 \right\| \right], \end{aligned}$$

where in the last line we used the fact that $(1 + \lambda)^{-\frac{1}{2}} \leq 1$. Substituting this into our earlier bound and simplifying,

$$\begin{aligned} &\|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} (1 + D_0^2 + \lambda)^{-1} - (1 + \lambda)(1 + \lambda + D_0^2)^{-1} A'_0 (1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{G}^{\frac{1}{2}}} \\ &\leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \left[\|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + \left\| \frac{A_t}{t} - A'_0 \right\| \right] \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}}. \end{aligned}$$

Let $u_t = \|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + \left\| \frac{A_t}{t} - A'_0 \right\|$. Since $\frac{A_t}{t} \rightarrow A'_0$ as $t \rightarrow 0$, $\left\{ \left\| \frac{A_t}{t} \right\| \right\}$ is bounded for t close enough to 0, so $A_t \rightarrow 0$ and $\frac{A_t}{t} \rightarrow A'_0$ as $t \rightarrow 0$ imply $u_t \rightarrow 0$ as $t \rightarrow 0$.

(c) Similarly to the previous part,

$$\begin{aligned} & \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} D_0(1 + D_0^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1} A'_0 D_0(1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \leq \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - D_0(1 + D_0^2 + \lambda)^{-1} A'_0\| \cdot \|D_0(1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \leq \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - D_0(1 + D_0^2 + \lambda)^{-1} A'_0\| \cdot (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}}. \end{aligned}$$

Use the triangle inequality to estimate the expression inside the operator norm

$$\begin{aligned} & \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} - D_0(1 + D_0^2 + \lambda)^{-1}\| \\ & \leq \|D_t(1 + D_t^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}\| \cdot \left\| \frac{A_t}{t} \right\| + \|D_0(1 + D_0^2 + \lambda)^{-1}\| \cdot \left\| \frac{A_t}{t} - A'_0 \right\| \\ & \leq (1 + \lambda)^{-1} \|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + (1 + \lambda)^{-\frac{1}{2}} \cdot \left\| \frac{A_t}{t} - A'_0 \right\| \leq (1 + \lambda)^{-\frac{1}{2}} \left[\|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + \left\| \frac{A_t}{t} - A'_0 \right\| \right], \end{aligned}$$

where in the last line we used the fact that $(1 + \lambda)^{-\frac{1}{2}} \leq 1$. Substituting back into our earlier inequality and simplifying,

$$\begin{aligned} & \|D_t(1 + D_t^2 + \lambda)^{-1} \frac{A_t}{t} D_0(1 + D_0^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1} A'_0 D_0(1 + D_0^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \left[\|A_t\| \cdot \left\| \frac{A_t}{t} \right\| + \left\| \frac{A_t}{t} - A'_0 \right\| \right] \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}}, \end{aligned}$$

which is the same upper bound as the one in (b).

(d) For $s, t \in [0, 1]$, using the triangle inequality, Lemma 3.5.13 and part (a), we have

$$\begin{aligned} & \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} B_t(1 + D_t^2 + \lambda)^{-1} - (1 + \lambda)(1 + D_s^2 + \lambda)^{-1} B_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \leq (1 + \lambda) \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_s^2 + \lambda)^{-1}\| \cdot \|B_t\| \cdot \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \quad + (1 + \lambda) \|(1 + D_s^2 + \lambda)^{-1}\| \cdot \|B_t - B_s\| \cdot \|(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \quad + (1 + \lambda) \|(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \cdot \|B_s\| \cdot \|(1 + D_t^2 + \lambda)^{-1} - (1 + D_s^2 + \lambda)^{-1}\| \\ & \leq (1 + \lambda)(1 + \lambda)^{-\frac{3}{2}} \cdot \|A_s - A_t\| \cdot \|B_t\| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \quad + (1 + \lambda) \cdot (1 + \lambda)^{-1} \cdot \|B_t - B_s\| \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \quad + (1 + \lambda) \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}} \cdot \|B_s\| \cdot (1 + \lambda)^{-\frac{3}{2}} \cdot \|A_s - A_t\| \\ & \leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}} \cdot (\|A_s - A_t\| \cdot \|B_t\| + \|B_t - B_s\| + \|B_s\| \cdot \|A_s - A_t\|). \end{aligned}$$

Let $v_{s,t} = \|A_s - A_t\| \cdot \|B_t\| + \|B_t - B_s\| + \|B_s\| \cdot \|A_s - A_t\|$. Fix $s \in [0, 1]$; since $\{B_t\}_{t \in [0,1]}$ and $\{A_s - A_t\}_{t \in [0,1]}$ are both continuous, $\{\|B_t\| : t \in [0, 1]\}$ and $\{\|A_s - A_t\| : t \in [0, 1]\}$ are bounded sets; so $A_s - A_t \rightarrow 0$ and $B_t - B_s \rightarrow 0$ as $t \rightarrow s$ is sufficient to ensure $v_{s,t} \rightarrow 0$ as $t \rightarrow s$. Hence

$$\begin{aligned} & \|(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} B_t(1 + D_t^2 + \lambda)^{-1} - (1 + \lambda)(1 + D_s^2 + \lambda)^{-1} B_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{L}^{\frac{1}{2}}} \\ & \leq K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{L}^{\frac{1}{2}}} \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot v_{s,t}, \end{aligned}$$

where $v_{s,t} \rightarrow 0$ as $s \rightarrow t$ and $v_{s,t}$ does not depend on λ .

(e) Use the triangle inequality and the various norm estimates established. We have to be careful in how we break up the expressions involved, as we did not establish a bound which depends on λ for $\|D_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}}$. For $s, t \in [0, 1]$ we have

$$\begin{aligned}
& \|D_t(1 + D_t^2 + \lambda)^{-1}B_tD_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}B_sD_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \leq \|D_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}\| \cdot \|B_t\| \cdot \|D_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \quad + \|D_s(1 + D_s^2 + \lambda)^{-1}\| \cdot \|B_t - B_s\| \cdot \|D_t(1 + D_t^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \quad + \|D_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot \|B_s\| \cdot \|D_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}\| \\
& \leq (1 + \lambda)^{-1}\|A_s - A_t\| \cdot \|B_t\| \cdot (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \quad + (1 + \lambda)^{-\frac{1}{2}}\|B_t - B_s\| \cdot (1 + \lambda)^{-\frac{\varepsilon}{2}} \cdot \|(1 + D_t^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \quad + (1 + \lambda)^{-\frac{\varepsilon}{2}}\|(1 + D_s^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot \|B_s\| \cdot (1 + \lambda)^{-1}\|A_s - A_t\|.
\end{aligned}$$

Use the bound in part (a) and simplify (keeping in mind that $(1 + \lambda)^{-\frac{1}{2}} \leq 1$) to get

$$\begin{aligned}
& \|D_t(1 + D_t^2 + \lambda)^{-1}B_tD_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}B_sD_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \leq (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot (\|A_s - A_t\| + \|B_t\| + \|B_t - B_s\| + \|B_s\| \cdot \|A_s - A_t\|).
\end{aligned}$$

This is the same expression as obtained in (d), giving us

$$\begin{aligned}
& \|D_t(1 + D_t^2 + \lambda)^{-1}B_tD_t(1 + D_t^2 + \lambda)^{-1} - D_s(1 + D_s^2 + \lambda)^{-1}B_sD_s(1 + D_s^2 + \lambda)^{-1}\|_{\mathcal{J}^{\frac{1}{2}}} \\
& \leq K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot \nu_{s,t},
\end{aligned}$$

as desired. ■

The two lemmas below give us expressions for $\frac{d}{dt}(F_t)$ and $(1 + D^2)^{-\frac{3}{2}}$ which are used in the proof of the equality of the two traces from the bounded and unbounded case (Lemma 3.5.17).

Lemma 3.5.15 ([14], Proposition 2.10) Suppose $\{D_t\}$ is a path in $D_0 + \mathcal{N}_{sa}$, with $(D_0^2 + 1)^{-1} \in \mathcal{J}$, and $D_t = D_0 + A_t$ with $\{A_t\}$ a C^1 path in \mathcal{N}_{sa} . If $F_t = D_t(1 + D_t^2)^{-\frac{1}{2}}$ and $\mathcal{J} = \mathcal{J}^{1-\varepsilon}$ for any $0 < \varepsilon < 1$ then

$$\begin{aligned}
\frac{d}{dt}(F_t) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1} \\
- D_t(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)D_t(1 + D_t^2 + \lambda)^{-1}] d\lambda
\end{aligned}$$

(where the integral converges in $\mathcal{J}^{\frac{1}{2}}$ -norm). It follows that $\{F_t\}$ is C^1 in $F_0 + \mathcal{L}_{F_0}$. ☞

PROOF In order to simplify the exposition, we introduce notation for two of the expressions appearing in our purporter formula for $\frac{d}{dt}(F_t)$. Namely, let

$$\begin{aligned}
L_t(\lambda) &= (1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1}, \text{ and} \\
R_t(\lambda) &= D_t(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)D_t(1 + D_t^2 + \lambda)^{-1}.
\end{aligned}$$

Apply Lemma 3.5.13 to show that L_t and R_t are continuous as functions of λ from \mathbb{R}_+ to $\mathcal{J}^{\frac{1}{2}}$ (part (e)), and $\|L_t(\lambda)\|_{\mathcal{J}^{\frac{1}{2}}}$ and $\|R_t(\lambda)\|_{\mathcal{J}^{\frac{1}{2}}}$ are each bounded by a constant multiple of $(1+\lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)}$ (part (b) of Lemma 3.5.13 applied with $s=t$ and $B_t = \frac{d}{dt}(A_t)$, along with part (a) of Lemma 3.5.14) – the constant is $\|B_t\| \cdot K \cdot \|(1+D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}}$, but we do not need its exact value. This is sufficient to prove that $\int_0^\infty \lambda^{-\frac{1}{2}} \cdot [L_t(\lambda) - R_t(\lambda)] d\lambda$ converges in $\mathcal{J}^{\frac{1}{2}}$ -norm.

We next want to show that the integral calculates $\frac{d}{dt}(F_t)$. It is sufficient to show that $\left. \frac{d}{dt} \right|_{t=0} (F_t) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [L_0(\lambda) - R_0(\lambda)] d\lambda$, as the proof is the same with 0 replaced by any other $s \in [0, 1]$. So our next step is to show that $\frac{1}{t} \times (F_{D_t} - F_{D_0})$ converges in $\mathcal{J}^{\frac{1}{2}}$ -norm to $\frac{1}{\pi} \cdot \int_0^\infty \lambda^{-\frac{1}{2}} \cdot [L_0(\lambda) - R_0(\lambda)] d\lambda$. As quoted in Theorem 3.5.11,

$$F_{D_t} - F_{D_0} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [(1+\lambda)(1+D_t^2+\lambda)^{-1} A_t (1+D_0^2+\lambda)^{-1} - D_t (1+D_t^2+\lambda)^{-1} A_t D_0 (1+D_0^2+\lambda)^{-1}] d\lambda.$$

Using Lemma 3.5.13 again (with $s=0$ and $B_t = A_t$), it can be shown quite similarly to above that this integral also converges in $\mathcal{J}^{\frac{1}{2}}$ -norm (use $s=0$ to get a constant which does not depend on t). Now, by Lemma 3.5.14.(d),

$$\|(1+\lambda)(1+D_t^2+\lambda)^{-1} \frac{A_t}{t} (1+D_0^2+\lambda)^{-1} - L_0(\lambda)\|_{\mathcal{J}^{\frac{1}{2}}} \leq (1+\lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot \|(1+D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot u_t,$$

where u_t does not depend on λ and $\lim_{t \rightarrow 0} u_t = 0$. Hence

$$\begin{aligned} & \left\| \int_0^\infty \lambda^{-\frac{1}{2}} \cdot [(1+\lambda)(1+D_t^2+\lambda)^{-1} \frac{A_t}{t} (1+D_0^2+\lambda)^{-1} - L_0(\lambda)] d\lambda \right\|_{\mathcal{J}^{\frac{1}{2}}} \\ & \leq \int_0^\infty \lambda^{-\frac{1}{2}} (1+\lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} \cdot \|(1+D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot u_t d\lambda \\ & = \|(1+D_0^2)^{-\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)}\|_{\mathcal{J}^{\frac{1}{2}}} \cdot u_t \cdot \int_0^\infty \lambda^{-\frac{1}{2}} (1+\lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} d\lambda. \end{aligned}$$

Since $\int_0^\infty \lambda^{-\frac{1}{2}} (1+\lambda)^{-\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)} d\lambda$ is finite, it follows that (in $\mathcal{J}^{\frac{1}{2}}$ norm)

$$\lim_{t \rightarrow 0} \int_0^\infty \lambda^{-\frac{1}{2}} \cdot [(1+\lambda)(1+D_t^2+\lambda)^{-1} \frac{A_t}{t} (1+D_0^2+\lambda)^{-1} - L_0(\lambda)] d\lambda = \int_0^\infty \lambda^{-\frac{1}{2}} L_0(\lambda) d\lambda.$$

Lemma 3.5.14 part (e) allows us to do the exact same thing for $D_t(1+D_t^2+\lambda)^{-1} A_t D_0(1+D_0^2+\lambda)^{-1}$ (the bound is even exactly the same), whence

$$\| \cdot \|_{\mathcal{J}^{\frac{1}{2}}} \lim_{t \rightarrow 0} \left[\frac{1}{t} (F_{D_t} - F_{D_0}) \right] = \frac{1}{\pi} \cdot \int_0^\infty \lambda^{-\frac{1}{2}} \cdot [L_0(\lambda) - R_0(\lambda)] d\lambda;$$

that is, the derivative at 0 is given by the claimed formula.

Finally, we have to show that $\frac{d}{dt}(F_t)$ is continuous in $\mathcal{J}^{\frac{1}{2}}$ norm. Fix $t_0 \in [0, 1]$. Using the integral formula just established for $\frac{d}{dt}(F_t)$, along with Lemma 3.5.14 (parts (d) and (e)), with

$B_t = A'_t$), for $s \in [0, 1]$ we get

$$\begin{aligned} \left\| \frac{d}{dt} \Big|_{t=t_0} (F_t) - \frac{d}{dt} \Big|_{t=s} (F_t) \right\|_{\mathcal{G}^{\frac{1}{2}}} &\leq 2 \int_0^\infty \lambda^{-\frac{1}{2}} \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \cdot K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}} \cdot v_{s, t_0} d\lambda \\ &= 2K \cdot \|(1 + D_0^2)^{-\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)}\|_{\mathcal{G}^{\frac{1}{2}}} \cdot v_{s, t_0} \cdot \left(\int_0^\infty \lambda^{-\frac{1}{2}} \cdot (1 + \lambda)^{-\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} d\lambda \right). \end{aligned}$$

Since the integral is finite, and $v_{s, t_0} \rightarrow 0$ as $s \rightarrow t_0$, continuity of $\frac{d}{dt}(F_t)$ at t_0 follows.

This concludes the proof that $\{F_t\}$ is C^1 in $\mathcal{G}^{\frac{1}{2}}$ -norm. By the comment preceding Theorem 3.5.10, this is sufficient to prove that $\{F_t\}$ is C^1 as a path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. Note that, since $\|\cdot\|_{\mathcal{S}\mathcal{G}_{F_0}} \geq \|\cdot\|_{\mathcal{G}^{\frac{1}{2}}}$ (for those elements which are in both spaces), and the derivative of $\{F_t\}$ exists with respect to both the $\mathcal{S}\mathcal{G}_{F_0}$ norm and the $\mathcal{G}^{\frac{1}{2}}$ norm, the two derivatives must be equal. In particular, $\frac{d}{dt}(F_t)$ calculated with respect to the $\mathcal{S}\mathcal{G}_{F_0}$ norm must be given by the integral in the statement of the theorem; we do not claim that the integral converges in $\mathcal{S}\mathcal{G}_{F_0}$ norm, just that the operator to which it converges in $\mathcal{G}^{\frac{1}{2}}$ norm is the derivative of $\frac{d}{dt}(F_t)$ with respect to $\mathcal{S}\mathcal{G}_{F_0}$ norm. ■

Lemma 3.5.16 ([14], Lemma 2.11)

$$(1 + D^2)^{-\frac{3}{2}} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [(1 + \lambda)(1 + D^2 + \lambda)^{-2} - D^2(1 + D^2 + \lambda)^{-2}] d\lambda$$

with the integral converging in operator norm. ☞

We use the above two formulas in our next proof. Proposition 2.12 of [14] proves this result for $h(x) = x^q$ with q a positive integer large enough so that $(1 - F_t^2)^q$ is trace class. In spite of being stated in slightly more generality below, the proof goes through in exactly the same manner.

Lemma 3.5.17 Let $\{D_t\}$ be a C^1 path in $D_0 + \mathcal{N}_{sa}$, with $(1 + D_0^2)^{-1} \in \mathcal{G}$. Let $\mathcal{G} = \mathcal{G}^{1-\varepsilon}$ for $0 < \varepsilon < 1$. If $\{F_t\}$ is the image of $\{D_t\}$ under the Riesz transform, then $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}\mathcal{G}_{F_0}$. Moreover, if h is a continuous function such that $h(1 - F_t^2) \in \mathcal{L}^1$, then

$$\tau \left(\frac{d}{dt} (D_t) (1 + D_t^2)^{-\frac{3}{2}} h((1 + D_t^2)^{-1}) \right) = \tau \left(\frac{d}{dt} (F_t) h(1 - F_t^2) \right). \quad \text{☞}$$

PROOF The fact that $\{F_t\}$ is C^1 is the content of Lemma 3.5.15; we only mention it again to emphasize that $\frac{d}{dt}(F_t)$ is calculated using the norm on (and as an element of) $\mathcal{S}\mathcal{G}_{F_0}$.

Using the formula for $\frac{d}{dt}(F_t)$ from Lemma 3.5.15 and the fact that $1 - F_t^2 = (1 + D_t^2)^{-1}$, we have $\tau(\frac{d}{dt}(F_t)h(1 - F_t^2))$ is equal to

$$\begin{aligned} \tau \left(\frac{1}{\pi} \left[\int_0^\infty \lambda^{-\frac{1}{2}} [(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1} \right. \right. \\ \left. \left. - D_t(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)D_t(1 + D_t^2 + \lambda)^{-1}] d\lambda \right] h((1 + D_t^2)^{-1}) \right). \end{aligned}$$

Since it is a bounded operator, we can pull $h((1 + D_t^2)^{-1})$ into the integral. Next, we will show that $\int \tau(|\text{integrand}|) < \infty$ and $\lambda \mapsto \text{integrand}$ is continuous in trace norm, and use Fubini's theorem to change the order of the trace and the integral.

Since t is fixed, $\frac{d}{dt}(A_t)$ and $h((1 + D_t^2)^{-1})$ are constants. From Lemma 3.5.7 we have the norm estimates

$$\begin{aligned} \|(1 + D_t^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-1} \text{ and} \\ \|D_t(1 + D_t^2 + \lambda)^{-1}\| &\leq (1 + \lambda)^{-\frac{1}{2}}, \end{aligned}$$

as well as the continuity of the functions $\lambda \mapsto (1 + D_t^2 + \lambda)^{-1}$ and $\lambda \mapsto D_t(1 + D_t^2 + \lambda)^{-1}$. Since $h((1 + D_t^2)^{-1})$ is trace-class and independent of λ , this is sufficient to show the continuity of the integrand (in trace norm) as a function of λ .

Next, we have to calculate trace-norm bounds for the integrand. For the first term, we have

$$\begin{aligned} &\lambda^{-\frac{1}{2}} \cdot \|(1 + D_t^2 + \lambda)^{-1}\| \cdot \left\| \frac{d}{dt}(A_t) \right\| \cdot \|(1 + D_t^2 + \lambda)^{-1}\| \cdot \|h((1 + D_t^2)^{-1})\|_1 \\ &\leq \lambda^{-\frac{1}{2}} \cdot (1 + \lambda) \cdot \frac{1}{(1 + \lambda)} \cdot \frac{1}{1 + \lambda} \cdot \underbrace{\|h((1 + D_t^2)^{-1})\|_1}_{\text{constant}}. \end{aligned}$$

We can split up $\int_0^\infty \lambda^{-\frac{1}{2}}(1 + \lambda)^{-1} d\lambda$ (from 0 to 1 and 1 to ∞) to conclude it converges. The second term is similar:

$$\begin{aligned} &\lambda^{-\frac{1}{2}} \cdot \|D_t(1 + D_t^2 + \lambda)^{-1}\| \cdot \left\| \frac{d}{dt}(A_t) \right\| \cdot \|D_t(1 + D_t^2 + \lambda)^{-1}\| \cdot \|h((1 + D_t^2)^{-1})\|_1 \\ &\leq \lambda^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{1 + \lambda}} \cdot \underbrace{\left\| \frac{d}{dt}(A_t) \right\|}_{\text{constant}} \cdot \frac{1}{\sqrt{1 + \lambda}} \cdot \underbrace{\|h((1 + D_t^2)^{-1})\|_1}_{\text{constant}}, \end{aligned}$$

so the integral of the \mathcal{L}^1 norm of the second term is bounded by a constant multiple of the same convergent integral.

So we can exchange the order in which we perform the integral and the trace. Moreover, since $h((1 + D_t^2)^{-1}) = h(1 - F_t^2)$ is trace class, we can split up the expression into two terms:

$$\begin{aligned} &\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[(1 + \lambda)(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1} h((1 + D_t^2)^{-1})] d\lambda \\ &\quad - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[D_t(1 + D_t^2 + \lambda)^{-1} \frac{d}{dt}(A_t) D_t(1 + D_t^2 + \lambda)^{-1} h((1 + D_t^2)^{-1})] d\lambda. \end{aligned}$$

Use the trace property to rearrange the above expression to

$$\begin{aligned} &\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[\frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1} h((1 + D_t^2)^{-1})(1 + \lambda)(1 + D_t^2 + \lambda)^{-1}] d\lambda \\ &\quad - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[\frac{d}{dt}(A_t) D_t(1 + D_t^2 + \lambda)^{-1} h((1 + D_t^2)^{-1}) D_t(1 + D_t^2 + \lambda)^{-1}] d\lambda. \end{aligned}$$

D_t is unbounded, but all the other various expressions in D_t are bounded and commute, so with a bit of care we can rearrange and recombine the two expressions to get

$$\begin{aligned} &\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[\frac{d}{dt}(A_t)(1 + \lambda)(1 + D_t^2 + \lambda)^{-2} h((1 + D_t^2)^{-1})] d\lambda \\ &\quad - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \tau[\frac{d}{dt}(A_t) D_t^2(1 + D_t^2 + \lambda)^{-2} h((1 + D_t^2)^{-1})] d\lambda. \end{aligned}$$

Once again, we switch the order of the integral and trace (we omit the check for continuity and for absolute convergence of the integral, as it is almost identical to the previous one), and recombine the two terms to get:

$$\begin{aligned}
&= \tau \left(\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \frac{d}{dt}(A_t) [(1 + \lambda)(1 + D_t^2 + \lambda)^{-2} - D_t^2(1 + D_t^2 + \lambda)^{-1}] h((1 + D_t^2)^{-1}) d\lambda \right) \\
&= \tau \left(\frac{d}{dt}(A_t) \left[\frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (1 + \lambda)(1 + D_t^2 + \lambda)^{-2} - D_t^2(1 + D_t^2 + \lambda)^{-1} d\lambda \right] h((1 + D_t^2)^{-1}) \right) \\
&= \tau \left(\frac{d}{dt}(A_t) (1 + D_t^2)^{-\frac{3}{2}} h((1 + D_t^2)^{-1}) \right),
\end{aligned}$$

concluding the proof. For the last equality we used the formula for $(1 + D_t^2)^{-\frac{3}{2}}$ given in Lemma 3.5.16. \blacksquare

Corollary 3.5.18 Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ such that $(1 + D_0^2)^{-1} \in \mathcal{S}$ for some small power invariant operator ideal \mathcal{S} (see Remark 3.3.1 for definition) and that k is a continuous function on $\mathbb{R} \setminus \{0\}$ which is non-zero on $(0, 1]$ and for which $\lim_{x \rightarrow 0} \frac{k(x)}{x^{3/2}} = 0$. We can define

$$h(x) = \begin{cases} x^{-\frac{3}{2}} k(x) & \text{for } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

which is in turn continuous on \mathbb{R} and non-zero on $(0, 1]$. Suppose h satisfies the remaining conditions of Theorem 3.5.4 for some $\mathcal{S} = \mathcal{S}^{1-\varepsilon}$ where $0 < \varepsilon < 1$; that is, $h(T)$ is trace class for all $T \in \mathcal{S}_{sa}$ and $\alpha_F : X \mapsto \tau(Xh(1 - F^2))$ is an exact one-form on $F_0 + \mathcal{S}\mathcal{S}_{F_0}$. Then

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t) \cdot k((1 + D_t^2)^{-1}) \right) dt + \beta(D_0) - \beta(D_1),$$

where $\tilde{C} = \int_{-1}^1 (1 - s^2)^{-\frac{3}{2}} k(1 - s^2) ds$ and $\beta(D) = \gamma(D(1 + D^2)^{-\frac{1}{2}})$ (with γ as defined in Theorem 3.5.4).

Note: if k is defined on $\mathbb{R}_+ \setminus \{0\}$ instead, we can replace x by $|x|$ in the definition of h in order to define h on all of \mathbb{R} . The rest of the conditions remain unchanged. \blacktriangledown

PROOF Lemma 3.5.15 tells us that $\{F_t\}$ is C^1 in $F_0 + \mathcal{S}\mathcal{S}_{F_0}$. By hypothesis, h satisfies all the requirements of Theorem 3.5.4, which gives us the formula

$$\text{sf}(\{F_t\}) = \frac{1}{C} \cdot \int_0^1 \tau \left(\frac{d}{dt}(F_t) h(1 - F_t^2) \right) dt + \gamma(F_1) - \gamma(F_0),$$

where $C = \int_0^1 h(1 - s^2) ds$ and $\gamma(F)$ is the integral of the one-form α on the straight-line path from F to \tilde{F} .

Since $(1 + D_t^2)^{-1}$ is a positive operator and for positive values of x we have $h(x) = x^{-\frac{3}{2}} k(x)$, Lemma 3.5.17 gives us that

$$\begin{aligned}
\tau \left(\frac{d}{dt}(F_t) h(1 - F_t^2) \right) &= \tau \left(\frac{d}{dt}(D_t) (1 + D_t^2)^{-3/2} \cdot h(1 - F_t^2) \right) \\
&= \tau \left(\frac{d}{dt}(D_t) (1 + D_t^2)^{-3/2} \cdot h((1 + D_t^2)^{-1}) \right) \\
&= \tau \left(\frac{d}{dt}(D_t) (1 + D_t^2)^{-3/2} \cdot (1 + D_t^2)^{3/2} \cdot k((1 + D_t^2)^{-1}) \right) \\
&= \tau \left(\frac{d}{dt}(D_t) \cdot k((1 + D_t^2)^{-1}) \right).
\end{aligned}$$

Since $(1+D_t^2)^{3/2}$ is unbounded, we stop for a second to worry about whether the expression in the second line simplifies indeed to the one in the third, but $h((1+D_t^2)^{-1}) = (1+D_t^2)^{3/2} \cdot k((1+D_t^2)^{-1})$ is a bounded operator, so the domain issues we might have expected do not materialize.

Finally, $\text{sf}(\{D_t\}) = \text{sf}(\{F_t\})$, so we can reach our desired conclusion,

$$\text{sf}(\{D_t\}) = \frac{1}{C} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)k((1+D_t^2)^{-1}) \right) dt + \gamma(D_1(1+D_1^2)^{-\frac{1}{2}}) - \gamma(D_0(1+D_0^2)^{-\frac{1}{2}}).$$

Note that $\tilde{C} = \int_{-1}^1 h(1-s^2) ds = \int_{-1}^1 (1-s^2)^{-\frac{3}{2}} k(1-s^2) ds$ and $\beta(D) = \gamma(D(1+D^2)^{-1/2})$. ■

Remark 3.5.19 Note that, if D_1 is unitarily equivalent to D_0 then $D_1(1+D_1^2)^{-1}$ and $D_0(1+D_0^2)^{-1}$ are likewise unitarily equivalent; so the correction terms cancel, as observed in Remark 3.5.5. •

Remark 3.5.20 We could try to generalize this approach even further, and consider functions $f(F_t)$ instead of $h(1-F_t^2)$. That is, we would like to integrate one-forms $X \mapsto \tau(Xf(F_t))$ on $F_0 + \mathcal{S}$ and relate this to spectral flow. To start with, we need $F_0 + \mathcal{S} \subset \mathcal{BF}_{sa}$, so that spectral flow can be calculated for paths in $F_0 + \mathcal{S}$, and probably some requirement that $f(F)$ be trace class for $F \in F_0 + \mathcal{S}$ so we can calculate the trace.

In order to have the proof of Lemma 3.5.3 go through, the conditions on f become rather technical. Namely, we need that if P and Q are projections such that $2P-1, 2Q-1 \in F_0 + \mathcal{S}$ then $f((2P-1) + 2t(Q-P)) = f_t(Q-P)$, where

- $f_t(x)$ is a continuous even function for $x \in [-1, 1]$ for each t
- $f_t(1) \neq 0$ for each t
- $f_t(Q-P)$ is trace class for each t ,
- $\int_0^1 f_t(1) dt$ exists.

This allows us to get the formula

$$\text{sf}(\{F_t\}) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(F_t) \cdot f(F_t) \right) dt,$$

for the case when $\{F_t\}$ is the straight line path from $2P-1$ to $2Q-1$ (here, $C = \int_0^1 2f_t(1) dt$).

If we now want to get the formula for general paths (step 2 in the above proof), we need $\text{sign}(F) \in F_0 + \mathcal{S}$ whenever $F \in F_0 + \mathcal{S}$, and $\alpha_F(X) = \tau(Xf(F))$ defines a closed one-form on $F_0 + \mathcal{S}$. This allows us to calculate the spectral flow for general paths by adding the correction terms which evaluate the integral from F_0 to \tilde{F}_0 and from F_1 to \tilde{F}_1 . However, the requirement that $\text{sign}(F) \in F_0 + \mathcal{S}$ seems to point us back towards the same kind of manifold that we have been considering.

For our purposes, it seemed unnecessary to clutter the presentation with such technical conditions, especially as it also makes the unbounded operators presentation messier. If \mathcal{S} is a

manifold of the type we were considering above and we were using the Riesz transform, one could probably show the slightly more general result

$$\tau \left(\frac{d}{dt} (D_t) (1 + D_t^2)^{-\frac{3}{2}} f (D_t (1 + D_t^2)^{-\frac{1}{2}}) \right) = \tau \left(\frac{d}{dt} (F_t) f (F_t) \right).$$

Eliminating the connection to the Riesz transform altogether makes it difficult to say what the corresponding unbounded operators formula might be. Presumably, for some different transform from unbounded operators to bounded operators which preserves spectral flow, if one could prove some version of Lemma 3.5.17, such a corresponding formula would follow. •

3.6 Analytic Continuation

We briefly outline the idea of the analytic continuation approach, as we do not try to address it in full generality, and it might apply in other situations not covered by this section. For a specific path $\{D_t\}$ (usually with unitarily equivalent endpoints) we have a family of formulas for spectral flow, say of the form

$$\text{sf}(\{D_t\}) = \frac{1}{C(m)} \cdot \int_0^1 \tau \left(\frac{d}{dt} (D_t) g(D_t)^m \right) dt.$$

We know that the equality holds for all $m \geq N$ for some number N , but we also know that we can calculate the integral on the right hand side for values smaller than N . Let q_0 be the infimum of all values m for which $\tau(\frac{d}{dt}(D_t)g(D_t)^m)$ is finite (and note that usually $\tau(\frac{d}{dt}(D_t)g(D_t)^{q_0})$ is not finite). We want to conclude that the spectral flow equality continues to hold for the numbers between q_0 and N . Rearranging the spectral flow formula, we get

$$C(m) \text{sf}(\{D_t\}) = \int_0^1 \tau \left(\frac{d}{dt} (D_t) g(D_t)^m \right) dt.$$

The two sides of the equality can be thought of as functions in m , where m is a real number; we would like to show that the two functions make sense if m is instead in some subset of the complex numbers, and use properties of complex functions to show that the spectral flow formula holds for all values of m for which the right hand side integral makes sense. For this, we need $z \mapsto C(z)$ and $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t)g(D_t)^z) dt$ to be analytic on the set $\{z \in \mathbb{C} : \Re(z) > q_0\}$. In this section, we address how one might go about proving this for the latter function. In Section 3.7, we apply the results from this section to the case when D_0 is p -summable.

To generalize this scenario further, one can replace $g(D_t)^m$ by an appropriate function $f(m, D_t)$. Lemma 3.6.3 provides a glimmer of how to proceed in that direction; however, such a level of abstraction seemed unnecessary for our situation.

In the beginning, we need only assume that D_0 is an unbounded self-adjoint operator, though of course in order to talk about spectral flow we will eventually need to add the condition that D_0 is Breuer-Fredholm and $D_0 + A$ is Breuer-Fredholm for any $A \in \mathcal{N}_{sa}$ (which holds if, for example, $(1 + D_0^2)^{-1} \in \mathcal{I}$ for some small power invariant operator ideal \mathcal{I} , as defined in Remark 3.3.1). Let $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ be a fixed C^1 path. We wish to consider the complex function

$$\varphi : z \mapsto \int_0^1 \tau \left(\frac{d}{dt} (D_t) g(D_t)^z \right) dt.$$

Sufficient restrictions on g to ensure that φ is well defined and analytic are covered in Lemma 3.6.1 and Lemma 3.6.5 respectively. The domain of φ is also established in the following lemma.

Lemma 3.6.1 Let $\{D_t\} \subset D_0 + \mathcal{N}_{sa}$ be a fixed C^1 path. Suppose g is a bounded continuous function $\mathbb{R} \rightarrow \mathbb{R}_+$ (ensuring that $g(D_t)$ is a bounded operator for all $t \in [0, 1]$) for which there exists a real number $q_0 > 0$ such that for any $m > q_0$ we have $g(D_t)^m \in \mathcal{L}^1$ and $t \mapsto g(D_t)^m$ is continuous in $\|\cdot\|_1$ -norm. Then for any fixed $z_0 \in \mathbb{C}$ with $\Re(z_0) > q_0$, the function $\varphi : z \mapsto \int_0^1 \tau \left(\frac{d}{dt}(D_t)g(D_t)^z \right) dt$ is defined at z_0 . \blacksquare

PROOF A quick word about the definition of $g(D_t)^z$. For λ a (strictly) positive number, λ^z can be defined via $e^{\log(\lambda) \times z}$. If z is a complex number with $\Re(z) > 0$ then $|\lambda^z| = \lambda^{\Re(z)}$, so $\lambda^z \rightarrow 0$ as $t \rightarrow 0$. Thus λ^z can be defined continuously on the spectrum of $g(D_t)$, and hence $g(D_t)^z$ makes sense and is a bounded operator.

Let $B_t = \frac{d}{dt}(D_t)$. Pick any $q \in \mathbb{R}$ with $q_0 < q < \Re(z_0)$, write $z_0 = (q + r) + is$ and note that $r \geq 0$ and $s \in \mathbb{R}$. We have

$$g(D_t)^{z_0} = g(D_t)^{q+r+is} = g(D_t)^q \cdot g(D_t)^r \cdot g(D_t)^{is}.$$

But $g(D_t)^q$ is trace class by hypothesis, $g(D_t)^r$ is bounded (since $r \geq 0$) and $g(D_t)^{is}$ is unitary, so it follows (from Hölder's inequality) that $B_t g(D_t)^{z_0}$ is trace class.

In order to show that $\int_0^1 \tau(B_t g(D_t)^{z_0}) dt$ is finite, since the interval of integration is compact, it is sufficient to show that the integrand is continuous in t on $[0, 1]$. We will show that $t \mapsto B_t g(D_t)^{z_0}$ is continuous at $t = 0$ in trace norm; the result for any t is similar. In order to show that $t \mapsto g(D_t)^{z_0}$ is continuous in $\|\cdot\|_1$ -norm note that, for t close to 0 we have

$$\begin{aligned} & \|g(D_t)^{z_0} - g(D_0)^{z_0}\|_1 \\ & \leq \|g(D_t)^q \cdot g(D_t)^r \cdot g(D_t)^{is} - g(D_0)^q \cdot g(D_0)^r \cdot g(D_0)^{is}\|_1 \\ & \leq \|g(D_t)^q - g(D_0)^q\|_1 \cdot \|g(D_t)^r\| \cdot \|g(D_t)^{is}\| \\ & \quad + \|g(D_0)^q\|_1 \cdot \|g(D_t)^r - g(D_0)^r\| \cdot \|g(D_t)^{is}\| \\ & \quad + \|g(D_0)^q\|_1 \cdot \|g(D_0)^r\| \cdot \|g(D_t)^{is} - g(D_0)^{is}\|. \end{aligned}$$

Since $t \mapsto g(D_t)^q$ is continuous in trace norm by hypothesis, this means $\|g(D_t)^q - g(D_0)^q\|_1 \rightarrow 0$. The fact that $\|g(D_t)^r - g(D_0)^r\|$ and $\|g(D_t)^{is} - g(D_0)^{is}\|$ converge to 0 as $t \rightarrow 0$ as well follows from the continuity of the functional calculus. Moreover, $\|g(D_t)^r\|$ is uniformly bounded for $t \in [0, 1]$ (since $t \mapsto \|g(D_t)^r\|$ is a continuous function defined on a compact set), and $\|g(D_t)^{is}\| = 1$ for all t ; this shows that $t \mapsto g(D_t)^{z_0}$ is continuous in trace norm.

Now we have

$$\begin{aligned} & \|B_t g(D_t)^{z_0} - B_0 g(D_0)^{z_0}\|_1 \\ & \leq \|B_t - B_0\| \cdot \|g(D_t)^{z_0}\|_1 + \|B_0\| \cdot \|g(D_t)^{z_0} - g(D_0)^{z_0}\|_1. \end{aligned}$$

Since $\{\|g(D_t)^{z_0}\|_1\}_{t \in [0, 1]}$ is a bounded set (by continuity of the function $g(D_t)^{z_0}$ in t), $t \mapsto B_t$ is continuous in norm (since $\{D_t\}$ is C^1), it follows that $t \mapsto B_t g(D_t)^{z_0}$ is continuous in trace norm. This concludes the proof that $\int_0^1 \tau \left(\frac{d}{dt}(D_t)g(D_t)^{z_0} \right) dt < \infty$.

Therefore, the domain of $\varphi : z \mapsto \int_0^1 \tau \left(\frac{d}{dt}(D_t)g(D_t)^z \right) dt$ contains $\mathcal{U} = \{z \in \mathbb{C} : \Re(z) \geq q\}$. \blacksquare

We would next like to show that φ is analytic. In the proof of this fact we will need the derivative of the function $z \mapsto T^z$ for fixed T appropriately chosen. As expected, the difference quotient at z_0 converges to $T^{z_0} \log(T)$; moreover, in Corollary 3.6.4, we use Taylor's Theorem to show that for T positive and of norm at most 1, this convergence happens at a rate which is independent of T .

Theorem 3.6.2 (Taylor's Theorem, [1], Section 3.1, Theorem 8) If $f(z)$ is analytic in a region Ω containing a , there exists a function $f_n(z)$ analytic in Ω such that

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

In fact, if C is a circle around a such that C and its interior are both contained in Ω , then $f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^n(w-z)} dw$ for z inside C . \clubsuit

Lemma 3.6.3 Consider $S \subset \mathbb{R}$ and an open set $\Omega \subset \mathbb{C}$. Suppose $f : S \times \Omega \rightarrow \mathbb{C}$ is such that

1. for each fixed $t_0 \in S$, $z \mapsto f(t_0, z)$ is analytic in Ω . Let $l(t, z) = \frac{\partial}{\partial z} f$ for $t, z \in S \times \Omega$.
2. for each fixed $z_0 \in \Omega$, $t \mapsto f(t, z_0)$ is continuous on S .
3. for each fixed $z_0 \in \Omega$, $t \mapsto l(t, z_0)$ is continuous on S .
4. $\{f(t, z) : t \in S, z \in \Omega\}$ is a bounded subset of \mathbb{C} .

Finally, suppose that T is a self-adjoint operator with $\sigma(T) \subset S$. Then, for each fixed $z_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ (depending on z_0 but not T) such that $|z - z_0| < \delta$, $z \neq z_0$ implies $z \in \Omega$ and

$$\left\| \frac{f(T, z) - f(T, z_0)}{z - z_0} - l(T, z_0) \right\| < \varepsilon. \quad \clubsuit$$

PROOF The proof relies on the Spectral Theorem and the complex analysis version of Taylor's Theorem (Theorem 3.6.2).

Given z_0 in the open set Ω , we can find a circle C around z_0 of some radius R such that C and its interior are contained in Ω . Let C_0 be the circle around z_0 of radius $\frac{R}{2}$.

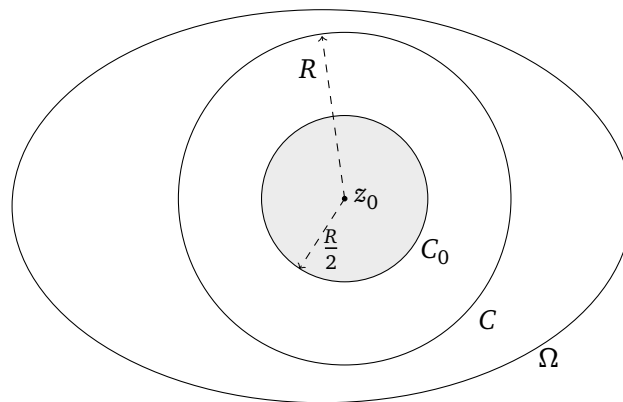


Figure 3.3: Setup for proof of Lemma 3.6.3.

Fix $t_0 \in S$, and let $h(z) = f(t_0, z)$. Then $h(z)$ is analytic in Ω (hypothesis 1) and by Taylor's Theorem (Theorem 3.6.2 with $n = 2$ and $a = z_0$),

$$h(z) = h(z_0) + h'(z_0) \cdot (z - z_0) + h_2(z) \cdot (z - z_0)^2,$$

where $h_2(z) = \frac{1}{2\pi i} \int_C \frac{h(w)}{(w-z_0)^2(w-z)} dw$ for z in the interior of C . If $z \neq z_0$ we can rearrange this equality to

$$\frac{h(z) - h(z_0)}{z - z_0} - h'(z_0) = h_2(z) \cdot (z - z_0).$$

However,

$$|h_2(z)| = \left| \frac{1}{2\pi i} \int_C \frac{h(w)}{(w-z_0)^2(w-z)} dw \right| \leq \left| \frac{1}{2\pi i} \right| \cdot \max_{w \in C} \left| \frac{h(w)}{(w-z_0)^2(w-z)} \right| \cdot \int_C |dw|.$$

Using $|w - z_0| = R$ (since w is on the circle C) and the fact that $\int_C |dw|$ is the circumference of C , namely $2\pi R$, we can continue from the last inequality proven to

$$\begin{aligned} |h_2(z)| &\leq \frac{1}{2\pi} \cdot \frac{1}{R^2} \frac{\max_{w \in C} |h(w)|}{\min_{w \in C} |w-z|} \cdot (2\pi R) \\ &= \frac{1}{R} \cdot \frac{\max_{w \in C} |h(w)|}{\min_{w \in C} |w-z|}. \end{aligned}$$

Going back to the notation used in the statement of the theorem, we have $h(z) = f(t_0, z)$ and $h'(z_0) = l(t_0, z_0)$; so we have shown that, for all z in the interior of C ,

$$\frac{f(t_0, z) - f(t_0, z_0)}{z - z_0} - l(t_0, z_0) = h_2(z) \cdot (z - z_0),$$

where $|h_2(z)| \leq \frac{1}{R} \cdot \frac{\max_{w \in C} |f(t_0, w)|}{\min_{w \in C} |w-z|}$.

By hypothesis 4, $\{f(t, z) : t \in S, z \in \Omega\}$ is bounded; say $|f(t, z)| < M$ for all $t \in S$ and $z \in \Omega$. If in addition we assume that z is in the interior of C_0 then the reverse triangle inequality gives us $|w - z| \geq \frac{R}{2}$ for w on the circle C , which implies $|h_2(z)| \leq \frac{1}{R} \cdot \frac{M}{R/2} = \frac{2M}{R^2}$. Hence, for any z in the interior of C_0 with $z \neq z_0$,

$$\left| \frac{f(t_0, z) - f(t_0, z_0)}{z - z_0} - l(t_0, z_0) \right| = |h_2(z)| \cdot |z - z_0| \leq \frac{2M}{R^2} \cdot |z - z_0|.$$

So, given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < \min\{\frac{R^2}{2M}, \frac{R}{2}\}$; note that the ball centered at z_0 of radius δ is contained in Ω . If $|z - z_0| < \delta$ with $z \neq z_0$, then z is in the interior of C_0 and for all $t \in S$ we have

$$\left| \frac{f(t, z) - f(t, z_0)}{z - z_0} - l(t, z_0) \right| \leq \frac{2M}{R^2} \cdot |z - z_0| < \varepsilon.$$

Finally, since $\sigma(T) \subset S$ the functional calculus gives us that, if $|z - z_0| < \delta$ with $z \neq z_0$, then

$$\left\| \frac{f(T, z) - f(T, z_0)}{z - z_0} - l(T, z_0) \right\| < \varepsilon.$$

Note in particular that δ depends on z_0 (for the choice of R) and the bound M on f , but does not depend on T . ■

Corollary 3.6.4 For T a positive operator with $\|T\| \leq 1$ and $z_0 \in \mathbb{C}$ such that $\Re(z_0) > 0$, for any $\varepsilon > 0$ there exists a δ (dependent on z_0 but not on T) such that if $|z - z_0| < \delta$ (but $z \neq z_0$) then

$$\left\| \frac{T^z - T^{z_0}}{z - z_0} - T^{z_0} \log T \right\| < \varepsilon. \quad \blacktriangleright$$

PROOF This is a straight-forward application of Lemma 3.6.3. Since T is positive and has norm at most 1, if $S = [0, 1]$ then $\sigma(T) \subset S$. Let $\Omega = \{z \in \mathbb{C} : \Re(z) > 0\}$ and

$$f(t, z) = \begin{cases} t^z & \text{if } t \neq 0, \text{ and} \\ 0 & \text{if } t = 0. \end{cases}$$

We check that this set-up satisfies the hypotheses of Lemma 3.6.3. Note that, for fixed $z_0 \in \Omega$, $t^{z_0} \rightarrow 0$ as $t \rightarrow 0$, so $f(t, z_0)$ is continuous on $[0, 1]$ (condition 2). Also, for fixed $t_0 \neq 0$, $z \mapsto t_0^z$ is analytic, as is the constant function $z \rightarrow 0$, so condition 1 holds. This leads to the definition

$$l(t, z) = \begin{cases} t^z \log t & \text{for } t \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $t \neq 0$ and $z_0 \in \Omega$ fixed, $t^{z_0} \log t \rightarrow 0$ as $t \rightarrow 0$, so $t \mapsto l(t, z_0)$ is continuous (condition 3).

Finally, we need to check that $\{f(t, z) : t \in S, z \in \Omega\}$ is a bounded subset of \mathbb{C} (condition 4). For $t \in \mathbb{R}_+ \setminus \{0\}$, $|t^z| = t^{\Re(z)}$ so when $\Re(z) > 0$ and $t \leq 1$ we have $|t^z| = t^{\Re(z)} \leq 1$. As $f(0, z) = 0$, $|f(t, z)| \leq 1$ for all $t \in S$ and $z \in \Omega$.

Therefore, $f : S \times \Omega \rightarrow \mathbb{C}$ satisfies the conditions of Lemma 3.6.3, allowing us to conclude that there exists a $\delta > 0$ for which, if $|z - z_0| < \delta$ and $z \neq z_0$, then $\left| \frac{T^z - T^{z_0}}{z - z_0} - T^{z_0} \log T \right| < \varepsilon$. \blacksquare

Lemma 3.6.5 Let $\{D_t\} \subset D_0 + \mathcal{A}_{sa}$ be a fixed C^1 -path. Suppose g is a continuous function $\mathbb{R} \rightarrow \mathbb{R}_+$ such that $\|g(D_t)\| \leq 1$ for all $t \in [0, 1]$. Moreover, suppose that there exists a real number $q_0 > 0$ such that $g(D_t)^m \in \mathcal{L}^1$ for all $m > q_0$, and $t \mapsto g(D_t)^m$ is continuous in trace norm for $m > q_0$. Then the function $\varphi : z \mapsto \int_0^1 \tau\left(\frac{d}{dt}(D_t)g(D_t)^z\right) dt$ is analytic at each z in the open half-plane $\{\Re(z) > q_0\}$. \blacktriangleright

PROOF From Lemma 3.6.1, we already know that φ is defined in the half-plane $\{\Re(z) > q_0\}$, so we need to show that $\frac{d}{dz} \int_0^1 \tau\left(\frac{d}{dt}(D_t)g(D_t)^z\right) dt$ exists at all z_0 with $\Re(z_0) > q_0$. Write $B_t = \frac{d}{dt}(D_t)$. Note that

$$\begin{aligned} \frac{d}{dz} \Big|_{z=z_0} \int_0^1 \tau(B_t g(D_t)^z) dt &= \lim_{z \rightarrow z_0} \frac{\int_0^1 \tau(B_t g(D_t)^z) dt - \int_0^1 \tau(B_t g(D_t)^{z_0}) dt}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \int_0^1 (z - z_0)^{-1} \cdot (\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0})) dt. \end{aligned}$$

We claim that

$$\lim_{z \rightarrow z_0} \int_0^1 (z - z_0)^{-1} \cdot (\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0})) dt = \int_0^1 \tau(B_t g(D_t)^{z_0} \log(g(D_t))) dt.$$

Since the interval over which we are integrating has finite measure, it is sufficient to show that the difference quotient converges uniformly to its limit; that is, given $\varepsilon > 0$, find a $\delta > 0$ such that $|z - z_0| < \delta$ implies that

$$\left| \frac{\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0})}{z - z_0} - \tau(B_t g(D_t)^{z_0} \log(g(D_t))) \right| < \varepsilon \text{ for all } t \in [0, 1].$$

To this end, use Corollary 3.6.4. Fix t ; by assumption, $g(D_t)$ is positive and has norm less than or equal to 1. Also fix $q \in \mathbb{R}$ such that $q_0 < q < \Re(z_0)$; then

$$\begin{aligned} & \left| \frac{\tau(B_t g(D_t)^z) - \tau(B_t g(D_t)^{z_0})}{z - z_0} - \tau(B_t g(D_t)^{z_0} \log(g(D_t))) \right| \\ &= \left| \tau \left(B_t \frac{(g(D_t)^z - g(D_t)^{z_0})}{z - z_0} - B_t g(D_t)^{z_0} \log(g(D_t)) \right) \right| \\ &= \left| \tau(B_t g(D_t)^q \cdot \left[\frac{g(D_t)^z - g(D_t)^{z_0}}{z - z_0} - g(D_t)^{z_0 - q} \log(g(D_t)) \right]) \right| \\ &\leq \|B_t\| \cdot \|g(D_t)^q\|_1 \cdot \left\| \frac{g(D_t)^z - g(D_t)^{z_0}}{z - z_0} - g(D_t)^{z_0 - q} \log(g(D_t)) \right\|. \end{aligned}$$

Use Corollary 3.6.4 with $z_0 - q$ instead of z_0 (recall that $\Re(z_0) > q$ by choice of q) to conclude that there exists $\delta > 0$ (independent of t) such that if $|z - z_0| = |(z - q) - (z_0 - q)| < \delta$ then

$$\left\| \frac{g(D_t)^z - g(D_t)^{z_0 - q}}{z - z_0} - g(D_t)^{z_0 - q} \log(g(D_t)) \right\| < \varepsilon.$$

Since $t \mapsto B_t$ is operator norm continuous (recall that $\{D_t\}$ is a C^1 -path), and $t \mapsto g(D_t)^q$ is trace-norm continuous, we know that $\|B_t\|$ and $\|g(D_t)^q\|_1$ are both uniformly bounded; hence, the difference quotient of the function $z \mapsto \tau(B_t g(D_t)^z)$ at z_0 converges uniformly for $t \in [0, 1]$ to $\tau(B_t g(D_t)^{z_0} \log(g(D_t)))$. Therefore,

$$\frac{d}{dz} \int_0^1 \tau(B_t g(D_t)^z) dt = \int_0^1 \tau(B_t g(D_t)^z \log(g(D_t))) dt.$$

This concludes the proof that $\varphi(z)$ is analytic. ■

Therefore we can use analytic continuation to find new formulas for spectral flow, as shown in Corollary 3.6.7 below.

Theorem 3.6.6 (Bak & Newman [4], p. 69) If two functions f and g , analytic in an open, connected set Ω , agree at a set of points with an accumulation point in Ω , then $f \equiv g$ through Ω . ☯

Corollary 3.6.7 Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$ for which the equality

$$C(m) \cdot \text{sf}(\{D_t\}) = \int_0^1 \tau \left(\frac{d}{dt}(D_t) g(D_t)^m \right) dt$$

holds for all $m \geq N$ (where $N \in \mathbb{R}_+$ is fixed). Let $q_0 = \inf\{m : \tau(g(D_t)^m) < \infty\}$. Suppose that $C(z)$ is analytic in the half-plane $\{\Re(z) > q_0\}$. Suppose further that $g : \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous, that $\|g(D_t)\| \leq 1$ for all t , and that for any $m > q_0$ $t \mapsto g(D_t)^m$ is continuous in trace norm. Then for any m between q_0 and N (not including q_0) we also have

$$C(m) \cdot \text{sf}(\{D_t\}) = \int_0^1 \tau \left(\frac{d}{dt}(D_t) g(D_t)^m \right) dt. \quad \text{☯}$$

PROOF By Lemma 3.6.5, $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t)g(D_t)^z) dt$ is analytic in the half-plane $\{\Re(z) > q_0\}$. The desired result follows immediately by applying Theorem 3.6.6: the complex functions $z \mapsto C(z) \cdot \text{sf}(\{D_t\})$ and $z \mapsto \int_0^1 \tau(\frac{d}{dt}(D_t)g(D_t)^z) dt$ are both analytic on the open connected set $\{\Re(z) > q_0\}$ and agree on the set $\{\Re(z) \geq N\}$, so they must be equal on the whole half-plane. ■

3.7 Proof of integral formula for p -summable unbounded operators

Our goal (the proof of Theorem 3.0.8) is to show that, if D_0 is a p -summable self-adjoint unbounded operator, then we can use the integral $\int \tau(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{p}{2}}) dt$ (suitably modified) to calculate spectral flow. We follow the program set out in Section 3.3: for each formula we want to show, figure out the corresponding bounded operators picture, and show that one can calculate the spectral flow in the bounded setting. At various points in the proof we will need extra maneuvering room, so we replace $\frac{p}{2}$ by a m sufficiently larger than $\frac{p}{2}$, and use analytic continuation to eventually get the result with $\frac{p}{2}$.

The corresponding bounded operators formula is of the form $\int \tau(\frac{d}{dt}(F_t)|1 - F_t^2|^q) dt$. To start with, we prove that our one-forms are closed; then go through the process of modifying them so they calculate spectral flow; and, finally, we use analytic continuation to prove the theorem stated at the beginning of the chapter. Showing the unbounded one-form is closed is not necessary for obtaining the spectral flow formula, as most of the work is accomplished in the bounded operators picture; however, it is needed in the proof of the Local Index Theorem, as outlined in Section 3.1.

3.7.1 Bounded Case Manifold and One-form

As this is the bounded picture for D_0 p -summable, we want to look at ideals $\mathcal{J} = (\mathcal{L}^{\frac{p}{2}})^{1-\varepsilon}$ for some $0 < \varepsilon < 1$, and the corresponding manifolds $F_0 + \mathcal{S}\mathcal{G}_{F_0}$ (where $D_0(1 + D_0^2)^{-\frac{1}{2}}$ will determine our choice of F_0). As \mathcal{J} is also an ideal of summable operators, we suppose $\mathcal{J} = \mathcal{L}^{\frac{p}{2}}$ and worry about the relation between the p used in this case and the p used in the unbounded case when it is time to put the various results together. The beginning of Section 3.5 introduces $\mathcal{S}\mathcal{G}_{F_0}$ and its properties (the corresponding specific results about $\mathcal{S}\mathcal{L}^{\frac{p}{2}}_{F_0}$ can be found in [14]); we recall that

$$\mathcal{S}\mathcal{L}^{\frac{p}{2}}_{F_0} = \{X \in \mathcal{L}^p_{sa} \mid 1 - (F_0 + X)^2 \in \mathcal{L}^{\frac{p}{2}}\},$$

and the norm on $\mathcal{S}\mathcal{L}^{\frac{p}{2}}_{F_0}$ is given by $\|X\|_{\mathcal{S}\mathcal{L}^{\frac{p}{2}}_{F_0}} = \|X\|_p + \|F_0X + XF_0\|_{\frac{p}{2}}$.

We review a few facts about the relationship between $\mathcal{S}\mathcal{G}_{F_0}$ and \mathcal{J} that, in our focus on paths, have not yet perhaps been made explicit. However, we will need to know that, for $F \in \mathcal{S}\mathcal{G}_{F_0}$ fixed, the set $\{\|1 - (F + X)^2\|_{\mathcal{J}} : \|X\|_{\mathcal{S}\mathcal{G}_{F_0}} \leq 1\}$ is bounded, so we discuss this in the following.

Remark 3.7.1 Fix $F \in F_0 + \mathcal{S}\mathcal{G}_{F_0}$ for some invariant operator ideal \mathcal{J} which has the Cauchy-Schwarz property. The map $X \mapsto 1 - (F + X)^2$ is continuous from $\mathcal{S}\mathcal{G}_{F_0}$ to \mathcal{J} (see [14], Lemma 1.2 part 2, for the original proof in the case when $\mathcal{J} = \mathcal{L}^p$). As $1 - (F + X)^2 = (1 - F^2) - \{F, X\} - X^2$, this claim is supported by the following:

- $X \mapsto \{F, X\}$ from $\mathcal{S}\mathcal{G}_{F_0}$ to \mathcal{G} is continuous. We show continuity at $X = 0$, as linearity in X allows us to conclude continuity at other values:

$$\begin{aligned} \|\{F, X\}\|_{\mathcal{G}} &= \|\{F_0 + (F - F_0), X\}\|_{\mathcal{G}} \leq \|\{F_0, X\}\|_{\mathcal{G}} + \|\{F - F_0, X\}\|_{\mathcal{G}} \\ &\leq \|\{F_0, X\}\|_{\mathcal{G}} + 2\|F - F_0\|_{\mathcal{G}^{\frac{1}{2}}} \cdot \|X\|_{\mathcal{G}^{\frac{1}{2}}} \\ &\leq \|X\|_{\mathcal{S}\mathcal{G}_{F_0}} + 2\|F - F_0\|_{\mathcal{G}^{\frac{1}{2}}} \cdot \|X\|_{\mathcal{S}\mathcal{G}_{F_0}} \\ &\leq \|X\|_{\mathcal{S}\mathcal{G}_{F_0}} (1 + 2\|F - F_0\|_{\mathcal{G}^{\frac{1}{2}}}). \end{aligned}$$

Note that for the above inequalities we used both the fact that \mathcal{G} has the Cauchy-Schwarz property and the definition of the norm on $\mathcal{S}\mathcal{G}_{F_0}$. The continuity of $X \mapsto \{F, X\}$ at $X = 0$ follows from the fact that the final expression on the right hand side (in which both F and F_0 are fixed) goes to zero as $\|X\|_{\mathcal{S}\mathcal{G}_{F_0}} \rightarrow 0$.

- $X \mapsto X^2$ from $\mathcal{S}\mathcal{G}_{F_0}$ to \mathcal{G} is continuous. Using the Cauchy-Schwarz property of \mathcal{G} , it is straight-forward to show that $X \mapsto X^2$ is continuous from $\mathcal{G}^{\frac{1}{2}}$ to \mathcal{G} , so the result follows from the fact that $\mathcal{S}\mathcal{G}_{F_0} \hookrightarrow \mathcal{G}^{\frac{1}{2}}$ is continuous.

We will need norm bounds for some of the above expressions in \mathcal{G} in the case when $\|X\|_{\mathcal{S}\mathcal{G}_{F_0}} \leq 1$. We have

- $\|X^2\|_{\mathcal{G}} \leq (\|X\|_{\mathcal{G}^{\frac{1}{2}}})^2 \leq (\|X\|_{\mathcal{S}\mathcal{G}_{F_0}})^2 \leq 1$ (using the fact that \mathcal{G} has the Cauchy-Schwarz property, and the definition of the norm on $\mathcal{S}\mathcal{G}_{F_0}$).
- $\|\{F, X\}\|_{\mathcal{G}} \leq 1 + 2\|F - F_0\|_{\mathcal{G}^{\frac{1}{2}}}$ (from the calculations at the beginning of the remark).
- $\|1 - (F + X)^2\|_{\mathcal{G}} \leq \|1 - F^2\|_{\mathcal{G}} + \|\{F, X\}\|_{\mathcal{G}} + \|X^2\|_{\mathcal{G}} \leq \|1 - F^2\|_{\mathcal{G}} + (1 + 2\|F - F_0\|_{\mathcal{G}^{\frac{1}{2}}}) + 1$ (combining the above two norms).

In particular we note that, if F is fixed, the set $\{\|1 - (F + X)^2\|_{\mathcal{G}} : \|X\|_{\mathcal{S}\mathcal{G}_{F_0}} \leq 1\}$ is bounded. •

Armed with the above calculations, we now tackle the task of defining a one-form on $\mathcal{L}^{\frac{p}{2}}_{F_0}$. Choose $r \in \mathbb{R}_+$ such that $r \geq p + 6$, and define the one-form $\alpha_F(X) = \frac{1}{C} \cdot \tau(X|1 - F^2|^r)$ for $X \in \mathcal{L}^{\frac{p}{2}}_{F_0}$ and C a constant. We would like to use Theorem 3.4.1 to conclude that α_F is a closed one-form on $F_0 + \mathcal{L}^{\frac{p}{2}}_{F_0}$. Let $g(F) = (1 - F^2)^2$; then $|1 - F^2|^r = g(F)^{r/2}$, and we can write the one-form as $\alpha_F(X) = \frac{1}{C} \cdot \tau(Xg(F)^{r/2})$. We have to show that g satisfies the properties outlined on page 85 (at the beginning of Section 3.4) with $q = \frac{r}{2}$.

We start by calculating the difference quotient, $d_{g,s}(F, X)$; in order to simplify the resulting algebraic expression, we will make use of the equality $A^2 - B^2 = \frac{1}{2}\{A - B, A + B\}$ (where $\{ \cdot, \cdot \}$

denotes the anti-commutator, that is $\{U, V\} = UV + VU$:

$$\begin{aligned}
d_{g,s}(F, X) &= \frac{1}{s} \cdot (g(F + sX) - g(F)) \\
&= \frac{1}{s} \cdot ((1 - (F + sX)^2)^2 - (1 - F^2)^2) \\
&= \frac{1}{s} \cdot \frac{1}{2} \{ (1 - (F + sX)^2) - (1 - F^2), (1 - (F + sX)^2) + (1 - F^2) \} \\
&= \frac{1}{2s} \cdot \{ F^2 - (F + sX)^2, (1 - (F + sX)^2) + (1 - F^2) \} \\
&= \frac{1}{s} \cdot \frac{1}{2} \{ \frac{1}{2} \{ F - (F + sX), F + (F + sX) \}, (1 - (F + sX)^2) + (1 - F^2) \} \\
&= \frac{1}{4s} \cdot (-s) \cdot \{ \{ X, 2F + sX \}, 2 - (F + sX)^2 - F^2 \} \\
&= -\frac{1}{4} \{ \{ X, 2F + sX \}, 2 - (F + sX)^2 - F^2 \} \\
&= -\frac{1}{2} \cdot \{ \{ X, F \} + sX^2, 2 - (F + sX)^2 - F^2 \}.
\end{aligned}$$

It should be clear that $d_{g,0} = \lim_{s \rightarrow 0} d_{g,s}$ exists and is equal to $(-1) \cdot \{ \{ X, F \}, 1 - F^2 \}$ (where the limit is calculated with respect to the operator norm). Note that the $\frac{1}{2}$ factor cancels out with the factor of 2 in $2(1 - F^2)$ after calculating the limit. This formula is specifically referenced in the conditions on g , so we will use it in the following.

1. For every F self-adjoint, $(1 - F^2)^2$ is positive and bounded, so $g(F)$ is positive and bounded for all $F \in F_0 + \mathcal{AL}^{\frac{p}{2}}_{F_0}$.

2. By definition of $\mathcal{AL}^{\frac{p}{2}}_{F_0}$, $1 - F^2 \in \mathcal{L}^{\frac{p}{2}}$ for any $F \in F_0 + \mathcal{AL}^{\frac{p}{2}}_{F_0}$. For $t \geq \frac{q-3}{2}$, we have $2t \geq q-3 = \frac{r}{2} - 3 \geq \frac{p}{2}$, so it follows that $\tau(|1 - F^2|^{2t}) < \infty$. Since $g(F)^t = |1 - F^2|^{2t}$, the desired property is satisfied.

3. Fix $F \in F_0 + \mathcal{AL}^{\frac{p}{2}}_{F_0}$. To show that $X \mapsto g(F + X)^t$ is continuous from $\mathcal{AL}^{\frac{p}{2}}_{F_0}$ to \mathcal{L}^1 for large enough t , show that g satisfies the alternate requirements outlined in Section 3.4.1. Fix $t \geq \frac{q-3}{2}$ (which, as shown earlier, implies $2t \geq \frac{p}{2}$).

- 3.(a) We need to show that $\{ \|(1 - (F + X)^2)^2\|_t : \|X\|_{\mathcal{AL}^{\frac{p}{2}}_{F_0}} \leq 1 \}$ is bounded. By Remark 3.7.1, the set $\{ \|1 - (F + X)^2\|_{\frac{p}{2}} : \|X\|_{\mathcal{AL}^{\frac{p}{2}}_{F_0}} \leq 1 \}$ is bounded; the properties of invariant operator ideals (see Definition 1.1.9) enable us to conclude that $\{ \|(1 - (F + X)^2)^2\|_{\frac{p}{2}} : \|X\|_{\mathcal{AL}^{\frac{p}{2}}_{F_0}} \leq 1 \}$ is likewise bounded (here, we need to recall the fact that $\|X\| \leq \|X\|_{\frac{p}{2}} \leq \|X\|_{\mathcal{AL}^{\frac{p}{2}}_{F_0}}$ for any $X \in \mathcal{AL}^{\frac{p}{2}}_{F_0}$). As $\|Y\|_t \leq \|Y\|_{\frac{p}{2}}$ for any $Y \in \mathcal{L}^{\frac{p}{2}}$ and $t \geq \frac{p}{2}$ (see Remark 1.1.15), result follows.
- 3.(b) We need to show that $\{ \|d_{g,s}(F, X)\|_t : s \in [-1, 1], \|X\|_{\mathcal{AL}^{\frac{p}{2}}_{F_0}} \leq 1 \}$ is bounded. Recall that, as calculated before we started in on checking the properties of g ,

$$d_{g,s}(F, X) = \frac{-1}{2} \cdot \{ \{ X, F \} + sX^2, 2 - (F + sX)^2 - F^2 \}.$$

From the definition of $\mathcal{AL}^{\frac{p}{2}}_{F_0}$ it follows easily that X^2 , $\{X, F\}$, $1 - (F + sX)^2$ and $1 - F^2$ are all in $\mathcal{L}^{\frac{p}{2}}$, so certainly $d_{g,s}(F, X) \in \mathcal{L}^{\frac{p}{2}}$ as well. From the norm calculations in Remark 3.7.1, we know that $\|1 - (F + sX)^2\|_{\frac{p}{2}} \leq \|1 - F^2\|_{\frac{p}{2}} + 2 + 2\|F - F_0\|_p$ and

$$\| \{ X, F \} + sX^2 \|_{\frac{p}{2}} \leq \| \{ X, F \} \|_{\frac{p}{2}} + |s| \cdot \| X^2 \|_{\frac{p}{2}} \leq 1 + 2\|F - F_0\|_p + 1 \cdot 1.$$

We can use these to calculate a bound on $\|d_{g,s}(F,X)\|_{\frac{p}{2}}$, and as in part (a) use the relationship between norms of powers of ideals (Remark 1.1.15) to reach the desired conclusion.

4. We want to show that the operator defined via the directional derivative $d_{g,0}$ is in fact the (Fréchet) derivative of $X \mapsto g(F+X)$. Recall that $d_{g,0}(F,X) = -\frac{1}{2}\{\{X,F\}, 1-F^2\}$. Define $L_X(Y) = d_{g,0}(F+X, Y)$. For X fixed, it is quite easy to see that L_X defines a continuous linear map from $\mathcal{L}^{\frac{p}{2}}_{F_0}$ to \mathcal{N} . We need to show that $X \mapsto L_X$ is continuous in a neighbourhood of $X=0$ in order to conclude that L_0 is in fact the derivative of $X \mapsto g(F+X)$ at $X=0$. To summarize, we want to show that

$$X \mapsto (Y \mapsto -\{\{Y, F+X\}, 1-(F+X)^2\})$$

is continuous as a map from $\mathcal{L}^{\frac{p}{2}}_{F_0}$ into the bounded linear maps from $\mathcal{L}^{\frac{p}{2}}_{F_0}$ to \mathcal{N} . This is straight-forward (if slightly tedious) to check.

5. Expanding the expression obtained for $d_{g,0}$, we get

$$-XF(1-F^2) - FX(1-F^2) - (1-F^2)XF - (1-F^2)FX;$$

so we can pair up the first term with the fourth and the second with the third in the desired manner. Moreover, $F \mapsto F$, $F \mapsto 1-F^2$ and $F \mapsto F(1-F^2)$ are easily seen to be continuous as functions from $\mathcal{L}^{\frac{p}{2}}_{F_0}$ to \mathcal{N}_{sa} .

So, with g as above, we get as a consequence of Theorem 3.4.1:

Corollary 3.7.2 Let $\mathcal{G} = \mathcal{L}^{\frac{p}{2}}$, and F_0 a self-adjoint Breuer-Fredholm operator with $1-F_0^2 \in \mathcal{G}$. Consider $r \in \mathbb{R}_+$ such that $r \geq p+6$, and let C be a non-zero constant. The one-form $\alpha_F(X) = \frac{1}{C} \cdot \tau(X|1-F^2|^r)$ is a closed one-form on $F_0 + \mathcal{G}_{F_0}$. \blacksquare

Since $\mathcal{L}^{\frac{p}{2}}_{F_0}$ is a Banach space, and the manifold under consideration is simply $F_0 + \mathcal{L}^{\frac{p}{2}}_{F_0}$, the fact that α is exact follows as in the Poincaré Lemma (see [46], Theorem V.4.1 for details). Namely, the form $\theta : (F_0 + \mathcal{L}^{\frac{p}{2}}_{F_0}) \rightarrow \mathbb{R}$ defined by $\theta(F) = \frac{1}{C} \int_0^1 \tau((F-F_0)|1-F_t^2|^r) dt$, where $F_t = F_0 + t(F-F_0)$ for $t \in [0, 1]$ (that is, $\{F_t\}$ is the straight-line path from F_0 to F), has the property that $d\theta = \alpha$. Using properties of integrals, it follows that α is independent of the path over which it is integrated. The fact that $d\theta = \alpha$ is proven in Appendix D, along with the fact that the integral of α is independent of path.

3.7.2 Unbounded Case Manifold and One-form

Assume given a self-adjoint unbounded operator D_0 for which $(1+D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$. In this case, our manifold is $D_0 + \mathcal{N}_{sa}$. For any $t \geq p+3$, consider the one-form $\alpha_D : X \mapsto \frac{1}{C} \cdot \tau(X(1+D^2)^{-t})$; we want to show this is closed and exact.

In order to use our earlier proof that certain one-forms are closed, we need to show that the function $g(D) = (1+D^2)^{-1}$ satisfies the required properties. We start out by reviewing a few facts about $(1+D^2)^{-1}$.

Lemma 3.7.3 ([14], **Corollary B.8**) Let D_0 be an unbounded self-adjoint operator affiliated with \mathcal{N} and suppose t is some positive real number for which $\tau((1 + D_0^2)^{-t}) < \infty$. Then $\tau((1 + (D_0 + A)^2)^{-t}) < \infty$ for any $A \in \mathcal{N}_{sa}$. \blacktriangleright

Lemma 3.7.4 ([14], **Proposition B.11**) If D_0 is an unbounded self-adjoint operator affiliated with \mathcal{N} and if $t \in \mathbb{R}$, $t \geq 1$ is such that $\tau((1 + D_0^2)^{-t}) < \infty$ then for $A \in \mathcal{N}_{sa}$ with $\|A\| \leq 1$ we have

$$\|(1 + D_0^2)^{-t} - (1 + (D_0 + A)^2)^{-t}\|_1 \leq 2^{\frac{t}{2}} (2\|A\|)^r [f(\|A\|)]^t (t + 1) \|(1 + D_0^2)^{-t}\|_1$$

where $r = \frac{t}{|t|+1}$ and $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + 4}$. \blacktriangleright

Lemma 3.7.5 ([14], **Lemma 2.9**) If D_0 is an unbounded self-adjoint operator on \mathcal{H} , A is bounded self-adjoint and $D = D_0 + A$, then for $x > 0$

$$(x + D^2)^{-1} - (x + D_0^2)^{-1} = -D_0(x + D_0^2)^{-1}A(x + D^2)^{-1} - (x + D_0^2)^{-1}AD(x + D^2)^{-1}. \quad \blacktriangleright$$

We can now show that $g(D) = (1 + D^2)^{-1}$ defined on $D_0 + \mathcal{N}_{sa}$ satisfies the properties laid out on page 85.

1. $(1 + D^2)^{-1}$ is a positive operator of norm at most one for any self-adjoint unbounded operator D (see section 118 of [59] for properties of $(1 + D^2)^{-1}$).

2. By assumption, D_0 is an operator for which $\tau((1 + D_0^2)^{-t}) < \infty$ for $t \geq \frac{p}{2}$. The fact that $\tau((1 + D^2)^{-t}) < \infty$ for all $D \in D_0 + \mathcal{N}_{sa}$ follows from Lemma 3.7.3.

3. Fix $D \in D_0 + \mathcal{N}_{sa}$. To show that $A \mapsto g(D + A)^{-t}$ is continuous at $A = 0$ as a map from \mathcal{N}_{sa} to \mathcal{L}^1 , we use the norm bound from Lemma 3.7.4. On the right hand side of the inequality, the only factors which depend on A are $(2\|A\|)^r$ and $f(\|A\|)^t$. By continuity of f , $f(\|A\|)^t$ has an upper bound for $\|A\| \leq 1$, and $r > 0$ gives us $(2\|A\|)^r \rightarrow 0$ as $\|A\| \rightarrow 0$, concluding the proof.

4. Fix $D \in D_0 + \mathcal{N}_{sa}$ and start out by calculating the directional derivative $d_{g,s}(D, A)$. Using the formula in Lemma 3.7.5,

$$\begin{aligned} d_{g,s}(D, A) &= \frac{(1 + (D + sA)^2)^{-1} - (1 + D^2)^{-1}}{s} \\ &= -D(1 + D^2)^{-1}A(1 + (D + sA)^2)^{-1} - (1 + D^2)^{-1}A(D + sA)(1 + (D + sA)^2)^{-1}. \end{aligned}$$

Lemma 3.5.7 part (v) tells us that, as $s \rightarrow 0$, $(1 + (D + sA)^2)^{-1} \rightarrow (1 + D^2)^{-1}$ and also $(D + sA)(1 + (D + sA)^2)^{-1} \rightarrow D(1 + D^2)^{-1}$. This allows us to conclude

$$\begin{aligned} d_{g,0}(D, A) &= \lim_{s \rightarrow 0} d_{g,s}(D, A) \\ &= -D(1 + D^2)^{-1}A(1 + D^2)^{-1} - (1 + D^2)^{-1}AD(1 + D^2)^{-1}. \end{aligned}$$

We want to show that the above actually gives us the derivative of $A \mapsto (1 + (D + A)^2)^{-1}$ at $A = 0$. Let $L_{A_0}(A) = -(D + A_0)(1 + (D + A_0)^2)^{-1}A(1 + (D + A_0)^2)^{-1} - (1 + (D + A_0)^2)^{-1}A(D + A_0)(1 + (D + A_0)^2)^{-1}$. It is clear that, for $A_0 \in \mathcal{N}_{sa}$ fixed, L_{A_0} is linear and continuous as a map from \mathcal{N}_{sa} to \mathcal{N} . The fact that $A \mapsto L_A$ is continuous on a neighbourhood of $A = 0$ follows from Lemma 3.5.7, allowing us to conclude that L_0 is in fact the derivative at $A = 0$.

5. As just calculated, $d_{g,0}(D, A)$ is equal to $-D(1 + D^2)^{-1}A(1 + D^2)^{-1} - (1 + D^2)^{-1}AD(1 + D^2)^{-1}$. Note that if $g_1(x) = -x(1 + x^2)^{-1}$ and $g_2(x) = (1 + x^2)^{-1}$, then the derivative has the form $g_1(D)Ag_2(D) + g_2(D)Ag_1(D)$, as desired (the continuity of g_1 and g_2 is given by Lemma 3.5.7).

Therefore, g satisfies the desired conditions and, by Theorem 3.4.1, α is a closed form. Again, we state the result for future reference.

Corollary 3.7.6 Let $\mathcal{S} = \mathcal{L}^{\frac{p}{2}}$. If D_0 is a self-adjoint unbounded operator such that $(1 + D_0^2)^{-1} \in \mathcal{S}$, and $t \geq p + 3$, then for any constant C , $\alpha_D(X) = \frac{1}{C} \cdot \tau(X(1 + D_2)^{-t})$ is a closed one-form on $D_0 + \mathcal{N}_{sa}$. \blacksquare

As previously noted (in the bounded case), in this context it means that α is also exact, so integrating α is independent of path.

3.7.3 Spectral Flow Formula for p -summable operators

If D_0 is p -summable, then $(1 + D_0^2)^{-1} \in \mathcal{L}^{\frac{p}{2}}$. Hence the invariant operator ideal that we want to consider is $\mathcal{S} = \mathcal{L}^{\frac{p}{2}}$, with the function $k(x) = x^{\frac{q}{2}}$ for some $q > p$. For the corresponding bounded operators picture, pick any $0 < \varepsilon < 1$ and let $\mathcal{S} = \mathcal{S}^{1-\varepsilon}$ (so \mathcal{S} is still an ideal of summable operators); the function which we need to use to calculate spectral flow for paths in $F_0 + \mathcal{S}_{F_0}$ is then $h(x) = |x|^{\frac{q-3}{2}}$.

We obtain the following as a corollary of Theorem 3.5.4. Note that here we have to make sure that the power used in the integral formula is large enough that we can use the result of Section 3.7.1 about closed one-forms (see Corollary 3.7.2).

Corollary 3.7.7 If $\{F_t\}$ is a C^1 path in $F_0 + \mathcal{S}_{F_0}$ with $\mathcal{S} = \mathcal{L}^q$ for some $q > 0$ then, if $r \geq 2q + 6$,

$$\text{sf}(\{F_t\}) = \frac{1}{C_r} \cdot \int_0^1 \tau \left(\frac{d}{dt}(F_t) |1 - F_t^2|^r \right) dt + \gamma_r(F_1) - \gamma_r(F_0).$$

Here $C_r = \int_{-1}^1 (1 - s^2)^{\frac{r}{2}} ds$ and $\gamma(F) = \frac{1}{C_r} \cdot \int_0^1 \tau \left(\frac{d}{dt}(G_t) |1 - G_t^2|^r \right) dt$, where $\{G_t\}$ is the straight-line path from F to $\text{sign}(F)$. \blacksquare

PROOF The function $h(x) = |x|^r$ is continuous on \mathbb{R} and non-zero on $(0, 1]$. Moreover, for $T \in \mathcal{L}^q$, $|T|^r = |T|^q \cdot |T|^{r-q} \in \mathcal{L}^1$ since $|T|^q \in \mathcal{L}^1$ (note that $r - q \geq q + 6 > 0$). By Corollary 3.7.2, $\alpha_F : X \mapsto \tau(X |1 - F^2|^r)$ is a closed one-form on $F_0 + \mathcal{S}_{F_0}$, so the integral is independent of path. As h satisfies all the conditions of Theorem 3.5.4, desired result follows. \blacksquare

We can similarly use Corollary 3.5.18 to obtain an unbounded formula. If additionally the endpoints are unitarily equivalent we can use analytic continuation to improve our formula. The constant in the unbounded formula will turn out to be related to the gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$; it will in fact be an instance of the beta function, introduced below.

Lemma 3.7.8 (Rudin [62], Theorem 8.20, p. 193) For $x > 0$ and $y > 0$ define the beta function $B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt$. Then $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. \blacksquare

We are now ready to state the formula for unbounded operators.

Theorem 3.7.9 Suppose $\{D_t\}$ is a C^1 path in $D_0 + \mathcal{N}_{sa}$. Define

$$q_0 = \inf \{p \in \mathbb{R}_+ : D_0 \text{ is } p\text{-summable}\}.$$

If there exists u a unitary operator such that $uD_0 - D_0u$ is bounded and $D_1 = uD_0u^*$ then for any $p > q_0$

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_{p/2}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{p}{2}} \right) dt.$$

The constant $\tilde{C}_{p/2}$ is equal to $\int_{-\infty}^{\infty} (1 + x^2)^{-p/2} dx$. Note that the formula works regardless of the path $\{D_t\}$ chosen from D_0 to D_1 .

If the endpoints of $\{D_t\}$ are not unitarily equivalent, the formula is weaker. Fix $p > q_0$. If $s > 2p + 15$, then

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_{\frac{s}{2}}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t)^{-\frac{s}{2}} \right) dt + \beta(D_1) - \beta(D_0).$$

Here $\beta(D) = \frac{1}{C} \int_0^1 \tau \left(\frac{d}{dt}(G_t)(1 - G_t^2)^{\frac{s-3}{2}} \right) dt$ with $\{G_t\}$ the straight line path from $F = D(1 + D^2)^{-\frac{1}{2}}$ to $\text{sign}(F)$. \blacktriangledown

PROOF We will start by proving the formula when the endpoints are not necessarily equivalent, and use analytic continuation in the case when the β terms cancel to obtain the formula with the better choice of exponent. The goal is to use Corollary 3.5.18, so we have to translate the given data to fit the requirements of the theorem. Denote by \mathcal{S} the ideal $\mathcal{L}^{\frac{p}{2}}$. For $s > 2p + 15$ we can find q such that $s > 2q + 15 > 2p + 15$. Then $\frac{p}{q} < 1$, so we can let $\varepsilon = 1 - \frac{p}{q}$; we will need $\mathcal{J} = \mathcal{S}^{1-\varepsilon}$, which is really $\mathcal{L}^{\frac{q}{2}}$.

Let $k(x) = x^{\frac{s}{2}}$ for $x \in \mathbb{R}_+$, which is continuous on \mathbb{R}_+ and non-zero on $(0, 1]$. As defined in Corollary 3.5.18, we then have $h(x) = |x|^{\frac{s-3}{2}}$, and we have to check that h satisfies the conditions of the theorem. If $T \in \mathcal{J}_{sa}$ then $|T|^{\frac{q}{2}} \in \mathcal{L}^1$, so it follows that $h(T) = |T|^{\frac{s-3}{2}} \in \mathcal{L}^1$ (note $\frac{s-3}{2} > q + 6 > \frac{q}{2}$). On the other hand, since $\frac{s-3}{2} \geq q + 6$, by Corollary 3.7.2, $X \mapsto \tau(Xh(1 - F^2))$ is a closed one-form on $F_0 + \mathcal{S}_{F_0}$. This gives us (as a consequence of Corollary 3.5.18) that

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_{s/2}} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-\frac{s}{2}} \right) dt + \beta(D_1) - \beta(D_0),$$

where $\tilde{C}_{s/2} = \int_{-1}^1 (1 - t^2)^{(s-3)/2} dt$.

Let us consider the constant \tilde{C}_m for $m > p + \frac{15}{2}$. So far, we have $\tilde{C}_m = \int_{-1}^1 (1 - t^2)^{(m-\frac{3}{2})} dt$. The change of variables $t = x(1 + x^2)^{-\frac{1}{2}}$ gives us $\tilde{C}_m = \int_{-\infty}^{\infty} (1 + x^2)^{-m} dx$. With the further change of variables $u = \frac{1}{1+x^2}$ we get $\tilde{C}_m = \int_0^1 u^{m-\frac{3}{2}}(1-u)^{-\frac{1}{2}} du$, which is now recognizable as an instance of the beta function. That is, from Lemma 3.7.8,

$$\tilde{C}_m = B \left(m - \frac{1}{2}, \frac{1}{2} \right) = \frac{\Gamma(m - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(m)}$$

(note that $m - \frac{1}{2} > 0$). It is known (see, for example, [62] for this result) that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We will have occasion to consider our constant C_m as a function of m , so we review a few facts about the Γ function. The Γ function can be extended to a function of complex numbers such that it is analytic everywhere in the plane except at the negative integers, where it has simple poles (see [4], Section 18.2, p. 215). Moreover, it is not zero for any complex number. It follows that the function $m \mapsto C_m$ can be extended to a complex function $z \mapsto C(z)$ (i.e. $C(z) = \frac{\Gamma(z - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(z)}$), which is analytic everywhere in the complex plane except at the negative half-integers (where it is not defined). In particular, $C(z)$ is analytic for all z in the half-plane $\Re(z) \geq \frac{p}{2}$.

Now suppose that the endpoints are unitarily equivalent; then, the β correction terms cancel (see Remark 3.5.19). We want to use analytic continuation with $g(D_t) = (1 + D_t^2)^{-1}$ and $N = p + 8$. We have shown above that, for all $m > p + \frac{15}{2}$,

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_m} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-m} \right) dt,$$

so the formula holds for $m \geq N$. The function $g(x) = (1 + x^2)^{-1}$ is continuous from \mathbb{R} to R_+ , and $\|g(D_t)\| \leq 1$ for all t . Moreover, for any $m > q_0$, D_0 is m -summable and so by Lemma 3.7.4,

$$\|(1 + (D_0 + A)^2)^{-m} - (1 + D_0^2)^{-m}\|_1 \leq 2^{\frac{1}{|m|}} (2 \cdot \|A\|)^r (f(\|A\|)^m (m + 1)) \|(1 + D_0^2)^{-m}\|_1,$$

where $r = \frac{m}{|m|+1}$ and $f(a) = 1 + \frac{1}{2} \cdot a^2 + \frac{1}{2} \cdot a \cdot \sqrt{a^2 + 4}$. From this, it easily follows that $g(D_t)^m$ is trace norm continuous for any $m > q_0$. Therefore, all the conditions of Corollary 3.6.7 are satisfied, and we can conclude that

$$\text{sf}(\{D_t\}) = \frac{1}{\tilde{C}_m} \cdot \int_0^1 \tau \left(\frac{d}{dt}(D_t)(1 + D_t^2)^{-m} \right) dt$$

for any $m > q_0$. ■

Remark 3.7.10 The first part of the above theorem was our final goal for this chapter – the p -summable formula from Theorem 3.0.8, proven using the steps outlined in Section 3.3. •

Chapter 4

Spectral Flow as an Intersection Number

In section 7 of [2], Atiyah, Patodi and Singer describe how one can view spectral flow as an intersection number. Given a path $\{F_t\}$ of self-adjoint Fredholm operators, consider the graph of the spectrum of $\{F_t\}$, that is, the set

$$\mathcal{S} = \{(t, \lambda) : \lambda \in \sigma(F_t)\} \subset [0, 1] \times \mathbb{R}.$$

The spectral flow is then given by the intersection number of \mathcal{S} with the line $\lambda = 0$ if both F_0 and F_1 are invertible, or with the line $\lambda = -\varepsilon$ for some suitably chosen $\varepsilon > 0$. Here is a visualization of this definition:

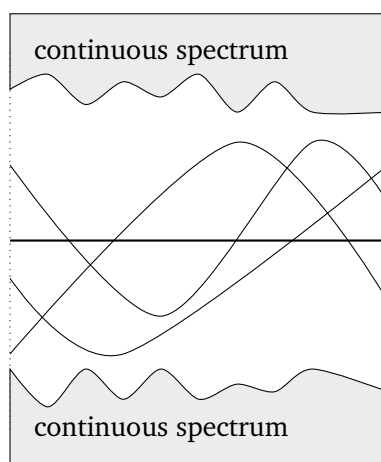


Figure 4.1: Spectral flow as the intersection of the spectrum with $\lambda = 0$ (in the above diagram, the spectral flow is 1)

In order to make sense of such an intersection number, the spectrum of the operators involved has to be discrete near zero. In Section 1.4 (the examples section), we have two diagrams showing the graph of the spectrum of two different paths (self-adjoint Fredholm operators, but in the unbounded setting); in the Type I case, the intersection number is explained in the accompanying text, but the spectral graph in the type II case does not allow for such an approach.

However, in [38], Getzler found a different way of looking at this idea (applied in a different setting); namely, an intersection with, say, $\lambda = 0$ signifies one eigenvalue being zero, that is, an operator on the path whose kernel has dimension one. This led to a presentation of spectral flow, in the context of bounded perturbations of a fixed unbounded operator, as an intersection between a path of operators and a manifold consisting of operators whose kernel has dimension one. In order to get this to work, one needs the unbounded operator being perturbed to have compact resolvents, as will be seen later. It was hoped that the technique could be adapted to the type II case; however, we did not find the proper approach for such an adaptation. We present Getzler's idea in the next section, explain the connection with integral formulas for spectral flow,

and then discuss the problems with the type II case. We do point out that some of the steps used by Getzler for the proof of the θ -summable formula (Theorem 2.6 of [38], quoted in this dissertation as Theorem 4.2.18) are insufficiently justified in the original paper, and occasionally based on incorrect assumptions; we discuss these at the appropriate time, before filling in the necessary details.

4.1 Type I Case

Fix \hat{T}_0 a self-adjoint unbounded Fredholm operator on \mathcal{H} , and consider bounded self-adjoint perturbations of \hat{T}_0 ; that is, the set

$$\Phi = \{\hat{T}_0 + A \mid A \in \mathcal{B}(\mathcal{H})_{sa}\}.$$

Note that Φ is a Banach manifold modelled on the real manifold of self-adjoint operators in $\mathcal{B}(\mathcal{H})$: at every $T_0 \in \Phi$ we can take the neighbourhood $\mathcal{U}_{T_0} = \{T_0 + A : A \in \mathcal{B}(\mathcal{H})_{sa}, \|A\| < 1\}$ and define the map $\varphi_{T_0} : (T_0 + A) \mapsto A$ from \mathcal{U}_{T_0} to the open ball of radius one centered at zero in $\mathcal{B}(\mathcal{H})_{sa}$. It is then easy to check that the collection $\{(U_T, \varphi_T) : T \in \Phi\}$ defines an atlas on Φ (see [46], Section II.1 for the definition of a smooth manifold modelled on a Banach space), with $\varphi_T \varphi_S^{-1}$ being a C^∞ isomorphism for every $T, S \in \Phi$. Recall that a chart (U, φ) is compatible with this atlas if $\varphi_T \varphi^{-1}$ is a C^∞ isomorphism. We also note that Φ is a metric space; since the operators on Φ are bounded perturbations of a fixed unbounded operator, the difference between any two operators in Φ is a bounded operator, allowing us to define a distance between two elements in Φ via the operator norm. The purpose of the following is to associate a number (the spectral flow) to each path in Φ with *invertible endpoints*. To this end, we will use the homology and cohomology groups of the manifold Φ . There are two cohomologies in which we are interested: singular cohomology and de Rham cohomology.

In order for the following approach to work, we need to add the additional restriction that \hat{T}_0 has compact resolvents; this will ensure that the bounded perturbations of \hat{T}_0 are also Fredholm (which is not true for a general unbounded Fredholm operator). To see this, we recall the following result.

Lemma 4.1.1 ([14], Lemma 6 of Appendix B) If D_0 is an unbounded self-adjoint operator, and A is bounded and self-adjoint, then

$$(1 + (D_0 + A)^2)^{-1} \leq f(\|A\|) \cdot (1 + D_0^2)^{-1},$$

where $f(a) = 1 + \frac{1}{2} \cdot (a^2 + a \cdot \sqrt{a^2 + 4})$. ☛

As shown in Remark 1.2.9, the fact that $(1 + D^2)^{-1}$ is compact is equivalent to D having compact resolvents. In our particular case, the fact that \hat{T}_0 has compact resolvents implies that $(1 + \hat{T}_0^2)^{-1}$ is compact. The inequality from Lemma 4.1.1 then shows that for any A bounded and self-adjoint $(1 + (\hat{T}_0 + A)^2)^{-1}$ is also compact (since its s -numbers must go to zero by Theorem 1.1.5, part (iv), which characterizes compact operators). This in turn means that $\hat{T}_0 + A$ has compact resolvents, which implies that $\hat{T}_0 + A$ is Fredholm. Note that, if \hat{T}_0 is part of a spectral triple, then the fact that \hat{T}_0 has compact resolvents is part of the definition of spectral triples, so this condition still allows a wide range of examples in which we are interested.

The fact that $\dim \ker T$ is finite for any $T \in \Phi$ allows us to define a stratification of Φ by

$$\Phi_k = \{T \in \Phi \mid \dim \ker(T) = k\}.$$

It will be shown in the following that each Φ_k is a submanifold of Φ . The submanifold Φ_1 will prove to be of particular importance; we will show that a path with endpoints in Φ_0 can be modified so it intersects Φ_1 at finitely many points and is in Φ_0 otherwise, allowing the definition of an intersection number between the path and Φ_1 . This intersection number will be shown to be the spectral flow.

Some notation and technical lemmas

A major role in the following presentation will be played by the map Ψ introduced in the lemma below; even though the result is very easy to prove, we need to refer to it often, so we record it here. The map Ψ was used by Getzler to show that Φ_k is a submanifold of Φ (see Theorem 2.1 of [38]).

Lemma 4.1.2 Suppose that $T \in \Phi$ is an operator which can be written as $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with respect to some decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$ of \mathcal{H} , in such a way that P has finite rank and D is invertible. Define

$$\Psi(T) = A - BD^{-1}B^*.$$

Then $\dim \ker T = \dim \ker \Psi(T)$. ☛

PROOF The assumption that P has finite rank ensures that A and B are bounded, eliminating any possible problems in evaluating Ψ . It is easy to check that $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if and only if $v = -D^{-1}B^*u$ and $(A - BD^{-1}B^*)u = 0$. Since v is determined by u , and $u \in \ker(A - BD^{-1}B^*)$, it follows immediately that $\dim \ker \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \dim \ker(A - BD^{-1}B^*)$. ■

The map Ψ can only be defined locally, on neighbourhoods on which it makes sense. The next order of business is to describe and name such neighbourhoods.

Definition 4.1.3 For any $T \in \Phi$, let P be the (finite-dimensional) projection onto $\ker T$, and write $T = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix}$ with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$. A **Ruget neighbourhood** of T is a neighbourhood \mathcal{U} of T (in Φ) such that, for all operators $S \in \mathcal{U}$, $P^\perp S P^\perp$ is invertible as an operator on $P^\perp\mathcal{H}$. If given a Ruget neighbourhood \mathcal{U} , we will say that \mathcal{U} is **centered at T** if we want to identify the operator whose kernel is used for the decomposition of \mathcal{H} . ☛

In other words, if \mathcal{U} is a Ruget neighbourhood of T , every $S \in \mathcal{U}$ can be written as $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with D invertible, so it follows from Lemma 4.1.2 that

$$\dim \ker S = \dim \ker(A - BD^{-1}B^*) \leq \text{rank } P = \dim \ker T.$$

In particular, if T is invertible, so is S . Such a neighbourhood was used by Ruget to prove Lemme 1 in Deuxième Partie of [63]; the idea was then built upon by Getzler in his paper [38]. Our next goal is to show that every $T_0 \in \Phi$ has a Ruget neighbourhood.

Lemma 4.1.4 For every $T_0 \in \Phi$ there exists an $\varepsilon > 0$ such that the ball of radius ε centered at T_0 is a Ruget ε -ball. \blacktriangleright

PROOF Suppose $T_0 \in \Phi_n$, and denote by P the projection onto the kernel of T_0 ; by definition of Φ_n , P has dimension n . Decompose \mathcal{H} as $P\mathcal{H} \oplus P^\perp\mathcal{H}$; then, with respect to this decomposition, T_0 looks like $\begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix}$, with C_0 invertible (as an unbounded operator on $P^\perp\mathcal{H}$). By Theorem 1.1.31, there exists $\varepsilon > 0$ (e.g. can take $\varepsilon = \|C_0^{-1}\|^{-1}$) such that, if E is a bounded operator on $P^\perp\mathcal{H}$ with $\|E\| < \varepsilon$ then $C_0 + E$ is also invertible. Now, if $T \in \Phi$ is such that $\|T - T_0\| < \varepsilon$ then, writing T as $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$ we have that $T - T_0$ is bounded and $\|D - C_0\| < \varepsilon$, whence D is invertible. Therefore the ball of radius ε centered at T_0 is a Ruget ε -ball. \blacksquare

Later proofs will rely heavily on gymnastics meant to reduce each question to Ruget neighbourhoods. As such, we also identify a **Ruget path**, which is a path $\rho(t)$ such that $\rho(0), \rho(1)$ are invertible operators and ρ is completely contained in a Ruget neighbourhood centered at some operator $T \in \Phi$. When we apply Ψ to a Ruget path we get a (continuous) path of finite-rank matrices, as indicated by the lemma below.

Lemma 4.1.5 Let \mathcal{U} be a Ruget neighbourhood centered at $T_0 = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix} \in \Phi_n$. Then the map from \mathcal{U} to $(\mathbb{M}_n)_{sa}$ defined by $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mapsto BD^{-1}B^*$ is continuous. It follows that if $\left\{ \begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix} \right\}$ is a Ruget path, then $t \mapsto B_t D_t^{-1} B_t^*$ is continuous. \blacktriangleright

PROOF From the definition of Φ , the difference between any two operators in $\mathcal{U} \subset \Phi$ is a bounded operator, which means B_t and B_t^* are bounded, as is $D - C_0$. The key fact we need is that $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mapsto D^{-1}$ is continuous on \mathcal{U} , which follows immediately from Theorem 1.1.31.

Using an $\varepsilon/3$ argument, this is sufficient to show that $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mapsto BD^{-1}B^*$ is continuous.

Finally, if $\left\{ \begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix} \right\}$ is a Ruget path, then $t \mapsto B_t D_t^{-1} B_t^*$ is a composition of continuous functions, so is also continuous. \blacksquare

4.1.1 Intersection number between a path in Φ and the manifold Φ_1

We use the map Ψ defined in the previous section to show that Φ_k is a submanifold of Φ . To recap, if \mathcal{U} is a Ruget neighbourhood of T_0 , with respect to the decomposition $\mathcal{H} = (\ker T_0) \oplus (\ker T_0)^\perp$ we can write all the operators in \mathcal{U} as $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with D invertible. Then $\Psi_{T_0}(T) = A - BD^{-1}B^*$. Henceforth, we use the subscript T_0 with Ψ whenever possible, in order to remind us of which operator is providing the decomposition, and to differentiate between different incarnations of Ψ acting on the same operator.

Proposition 4.1.6 (Theorem 2.1 of [38]) For $n \geq 0$, Φ_n is a submanifold of Φ with codimension $n(n+1)/2$. \blacktriangledown

PROOF Fix $T_0 \in \Phi_n$ and let \mathcal{U} be a Ruget ε -ball centered at T_0 (such a \mathcal{U} exists by Lemma 4.1.4). Then, for all $T \in \mathcal{U}$, define the map $\Psi_{T_0} : \mathcal{U} \rightarrow (\mathbb{M}_n)_{sa}$ by $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \mapsto A - BD^{-1}B^*$ as already described.

Recall (Lemma 4.1.2) that $\dim \ker T = \dim \ker(A - BD^{-1}B^*)$, so (since $A - BD^{-1}B^* \in (\mathbb{M}_n)_{sa}$) $\dim \ker T \leq n$. In fact, $T \in \Phi_n$ (i.e. $\dim \ker T = n$) if and only if $\Psi_{T_0}(T)$, which is equal to $(A - BD^{-1}B^*)$, is the zero matrix.

Define φ on \mathcal{U} by $\varphi \left(\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \right) = (B, D, A - BD^{-1}B^*)$. It is clear that the inverse of φ is $(B, D, M) \mapsto \begin{bmatrix} M + BD^{-1}B^* & B \\ B^* & D \end{bmatrix}$, and φ and φ^{-1} are both continuous (see Lemma 4.1.5).

We claim in fact that (\mathcal{U}, φ) is a chart at T_0 compatible with the manifold structure on Φ . Recall that, when we introduced Φ and its manifold structure (at the beginning of Section 4.1), at each $T \in \Phi$ we defined a chart $(\mathcal{U}_T, \varphi_T)$; we need to check that $\varphi_T \varphi^{-1}$ is C^∞ as a map between the open sets $\varphi(\mathcal{U} \cap \mathcal{U}_T)$ and $\varphi_T(\mathcal{U} \cap \mathcal{U}_T)$. The crux of the proof is that, in a neighbourhood of an invertible operator X_0 (consisting of bounded perturbations $X_0 + A$ for $\|A\|$ sufficiently small), the map $X \mapsto X^{-1}$ is C^1 , with derivative at X_0 given by $X \mapsto X_0^{-1}XX_0^{-1}$; the details are tedious, and we omit them.

As $\varphi|_{\mathcal{U} \cap \Phi_n} = \{(B, D, 0)\}$, and $\varphi|_{\mathcal{U} \cap \Phi_n}$ is a bijection, it follows that Φ_n is a submanifold of Φ . Moreover, since $(\mathbb{M}_n)_{sa}$ has dimension $\frac{n(n+1)}{2}$, this proves the codimension part of the result. \blacksquare

Lemma 4.1.7 For each integer $n \geq 1$, $\bar{\Phi}_n = \bigcup_{i=n}^{\infty} \Phi_i$. \blacktriangledown

PROOF Since $\Phi_n \subset \bar{\Phi}_n$, it remains to show that $\Phi_i \subset \bar{\Phi}_n$ for each $i > n$ but $\Phi_i \cap \bar{\Phi}_n = \emptyset$ for each $i < n$.

Suppose $i > n$ and pick any $T \in \Phi_i$. Then $\dim \ker T = i = n + k$ for some $k > 0$. Write $\ker T = \mathcal{A} \oplus \mathcal{B}$ for some subspaces \mathcal{A} and \mathcal{B} of dimension n and k , respectively. With respect to the decomposition $\mathcal{H} = \mathcal{A} \oplus \mathcal{B} \oplus (\ker T)^\perp$, T looks like $0 \oplus 0 \oplus D$, with D invertible. Then define for each $j \geq 1$ the operator $T_j = 0 \oplus \frac{1}{j} \cdot I_k \oplus D$; clearly, $\{T_j\}_{j=1}^{\infty}$ is a sequence in Φ_n which converges to T . Therefore, $T \in \bar{\Phi}_n$. Hence, since T was an arbitrary operator in some Φ_i with $i > n$, we can conclude that $\Phi_i \subset \bar{\Phi}_n$ for any $i > n$.

On the other hand, suppose $i < n$ and pick any $S \in \Phi_i$. Lemma 4.1.4 gives us a Ruget neighbourhood \mathcal{U} of S , which means (by the comment following Definition 4.1.3) that for all $R \in \mathcal{U}$ we have $\dim \ker R \leq i < n$, and so $\mathcal{U} \cap \Phi_n = \emptyset$. Therefore, $S \notin \bar{\Phi}_n$, and so $\Phi_i \cap \bar{\Phi}_n = \emptyset$ for any $i < n$. \blacksquare

Remark 4.1.8 From the above, $\bar{\Phi}_0 = \Phi$, but we can say something even stronger. Given any $T \in \Phi$ and $\varepsilon > 0$, there exists a path ρ such that $\rho(0) = T$, $\rho(t) \in \Phi_0$ for $t \neq 0$, and $\|\rho(t) - T\| < \varepsilon$.

To see this, let P be the projection onto the kernel of T . Then $\{T + t \cdot \frac{\varepsilon}{2}P\}$ is a path from T to $T + \frac{\varepsilon}{2}P$, and $T + t \cdot \frac{\varepsilon}{2}P$ is invertible for all $t > 0$. Recall that this is the construction used in the proof of Lemma 2.3.1. •

We are mainly interested in Φ_0 (which consists of the invertible operators in Φ), and Φ_1 (which consists of operators of Φ for which 0 is an eigenvalue of multiplicity 1). The goal is to show that $H^1(\Phi, \Phi_0) \cong H^0(\Phi_1)$; as a consequence, whenever we have a path in Φ with endpoints in Φ_0 , we can define an intersection number of the path with Φ_1 . The proof works because $\Phi_0 \cup \Phi_1$ has codimension greater than 1 in Φ and Φ_1 has codimension 1 in $\Phi_0 \cup \Phi_1$. Getzler's proof uses results from [63] to achieve this goal (the discussion leading to Theorem 2.3 of [63]). In order to get to the heart of the matter, and hopefully glimpse something which would be helpful to deal with the type II case, we unravel the meaning of these results to our specific circumstances and prove things directly.

We take a quick detour to review the notation for singular homology and cohomology. For X a topological space and A a subspace of X , we will use $C_n(X, A) := C_n(X)/C_n(A)$ for the n -chains of X relative to A . The relative cycles are then the chains of X whose boundary is a chain of A , and the relative boundaries are the chains in X which are homologous to chains in A ; the relative homology $H_n(X, A)$ can be thought of as the quotient of relative cycles by relative boundaries. The relative cochains $C^n(X, A)$ are the cochains in $C^n(X)$ which are zero on chains in A . We will mostly use $n = 1$, for which we have $H^1(X, A) \cong \text{Hom}(H_1(X, A), \mathbb{Z})$, allowing us to define cochains by describing their action on chains whose boundary is in A . An in-depth discussion of singular homology and cohomology (including the relative case) can be found in [40]. In order to lighten the notation, if ρ is a path which also happens to be a relative cycle and $v \in H^1(X, A)$, we will often write $v(\rho)$ instead of $v([\rho])$, with the understanding that $v(\rho)$ means that we evaluate v on the equivalence class of ρ in $H_1(X, A)$.

A recurring idea of the arguments will be to split up the path in such a way that we can apply our map Ψ to get a path of finite matrices, which we can then examine / manipulate more leisurely. The following lemma will allow us to get back to our original picture from the result of applying our map Ψ .

Lemma 4.1.9 Suppose $\left\{ \begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix} : t \in [0, 1] \right\}$ is a Ruget path (that is, the endpoints are invertible, and D_t is invertible for all t). Let $M_t = A_t - B_t D_t^{-1} B_t^*$; then by Lemma 4.1.5, $\{M_t\}$ is a path in $(\mathbb{M}_n)_{sa}$ for some positive integer n . Suppose $H_s(t)$ is a homotopy (with endpoints fixed) in $(\mathbb{M}_n)_{sa}$ from $\{M_t\}$ to some other path. Then $G_s(t) = \left\{ \begin{bmatrix} H_s(t) + B_t D_t^{-1} B_t^* & B_t \\ B_t^* & D_t \end{bmatrix} \right\}$ is a homotopy (with endpoints fixed) in $\cup_{k \leq n} \Phi_k$. •

PROOF We know $(s, t) \mapsto H_s(t)$ is continuous since $H_s(t)$ is a homotopy, $t \mapsto B_t D_t^{-1} B_t^*$ is continuous by Lemma 4.1.5, and finally $t \mapsto B_t$, $t \mapsto B_t^*$, $t \mapsto D_t$ are all continuous (with respect to the appropriate norms) because $\left\{ \begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix} \right\}_{t \in [0, 1]}$ is. It follows that $G_s(t)$ is continuous.

Additionally, $\dim \ker G_s(t) = \dim \ker H_s(t)$ by Lemma 4.1.2; this shows, since $H_s(t) \in (\mathbb{M}_n)_{sa}$, that $\dim \ker G_s(t) \leq n$, and so $G_s(t) \in \cup_{k \leq n} \Phi_k$ for all s, t . ■

The next goal is to show that $H^1(\Phi, \Phi_0) \cong H^1(\Phi_1 \cup \Phi_0, \Phi_0)$ (see Lemma 4.1.12), in preparation for which we need the following two technical results.

Lemma 4.1.10 Suppose $\{M_t\}$ is a path in $(\mathbb{M}_n)_{sa}$ such that M_0, M_1 are invertible. Then there exists a homotopy $H_s(t)$ in $(\mathbb{M}_n)_{sa}$ from $\{M_t\}$ to a path $H_1(t)$ such that $\dim \ker H_1(t) \leq 1$ for all $t \in [0, 1]$. \blacktriangleright

PROOF By Proposition 5.5 of [47], the path components of the self-adjoint invertible operators on a finite dimensional Hilbert space are labelled by the rank of the projection onto the positive eigenspace (which is necessarily an integer between zero and n , inclusive).

If $\{M_t\}$ and $\{R_t\}$ are two paths with the same endpoints, then $H_s(t) = (1-s)M_t + sR_t$ is a homotopy between them. Therefore, it is sufficient to find a path $\{R_t\}$ from M_0 to M_1 such that $\dim \ker R_t \leq 1$ for all t . Let k be the rank of the projection onto the positive eigenspace of M_0 ; by the aforementioned Proposition 5.5 of [47], there is a path of invertibles from M_0 to $2P - 1$ for any projection P of rank k . Similarly, let l be the rank of the projection onto the positive eigenspace of M_1 . If Q is any projection of rank l , then $2Q - 1$ can be connected via a path of invertible operators to M_1 , so it is sufficient to construct a path from $2P - 1$ to $2Q - 1$ that satisfies the requirements. Assume without loss of generality that $l \geq k$, and choose Q such that $P \leq Q$.

If $l - k = 1$, we can write $Q = P + R$ with P and R mutually orthogonal and R a projection of dimension one. With respect to the decomposition $\mathcal{H} = R\mathcal{H} \oplus P\mathcal{H} \oplus Q^\perp\mathcal{H}$, $2P - 1 = -1 \oplus 1 \oplus -1$, while $2Q - 1 = 1 \oplus 1 \oplus -1$. It is easy to see that $\{(2t - 1) \oplus 1 \oplus -1\}$ is then a path from $2P - 1$ to $2Q - 1$, and the operators along the path are invertible except when $t = \frac{1}{2}$, when the dimension of the kernel is one. On the other hand, if $l - k > 1$, then we can write $Q = P + R_1 + R_2 + \dots + R_{l-k}$, a sum of mutually orthogonal projections with each R_i one-dimensional, and use concatenation of the $l - k$ paths from $2(P + R_1 + \dots + R_i) - 1$ to $2(P + R_1 + \dots + R_{i+1}) - 1$ (for $i \geq 1$) to get a path from $2P - 1$ to $2Q - 1$.

Thus we can construct a path $\{R_t\}$ from M_0 to M_1 by concatenating a path from M_0 to $2P - 1$ which consists of invertible operators with a path from $2P - 1$ to $2Q - 1$ which consists of operators with kernel dimension at most one, and with a path from $2Q - 1$ to M_1 . Since $\{R_t\}$ and $\{M_t\}$ are homotopic, this concludes the proof. \blacksquare

Lemma 4.1.11 Suppose $\rho : [0, 1] \rightarrow \Phi$ is a path with $\rho(0), \rho(1) \in \Phi_0$. Then there exists $\xi : [0, 1] \rightarrow \Phi_0 \cup \Phi_1$ such that ρ and ξ are homotopic (with endpoints fixed). \blacktriangleright

PROOF First show that ρ is homotopic to a concatenation of Ruget paths. At every $\rho(t)$ there is a Ruget ε -ball (by Lemma 4.1.4), and by compactness we can cover ρ by finitely many of these balls. Thus we can find indices r_1, \dots, r_k such that $\rho|_{[r_i, r_{i+1}]}$ is completely contained in an ε -ball. Consider r_1 ; there exists a $\delta > 0$ such that the ball centered at $\rho(r_1)$ of radius δ is contained in the intersection of the Ruget ε -balls which contain $\rho(r_1)$. Construct a path $\tilde{\rho}$ such that $\tilde{\rho}(0) = \rho(r_1)$, all operators along this path are within δ of $\rho(r_1)$, and $\tilde{\rho}(t)$ is invertible for all $t \neq 0$ (for details of how this is done, see Remark 4.1.8). Then ρ is homotopic to $(\rho|_{[0, r_1]} * \tilde{\rho}) * (-\tilde{\rho} * \rho|_{[r_1, 1]})$ (this is shown as in Lemma 2.3.1), and $(\rho|_{[0, r_1]} * \tilde{\rho})$ is a Ruget path, while $-\tilde{\rho} * \rho|_{[r_1, 1]}$ has invertible endpoints. Continue in the same manner for each $i = 2, \dots, k$ to show that ρ is homotopic to a concatenation of Ruget paths.

Next, we show that every Ruget path is homotopic to a path in $\Phi_0 \cup \Phi_1$. Since our path is contained in a Ruget neighbourhood, we can write each matrix along the path as $\begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix}$ with respect to some appropriate decomposition such that D_t is invertible; let $M_t = A_t - B_t D_t^{-1} B_t^*$. Apply Lemma 4.1.10 to $\{M_t\}$ to get a homotopy $H_s(t)$ from $\{M_t\}$ to a path $H_1(t)$ such that $\dim \ker H_1(t) \leq 1$ for all t . Define $G_s(t) = \begin{bmatrix} H_s(t) + B_t D_t^{-1} B_t^* & B_t \\ B_t^* & D_t \end{bmatrix}$; then $G_s(t)$ is a homotopy (Lemma 4.1.9), and $\dim \ker G_s(t) = \dim \ker H_s(t) \leq 1$ (Lemma 4.1.2), concluding the proof. ■

Lemma 4.1.12 $H^1(\Phi, \Phi_0) \cong H^1(\Phi_0 \cup \Phi_1, \Phi_0)$. ■

PROOF From the long exact sequence for $\Phi_0 \subset \Phi_0 \cup \Phi_1 \subset \Phi$, we already have a map $i^* : H^1(\Phi, \Phi_0) \rightarrow H^1(\Phi_0 \cup \Phi_1, \Phi_0)$. Namely, the inclusion $i : C_1(\Phi_0 \cup \Phi_1, \Phi_0) \rightarrow C_1(\Phi, \Phi_0)$ induces a map $i^* : C^1(\Phi, \Phi_0) \rightarrow C^1(\Phi_0 \cup \Phi_1, \Phi_0)$ defined by $[i^*(\mu)](\rho) = \mu(i\rho)$. This map is well behaved with respect to the boundary map, so induces in turn a map $H^1(\Phi, \Phi_0) \rightarrow H^1(\Phi_0 \cup \Phi_1, \Phi_0)$, which we will of course also call i^* . This map between cohomology groups exists whenever we have an inclusion of one space into another. The claim in this particular case, however, is that i^* is an isomorphism from $H^1(\Phi, \Phi_0)$ to $H^1(\Phi_0 \cup \Phi_1, \Phi_0)$. The key to this argument is Lemma 4.1.11, which assures us that we can find a homotopy from a path in Φ with endpoints in Φ_0 to a path in $\Phi_0 \cup \Phi_1$ with endpoints in Φ_0 , allowing us to reverse the action of i^* and exhibit an inverse.

Since we are working with $H^1(\Phi, \Phi_0) \cong \text{Hom}(H_1(\Phi, \Phi_0), \mathbb{Z})$, it is sufficient to describe the action of the inverse on paths $\rho : [0, 1] \rightarrow \Phi$ with endpoints in Φ_0 (i.e. ρ is a relative cycle). By Lemma 4.1.11, we know that we can find a homotopic path $\tilde{\rho} : [0, 1] \rightarrow \Phi_0 \cup \Phi_1$, and since the endpoints do not change along the homotopy, $\tilde{\rho}(0), \tilde{\rho}(1) \in \Phi_0$, so $\tilde{\rho}$ is a cycle of $\Phi_0 \cup \Phi_1$ relative to Φ_0 . What we would like to do is define

$$k : H^1(\Phi_0 \cup \Phi_1, \Phi_0) \rightarrow H^1(\Phi, \Phi_0) \text{ such that } [k(\mu)](\rho) = \mu(\tilde{\rho})$$

whenever ρ is a path in Φ with endpoints in Φ_0 . We need to check that, if ρ is equivalent to zero in $H_1(\Phi, \Phi_0)$ then $[k(\mu)](\rho) = 0$. However, as an element of $H^1(\Phi_0 \cup \Phi_1, \Phi_0)$, μ evaluates to zero on chains contained in Φ_0 , so it is easy to see that if ρ is a relative boundary then $k(\mu)(\rho) = 0$. It follows that we can extend the definition of $k(\mu)$ to all of $H_1(\Phi_0 \cup \Phi_1, \Phi_0)$ by linearity. Using the definition of k and the properties of cohomology classes, it is not hard to check that $\mu \mapsto k(\mu)$ is a well-defined homomorphism at the level of cohomology.

We now check that k is an inverse of i^* . What is really making this work is that homotopic paths are homologous. We have

$$[ki^*\mu](\rho) = k\mu(i\rho) = \mu(\widetilde{i\rho}),$$

however, since ρ is already in $\Phi_0 \cup \Phi_1$ we do not need to modify it in any way, so we can choose $\widetilde{i\rho} = \rho$. Hence, $(ki^*)\mu = \mu$ for $\mu \in H^1(\Phi, \Phi_0)$. Conversely, for $\nu \in H^1(\Phi_0 \cup \Phi_1, \Phi_0)$,

$$[i^*k\nu](\xi) = i^*\nu(\tilde{\xi}) = \nu(i\tilde{\xi}),$$

which is equal to $\nu(\xi)$ since ξ and $\tilde{\xi}$ are homotopic. So $(i^*k)\nu = \nu$ as well, concluding the proof that k and i^* are inverses of each other. ■

Our next goal is to show that $H^1(\Phi_0 \cup \Phi_1, \Phi_0)$ is isomorphic to $H^0(\Phi_0)$. In order to be able to define the intersection number of a path with Φ_1 , we need to be able to homotope a path so it intersects Φ_1 finitely many times, and to assign an orientation to each intersection number in a consistent manner. This is where Ψ will show its mettle. The key ingredient is that, when applied to a Ruget neighbourhood contained in $\Phi_0 \cup \Phi_1$, Ψ returns a real number; the sign of this real number will give us our desired orientation.

Remark 4.1.13 Our goal is to use the map Ψ to calculate an intersection number, and particularly to decide on the sign assigned to each intersection point. However, the definition of Ψ depends very much on the decomposition of \mathcal{H} used before Ψ is applied; for example, we can apply Ψ to two different decompositions of the same operator, and the result need not even have the same sign. This is presumably not a surprise, but nonetheless here's a simple example to illustrate this

point. Suppose we have an operator which looks like $\begin{bmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & D \end{bmatrix}$ (where D is invertible)

with respect to some decomposition of \mathcal{H} . Using the top left corner as our 'a' entry, when we apply Ψ we get

$$(-1) - \begin{bmatrix} 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & D \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = -2 < 0.$$

Next, a simple change of basis allows us to write the same operator as $\begin{bmatrix} 4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & D \end{bmatrix}$; applying Ψ to this decomposition (again with the top left corner as our 'a' value) gives us

$$4 - \begin{bmatrix} 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & D \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 8 > 0.$$

So in vacuum Ψ does not offer a way of assigning a positive or negative value to an invertible operator, even if we can find a decomposition in which the 'bottom right corner' is invertible. However, Ψ becomes relevant when we apply it to a neighbourhood centered at $T = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix} \in \Phi_1$.

The idea is that, since zero is an isolated eigenvalue of T of dimension 1, we can choose a small interval around zero such that, if S is an invertible operator which is close to T , then S has exactly one eigenvalue λ in this interval (see [44], the discussion in Section V.3 and Theorem V.4.10); what we want to show is that $\lambda > 0$ if and only if $\Psi(S) > 0$. This is made precise in the Lemma 4.1.14 below. •

Lemma 4.1.14 Let $T \in \Phi_1$. There exists an $\varepsilon > 0$ and an interval $[-r, r]$ such that, if $\|S - T\| < \varepsilon$ and S is invertible, then S has exactly one eigenvalue λ in $[-r, r]$ and $\lambda > 0$ if and only if $\Psi(S) > 0$. Moreover, the ball centered at T of radius ε is a Ruget ε -ball. We will call this type of neighbourhood a **tight Ruget ε -ball**. ☛

PROOF Suppose $T = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix}$ with respect to the decomposition $\ker T \oplus (\ker T)^\perp$. Let $\delta_1 > 0$ be such that, if $\|D - C_0\| < \delta_1$, then $\|D^{-1}\| < 2\|C_0^{-1}\|$ (see Theorem 1.1.31). We know T has an eigenvalue at 0, but suppose T has no other eigenvalues in $[-3r, 3r]$ for some $0 < r < \delta_1/2$

(such an r exists since T is self-adjoint Fredholm, so its spectrum is discrete near 0). In that case, C_0 also has no eigenvalues in $[-3r, 3r]$. Next, choose δ_2 such that if $\|S - T\| < \delta_2$ then S has an eigenvalue in $[-r, r]$ but no other eigenvalues in $[-2r, 2r]$, and if $\|D - C_0\| < \delta_2$ then D has no eigenvalues in $[-2r, 2r]$ – that is, the distance between $\sigma(S)$ and $\sigma(T)$ is less than r and similarly for $\sigma(D)$ and $\sigma(C_0)$ (one can use Theorem 2.4.4 to find δ_2 ; see [44], the discussion in Section V.3 and Theorem V.4.10 for more details).

Let $\varepsilon = \min\{\delta_1/2, \delta_2, \frac{1}{2} \cdot \|C_0^{-1}\|^{-1}\}$, and suppose $S \in \Phi_0$ is such that $\|S - T\| < \varepsilon$. Let λ be the eigenvalue of S in $[-r, r]$; then $0 < |\lambda| < r$. Write $S = \begin{bmatrix} a & B \\ B^* & D \end{bmatrix}$ with respect to the decomposition $\ker T \oplus (\ker T)^\perp$; we use a small letter for a to remind ourselves that, since $\ker T$ is a one-dimensional space, $a \in \mathbb{R}$. The claim is that $\lambda > 0$ if and only if $\Psi(S) = a - BD^{-1}B^* > 0$.

Now $S - \lambda = \begin{bmatrix} a - \lambda & B \\ B^* & D - \lambda \end{bmatrix} \in \Phi_1$. Moreover, λ is not an eigenvalue of D , since $\lambda \in [-r, r]$ and D has no eigenvalues in $[-2r, 2r]$; hence, $D - \lambda$ is invertible. By Lemma 4.1.2, $S - \lambda \in \Phi_1$ is equivalent to $\Psi_T(S - \lambda) = 0$; thus we have $(a - \lambda) - B(D - \lambda)^{-1}B^* = 0$, i.e. $a = \lambda + B(D - \lambda)^{-1}B^*$. It follows that

$$\begin{aligned} a - BD^{-1}B^* &= \lambda + B(D - \lambda)^{-1}B^* - BD^{-1}B^* \\ &= \lambda + BD^{-1} \cdot [D - (D - \lambda)] \cdot (D - \lambda)^{-1}B^* \\ &= \lambda \cdot (1 + BD^{-1}(D - \lambda)^{-1}B^*). \end{aligned}$$

We have $\|D - C_0\| < \varepsilon \leq \frac{\delta_1}{2}$, and so $\|(D - \lambda) - C_0\| \leq \|D - C_0\| + |\lambda| < \frac{\delta_1}{2} + r < \delta_1$ (recall we chose $r < \frac{\delta_1}{2}$). Our choice of δ_1 ensures that $\|D^{-1}\|$ and $\|(D - \lambda)^{-1}\|$ are both less than $2\|C_0^{-1}\|$, whence the real number $BD^{-1}(D - \lambda)^{-1}B^*$ has absolute value less than 1 (since each of $\|B\|$, $\|B^*\|$ is less than $\frac{1}{2}\|C_0^{-1}\|^{-1}$). Therefore, $1 + BD^{-1}(D - \lambda)^{-1}B^* > 0$, and hence $a - BD^{-1}B^* > 0$ if and only if $\lambda > 0$.

By Lemma 4.1.4, we would just need to decrease ε if necessary to get a Ruget neighbourhood; however, a look at the two proofs assures us that the ε found above should already be small enough to give us a Ruget neighbourhood. ■

Corollary 4.1.15 If S is invertible, and $T_0, T_1 \in \Phi_1$ are such that $S \in (\mathcal{U}_{T_0} \cap \mathcal{U}_{T_1})$ for \mathcal{U}_{T_0} and \mathcal{U}_{T_1} tight Ruget ε -balls centered at T_0 and T_1 respectively, then $\Psi_{T_0}(S)$ and $\Psi_{T_1}(S)$ have the same sign. 🗑️

PROOF Suppose $\Psi_{T_0}(S) > 0$; then, according to Lemma 4.1.14, S contains a positive eigenvalue in some interval $[r_-, r_+]$, and no other eigenvalues in that interval. If $S \in \mathcal{U}_{T_1}$ as well, then the eigenvalue must be in the corresponding interval for T_1 , but would still be positive, so $\Psi_{T_1}(S) > 0$ as well. A similar argument works if the eigenvalue is negative instead (in which case $\Psi_{T_0}(S) < 0$). ■

It is not easy to get a handle on the path components of Φ_1 ; it seems unlikely that Φ_1 is in general connected. However, we make a note of the fact that two operators in Φ_1 which are in the same Ruget ε -ball (necessarily centered at an operator in Φ_1 , as Ruget ε -balls centered at invertible operators contain only invertible operators) can be connected by a path contained in Φ_1 .

Lemma 4.1.16 Consider a Ruget ε -ball \mathcal{U} centered at $T = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix}$ (where the upper left hand corner is one-dimensional). If $S = \begin{bmatrix} a & B \\ B^* & C_0 + E \end{bmatrix}$ is an operator in $\mathcal{U} \cap \Phi_1$, then S and T are in the same path component of Φ_1 . \blacksquare

PROOF By the way \mathcal{U} is defined, we know $\begin{bmatrix} a & B \\ B^* & E \end{bmatrix}$ is bounded, with norm say less than ε . Moreover, since $S \in \Phi_1$, $a = B(C_0 + E)^{-1}B^*$ (by Lemma 4.1.2). We claim that

$$\rho(t) = \left\{ \begin{bmatrix} t^2 B(C_0 + tE)^{-1} B^* & tB \\ tB^* & C_0 + tE \end{bmatrix} \right\}_{t \in [0,1]}$$

defines a path in Φ_1 between S and T . First, for any $t \in [0,1]$, $\|C_0 + tE - C_0\| \leq t\|E\| \leq \|E\| < \varepsilon$, so $C_0 + tE$ is invertible. Using the additional fact that $S - T$ is bounded, it follows that $\rho(t) - T$ is a bounded operator, so $\rho(t) \in \Phi$ for all t . The continuity of ρ follows from Lemma 4.1.5, so ρ is a path in Φ . Next, it is easy to verify that

$$\Psi_T(\rho(t)) = t^2 B(C_0 + tE)^{-1} B^* - (tB)(C_0 + tE)^{-1}(tB^*) = 0,$$

which means $\rho(t) \in \Phi_1$ for all t . Moreover, $\rho(0) = T$ and $\rho(1) = S$. This shows that S and T are in the same path component of Φ_1 . \blacksquare

Lemma 4.1.17 Consider a Ruget ε -ball \mathcal{U} centered at $T = \begin{bmatrix} 0 & 0 \\ 0 & C_0 \end{bmatrix}$ (where the upper left hand corner is one-dimensional). If ρ and ξ are two paths contained in \mathcal{U} such that

- $\rho(t)$ is invertible for all t except for some t_0 , $\xi(t)$ is similarly invertible for all t except for some t_1 , and $\rho(t_0) = \xi(t_1) = T$, and
- $\Psi(\rho(0)), \Psi(\xi(0)) < 0$, while $\Psi(\rho(1)), \Psi(\xi(1)) > 0$.

Then ρ is homotopic to ξ (the endpoints are not necessarily fixed, but remain invertible throughout the homotopy). \blacksquare

PROOF First note that, by reparametrizing ξ if necessary, we can assume without loss of generality that $t_1 = t_0$.

With respect to the same decomposition as T , suppose $\rho(t) = \begin{bmatrix} a_t & B_t \\ B_t^* & D_t \end{bmatrix}$. Then, via $H_s(t) = \begin{bmatrix} a_t - [s(2-s)]B_t D_t^{-1} B_t^* & (1-s)B_t \\ (1-s)B_t^* & D_t \end{bmatrix}$, ρ is homotopic to $\left\{ \begin{bmatrix} a_t - B_t D_t^{-1} B_t^* & 0 \\ 0 & D_t \end{bmatrix} \right\}$ (use Lemma 4.1.5 to prove continuity of $H_s(t)$ in both s and t).

Since $\|\rho(t) - T\| < \varepsilon$, we can write $D_t = C_0 + E_t$ for some E_t with $\|E_t\| < \varepsilon$. For any $s \leq 1$, we have $\|(C_0 + sE_t) - C_0\| \leq \|E_t\| < \varepsilon$, and so $C_0 + sE_t$ is invertible. It follows that

$G_s(t) = \begin{bmatrix} a_t - B_t D_t^{-1} B_t^* & 0 \\ 0 & C_0 + (1-s)E_t \end{bmatrix}$ defines a homotopy from $H_1(t)$ to the path $\tilde{\rho}(t) := \begin{bmatrix} a_t - B_t D_t^{-1} B_t^* & 0 \\ 0 & C_0 \end{bmatrix}$; note that we can also write $\tilde{\rho}(t)$ as $\begin{bmatrix} \Psi(\rho(t)) & 0 \\ 0 & C_0 \end{bmatrix}$.

Similarly, we can find a homotopy from ξ to $\tilde{\xi} = \begin{bmatrix} \Psi(\xi(t)) & 0 \\ 0 & C_0 \end{bmatrix}$. Finally, $\tilde{\rho}(t)$ and $\tilde{\xi}(t)$ are homotopic by changing the top left corner linearly from one into the other; since $\Psi(\rho(t))$ and $\Psi(\tilde{\xi}(t))$ have the same sign at $t = 0$ and $t = 1$, the endpoints of the path stay invertible along this homotopy. ■

Lemma 4.1.18 Let ρ be a path in $\Phi_0 \cup \Phi_1$ with endpoints in Φ_0 . We can find a path ξ in $\Phi_0 \cup \Phi_1$ such that ξ is homotopic to ρ with endpoints fixed, and ξ intersects Φ_1 finitely many times. Moreover, if $\xi(t_0) \in \Phi_1$ and $t_- < t_0 < t_+$ are such that $\xi|_{[t_-, t_+]}$ is contained in a Ruget neighbourhood of $\xi(t_0)$ then $\Psi(\xi(t_-))$ and $\Psi(\xi(t_+))$ have different signs. 🖱

PROOF As in the beginning of the proof of Lemma 4.1.11, we can find a homotopy from ρ to a concatenation of Ruget paths, so it is sufficient to show that the result holds if ρ is a Ruget path.

Suppose ρ is contained in a Ruget neighbourhood \mathcal{U} centered at some $\rho(t_0)$. Since $\rho \subset \Phi_0 \cup \Phi_1$, $\dim \ker \rho(t_0) \leq 1$. However, if $\rho(t_0)$ is invertible then all the operators in the Ruget neighbourhood are invertible so, since ρ does not intersect Φ_1 , there is nothing further to be done. Assume thus that $\rho(t_0)$ has kernel dimension one. Let P be the projection onto the kernel of $\rho(t_0)$ and, with respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, write $\rho(t) = \begin{bmatrix} a_t & B_t \\ B_t^* & D_t \end{bmatrix}$ (we use a small letter for a_t to remind ourselves that, since P has dimension one, a_t is simply a real number). Let $m_t = a_t - B_t D_t^{-1} B_t^*$; then $m_t \in \mathbb{R}$. Let $r_t = (1-t)m_0 + t m_1$ (the straight line path from m_0 to m_1), and note that r_t is zero at most once (if the signs of m_0 and m_1 are different). Then $H_s(t) = (1-s)m_t + s r_t$ is a homotopy from $\{m_t\}$ to $\{r_t\}$. Consider $G_s(t) = \begin{bmatrix} H_s(t) + B_t D_t^{-1} B_t & B_t \\ B_t^* & D_t \end{bmatrix}$; this is a homotopy (by Lemma 4.1.9) to the path $G_1(t) = \begin{bmatrix} r_t + B_t D_t^{-1} B_t & B_t \\ B_t^* & D_t \end{bmatrix}$. Recall that $\dim \ker G_1(t) = \dim \ker r_t$ (Lemma 4.1.2), whence $\dim \ker G_1(t) = 1$ at most once (by construction of $\{r_t\}$); it follows that $G_1(t)$ intersects Φ_1 at most once.

If t_0 is such that $\rho(t_0) \in \Phi_1$, with t_-, t_+ defined as in the statement of the theorem, if $\Psi(\rho(t_-))$ and $\Psi(\rho(t_+))$ have the same sign then we can perform the same construction as above to get a homotopic path between $\rho(t_-)$ and $\rho(t_+)$ which does not intersect Φ_1 at all (since r_t would be either positive for all t or negative for all t). ■

Lemma 4.1.19 $H^1(\Phi_0 \cup \Phi_1, \Phi_0) \cong H^0(\Phi_1)$. 🖱

PROOF Define a map from $H^0(\Phi_1)$ to $H^1(\Phi_0 \cup \Phi_1, \Phi_0)$, a map going in the opposite direction, and then show they are inverses of each other.

Define a map $k : H^0(\Phi_1) \rightarrow H^1(\Phi_0 \cup \Phi_1, \Phi_0)$

An element $\mu \in H^0(\Phi_1)$ assigns an integer to each path component of Φ_1 . We want to define a map v_μ that assigns an integer to each equivalence class in $H_1(\Phi_0 \cup \Phi_1, \Phi_0)$.

Start with a path $\rho : [0, 1] \rightarrow \Phi_0 \cup \Phi_1$ with endpoints in Φ_0 . At every $t \in [0, 1]$ we can put a Ruget ε -ball (as shown in Lemma 4.1.4), in fact, if $\rho(t) \in \Phi_1$ then use a tight Ruget ε -ball (as

defined in Lemma 4.1.14). It follows that we can get division points $0 = t_0 < t_1 < \dots < t_n = 1$ such that $\rho|_{[t_i, t_{i+1}]} \subset \mathcal{U}_{t_i}$ which is centered at some $T_i \in \Phi_0 \cup \Phi_1$.

We want to assign an integer value to ρ . Start by assigning an integer to each subdivision of ρ , as follows:

- If $\mathcal{U}_{t_i} \subset \Phi_0$ then $a(\rho, t_i, t_{i+1}) = 0$.
- If $\mathcal{U}_{t_i} \not\subset \Phi_0$, this automatically means $T_i \in \Phi_1$, so suppose T_i belongs to the path component \mathcal{V} of Φ_1 . Define

$$a(\rho, t_i, t_{i+1}) = \begin{cases} 1 \cdot \mu(\mathcal{V}) & \text{if } \Psi_{T_i}(\rho(t_i)) < 0 \text{ and } \Psi_{T_i}(\rho(t_{i+1})) \geq 0 \\ -1 \cdot \mu(\mathcal{V}) & \text{if } \Psi_{T_i}(\rho(t_i)) \geq 0 \text{ and } \Psi_{T_i}(\rho(t_{i+1})) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Even though it is not mentioned in the parameters list, in order to calculate $a(\rho, t_i, t_{i+1})$ we assume that we have a matching Ruget ε -ball \mathcal{U} centered at some T_i such that $\rho|_{[t_i, t_{i+1}]} \subset \mathcal{U}$; the choice of ε -ball will be clear from context, as the division points will be chosen based on some fixed cover of the path by ε -balls. We do make the observation that, if $\rho|_{[t_i, t_{i+1}]}$ is contained in two ε -balls, \mathcal{U}_i centered at T_i and \mathcal{U} centered at T , then the value calculated for $a(\rho, t_i, t_{i+1})$ does not depend on whether we use \mathcal{U}_i or \mathcal{U} to calculate it. For the proof of this statement only, for clarity, we write $a_{\mathcal{U}}(a, t_i, t_{i+1})$ and $a_{\mathcal{U}_i}(a, t_i, t_{i+1})$ to keep track of which ε -ball is used to calculate the value of a . There are multiple possibilities to consider, depending on the type of ε -balls:

- T_i and T are both invertible, which means $\mathcal{U}_i, \mathcal{U} \subset \Phi_0$. Then $a(\rho, t_i, t_{i+1}) = 0$ by definition, in both cases.
- Only one of T_i, T is invertible; say, without loss of generality, T_i is invertible. Then $\mathcal{U}_i \subset \Phi_0$; this means that $\rho|_{[t_i, t_{i+1}]} \subset \Phi_0$. To calculate $a(\rho, t_i, t_{i+1})$ using \mathcal{U} we calculate $\Psi_T(\rho(t_i))$ and $\Psi_T(\rho(t_{i+1}))$. Since Ψ_T is continuous on \mathcal{U} and $\Psi_T \neq 0$ on Φ_0 , it follows that $\Psi_T(\rho(t_i))$ and $\Psi_T(\rho(t_{i+1}))$ are either both positive or both negative, so we still have $a(\rho, t_i, t_{i+1}) = 0$ when we use the definition for \mathcal{U} .
- Neither T_i nor T is invertible. We consider two cases, depending on whether or not $\rho|_{[t_i, t_{i+1}]} \subset \Phi_0$.

Suppose first that there exists an $r \in [t_i, t_{i+1}]$ such that $\rho(r) \in \Phi_1$. It follows from Lemma 4.1.16 that T_i and T must be in the same path component of Φ_1 (as $T_i, \rho(r) \in \mathcal{U}_i$ and $\rho(r), T \in \mathcal{U}$ are elements of Φ_1 which are in the same Ruget ε -ball). Moreover, by Lemma 4.1.14, $\Psi_{T_i}(\rho(t_i))$ and $\Psi_T(\rho(t_i))$ have the same sign, and the same is true about $\Psi_{T_i}(\rho(t_{i+1}))$ and $\Psi_T(\rho(t_{i+1}))$; which is what we need in order to conclude that $a_{\mathcal{U}_i}(a, t_i, t_{i+1}) = a_{\mathcal{U}}(a, t_i, t_{i+1})$.

Now suppose that $\rho|_{[t_i, t_{i+1}]} \subset \Phi_0$. Then, by continuity of Ψ_T , and since $\Psi_T \neq 0$ on Φ_0 , the sign of $\Psi_T(\rho(t_i))$ must be the same as $\Psi_T(\rho(t_{i+1}))$, and so $a_{\mathcal{U}}(a, t_i, t_{i+1}) = 0$. Similarly, $\Psi_{T_i}(a, t_i, t_{i+1}) = 0$ as well.

Therefore, $a_{\mathcal{U}}(\rho, t_i, t_{i+1}) = a_{\mathcal{U}_i}(\rho, t_i, t_{i+1})$. Henceforth, we no longer keep track of the ε -ball used to calculate a ; though, for every mention of a , it is understood that there is a Ruget ε -ball \mathcal{U} centered at some T (a 'tight' Ruget ε -ball if $T \in \Phi_1$) such that $\rho|_{[t_i, t_{i+1}]} \subset \mathcal{U}$, and this ε -ball is used to calculate the value of $a(\rho, t_i, t_{i+1})$ according to the definition given.

We would like to define $v_\mu(\rho) = \sum_{i=0}^{n-1} a(\rho, t_i, t_{i+1})$. First we need to show that this is well-defined at the level of ρ – that is, the number we get is independent of the division points and of the choice of neighbourhoods. Note that, for the sake of later applications of this definition, we are careful not to require the ε -balls covering our path to be centered at points on the path (though we do require ε -balls centered at operators in Φ_1 to be tight).

If $0 = s_0 < s_1 < \dots < s_m = 1$ is a different partition of our path, we can find a common refinement of $\{s_0, s_1, \dots, s_m\}$ and $\{t_0, t_1, \dots, t_n\}$, so it is sufficient to show that we get the same result when we consider a refinement of a partition. Even more, since such a refinement is obtained by adding a finite collection of points, we need only show that the answer does not change when we add a single point. Add $s \in [t_i, t_{i+1}]$ and replace the neighbourhood \mathcal{U}_{t_i} by \mathcal{U}_{s-} (covering $[t_i, s]$ and centered at S_-) and \mathcal{U}_{s+} (covering $[s, t_{i+1}]$ and centered at S_+). The sum used in the calculation of v_μ is the same, except $a(\rho, t_i, t_{i+1})$ is removed and replaced by the sum of $a(\rho, t_i, s)$ and $a(\rho, s, t_{i+1})$. As already shown, we can ignore \mathcal{U}_{s-} and \mathcal{U}_{s+} and calculate a using the neighbourhood \mathcal{U}_{t_i} for all three segments without in any way affecting the result. We need to show that $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) + a(\rho, s, t_{i+1})$, which we do by considering possibilities:

- $T_i \in \Phi_0$, so $\mathcal{U}_{t_i} \subseteq \Phi_0$ and by definition $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) = a(\rho, s, t_{i+1}) = 0$. It is thus clear that in this case $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) + a(\rho, s, t_{i+1})$.
- $T_i \in \Phi_1$, so a is defined using Ψ_{T_i} . Split again into cases, depending on whether or not the path segments consist of invertible operators.

Suppose $\rho|_{[t_i, s]}$ and $\rho|_{[s, t_{i+1}]}$ $\subset \Phi_0$. Since Ψ_{T_i} is continuous on \mathcal{U}_{t_i} and $\Psi_{T_i}(S)$ is only zero if $S \in \Phi_1$, the sign of $\Psi(\rho(t_i))$ and $\Psi(\rho(t_{i+1}))$ must be the same, so $a(\rho, t_i, t_{i+1}) = 0$. Similarly, $a(\rho, t_i, s) = 0$ and $a(\rho, s, t_{i+1}) = 0$; thus $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) + a(\rho, s, t_{i+1})$.

Suppose $\rho|_{[t_i, s]} \subset \Phi_0$, but $\rho|_{[s, t_{i+1}]}$ intersects Φ_1 . As above, since $\rho|_{[t_i, s]} \cap \Phi_1 = \emptyset$, we know $\Psi_{T_i}(\rho(t_i))$ and $\Psi_{T_i}(\rho(s))$ have the same sign; so $a(\rho, t_i, s) = 0$, and also $a(\rho, s, t_{i+1}) = a(\rho, t_i, t_{i+1})$. The case when $\rho|_{[t_i, s]}$ intersects Φ_1 but $\rho|_{[s, t_{i+1}]} \subset \Phi_0$ is similar, with $a(\rho, t_i, s) = a(\rho, t_i, t_{i+1})$ and $a(\rho, s, t_{i+1}) = 0$. Once again it follows that $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) + a(\rho, s, t_{i+1})$.

Suppose neither $\rho|_{[t_i, s]}$ nor $\rho|_{[s, t_{i+1}]}$ is fully contained in Φ_0 . By necessity, $\Psi(\rho(s))$ has the same sign as at least one of $\Psi(\rho(t_i))$ and $\Psi(\rho(t_{i+1}))$ (where, due to the definition of a , we consider zero to have positive sign). If $\Psi(\rho(s))$ has the same sign as $\Psi(\rho(t_i))$ then $a(\rho, t_i, s) = 0$ and $a(\rho, s, t_{i+1}) = a(\rho, t_i, t_{i+1})$; otherwise, $a(\rho, t_i, s) = a(\rho, t_i, t_{i+1})$ and $a(\rho, s, t_{i+1}) = 0$. So $a(\rho, t_i, t_{i+1}) = a(\rho, t_i, s) + a(\rho, s, t_{i+1})$, as desired.

Next, we have to show that this map on paths can be used to define a map at the level of homology. To this end, show that if ρ and ξ are two homotopic paths, then $v_\mu(\rho) = v_\mu(\xi)$.

We first show that this holds if ρ and ξ are 'close enough', and then extend this to a general statement.

Claim For each path ρ there exists a $\delta > 0$ such that, whenever $H_s(t)$ is a homotopy between ρ and some other path $\xi = H_1(t)$ for which $\|\rho(t) - H_s(t)\| < \delta$ for all $s, t \in [0, 1]$, we have $v_\mu(\rho) = v_\mu(\xi)$.

PROOF Once again, this relies on the fact that on each Ruget neighbourhood Ψ is continuous, so since Ψ is only ever zero at points in Φ_1 , it cannot switch sign when calculated over a path of invertible operators. As usual, start by covering our path ρ in Ruget ε -neighbourhoods, with the caveat that the neighbourhoods should be tight when centered at elements of Φ_1 (see Lemma 4.1.14 for a reminder of what we mean by 'tight'). We can then get a partition of our path $0 = t_0 < t_1 < \dots < t_n = 1$ such that $\rho|_{[t_i, t_{i+1}]} \subset \mathcal{U}_{t_i}$, where \mathcal{U}_{t_i} is a Ruget ε -ball centered at some $T_i \in \Phi_0 \cup \Phi_1$. Choose $\delta > 0$ such that, if γ is any path satisfying $\|\rho(t) - \gamma(t)\| < \delta$, then $\gamma|_{[t_i, t_{i+1}]} \subset \mathcal{U}_{t_i}$. In particular, $H_s([t_i, t_{i+1}]) \subset \mathcal{U}_{t_i}$ for all $s \in [0, 1]$.

Use the partition $\{t_0, \dots, t_n\}$ and the neighbourhoods \mathcal{U}_{t_i} to calculate both $v_\mu(\rho)$ and $v_\mu(\xi)$ (we've already shown that the result is independent of the choice of partition and neighbourhoods). By definition, if $\mathcal{U}_{t_i} \subset \Phi_0$ for some i then $a(\rho, t_i, t_{i+1}) = a(\xi, t_i, t_{i+1}) = 0$. However, it need not be true that $a(\rho, t_i, t_{i+1}) = a(\xi, t_i, t_{i+1})$ for all i .

Suppose there is a neighbourhood U_r which is not contained in Φ_0 . Since \mathcal{U}_{t_0} and \mathcal{U}_{t_n} consist of invertible operators, there exist some integers $i, j \in \{1, \dots, n - 1\}$ with $i \leq r \leq k$ such that $\mathcal{U}_{t_{i-1}}$ and $\mathcal{U}_{t_{k+1}}$ are both contained in Φ_0 , but for all j with $i \leq j \leq k$ we have $\mathcal{U}_{t_j} \not\subset \Phi_0$. We will show that the sum of the $a(\rho, t_i, t_{i+1})$'s over such a sequence of neighbourhoods has to be equal to the sum of $a(\xi, t_i, t_{i+1})$'s. In order to make explanations easier, suppose $k = i + 1$; however, it is possible to adapt this explanation to any number of consecutive such neighbourhoods.

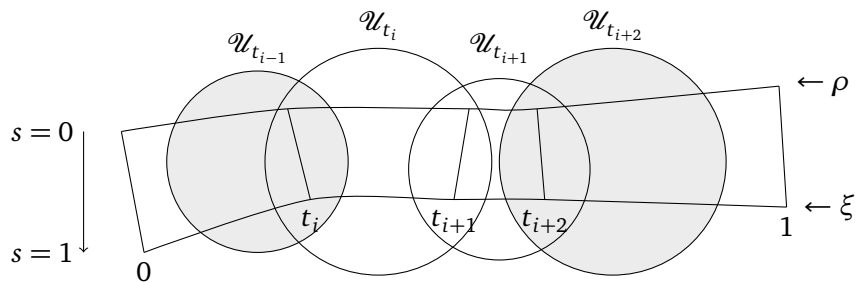


Figure 4.2: The homotopy between ρ and ξ in relation to the two neighbourhoods we are considering. The important values of t are marked along ξ . The shaded neighbourhoods are contained in Φ_0 by hypothesis.

From the fact that $H_s(t_i) \subset \mathcal{U}_{t_{i-1}} \subset \Phi_0$, we know that $H_s(t_i)$ is invertible for all $s \in [0, 1]$. It follows that $\Psi_{T_i}(\rho(t_i))$ and $\Psi_{T_i}(\xi(t_i))$ have the same sign (since Ψ is continuous, and only zero at elements of Φ_1). Similarly, $H_s(t_{i+2})$ is invertible for all s , and so $\Psi_{T_{i+2}}(\rho(t_{i+2}))$ and $\Psi_{T_{i+2}}(\xi(t_{i+2}))$ have the same sign. We now consider two cases, depending on what happens along $H_s(t_{i+1})$.

If $H_s(t_{i+1}) \subset \Phi_0$, then it is not hard to check that $a(\rho, t_i, t_{i+1}) = a(\xi, t_i, t_{i+1})$ (as the sign of $\Psi_{T_i}(H_0(t_{i+1}))$ must be the same as that of $\Psi_{T_i}(H_1(t_{i+1}))$), and the path component of T_i is used

to calculate the value of a for both ρ and ξ) and similarly $a(\rho, t_{i+1}, t_{i+2}) = a(\xi, t_{i+1}, t_{i+2})$. On the other hand, if $H_s(t_{i+1}) \notin \Phi_0$ then there exists an $r \in [0, 1]$ such that $H_r(t_{i+1}) \in \Phi_1$. It follows from Lemma 4.1.16 that $T_i, H_r(t_{i+1})$ (which are in the same ε -ball $\mathcal{U}_{t_{i+1}}$) are in the same path component of Φ_1 , and $H_r(t_{i+1}), T_{i+1}$ are also in the same path component of Φ_1 (as they are both in the ε -ball $\mathcal{U}_{t_{i+2}}$). This enables us to conclude that T_i, T_{i+1} are in the same path component of Φ_1 , so we just need to check the change in sign. We know that $\Psi_{T_i}(\xi(t_{i+1}))$ and $\Psi_{T_{i+1}}(\xi(t_{i+1}))$ are both either non-negative or both negative, so we write $\Psi(\xi(t_{i+1}))$ to remind ourselves we don't care which decomposition is used (as long as we just need to worry about the sign and not the actual value). Now $\Psi(\xi(t_{i+1}))$ need not have the same sign as $\Psi(\rho(t_{i+1}))$; however, going through all possible choices of sign and the corresponding values of $a(\rho, t_i, t_{i+1})$, etc. it should be easy to check that $a(\rho, t_i, t_{i+1}) + a(\rho, t_{i+1}, t_{i+2}) = a(\xi, t_i, t_{i+1}) + a(\xi, t_{i+1}, t_{i+2})$. \diamond

By compactness, we can split up a homotopy $H_s(t)$ into 'small enough' sections so that if H_{s_1} and H_{s_2} are within δ of each other then $\nu_\mu(H_{s_1}(t)) = \nu_\mu(H_{s_2}(t))$, allowing us to conclude that $\nu_\mu(H_0(t)) = \nu_\mu(H_1(t))$.

Define a map $l : H^1(\Phi_0 \cup \Phi_1, \Phi_0) \rightarrow H^0(\Phi_1)$

Start with $v \in H^1(\Phi_0 \cup \Phi_1, \Phi_0)$; for any path $\xi \subset \Phi_0 \cup \Phi_1$ with endpoints in Φ_0 we can calculate $v(\xi)$ by evaluating v on the homology class of ξ . We want to define a function μ_v which assigns an integer to each path component of Φ_1 (and hence is an element of $H^0(\Phi_1)$).

Suppose \mathcal{V} is a path component of Φ_1 , and pick any $T \in \mathcal{V}$. Let P be the projection onto the kernel of T (and note for future reference that $T|_{p^\perp \mathcal{H}}$ is invertible). For any $\delta > 0$, we can define the path $\xi_T(t) = T + (2t - 1)\delta P$; then $\xi_T(t)$ is invertible except when $t = \frac{1}{2}$. Define $\mu_v(\mathcal{V}) = v(\xi_T)$, and note that the definition does not depend on the choice of δ as v evaluates to zero on paths consisting of invertible operators – so we can “chop off” the ends of ξ_T with no effect; as such, we do not include δ in the notation for ξ_T , and simply assume that δ is chosen small enough to suit our purposes (e.g. we often need to ensure that ξ_T is contained in some specific neighbourhood of T). To show that μ_v is well-defined, we need to check that v evaluates to the same integer if we were to pick some other $S \in \mathcal{V} \subset \Phi_1$, perform this construction, and calculate $v(\xi_S)$. Since S and T are in the same path component, there is a path $\rho \subset \Phi_1$ from T to S .

Place a tight Rugey neighbourhood \mathcal{U}_T at T and choose $r_T > 0$ (as described in Lemma 4.1.14) so that any $R \in \mathcal{U}_T \cap \Phi_0$ has exactly one eigenvalue in $[-r_T, r_T]$ and the sign of the eigenvalue is the same as the sign of $\Psi_T(R)$ (as $R \in \Phi_0$, neither the eigenvalue nor $\Psi_T(R)$ can be zero).

Suppose first that $S \in \mathcal{U}_T$. Then we can ensure that $\xi_S(T) \subset \mathcal{U}_T$ (by choosing δ small enough); we can simultaneously ensure $\delta < r_T$. By design, $\xi_S(t)$ has $\delta \cdot (2t - 1)$ as an eigenvalue (negative when $t < \frac{1}{2}$, positive when $t > \frac{1}{2}$), so Lemma 4.1.14 can be used to conclude that $\Psi(\xi_S(0)) < 0$ and $\Psi(\xi_S(1)) > 0$. Now ξ_T and ξ_S are both contained in the Rugey ε -ball \mathcal{U}_T , and the sign of Ψ evaluated on ξ_T is similarly negative at the end $t = 0$ and positive at the end $t = 1$, so we can use Lemma 4.1.17 to conclude ξ_T and ξ_S are homotopic, and hence $v(\xi_T) = v(\xi_S)$.

On the other hand, if $S \notin \mathcal{U}_T$, then we can cover ρ by open balls $\{\mathcal{U}_{\rho(t)} : t \in [0, 1]\}$ and find a division $0 = t_0 < t_1 < \dots < t_{2n} = 1$ such that $\rho|_{[t_{2i}, t_{2i+2}]} \subseteq \mathcal{U}_{\rho(t_{2i+1})}$. Then, as in the previous case, $v(\xi_{\rho(t_{2i})}) = v(\xi_{\rho(t_{2i+2})})$ for all $i = 0, \dots, n$, whence we can conclude $v(\xi_{\rho(0)}) = v(\xi_{\rho(1)})$, i.e. $v(\xi_T) = v(\xi_S)$, as desired. Therefore μ_v is well-defined.

Show k and l are inverses of each other

If $\mu \in H^0(\Phi_1)$ and \mathcal{V} is a path component of Φ_1 , then $(lk\mu)(\mathcal{V}) = (lv_\mu)(\mathcal{V})$. Apply the construction in the definition of l : let ξ be a Ruget path such that $\xi(t)$ is invertible except at one point t_0 for which $\xi(t_0) \in \mathcal{V}$ and such that $\Psi_{\xi(t_0)}(\xi(0)) < 0$, $\Psi_{\xi(t_0)}(\xi(1)) > 0$ (the precise way to construct ξ is given when the map l is defined). By definition of l and v_μ , $(lv_\mu)(\mathcal{V}) = v_\mu(\xi) = \mu(\text{path component of } \xi(t_0)) = \mu(\mathcal{V})$. Therefore $lk\mu = \mu$.

Now start with $v \in H^1(\Phi_0 \cup \Phi_1, \Phi_0)$ and a path $\xi \subset \Phi_0 \cup \Phi_1$ with endpoints in Φ_0 ; we want to show that $(klv)(\xi) = v(\xi)$. Since any path is homotopic to a concatenation of Ruget paths (as described in Lemma 4.1.18), we can assume without loss of generality that ξ is a Ruget path that intersects Φ_1 exactly once or not at all.

Suppose first that ξ does not intersect Φ_1 , which necessarily means that $\xi \subset \Phi_0$. The equivalence class of ξ in $C_1(\Phi_0 \cup \Phi_1)/C_1(\Phi_0)$ is then zero, whence $v(\xi) = 0$. On the other hand, $(k\mu)(\xi) = a(\xi, 0, 1) = 0$ by definition regardless of which $\mu \in H^0(\Phi_1)$ is being used to calculate a . In particular for $\mu = lv = \mu_v$ we have $(k\mu_v)(\xi) = 0$, giving us that $(klv)(\xi) = v(\xi)$, and hence in general $klv = v$.

Next, suppose ξ intersects Φ_1 at t_0 , and $\xi(t_0)$ is in the path component \mathcal{V} of Φ_1 . Place a Ruget ε -ball \mathcal{U} around $\xi(t_0)$. Let $t_- < t_0 < t_+$ be such that $\xi|_{[t_-, t_+]} \subset \mathcal{U}$; it follows that $v(\xi|_{[0, t_-]}) = v(\xi|_{[t_+, 1]}) = 0$ (since $\xi|_{[0, t_-]}, \xi|_{[t_+, 1]} \subset \Phi_0$) and thus $v(\xi) = v(\xi|_{[t_-, t_+]})$. That is, we can assume without loss of generality that $\xi \subset \mathcal{U}$.

Let

$$\delta = \begin{cases} 1 & \text{if } \Psi(\xi(0)) < 0 \text{ and } \Psi(\xi(1)) > 0 \\ -1 & \text{if } \Psi(\xi(0)) > 0 \text{ and } \Psi(\xi(1)) < 0 \\ 0 & \text{otherwise;} \end{cases}$$

note that, going back to the notation used to define k , for fixed $\mu \in H^0(\Phi_1)$ we had defined $a(\xi, 0, 1) = \delta \cdot \mu(\mathcal{V})$ (note that neither $\Psi(\xi(0))$ nor $\Psi(\xi(1))$ can be zero, which is why we write ' $>$ ' instead of ' \geq ' in the definition of δ), and this a was then used to calculate v_μ .

Now $(klv)(\xi) = (k\mu_v)(\xi) = \mu_v(\mathcal{V}) \cdot \delta$. In order to calculate $\mu_v(\mathcal{V})$, construct the path $\rho(t) = \xi(t_0) + \frac{\varepsilon}{2}(2t - 1)P_{\ker \xi(t_0)}$ (that is, $\rho(t)$ is a path through $\xi(t_0)$ with $\rho(t) \subset \mathcal{U}$) and recall that we had defined $\mu_v(\mathcal{V}) = v(\rho)$. It is easy to check that, since \mathcal{U} is centered at $\xi(t_0)$, $\Psi(\rho(0)) < 0$ and $\Psi(\rho(1)) > 0$. We now use Lemma 4.1.17: if $\delta > 0$ then ρ and ξ are homotopic, and if $\delta < 0$ then $-\rho$ and ξ are homotopic. Hence $v(\rho) = v(\xi) \cdot \delta$, and

$$(klv)(\xi) = \mu_v(\mathcal{V}) \cdot \delta = v(\rho) \cdot \delta = (v(\xi) \cdot \delta) \cdot \delta = v(\xi) \cdot \delta^2 = v(\xi).$$

This concludes the proof that k and l are inverses of each other. ■

We are now in position to fit in spectral flow in this cohomology setting.

Corollary 4.1.20 Let μ the the element of $H^0(\Phi_1)$ which assigns 1 to every connected component of Φ_1 . Denote by $[\Phi_1]$ the image of μ under the isomorphism $H^0(\Phi_1) \xrightarrow{\cong} H^1(\Phi, \Phi_0)$. For $\rho \in \Omega(\Phi, \Phi_0)$ a path, $[\Phi_1](\rho) = \text{sf}(\rho)$. ▮

PROOF The isomorphism $H^0(\Phi_1) \xrightarrow{\cong} H^1(\Phi, \Phi_0)$ mentioned above is obtained by composing the two isomorphisms $H^1(\Phi_0) \cong H^1(\Phi_0 \cup \Phi_1, \Phi_0)$ (Lemma 4.1.19) and $H^1(\Phi_0 \cup \Phi_1, \Phi_0) \cong H^1(\Phi, \Phi_0)$ (Lemma 4.1.12). For the given $\mu \in H^0(\Phi_1)$, denote by $\tilde{\mu}$ its image in $H^1(\Phi_0 \cup \Phi_1, \Phi_0)$ under the first isomorphism, which means that by hypothesis the image of $\tilde{\mu}$ under the second isomorphism is $[\Phi_1]$.

Now consider $\rho \in \Omega(\Phi, \Phi_0)$, and let us check that $[\Phi_1](\rho) = \text{sf}(\rho)$. The value of $[\Phi_1](\rho)$ can be calculated by the expedient of homotoping ρ to a representative $\tilde{\rho}$ of a class in $H_1(\Phi_0 \cup \Phi_1, \Phi_0)$ and evaluating $\tilde{\mu}$ on $\tilde{\rho}$. We claim that the resulting integer value is the spectral flow along $\tilde{\rho}$. To see this, recall first that $\tilde{\rho}$ can be ensured to consist of invertible operators, except at finitely many points where it intersects Φ_1 . Moreover, if we restrict $\tilde{\rho}$ to a neighbourhood of each intersection point, $\tilde{\mu}$ evaluates to 1 if there is an eigenvalue changing sign from negative to positive, and to -1 if there is an eigenvalue changing sign from positive to negative (see Remark 4.1.13 and the proof of Lemma 4.1.19). Therefore, $\tilde{\mu}(\tilde{\rho})$ is indeed counting the net number of eigenvalues changing sign, i.e. the spectral flow of $\tilde{\rho}$. Since $\text{sf}(\tilde{\rho}) = \text{sf}(\rho)$, it follows that $[\Phi_1](\rho) = \text{sf}(\rho)$. ■

The above result means that, geometrically, spectral flow is the intersection number of the path ρ with the submanifold Φ_1 of Φ .

4.2 Intersection Number and Integral Formulas for Spectral Flow

We started out by proving that Φ , Φ_0 and Φ_1 are manifolds (introduction of Section 4.1 and Proposition 4.1.6), but then ignored all the corresponding structure by working with singular homology and cohomology, which can be defined very well just for topological spaces. However, we now wish to get back to a point of view which does require our spaces to be manifolds. Namely, the differential structure allows us to replace singular cohomology by de Rham cohomology. In order to tell the various theories apart, denote the de Rham cohomology groups of X by $H_{dR}^n(X)$. It is known that, in cases in which both can be calculated, $H_{dR}^n(X) \cong H^n(X) \otimes \mathbb{R}$ (where the latter is isomorphic to the singular cohomology of X with real coefficients). The isomorphism from de Rham cohomology to singular cohomology works as follows for $n = 1$: given a one-form ω , construct a map γ_ω by defining $\gamma_\omega(\rho) = \int_\rho \omega$ for any path $\rho \subset X$ and extending to one-chains by linearity.

Our result in the previous section was that there is a class in $H^1(\Phi, \Phi_0)$, denoted $[\Phi_1]$, such that if ρ is a path with endpoints in Φ_0 then $[\Phi_1](\rho) = \text{spectral flow of } \rho$ (Corollary 4.1.20). We want to now change our viewpoint to de Rham cohomology.

There are two equivalent pictures of relative de Rham cohomology. If N is a submanifold of M then we let $\Omega^k(M, N)$ be the k -forms α on M such that $i^*\alpha = 0$, where i^* is the map induced by the inclusion $i : N \hookrightarrow M$. Use the usual differential d and take the quotient groups to get $H^k(M, N)$. The other view point is to let $\Omega^k(M, N) = \Omega^k(M) \oplus \Omega^{k-1}(N)$, with differential $d(\omega_1 \oplus \omega_2) = d\omega_1 \oplus (i^*\omega_1 - d\omega_2)$, where $i : N \rightarrow M$ is the inclusion map. The map between the first picture and the second is given by $\omega \mapsto \omega \oplus 0$. The second picture will be more useful to us (see [10], Chapter I.6, as a brief reference for this definition). For relative cohomology, we also have $H_{dR}^k(M, N) \cong H^k(M, N; \mathbb{R})$. Note that, given $\omega_1 \oplus \omega_2 \in \Omega^1(M) \oplus \Omega^0(N)$, the image of $\omega_1 \oplus \omega_2$ under this isomorphism is the cohomology class of $[\gamma_\omega]$, where γ_ω pairs with paths

contained in M whose endpoints are in N via $\gamma_{\omega_1 \oplus \omega_2}(\xi) = \int_{\xi} \omega_1 - (\omega_2(\xi(1)) - \omega_2(\xi(0)))$; this definition ensures that, for paths $\xi \subset N$, we get $\gamma_{\omega}(\xi) = 0$.

In this section, D is a self-adjoint, unbounded operator which is in addition θ -summable, ensuring that $\text{Tr}(e^{-tD^2}) < \infty$ for any $t > 0$. In particular, it means that $e^{-tD^2} \in \mathcal{K}(\mathcal{H})$, which is equivalent to the condition that D has compact resolvents; thus, as described in Section 4.1, we can construct a manifold Φ consisting of bounded perturbations of D . Recall that Φ_0 is then a submanifold of Φ consisting of invertible operators. Define a one-form α_{ε} on Φ and a zero-form on Φ_0 by

$$\begin{aligned}\alpha_{\varepsilon}(X) &= \left(\frac{\varepsilon}{\pi}\right)^{1/2} \text{Tr}(X e^{-\varepsilon D^2}) \text{ for } X \in T_D \Phi \cong \mathcal{B}(\mathcal{H})_{sa}, \\ \eta_{\varepsilon}(D) &= \frac{1}{\sqrt{\pi}} \int_{\varepsilon}^{\infty} \text{Tr}(D e^{-tD^2}) t^{-\frac{1}{2}} dt.\end{aligned}$$

It is not immediately evident that the integral used to define η_{ε} even converges, but it does; for a proof of this, see [15], Corollary 8.4 (where it is proven that the integral converges even if D is not invertible). A key step in the proof of this result is the lemma cited below, which we will also use later to prove various trace norm results.

Theorem 4.2.1 (Lemma 8.2 of [15]) Let D be an unbounded self-adjoint operator affiliated with a von Neumann algebra \mathcal{N} such that $(1 + D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$. Let $\{E_{\lambda}\}$ denote the spectral resolution of $|D|$ and suppose $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and $f(|D|)$ is trace class. Let $\varphi_{\lambda} = \tau(E_{\lambda})$ for each λ ; then $\tau(f(|D|)) = \int_0^{\infty} f(\lambda) d\varphi_{\lambda}$. \blacksquare

The claim is that $\alpha \oplus -\frac{1}{2}\eta$ is a representative of a class in $H_{dR}^1(\Phi, \Phi_0)$; moreover, evaluating this one-form on one-chains gives us one of the elements of $H^1(\Phi, \Phi_0; \mathbb{R})$ which calculates spectral flow. We should mention that the proof given by Getzler of these facts (Proposition 2.5 and Theorem 2.6 in his paper) seems to at the very least omit checking some details, and contains the occasional error. For example, a step in the proof of Lemma 2.4 of [38] (which is used later to find norm upper bounds) relies on the function $x \mapsto e^x$ being operator monotone, which is not true. Another possible issue is the use of the formula

$$\frac{d}{dt}(e^{z(t)}) = \int_0^1 e^{uz(t)} \cdot \frac{d}{dt}(z(t)) \cdot e^{(1-u)z(t)} dt.$$

See, for example, [67], formula (B5) in the Appendix, for a way to prove this result: expand the exponential as an infinite sum, rearrange the summation, use the integral definition of the beta function, and regroup the terms to obtain the desired final expression. The proof of Theorem 2.5 in [38] seems to use this result twice; once (when calculating $d\alpha$) to conclude that

$$\left. \frac{d}{dh} \right|_{h=0} \text{Tr}(X e^{-(D+hY)^2}) = - \int_0^1 \text{Tr}(X e^{-uD^2} \cdot (DY + YD) \cdot e^{-(1-u)D^2}) du,$$

and later (when calculating $d\eta$) to write

$$\left. \frac{d}{dh} \right|_{h=0} \text{Tr}(D e^{-(D+hX)^2}) = - \int_0^1 \text{Tr}(D e^{-uD^2} t(DX + XD) e^{-(1-u)tD^2}) du.$$

However, since D is an unbounded operator and Y can be any self-adjoint bounded operator, it is not obvious that $\frac{d}{dt}((D + tY)^2)$ has any meaning – the domain of the unbounded operators $(D + tX)^2$ and D^2 need not even be the same. In the integrand of the final expression, the domain

of $DY + YD$ (which is equal to $\frac{d}{dt}((D + tY)^2)$ when D is bounded) might in the worst case just consist of the zero vector. Moreover, one has to justify the convergence in trace norm separately from the operator norm convergence of the integral, an issue which is also not addressed in [38]. In the next section, we present the steps followed by Getzler to prove the integral spectral flow formula in [38], filling out the necessary justifications for them. For example, one can obtain an integral formula for $e^{-(D+hY)^2} - e^{-D^2}$ (see Lemma 4.2.4, part (ii) below), and use this formula to show that taking the limit as $h \rightarrow 0$ in the calculation of $d\alpha$ gives us a rearrangement of the first integral (namely, $\int_0^1 \text{Tr}(XDe^{-uD^2}Ye^{-(1-u)D^2} + Xe^{-uD^2}YDe^{-(1-u)D^2}) du$) in which the domain issues are no longer present (see Proposition 4.2.5 for the calculation of $d\alpha$).

Exponential formulas and norm estimates

In order to justify the steps used by Getzler for the proof of $d\alpha = 0$ and $d\eta = -2\alpha$, we need to verify various derivative formulas for expressions involving unbounded operators. We start by stating some easy results for norm bounds of various expressions involving exponentials. The key step is Lemma 4.2.4, which gives us an integral formula for a difference of exponentials and allows us to calculate $d\alpha$ and $d\eta$ later on. We introduce the constants $c_{n,s}$ which will appear in some of our norm bound expressions.

Definition 4.2.2 For $s > 0$ a fixed real number define the sequence

$$c_{n,s} = \begin{cases} 1 & \text{if } n = 0 \\ e^{-\frac{n}{2}} \cdot \left(\frac{n}{2}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{s}\right)^{\frac{n}{2}} & \text{if } n \geq 1. \end{cases}$$

Note that, for fixed $n > 0$, $s \rightarrow c_{n,s}$ is a continuous decreasing function on $(0, \infty)$ with $\lim_{s \rightarrow \infty} c_{n,s} = 0$. These bounds will appear almost exclusively with $n = 0$ and $n = 1$; for quick reference, we make note of a few calculations involving the $c_{n,s}$:

- for $n \geq 1$ and $u \neq 0$, $c_{n,us} = c_{n,s} \cdot u^{-\frac{n}{2}}$; in particular, with $u = \frac{1}{2}$, get $c_{n,\frac{s}{2}} = c_{n,s} \cdot 2^{\frac{n}{2}}$. Note that for $n = 0$ and $u \in (0, 1)$ we could write $c_{n,us} \leq u^{-\frac{n}{2}} \cdot c_{n,s}$ as $c_{0,\cdot} = 1$.
- $c_{1,s} = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{s}}$. ☛

First deal with e^{-sD^2} when D is a fixed unbounded operator and s is allowed to change – later we will have to see what happens when s is fixed and D is allowed to change by bounded perturbations.

Lemma 4.2.3 Suppose D is an unbounded self-adjoint operator such that $\text{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$.

- (i) For any integer $n \geq 0$, and any real number $s > 0$, $\|D^n \cdot e^{-sD^2}\| \leq c_{n,s}$.
- (ii) If $0 < r \leq s$ then $\text{Tr}(e^{-rD^2}) \geq \text{Tr}(e^{-sD^2})$.
- (iii) For any integer $n \geq 0$, and any real number $s > 0$, $\|D^n \cdot e^{-sD^2}\|_1 \leq c_{n,\frac{s}{2}} \cdot \|e^{-\frac{s}{2}D^2}\|_1$.
- (iv) For any $n \geq 0$ the function $s \mapsto D^n e^{-sD^2}$ is continuous on $(0, \infty)$ in both operator norm and trace norm. ☛

PROOF (i) Consider the function $f(x) = x^n \cdot e^{-sx^2}$. When $n = 0$, it is clear that $0 < f(x) \leq 1$, so suppose that $n \neq 0$. It is not hard to check that the function has local extrema at $x = \pm \sqrt{\frac{n}{2s}}$, and verify that $|f(x)| \leq e^{-\frac{n}{s}} \cdot \left(\frac{n}{2s}\right)^{\frac{n}{2}} = c_{n,s}$. The Spectral Theorem then gives us the desired result.

(ii) Use Theorem 4.2.1 to get a formula for the trace; that is, let $\{E_\lambda\}$ denote the spectral resolution of $|D|$, and $\varphi_\lambda = \text{Tr}(E_\lambda)$. For any $t \in (0, \infty)$, we can define a function $f_t : \lambda \mapsto e^{-t\lambda^2}$. Then f_t is continuous and, by the hypotheses on D , $f_t(|D|) = e^{-tD^2}$ is trace class. By Theorem 4.2.1, we have that $\text{Tr}(f_t(|D|)) = \int_0^\infty f_t(\lambda) d\varphi_\lambda$. If $r \leq s$ then $0 < f_s(\lambda) \leq f_r(\lambda)$ for each $\lambda \in (0, \infty)$, allowing us to conclude that $\text{Tr}(f_s(|D|)) \leq \text{Tr}(f_r(|D|))$, which translates to our desired conclusion.

(iii) This is a simple matter of writing $D^n \cdot e^{-sD^2} = D^n e^{-\frac{s}{2}D^2} \cdot e^{-\frac{s}{2}D^2}$; Hölder's inequality, together with the bound in part (i), give us

$$\|D^n \cdot e^{-sD^2}\|_1 \leq \|D^n e^{-\frac{s}{2}D^2}\| \cdot \|e^{-\frac{s}{2}D^2}\|_1 \leq c_{n,\frac{s}{2}} \cdot \|e^{-\frac{s}{2}D^2}\|_1.$$

(iv) Fix $s_0 \in (0, \infty)$. Our first goal is to establish an upper bound on $|e^{-s_0x^2} - e^{-sx^2}|$; this will allow us to use the Spectral Theorem in the operator norm case, and Theorem 4.2.1 in the trace norm case to obtain the desired result. We will first establish the desired bound in the claim below; then discuss the operator norm continuity of the function, the trace norm continuity when $n = 0$, and finally the trace norm continuity for $n > 0$. Note that the proof of the claim could also be achieved (with somewhat less stress) via an application of the Mean Value Theorem.

Claim Recall $s_0 \in (0, \infty)$ is fixed. Suppose $s \in (0, \infty)$ is such that $|s - s_0| \leq \frac{s_0}{2}$. Then, for all $x \in \mathbb{R}$, $|e^{-s_0x^2} - e^{-sx^2}| \leq |s - s_0| \cdot (e^{-\frac{s_0}{2}x^2} x^2)$.

PROOF If t is any non-negative real number then $1 - e^{-tx^2} \leq tx^2$ (this is a rearrangement of the inequality in Lemma B.0.6, part (i)).

Suppose first that $s_0 \leq s$. Then $s - s_0 \geq 0$, and

$$|e^{-s_0x^2} - e^{-sx^2}| = e^{-s_0x^2} - e^{-sx^2} = e^{-s_0x^2} (1 - e^{-(s-s_0)x^2}) \leq e^{-s_0x^2} (s - s_0) \cdot x^2.$$

Since $s_0 \geq \frac{s_0}{2}$ we have $e^{-s_0x^2} \leq e^{-\frac{s_0}{2}x^2}$ for all $x \in \mathbb{R}$, which gives us the desired inequality in this case.

Now suppose $s_0 > s$. Exchanging s and s_0 in the previous calculation

$$|e^{-s_0x^2} - e^{-sx^2}| = e^{-sx^2} - e^{-s_0x^2} \leq e^{-sx^2} (s_0 - s) \cdot x^2.$$

In order to get rid of the dependence on s , recall that, by hypothesis, $s \geq \frac{s_0}{2}$, and so $e^{-sx^2} \leq e^{-\frac{s_0}{2}x^2}$; so once again we've reached the desired result. Therefore, for any $x \in \mathbb{R}$,

$$|e^{-sx^2} - e^{-s_0x^2}| \leq |s_0 - s| \cdot x^2 e^{-\frac{s_0}{2}x^2},$$

as long as $|s - s_0| \leq \frac{s_0}{2}$. ◇

Armed with the above claim, we are ready to show that $s \mapsto D^n e^{-sD^2}$ is continuous at s_0 (in operator norm) for any $n \geq 0$. From the proven inequality it follows that, for any $x \in \mathbb{R}$, if $|s - s_0| \leq \frac{s_0}{2}$ then

$$\begin{aligned} |x^n e^{-s_0 x^2} - x^n e^{-s x^2}| &\leq |x^n| |s_0 - s| \cdot x^2 e^{-\frac{s_0}{2} x^2} \\ &\leq |s_0 - s| |x^{n+2} e^{-\frac{s_0}{2} x^2}| \\ &\leq |s_0 - s| c_{n+2, \frac{s_0}{2}}, \end{aligned}$$

where in the last line we used the fact that, if $f(x) = x^n e^{-s x^2}$, then $|f(x)| \leq c_{n,s}$, as shown in the proof of (i) above. Since the right hand side does not depend on x we can conclude that $\|D^n e^{-s_0 D^2} - D^n e^{-s D^2}\| \leq |s_0 - s| \cdot c_{n+2, \frac{s_0}{2}}$ by the Spectral Theorem. Finally, since $c_{n+2, \frac{s_0}{2}}$ does not depend on s , it follows that $\lim_{s \rightarrow s_0} \|D^n e^{-s_0 D^2} - D^n e^{-s D^2}\| = 0$, concluding the proof that $s \mapsto D^n e^{-s D^2}$ is continuous (in operator norm) at $s = s_0$.

Next, we prove the continuity of $s \mapsto D^n e^{-s D^2}$ in trace norm. Suppose first that $n = 0$; and, in order to use the result in the claim, assume $|s - s_0| \leq \frac{s_0}{2}$. With E_λ the spectral resolution of $|D|$ and $\varphi_\lambda = \text{Tr}(E_\lambda)$, use Theorem 4.2.1 to write

$$\begin{aligned} \|e^{-s_0 D^2} - e^{-s D^2}\|_1 &= \text{Tr}(|e^{-s_0 D^2} - e^{-s D^2}|) \\ &= \int_0^\infty |e^{-s_0 \lambda^2} - e^{-s \lambda^2}| d\varphi_\lambda \\ &\leq \int_0^\infty |s - s_0| \cdot \lambda^2 e^{-\frac{s_0}{2} \lambda^2} d\varphi_\lambda \\ &= |s - s_0| \cdot \text{Tr}(D^2 e^{-\frac{s_0}{2} D^2}). \end{aligned}$$

Since $\text{Tr}(D^2 e^{-\frac{s_0}{2} D^2})$ depends on s_0 only, the continuity at $s = s_0$ of $s \mapsto e^{-s D^2}$, in trace norm, follows.

Finally, suppose $n > 0$. We can write $D^n e^{-s D^2} = D^n e^{-\frac{s}{2} D^2} \cdot e^{-\frac{s}{2} D^2}$. Since, as already proven, the first factor is continuous in operator norm and the second in trace norm, continuity of $s \mapsto D^n e^{-s D^2}$ in trace norm follows from Hölder's inequality. ■

Part (ii) of the following result is a key part of the proof that $d\alpha = 0$, as it gives us a formula for the difference of exponentials which can be appropriately manipulated.

Lemma 4.2.4 In the following, D is a fixed unbounded self-adjoint operator, and A is a bounded self-adjoint operator.

(i) For any $s_0 > 0$,

$$\left. \frac{d}{ds} \right|_{s=s_0} (e^{-s D^2}) = -D^2 \cdot e^{-s_0 D^2}.$$

(ii) For any fixed $s > 0$,

$$e^{-s(D+A)^2} - e^{-s D^2} = -s \cdot \int_0^1 D e^{-us D^2} A e^{-(1-u)s(D+A)^2} + e^{-us D^2} A (D+A) e^{-(1-u)s(D+A)^2} du. \quad \blacktriangleright$$

PROOF (i) The derivative agrees with what we would expect formally, but care needs to be taken since D^2 is an unbounded operator – although, of course, the condition that $s_0 > 0$ ensures

that e^{-sD^2} is bounded in a neighbourhood of s_0 , and $D^2e^{-s_0D^2}$ is likewise bounded. In order to prove that the derivative is as expected, we need to show that

$$\lim_{h \rightarrow 0} \left\| \frac{e^{-(s_0+h)D^2} - e^{-s_0D^2}}{h} + D^2e^{-s_0D^2} \right\| = 0.$$

By the Spectral theorem, it suffices to find a bound for $\left\{ \left| \frac{e^{-(s_0+h)\lambda^2} - e^{-s_0\lambda^2}}{h} + \lambda^2 e^{-s_0\lambda^2} \right| \mid \lambda \in \mathbb{R} \right\}$ which can be shown to go to zero as $h \rightarrow 0$. The proof uses the exponential bounds in Lemma B.0.7 and Lemma B.0.6, which require us to consider the cases $h > 0$ and $h < 0$ separately.

If $h > 0$, let $h = a > 0$. Then, using the results of Lemma B.0.6,

$$\begin{aligned} \left| \frac{e^{-(s_0+h)\lambda^2} - e^{-s_0\lambda^2}}{h} + \lambda^2 e^{-s_0\lambda^2} \right| &= \left| \frac{e^{-(s_0+a)\lambda^2} - e^{-s_0\lambda^2}}{a} + \lambda^2 e^{-s_0\lambda^2} \right| \\ &= e^{-s_0\lambda^2} \cdot \left(\frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \right) \\ &\leq e^{-s_0\lambda^2} \cdot a \cdot \frac{1}{2} \cdot \lambda^4. \end{aligned}$$

As shown in the proof of Lemma 4.2.3, $\lambda^4 e^{-s_0\lambda^2} \leq c_{4,s_0}$, where c_{4,s_0} is a constant whose value depends only on s_0 . Since $\lim_{a \rightarrow 0^+} a \cdot \frac{1}{2} \cdot c_{4,s_0} = 0$, this shows the desired result for $h > 0$.

If $h < 0$, let $h = -a$ for $a > 0$. In addition, pick n a positive integer for which $s_0 - \frac{1}{n} > 0$. As we are interested in the limit as $h = -a$ goes to zero, assume without loss of generality that $a < \frac{1}{n}$. Then, using the results of Lemma B.0.7,

$$\begin{aligned} \left| \frac{e^{-(s_0+h)\lambda^2} - e^{-s_0\lambda^2}}{h} + \lambda^2 e^{-s_0\lambda^2} \right| &= \left| \frac{e^{-(s_0-a)\lambda^2} - e^{-s_0\lambda^2}}{-a} + \lambda^2 e^{-s_0\lambda^2} \right| \\ &= e^{-s_0\lambda^2} \cdot \left(\frac{e^{a\lambda^2} - 1}{a} - \lambda^2 \right) \\ &\leq e^{-s_0\lambda^2} \cdot a \cdot n^2 \cdot e^{\frac{1}{n}\lambda^2}. \end{aligned}$$

As $e^{-s_0\lambda^2} \cdot e^{\frac{1}{n}\lambda^2} = e^{-(s_0 - \frac{1}{n})\lambda^2} \leq 1$ (it is here that we need the fact that $s_0 > \frac{1}{n}$), and n does not depend on a , $\lim_{a \rightarrow 0^+} a \cdot n^2 \cdot 1 = 0$, showing the desired result for $h < 0$.

Therefore, $\lim_{h \rightarrow 0} \left\| \frac{e^{-(s_0+h)D^2} - e^{-s_0D^2}}{h} + D^2e^{-s_0D^2} \right\| = 0$, showing that the derivative of $s \mapsto e^{-sD^2}$ for $s > 0$ has the expected formula (namely, $-D^2e^{-sD^2}$).

(ii) Let $F(u) = e^{-usD^2} e^{-(1-u)s(D+A)^2}$; then, as the product of two differentiable functions (by part (i)), $F(u)$ is differentiable on $(0, 1)$ (but not necessarily at $u = 0, 1$). Note that $F(0) = e^{-s(D+A)^2}$ and $F(1) = e^{-sD^2}$. In other words (as in the beginning of Section 10.3 of [15]), we have

$$e^{-s(D+A)^2} - e^{-sD^2} = - \int_0^1 \frac{d}{du} (F(u)) du.$$

Note that $F(u)$ is not norm continuous at $u = 0$ and $u = 1$, although $\text{SO-}\lim_{u \rightarrow 0} F(u) = F(0)$. We have to use this observation, combined with the fact that the integral converges in operator norm (see Lemma 4.2.8 and the beginning of the proof of Lemma 4.2.9 for how to show this), to conclude that the bounded operators on the left hand side and the right hand side of the equality must be equal.

Using the formula for the derivative from part (i), and standard differentiation rules,

$$\frac{d}{du}F(u) = -sD^2e^{-usD^2}e^{-(1-u)s(D+A)^2} + e^{-usD^2}s(D+A)^2e^{-(1-u)s(D+A)^2}.$$

In the case when $A\text{Dom}(D) \subset \text{Dom}D$ we have that $\text{Dom}(D+A)^2 = \text{Dom}D^2$, so there are no domain issues in the following calculations

$$\begin{aligned} & -sD^2e^{-usD^2}e^{-(1-u)s(D+A)^2} + e^{-usD^2}s(D+A)^2e^{-(1-u)s(D+A)^2} \\ &= se^{-usD^2}[(D+A)^2 - D^2]e^{-(1-u)s(D+A)^2} \\ &= se^{-usD^2}[DA + A(D+A)]e^{-(1-u)s(D+A)^2} \\ &= sDe^{-usD^2}Ae^{-(1-u)s(D+A)^2} + se^{-usD^2}A(D+A)e^{-(1-u)s(D+A)^2}. \end{aligned}$$

For a general A bounded and self-adjoint, equality might not hold in the middle steps; however, the final expression still makes sense, and we would like to show that

$$\begin{aligned} & -sD^2e^{-usD^2}e^{-(1-u)s(D+A)^2} + e^{-usD^2}s(D+A)^2e^{-(1-u)s(D+A)^2} \\ &= sDe^{-usD^2}Ae^{-(1-u)s(D+A)^2} + se^{-usD^2}A(D+A)e^{-(1-u)s(D+A)^2}. \end{aligned}$$

continues to hold. We follow the approach of Lemma 6 in Appendix B of [14], and approximate A by better-suited operators. Let $E_n = \chi_{[-n,n]}(D)$, and $A_n = E_n A E_n$. Then $A_n \rightarrow A$ strongly, and $A_n \text{Dom}(D) \subset \text{Dom}D$. To finish the proof, we need the following result of Rellich (quoted as Exercise XII.9.38 in [34], and slightly simplified below to the form we need):

Let T_n be self-adjoint for all $n \geq 1$ and suppose $T_n x \rightarrow T x$ for all $x \in \text{Dom}(T)$. If T is self-adjoint then $f(T_n) \rightarrow f(T)$ strongly for each bounded continuous function f of a real variable.

It is easy to check that $T_n = D + A_n$ and $T = D + A$ satisfy the hypotheses. For $k > 0$ a constant, define $f_j(x) = x^j e^{-kx^2}$ for $j = 0, 1, 2$, and apply the result in turn for the three f_j 's to conclude that $(D + A_n)^j e^{-k(D+A_n)^2} \rightarrow (D + A)^j e^{-k(D+A)^2}$ strongly for $j = 0, 1, 2$. We can then write

$$\begin{aligned} & -sD^2e^{-usD^2}e^{-(1-u)s(D+A)^2} + e^{-usD^2}s(D+A)^2e^{-(1-u)s(D+A)^2} \\ &= \text{SO-}\lim_{n \rightarrow \infty} \left(-sD^2e^{-usD^2}e^{-(1-u)s(D+A_n)^2} + e^{-usD^2}s(D+A_n)^2e^{-(1-u)s(D+A_n)^2} \right) \\ &= \text{SO-}\lim_{n \rightarrow \infty} \left(sDe^{-usD^2}A_n e^{-(1-u)s(D+A_n)^2} + se^{-usD^2}A_n(D+A_n)e^{-(1-u)s(D+A_n)^2} \right) \\ &= sDe^{-usD^2}Ae^{-(1-u)s(D+A)^2} + se^{-usD^2}A(D+A)e^{-(1-u)s(D+A)^2}. \end{aligned}$$

For the sake of completeness, we offer the following two comments on the above calculations, addressing the fact that multiplication is not strong operator continuous:

- The operators $D^2 e^{-usD^2}$, e^{-usD^2} and De^{-usD^2} are all bounded. Thus, for example, the fact that $e^{-(1-u)s(D+A_n)^2} \xrightarrow{\text{SO}} e^{-(1-u)s(D+A)}$ is sufficient to ensure that

$$D^2 e^{-usD^2} e^{-(1-u)s(D+A_n)^2} \xrightarrow{\text{SO}} D^2 e^{-usD^2} e^{-(1-u)s(D+A)}.$$

- In spite of the fact that $A_n \rightarrow A$ and $e^{-(1-u)s(D+A_n)^2} \rightarrow e^{-(1-u)s(D+A)^2}$ only in the strong operator topology, it is nonetheless true that $A_n e^{-(1-u)s(D+A_n)^2} \rightarrow A e^{-(1-u)s(D+A)^2}$ in the strong operator topology as well. Here we need to recall the fact that $A_n = E_n A E_n$, so E_n (which was by definition a spectral projection of D) commutes with both D and A_n . Hence $A_n e^{-(1-u)s(D+A_n)^2} = E_n A e^{-(1-u)s(D+A_n)^2} E_n$. For any $v \in \mathcal{H}$ we have

$$\begin{aligned} & \|E_n A e^{-(1-u)s(D+A_n)^2} E_n v - A e^{-(1-u)s(D+A)^2} v\| \\ & \leq \|E_n\| \cdot \|A\| \cdot \|e^{-(1-u)s(D+A_n)^2}\| \cdot \|E_n v - v\| \\ & \quad + \|E_n\| \cdot \|A\| \cdot \|e^{-(1-u)s(D+A_n)^2} v - e^{-(1-u)s(D+A)^2} v\| \\ & \quad + \|E_n A e^{-(1-u)s(D+A)^2} v - A e^{-(1-u)s(D+A)^2} v\|. \end{aligned}$$

The fact that $E_n \rightarrow 1$ and $e^{-(1-u)s(D+A_n)^2} \rightarrow e^{-(1-u)s(D+A)^2}$ in the strong operator topology, combined with the bounds $\|E_n\| \leq 1$ and $\|e^{-(1-u)s(D+A_n)^2}\| \leq 1$ for all n , allows us to conclude that $A_n e^{-(1-u)s(D+A_n)^2} \xrightarrow{\text{SO}} A e^{-(1-u)s(D+A)^2}$.

Therefore, for any D unbounded and self-adjoint and A bounded and self-adjoint we have

$$\begin{aligned} & -sD^2 e^{-usD^2} e^{-(1-u)s(D+A)^2} + e^{-usD^2} s(D+A)^2 e^{-(1-u)s(D+A)^2} \\ & = sD e^{-usD^2} A e^{-(1-u)s(D+A)^2} + s e^{-usD^2} A(D+A) e^{-(1-u)s(D+A)^2}. \end{aligned}$$

Substituting into the earlier integral formula we get

$$e^{-s(D+A)^2} - e^{-sD^2} = -s \cdot \int_0^1 D e^{-usD^2} A e^{-(1-u)s(D+A)^2} + e^{-usD^2} A(D+A) e^{-(1-u)s(D+A)^2} du,$$

as desired. ■

We state the next result here so we can reference it when necessary; however, the proof is farmed out to Lemma 4.2.12 and Lemma 4.2.16.

Proposition 4.2.5 $\alpha_\varepsilon \oplus -\frac{1}{2}\eta_\varepsilon$ is a representative of a class in $H_{dR}^1(\Phi, \Phi_0)$; that is,

- (i) $d\alpha_\varepsilon = 0$, and
- (ii) $d\eta_\varepsilon = -2\alpha_\varepsilon$ on Φ_0 . ☞

4.2.1 Show $d\alpha_\varepsilon = 0$

Recall $\alpha_\varepsilon = \left(\frac{\varepsilon}{\pi}\right)^{1/2} \text{Tr}(X e^{-\varepsilon D^2})$ for $X \in T_D(\Phi)$. The calculation of $d\alpha$ will require us to deal with differences of the type $e^{-t(D+hY)^2} - e^{-tD^2}$, which we will handle through the use of Lemma 4.2.4 to get an integral formula. As the goal is then to take the limit as $h \rightarrow 0$, we need to establish various norm results for expressions appearing in the integral formula.

Lemma 4.2.6 ([14], **Corollary 8, part (2) of Appendix B**) Suppose D is an unbounded self-adjoint operator, with $\text{Tr}(e^{-tD^2}) < \infty$ for $t > 0$. Suppose A is a bounded self-adjoint operator. With f our old friend $f(a) = 1 + \frac{1}{2}a(a + \sqrt{a^2 + 4})$, $\text{Tr}(e^{-t(D+A)^2}) \leq e^{1 - \frac{1}{f(t^{1/2} \cdot \|A\|)}} \cdot \text{Tr}\left(e^{-tD^2 \cdot \frac{1}{f(t^{1/2} \cdot \|A\|)}}\right)$. \blacktriangleright

Remark 4.2.7 The expressions involved in the above result look quite complex, so we point out some simplifications. To start with, note, since $f(t^{1/2} \cdot \|A\|) > 0$, that $e^{1 - \frac{1}{f(t^{1/2} \cdot \|A\|)}} \leq e$.

If $\|A\| \leq 1$, and $t \geq \frac{1}{n}$, then $\frac{t}{f(\sqrt{t}\|A\|)} \geq \frac{1}{3n+1}$. To see this, use the fact that f is an increasing function, so

$$\begin{aligned} f(\sqrt{t}\|A\|) &\leq f(\sqrt{t}) = 1 + \frac{1}{2}t + \frac{1}{2}\sqrt{t}\sqrt{t+4} \leq 1 + \frac{1}{2}t + \frac{1}{2}(t+4) \\ &= 3 + t \leq 3nt + t = (3n+1)t. \end{aligned}$$

Rearrange to get the claimed inequality. This observation allows us, when these two conditions are satisfied, to write $\text{Tr}\left(e^{-tD^2 \cdot \frac{1}{f(t^{1/2} \cdot \|A\|)}}\right) \leq \text{Tr}\left(e^{-\frac{1}{3n+1}D^2}\right)$ (see Lemma 4.2.3, part (ii)), removing the dependence on A and some of the dependence on t from the right hand side of the inequality. \bullet

Lemma 4.2.8 Suppose $u \in (0, 1)$, $t > 0$ and $A, B, C \in \mathcal{B}(\mathcal{H})_{sa}$. Then

- (i) $\|BDe^{-tuD^2}Ce^{-t(1-u)(D+A)^2}\| \leq \|B\| \cdot \|C\| \cdot \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{ut}}$.
- (ii) $\|Be^{-tuD^2}C(D+A)e^{-t(1-u)(D+A)^2}\| \leq \|B\| \cdot \|C\| \cdot \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{(1-u)t}}$.

Further assume that $\|A\| \leq 1$, and that n is a positive integer such that $t \geq \frac{1}{n}$. Then

- (iii) $\|BDe^{-tuD^2}Ce^{-t(1-u)(D+A)^2}\|_1 \leq \|B\| \cdot \|C\| \cdot \sqrt{e} \cdot \frac{1}{\sqrt{ut}} \cdot \left\|e^{-\frac{1}{12n+1}D^2}\right\|_1$.
- (iv) $\|Be^{-tuD^2}C(D+A)e^{-t(1-u)(D+A)^2}\|_1 \leq \|B\| \cdot \|C\| \cdot \sqrt{e} \cdot \frac{1}{\sqrt{(1-u)t}} \cdot \left\|e^{-\frac{1}{12n+1}D^2}\right\|_1$. \blacktriangleright

PROOF Using the norm bounds in Lemma 4.2.3, we have, with $n_1 = 1$ and $n_2 = 0$ or vice-versa,

$$\begin{aligned} \|BD^{n_1}e^{-tuD^2} \cdot C \cdot (D+A)^{n_2}e^{-t(1-u)(D+A)^2}\| &\leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\| \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\| \\ &\leq \|B\| \cdot c_{n_1,ut} \cdot \|C\| \cdot c_{n_2,t(1-u)}. \end{aligned}$$

When $n_1 = 1$ and $n_2 = 0$, we have $c_{1,ut} = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{ut}}$ and $c_{0,t(1-u)} = 1$, and the inequality simplifies to the expression given in (i). When $n_1 = 0$ and $n_2 = 1$, we have $c_{0,ut} = 1$ and $c_{1,t(1-u)} = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{(1-u)t}}$, and the inequality simplifies to the expression given in (ii).

To establish the trace norm bounds, we have to be a bit more subtle in order to get the desired expressions; we consider two cases, depending on the value of u , and in each case we split up the expression differently. Again, to minimize the amount of work, use n_1 for the power of D and n_2 for the power of $D+A$, and substitute the appropriate values of one and zero at the end.

First suppose that $u \in (0, \frac{1}{2}]$. Apply Hölder's inequality, followed by Lemma 4.2.3 and Lemma 4.2.6:

$$\begin{aligned} \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 &\leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\| \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 \\ &\leq \|B\| \cdot c_{n_1,tu} \cdot \|C\| \cdot c_{n_2,\frac{t(1-u)}{2}} \cdot \|e^{-\frac{t(1-u)}{2}(D+A)^2}\|_1 \\ &\leq \|B\| \cdot c_{n_1,tu} \cdot \|C\| \cdot c_{n_2,\frac{t(1-u)}{2}} \cdot e \cdot \|e^{-\frac{t(1-u)}{2}D^2}\|_1. \end{aligned}$$

To simplify this expression somewhat, recall that $c_{n_2,\frac{1}{2}t(1-u)} \leq 2^{n_2/2}c_{n_2,t(1-u)}$ (equality holds if $n_2 \neq 0$). Further note that, since $u \in (0, \frac{1}{2}]$, $\frac{1}{4} \leq \frac{1-u}{2} \leq \frac{1}{2}$; so in particular, given the assumption $t > \frac{1}{n}$, we have $\frac{1}{2}t(1-u) \geq \frac{1}{4n}$. As pointed out in Remark 4.2.7, this means $\|e^{-\frac{t(1-u)}{2}D^2}\|_1 \leq \|e^{-\frac{1}{12n+1}D^2}\|_1$. Hence, if $u \in (0, \frac{1}{2}]$ we get

$$\|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 \leq 2^{n_2/2} \cdot e \cdot c_{n_1,tu} \cdot c_{n_2,t(1-u)} \cdot \|B\| \cdot \|C\| \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1.$$

Now suppose that $u \in (\frac{1}{2}, 1)$. Then, appealing to the same results as earlier for our norm bounds, we have:

$$\begin{aligned} \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 &\leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\|_1 \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 \\ &\leq \|B\| \cdot c_{n_1,\frac{tu}{2}} \cdot \|e^{-\frac{tu}{2}D^2}\|_1 \cdot \|C\| \cdot c_{n_2,t(1-u)}. \end{aligned}$$

Once again, use the fact that $c_{n_1,\frac{tu}{2}} \leq 2^{n_1/2}c_{n_1,tu}$. Here, since $u \in (\frac{1}{2}, 1)$ we have $\frac{tu}{2} \geq \frac{t}{4} \geq \frac{1}{4n}$, so $\|e^{-\frac{tu}{2}D^2}\|_1 \leq \|e^{-\frac{1}{4n}D^2}\|_1$. Thus, for $u \in (\frac{1}{2}, 1)$,

$$\|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 \leq 2^{n_1/2} \cdot c_{n_1,tu} \cdot c_{n_2,t(1-u)} \cdot \|B\| \cdot \|C\| \cdot \|e^{-\frac{1}{4n}D^2}\|_1.$$

Combining the two bounds by choosing the largest of each factor; that is, for any $u \in (0, 1)$ conclude that

$$\|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2}\|_1 \leq 2^{\frac{\max\{n_1,n_2\}}{2}} \cdot e \cdot c_{n_1,tu} \cdot c_{n_2,t(1-u)} \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1.$$

Here we use the fact that $\frac{1}{4n} \geq \frac{1}{12n+1}$ to decide which of the trace norms is larger. Finally, when $n_1 = 1$ and $n_2 = 0$ we have $c_{1,tu} = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{ut}}$ and $c_{0,t(1-u)} = 1$; simplifying gives us the inequality in (iii). When $n_1 = 0$ and $n_2 = 1$ we have $c_{0,tu} = 1$ and $c_{n_2,t(1-u)} = \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{t(1-u)}}$; simplifying gives the inequality claimed in (iv). ■

We now tackle the question of finding norm bounds for differences of the type of expressions appearing in the integral formula of Lemma 4.2.4; we will need these bounds to prove continuity results and convergence of limits involving integrals. A different bound for $\|e^{-t(D+A)^2} - e^{-tD^2}\|$ (though similar in magnitude) and a completely different proof for said bound than the one presented below can be found in [14], Corollary 18 of Appendix A.

Lemma 4.2.9 Suppose D is an unbounded self-adjoint operator, with $\text{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$. If A is a bounded self-adjoint operator such that $\|A\| \leq 1$, and n is an integer for which $s \geq \frac{1}{n}$, we have the following norm estimates:

- (i) $\|e^{-s(D+A)^2} - e^{-sD^2}\| \leq \|A\| \cdot s^{\frac{1}{2}} \cdot \left(\frac{2\sqrt{2}}{\sqrt{e}}\right)$ and
 $\|e^{-s(D+A)^2} - e^{-sD^2}\|_1 \leq \|A\| \cdot s^{\frac{1}{2}} \cdot (4\sqrt{e}) \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1$
- (ii) $\|(D+A)e^{-s(D+A)^2} - De^{-sD^2}\| \leq \|A\| \cdot \left(\frac{4}{e} + 1\right)$ and
 $\|(D+A)e^{-s(D+A)^2} - De^{-sD^2}\|_1 \leq \|A\| \cdot (4\sqrt{2} + 1) \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1$ ☛

PROOF (i) Use the formula from Lemma 4.2.4 (part (ii)) for the difference of exponentials; that is,

$$e^{-s(D+A)^2} - e^{-sD^2} = -s \cdot \left(\int_0^1 De^{-usD^2} Ae^{-(1-u)s(D+A)^2} + e^{-usD^2} A(D+A)e^{-(1-u)s(D+A)^2} du \right).$$

Hence we can estimate the operator norm via

$$\begin{aligned} & \|e^{-s(D+A)^2} - e^{-sD^2}\| \\ & \leq s \left(\int_0^1 \|De^{-usD^2} Ae^{-(1-u)s(D+A)^2}\| du + \int_0^1 \|e^{-usD^2} A(D+A)e^{-(1-u)s(D+A)^2}\| du \right). \end{aligned}$$

From Lemma 4.2.8 (used with $B = 1$ and $C = A$),

$$\begin{aligned} \|De^{-usD^2} Ae^{-(1-u)s(D+A)^2}\| & \leq \|A\| \cdot \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{us}}, \text{ and} \\ \|e^{-usD^2} A(D+A)e^{-(1-u)s(D+A)^2}\| & \leq \|A\| \cdot \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{(1-u)s}}. \end{aligned}$$

The fact that $\int_0^1 \frac{1}{\sqrt{u}} du = 2$ and $\int_0^1 \frac{1}{\sqrt{1-u}} du = 2$ allows us to conclude that

$$\|e^{-s(D+A)^2} - e^{-sD^2}\| \leq s \cdot \|A\| \cdot \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{s}} \cdot (2 + 2) = \|A\| \cdot \frac{2\sqrt{2}}{\sqrt{e}} \cdot \sqrt{s},$$

as claimed.

Similarly use the trace norms from Lemma 4.2.8. To start with,

$$\begin{aligned} & \|e^{-s(D+A)^2} - e^{-sD^2}\|_1 \\ & \leq s \cdot \left(\int_0^1 \|De^{-usD^2} Ae^{-(1-u)s(D+A)^2}\|_1 du + \int_0^1 \|e^{-usD^2} A(D+A)e^{-(1-u)s(D+A)^2}\|_1 du \right). \end{aligned}$$

Recall that n is such that $s \geq \frac{1}{n}$ and does not depend on u ; since

$$\begin{aligned} \|De^{-usD^2} Ae^{-(1-u)s(D+A)^2}\|_1 & \leq \sqrt{e} \cdot \|A\| \cdot \frac{1}{\sqrt{su}} \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1, \text{ and} \\ \|e^{-usD^2} A(D+A)e^{-(1-u)s(D+A)^2}\|_1 & \leq \sqrt{e} \cdot \|A\| \cdot \frac{1}{\sqrt{s(1-u)}} \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1, \end{aligned}$$

we can again integrate the expressions depending on u and simplify to get

$$\|e^{-s(D+A)^2} - e^{-sD^2}\|_1 \leq 4\sqrt{e} \cdot \sqrt{s} \cdot \|A\| \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1.$$

(ii) Split up the difference $(D+A)e^{-s(D+A)^2} - De^{-sD^2}$ in such a way that we do not have to deal with unbounded operators, and can use the bounds for the exponential difference computed above. Namely (and the calculation is performed in a bit more detail in the beginning of the proof of Lemma 4.2.16) write

$$(D+A)e^{-s(D+A)^2} - De^{-sD^2} = (D+A)e^{-\frac{s}{2}(D+A)^2} (e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2}) \\ + e^{-\frac{s}{2}(D+A)^2} A e^{-\frac{s}{2}D^2} + (e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2}) D e^{-\frac{s}{2}D^2},$$

then use the norm bounds from Lemma 4.2.3 and part (i).

For the operator norm bound:

$$\|(D+A)e^{-s(D+A)^2} - De^{-sD^2}\| \\ \leq \|(D+A)e^{-\frac{s}{2}(D+A)^2}\| \cdot \|e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2}\| \\ + \|e^{-\frac{s}{2}(D+A)^2}\| \cdot \|A\| \cdot \|e^{-\frac{s}{2}D^2}\| + \|(e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2})\| \cdot \|De^{-\frac{s}{2}D^2}\| \\ \leq c_{1,\frac{s}{2}} \cdot \|A\| \cdot \frac{\sqrt{s}}{2} \cdot \frac{2\sqrt{2}}{\sqrt{e}} + \|A\| + \|A\| \cdot \frac{\sqrt{s}}{2} \cdot \frac{2\sqrt{2}}{\sqrt{e}} \cdot c_{1,\frac{s}{2}}.$$

Since $c_{1,\frac{s}{2}} = \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{2}{s}}$, the expression simplifies to the desired result.

For the trace norm bound, use part (i) above and Lemma 4.2.3 (note that, by assumption, $\frac{s}{2} \geq \frac{1}{2n}$, which gives us the trace term):

$$\|(D+A)e^{-s(D+A)^2} - De^{-sD^2}\|_1 \\ \leq \|(D+A)e^{-\frac{s}{2}(D+A)^2}\| \cdot \|e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2}\|_1 \\ + \|e^{-\frac{s}{2}(D+A)^2}\| \cdot \|A\| \cdot \|e^{-\frac{s}{2}D^2}\|_1 + \|(e^{-\frac{s}{2}(D+A)^2} - e^{-\frac{s}{2}D^2})\|_1 \cdot \|De^{-\frac{s}{2}D^2}\|_1 \\ \leq c_{1,\frac{s}{2}} \cdot \sqrt{\frac{s}{2}} \cdot \|A\| \cdot 4\sqrt{e} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1 + 1 \cdot \|A\| \cdot \|e^{-\frac{1}{2n}D^2}\|_1 + \sqrt{\frac{s}{2}} \cdot \|A\| \cdot 4\sqrt{e} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1 \cdot c_{1,\frac{s}{2}}.$$

Again, $c_{1,\frac{s}{2}} = \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{2}{s}}$; moreover, $\|e^{-\frac{1}{2n}D^2}\|_1 \leq \|e^{-\frac{1}{24n+1}D^2}\|_1$ (by Lemma 4.2.3, part (ii)). We can thus simplify the last expression to reach the desired final bound. ■

Lemma 4.2.10 Suppose that $u \in (0, 1)$, $t > 0$, and $A, B, C \in \mathcal{B}(\mathcal{H})$ with $\|A\| \leq 1$.

(i) Operator norm bounds:

$$\|BDe^{-tuD^2}Ce^{-t(1-u)(D+A)^2} - BDe^{-tuD^2}Ce^{-t(1-u)D^2}\| \leq \|A\| \cdot \|B\| \cdot \|C\| \cdot \frac{2}{e} \cdot u^{-\frac{1}{2}}; \\ \|Be^{-tuD^2}C(D+A)e^{-t(1-u)(D+A)^2} - Be^{-tuD^2}CDe^{-t(1-u)D^2}\| \leq \|A\| \cdot \|B\| \cdot \|C\| \cdot \left(\frac{4}{e} + 1\right).$$

(ii) Suppose further that n is a positive integer such that $t \geq \frac{1}{n}$. Trace norm bounds:

$$\|BDe^{-tuD^2}Ce^{-t(1-u)(D+A)^2} - BDe^{-tuD^2}Ce^{-t(1-u)D^2}\|_1 \\ \leq \|A\| \cdot \|B\| \cdot \|C\| \cdot 2\sqrt{2} \cdot u^{-\frac{1}{2}} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1, \text{ and} \\ \|Be^{-tuD^2}C(D+A)e^{-t(1-u)(D+A)^2} - Be^{-tuD^2}CDe^{-t(1-u)D^2}\|_1 \\ \leq \|A\| \cdot \|B\| \cdot \|C\| \cdot (4\sqrt{2} + 1) \cdot \|e^{-\frac{1}{48n+1}D^2}\|_1.$$

PROOF (i) For any n_1, n_2 non-negative integers, we have

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\| \\ & \leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\| \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\| \\ & \leq \|B\| \cdot c_{n_1,ut} \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\|. \end{aligned}$$

For the difference, use the operator norm bound from Lemma 4.2.9. For $n_2 = 0$,

$$\|e^{-t(1-u)(D+A)^2} - e^{-t(1-u)D^2}\| \leq \|A\| \cdot \sqrt{t(1-u)} \cdot \frac{2\sqrt{2}}{\sqrt{e}} \leq \|A\| \cdot \sqrt{t} \cdot \frac{2\sqrt{2}}{\sqrt{e}}.$$

Combine this with the fact that (for $n_1 = 1$) $c_{1,ut} = \frac{1}{\sqrt{2e}}u^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}$ to obtain the first inequality. For $n_1 = 0$ and $n_2 = 1$, we have $c_{0,ut} = 1$ and $\|(D+A)e^{-t(1-u)(D+A)^2} - De^{-t(1-u)D^2}\| \leq \|A\| \cdot \left(\frac{4}{e} + 1\right)$; simplifying, we get the second inequality.

(ii) Again, suppose first that n_1 and n_2 are any non-negative integers, then simplify the upper bound formula for the cases when n_1 and n_2 have values zero and one. Because of the lower bound on the value of the coefficient s of the exponent in the trace norm formulas, we consider two cases, based on the value of u .

For $u \in (0, \frac{1}{2}]$,

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\| \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot c_{n_1,ut} \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\|_1. \end{aligned}$$

Since $u \in (0, \frac{1}{2}]$, the original assumption that $t \geq \frac{1}{n}$ gives $t(1-u) \geq \frac{1}{2n}$. When $n_1 = 1$ and $n_2 = 0$, we have $c_{1,ut} = \frac{1}{\sqrt{2e}\sqrt{ut}}$ and $\|e^{-t(1-u)(D+A)^2} - e^{-t(1-u)D^2}\|_1 \leq [t(1-u)]^{\frac{1}{2}} \cdot \|A\| \cdot 4\sqrt{e} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1$. Simplifying, when $n_1 = 1$, $n_2 = 0$ and $u \in (0, \frac{1}{2}]$ we get

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|C\| \cdot \|A\| \cdot 2\sqrt{2} \cdot u^{-\frac{1}{2}} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1. \end{aligned}$$

When $n_1 = 0$ and $n_2 = 1$, we have $c_{0,ut} = 1$ and

$$\|(D+A)e^{-t(1-u)(D+A)^2} - De^{-t(1-u)D^2}\|_1 \leq \|A\| \cdot (4\sqrt{2} + 1) \cdot \|e^{-\frac{1}{48n+1}D^2}\|_1.$$

Simplifying, when $n_1 = 0$, $n_2 = 1$ and $u \in (0, \frac{1}{2}]$ we get

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|C\| \cdot \|A\| \cdot (4\sqrt{2} + 1) \cdot \|e^{-\frac{1}{48n+1}D^2}\|_1. \end{aligned}$$

For $u \in (\frac{1}{2}, 1)$,

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|D^{n_1}e^{-tuD^2}\|_1 \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot c_{n_1, \frac{ut}{2}} \cdot \|e^{-\frac{tu}{2}D^2}\|_1 \cdot \|C\| \cdot \|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\|_1. \end{aligned}$$

In this case, we have $\frac{tu}{2} \geq \frac{1}{4n}$, so $\|e^{-\frac{tu}{2}D^2}\|_1 \leq \|e^{-\frac{1}{4n}D^2}\|_1$. We now simplify the expression for our desired values of n_1 and n_2 . When $n_1 = 1$ and $n_2 = 0$, $c_{n_1, \frac{tu}{2}} = \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{2}{ut}}$, and $\|e^{-t(1-u)(D+A)^2} - e^{-t(1-u)D^2}\| \leq \|A\| \cdot \sqrt{t(1-u)} \cdot \frac{2\sqrt{2}}{\sqrt{e}} \leq \|A\| \cdot \sqrt{t} \cdot \frac{2\sqrt{2}}{\sqrt{e}}$. Thus, when $n_1 = 1$, $n_2 = 0$ and $u \in (\frac{1}{2}, 1)$, we get

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|C\| \cdot \|A\| \cdot \frac{2\sqrt{2}}{e} \cdot u^{-\frac{1}{2}} \cdot \|e^{-\frac{1}{4n}D^2}\|_1. \end{aligned}$$

When $n_1 = 0$ and $n_2 = 1$, $c_{0, \frac{tu}{2}} = 1$ and $\|(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - D^{n_2}e^{-t(1-u)D^2}\| \leq \|A\| \cdot (\frac{4}{e} + 1)$. This gives us the bound for $n_1 = 0$, $n_2 = 1$ and $u \in (\frac{1}{2}, 1)$

$$\begin{aligned} & \|BD^{n_1}e^{-tuD^2}C(D+A)^{n_2}e^{-t(1-u)(D+A)^2} - BD^{n_1}e^{-tuD^2}CD^{n_2}e^{-t(1-u)D^2}\|_1 \\ & \leq \|B\| \cdot \|C\| \cdot \|A\| \cdot (\frac{4}{e} + 1) \cdot \|e^{-\frac{1}{4n}D^2}\|_1. \end{aligned}$$

Finally, we have to reconcile the different bounds for $u \in (0, \frac{1}{2}]$ and $u \in (\frac{1}{2}, 1)$. Comparing the different factors appearing, and observing that $\|e^{-\frac{1}{4n}D^2}\|_1 \leq \|e^{-\frac{1}{24n+1}D^2}\|_1 \leq \|e^{-\frac{1}{48n+1}D^2}\|_1$ allows us to get the final formulas. ■

The following result is the culmination of all these norm bounds, and is used in both the calculation of $d\alpha$ and that of $d\eta$.

Lemma 4.2.11 Suppose D is a self-adjoint unbounded operator such that $\text{Tr}(e^{-tD^2}) < \infty$ for any $t > 0$, and $X \in \mathcal{B}(\mathcal{H})_{sa}$. For any fixed $t > 0$,

$$\|\cdot\|_* \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left(e^{-t(D+hX)^2} - e^{-tD^2} \right) = -t \cdot \left(\int_0^1 (De^{-utD^2}Xe^{-(1-u)tD^2} + e^{-utD^2}XDe^{-(1-u)tD^2}) du \right),$$

where $\|\cdot\|_*$ can stand for either the operator norm or the trace norm. ☞

PROOF Using the formula for the difference of exponential expressions in Lemma 4.2.4 with $A = hX$ we get

$$\begin{aligned} & \frac{1}{h} \cdot (e^{-t(D+hX)^2} - e^{-tD^2}) \\ & = -t \cdot \left(\int_0^1 De^{-utD^2}Xe^{-(1-u)t(D+hX)^2} + e^{-utD^2}X(D+hX)e^{-(1-u)t(D+hX)^2} du \right). \end{aligned}$$

To obtain the desired result, it is sufficient to show that

$$\begin{aligned} \|\cdot\|_* - \lim_{h \rightarrow 0} \int_0^1 De^{-utD^2}Xe^{-(1-u)t(D+hX)^2} du &= \int_0^1 De^{-utD^2}Xe^{-(1-u)tD^2}, \text{ and} \\ \|\cdot\|_* - \lim_{h \rightarrow 0} \int_0^1 e^{-utD^2}X(D+hX)e^{-(1-u)t(D+hX)^2} du &= \int_0^1 e^{-utD^2}XDe^{-(1-u)tD^2} du. \end{aligned}$$

From Lemma 4.2.3 we can easily conclude that $u \mapsto e^{-utD^2}$, $u \mapsto De^{-utD^2}$ are continuous in u (for $u \in (0, 1)$) in both operator norm and trace norm, and with $A = hX$ or $A = 0$ so are $u \mapsto e^{-(1-u)t(D+hX)^2}$ and $u \mapsto (D+hX)e^{-(1-u)t(D+hX)^2}$; thus, the integrands are all continuous as functions of u .

We show the first limit; the second is similar (the bounds used for the second limit can be found in the same lemmas as those used for the first). Since we want to calculate the limit

as $h \rightarrow 0$, we can assume without loss of generality that $\|hX\| < 1$. Since $t > 0$, there exists a positive integer n such that $t > \frac{1}{n}$. By Lemma 4.2.8, for $h \geq 0$ but small enough to ensure $\|hX\| < 1$, we have

$$\begin{aligned} \|De^{-utD^2}Xe^{-(1-u)t(D+hX)^2}\| &\leq \frac{1}{\sqrt{2e}} \cdot \|X\| \cdot \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{u}} \\ \|De^{-utD^2}Xe^{-(1-u)t(D+hX)^2}\|_1 &\leq \sqrt{e} \cdot \|X\| \cdot \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{u}} \cdot \|e^{-\frac{1}{12n+1}D^2}\|_1. \end{aligned}$$

As $\frac{1}{\sqrt{u}}$ is the only factor which depends on u , and $\int_0^1 u^{-\frac{1}{2}} du = 2 < \infty$, the functions we are considering have finite integrals.

Finally, we want to show that the limit is indeed as claimed. By Lemma 4.2.10,

$$\begin{aligned} \int_0^1 \|De^{-usD^2}Xe^{-(1-u)s(D+hX)^2} - De^{-usD^2}Xe^{-(1-u)sD^2}\| du &\leq \|X\| \cdot \|hX\| \cdot \frac{2}{e} \cdot \int_0^1 u^{-\frac{1}{2}} du, \text{ and} \\ \int_0^1 \|De^{-usD^2}Xe^{-(1-u)s(D+hX)^2} - De^{-usD^2}Xe^{-(1-u)sD^2}\|_1 du &\leq \|X\| \cdot \|hX\| \cdot 2\sqrt{2} \cdot \|e^{-\frac{1}{24n+1}D^2}\|_1 \cdot \int_0^1 u^{-\frac{1}{2}} du. \end{aligned}$$

This shows that $\lim_{h \rightarrow 0} \left\| \int_0^1 De^{-usD^2}Xe^{-(1-u)s(D+hX)^2} du - \int_0^1 De^{-usD^2}Xe^{-(1-u)sD^2} du \right\|_* = 0$, concluding the proof of the first limit. The other integral is dealt with similarly, allowing us to obtain the desired formula for $\lim_{h \rightarrow 0} \frac{1}{h} \cdot (e^{-t(D+hX)^2} - e^{-tD^2})$. ■

Lemma 4.2.12 $d\alpha_\varepsilon = 0$. ☞

PROOF We need only show that $d\alpha_{\varepsilon,D}(X) = 0$ for $\varepsilon = 1$; since $\alpha_{\varepsilon,D}(X) = \sqrt{\varepsilon} \cdot \alpha_{1,\varepsilon D}(X)$ result follows for any $\varepsilon > 0$. We have $d\alpha_{1,D}(X, Y) = X\alpha_1(Y) - Y\alpha_1(X) - \alpha([X, Y])$ by definition, but the last term is zero (as in the proof of Theorem 3.4.1), so we just have to show that $X\alpha_1(Y) = Y\alpha_1(X)$. We have

$$Y\alpha_1(X) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \sqrt{\frac{1}{\pi}} \cdot (\text{Tr}(Xe^{-(D+hY)^2} - Xe^{-D^2})).$$

Recall (Lemma 4.2.11, used with $t = 1$ and Y instead of X) that

$\|\cdot\|_1 - \lim_{h \rightarrow 0} \frac{1}{h}(e^{-(D+hY)^2} - e^{-D^2}) = \int_0^1 (De^{-utD^2}Ye^{-(1-u)tD^2} + e^{-utD^2}YDe^{-(1-u)tD^2}) du$; since X is bounded, we can thus show that

$$\begin{aligned} Y\alpha_1(X) &= \text{Tr}\left(\int_0^1 XDe^{-uD^2}Ye^{-(1-u)D^2} + Xe^{-uD^2}YDe^{-(1-u)D^2} du\right) \\ &= \int_0^1 \text{Tr}(XDe^{-uD^2}Ye^{-(1-u)D^2}) + \text{Tr}(Xe^{-uD^2}YDe^{-(1-u)D^2}) du, \end{aligned}$$

where for the last equality we justify switching the integral and trace by the fact that the integrand is continuous in trace norm and $\int_0^1 \|XDe^{-uD^2}Ye^{-(1-u)D^2} + Xe^{-uD^2}YDe^{-(1-u)D^2}\|_1 du < \infty$.

By symmetry, $X\alpha_1(Y) = \int_0^1 \text{Tr}(YDe^{-uD^2}Xe^{-(1-u)D^2} + Ye^{-uD^2}XDe^{-(1-u)D^2}) du$. Use the trace property to show that $Y\alpha_1(X) = X\alpha_1(Y)$; namely, since $Ye^{-(1-u)D^2} \in \mathcal{L}^1$ and $Xe^{-uD^2} \in \mathcal{L}^1$ for $u \neq 0, 1$ we have

$$\begin{aligned} Y\alpha_1(X) &= \int_0^1 \text{Tr}(XDe^{-uD^2}Ye^{-(1-u)D^2}) + \text{Tr}(Xe^{-uD^2}YDe^{-(1-u)D^2}) du \\ &= \int_0^1 \text{Tr}(Ye^{-(1-u)D^2}XDe^{-uD^2}) + \text{Tr}(YDe^{-(1-u)D^2}Xe^{-uD^2}) du. \end{aligned}$$

The change of variables $v = 1 - u$ then shows that this final expression is equal to $X\alpha_1(Y)$, concluding the proof that $d\alpha = 0$. ■

4.2.2 $d\eta_\varepsilon = -2\alpha_\varepsilon$ on Φ_0

Recall that $\eta_\varepsilon(D) = \frac{1}{\sqrt{\pi}} \cdot \int_\varepsilon^\infty \text{Tr}(De^{-tD^2})t^{-\frac{1}{2}} dt$. The beginning of the calculation of $d\eta$ is not dissimilar to the calculation for $d\alpha$; except here we have to worry about the extra integral and t parameter. On the other hand, we have the advantage that the operators in Φ_0 are by definition invertible (as unbounded operators) – that is, if $D \in \Phi_0$, there exists a bounded operator (with domain \mathcal{H}), which we denote D^{-1} , such that $D^{-1}D = 1$ on the domain of D and $DD^{-1} = 1$ on \mathcal{H} .

Lemma 4.2.13 If, in addition to our standing assumptions that D is a self-adjoint θ -summable unbounded operator, D is also invertible, then for any $-1 < r < 0$ we have that $\int_\varepsilon^\infty \text{Tr}(e^{-tD^2})t^r dt$ converges. \blacksquare

PROOF In Lemma 8.3 of [15], Phillips and Carey prove (under similar assumptions, except the fact that D is invertible is not needed) that $\int_1^\infty \text{Tr}(|D|e^{-tD^2}) dt < \infty$; we simply follow their main steps, adapted for our different function. The change of variables $s = \frac{t}{\varepsilon}$ shows that $\int_\varepsilon^\infty \text{Tr}(e^{-tD^2})t^r dt = \varepsilon^{1+r} \cdot \int_1^\infty \text{Tr}(e^{-s(\sqrt{\varepsilon}D)^2})s^r ds$, so it is sufficient to prove the result for $\varepsilon = 1$.

By Theorem 4.2.1, if we denote by E_λ the spectral resolution of $|D|$, with $\varphi_\lambda = \text{Tr}(E_\lambda)$ we have $\text{Tr}(e^{-tD^2}) = \int_0^\infty e^{-t\lambda^2} d\varphi_\lambda$. Hence

$$\begin{aligned} \int_1^\infty \text{Tr}(e^{-tD^2})t^r dt &= \int_1^\infty \left(\int_0^\infty e^{-t\lambda^2} d\varphi_\lambda \right) \cdot t^r dt \\ &= \int_0^\infty \int_1^\infty e^{-t\lambda^2} \cdot t^r dt d\varphi_\lambda \\ &= \int_0^\infty e^{-\lambda^2} \int_1^\infty e^{-(t-1)\lambda^2} \cdot t^r dt d\varphi_\lambda. \end{aligned}$$

Perform the change of variables $s = (t-1)\lambda^2$ to get

$$\begin{aligned} \int_1^\infty \text{Tr}(e^{-tD^2})t^r dt &= \int_0^\infty \left(e^{-\lambda^2} \int_0^\infty e^{-s} \left(\frac{s+\lambda^2}{\lambda^2} \right)^r \lambda^{-2} ds \right) d\varphi_\lambda \\ &= \int_0^\infty \left(e^{-\lambda^2} \lambda^{-(2+2r)} \int_0^\infty e^{-s} (s+\lambda^2)^r ds \right) d\varphi_\lambda. \end{aligned}$$

Since $\lambda \geq 0$ and $r \leq 0$, $e^{-s}(s+\lambda^2)^r \leq e^{-s}s^r$, so

$$\begin{aligned} \int_0^\infty \left(e^{-\lambda^2} \lambda^{-(2+2r)} \int_0^\infty e^{-s} (s+\lambda^2)^r ds \right) d\varphi_\lambda &\leq \int_0^\infty e^{-\lambda^2} \lambda^{-(2+2r)} d\varphi_\lambda \cdot \int_0^\infty e^{-s}s^r ds \\ &\leq \text{Tr}(|D|^{-(2+2r)}e^{-D^2}) \cdot \Gamma(r). \end{aligned}$$

Note that here we use the fact that D is invertible to replace $\lambda^{-(2+2r)}$ (which is not continuous at zero) by a function in λ which is continuous on $[0, \infty)$ and which is equal to it on the spectrum of $|D|$, allowing us to satisfy the hypotheses of Theorem 4.2.1. Also note that the requirement that $-1 < r < 0$ ensures that $\Gamma(r)$ is defined. Therefore, we can conclude that $\int_1^\infty \text{Tr}(e^{-tD^2})t^r dt < \infty$. \blacksquare

Lemma 4.2.14 As per our standing assumptions in this section, D is a self-adjoint unbounded operator such that $\text{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$. Suppose that B is any self-adjoint bounded operator; then $\frac{d}{dt} \text{Tr}(Be^{-tD^2}) = -\text{Tr}(BD^2e^{-tD^2})$ for $t \in (0, \infty)$. \blacksquare

PROOF By definition,

$$\frac{d}{dt} \operatorname{Tr}(B e^{-tD^2}) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \operatorname{Tr}(B e^{-(t+h)D^2} - B e^{-tD^2}),$$

and we want to show that this limit is equal to $-\operatorname{Tr}(BD^2 e^{-tD^2})$; in other words, we claim that $|\operatorname{Tr}(\frac{B e^{-(t+h)D^2} - B e^{-tD^2}}{h} + BD^2 e^{-tD^2})| \rightarrow 0$ as $h \rightarrow 0$ (where $t \in (0, \infty)$ is henceforth fixed).

It is sufficient to show that $\|\frac{e^{-(t+h)D^2} - e^{-tD^2}}{h} + D^2 e^{-tD^2}\|_1 \rightarrow 0$, as Hölder's inequality (since B is bounded) will allow us to reach the desired conclusion. This is yet another application of Theorem 4.2.1. Namely, with E_λ the spectral resolution of $|D|$ and $\varphi_\lambda = \operatorname{Tr}(E_\lambda)$, for any fixed $h \neq 0$ we have

$$\begin{aligned} \operatorname{Tr}(|\frac{e^{-(t+h)D^2} - e^{-tD^2}}{h} + D^2 e^{-tD^2}|) &= \int_0^\infty \left| \frac{e^{-(t+h)\lambda^2} - e^{-t\lambda^2}}{h} + \lambda^2 e^{-t\lambda^2} \right| d\varphi_\lambda \\ &= \int_0^\infty e^{-t\lambda^2} \cdot \left| \frac{e^{-h\lambda^2} - 1}{h} + \lambda^2 \right| d\varphi_\lambda. \end{aligned}$$

The problem now reduces to finding appropriate bounds for the integrand to be able to show that $\lim_{h \rightarrow 0} \int_0^\infty |\frac{e^{-(t+h)\lambda^2} - e^{-t\lambda^2}}{h} + \lambda^2 e^{-t\lambda^2}| d\varphi_\lambda = 0$. We will treat the h positive and h negative cases separately.

So assume first that $h > 0$. In order to keep track of the sign, write $h = a$ for some $a > 0$. It is straight-forward to check that $\frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \geq 0$ (so we can drop the absolute value in the integrand), and that $\frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \leq a \cdot \frac{\lambda^4}{2}$ (see Lemma B.0.6 for calculations). This gives us that

$$0 \leq e^{-t\lambda^2} \cdot \left(\frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \right) \leq a \cdot \frac{\lambda^4}{2} \cdot e^{-t\lambda^2}.$$

But $\int_0^\infty a \cdot \frac{\lambda^4}{2} \cdot e^{-t\lambda^2} d\varphi_\lambda = a \cdot \operatorname{Tr}(D^4 e^{-tD^2})$. Since $\operatorname{Tr}(D^4 e^{-tD^2}) < \infty$, this expression goes to zero as $a \rightarrow 0^+$, whence $\lim_{h \rightarrow 0^+} \int_0^\infty \left| \frac{e^{-(t+h)\lambda^2} - e^{-t\lambda^2}}{h} + \lambda^2 e^{-t\lambda^2} \right| d\varphi_\lambda = 0$.

Now suppose that $h < 0$, and write $h = -a$ for $a > 0$. Pick any positive integer n such that $t - \frac{1}{n} > 0$ (recall that, for the purposes of this calculation, $t > 0$ is fixed, so such an n certainly exists). Now, since we are interested in taking the limit as $a \rightarrow 0$, we can further assume that $a < \frac{1}{n}$. In this case (see Lemma B.0.7 for calculation details) we have that

$$\left| \frac{e^{-h\lambda^2} - 1}{h} + \lambda^2 \right| = \left| \frac{e^{a\lambda^2} - 1}{-a} + \lambda^2 \right| = \frac{e^{a\lambda^2} - 1}{a} - \lambda^2,$$

and further that $0 \leq \frac{e^{a\lambda^2} - 1}{a} - \lambda^2 \leq a \cdot n^2 \cdot e^{\frac{1}{n}\lambda^2}$. However, $\int_0^\infty a \cdot n^2 \cdot e^{\frac{1}{n}\lambda^2} \cdot e^{-t\lambda^2} d\varphi_\lambda$ is equal to $a \cdot n^2 \cdot \operatorname{Tr}(e^{-(t-\frac{1}{n})D^2})$, which approaches 0 as $a \rightarrow 0^+$ (note that by hypothesis $t - \frac{1}{n} > 0$, and hence $\operatorname{Tr}(e^{-(t-\frac{1}{n})D^2}) < \infty$). It follows that $\int_0^\infty e^{-t\lambda^2} \cdot (\frac{e^{a\lambda^2} - 1}{a} - \lambda^2) d\varphi_\lambda$ also approaches zero as $a \rightarrow 0^+$, and hence $\lim_{h \rightarrow 0^-} \int_0^\infty \left| \frac{e^{-(t+h)\lambda^2} - e^{-t\lambda^2}}{h} + \lambda^2 e^{-t\lambda^2} \right| d\varphi_\lambda = 0$. ■

In Section 4.2.1 we found an upper bound for $\|e^{-t(D+A)^2} - e^{-tD^2}\|$ (see Lemma 4.2.9); however, as this kind of expression is now inside an integral, the bound established there is not good enough for our purposes once we have to integrate over $t \in (1, \infty)$. One possible way of improving the norm bound is to use the fact that D and $D + A$ can be considered to both be invertible.

Lemma 4.2.15 Suppose that the spectrum of both D and $D + hX$ does not intersect $[-\lambda_0, \lambda_0]$ for some $\lambda_0 > 0$. Then we can show that, for any fixed $t > 0$,

$$t^{-\frac{1}{2}} \left\| \frac{1}{h} (e^{-t(D+hX)^2} - e^{-tD^2}) \right\| \leq \frac{4}{\sqrt{e}} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2}.$$

Note, moreover, that $\int_{\varepsilon}^{\infty} e^{-\frac{1}{2}t\lambda_0^2} dt < \infty$ for any $\varepsilon \geq 0$. ▀

PROOF The fact that the spectrum of D does not intersect $[-\lambda_0, \lambda_0]$ means, for example, that $\|e^{-tD^2}\| \leq e^{-t\lambda_0^2}$ (improving on the bound of 1 that we used earlier); a similar consideration applies to $D + hX$. Using the expression in Lemma 4.2.4,

$$\begin{aligned} & \frac{1}{h} (e^{-t(D+hX)^2} - e^{-tD^2}) \\ &= -\frac{t}{h} \cdot \int_0^1 (De^{-utD^2} hX e^{-(1-u)t(D+hX)^2} + e^{-utD^2} hX (D+hX) e^{-(1-u)t(D+hX)^2}) du \\ &= \left(\int_0^1 -tDe^{-utD^2} X e^{-(1-u)t(D+hX)^2} du + \int_0^1 -te^{-utD^2} X (D+hX) e^{-(1-u)t(D+hX)^2} du \right). \end{aligned}$$

The calculations are symmetric, so we only show the first integral. For fixed $u \in (0, 1)$ we have

$$\begin{aligned} \| -tDe^{-utD^2} X e^{-(1-u)t(D+hX)^2} \| &\leq t \cdot \|De^{-\frac{1}{2}utD^2}\| \cdot \|e^{-\frac{1}{2}utD^2}\| \cdot \|X\| \cdot \|e^{-(1-u)t(D+hX)^2}\| \\ &\leq t \cdot c_{1, \frac{ut}{2}} \cdot e^{-\frac{1}{2}ut\lambda_0^2} \cdot \|X\| \cdot e^{-(1-u)t\lambda_0^2} \\ &= t \cdot \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{2}{ut}} \cdot \|X\| \cdot e^{-t\lambda_0^2 \cdot (\frac{1}{2}u+1-u)}. \end{aligned}$$

As $u \in (0, 1)$, it is easy to check that $\frac{1}{2} < 1 - \frac{1}{2}u < 1$, so $e^{-t\lambda_0^2 \cdot (\frac{1}{2}u+1-u)} \leq e^{-\frac{1}{2}t\lambda_0^2}$. Simplifying the final expression gives us that

$$\| -tDe^{-utD^2} X e^{-(1-u)t(D+hX)^2} \| \leq e^{-\frac{1}{2}t\lambda_0^2} \cdot t^{\frac{1}{2}} \cdot u^{-\frac{1}{2}} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2},$$

which, using the fact that $\int_0^1 u^{-\frac{1}{2}} du = (2\sqrt{u})|_0^1 = 2$, allows us to conclude that

$$\| \int_0^1 -tDe^{-utD^2} X e^{-(1-u)t(D+hX)^2} du \| \leq 2e^{-\frac{1}{2}t\lambda_0^2} \cdot t^{\frac{1}{2}} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2}.$$

Similarly, $\| \int_0^1 -te^{-utD^2} X (D+hX) e^{-(1-u)t(D+hX)^2} du \| \leq 2e^{-\frac{1}{2}t\lambda_0^2} \cdot t^{\frac{1}{2}} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2}$. The triangle inequality thus gives us that $\left\| \frac{1}{h} (e^{-t(D+hX)^2} - e^{-tD^2}) \right\| \leq 4e^{-\frac{1}{2}t\lambda_0^2} \cdot t^{\frac{1}{2}} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2}$, and finally multiplying both sides by $t^{-\frac{1}{2}}$ finishes the proof.

The fact that $\int_{\varepsilon}^{\infty} e^{-\frac{1}{2}t\lambda_0^2} dt$ is finite can be easily established by calculating the integral. Namely,

$$\begin{aligned} \int_{\varepsilon}^{\infty} e^{-\frac{1}{2}t\lambda_0^2} dt &= \lim_{r \rightarrow \infty} \left(-\frac{2}{\lambda_0^2} e^{-\frac{1}{2}t\lambda_0^2} \right) \Big|_{\varepsilon}^r \\ &= \frac{2}{\lambda_0^2} \cdot \lim_{r \rightarrow \infty} \left(-e^{-\frac{1}{2}r\lambda_0^2} + e^{-\frac{1}{2}\varepsilon\lambda_0^2} \right) \\ &= \frac{2}{\lambda_0^2} \cdot (-0 + e^{-\frac{1}{2}\varepsilon\lambda_0^2}) \\ &= \frac{2e^{-\frac{1}{2}\varepsilon\lambda_0^2}}{\lambda_0^2}. \end{aligned}$$

Note that this implies that $\int_{\varepsilon}^{\infty} t^{-\frac{1}{2}} \left(\frac{1}{h} \cdot (e^{-t(D+hX)^2} - e^{-tD^2}) \right) dt$ converges, an observation which will be used in the calculation of $d\eta$, in Lemma 4.2.16. \blacksquare

Lemma 4.2.16 Recall that we had the definitions $\alpha_{\varepsilon}(X) = \left(\frac{\varepsilon}{\pi} \right)^{1/2} \cdot \text{Tr}(Xe^{-\varepsilon D^2})$ (a one-form on Φ) and $\eta_{\varepsilon}(D) = \frac{1}{\sqrt{\pi}} \int_{\varepsilon}^{\infty} \text{Tr}(De^{-tD^2}) t^{-\frac{1}{2}} dt$ (a zero-form on Φ_0). Then $d\eta_{\varepsilon} = -2\alpha_{\varepsilon}$ on Φ_0 . \blacksquare

PROOF In order to simplify the calculations, assume without loss of generality that $\varepsilon = 1$; the change of variables $s = \frac{t}{\varepsilon}$ shows that $\eta_{\varepsilon}(D) = \sqrt{\varepsilon} \eta_1(\sqrt{\varepsilon}D)$ and, as $\alpha_{\varepsilon,D}(X) = \sqrt{\varepsilon} \alpha_{1,\sqrt{\varepsilon}D}(X)$, result follows for all $\varepsilon > 0$. Start from the definition:

$$d\eta_{1,D}(X) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \left[\frac{1}{\sqrt{\pi}} \cdot \int_1^{\infty} (\text{Tr}((D+hX)e^{-t(D+hX)^2}) - \text{Tr}(De^{-tD^2})) \cdot t^{-\frac{1}{2}} dt \right].$$

We want to show this is equal to $-2\alpha_{1,D}(X)$. Note for future reference that the integrand is continuous in t , by Lemma 4.2.3. In order to avoid dealing with unbounded operators, and to be able to use some of the already established limits, we split up $(D+hX)e^{-t(D+hX)^2} - De^{-tD^2}$ into three parts as follows:

$$\begin{aligned} (D+hX)e^{-t(D+hX)^2} - De^{-tD^2} &= (D+hX)e^{-\frac{t}{2}(D+hX)^2} e^{-\frac{t}{2}(D+hX)^2} - (D+hX)e^{-\frac{t}{2}(D+hX)^2} e^{-\frac{t}{2}D^2} \\ &\quad + e^{-\frac{t}{2}(D+hX)^2} (D+hX)e^{-\frac{t}{2}D^2} - e^{-\frac{t}{2}(D+hX)^2} De^{-\frac{t}{2}D^2} \\ &\quad + e^{-\frac{t}{2}(D+hX)^2} De^{-\frac{t}{2}D^2} - e^{-\frac{t}{2}D^2} De^{-\frac{t}{2}D^2} \\ &= (D+hX)e^{-\frac{t}{2}(D+hX)^2} \cdot \left(e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2} \right) \\ &\quad + e^{-\frac{t}{2}(D+hX)^2} hXe^{-\frac{t}{2}D^2} \\ &\quad + \left(e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2} \right) \cdot De^{-\frac{t}{2}D^2}. \end{aligned}$$

Note that, in order to write the first equality, we used the fact that $D+hX$ and D have the same domain. We can now consider separately (note that we leave out for now the factor of $\frac{1}{\sqrt{\pi}}$, which we should remember to throw back in at the end) the three limits

- $\lim_{h \rightarrow 0} \int_1^{\infty} \text{Tr} \left((D+hX)e^{-\frac{t}{2}(D+hX)^2} \cdot \frac{1}{h} (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \right) \cdot t^{-\frac{1}{2}} dt,$
- $\lim_{h \rightarrow 0} \int_1^{\infty} \text{Tr} (e^{-\frac{t}{2}(D+hX)^2} hXe^{-\frac{t}{2}D^2}) \cdot t^{-\frac{1}{2}} dt,$ and
- $\lim_{h \rightarrow 0} \int_1^{\infty} \text{Tr} \left(\frac{1}{h} (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \cdot De^{-\frac{t}{2}D^2} \right) \cdot t^{-\frac{1}{2}} dt.$

If we can calculate each of these limits, we simply add them together and multiply by $\frac{1}{\sqrt{\pi}}$ to get a first expression for $d\eta$. Use the Dominated Convergence Theorem to evaluate the limits; first find an expression to which each integrand converges pointwise, then show that each integrand is uniformly bounded (for h small enough) by an integrable function. To make precise what we mean by 'small enough', choose $\delta > 0$ as follows:

- (A) To start with, D is invertible, so there exists a $\lambda_0 > 0$ such that $[-\lambda_0, \lambda_0] \cap \sigma(D) = \emptyset$. By Theorem 2.4.4, there exists a $\delta_1 > 0$ such that if $\|A\| < \delta_1$ then $\sigma(D+A) \cap [-\lambda_0, \lambda_0] = \emptyset$ as well. Let $\delta = \frac{1}{\|X\|} \cdot \min\{1, \delta_1\}$ (which means both that $\|hX\| \leq 1$ and $\|hX\| \leq \delta_1$). If $h < \delta$ then $\sigma(D+hX) \cap [-\lambda_0, \lambda_0] = \emptyset$ (which in particular also means that $D+hX$ is invertible).

Claim Fix $t \geq 1$. Then, as $h \rightarrow 0$,

- (1) $\text{Tr}((D + hX)e^{-\frac{t}{2}(D+hX)^2} \cdot \frac{1}{h}(e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2})) \cdot t^{-\frac{1}{2}}$ converges to $-\frac{\sqrt{t}}{2} \cdot \text{Tr}(De^{-\frac{t}{2}D^2} \cdot \int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du)$.
- (2) $\text{Tr}(e^{-\frac{t}{2}(D+hX)^2} X e^{-\frac{t}{2}D^2}) \cdot t^{-\frac{1}{2}} \rightarrow \text{Tr}(e^{-\frac{t}{2}D^2} X e^{-\frac{t}{2}D^2}) t^{-\frac{1}{2}}$.
- (3) $\text{Tr}(\frac{1}{h}(e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \cdot De^{-\frac{t}{2}D^2}) \cdot t^{-\frac{1}{2}}$ converges to $-\frac{\sqrt{t}}{2} \cdot \text{Tr}((\int_0^1 De^{-utD^2} X e^{-(1-u)tD^2} + e^{-utD^2} X De^{-(1-u)tD^2} du) \cdot De^{-\frac{t}{2}D^2})$.

PROOF By Lemma 4.2.11,

$$\|\cdot\|_1 - \lim_{h \rightarrow 0} (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) = -\frac{t}{2} \cdot \int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du.$$

Combined with the fact that $\|(D + hX)e^{-\frac{t}{2}(D+hX)^2}\| \leq c_{1, \frac{t}{2}}$ is uniformly bounded for all h (Lemma 4.2.3), and $(D + hX)e^{-\frac{t}{2}(D+hX)^2} \rightarrow De^{-\frac{t}{2}D^2}$ (see Lemma 4.2.9, part (ii)), this is sufficient to prove (1). Note that (3) is similar, but slightly easier as $De^{-\frac{t}{2}D^2}$ does not depend on h .

To show the convergence in (2), pick n an integer such that $t \geq \frac{1}{n}$, and use Hölder's inequality along with the bounds in Lemma 4.2.3 and Lemma 4.2.9:

$$\begin{aligned} & |\text{Tr}(e^{-\frac{1}{2}t(D+hX)^2} X e^{-\frac{1}{2}tD^2}) - \text{Tr}(e^{-\frac{1}{2}tD^2} X e^{-\frac{1}{2}tD^2})| \\ & \leq \|(e^{-\frac{1}{2}t(D+hX)^2} - e^{-\frac{1}{2}tD^2}) \cdot X \cdot e^{-\frac{1}{2}tD^2}\|_1 \\ & \leq \|e^{-\frac{1}{2}t(D+hX)^2} - e^{-\frac{1}{2}tD^2}\| \cdot \|X\| \cdot \|e^{-\frac{1}{2}tD^2}\|_1 \\ & \leq \|hX\| \cdot \sqrt{\frac{t}{2}} \cdot \frac{2\sqrt{2}}{e} \cdot \|X\| \cdot \|e^{-\frac{1}{2}tD^2}\|_1. \end{aligned}$$

The only factor that depends on h is $\|hX\|$, which goes to zero as $h \rightarrow 0$, concluding the limit proof. \diamond

Claim Suppose $|h| < \delta$, where δ is defined as in (A) (that is, for all such h we have $\|hX\| \leq 1$, $D + hX$ is invertible, and the spectrum of $D + hX$ does not intersect $[-\lambda_0, \lambda_0]$ for some $\lambda_0 > 0$ which is the same for all the h 's). Then each of our functions is bounded by a positive function in t which is integrable from 1 to ∞ ; namely,

- (1) $\left| \text{Tr} \left((D + hX)e^{-\frac{t}{2}(D+hX)^2} \cdot \frac{1}{h}(e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \right) \cdot t^{-\frac{1}{2}} \right|$ is bounded by the integrable function $\|e^{-\frac{1}{13}D^2}\|_1 \cdot 4 \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2}$.
- (2) $|\text{Tr}(e^{-\frac{t}{2}(D+hX)^2} \cdot X \cdot e^{-\frac{t}{2}D^2}) \cdot t^{-\frac{1}{2}}|$ is bounded by the integrable function $\|X\| \cdot \text{Tr}(e^{-\frac{1}{2}tD^2}) t^{-\frac{1}{2}}$.
- (3) $\left| \text{Tr} \left(\frac{1}{h} \cdot (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \cdot De^{-\frac{t}{2}D^2} \right) \cdot t^{-\frac{1}{2}} \right|$ is bounded by the integrable function $\frac{4}{e} \cdot \|X\| \cdot e^{-\frac{1}{2}t\lambda_0^2} \cdot \|e^{-\frac{1}{4}D^2}\|_1$.

PROOF From Lemma 4.2.15, $t^{-\frac{1}{2}} \cdot \left\| \frac{1}{h} \cdot (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \right\| \leq \frac{4}{\sqrt{2e}} \cdot \|X\| \cdot e^{-\frac{t}{4}\lambda_0^2}$ (and the integral of the function on the right hand-side, calculated from 1 to ∞ , converges). To complete the picture for (1), we note that

$$\begin{aligned} \|(D + hX)e^{-\frac{t}{2}(D+hX)^2}\|_1 & \leq c_{1, \frac{t}{4}} \cdot \|e^{-\frac{t}{4}(D+hX)^2}\|_1 \\ & \leq \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{4}{t}} \cdot e \cdot \left\| e^{-\frac{t}{4}D^2} \frac{1}{f(\sqrt{\frac{t}{4}\|hX\|})} \right\|_1. \end{aligned}$$

For $t \geq 1$ we have $\sqrt{\frac{4}{t}} \leq 2$, and so $\frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{4}{t}} \cdot e \leq \sqrt{2e}$. By Remark 4.2.7, since $\frac{t}{4} \geq \frac{1}{4}$,

$$\left\| e^{-\frac{t}{4}D^2} \frac{1}{f(\sqrt{\frac{t}{4}\|hX\|})} \right\|_1 \leq \|e^{-\frac{1}{13}D^2}\|_1. \quad (1).$$
The Hölder inequality allows us to get the function inequality in

For (3), we simply note that $\frac{t}{4} \geq \frac{1}{4}$, and apply a similar treatment to the above to get an upper bound for $\|De^{-\frac{t}{2}D^2}\|_1$. That is,

$$\begin{aligned} \left| \text{Tr} \left(\frac{1}{h} (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \cdot De^{-\frac{t}{2}D^2} \right) \cdot t^{-\frac{1}{2}} \right| &\leq \|t^{-\frac{1}{2}} \cdot \left(\frac{1}{h} \cdot (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \right) \cdot De^{-\frac{t}{2}D^2}\|_1 \\ &\leq t^{-\frac{1}{2}} \cdot \left\| \frac{1}{h} (e^{-\frac{t}{2}(D+hX)^2} - e^{-\frac{t}{2}D^2}) \right\| \cdot \|De^{-\frac{t}{2}D^2}\|_1 \\ &\leq \left(\frac{4}{\sqrt{2e}} \cdot \|X\| \cdot e^{-\frac{t}{4}\lambda_0^2} \right) \cdot (c_{1,\frac{t}{4}} \cdot \|e^{-\frac{t}{4}D^2}\|_1) \\ &\leq \left(\frac{4}{\sqrt{2e}} \cdot \|X\| \cdot e^{-\frac{t}{4}\lambda_0^2} \right) \cdot \frac{1}{\sqrt{2e}} \cdot \sqrt{\frac{4}{t}} \cdot \|e^{-\frac{1}{4}D^2}\|_1, \end{aligned}$$

which simplifies to prove the desired function inequality.

Finally, let us consider the function in (2). We have

$$\begin{aligned} |\text{Tr}(e^{-\frac{1}{2}t(D+hX)^2} X e^{-\frac{1}{2}tD^2})| &\leq \|e^{-\frac{1}{2}t(D+hX)^2} X e^{-\frac{1}{2}tD^2}\|_1 \\ &\leq \|e^{-\frac{1}{2}t(D+hX)^2}\| \cdot \|X\| \cdot \|e^{-\frac{1}{2}tD^2}\|_1 \\ &\leq 1 \cdot \|X\| \cdot \|e^{-\frac{1}{2}tD^2}\|_1. \end{aligned}$$

Since $\int_1^\infty \|e^{-t(D/\sqrt{2})^2}\|_1 \cdot t^{-\frac{1}{2}} dt = \int_1^\infty \text{Tr}(e^{-t(D/\sqrt{2})^2}) \cdot t^{-\frac{1}{2}} dt$ converges by Lemma 4.2.13, this concludes the proof of (2). \diamond

The above two claims assure us that we can exchange the limit and integral in each term, and then evaluate the pointwise limit, to get

$$\begin{aligned} d\eta_{1,D}(X) &= \frac{1}{\sqrt{\pi}} \cdot \left(\int_1^\infty -\frac{\sqrt{t}}{2} \text{Tr}(De^{-\frac{t}{2}D^2}) \cdot \int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du \right) dt \\ &\quad + \int_1^\infty \text{Tr}(e^{-\frac{t}{2}D^2} X e^{-\frac{t}{2}D^2}) \cdot t^{-\frac{1}{2}} dt \\ &\quad + \int_1^\infty -\frac{\sqrt{t}}{2} \text{Tr} \left(\int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du \cdot De^{-\frac{t}{2}D^2} \right) dt. \end{aligned}$$

This isn't quite looking like $-2\alpha_1$ yet, so we simplify it a bit further. Apply the trace property to the third term to make it the same as the first, allowing us to combine them, and apply it to the second term to group the expressions in D together:

$$\begin{aligned} d\eta_{1,D}(X) &= \frac{1}{\sqrt{\pi}} \cdot \left(\int_1^\infty -\sqrt{t} \cdot \text{Tr}(De^{-\frac{t}{2}D^2}) \cdot \int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du \right) dt \\ &\quad + \int_1^\infty \text{Tr}(X e^{-tD^2}) \cdot t^{-\frac{1}{2}} dt. \end{aligned}$$

Next, get rid of the inner integral in the first term.

$$\text{Claim } \text{Tr}(De^{-\frac{t}{2}D^2}) \cdot \int_0^1 De^{-u\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2} + \int_0^1 e^{-u\frac{t}{2}D^2} X De^{-(1-u)\frac{t}{2}D^2} du = 2 \text{Tr}(XD^2 e^{-tD^2}).$$

PROOF From the proof of Lemma 4.2.11, we know that each of the two integrals converges in trace norm, so since $De^{-\frac{t}{2}D^2}$ is bounded, we can pull it into the integral and exchange the trace and integral to get

$$\begin{aligned} & \int_0^1 \text{Tr}(D^2 e^{-(1+u)\frac{t}{2}D^2} X e^{-(1-u)\frac{t}{2}D^2}) du + \int_0^1 \text{Tr}(D e^{-(1+u)\frac{t}{2}D^2} X D e^{-(1-u)\frac{t}{2}D^2}) du \\ &= \int_0^1 \text{Tr}(X D^2 e^{-tD^2}) du + \int_0^1 \text{Tr}(X D^2 e^{-tD^2}) du \\ &= 2 \text{Tr}(X D^2 e^{-tD^2}) \int_0^1 1 du \\ &= 2 \text{Tr}(X D^2 e^{-tD^2}), \end{aligned}$$

as claimed. \diamond

Therefore,

$$d\eta_{1,D}(X) = \frac{1}{\sqrt{\pi}} \cdot \int_1^\infty -2\sqrt{t} \text{Tr}(X D^2 e^{-tD^2}) dt + \frac{1}{\sqrt{\pi}} \cdot \int_1^\infty \text{Tr}(X e^{-tD^2}) \cdot t^{-\frac{1}{2}} dt.$$

By Lemma 4.2.14, $\text{Tr}(X D^2 e^{-tD^2}) = \frac{d}{dt} \text{Tr}(-X e^{-tD^2})$. The rest of the proof is exactly as in [38] and adds no new details, but does correct a sign error, as $d\eta = -2\alpha$, not 2α as claimed in [38]. Using integration by parts, the first term in the above formula, without the $\frac{1}{\sqrt{\pi}}$ factor, is equal to

$$\begin{aligned} \int_1^\infty -2\sqrt{t} \text{Tr}(X D^2 e^{-tD^2}) dt &= \int_1^\infty -2\sqrt{t} \cdot \frac{d}{dt} \text{Tr}(-X e^{-tD^2}) dt \\ &= \text{Tr}(-X e^{-tD^2})(-2\sqrt{t}) \Big|_1^\infty - \int_1^\infty \text{Tr}(-X e^{-tD^2}) \frac{d}{dt}(-2\sqrt{t}) dt \\ &= 2 \text{Tr}(X e^{-tD^2}) \sqrt{t} \Big|_1^\infty - \int_1^\infty \text{Tr}(X e^{-tD^2}) t^{-\frac{1}{2}} dt. \end{aligned}$$

Hence

$$\begin{aligned} d\eta_{1,D}(X) &= \frac{1}{\sqrt{\pi}} \left(2 \text{Tr}(X e^{-tD^2}) \sqrt{t} \Big|_1^\infty - \int_1^\infty \text{Tr}(X e^{-tD^2}) t^{-\frac{1}{2}} dt. \right) + \frac{1}{\sqrt{\pi}} \cdot \int_1^\infty \text{Tr}(X e^{-tD^2}) \cdot t^{-\frac{1}{2}} dt \\ &= \left[\frac{2}{\sqrt{\pi}} \cdot \text{Tr}(X e^{-tD^2}) t^{\frac{1}{2}} \right] \Big|_1^\infty \\ &= \frac{2}{\sqrt{\pi}} \cdot [\lim_{t \rightarrow \infty} \text{Tr}(X e^{-tD^2}) t^{\frac{1}{2}} - \text{Tr}(X e^{-D^2})] \\ &= \frac{2}{\sqrt{\pi}} \cdot \lim_{t \rightarrow \infty} \text{Tr}(X e^{-tD^2}) t^{\frac{1}{2}} - 2\alpha_{1,D}(X). \end{aligned}$$

To finish the proof we just have to show that $\lim_{t \rightarrow \infty} \text{Tr}(X e^{-tD^2}) t^{\frac{1}{2}} = 0$. As $t \geq 1$, and $\lambda_0 > 0$ was chosen such that $[-\lambda_0, \lambda_0] \cap \sigma(D) = \emptyset$, we have

$$\begin{aligned} \|X e^{-tD^2} t^{\frac{1}{2}}\|_1 &\leq t^{\frac{1}{2}} \cdot \|X\| \cdot \|e^{-\frac{t}{2}D^2}\| \cdot \|e^{-\frac{t}{2}D^2}\|_1 \\ &\leq t^{\frac{1}{2}} \cdot \|X\| \cdot e^{-\frac{t}{2}\lambda_0^2} \cdot \|e^{-\frac{t}{2}D^2}\|_1; \end{aligned}$$

since $\lim_{t \rightarrow \infty} \frac{\sqrt{t}}{e^{\frac{t}{2}\lambda_0^2}} = 0$, it follows $\lim_{t \rightarrow \infty} \text{Tr}(X e^{-tD^2}) t^{\frac{1}{2}} = 0$ as claimed, and so $d\eta = -2\alpha$ on Φ_0 . \blacksquare

4.2.3 $\alpha_\varepsilon \oplus -\frac{1}{2}\eta_\varepsilon$ calculates spectral flow

As stated in Proposition 4.2.5, $\alpha \oplus -\frac{1}{2}\eta$ is a representative of a class in $H_{dR}^1(\Phi, \Phi_0)$. Our next goal is to show that its pairing with paths in $\Omega(\Phi, \Phi_0)$ calculates spectral flow. Denote by $\gamma_{\alpha \oplus -\frac{1}{2}\eta}$ the corresponding element of $H^1(\Phi, \Phi_0; \mathbb{R})$. Recall that in Section 4.1.1 we exhibited an element $[\Phi_1]$ of $H^1(\Phi, \Phi_0)$ for which $[\Phi_1](\rho) = \text{sf}(\rho)$. We will have occasion to use the intersection number definition of $[\Phi_1]$ established in Corollary 4.1.20.

Proposition 4.2.17 Suppose $\rho = \{D_t\} \subset \Phi_0 \cup \Phi_1$ is such that $\rho(t)$ is invertible except when $t = t_0$, and $\rho(t_0) \in \Phi_1$. Then

$$[\Phi_1](\rho) = \frac{1}{2} \cdot \left(\lim_{t \rightarrow t_0^+} \eta_\varepsilon(D_t) - \lim_{t \rightarrow t_0^-} \eta_\varepsilon(D_t) \right). \quad \blacksquare$$

PROOF Recall that, by definition, $\eta_\varepsilon(D) = \frac{1}{\sqrt{\pi}} \cdot \int_\varepsilon^\infty \text{Tr}(D e^{-tD^2}) t^{-\frac{1}{2}} dt$.

Consider the real function $N(x) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty x e^{-tx^2} t^{-\frac{1}{2}} dt$. We want to calculate the limit as $x \rightarrow 0$ of $N(x)$; the limit from the left and the limit from the right will turn out to be different. For $x \neq 0$ we can perform the substitution $v = x^2 t$ to get $N(x) = \frac{1}{\sqrt{\pi}} \int_{\varepsilon x^2}^\infty x \cdot e^{-v|x|} \cdot v^{-\frac{1}{2}} \cdot \frac{1}{x^2} dv$. Write $\text{sign}(x)$ for $\frac{|x|}{x}$ (which is equal to 1 for $x > 0$ and -1 for $x < 0$); with this notation, $N(x)$ is equal to $\frac{1}{\sqrt{\pi}} \cdot \text{sign}(x) \cdot \int_{\varepsilon x^2}^\infty e^{-v} v^{-\frac{1}{2}} dv$. The integral is now recognizable as an instance of the incomplete gamma function, defined by $\Gamma(z, s) = \int_s^\infty e^{-t} t^{z-1} dt$ (note that the gamma function is equal to $\Gamma(z, 0)$). That is, $N(x) = \frac{1}{\sqrt{\pi}} \text{sign}(x) \cdot \Gamma(\frac{1}{2}, \varepsilon x^2)$. For $z = \frac{1}{2}$ one can perform the substitution $u = t^{\frac{1}{2}}$ and use the equality $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ to show that $\Gamma(\frac{1}{2}, s) = \sqrt{\pi} (1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{s}} e^{-t^2} dt)$. Hence

$$N(x) = \frac{1}{\sqrt{\pi}} \cdot \text{sign}(x) \cdot \Gamma\left(\frac{1}{2}, x^2 \cdot \varepsilon\right) = \text{sign}(x) \cdot \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{|x|\sqrt{\varepsilon}} e^{-t^2} dt\right).$$

It should thus be clear that $\lim_{x \rightarrow 0^-} N(x) = -1$, whereas $\lim_{x \rightarrow 0^+} N(x) = 1$.

As a zero form on Φ_0 , η_ε is continuous over invertible operators. Place a tight Ruget neighbourhood \mathcal{U} around D_{t_0} (see Definition 4.1.3 and Lemma 4.1.14 for a review of this terminology). Each $D_t \in \mathcal{U}$ is unitarily equivalent to $\begin{bmatrix} \lambda_t & 0 \\ 0 & C_t \end{bmatrix}$, where C_t is invertible and λ_t is an eigenvalue of D_t . It follows that $\eta_\varepsilon(D_t) = N(\lambda_t) + \eta_\varepsilon(C_t)$. As $t \rightarrow t_0^+$, this expression approaches $\lim_{t \rightarrow t_0^+} N(\lambda_t) + \eta_\varepsilon(C_{t_0})$. Similarly, as $t \rightarrow t_0^-$, the expression approaches $\lim_{t \rightarrow t_0^-} N(\lambda_t) + \eta_\varepsilon(C_{t_0})$, so when we take the difference the $\eta_\varepsilon(C_{t_0})$ term cancels out. Moreover,

$$\lim_{t \rightarrow t_0^+} N(\lambda_t) = \begin{cases} 1 & \text{if } \lambda_t > 0 \text{ for } t > t_0 \\ -1 & \text{if } \lambda_t < 0 \text{ for } t > t_0. \end{cases}$$

Similarly, $\lim_{t \rightarrow t_0^-} N(\lambda_t)$ mirrors the sign of λ_t for $t < t_0$; that is, $\lim_{t \rightarrow t_0^-} N(\lambda_t) = \lim_{t \rightarrow t_0^+} N(\lambda_t)$ if λ_t has the same sign on both sides of t_0 , and $\lim_{t \rightarrow t_0^-} N(\lambda_t) = -\lim_{t \rightarrow t_0^+} N(\lambda_t)$ if the sign changes. Since

$$[\Phi_1](\rho) = \begin{cases} 1 & \text{if } \lambda_t < 0 \text{ for } t < t_0 \text{ and } \lambda_t > 0 \text{ for } t > t_0 \\ -1 & \text{if } \lambda_t > 0 \text{ for } t < t_0 \text{ and } \lambda_t < 0 \text{ for } t > t_0 \\ 0 & \text{otherwise,} \end{cases}$$

the formula $[\Phi_1](\rho) = \frac{1}{2} \cdot \left(\lim_{t \rightarrow t_0^+} \eta_\varepsilon(D_t) - \lim_{t \rightarrow t_0^-} \eta_\varepsilon(D_t) \right)$ follows easily. \blacksquare

Theorem 4.2.18 (Theorem 2.6 of [38]) Let $\{\rho(t) = D_t\} \subset \Phi$ be a (differentiable) path for which D_0 and D_1 are invertible. Then

$$sf(\rho) = - \int_{\rho} \alpha_{\varepsilon} + \frac{1}{2} \eta_{\varepsilon}(D_1) - \frac{1}{2} \eta_{\varepsilon}(D_0);$$

that is, the class of $\alpha \oplus -\frac{1}{2}\eta$ in $H_{dR}^1(\Phi, \Phi_0)$ can be used to calculate spectral flow. \blacktriangleright

PROOF From the proof of Lemma 4.1.18, it should be clear that via a homotopy with endpoints fixed we could ensure that ρ is transversal to Φ_1 if we wish; that is, the path intersects Φ_1 finitely many times, and Ψ changes sign at each intersection. Hence, we can assume without loss of generality that ρ is transversal to Φ_1 (as α_{ε} is a closed form, $\int_{\rho} \alpha_{\varepsilon}$ remains unchanged). Suppose ρ intersects Φ_1 at $0 < t_1 < \dots < t_k < 1$, and in addition let $t_0 = 0$ and $t_{k+1} = 1$. From Proposition 4.2.17, we get

$$sf(\rho) = [\Phi_1](\rho) = \frac{1}{2} \sum_{i=1}^k (\lim_{t \rightarrow t_i^+} \eta_{\varepsilon}(D_t) - \lim_{t \rightarrow t_i^-} \eta_{\varepsilon}(D_t)).$$

On the other hand, by the Fundamental Theorem of Calculus (and using $\eta_{\varepsilon}(D_0) = \lim_{t \rightarrow 0} \eta_{\varepsilon}(D_t)$, and similarly for D_1),

$$\int_{\rho} d\eta_{\varepsilon} = \sum_{i=0}^k \int_{\rho|_{[t_i, t_{i+1}]}} d\eta_{\varepsilon} = \sum_{i=0}^k (\lim_{t \rightarrow t_{i+1}^-} \eta_{\varepsilon}(D_t) - \lim_{t \rightarrow t_i^+} \eta_{\varepsilon}(D_t));$$

hence, we can rewrite the spectral flow in terms of η as

$$sf(\rho) = \frac{1}{2} (- \int_{\rho} d\eta_{\varepsilon} + \eta_{\varepsilon}(D_1) - \eta_{\varepsilon}(D_0)) = \frac{1}{2} (\int_{\rho} 2\alpha_{\varepsilon} + \eta_{\varepsilon}(D_1) - \eta_{\varepsilon}(D_0)),$$

where for the last equality we used the fact that $d\eta_{\varepsilon} = -2\alpha_{\varepsilon}$. This simplifies to the desired formula, and agrees with the pairing of $\alpha \oplus -\frac{1}{2}\eta$ and ρ . \blacksquare

4.3 Type II Case

In this section, we investigate a possible avenue towards applying this approach in the type II case; the reader is warned that the presentation is stitched together with suppositions and conjectures. Our setup consists of a type II factor \mathcal{N} with semifinite trace τ , and D a self-adjoint unbounded operator affiliated with \mathcal{N} . As in Section 4.1, we would like $D + A$ to be Breuer-Fredholm for $A \in \mathcal{N}_{sa}$, so we ask that $(1 + D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ (or, equivalently, the resolvents of D are τ -compact); we then consider $\Phi = D + \mathcal{N}_{sa}$, and set our goal a geometric interpretation of spectral flow for paths in Φ whose endpoints are invertible operators.

Certainly Φ is a manifold, but we now encounter our first snag. Recall that in the type I case, we considered submanifolds Φ_n , consisting of operators $T \in \Phi$ with kernel dimension n . The spectral flow (an integer), viewed as a pairing of a specific cohomology class in $H^1(\Phi, \Phi_0)$ with equivalence classes of paths in $\Omega(\Phi, \Phi_0)$, could then be calculated as an intersection number of the path with Φ_1 . That is, the two submanifolds of importance were Φ_0 , consisting of the invertible operators, and Φ_1 , consisting of operators with kernel dimension one. The question

then is what should play the role of Φ_0 and Φ_1 in the type II case, assuming such an easy parallel even exists.

A first guess might be that we could separate the operators by the dimension of the kernel, just as in the type I case. This should be immediately suspicious though, as we now have to deal with operators where zero is not an eigenvalue but is nonetheless in the (continuous) spectrum, so this partition of the operators does not identify the invertible operators. We look instead at

$$\Phi_r = \{T \in \Phi : \text{with } P = \chi_0(T), \tau(P) = r \text{ and } T|_{P^\perp \mathcal{H}} \text{ is invertible}\}.$$

Now, $T \in \Phi_0$ means that T is invertible, but it also means that if $0 \in \sigma(T)$ then zero is an eigenvalue of T . Hence, we now clearly have $\bigcup_{r \in \mathbb{R}} \Phi_r \subsetneq \Phi$; however, as explained in the remark below, we have a chance of homotoping our path to a path in $\bigcup_{r \in \mathbb{R}} \Phi_r$. Unfortunately, we now hit our second snag, which is that it is clear there is no finite collection of Φ_r 's which could possibly play the role played by Φ_1 in the type I case.

Remark 4.3.1 Suppose that $\{T_t\}$ is a path of operators with invertible endpoints and for which there exists some decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ with $\tau(P) < \infty$ such that $T_t = \begin{bmatrix} A_t & B_t \\ B_t^* & D_t \end{bmatrix}$ with D_t invertible (i.e. if we were in the type I case, $\{T_t\}$ would look like a Ruget path). Then there is no reason why we cannot apply our map $\Psi : T_t \mapsto A_t - B_t D_t^{-1} B_t^*$, which is continuous just as in the type I case. Let $M_t = A_t - B_t D_t^{-1} B_t^*$. Lemma 2.3.3 in Chapter 2 was stated for bounded operators in order to avoid domain issues; however, since all our unbounded operators have a common domain, we can prove a similar result that will cover this case, and via some result similar to Lemma 2.3.4, $\begin{bmatrix} A_t - (2t - t^2)B_t D_t^{-1} C_t & (1 - t)B_t \\ (1 - t)B_t^* & D_t \end{bmatrix}$ would provide a homotopy (with endpoints in the invertibles) to a path of operators which are block diagonal.

As a consequence, our initial path is homotopic to $\begin{bmatrix} M_t & 0 \\ 0 & D_t \end{bmatrix}$; moreover, $\begin{bmatrix} M_0 & 0 \\ 0 & D_0 \end{bmatrix}$ and $\begin{bmatrix} M_1 & 0 \\ 0 & D_1 \end{bmatrix}$ are invertible (M_0 and M_1 are invertible operators in the II_1 factor $P\mathcal{N}P$). Let $Q_0 = \chi_{[0, \infty)}(M_0)$ and $Q_1 = \chi_{[0, \infty)}(M_1)$. We can connect M_0 to $2Q_0 - 1$ via a path of invertible operators, and similarly connect M_1 to $2Q_1 - 1$. As in the type I case, because the action is happening on a finite trace corner, any two paths with the same endpoints and where the D_t corner does not change can be homotoped by connecting them with straight lines. So it is sufficient to construct a path from $2Q_0 - 1$ to $2Q_1 - 1$ with appropriate characteristics.

Now, as in Remark 1.1.29, with the decomposition of \mathcal{H} given by

$$(\text{ran } P \cap \ker Q) \oplus (\text{ran } Q \cap \ker P) \oplus [(\text{ran } P \cap \ker Q) \oplus (\text{ran } Q \cap \ker P)]^\perp$$

we can write

$$\begin{aligned} 2Q_0 - 1 &= 1 \oplus -1 \oplus (2\tilde{Q}_0 - 1), \text{ and} \\ 2Q_1 - 1 &= -1 \oplus 1 \oplus (2\tilde{Q}_1 - 1). \end{aligned}$$

Recall moreover that \tilde{Q}_0 and \tilde{Q}_1 are then unitarily equivalent projections. We can construct a path from $2Q_0 - 1$ to $2Q_1 - 1$ by concatenating three paths: $\{(1 - 2t) \oplus -1 \oplus (2\tilde{Q}_0 - 1)\}_{t \in [0, 1]}$

(which changes the first block matrix from 1 to -1), $\{-1 \oplus (2s - 1) \oplus (2\tilde{Q}_0 - 1)\}_{s \in [0,1]}$ (which changes the middle block matrix from -1 to 1), and the third path from $\{-1 \oplus 1 \oplus 2\tilde{Q}_0 - 1\}$ to $\{-1 \oplus 1 \oplus 2\tilde{Q}_1 - 1\}$ constructed using the unitary equivalence between \tilde{Q}_0 and \tilde{Q}_1 . The resulting path thus intersects Φ_r and Φ_s and is invertible everywhere else.

Here we have a bit of a pause; r and s can be any real numbers, depending on the path with which we started. In the type I case, r and s are both integers, and since the integers are generated by 1, we have a chance of further manipulating the path to obtain intersections with Φ_1 . The real numbers are not finitely generated over the integers, so there is no obvious Φ_r , fixed for all paths, with which we might define an intersection number (or even a finite collection of such Φ_r 's). In fact, since the spectral flow in the type II case can be any real number, and an intersection number is necessarily an integer, this seems to suggest that at best we might be able to obtain spectral flow as some sort of limit of (suitably modified) intersection numbers. •

The above remark suggests that we could split up the path into sections, ensure the endpoints at each section are invertible, and then homotope each section so the operators along it are in Φ_0 (so invertible), except for finitely many which are in Φ_{r_i} for some collection of r_i 's. This suggests that, locally, our path is in $\Phi_0 \cup \Phi_r$ (for some r), so we need to think about how Φ_r fits inside $\Phi_0 \cup \Phi_r$. Recall, in the type I case, we could restrict ourselves to $r = 1$, and since Φ_1 had codimension one in $\Phi_0 \cup \Phi_1$, any path in $\Phi_0 \cup \Phi_1$ could be homotoped to intersect Φ_1 transversally, allowing us to define an intersection number; this intersection number was equal to the spectral flow of the path. It should be clear from the description that $r = 1$ was special even in the type I case. For example, if we were to look at $\Phi_0 \cup \Phi_2$ instead, Φ_2 has codimension three, so since a path has dimension one there is no way to get it to be transversal to Φ_2 . To belabor the point, an intersection with Φ_2 might not say anything about spectral flow. For example, pick P a projection onto a 2-dimensional subspace of \mathcal{H} , and D an invertible suitable operator on $P^\perp \mathcal{H}$; the path $\rho(t) = \left\{ \left[\begin{array}{cc} 2t - 1 & 0 \\ 0 & 1/4 - t^2 \end{array} \right] \oplus D : t \in [0, 1] \right\}$ is invertible everywhere except when $t = \frac{1}{2}$ (when it intersects Φ_2), but the spectral flow is 1. On the other hand, thinking about what happens on a path like $\left\{ \left[\begin{array}{cc} 2t - 1 & 0 \\ 0 & 2t - 1 \end{array} \right] \oplus D \right\}$ (for which both eigenvalues change sign from negative to positive at $t = \frac{1}{2}$) suggests that, if we are willing to restrict the types of paths we are willing to consider, there might be a setting in which we can define some sort of intersection number with Φ_2 .

We now apply these observations to the type II case. First, note that no Φ_r has any hope of having codimension one in $\Phi_r \cup \Phi_0$. As in Proposition 4.1.6 and the preceding material, at each $T_0 \in \Phi_r$ ($r \neq 0$) we can place a Ruget neighbourhood \mathcal{U} , and define the map $\Psi : \left[\begin{array}{cc} A & B \\ B^* & D \end{array} \right] \mapsto A - BD^{-1}B$ on \mathcal{U} . We still have $\Psi(S) = 0$ if and only if $S \in \Phi_r$; however, we now have $A - BD^{-1}B^* \in (P\mathcal{N}P)^{sa}$, and since $P\mathcal{N}P$ is a type II factor, it cannot be finite-dimensional.

Instead, we modify our approach to reflect the comments we made about Φ_2 ; namely, for each Φ_r we consider a subset $\Phi_{0,r}$ of Φ_0 so that for paths in $\Phi_r \cup \Phi_{0,r}$ we have a chance of

defining an intersection number with Φ_r . Define

$$\begin{aligned} \Phi_{0,r} = \{T \in \Phi_0 : \text{there exists a decomposition } \mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H} \\ \text{such that } \tau(P) = r \text{ and, with respect to this decomposition,} \\ T = \begin{bmatrix} k \cdot 1 & 0 \\ 0 & D \end{bmatrix} \text{ for some } k \in \mathbb{R} \text{ and some } D : P^\perp\mathcal{H} \rightarrow P^\perp\mathcal{H} \text{ invertible}\}. \end{aligned}$$

Note that since, by assumption, T is invertible, it means that zero is not an eigenvalue of T , and so $k \neq 0$ in the decomposition.

It seems unlikely that $\Phi_{0,r}$ is a submanifold of Φ . In fact, it is not quite clear how to tell, given $T_0 = \begin{bmatrix} 0 & 0 \\ 0 & D_0 \end{bmatrix} \in \Phi_r$ and some $S = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ in a neighbourhood of T_0 , whether or not S is in $\Phi_{0,r}$ (without being able to examine the spectrum of S). Nonetheless, we can consider $\Phi_{0,r} \cup \Phi_r$ as a topological subspace of Φ , so it still makes sense to talk about its singular homology and cohomology groups. Since we somehow want to get spectral flow out of these groups (and spectral flow is a real number in this case), it also makes sense to consider cohomology with coefficients in \mathbb{R} . In keeping with earlier notation (e.g. Chapter 2), we denote by $\Omega(X, Y)$ the set of paths contained in a set X such that the endpoints of the path are in Y .

We would like the following to be true; however, our hopes are based on homotopies at the level of $\Omega(\Phi, \Phi_0)$, and it is not certain if these constructions really translate to maps from or to $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r})$ as necessary.

- $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r}) \cong H^0(\Phi_r)$ follows just like in Lemma 4.1.19 – due to the fact that an element of $\Phi_{0,r}$ has an eigenvalue at k , we can make the tight Ruget neighbourhood definition of Lemma 4.1.14 work, and Ψ will return a positive operator if and only if $k > 0$. The proof of Lemma 4.1.19 does rely on the fact that the path can be homotoped within $\Phi_0 \cup \Phi_1$ to intersect Φ_1 finitely many times; it is unclear whether this is true for Φ_r , or if the proof can be suitably modified.
- If $s = j \cdot r$ for some positive integer j then we can define a map $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r})$ to $H^1(\Phi_s \cup \Phi_{0,s}, \Phi_{0,s})$. Note that it is certainly true that $\Phi_{0,s} \subset \Phi_{0,r}$ for every $s > r$; however, Φ_r and Φ_s are disjoint, so there is no obvious such map. However, we do have something in mind when expressing this desire. To make the explanation easier, suppose $s = 2r$. Suppose moreover that ρ is a path in $\Phi_s \cup \Phi_{0,s}$ with endpoints in $\Phi_{0,s}$, that ρ intersects Φ_s once, at t_0 , and is contained in a tight Ruget neighbourhood centered at $\rho(t_0)$. We perform our usual construction by letting $P = \ker \rho(t_0)$, and noting $\tau(P) = s$. With respect to the decomposition $P\mathcal{H} \oplus P^\perp\mathcal{H}$, $\rho(0)$ looks like $\begin{bmatrix} A_0 & B_0 \\ B_0^* & D_0 \end{bmatrix}$ with D_0 invertible.

We can connect this to $\begin{bmatrix} A_0 - B_0 D_0^{-1} B_0^* & 0 \\ 0 & D_0 \end{bmatrix}$ in Φ , and if $A_0 - B_0 D_0^{-1} B_0^* > 0$ we can connect it to $\begin{bmatrix} 1 & 0 \\ 0 & D_0 \end{bmatrix}$. A similar construction at the other endpoint, $\rho(1)$, connects it to $\begin{bmatrix} \pm 1 & 0 \\ 0 & D_1 \end{bmatrix}$; suppose for explanation purposes that the top left corner is -1 , as

this is the harder case. We can now construct a path in $\Phi_r \cup \Phi_{0,r}$ from $\begin{bmatrix} 1 & 0 \\ 0 & D_0 \end{bmatrix}$ to $\begin{bmatrix} -1 & 0 \\ 0 & D_1 \end{bmatrix}$ by writing $P = P_1 + P_2$ where $\tau(P_1) = \tau(P_2) = r$, changing 1 to -1 on the $P_1 \mathcal{H}$ corner first, then on the $P_2 \mathcal{H}$ corner, and finally changing D_0 to D_1 via the D_t 's in the original path. The result is a path from $\begin{bmatrix} 1 & 0 \\ 0 & D_0 \end{bmatrix}$ to $\begin{bmatrix} -1 & 0 \\ 0 & D_1 \end{bmatrix}$ which is contained in $\Phi_r \cup \Phi_{0,r}$. So given an element of $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r})$ we can evaluate it on this new path and obtain an integer; though whether this does actually give us a map at the level of cohomology is not certain.

- Assume the above construction can be used to get a map κ_{sr} from $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r}; \mathbb{R})$ to $H^1(\Phi_s \cup \Phi_{0,s}, \Phi_{0,s}; \mathbb{R})$ for $s = j \cdot r$ (note that we have now moved on to cohomology with real coefficients). We can place a partial order on \mathbb{R} defined by $r \prec s$ if $s = r \cdot j$ for j a positive integer. Even though under these circumstances \mathbb{R} is not a directed set, we can nevertheless view $\{H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r}; \mathbb{R}); \kappa_{sr}\}$ as an inverse system. The next wild hope is that the inverse limit of this system is $H^1(\Phi, \Phi_0; \mathbb{R})$. We already have candidates for the maps from $H^1(\Phi, \Phi_0; \mathbb{R})$ to $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r}; \mathbb{R})$ – since $\Phi_r \cup \Phi_{0,r} \subset \Phi$ and $\Phi_{0,r} \subset \Phi_0$, there is a map between the two cohomologies induced by the inclusion. Moreover, we can apply homotopies to each path in $\Omega(\Phi, \Phi_0)$ to obtain a concatenation of paths, each of which consists entirely of invertible operators or is contained in $\Phi_r \cup \Phi_{0,r}$ for some r . To start with, given a path ρ in $\Omega(\Phi, \Phi_0)$ we can cover it by Ruget neighbourhoods, and homotope it to a concatenation of paths, each of which has invertible endpoints and is contained in a Ruget neighbourhood (see e.g. Lemma 2.3.1 for this type of argument). We can then apply the construction of Remark 4.3.1 to each path contained in a Ruget neighbourhood to conclude that ρ can be homotoped (in $\Omega(\Phi, \Phi_0)$) to a concatenation $\rho_1 * \rho_2 * \dots * \rho_n$ where either ρ_i consists of invertible operators or $\rho_i \in \Omega(\Phi_{r_i} \cup \Phi_{0,r_i}, \Phi_{0,r_i})$ (in the construction, the paths from $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ to $\begin{bmatrix} 2P - 1 & 0 \\ 0 & D \end{bmatrix}$ consist of invertible operators but are not necessarily in $\Phi_{0,r}$ for any r).

If all the above are true, then we can identify an element of $H^1(\Phi, \Phi_0; \mathbb{R})$ which calculates spectral flow. Namely, consider the element of $H^0(\Phi_r; \mathbb{R})$ which assigns the value r to each path component of Φ_r . Certainly its image $[\Phi_r]$ in $H^1(\Phi_r \cup \Phi_{0,r}, \Phi_{0,r}; \mathbb{R})$ would then calculate the intersection number with Φ_r , appropriately scaled by r , reflecting the spectral flow of the restricted paths under consideration. Finally, $\{[\Phi_r]\}_{r \in \mathbb{R}}$ in $H^1(\Phi, \Phi_0; \mathbb{R})$ would calculate spectral flow of paths in $\Omega(\Phi, \Phi_0)$.

The manipulations in this section are not dissimilar to those used to prove the uniqueness of spectral flow in Chapter 2, so there is a suggestion that some path can be picked out from under all these conjectures. It is certainly likely that further modifications would have to be made (to Φ_r , $\Phi_{0,r}$ or to the spaces on which we are considering the cohomology), or that the construction can be greatly simplified (for example, is it really necessary to take the inverse limit over all real numbers, or would a suitable subset do?). As we know that integral formulas exist for spectral flow in this situation, one feels that there should be a suitable geometric interpretation; though the complications attendant on the type II setting make it likely to be somewhat unwieldy.


Appendix A

Comparison of Integrals for Real-valued Functions


The goal of this section is to prove the following inequality of integrals:

If f, g are non-negative, decreasing functions such that $\int_0^s f dx \leq \int_0^s g dx$ for all $s \in \mathbb{R}_+$, then for any $p \geq 1$ we have $\int_0^\infty f^p dx \leq \int_0^\infty g^p dx$.

It is relegated to the appendix as I feel that it should be a known result, I just could not find a reference for it. The result is used in Section 3.4.1 to prove an operator inequality using generalized s -numbers. The proof is an approximation by simple functions argument (Lemma A.0.5 below), which uses the following standard result from functional analysis.

Theorem A.0.2 (Folland [37], p. 47, Theorem 2.10, 2.14) (X, \mathcal{M}) measure space, $f : X \rightarrow [0, \infty]$ measurable. Then there exists a sequence $\{\varphi_n\}$ of simple functions such that $0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq f$ and $\varphi_n \rightarrow f$ pointwise. [Monotone Convergence Theorem - Theorem 2.14] It follows that $\int f = \lim_{n \rightarrow \infty} \int \varphi_n$. 

First prove an inequality for real numbers, which will later be used to prove the corresponding inequality for integrals of simple functions. The inequality we need is a slight variation on the following result:

Lemma A.0.3 ([61], Lemma 6.3) Let $1 \leq p < \infty$. Then for a, b, t non-negative we have $(a + tb)^p \geq a^p + ptba^{p-1}$. 

Going through the proof, the only reason for the requirement that t is positive is that $(a + tb)^p$ might not be defined if $a + tb$ is a negative number; however, if we avoid this pitfall, the result would still hold for suitably chosen negative values of t . The inequality we need, stated and proven below, can be derived from the above by setting first $t = 1$ and then $t = -1$ and performing a change of variable (different in each case). The proof follows the same steps as in [61].

Lemma A.0.4 If $a, b \geq 0$ and $p > 1$ the following inequality holds:

$$pa^{p-1}(b - a) \leq b^p - a^p \leq pb^{p-1}(b - a). \quad \text{img alt="hand icon" data-bbox="865 704 882 717" style="float: right;"/>$$

PROOF We can suppose without loss of generality that $b \geq a$, since if a is larger we can multiply the corresponding chain of inequalities for a by -1 to get the above result.

Consider the leftmost inequality first. Let $\varphi(t) = [a + t(b - a)]^p - a^p - tpa^{p-1}(b - a)$ for $t \in [0, 1]$. Note that $\varphi(0) = 0$ and $\varphi(1) = b^p - a^p - pa^{p-1}(b - a)$, so in order to prove the leftmost inequality, it is sufficient to prove that φ is increasing on $[0, 1]$. However,

$$\varphi'(t) = p[a + t(b - a)]^{p-1}(b - a) - pa^{p-1}(b - a) = p(b - a)\{[a + t(b - a)]^{p-1} - a^{p-1}\}.$$

Since $b \geq a$ by assumption, $t \geq 0$ and for $p > 1$ the function $z \mapsto z^{p-1}$ is increasing, we get $\varphi'(t) \geq 0$. This proves that φ is increasing, hence $\varphi(1) \geq 0$. That is, $pa^{p-1}(b - a) \leq b^p - a^p$.

Now let us show that $b^p - a^p \leq pb^{p-1}(b-a)$. Let $\psi(t) = tpb^{p-1}(b-a) + a^p - [a + t(b-a)]^p$ for $t \in [0, 1]$. Then, $\psi(0) = 0$ and $\psi(1) = pb^{p-1}(b-a) - b^p + a^p$. Hence, once again, it is sufficient to prove that ψ is increasing. However,

$$\psi'(t) = pb^{p-1}(b-a) - [a + t(b-a)]^{p-1}(b-a) = p(b-a)\{b^{p-1} - [a + t(b-a)]^{p-1}\},$$

which is greater than or equal to 0. Therefore, $b^p - a^p \leq pb^{p-1}(b-a)$. ■

Lemma A.0.5 If f, g are non-negative, decreasing functions such that $\int_0^s f dx \leq \int_0^s g dx$ for all $s \in \mathbb{R}_+$, then for any $p \geq 1$ we have $\int_0^\infty f^p dx \leq \int_0^\infty g^p dx$. ■

PROOF First prove the case when f and g are simple functions, then for the general case approximate f and g by simple functions.

Consider f and g simple functions, say $f = a_1\chi_{E_1} + \dots + a_n\chi_{E_n}$ and $g = b_1\chi_{F_1} + \dots + b_m\chi_{F_m}$, where the E_i 's and F_j 's are disjoint. By replacing the E_i 's and F_j 's by all the non-empty intersections $E_i \cap F_j$, we can assume without loss of generality that $m = n$ and $E_i = F_i$ for all i . Let $\mu(E_i) = \alpha_i \geq 0$. Since f and g are non-negative and decreasing we know that $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \dots \geq b_n \geq 0$.

Since $f^p = \sum_{i=1}^n a_i^p \chi_{E_i}$ we have $\int_0^\infty f^p dx = \sum_{i=1}^n \alpha_i a_i^p$ (and similarly for g). Hence, we need to show that

$$\alpha_1 a_1^p + \alpha_2 a_2^p + \dots + \alpha_n a_n^p \leq \alpha_1 b_1^p + \alpha_2 b_2^p + \dots + \alpha_n b_n^p$$

By Lemma A.0.4 we have

$$\alpha_i(b_i^p - a_i^p) \geq pa_i^p \alpha_i(b_i - a_i) \geq pa_n^p \alpha_i(b_i - a_i)$$

(where the second inequality is true since $a_i \geq a_n$ for any i); hence,

$$\sum_{i=1}^n \alpha_i(b_i^p - a_i^p) \geq \sum_{i=1}^n [pa_n^p \alpha_i(b_i - a_i)] = pa_n^p \sum_{i=1}^n [\alpha_i(b_i - a_i)]$$

But $\sum_{i=1}^n [\alpha_i(b_i - a_i)] \geq 0$ (since $\int_0^\infty f \leq \int_0^\infty g$), and hence the right hand side is non-negative (since $a_n \geq 0$ as well). This proves the result for simple functions.

Since g is decreasing, it is bounded by $g(0)$ so we can find a sequence of simple functions $\{\psi_n\}$ decreasing to g (by approximating $g(0) - g$). For any s , $\int_0^s \varphi_n \leq \int_0^s f \leq \int_0^s g \leq \int_0^s \psi_n$. Using the result for simple functions $\int_0^\infty \varphi_n^p \leq \int_0^\infty \psi_n^p$. But $\varphi_1^p \leq \varphi_2^p \leq \dots \leq f^p$ (since $z \mapsto z^p$ is an increasing function) and $\{\varphi_n^p\}$ converges pointwise to f^p (since $z \mapsto z^p$ is continuous, and $\varphi_n \rightarrow f$ pointwise). Therefore, the Monotone Convergence Theorem applies and, by taking limits of both sides, we get $\int_0^\infty f^p \leq \int_0^\infty g^p$. ■

Appendix B

Some Function Inequalities with Exponentials

We felt that the proof of certain function inequalities used in Chapter 4 would derail the narrative at that point, as the work is very routine; nonetheless, it is nice to have it for future reference, so we include it here. The proof will be the same for every one of the following; namely, to show that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, show that $g(0) - f(0) \geq 0$, and $g(x) - f(x)$ is a continuous function which is decreasing for $x < 0$ and increasing for $x > 0$.

Lemma B.0.6 Suppose $a > 0$ is fixed. Then, for any $\lambda \in \mathbb{R}$,

$$(i) \quad \frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \geq 0$$

$$(ii) \quad \frac{e^{-a\lambda^2} - 1}{a} + \lambda^2 \leq a \cdot \frac{\lambda^4}{2}$$



PROOF To make the calculations easier, multiply through by a (which doesn't change the direction of the inequality since $a > 0$); then, follow the recipe indicated at the beginning of this section.

(i) Let $f(\lambda) = e^{-a\lambda^2} - 1 + a\lambda^2$. Then $f(0) = 0$ and $f'(\lambda) = -2a\lambda e^{-a\lambda^2} + 2a\lambda = 2a\lambda(1 - e^{-a\lambda^2})$. Since $a\lambda^2 \geq 0$, $e^{-a\lambda^2} \leq 1$ (with equality when $\lambda = 0$), which gives us that $f'(\lambda) < 0$ if $\lambda < 0$ and $f'(\lambda) > 0$ if $\lambda > 0$. This establishes the fact that f is decreasing for λ negative and is increasing for λ positive, which, combined with the fact that $f(0) = 0$, gives that $f(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$.

(ii) This is equivalent to showing that, if $g(\lambda) = a^2 \cdot \frac{\lambda^4}{2} - e^{-a\lambda^2} - a\lambda^2 + 1$, then $g(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Calculate that $g(0) = 0$ and $g'(\lambda) = a^2 \cdot 4 \cdot \frac{\lambda^3}{2} - e^{-a\lambda^2} \cdot (-2a\lambda) - 2a\lambda = 2a\lambda(a\lambda^2 + e^{-a\lambda^2} - 1)$. We know $a\lambda^2 + e^{-a\lambda^2} - 1$ is greater than or equal to zero (this is the function $f(\lambda)$ from the previous part); it follows that $g'(\lambda) < 0$ when $\lambda < 0$ and $g'(\lambda) > 0$ when $\lambda > 0$, once again concluding the proof. ■

Lemma B.0.7 Suppose $a > 0$ is fixed. Then, for any $\lambda \in \mathbb{R}$,

$$(i) \quad \frac{e^{a\lambda^2} - 1}{a} - \lambda^2 \geq 0.$$

$$(ii) \quad \text{if } n \text{ is a positive integer such that } a < \frac{1}{n}, \text{ then } ane^{\frac{1}{n}\lambda^2} - e^{a\lambda^2} + 1 > 0.$$

$$(iii) \quad \text{if } n \text{ is a positive integer such that } a < \frac{1}{n}, \text{ then } \frac{e^{a\lambda^2} - 1}{a} - \lambda^2 \leq a \cdot n^2 \cdot e^{\frac{1}{n}\lambda^2}$$



PROOF Get rid of fractions by multiplying each inequality through by $a > 0$, if necessary.

(i) Let $f(\lambda) = e^{a\lambda^2} - 1 - a\lambda^2$. Then $f(0) = 0$ and $f'(\lambda) = 2a\lambda e^{a\lambda^2} - 2a\lambda = 2a\lambda(e^{a\lambda^2} - 1)$. Since $a\lambda^2 \geq 0$, $e^{a\lambda^2} - 1 \geq 0$ (with equality when $\lambda = 0$), so f is decreasing for $\lambda < 0$ and increasing for $\lambda > 0$. Since $f(0) = 0$, this concludes the proof of the inequality.

(ii) Let $h(\lambda) = ane^{\frac{1}{n}\lambda^2} - e^{a\lambda^2} + 1$. Then $h(0) = an - 1 + 1 = an > 0$. Next,

$$h'(\lambda) = an \cdot \frac{1}{n} \cdot 2\lambda \cdot e^{\frac{1}{n}\lambda^2} - 2a\lambda \cdot e^{a\lambda^2} = 2a\lambda(e^{\frac{1}{n}\lambda^2} - e^{a\lambda^2}).$$

Since $a < \frac{1}{n}$, $e^{\frac{1}{n}\lambda^2} - e^{a\lambda^2} > 0$, so h is decreasing for $\lambda < 0$ and increasing for $\lambda > 0$, allowing us to conclude that $h(\lambda) \geq h(0) = an > 0$ for all λ .

(iii) Let $g(\lambda) = a^2n^2e^{\frac{1}{n}\lambda^2} - (e^{a\lambda^2} - 1 - a\lambda^2)$. Note that $g(0) = a^2n^2 - (1 - 1 - 0) = a^2n^2 > 0$. Next, calculate $g'(\lambda) = a^2n^2(\frac{1}{n} \cdot 2\lambda) \cdot e^{\frac{1}{n}\lambda^2} - 2a\lambda \cdot e^{a\lambda^2} + 2a\lambda = 2a\lambda(ane^{\frac{1}{n}\lambda^2} - e^{a\lambda^2} + 1)$. As shown above, $ane^{\frac{1}{n}\lambda^2} - e^{a\lambda^2} + 1 > 0$, so as in all the other proofs we can now conclude that $g(\lambda) \geq g(0) = a^2n^2 > 0$ for all λ . ■

Appendix C

Matrix Multiplications

In order to not clutter up the presentation in 2, we postponed details of some matrix multiplications to this appendix. These results are used in the proof of Lemma 2.4.3. Without further ado, here they are:

Lemma C.0.8 Suppose U and Z are bounded operators on \mathcal{H} such that, with respect to some decomposition of \mathcal{H} , we have $U = \begin{bmatrix} X & V \\ W & Y \end{bmatrix}$, and

$$Z = \begin{bmatrix} X - tV(tY + 1)^{-1}W & \sqrt{1 - t^2} \cdot V(tY + 1)^{-1} \\ \sqrt{1 - t^2} \cdot (tY + 1)^{-1}W & (Y + t)(tY + 1)^{-1} \end{bmatrix},$$

where t is some real number in $[0, 1]$. If $UU^* = 1$ then $ZZ^* = 1$. ☞

PROOF Block matrix multiplication for $UU^* = 1$ gives us the following equalities:

$$\begin{aligned} XX^* + VV^* &= 1 \\ XW^* + VY^* &= 0 \\ WX^* + YV^* &= 0 \\ WW^* + YY^* &= 1 \end{aligned}$$

We use these to verify that $ZZ^* = 1$. Note that, with Z as given in the statement of the lemma,

$$Z^* = \begin{bmatrix} X^* - tW^*(tY^* + 1)^{-1}V^* & \sqrt{1 - t^2} \cdot W^*(tY^* + 1)^{-1} \\ \sqrt{1 - t^2} \cdot (tY^* + 1)^{-1}V^* & (tY^* + 1)^{-1}(Y^* + t) \end{bmatrix}.$$

Before we get started, we note the following equality:

$$A(A + 1)^{-1} + B(B + 1)^{-1} - A(A + 1)^{-1}B(B + 1)^{-1} + (A + 1)^{-1}(B + 1)^{-1} = 1. (\dagger)$$

To see that this holds, note that $A(A + 1)^{-1} = 1 - (A + 1)^{-1}$, so the left-hand side simplifies to

$$\begin{aligned} &1 - (A + 1)^{-1} + B(B + 1)^{-1} - (1 - (A + 1)^{-1})B(B + 1)^{-1} + (A + 1)^{-1}(B + 1)^{-1} \\ &= 1 - (A + 1)^{-1} + B(B + 1)^{-1} - B(B + 1)^{-1} + (A + 1)^{-1}B(B + 1)^{-1} + (A + 1)^{-1}(B + 1)^{-1} \\ &= 1 - (A + 1)^{-1}(1 - B(B + 1)^{-1} - (B + 1)^{-1}), \end{aligned}$$

which simplifies to 1 since $1 - B(B + 1)^{-1} - (B + 1)^{-1}$ is equal to 0.

Perform the block matrix multiplication for ZZ^* :

- Multiply the first row by the first column.

$$\begin{aligned} &(X - tV(tY + 1)^{-1}W)(X^* - tW^*(tY^* + 1)^{-1}V^*) + (1 - t^2)V(tY + 1)^{-1}(tY^* + 1)^{-1}V^* \\ &= XX^* - tXW^*(tY^* + 1)^{-1}V^* - tV(tY + 1)^{-1}WX^* + \\ &\quad t^2V(tY + 1)^{-1}WW^*(tY^* + 1)^{-1}V^* + (1 - t^2)V(tY + 1)^{-1}(tY^* + 1)^{-1}V^*. \end{aligned}$$

Using the substitutions $XW^* = -VY^*$ and $WX^* = -YV^*$, the above expression becomes

$$\begin{aligned} & XX^* + tVY^*(tY^* + 1)^{-1}V^* + tV(tY + 1)^{-1}YV^* + t^2V(tY + 1)^{-1}WW^*(tY^* + 1)^{-1}V^* \\ & \quad + V(tY + 1)^{-1}(tY^* + 1)^{-1}V^* - t^2V(tY + 1)^{-1}(tY^* + 1)^{-1}V^* \\ & = XX^* + V[tY^*(tY^* + 1)^{-1} + t(tY + 1)^{-1}Y + t^2(tY + 1)^{-1}(WW^* - 1)(tY^* + 1)^{-1} + \\ & \quad + (tY + 1)^{-1}(tY^* + 1)^{-1}]V^*. \end{aligned}$$

Since $WW^* - 1 = -YY^*$ the expression in brackets simplifies to

$$tY^*(tY^* + 1)^{-1} + t(tY + 1)^{-1}Y - t^2(tY + 1)^{-1}YY^*(tY^* + 1)^{-1} + (tY + 1)^{-1}(tY^* + 1)^{-1},$$

which is 1 by (†) (with $A = tY$ and $B = tY^*$). Therefore the entry in the 1,1 location of the result is

$$XX^* + V \cdot 1 \cdot V^* = 1.$$

- Multiply the first row by the second column.

$$\begin{aligned} & (X - tV(tY + 1)^{-1}W)(\sqrt{1 - t^2} \cdot W^*(tY^* + 1)^{-1}) \\ & \quad + \sqrt{1 - t^2} \cdot V(tY + 1)^{-1}(tY^* + 1)^{-1}(Y^* + t) \\ & = \sqrt{1 - t^2} \cdot [XW^*(tY^* + 1)^{-1} - tV(tY + 1)^{-1}WW^*(tY^* + 1)^{-1} \\ & \quad + V(tY + 1)^{-1}(tY^* + 1)^{-1}(Y^* + t)]. \end{aligned}$$

Substituting $XW^* = -VY^*$ and $WW^* = 1 - YY^*$, get

$$\begin{aligned} & \sqrt{1 - t^2} \cdot [-VY^*(tY^* + 1)^{-1} - tV(tY + 1)^{-1}(tY^* + 1)^{-1} \\ & \quad + tV(tY + 1)^{-1}YY^*(tY^* + 1)^{-1} + V(tY + 1)^{-1}(tY^* + 1)^{-1}(Y^* + t)] \\ & = \sqrt{1 - t^2} \cdot V[-Y^*(tY^* + 1)^{-1} + (tY + 1)^{-1}(-t + tYY^* + Y^* + t)(tY^* + 1)^{-1}] \\ & = \sqrt{1 - t^2} \cdot V[-Y^*(tY^* + 1)^{-1} + (tY + 1)^{-1}(tY + 1)Y^*(tY^* + 1)^{-1}]. \end{aligned}$$

which simplifies to 0.

- Multiply the second row by the first column.

$$\begin{aligned} & \sqrt{1 - t^2} \cdot (tY + 1)^{-1}W(X^* - tW^*(tY^* + 1)^{-1}V^*) \\ & \quad + (Y + t)(tY + 1)^{-1} \cdot \sqrt{1 - t^2} \cdot (tY^* + 1)^{-1}V^* \\ & = \sqrt{1 - t^2} \cdot (tY + 1)^{-1}[WX^* - tWW^*(tY^* + 1)^{-1}V^* + (Y + t)(tY^* + 1)^{-1}V^*]. \end{aligned}$$

Substitute $WX^* = -YV^*$ and $WW^* = 1 - YY^*$,

$$\begin{aligned} & \sqrt{1 - t^2} \cdot (tY + 1)^{-1}[-YV^* - t(tY^* + 1)^{-1}V^* + tYY^*(tY^* + 1)^{-1}V^* + (Y + t)(tY^* + 1)^{-1}V^*] \\ & = \sqrt{1 - t^2} \cdot (tY + 1)^{-1}[-Y - (-t + tYY^* + Y + t)(tY^* + 1)^{-1}]V^* \\ & = \sqrt{1 - t^2} \cdot (tY + 1)^{-1}[-Y - Y(tY^* + 1)(tY^* + 1)^{-1}]V^*. \end{aligned}$$

which simplifies to 0.

- Multiply the second row by the second column.

$$\begin{aligned} & \sqrt{1-t^2} \cdot (tY+1)^{-1}W \cdot \sqrt{1-t^2} \cdot W^*(tY^*+1)^{-1} + (Y+t)(tY+1)^{-1}(Y^*+t)(tY^*+1)^{-1} \\ &= (tY+1)^{-1} \cdot [(1-t^2)WW^* + (Y+t)(Y^*+t)] \cdot (tY^*+1)^{-1}. \end{aligned}$$

Substitute $WW^* = 1 - YY^*$:

$$\begin{aligned} &= (tY+1)^{-1} \cdot [(1-t^2)(1-YY^*) + YY^* + tY + tY^* + t^2] \cdot (tY^*+1)^{-1} \\ &= (tY+1)^{-1} \cdot [1-YY^* - t^2 + t^2YY^* + YY^* + tY + tY^* + t^2] \cdot (tY^*+1)^{-1} \\ &= (tY+1)^{-1} \cdot [(1+tY) + (tY+1)tY^*] \cdot (tY^*+1)^{-1} \\ &= (tY+1)^{-1} \cdot (tY+1) \cdot (1+tY^*) \cdot (tY^*+1)^{-1}, \end{aligned}$$

which simplifies to 1.

Therefore, $ZZ^* = 1$, as claimed. ■

Lemma C.0.9 Suppose U and Z are bounded operators on \mathcal{H} such that, with respect to some decomposition of \mathcal{H} , we have $U = \begin{bmatrix} X & V \\ W & Y \end{bmatrix}$, and

$$Z = \begin{bmatrix} X - tV(tY+1)^{-1}W & \sqrt{1-t^2} \cdot V(tY+1)^{-1} \\ \sqrt{1-t^2} \cdot (tY+1)^{-1}W & (Y+t)(tY+1)^{-1} \end{bmatrix},$$

where t is some real number in $[0, 1]$. Suppose further λ is either 1 or -1 and that

$$(U - \lambda)^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then, except in the case when λ and t are both 1,

$$(Z - \lambda)^{-1} = \begin{bmatrix} A & \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}B \\ \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}C & \frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t}D \end{bmatrix}. \quad \text{☞}$$

PROOF Suppose that λ is either plus or minus 1. The special properties of such a λ that are not shared by other complex units are that $\lambda^2 = 1$, and $\sqrt{1-t^2} = \sqrt{1-\lambda t} \cdot \sqrt{1+\lambda t}$ (note here that, since $\lambda = \pm 1$ and $t \in [0, 1]$, $1-\lambda t$ and $1+\lambda t$ are both non-negative integers). These two facts will cause some terms to cancel out in the calculations below (and will be used without further mention).

Since $U - \lambda$ has an inverse, block matrix multiplication gives us the following equalities:

$$\begin{aligned} (X - \lambda)A + VC &= 1 \\ (X - \lambda)B + VD &= 0 \\ WA + (Y - \lambda)C &= 0 \\ WB + (Y - \lambda)D &= 1 \end{aligned}$$

Now multiply $Z - \lambda$ by its purported inverse, and verify that we get the identity:

- Multiply the first row by the first column, and use the equality $WA = -(Y - \lambda)C$.

$$\begin{aligned} & (X - \lambda)A - tV(tY + 1)^{-1}WA + \sqrt{1 - t^2} \cdot \sqrt{\frac{1+\lambda t}{1-\lambda t}} V(tY + 1)^{-1}C = \\ & (X - \lambda)A - tV(tY + 1)^{-1}(-(Y - \lambda)C) + \sqrt{1 - t^2} \cdot \sqrt{\frac{1+\lambda t}{1-\lambda t}} V(tY + 1)^{-1}C = \\ & (X - \lambda)A + V(tY + 1)^{-1}[t(Y - \lambda) + \sqrt{1 - t^2} \cdot \sqrt{\frac{1+\lambda t}{1-\lambda t}}]C. \end{aligned}$$

Since $\sqrt{1 - t^2} = \sqrt{1 - \lambda t} \cdot \sqrt{1 + \lambda t}$ for $\lambda = \pm 1$, $\sqrt{1 - t^2} \cdot \sqrt{\frac{1+\lambda t}{1-\lambda t}} = 1 + \lambda t$; and so the square bracket above simplifies to $tY - t\lambda + 1 + \lambda t = tY + 1$; that is, we get

$$(X - \lambda)A + V(tY + 1)^{-1}(tY + 1)C = (X - \lambda)A + VC = 1$$

- Multiply the first row by the second column.

$$\begin{aligned} & \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot (X - \lambda)B - \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot tV(tY + 1)^{-1}WB + \\ & \quad (\sqrt{1 - \lambda t} \sqrt{1 + \lambda t}) \cdot V(tY + 1)^{-1} \cdot \left(\frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t} \cdot D\right) = \\ & \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot (X - \lambda)B - \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot tV(tY + 1)^{-1}WB + \\ & \quad \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot V(tY + 1)^{-1}(t + (1 + \lambda t)D) = \\ & \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} \cdot [(X - \lambda)B - tV(tY + 1)^{-1}WB + V(tY + 1)^{-1}(t + (1 + \lambda t)D)]. \end{aligned}$$

The expression in the square brackets can be simplified to

$$\begin{aligned} & (X - \lambda)B - V(tY + 1)^{-1}(tWB - (t + (1 + \lambda t)D)) = \\ & (X - \lambda)B - V(tY + 1)^{-1}(t(1 - YD + \lambda D) - t - D - \lambda tD) = \\ & (X - \lambda)B + V(tY + 1)^{-1}(tY + 1)D = \\ & (X - \lambda)B + VD, \end{aligned}$$

which is 0. Therefore, the entry in the first row, second column of the result is the 0 operator.

- Multiply the second row by the first column.

$$\sqrt{1 - t^2}(tY + 1)^{-1}WA + ((Y + t)(tY + 1)^{-1} - \lambda) \cdot \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}C.$$

Replace WA by $-(Y - \lambda)C$

$$-\sqrt{1 - t^2}(tY + 1)^{-1}(Y - \lambda)C + (Y + t)(tY + 1)^{-1}C \cdot \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}C - \lambda \cdot \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}C.$$

Since $\lambda = \pm 1$ and $t \in [0, 1]$, $\sqrt{1 - t^2} = \sqrt{1 - \lambda t} \cdot \sqrt{1 + \lambda t} = \frac{(1 - \lambda t)\sqrt{1 + \lambda t}}{\sqrt{1 - \lambda t}}$, so we can factor out $\frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}$ from each term

$$\frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} [-(1 - \lambda t)(tY + 1)^{-1}(Y - \lambda) + (Y + t)(tY + 1)^{-1} - \lambda]C.$$

Let us simplify the expression inside the square brackets first; note that we will need to use the fact that $\lambda^2 = 1$:

$$\begin{aligned} & (tY + 1)^{-1}(-(1 - \lambda t)(Y - \lambda) + Y + t) - \lambda = \\ & (tY + 1)^{-1}(-Y + \lambda + \lambda tY - t + Y + t) - \lambda = \\ & (tY + 1)^{-1}\lambda(tY + 1) - \lambda, \end{aligned}$$

which is zero. Since the expression inside the square brackets is 0, we get

$$\frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}} [-(1-\lambda t)(tY+1)^{-1}(Y-\lambda) + (Y+t)(tY+1)^{-1} - \lambda]C = 0,$$

and hence the entry in the second row, first column of the result is also zero.

- Multiply the second row by the second column.

$$\begin{aligned} & (1-t^2)(tY+1)^{-1}W \cdot \frac{\sqrt{1+\lambda t}}{\sqrt{1-\lambda t}}B + [(Y+t)(tY+1)^{-1} - \lambda] \left[\frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t}D \right] = \\ & (1+\lambda t)(tY+1)^{-1}(1-YD + \lambda D) + (tY+1)^{-1}(Y+t - \lambda(tY+1)) \left(\frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t}D \right) = \\ & (tY+1)^{-1} \left[(1+\lambda t)(1-YD + \lambda D) + [((1-\lambda t)Y + (t-\lambda)) \left(\frac{t}{1-\lambda t} + \frac{1+\lambda t}{1-\lambda t}D \right)] \right] = \\ & (tY+1)^{-1} \left[(1+\lambda t) - (1+\lambda t)YD + (1+\lambda t)\lambda D + \right. \\ & \quad \left. [tY + (1+\lambda t)YD + \frac{t(t-\lambda)}{1-\lambda t} + \frac{(t-\lambda)(1+\lambda t)}{1-\lambda t}D] \right] = \\ & (tY+1)^{-1} \left[tY + (1+\lambda t + \frac{t(t-\lambda)}{1-\lambda t}) + D(1+\lambda t)(\lambda + \frac{t-\lambda}{1-\lambda t}) \right]. \end{aligned}$$

It is easy to check that, when $\lambda = \pm 1$, the coefficient of D becomes 0, and $(1 + \lambda t + \frac{t(t-\lambda)}{1-\lambda t})$ simplifies to 1. In other words, the whole expression becomes

$$(tY+1)^{-1}(tY+1 + D \cdot 0) = 1.$$

The verification that multiplying the matrices in the opposite order also gives the identity is similar. ■

Appendix D

Path Independence of Integral

In Section 3.7.1 we showed that, if q is large enough, $\alpha_F : X \mapsto \tau(X|1 - F^2|^q)$ is a closed one-form on $\mathcal{L}^{\frac{p}{2}}_{F_0}$. The purpose of this appendix is to show that the integral of this one-form is independent of the path of integration; the basis of the approach is the standard proof of Poincaré's Lemma. The proof that α is exact (ie. there is a θ for which $d\theta = \alpha$) is modelled on Proposition 1.4 of [14], in which the equivalent result is proven for a different definition of θ and α , and the fact that any path can be suitably approximated by a piecewise linear path (Lemma D.0.11) is a combination of the techniques used in Remark 1.8 and Proposition 1.5 of [14].

Recall that

$$\mathcal{L}^{\frac{p}{2}}_{F_0} = \{X \in \mathcal{L}^p_{sa} : 1 - (F_0 + X)^2 \in \mathcal{L}^{\frac{p}{2}}\}.$$

Lemma D.0.10 Define $\theta : (F_0 + \mathcal{L}^{\frac{p}{2}}_{F_0}) \rightarrow \mathbb{R}$ by $\theta(F) = \frac{1}{C_q} \int_0^1 \tau((F - F_0)|1 - F_t^2|^q) dt$, where $F_t = F_0 + t(F - F_0)$ for $t \in [0, 1]$ (that is, $\{F_t\}$ is the straight-line path from F_0 to F). Then $d\theta = \alpha$. ♣

PROOF By definition,

$$d\theta_F(X) = \left. \frac{d}{ds} \right|_{s=0} \theta(F + sX) = \lim_{s \rightarrow 0} \frac{\theta(F + sX) - \theta(F)}{s}.$$

Evaluating θ at $F + sX$ requires the straight line path from $F + sX$ to F_0 , which we denote by $F_{s,t}$; that is, $F_{s,t} = (1 - t)F_0 + t(F + sX)$, defined for $t \in [0, 1]$ and s close to 0. For purposes of readability, we also let $N_{s,t} = 1 - F_{s,t}^2$. Then

$$\theta(F + sX) = \frac{1}{C_q} \int_0^1 \tau([(F + sX) - F_0]|N_{s,t}|^q) dt,$$

so, continuing from where we left off above, we have

$$\begin{aligned} d\theta_F(X) &= \lim_{s \rightarrow 0} \frac{1}{C_q} \cdot \frac{1}{s} \int_0^1 \tau((F + sX - F_0)|N_{s,t}|^q - (F - F_0)|N_{0,t}|^q) dt \\ &= \frac{1}{C_q} \lim_{s \rightarrow 0} \int_0^1 \frac{1}{s} \cdot \tau((F - F_0)(|N_{s,t}|^q - |N_{0,t}|^q) + sX|N_{s,t}|^q) dt \\ &= \frac{1}{C_q} \lim_{s \rightarrow 0} \int_0^1 \tau((F - F_0) \left[\frac{|N_{s,t}|^q - |N_{0,t}|^q}{s} \right] + \frac{s}{s} X|N_{s,t}|^q) dt \\ &= \frac{1}{C_q} \int_0^1 \left[\lim_{s \rightarrow 0} \tau((F - F_0) \frac{|N_{s,t}|^q - |N_{0,t}|^q}{s}) + \lim_{s \rightarrow 0} \tau(X|N_{s,t}|^q) \right] dt \\ &\quad \text{(where we used the continuity of the integrand and the fact that the interval of} \\ &\quad \text{integration is compact to pull the limit into the integral).} \end{aligned}$$

In Section 3.4 we showed that

$$\lim_{s \rightarrow 0} \tau(Yg(T + sX)^q) = \sum_{i=0}^n \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{-r} \tau(Yg(T)^{r(n-i)} R_\lambda g(T) d_{g,0}(T, X) R_\lambda g(T) g(T)^{ri} d\lambda,$$

where $n = \lfloor q \rfloor$ and $r = \frac{q}{n+1}$. In Section 3.7.1 we applied this formula with $g(F) = |1 - F^2|^q$. Using the same approach as in these two sections, we get

$$\begin{aligned} & \lim_{s \rightarrow 0} \tau((F - F_0) \frac{|N_{s,t}|^q - |N_{0,t}|^q}{s}) \\ &= -\frac{\sin(\frac{\pi r}{2})}{\pi} \sum_{i=0}^n \left[\int_0^\infty \lambda^{-\frac{r}{2}} \tau((F - F_0) |N_{0,t}^{ri}| (1 + \lambda N_{0,t}^2)^{-1} \right. \\ & \quad \left. \{ \{(1-t)F_0 + tF, tX\}, N_{0,t} \} (1 + \lambda N_{0,t}^2)^{-1} |N_{0,t}|^{r(n-i)} \} \right] d\lambda. \end{aligned}$$

Moreover, $\lim_{s \rightarrow 0} \tau(X |N_{s,t}|^q) = \tau(X |N_{0,t}|^q)$, as in part 3 of Section 3.7.1. This concludes our calculation of $d\theta$, and we now wish to compare it with α . We have that

$$\begin{aligned} \alpha_F(X) &= \tau(X |1 - F^2|^q) = \tau(X |N_{0,1}|^q) \\ &= \int_0^1 \frac{d}{dt} [t \tau(X |N_{0,t}|^q)] dt \\ &= \int_0^1 t \cdot \frac{d}{dt} [\tau(X |N_{0,t}|^q)] + \frac{d}{dt} [t] \cdot \tau(X |N_{0,t}|^q) dt \\ &= \int_0^1 t \lim_{s \rightarrow 0} \frac{\tau(X |N_{0,t+s}|^q - X |N_{s,t}|^q)}{s} + \tau(X |N_{0,t}|^q) dt. \end{aligned}$$

This is similar-looking to what we got for $d\theta$, so we use the same method of expansion to get

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{\tau(X |N_{0,t+s}|^q - X |N_{s,t}|^q)}{s} \\ &= -\frac{\sin(\frac{\pi r}{2})}{\pi} \sum_{i=0}^n \left[\int_0^\infty \lambda^{-\frac{r}{2}} \tau(X |N_{0,t}|^i (1 + \lambda N_{0,t}^2)^{-1} \right. \\ & \quad \left. \{ \{(1-t)F_0 + tF, (F - F_0)\}, N_{0,t} \} (1 + \lambda N_{0,t}^2)^{-1} |N_{0,t}|^{r(n-i)} \} \right] d\lambda. \end{aligned}$$

By using the cyclic property of trace and rearranging, the two formulas obtained above can be shown to be the same. Therefore, $d\theta = \alpha$ as claimed. \blacksquare

Lemma D.0.11 Suppose $\{F_t\}$ is C^1 -path in $\mathcal{M} = F_0 + \mathcal{L}^{\frac{p}{2}}_{F_0}$. Suppose moreover that $F \rightarrow g(F)$ is a continuous function from \mathcal{M} to \mathcal{L}^1 . Given $\varepsilon > 0$, we can find a piecewise linear path $\{G_t\}$ from F_0 to F such that

$$\left| \int_0^1 \tau(F'_t g(F_t)) dt - \int_0^1 \tau(G'_t g(G_t)) dt \right| < \varepsilon. \quad \blacktriangleright$$

PROOF We can assume without loss of generality that $\varepsilon < 1$. To differentiate it from the operator norm, denote by $\|\cdot\|_{\mathcal{I}}$ the norm on $\mathcal{L}^{\frac{p}{2}}_{F_0}$. Recall that, for any small power invariant operator ideal \mathcal{I} , we had $\|X\|_{\mathcal{I}_{F_0}} = \|X\|_{\mathcal{I}^{\frac{1}{2}}} + \|XF_0 + F_0X\|_{\mathcal{I}}$ (see Theorem 3.5.1 for this definition); since $\mathcal{I}^{\frac{1}{2}}$ is itself an invariant operator ideal, we have $\|X\|_{\mathcal{I}^{\frac{1}{2}}} \geq \|X\|$, whence $\|X\|_{\mathcal{I}_{F_0}} \geq \|X\|$.

Let $M = \sup_{t \in [0,1]} \|F'_t\|_{\mathcal{I}}$ and $N = \sup_{t \in [0,1]} \|g(F_t)\|_1$ (both these suprema are finite since $t \mapsto F'_t$ and $t \mapsto g(F_t)$ are both continuous in their respective norms). Increase M and N by a small amount if necessary to ensure that neither is zero.

Fix any $r \in [0, 1]$; we want to place an open ball around r such that in that open ball we can find upper bounds for the norm of various differences:

- Pick $\nu_r > 0$ so that, if $|s - r| < \nu_r$, $\| \frac{F_s - F_r}{s - r} - F'_r \|_{\mathcal{S}} < \frac{\varepsilon}{2N}$ (such a ν_r exists since $F'_r = \lim_{s \rightarrow r} \frac{F_s - F_r}{s - r}$, with the limit calculated in $\mathcal{AL}^{\frac{p}{2}}_{F_0}$).
- Pick $\mu_r > 0$ $|t - r| < \mu_r$ then $\|F'_t - F'_r\| < \frac{\varepsilon}{4N}$ (use the continuity of $t \mapsto F'_t$).
- Since $F \mapsto g(F)$ is continuous, we can find a α_r such that if $\|F - F_r\|_{\mathcal{S}} < \alpha_r$ then $\|g(F) - g(F_r)\|_1 < \frac{\varepsilon \cdot N}{2 \cdot (2MN + 1)}$. In particular, since $t \mapsto F_t$ is continuous, we can find ρ_r such that if $|t - r| < \rho_r$ then $\|F_t - F_r\|_{\mathcal{S}} < \alpha_r$, and so $\|g(F_t) - g(F_r)\|_1 < \frac{\varepsilon \cdot N}{2 \cdot (2MN + 1)}$.

Let $0 < \delta_r < \min\{\mu_r, \nu_r, \rho_r\}$. The open balls $B(r, \delta_r)$ form an open cover of $[0, 1]$; hence, using the compactness of $[0, 1]$, we can find an even number of points $0 = r_0 < r_1 < r_2 < \dots < r_{2n} = 1$ such that

- for $i = 0, \dots, n$, we have $B(r_{2i}, \delta_{r_{2i}})$ cover $[0, 1]$
- $\{F_t\}_{[r_{2i-1}, r_{2i+1}]} \subset B(r_{2i}, \delta_{r_{2i}})$.

Define a new path from F_0 to F_1 , denoted $\{G_t\}$, by joining consecutive F_{r_i} 's with straight-line paths; that is, $F_0 = F_{r_0} \circ \dots \circ F_{r_1} \circ \dots \circ F_{r_{2n}} = F_1$. Each $t \in [0, 1]$ lies in $[r_k, r_{k+1}]$ for some k . Then the restriction of $\{F_t\}$ to the interval $[r_k, r_{k+1}]$ is contained in one of the open balls in our collection; the open ball is centered at r_k if k is even, and at r_{k+1} if k is odd. This suggests that the cases k even and k odd need to be treated separately; however, as the difference is purely in the indices, we only show the case when k is odd. So, suppose $k + 1$ is even and equal to $2i$. By construction, $\{F_t\}_{[r_{2i-1}, r_{2i+1}]} \subset B(r_{2i}, \delta_{r_{2i}})$, and since $\delta_{r_{2i}} < \min\{\mu_{r_{2i}}, \nu_{r_{2i}}, \rho_{r_{2i}}\}$ we can perform some norm estimates.

- We want to find an upper bound for $\|G'_t\|$ and $\|G'_t - F'_t\|$. Since $|r_{2i} - r_{2i-1}| < \nu_{r_{2i}}$, $\| \frac{F_{r_{2i-1}} - F_{r_{2i}}}{r_{2i-1} - r_{2i}} - F'_{r_{2i}} \|_{\mathcal{S}} < \frac{\varepsilon}{4N}$. Now $\{G_t\}$ is piecewise linear, so for any $t \in [r_{2i-1}, r_{2i}]$ we have $G'_t = \frac{G_{r_{2i}} - G_{r_{2i-1}}}{r_{2i} - r_{2i-1}}$, which in turn is equal to $\frac{F_{r_{2i}} - F_{r_{2i-1}}}{r_{2i} - r_{2i-1}}$ (by construction, $G_{r_j} = F_{r_j}$ for all $j = 0, \dots, 2n$). It follows that $\|G'_t - F'_{r_{2i}}\|_{\mathcal{S}} = \| \frac{F_{r_{2i}} - F_{r_{2i-1}}}{r_{2i} - r_{2i-1}} - F'_{r_{2i}} \|_{\mathcal{S}} < \frac{\varepsilon}{4N}$.

On the other hand, since $|t - r_{2i}| < \mu_{r_{2i}}$ we also have $\|F'_t - F'_{r_{2i}}\|_{\mathcal{S}} < \frac{\varepsilon}{4N}$. It follows that $\|G'_t - F'_t\|_{\mathcal{S}} \leq \|G'_t - F'_{r_{2i}}\|_{\mathcal{S}} + \|F'_{r_{2i}} - F'_t\|_{\mathcal{S}} < \frac{\varepsilon}{4N} + \frac{\varepsilon}{4N} = \frac{\varepsilon}{2N}$. Applying the reverse triangle inequality to this result, we also get $\|G'_t\|_{\mathcal{S}} < M + \frac{1}{2N}$ (since $\|F'_t\|_{\mathcal{S}} \leq M$ and $\varepsilon < 1$). Finally, using the relationship between $\|\cdot\|_{\mathcal{S}}$ and the operator norm, we also get the operator norm bounds $\|G'_t\| < \frac{2NM+1}{2N}$ and $\|G'_t - F'_t\| < \frac{\varepsilon}{2n}$ for all $t \in [r_{2i-1}, r_{2i}]$.

- Find a bound for $\|g(G_t) - g(F_t)\|_1$. Since $|t - r_{2i}| < \rho_{r_{2i}}$, $\|g(F_t) - g(F_{r_{2i}})\|_1 < \frac{\varepsilon \cdot N}{2 \cdot (2MN + 1)}$. On the other hand, $G_t = (1 - t)F_{r_{2i-1}} + tF_{r_{2i}}$, so

$$\|G_t - F_{r_{2i}}\|_{\mathcal{S}} = \|(1 - t)(F_{r_{2i-1}} - F_{r_{2i}})\|_{\mathcal{S}} \leq \|F_{r_{2i-1}} - F_{r_{2i}}\|_{\mathcal{S}} < \alpha_{r_{2i}},$$

hence $\|g(G_t) - g(F_{r_{2i}})\|_1 < \frac{\varepsilon \cdot N}{2 \cdot (2MN + 1)}$. Therefore,

$$\|g(G_t) - g(F_t)\|_1 < \|g(G_t) - g(F_{r_{2i}})\|_1 + \|g(F_{r_{2i}}) - g(F_t)\|_1 < 2 \cdot \frac{\varepsilon \cdot N}{2 \cdot (2MN + 1)} = \frac{\varepsilon \cdot N}{2MN + 1}.$$

Hence, for the path $\{G_t\}$, it follows from the above inequalities that

$$\begin{aligned} \|G'_t g(G_t) - F'_t g(F_t)\|_1 &\leq \|G'_t\| \|g(G_t) - g(F_t)\|_1 + \|G'_t - F'_t\| \|g(F_t)\|_1 \\ &< \frac{2MN+1}{2N} \cdot \frac{\varepsilon \cdot N}{2MN+1} + \frac{\varepsilon}{2N} \cdot N = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which allows us to write

$$\begin{aligned} \left| \int_0^1 \tau\left(\frac{d}{dt}(G_t)\right) g(G_t) dt - \int_0^1 \tau\left(\frac{d}{dt}(F_t)\right) g(F_t) dt \right| &\leq \int_0^1 |\tau(G'_t g(G_t) - F'_t g(F_t))| dt \\ &\leq \int_0^1 \|G'_t g(G_t) - F'_t g(F_t)\|_1 dt \\ &\leq \int_0^1 \varepsilon dt = \varepsilon. \end{aligned}$$

Therefore, $\{G_t\}$ is a piecewise linear path which approximates $\{F_t\}$, as desired. \blacksquare

In particular, we can apply the above lemma to $g(F) = |1 - F^2|^q$, allowing us to show first that the result does not change when we evaluate the integral over two different piecewise linear paths, and then conclude that the integral does not depend on the path. This is the content of the lemma below.

Lemma D.0.12 α is independent of the path over which it is integrated. \blacklozenge

PROOF Consider first piecewise linear paths. Suppose $G_0 \in \mathcal{AL}_{\frac{p}{2} F_0}$. It is sufficient to show that the integral along $F_0 \circ \rightarrow F$ is the same as the integral along $F_0 \circ \rightarrow G_0 \circ \rightarrow F$ (where $\circ \rightarrow$ represents the straight-line path); induction will then give us that the integral is the same along any piecewise linear path from F_0 to F . Since α does not depend on F_0 , if we define $\psi(F) = \frac{1}{C_q} \int_0^1 \tau((F - G_0)|1 - G_t^2|^q) dt$ (where $G_t = G_0 + t(F - G_0)$ for $t \in [0, 1]$), it follows as above that $d\psi = \alpha$ as well. But then $d(\theta - \psi) = 0$, so $\theta - \psi$ is a constant, say $(\theta - \psi)(F) = C$ for any F . In particular, using $F = G_0$ we get $C = \theta(G_0)$ (since $\psi(G_0) = 0$). Hence, in general, $\theta(F) = \psi(F) + \theta(G_0)$. But $\theta(F)$ is the integral along the straight path from F_0 to F , $\theta(G_0)$ is the integral along the straight-line path from F_0 to G_0 , and $\psi(F)$ is the integral along the straight-line path from G_0 to F .

Now consider any path from F_0 to F . By Lemma D.0.11, we can find a piecewise linear path such that the integral from F_0 to F along the given path is within ε of the integral from F_0 to F along the piecewise linear path. Since the value of the integral is the same for any piecewise linear path, by letting ε go to zero it follows that the integral along $\{F_t\}$ must be equal to the integral along any piecewise linear path. This proves the desired result. \blacksquare

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